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An Approach to Social Choice Theory

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Abstract

This work is an overview of Social Choice Theory. Social Choice Theory is a theoretical framework to analyse the combination of individual preferences and interests of a collectivity to reach a collective decision. For example, we could have a group of people choosing between budgets, or electing a candidate for an organization.

We deep dive into Arrow and Gibbard-Satterthwaite theorems, which give us two impossibility results. They give some conditions for which it is impossible to implement a "reasonable" voting system. We also look at the simple majority rule and single-peaked preferences. That gives a sense of the conditions we should have so the voting system provides a fairer output.

Resum

Aquest treball és una introducció a la Social Choice Theory. Un marc teòric on analitzar la combinació de preferencies, el benestar i els interessos entre individus per a arribar a prendre una decisió col·lectiva. Per exemple, un grup de persones que ha de decidir quin presupost aprovar o quin candidat escollir per a representar una organització.

Profunditzem en els teoremes d'Arrow i de Gibbard-Satterthwaite, els quals ens caracteritzen dos resultats d'impossibilitat. Ens donen unes condicions per a les quals no podem implementar un sistema de votació que sigui "raonable". També estudiem breument la regla de majoria simple i les preferencies "single-peak" (de pic únic). Amb això, tindrem un conjunt de condicions que hauriem de exigir al sistema per tal que les votacions tinguin un resultat més just.

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Chapter 0

Introduction

0.1 Brief look into the past

According to McLean and London (1990) [8], the roots of the theory of social choice bring us back to the thirteenth century. There, Ramon Llull[1] designed two voting procedures that are similar to what is now known as the Borda count and the Condorcet procedure.

Lull proposed a method consisting of exhaustive pairwise comparisons; each candidate is compared to every other candidate under consideration. Lull advocated the choice of the candidate who receives the highest number of votes in the aggregate of the pairwise comparisons. This procedure is identical to a method suggested by Borda in 1770 which, as was demonstrated by Borda (1781) himself.

Lull devised a second procedure in 1299. He published it in his treatise *De Arte Eleccionis*. A successive voting rule is proposed that ends up with a so-called Condorcet winner, if there exists one. However, this method does not necessarily detect possible cycles, since not every logically possible pairwise comparison is made in determining the winner.

Also, McLean and London also refer to the Letters of Pliny the Younger (around ad 90) who described secret ballots in the Roman senate. Pliny, in one of his letters, discussed a case of manipulation of preferences in a voting situation.

There is evidence that Nicolaus Cusanus (1434) had studied *De Arte Eleccionis* so that he knew about Lull's Condorcet procedure of pairwise comparisons. However, Cusanus rejected it and proposed a Borda rank-order method with se-

[1]Ramon Llull, 1232-1315, was a mathematician, philosopher, logician, Franciscan tertiary and writer from the Kingdom of Majorca. Most of his recognition comes from writing the first major work of Catalan literature. Some manuscripts found recently, shown that he did significant progress on election theory, several centuries before what was thought it started. By his influence on Leibniz, he is considered somehow as a pioneer on computation theory.

cret voting instead; secret voting because, otherwise, there would be too many opportunities and incentives for strategic voting. McLean and London indicate that Cusanus rejected Lull's Condorcet method.

In 1672 Pufendorf published his magnum opus *De Jure Naturae et Gentium* (The Law of Nature and of Nations) where he discussed, among other things, weighted voting, qualified majorities and, very surprisingly, a preference structure that in the middle of the twentieth century has become known as single-peaked preferences. Pufendorf was also very much aware of manipulative voting strategies. He mentioned an instance of manipulation of agendas, reported by the Greek historian Polybios, which was similar to the one discussed by Pliny, but considerably earlier in time.

Much better known than the writings of Lull, Cusanus, and Pufendorf is the scientific work by de Borda (1781) and the Marquis de Condorcet (1785). Condorcet strongly advocated a binary notion, i.e. pairwise comparisons of candidates, whereas Borda focused on a positional approach where the positions of candidates in the individual preference orderings matter. Condorcet extensively discussed the election of candidates under the majority rule, and he was probably the first to demonstrate the existence of cyclical majorities for particular profiles of individual preferences.

0.2 So, what is Social Choice Theory?

Social choice theory is the analysis of collective decision making. From Gaertner (2006) [6], we see that the theory of social choice starts from the values of the agents of a society. We assume they have a preference profile over the possible states of the society. For example, we can think of profiles as a group of candidates of an election, but also a set of possible budgets for a project, or the allocation of costs of a public good. When faced to a set of alternatives, each agent chooses a subset of it. And then, they attempt to derive a collective preference ordering, which we will call *social welfare function* (SWF). Finally, we get the most preferred alternative given by that function, which is the outcome of the *social choice function* (SCF). The main difference is that a SCF chooses one or some winners, or maybe no winner, whereas the SWF chooses a preference profile. Obviously, there is a relationship between SCF and SWF.

But, leaving in an era where the market is the predominant mechanism, maybe it's superfluous to invest time understanding collective decision making. Far than that, we have many examples in which decision making is done collectively (and not through the market). Think of decisions on defence outlays, investments in education or health-care. We can think also in the election of candidates for an organization (a government or a club for example). Those decisions are a fundamental part of our modern society. Also, consider the impact of externalities such as air pollution or noise. So we have to internalise them through policy measures. Often, these measures will be decided collectively (probably within a committee).

As we know, these decisions are complicated as they may affect in favour and against some group or another. So, how can such decisions be made in a transparent and rational way? Is there a handy criterion to face that problem? That is what we will try to address in the following pages.

Structure of the work

In the first chapter we introduce three basic and key concepts in social choice theory. These are: the *choice rule*, the *social welfare function* (SWF) and the *social choice function* (SCF). We see also how the two last concepts relate each other.

On the second and fourth chapters we present two important impossibility results. Arrow and Gibbard–Satterthwaite theorems. On Arrow's part, besides stating the theorem, we also give three different proves. One may think that with one single proof it's enough. However we wanted to do them as in each one we are capable of highlighting different aspects of the result.

Then, we get that the conclusion of Arrow's theorem is that, under certain conditions we can not design a voting system that satisfies some "rational" criteria. For that, we wanted to add another chapter, the third one, in order to show some possibility results. We try to see which are the basic properties that a social aggregation rule (used in real life) should reasonably satisfy. Mainly, we will take a look at the absolute majority rule and the single peak preferences.

Chapter 1

Social Choice Theory

In this chapter we introduce three concepts, the choice rule, the social welfare function (SWF) and the social choice function (SCF). First we analyze preferences and how the individuals select their preferred alternative. Next we define the social welfare function, that gives for a preference profile of the society which is the aggregated choice. We give some properties that are important in the sequel. Finally we define the social choice function and some procedures to select one (or more in the case of ties) alternative that the society values at most.

Just as a preliminary for the reader, these will help us to go the the next chapter, the Arrow's impossibility theorem, in which we widely use the social welfare function. And one of the last chapters, the Gibbard–Satterthwaite theorem, in which we use the social choice function.

We can take as an example the representative democracy. There, the agents of the society have preferences among the candidates, their *choice rule*. Then, there's a voting mechanism that takes all preferences of agents and derives in a collective preference ordering, the *social welfare function*, which normally ends up with a final elected candidate, which is indeed the outcome of a *social choice function*.

1.1 Preferences and choice rule

We will follow Gaertner (2006) [6] to take a formal approach to what we just introduced.

Let $X = \{x, y, z, \ldots\}$ denote the set of all conceivable social states. To put two examples, those conceivable social states can be a group of candidates of an election or different budgets for a project. Let $N = \{1, \ldots, n\}$ denote a finite set of individuals, with $n \geq 2$. Let now denote R for the preference relation on X, so R is a subset of ordered pairs of the product $X \times X$, for example xRy. Now, when we speak about individual i's preference, $i \in N$, we just write R_i. A binary relation gives just a subset of the ordered pairs of the product $X \times X$.

Definition 1.1. (Preference relation). *We can call R a* preference relation *on X if R holds the following properties:*

i) R is reflexive *if $\forall x \in X$, then xRx.*

ii) R is complete *if $\forall x, y \in X, x \neq y$, then $xRy \lor yRx$.*

iii) R is transitive *if $\forall x, y, z \in X$ such that $xRy \land yRz$, then xRz.*

We will read xRy as "x is at least as good as y". Let's now give two related relations to a preference relation R.

Definition 1.2. (Strict preference relation). *We call P the* strict preference relation *(the asymmetric part of R) if $\forall x, y \in X, xPy$ if and only if $xRy \land \neg yRx$.*

We will read xPy as "x is strictly preferred to y".

Definition 1.3. (Indifference relation). *We call I the* indifference relation *(the symmetric part of R) if $\forall x, y \in X, xIy$ if and only if $xRy \land yRx$.*

To end with, we will say that R is *acyclical* if, $\forall\{x_1, \dots, x_n\}$ finite sequences from X, it is not the case that $x_1Px_2 \ \land \ x_2Px_3 \ \land \cdots \land \ x_{n-1}Px_n \ \land \ x_nPx_1$. Also, let us add here the definitions of partial and total orders.

Definition 1.4. (Order/Partial order). *We say that a binary relation R is an* order *if it holds the following properties:*

i) R is reflexive, *$\forall x \in X$, then xRx.*

ii) R is antisymmetric, *$\forall x, y, \in X$ such that $xRy \land yRx$, then $x = y$.*

iii) R is transitive, *$\forall x, y, z \in X$ such that $xRy \land yRz$, then xRz.*

Whenever this properties hold, but not completeness, we speak of a *partial order*.

Definition 1.5. (Complete). *We will say that R is* complete *if $\forall x, y \in X$, we have xRy, yRx or both.*

Definition 1.6. (Total order). *We say that a preference relation R is a* total order *if it holds the properties of a partial order and we have that R is complete. This means that $\forall x, y \in X, xRy \ \lor \ yRx$.*

Definition 1.7. (Maximal set). *An element $x \in S$ is a* maximal element *of S with respect to a preference relation R if and only if $\nexists\, y \in S$ such that yPx. The set of maximal elements in S will be called its* maximal set, *denoted by $M(S, R)$.*

The maximal elements of a set S with respect to a binary relation R are those elements which are not dominated via the strict preference relation P by any other element in S.

Definition 1.8. (Best element). *An element $x \in S$ is a* best element *of S with respect to a binary relation R if and only if $\forall y \in S, xRy$ holds.*

Definition 1.9. (Choice set). *The set of best elements in S will be called its* choice set, *denoted by $C(S, R)$.*

Best elements of a set S have the property that they are at least as good as every other element of S with respect to the given relation R. Notice that a best element is always a maximal element, but the opposite does not hold. For example let's consider the set $S = \{x, y\}$ and neither xRy nor yRx holds[1]. Now, both elements x, y are *maximal elements* of the given set, but none of them is a *best element*. Then we can easily see that

$$C(S, R) \subset M(S, R).$$

Another interesting situation to recall is that both $C(S, R)$ and $M(S, R)$ may be empty subsets. Consider the case[2] in which xPy, yPz, and zPx. Here, there is neither a best element nor a any element that is not dominated by some other via the relation P. Lastly, recall that if S is finite and R is an ordering, it is always the case that $C(S, R) = M(S, R) \neq \emptyset$.

Now, following Mas-Colell et al. (1995) [7], we see that we can represent choice behavior by means of a *choice structure*. A choice structure $(K, C(\cdot))$ consists of two ingredients:

- K is a family of non-empty subsets of X. That is, every element of K is a set $K \subset X$. The sets in K should be thought of as an exhaustive listing of all the choice experiments that the institutionally, physically, or otherwise restricted social situation can conceivably pose to the decision maker. It need not, however, include all possible subsets of X.
- Let X be a set of feasible alternatives and let $\mathcal{K}$ be the set of all non-empty subsets of X. A *choice rule* $C : \mathcal{K} \mapsto \mathcal{K}$ assigns a non-empty subset $C(S)$ of S to every $S \in K$. When $C(S)$ contains a single element, that element

[1] Here the completeness property is not satisfied.

[2] Here we have that P is *cyclical*. Example 1.12 shows such a situation.

is the individual's choice from among the alternatives in S. The set $C(S)$ may, however, contain more than one element. When it does, the elements of $C(S)$ are the alternatives in S that the decision maker *might* choose. That is, they are her *acceptable alternatives* in S. In this case, the set $C(S)$ can be thought of as containing those alternatives that we would actually see chosen if the decision maker were repeatedly to face the problem of choosing an alternative from set S.

We finally consider two consistency conditions of choice.

i) **(Contraction consistency).** For all $x \in S \subseteq T$, if $x \in C(T)$, then $x \in C(S)$.

ii) **(Expansion consistency).** For all $x, y \in S$, if $x, y \in C(S)$ and $S \subseteq T$, then $x \in C(T)$ if and only if $y \in C(T)$.

Example 1.10. Suppose that $X = \{x, y, z\}$ and $K = \{\{x, y\}, \{x, y, z\}\}$. One possible choice structure is $(K, C_1(\cdot))$, where the choice rule $C_1(\cdot)$ is: $C_1(\{x, y, z\}) = \{x\}$. Then, it's clear that we will have $C_1(\{x, y\}) = \{x\}$. In this case, we see x chosen no matter what budget the decision maker faces. And this is which the *contraction consistency* tries to express.

1.2 Social Welfare Function

We introduce an important concept, the social welfare function. What it does is take as an input the preferences of agents and give as a result another preference relation, which is supposed to be the one that fits the most the preferences of society.

Let $\mathcal{E}$ denote the set of preference orderings on X, and let $\mathcal{E}'$ stand for a subset of orderings that satisfies a particular restriction. $\mathcal{E}'^n$ will denote the cartesian product $\mathcal{E}' \times \cdots \times \mathcal{E}'$, n-times. An element of $\mathcal{E}'^n$ is an n-tuple of preference orderings $(R_1, \ldots, R_n)$ or the profile of an n-member society consisting of preference orderings. Now, with the notation just defined, we have the following definition.

Definition 1.11. (Social welfare function): *A* social welfare function *in the sense of Arrow is a mapping such that*

$$f : \mathcal{E}'^n \mapsto \mathcal{E}'.$$

In other words, a *social welfare function* takes the preferences of the agents of a society and gives us as an outcome one preference ordering. In general, this preference ordering that we have as an output is of the same type of the preference ordering of the society. However, if could be a different one. For example, we could find out that agents are not indifferent between their alternatives, but the output of the *social welfare function* has some indifference. We use xRy for two alternatives $x, y \in X$ to denote $xf(R_1, \ldots, R_n)y$.

Let's give now some of the desired properties for the social welfare function:

i) **Unrestricted domain (*U*).** The domain of the mapping f defined above includes all logically possible n-tuples of individual orderings on X. That is $\mathcal{E}' = \mathcal{E}$.

   This condition requires not to exclude any individual preference ordering a priori, even the most odd ones.

ii) **Weak Pareto principle (*P*).** For any $x, y \in X$, if everyone in society strictly prefers x to y, that is $xP_iy\ \forall i \in N$, then xPy.

   This states that if all individuals strictly prefer x to some other alternative y, then the same should hold for the society's preference.

iii) **Independence of irrelevant alternatives (*I*).** Given two profiles of individual orderings $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$, and in them every individual in society has exactly the same preference with respect to two alternatives x, y. Then, the social preference given by $f(R_1, \ldots, R_n)$ or $f(R'_1, \ldots, R'_n)$ with respect those two alternatives must be the same for the two profiles.

   What this condition states is that if society has to take a decision with respect to some pair of alternatives (x, y), only the individuals' preferences with respect to this pair should be taken into consideration, and no more.

iv) **Non-dictatorship (*D*).** There is no individual in society such that for all profiles in the domain of f and for all pairs of alternatives $x, y \in X$, if xP_iy then xPy.

   This states that there should be no individual in society whose preferences end up being the society preferences. This person would have dictatorial power and we want to exclude that.

Let's give an example of an implementation of a social welfare function, the Condorcet winner. This will serve us to show also the problems we may face when trying to reach a final desired outcome. This Condorcet paradox is a situation in which collective preferences can be cyclic, even if the individual preferences are not. Which is something one may not expect.

Example 1.12. (Condorcet[3] Paradox): Let's suppose we have three candidates, A, B, C, this is our set X of *feasible alternatives* is given by $X = \{A, B, C\}$. Let's suppose we have three voters, our *agents*, who reveal their preferences,

Preferences	Voter 1	Voter 2	Voter 3
1st	A	B	C
2nd	B	C	A
3rd	C	A	B

The preferences of the voters are orderings. The Condorcet procedure asks the voter to choose between only 2 alternatives. The society's preferences are given by how many of the voters favor one alternative over the other. Then,

i) We could argue that ARB since voters 1 and 3 prefer that.

ii) We could argue that BRC since voters 1 and 2 prefer that.

iii) We could argue that CRA since voters 2 and 3 prefer that.

Now we could have as an outcome of our social welfare function: ARB $\wedge$ BRC $\wedge$ CRA. So we get a cycle. It's interesting to see here the expectation that the *transitivity* on the individual preferences should imply *transitivity* on the societal preferences. But here this does not hold. This situation is called fallacy of composition[4].

In this example, there's no any pair of alternatives for which all the agents have the same preference, for that reason we cannot say anything about the *weak Pareto principle*. Also, we can see that there's no agent whose preferences end up being the society preferences no matter the other agents' preferences. So *non-dictatorship* property holds.

Example 1.13. (Borda[5] Count): This is a voting method, developed in 1770. Here, we assign points, our *preferences*, to candidates that are ranked by the electors. The points assigned to candidates are based on their ranking. The point values for all ballots are totaled, and the result is the list of candidates sorted by their

[3]Marquis of Condorcet, (1743 – 1794), known as Nicolas de Condorcet, was a French philosopher and mathematician. His ideas, including support for a liberal economy, free and equal public instruction, constitutional government, and equal rights for women and people of all races, have been said to embody the ideals of the Age of Enlightenment and Enlightenment rationalism.

[4]Arises when one infers that something is true of the whole from the fact that is true for some part of the whole.

[5]Jean-Charles, chevalier de Borda, (1733 – 1799), was a French mathematician and physicist.

number of points, the outcome of our *social welfare function*. Finally, the winner is the candidate with the highest number of points.

1.3 Social Choice Function

As we were saying at the beginning of this chapter, we first have the *choice rule* which are the preferences of an agent over the different alternatives that he/she may have. Later, we have the *social welfare function*, which from the preferences of a set of agents, give us as an outcome, one preference ordering. Which is supposed to be the one that fits the most the overall preferences of agents. Lastly, we have the *social choice function*, which from that outcome from the social welfare function or from the choice rule of every agent, gives us the most preferred alternative for the society (in the case of the social welfare function) or for every agent (in case of the choice rule). Let's proceed now with Taylor and Pacelli (2009) [10].

Let's denote a finite set of alternatives by X. $\sum$ stands for the set of all strict linear orderings (or total ordering) on X. Let there be n individuals. A member of $\sum^n$ is called a profile of linear orderings with its ith component being individual i's strict ranking that we shall denote by P_i. Now, with this notation we define the social choice function.

Definition 1.14. (Social choice function) *A correspondence $h : \sum^n \mapsto X$, will be called a* social choice function.

This correspondence selects a subset of X, eventually empty. This subset $h(P_1, \ldots, P_n)$ is called the *social choice set.*

Let's give now some of the properties that we can find in on our social choice functions. A social choice function $h : \sum^n \mapsto X$ is:

i) **Pareto efficient** if whenever alternative a is at the top of every individual i's ordering $P_i \quad \forall i \in N$, then $h(P_1, \ldots, P_n) = a$.

   If we were not allowing ties, we could have said "the" social choice instead of "a" social choice in the statement of the Pareto condition. With ties, however, what we are saying is that if everyone finds x strictly preferable to y (recall that we are not allowing ties in the individual preference lists), then alternative y should not be the social choice and should not even be among the social choices if there is a tie.

ii) **Monotonic** if whenever $h(P_1, \ldots, P_n) = a$ and for every individual i and every alternative b the ordering P'_i ranks a above b if P_i does, then $h(P'_1, \ldots, P'_n) = a$.

The intuition behind the monotonicity condition is that if x is the social choice and someone changes his or her list in a way that is favorable to x (but not favorable to any other alternative) then x should remain the social choice. Monotonicity has also been called ”nonperversity” in the literature. Indeed, a social choice procedure that is not monotone might well be regarded as perverse.

iii) **Independence of irrelevant alternatives** if the social choice set includes x but not y, and one or more voters change their preferences, but no one changes his or her mind about whether x is preferred to y or y to x, then the social choice set should not change so as to include y.

The point here is that if a preference list is changed but the relative positions of x and y to each other are not changed, then the new list can be described as arising from upward and downward shifts of alternatives other than x and y. Changing preferences toward these other alternatives should, intuitively, be irrelevant to the question of social preference of x to y or y to x.

Of course, if we start with x a winner and y a non-winner, and people move some other alternative z around, then we cannot hope to conclude that x is still a winner. After all, everyone may have moved z to the top of their list. In this case, independence of irrelevant alternatives is simply saying that y should remain a non-winner.

iv) **Dictatorial** if there exists an individual i such that $h(P_1, \ldots, P_n) = a$ if and only if a is at the top of i's ordering P_i.

Let's give now some examples from Taylor and Pacelli (2009) [10] to have different possible social choice functions. The first one is the Plurality voting.

Example 1.15. (Plurality voting): Plurality voting is the social choice procedure that most directly generalizes the idea of simple majority vote. Both from the easy case of two alternatives to the complicated case of three or more alternatives. The idea is simply to declare as the social choice(s) the alternative(s) with the largest number of first-place rankings in the individual preference lists.

This is a procedure widely used, for example in the election of the Mexican president.

Another example we can look at is the Hare procedure, introduced by Thomas Hare in 1861. In 1862, John Stuart Mill spoke of it as being ”among the greatest improvements yet made in the theory and practice of government”. Today, it is used to elect public officials in Australia, Malta, the Republic of Ireland, and Northern Ireland.

Example 1.16. (Hare procedure): This social choice procedure is also known by names such as the ”single transferable vote system” or ”instant runoff voting”.

The Hare system is based on the idea of arriving at a social choice by successive deletions of less desirable alternatives. More precisely, the procedure is as follows. We begin by deleting the alternative or alternatives occurring on top of the fewest lists. At this stage we have lists that are at least one alternative shorter than that with which we started. Now, we simply repeat this process of deleting the least desirable alternative or alternatives (as measured by the number of lists on top of which it, or they, appear). The alternative(s) deleted last is declared the winner.

Notice that if, at any stage, some alternative occurs at the top of more than half the lists, then that alternative will turn out to be the unique winner. However, an alternative occurring at the top of exactly half the lists -even if it is the only one doing so- is not necessarily the unique winner (although it must be among the winners).

Now, we would like to present an example of a situation having agents, alternatives and preferences and see which would each social choice function presented bring as an output.

Example 1.17. Let's suppose we have 5 alternatives, a, b, c, d and e. Let's suppose we have 7 people who have individual preference lists as follows:

a	a	a	c	c	b	e
b	d	d	b	d	c	c
c	b	b	d	b	d	d
d	e	e	e	a	a	b
e	c	c	a	e	e	a

Table 1.1

For each of our procedures presented, we shall calculate what the resulting social choice is.

Condorcet's method: If we look at a one-on-one contest between alternatives a and b, we see that a occurs over b on the first three ballots and b occurs over a on the last four ballots. Thus, alternative b would defeat alternative a by a vote of 4 to 3 if they were pitted against each other. Similarly, alternative b would defeat alternative c (4 to 3, again) and alternative e (6 to 1). Thus, we have so far determined that neither a nor c nor e is the winner with Condorcet's method. But alternative d would defeat alternative b by a score of 4 to 3, and so b is not a winner either. This leaves only alternative d as a possibility for a winner. But alternative c handily defeats alternative d (5 to 2) and so d is also a non-winner. Hence, there is no winner with Condorcet's method.

Plurality: Since a occurs at the top of the most lists (three), it is the social choice when the plurality method is used.

Borda count: One way to find the Borda winner is to actually make a vertical column of values 4, 3, 2, 1, 0 to the left of the preference rankings. (Another way is to count the number of symbols occurring below the alternative whose Borda score is being calculated.) For example, alternative a receives a total of 14 points in the Borda system: four each for being in first place on the first three lists, none for being in last place on the fourth list and the seventh list, and one each for being in next to last place on the fifth and sixth lists. (Or, scanning the columns from left to right, we see that the number of symbols below a is $4 + 4 + 4 + 0 + 1 + 1 + 0$.) Similar calculations, again left for the reader, show that b gets 17 points, c and d each gets 16 points, and e gets only 7 points. Thus, the social choice is b when the Borda count is used.

Hare system: We decide which alternative occurs on the top of the fewest lists and delete it from all the lists. Since d is the only alternative not occurring at the top of any list, it is deleted from each list leaving the following:

a	*a*	*a*	*c*	*c*	*b*	*e*
b	*b*	*b*	*b*	*b*	*c*	*c*
c	*e*	*e*	*e*	*a*	*a*	*b*
e	*c*	*c*	*a*	*e*	*e*	*a*

Table 1.2

Here, b and e are tied, each appearing on top of a single list, and so we now delete both of these from each list leaving the following:

a	*a*	*a*	*c*	*c*	*c*	*c*
c	*c*	*c*	*a*	*a*	*a*	*a*

Now, a occurs on top of only three of the seven lists, and thus is eliminated. Hence, c is the social choice when the Hare system is used.

From the previous pages we have at hand four social choice procedures (Condorcet's method, plurality, Borda, Hare) and three properties (Pareto, monotonicity, and independence of irrelevant alternatives) pertaining to such procedures.

Which procedures satisfy which properties? The answer is given in the following table (where a ”yes” indicates the property holds for the given procedure).

	Pareto	Mono	IIA
Condorcet	Yes	Yes	Yes
Plurality	Yes	Yes	
Borda	Yes	Yes	
Hare	Yes		

Table 1.3

Now, we will prove some of this positive results of the Table 1.3. Indeed, we will prove one result for each social choice procedures presented. For the results we don’t prove, the reader can look at Taylor and Pacelli (2009) [10].

Proposition 1.18. *Condorcet’s method satisfies independence of irrelevant alternatives.*

Proof. Assume the social choice procedure being used is Condorcet’s method and that we have an arbitrary sequence of individual preference lists yielding x as a winner and y as a non-winner. Thus x defeats every other alternative in a one-on-one contest. Now suppose that preference lists are changed but no one changes his or her mind about whether x is preferred to y or y to x. We want to show that y is not among the social choices, which simply means that y does not defeat every other alternative in a one-on-one contest. But because no one who had x over y changed this to y over x, we still have y losing to x in a one-on-one contest. Hence, y is not a social choice using Condorcet’s method. □

Proposition 1.19. *The plurality count satisfies monotonicity.*

Proof. Assume the social choice procedure being used is the plurality procedure and assume that we have an arbitrary sequence of preference lists yielding x as a social choice. Now assume that someone exchanges x’s position with that of the alternative above x on his or her list. We want to show that x is still a social choice. But since x was originally a social choice, x was at least tied for being on top of the most lists. The change in the single list described above neither decreases the number of lists that x is on top of nor increases the number of lists that any other alternative is on top of. Thus, x is still among the social choices. □

Proposition 1.20. *The Borda count satisfies monotonicity.*

Proof. Assume the social choice procedure being used is the Borda count and assume that we have an arbitrary sequence of preference lists yielding x as a social choice. Now assume that someone exchanges x's position with that of the alternative above x on his or her list. We want to show that x is still a social choice. But the change in the single list described above simply adds one point to x's total, subtracts one point from that of the other alternative involved, and leaves the scores of all the other alternatives unchanged. Thus, x is still a social choice. □

Proposition 1.21. *The Hare system satisfies the Pareto condition.*

Proof. Assume the social choice procedure being used is the Hare system and assume that we have an arbitrary sequence of preference lists where everyone prefers alternative x to alternative y. We must show that y is not a social choice. Notice again that y is not on top of any list. Thus, y is among the alternatives immediately deleted, since it occurs at the top of no lists and that is as few as you can get. This shows that y is not a social choice. □

Relationship between SCF and SWF

Once explained what the *social choice function* and *social welfare function* are, let's see the relationship between the two concepts. We will follow Taylor and Pacelli (2009) [10].

Proposition 1.22. *Every SWF gives rise to a SCF (for that choice of voters and alternatives). Moreover , every social choice procedure that always produces a winner gives rise to a social welfare function.*

Proof. Having a function that produces as output a listing of all the alternatives, we can take whichever alternative or alternatives are at the top of the list as the social choice. So, it's clear that every SWF give us a SCF for that choice of voters and alternatives. Now, let's face the other part of the proof.

Let's suppose we have a SCF. Then, this procedure give us the most preferred alternatives. But we need the listing of all the alternatives, not only the winners. So we apply again the SCF, but taking out from the domain those winners we got. Then the output of the SCF will be again a winner (or winners), which is (are) indeed the one (ones) of the second place in the order we are looking for (the order that the SWF would give us). Iterating this process, we get the entire listing we were looking for. That listing is the one that the SWF would give us. Therefore, we see that the SWF gives as SCF and vice versa. □

Chapter 2

Arrow's Impossibility Theorem

In 1951, Kenneth Arrow[1] showed the general impossibility of the existence of a social welfare function. He showed that result in his doctoral thesis while later publishing it in his 1951 book Social Choice and Individual Values, Arrow (1951) [1]. However, some welfare economists were confused with the idea of Arrow. Bergson and Samuelson, for example, had works in which they tried to prove that the result was possible.

In this chapter, we will state Arrow's theorem. Later we will give three different proofs that will show us different aspects within his impossibility result. All this results will be based on Gaertner (2006) [6].

2.1 Introduction

The theorem states that when agents have three or more alternatives, there is no ordinal voting system that can convert the ordered preferences of those agents into a social order, complete and transitive, also meeting a specified set of criteria.

We are going to take a look to few examples, which we are also going to take most of the results from. Let's consider we have n members of a society which are constantly expressing their preferences and there is one of those members that has preferences which society view as unacceptable or strange. So if that member prefers alternative a to b, then the society prefers b to a. Let's suppose now that given alternatives c, d there's complete unanimity on the preferences, let's say they prefer c to d. Now the question is, should society prefer d to c? But this would

[1]Kenneth Joseph Arrow, (1921 – 2017), was an American economist, mathematician, writer, and political theorist. He was the joint winner of the Nobel Memorial Prize in Economic Sciences with John Hicks in 1972 for pioneering contributions to general equilibrium theory and welfare theory.

violate one of the basic properties we want to have, the weak Pareto principle (defined on the following lines).

Another aggregation rule could say that when an alternative a is considered by the society, then a should be preferred among other alternatives. But if one requires this rule to be applied at any given set of individual preferences, then we will have another clash with the Pareto principle.

Conditions and Arrow's theorem

What Arrow's result says is that there does not exist a social welfare function if the mapping $f(R_1, \ldots, R_n)$ satisfies the following four conditions (stated in the previous chapter).

i) **Unrestricted domain (U).** The domain of the mapping f defined above includes all logically possible n-tuples of individual orderings on X. That is $\mathcal{E}' = \mathcal{E}$.

ii) **Weak Pareto principle (P).** For any $x, y \in X$, if everyone in society strictly prefers x to y, then xPy.

iii) **Independence of irrelevant alternatives (I).** Given two profiles of individual orderings $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$, and every individual in society has exactly the same preference with respect to two alternatives x, y, then the social preference with respect those two alternatives must be the same for the two profiles.

iv) **Non-dictatorship (*D*).** There is no individual in society such that for all profiles in the domain of f and for all pairs of alternatives $x, y \in X$, if xP_iy then xPy.

Arrow set these four conditions on f in order to express the "doctrines of citizens" sovereignty and rationality in a very general form.

Theorem 2.1. (Arrow's general possibility theorem (1951)). *For a finite number of individuals and at least three distinct social alternatives, there is no social welfare function f satisfying the conditions $(U), (P), (I)$ and (D).*

2.2 The original proof of Arrow's Theorem

This proof follows closely Arrow's own proof from the 1963 edition of his book. There we can see in a transparent way how decisiveness over some pair of social alternatives spreads to decisiveness over all pairs of alternatives (which belong to a finite set of alternatives). This phenomenon has sometimes been called a *contagion property*. We are going to see the definition of these concepts in the following lines.

In order to successfully prove the result, we have to start with two definitions that will help us on the argument.

Definition 2.2. (Almost decisive). *A set of individuals $V \subset N$ is* almost decisive *for (some) x against (some) y if, whenever $xP_iy, \forall i \in V$ and $yP_ix, \forall i \notin V$, x is socially preferred to y, i.e. xPy.*

Definition 2.3. (Decisive). *A set of individuals $V \subset N$ is* decisive *for some x against some y if, whenever $xP_iy, \forall i \in V, xPy$.*

We will denote the set of individuals that is *almost decisive* for x against y by $D(x,y)$. And the set of individuals that is *decisive* for x against y by $\overline{D}(x,y)$. Recall that $\overline{D}(x,y) \Rightarrow D(x,y)$, that is if a set of individuals is decisive (of x against y) then this set is almost decisive.

Now, we state and prove a Lemma that will help us on the proof of Arrow's theorem.

Lemma 2.4. *If there is some individual $j \in N$ who is almost decisive for some ordered pair of alternatives $\{x,y\}$, an Arrovian social welfare function f satisfying conditions (U), (P) and (I) implies that j must have a dictatorial power.*

Proof. To prove this lemma we will take two steps which we later iterate to consider all the different cases and finally demonstrate the statement of the lemma.

First Step: Assume that j is *almost decisive* for some x against another alternative y, that is $j \in D(x,y)$. Consider now another alternative $z \in X$ and let $i \in N$ denote any other agent, $\forall i \in N \setminus \{j\}$. Recall that with condition (U) we are free to choose any of the logically possible preference profiles for this society, and suppose

$$\begin{aligned} &xP_jy, \quad yP_jz \quad \textit{and} \\ &yP_ix, \quad yP_iz \quad \forall i \in N \setminus \{j\}. \end{aligned}$$

Since j is almost decisive, j is $D(x,y)$ we obtain xPy. Because yP_jz and yP_iz, the weak Pareto principle implies yPz. For how f is defined, by transitivity we have that xPy and yPz imply xPz. We have assumed that yP_ix and yP_iz, but according to condition (I), these preferences don't play any role in the social decision between x and z. So then, xPz has to be a consequence of xP_jz alone, regardless the other orderings. But this makes that j is decisive for x against z and we obtain

$$D(x,y) \rightarrow \overline{D}(x,z).$$

Second Step: Now we consider that j is almost decisive for x over y (i.e. $j \in D(x,y)$) too and the following preferences in society

$$\begin{aligned} &zP_jx, \quad xP_jy \quad and \\ &zP_ix, \quad yP_ix. \quad \forall i \in N \setminus \{j\}. \end{aligned}$$

From $D(x,y)$ we get xPy, and from the condition (P) we get zPx. And because of the transitivity we have zPy. Similarly to the logic used on the first step and with the independence condition we get that zPy has to be a consequence of zP_jy alone, and we obtain:

$$D(x,y) \rightarrow \overline{D}(z,y).$$

To successfully prove the *contagion phenomenon*[2] we have mentioned at the beginning of this section we will argue via permutations of alternatives. We will do it in schematic way. So we will write $x \rightarrow y$ if x is preferred to y, and $x \leftarrow y$ if y is preferred to x. A generic agent i stands for all agents different from j.

1. $j : x \rightarrow y \rightarrow x$

$$xPy, yPz \rightarrow xPz$$

   $i : x \leftarrow y \rightarrow z$

$$D(x,y) \rightarrow \overline{D}(x,z) \rightarrow D(x,z)$$

2. $j : z \rightarrow x \rightarrow y$

$$zPx, xPy \rightarrow zPy$$

   $i : z \rightarrow x \leftarrow y$

$$D(x,y) \rightarrow \overline{D}(z,y) \rightarrow D(z,y)$$

3. $j : y \rightarrow x \rightarrow z$

$$yPx, xPz \rightarrow yPz$$

   $i : y \rightarrow x \leftarrow z$

$$D(x,z) \rightarrow \overline{D}(y,z) \rightarrow D(y,z)$$

[2]That decisiveness over some pair of social alternatives spreads to decisiveness over all pairs of alternatives.

4. $j : y \to z \to x$

$$yPz, zPx \to yPx$$

$i : y \leftarrow z \to x$

$$D(y,z) \to \overline{D}(y,x) \to D(y,x)$$

5. $j : z \to y \to x$

$$zPy, yPx \to zPx$$

$i : z \to y \leftarrow x$

$$D(y,x) \to \overline{D}(z,x) \to D(z,x)$$

6. $j : x \to z \to y$

$$xPz, zPy \to xPy$$

$i : x \leftarrow z \to y$

$$D(x,z) \to \overline{D}(x,y) \to D(x,y).$$

Recall that the first two iterations are the ones shown on the two steps that we managed with detail.

With the scheme we have shown that starting from $D(x,y)$, individual j is decisive (and therefore almost decisive) for every ordered pair from the triple of alternatives $\{x, y, z\}$, having the conditions (U), (P) and (I) (Unrestricted domain, Weak Pareto principle, Independence of irrelevant alternatives). Then, we have that the individual j is a dictator for any three alternatives that contain x and y.

Now, what happens with cases that have more than three alternatives? Even though we are not going to provide a full argument we are showing how the reasoning works.

Let's consider four different elements, x, y, u and v, where u and v are different from x and y. We start by the triplet $\{x, y, u\}$. Due to the result we just proven and condition (U), we have that $\overline{D}(x,u)$ and $D(x,u)$. Now, we consider the triplet (x, y, v). Since $D(x,u)$, the argument above show us that $\overline{D}(u,v)$ and $\overline{D}(v,u)$ follow. Then, for some pair x and y, $D(x,y)$ implies $\overline{D}(u,v)$ for all possible ordered pairs (u,v). So the contagion result we wanted to prove holds any finite number of alternatives, therefore the lemma is proved. □

What we get from the lemma is that we can not allow an individual to be almost decisive over some ordered pair of alternatives. If not, this would break our non-dictatorship condition set at the beginning of the chapter. To follow with the proof, let's assume that there is no *almost decisive* individual and we will get a contradiction.

Proof. (Arrow's Theorem)
With the condition (P) we see that there is at least one decisive set for any ordered pair (x, y), the whole society. So there also exists at least one almost decisive set. Among all of them, the almost decisive sets for any pair of alternatives, let us choose the smallest one, which is not necessarily unique. Recall that because of the lemma we have that this set has to contain at least two individuals. If not, if there is on almost decisive person, this would yield to a dictatorship, and the proof would be complete.

We will call this set V. So V is decisive for (x, y). Now we divide V into two parts: V_1 containing only one single individual and V_2 containing the other individuals of V. We will write V_3 for the individuals outside of V. Through the condition (U) we postulate the following profiles:

a) For i in V_1: xP_iy and yP_iz.

b) For j in V_2: zP_jx and xP_jy.

c) For k in V_3: yP_kz and zP_kx.

Since V is almost decisive for (x, y) we obtain xPy. Now, can zPy hold? If so, V_2 would be almost decisive for (z, y), since zP_jy and all other individuals (of V_1 and V_3) prefer y to z. But with our assumption, V is a smallest decisive set, and $V_2 \subsetneq V$. Then, zPy is impossible and therefore yRz.

Now, the transitivity property yields to xPz. But this means that the single member of V_1 would be almost decisive, in contradiction with the assumption.

The impossibility result we wanted to prove now follows from the lemma. □

2.3 A second proof

Now we are about to give a second proof of the Arrow's impossibility theorem. While the first one was bringing out the contagion property of decisiveness, this second proof gives us clearly the function of Arrow's independence of irrelevant alternatives condition.

Let's consider a finite set of alternatives X and n individuals, who have strict orderings over these alternatives. We assume that social ordering to be a weak order. Now we pick two distinct alternatives $a, b \in X$ and we consider the following steps.

First Step: Here alternative a is ranked highest and b lowest by all the agents $i \in \{1, \ldots, n\}$. By condition (P) we know that a is at the top of the social ordering. Now, suppose that b is raised to the top of individual 1's ordering, while keeping the other alternatives unchanged. Thanks to the independence condition we only have two cases, whether a holds on the top of the social order or whether is replaced by b. If a remains, let's apply the same change, raising b to the top, on the following agent. Then iterate the process. We know that there will be some agent, assume it is agent m, in which this change on her preferences will change the social order by placing b on top of the rank. We can see this on the following Tables.

R_1	$\ldots$	R_{m-1}	R_m	R_{m+1}	$\ldots$	R_n	Social order R
b	$\ldots$	b	a	a	$\ldots$	a	a
a		a	b	.		.	.
.		.	.	.		.	b
.		.	.	.		.	.
.		.	.	b		b	.

Table 1

R_1	$\ldots$	R_{m-1}	R_m	R_{m+1}	$\ldots$	R_n	Social order R
b	$\ldots$	b	b	a	$\ldots$	a	b
a		a	a	.		.	a
.		.	.	.		.	.
.		.	.	.		.	.
.		.	.	b		b	.

Table 2

Second Step: Let's make some changes into tables 1 and 2. We move the alternative a to the lowest position of individual's i ordering for $i < m$ and move a to the second lowest position for $i > m$. Recall here that this change, with respect to the Table 2, does not alter at anything in the relationship between b and any other alternative. Due to the condition (I), b must remain on the top on the social ordering.

Now, we have made a change to each of the two tables, let's call the tables changed $1'$ and $2'$. We see that the only difference between the tables changed lies in m's ranking of alternatives a and b. Due to condition (I), b must in situation $1'$ remain socially ordered above every alternative but possibly a. But in this situation, due to condition (I), b would be socially ranked at least as high as a in table

1. But this would be in contradiction of what we obtained in step 1. Therefore, in $1'$, a is top-ranked socially.

R_1	$\dots$	R_{m-1}	R_m	R_{m+1}	$\dots$	R_n	Social order R
b	$\dots$	b	a	.	$\dots$	a	a
.		.	b	.		.	.
.		.	.	.		.	b
.		.	.	a		a	.
a		a	.	b		b	.

Table $1'$

R_1	$\dots$	R_{m-1}	R_m	R_{m+1}	$\dots$	R_n	Social order R
b	$\dots$	b	b	.	$\dots$	.	b
.		.	a	.		.	a
.		.	.	.		.	.
.		.	.	a		a	.
a		a	.	b		b	.

Table $2'$

Third Step: Let's take now another alternative $c \in X$, distinct from a and b. We construct a profile in the next table. This one makes the ranking of a to remain the same as in situation $1'$ in relation to any alternative in any individual's ordering. Here, all agents ordered c above b. The important point here is that, again due to condition (I), alternative a must again be top-ranked socially.

R_1	$\dots$	R_{m-1}	R_m	R_{m+1}	$\dots$	R_n	Social order R
.		.	a	.		.	a
.		.	c	.		.	.
c		c	b	c		c	.
b		b	.	a		a	.
a		a	.	b		b	.

Table 3

Fourth Step: Let's take the last preference profile, the one in Table 3. Now we take the the agents with $i > m$ and we reverse a and b on their rankings. Because condition (I), the social ranking of a versus the other alternatives keeps the same except, maybe, for b.

But here b cannot be top-ranked since c must be socially preferred to b due to the Pareto condition. Therefore, a holds on the top of the social ordering and c is socially ranked above b.

Fifth Step: Here we construct an arbitrary profile of orderings with a above b for individual m, for example the one listed below, in the Table 4. Here we have c between a and b for the individual m, and c on the top for all the other individuals. Because condition (I), c can not have any effect on the social ranking between a and b. Due to the step 4, a must be ranked above c due to condition (I), and c is Pareto-preferred to b. Now, by transitivity of the social relation, a is preferred to b, and this holds whenever person m orders a above b.

R_1	...	R_{m-1}	R_m	R_{m+1}	...	R_n	Social order R
c		c	a	c		c	a
.		.	c	.		.	.
.		.	b	.		.	c
b		b	.	b		b	.
a		a	.	a		a	b

Table 4

By permuting alternatives b and c on the logic we used above we obtain the same qualitative result. a is above c when person m orders a above that alternative. An this holds for every alternative distinct from a. So we saw that agent m has dictatorial power over a versus any other alternative. And since a was chosen arbitrarily on the first step, it's clear that it holds for every $a \in X$. Therefore, there can only be one dictator for all elements from X.

2.4 A third proof

We will now give the third and last proof of the Arrow's impossibility theorem. This one gives us a diagrammatic representation of the result, that was introduced by Blackorby et al (1984) [4]. In order to keep the diagrams two-dimensional, we consider only two agents, although this can be extended to more persons. This proof unfolds in the utility space.

First, we have to remember that a preference ordering can be transformed into a utility function, if continuity is postulated in addition to the other properties that turn a binary preference relation into an ordering[3].

[3]With this we mean that the better-than-or-indifferent set and the worse -than-or-indifferent set

Second, we have to remember also that given any preference ordering and its corresponding utility function, any other utility function generated applying a strictly monotone transformation to the original function has the same informational content as the original. Recall here that any 'degree' of comparability of utilities across individuals is excluded.

Now, we take the social welfare function previously defined and we turn it into a *social evaluation functional* F. Its domain is on $U = (u_1, \ldots, u_n)$, sets of n-tuples of individual utility functions. From the social states $x \in X$, each individual $i \in \{1, \ldots, n\}$ evaluates that in terms of utility function $u_i(x)$. Here, all possible n-tuples of utility functions are admissible, so be have an *unrestricted domain*. Then we have

$$\begin{aligned} F\colon \mathcal{U} &\longrightarrow \mathcal{E} \\ (u_1, \ldots, u_n) &\longmapsto F(U) = R_U. \end{aligned}$$

A function which maps the set of all logically possible n-tuples or profile of utility function into the set of all orderings of X, which we denote by $\mathcal{E}$. Then, $F(U) = R_U$ is the ordering generated by F, when the utility profile is U.

If the first condition we asked to F was to have unrestricted domains, now we ask the *independence of irrelevant alternatives*, I. This means that if given $x, y \in X$ alternatives, and U', U'' utility profiles, both x, y obtain the same n-tuple of utilities in U' and U'', then $R_{U'}$ and $R_{U''}$ must coincide on $\{x, y\}$.

We will now introduce another condition which we want F to have, the *Pareto indifference*. This requires that if all members of the society are indifferent between a pair of alternatives, the same should hold for society's preference over this pair.

Condition (PI) (Pareto indifference). For all $x, y \in X$ and for all U from the (unrestricted) domain, if $U(x) = U(y)$, then xI_Uy.

Recall here that xI_Uy means that xR_Uy and yR_Ux, and $U(x) = U(y)$ means that $u_i(x) = u_i(y)$ for all $i \in \{1, \ldots, n\}$. Also, we imposed three conditions to our function F. (U), (I) and (PI). These three conditions are equivalent to the property called *strong neutrality*. This requires that the *social evaluation functional F* ignores all non-utility information with respect to the alternatives. So, the only information that counts is the vector of individual utilities associated with any social alternatives[4].

with reference to any point in Euclidian space are assumed to be closed sets.

[4]This 'fact' has been termed 'welfarism' in the literature of social choice theory.

This 'welfaristic' set-up has a great advantage considering our situation. Instead of considering the orderings R_U generated by F, we can focus on an ordering R^* of $\mathbb{R}^n$. This is, the space of utility n-tuples, that orders vectors of individual utilities which correspond to the social alternatives from the given set X. The result which Blackorby et al. (1984) [4] state (mainly based on D'Aspremont (1977) [5]) is the following: If the social evaluation functional F satisfies the three axioms of welfarism that we just presented, there exists an ordering R^* of $\mathbb{R}^n$ such that for all $x, y \in X$ and all logically possible utility profiles $U, xR_Uy \leftrightarrow \bar{u}R^*\bar{\bar{u}}$, where $\bar{u} = U(x)$ and $\bar{\bar{u}} = U(y)$.

D'Aspremont et al. (1977) [5] show that when F holds the three axioms of welfarism, the ordering R^* in utility space inherits these properties. Now, on the new framework, we will show diagrammatically that the ordering R^* is a dictatorship if and only if there exists an individual $i \in \{1, \ldots, n\}$ such that for all $\bar{u}, \bar{\bar{u}} \in \mathbb{R}^n$, if $\bar{u}_i > \bar{\bar{u}}_i$, then $\bar{u}P^*\bar{\bar{u}}$.

Arrived to this point, what we will prove diagrammatically is that the social evaluation functional F satisfies the three axioms of welfarism, the framework of ordinally measurable and non-comparable utilities together with the weak Pareto rule are necessary and sufficient for the social ordering R^* to be a *dictatorship*.

Let's consider the Figure 2.1. $\bar{u}$ in $\mathbb{R}^2$ will be our point of reference. Although we divided the plan into four regions, we do not consider their boundaries for now but only their interior. Because of the weak Pareto principle, we can see that all utility vectors in region I are socially preferred to our reference point $\bar{u}$, and that the latter is preferred to all utility vectors in region III.

Now, it's interesting to analyze the case of the utility vectors in regions II and IV. We want to show that either one of the two cases is true:

i) All points in region II are preferred to $\bar{u}$ and the latter is preferred to all points in region IV.

ii) All points in region IV are preferred to $\bar{u}$ and the latter is preferred to all points in region II.

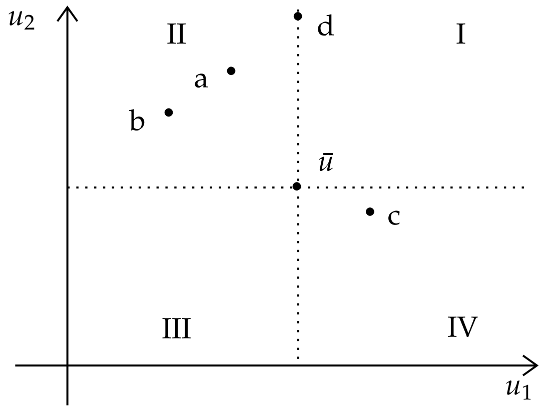

Figure 2.1

Now we prove that all the points on the region II (region IV) must be ranked identically against $\bar{u}$. Recall here that points in II are such that $u_1 < \bar{u}_1$ and $u_2 > \bar{u}_2$. Given a, b of the region II, we can assume that $aP^*\bar{u}$, and then we can argue that $bP^*\bar{u}$. This is because each of two persons is totally free to map his or her utility scale into another one by a strictly increasing transformation. So then, we can find a transformation that maps a_1 into b_1 and $\bar{u}_1$ into $\bar{u}_1$. We can do it similar for a_2, b_2 and $\bar{u}_2$. On Figures 2.2 (a) and (b) we can find those transformations.

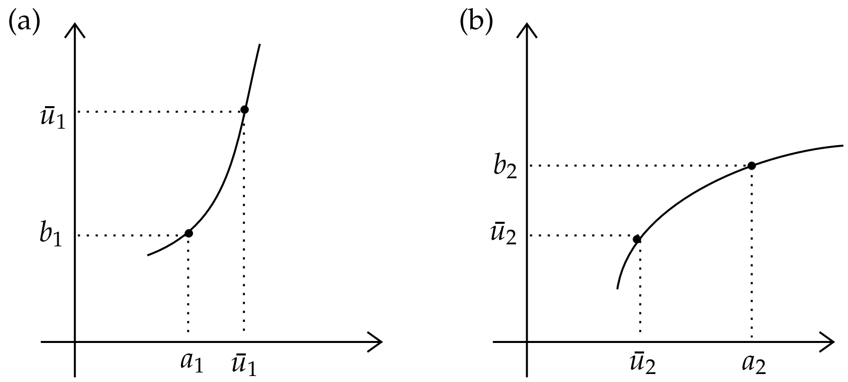

Figure 2.2

As we are in a framework of ordinal and non-comparable utilities, as said, those transformations do not affect the rankings of the two persons. Therefore, if $aP^*\bar{u}$, then $bP^*\bar{u}$, and this holds for any points in the interior of region II. So, we can say that any two points on the interior of II are ranked identically with respect to reference point $\bar{u}$. And this hold analogously for points in region IV.

As R^* is an ordering, we can have three possible cases when ranking points in region II against $\bar{u}$: the points in II could be preferred, indifferent or worse. In our steps above, we have assumed a strict preference against $\bar{u}$ (in the case of the points in II being worse against $\bar{u}$ the logic would be analogous). However, if we had indifference between points in II and $\bar{u}$ this would lead to a contradiction. Let's suppose we had $aI^*\bar{u}$ and $bI^*\bar{u}$. But because R^* being an ordering, we would also had aI^*b. But as we can see in Figure 2.1, a must be Pareto-preferred to point b. And so, indifference can not hold.

We now want to show that the ranking of points of II against $\bar{u}$ must be the opposite to the ranking of the points of IV against $\bar{u}$. Here we use again that strictly monotone transformations of individual utility scales d not change the informational content.

We keep considering that points in II are preferred against $\bar{u}$, more concretely, $(a_1, a_2)P^*(\bar{u}_1, \bar{u}_2)$. Consider now the transformations such that we map a into $\bar{u}$ and $\bar{u}$ into c. By assumption, we had a preferred to $\bar{u}$ and this holds after the transformation (also with $\bar{u}$ preferred to c). From our previous steps, we infer that $\bar{u}$ is preferred to all points in region II.

The proof is almost complete, except for the cases of points in the boundaries. Let's consider now the case of d on Figure 2.1. Assume that region II is preferred to $\bar{u}$. Then, there always exists a point in II, in this case a, that is Pareto-inferior to d. Therefore, we have that dP^*a and $aP^*\bar{u}$. Because transitivity of R^* we get $dP^*\bar{u}$. As this holds for any choice of d, we can say that if two adjacent regions have the same preference relationship to $\bar{u}$, then the same ranking holds for any point on their common boundary.

To end up with, let's see what we just shown. There are two possible cases that we can see on the Figures 2.3 (a) and (b).

i) If region II is preferred to $\bar{u}$, then regions I and II and their common boundary are preferred to $\bar{u}$. In this case, we can see that the direction of social preference is vertical, and person 2 is a dictator in the sense defined.

ii) If region IV is preferred to $\bar{u}$, then the regions I and IV and their common boundary are preferred to $\bar{u}$. In this case, the direction of social preference is horizontal, and person 1 is a dictator.

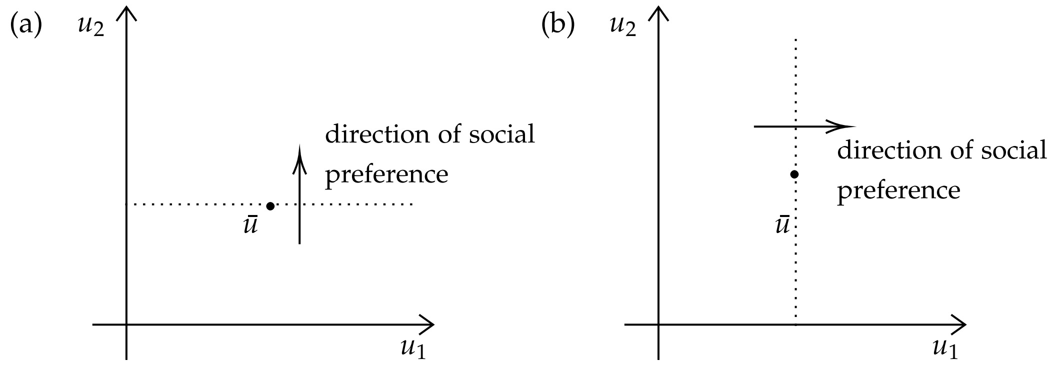

Figure 2.3

Conclusions

In this chapter, we presented the Arrow's impossibility theorem with 3 possible proofs. The reasons for it have been to show in a transparent way, the implications and generality that the theorem has. On the first one, we saw that the decisiveness over some pair of social alternatives spreads to decisiveness over all pairs, and that this phenomenon has been called *contagion property*. On the second one, we used the Arrow's independence condition. Lastly, the third proof unfolds in utility space and provides the result in a diagrammatic representation.

Chapter 3

Some Possibility Results

What Arrow tried to do with the requirements for his social welfare function was to express both rationality and doctrines of citizens' sovereignty in a general way. And we have seen that *any* social welfare function following those conditions is doomed to be dictatorial. But, what happens here? Because the consequences of Arrow's theorem are not very encouraging, is that there is not any solution?

In the first part of the chapter we will try to ask which are the basic properties that a social aggregation rule used in real life should reasonably satisfy. We will give an approach to the simple and absolute majority rules.

In the second part of the chapter, we will see that the simple majority rule may have problems regarding some preference patterns. And so, we will try to find out if there is a way to depict those patterns which do not cause problems. One of these ones are the *single peak* ones. For that, we keep following Gaertner (2006) [6] on the next lines.

3.1 Conditions

In democratic voting procedures, one of the basic properties that those systems have is the *anonymity*. Recall that this property requires in social decision making that all individuals should count the same.

Let's follow with three conditions:

i) **Condition A (Anonymity).** If $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$ are two profiles of preferences that are permutations of one another via some one-to-one correspondence (that is, agents interchange their "names"), then for any $x, y \in X, xRy \leftrightarrow xR'y$, where R and R' are the social relations corresponding to the two profiles.

Recall here that *anonymity* is a stronger condition than Arrow's non-dictatorship condition.

ii) **Condition N (Neutrality).** For any $x, y, z, w \in X$, if for all $i \in N$,

$(xR_iy \Leftrightarrow zR'_iw)$ and $(yR_ix \Leftrightarrow wR'_iz)$, then

$(xRy \Leftrightarrow zR'w)$ and $(yRx \Leftrightarrow wR'z)$.

where for all i, R_i and R'_i belong to two profiles of individual preferences and R and R' are the corresponding social relationships.

In other words, demand to the social decision mechanism to treat alternatives equally. Recall that this condition is stronger than Arrow's independence axiom.

iii) **Condition *PR* (Positive responsiveness).** For any two profiles $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$ and any $x, y \in X$, if individual preferences are such that for all $i \in N$, $(xP'_iy$ whenever $xP_iy)$, and $(xR'_iy$ whenever $xI_iy)$ and $\exists k \in N$ such that either $(xP'_ky$ whenever $xI_ky)$ or $(xR'_ky$ whenever $yP_kx)$. Then xRy under $(R_1, \ldots, R_n)$ implies $xP'y$ under $(R'_1, \ldots, R'_n)$.

In other words, positive responsiveness considers the effect on the social preference relation when an individual k expresses a change in favour of x.

3.2 Simple and Absolute Majority Rule

Now, let's consider two rules that are widely used and accepted in the real world. The *simple majority rule* and the *absolute majority rule*. But before, we will give now the notation of three terms we are going to use:

i) Denote by $N(xP_iy)$ the number of voters with xP_iy.

ii) Denote by $N(xR_iy)$ the number of voters with xR_iy.

iii) Denote by $|N|$ the total number of voters.

Definition 3.1. (Simple majority rule). *For all* $(R_1, \ldots, R_n)$ *and for any* $x, y \in X$: $xRy \Leftrightarrow [N(xP_iy) \geq N(yP_ix)]$.

Recall here that it would be equivalent to change P_i by R_i.

Definition 3.2. (Absolute majority rule). *For all* $(R_1, \ldots, R_n)$ *and for any* $x, y \in X$: $xPy \Leftrightarrow [N(xP_iy) > \frac{1}{2} \cdot |N|], yPx \Leftrightarrow [N(yP_ix) > \frac{1}{2} \cdot |N|]$, *and* xIy *otherwise.*

Let's consider now an example in which positive responsiveness is satisfied by the simple majority, but not the absolute one. Recall that this example uses indifferences, which we do not pay too much attention in this work.

Example 3.3. Lets take a society that has 7 individuals. Two of them prefer x to y, another two prefer y to x and the rest is indifferent between those two alternatives. Now, both the simple and absolute majority rule yield to social indifference, that is to write xIy. However, consider that an individual from the group that was indifferent declares that prefers x over y while everyone staying constant. Now, positive responsiveness requires xPy, which is what the simple majority rule brings, but not the absolute one.

Therefore, we can see that *simple majority rule* is more sensitive to the changes on preferences that society may have.
Let's give now a result first proved by Kenneth O. May (1951) [1].

Theorem 3.4. (May's characterization of simple majority voting (1952)). *The conditions of unrestricted domain, anonymity, neutrality and positive responsiveness are necessary and sufficient for a social aggregation rule to be the simple majority rule.*

We can find a proof in Gaertner (2006) [6].

3.3 Single Peaked Preferences

As we were saying at the beginning, we found out that the simple majority rule has some problems for some preference patterns. So our objective will be to depict the preference profiles in a useful way while adding a reasonable interpretation.

Let's consider some interesting cases. For example, a student that may want to live as close as possible to his/her university. Or in other case, a frequent traveller, who may want to live as close as possible to the train station. These individuals will have increasing preferences for locations closer to their preferred locations. Also, we can consider three individuals, with different purchasing power and with the intention of buying a car. Suppose there are three categories of cars, the expensive big car, the medium-priced average car and the low-priced small car. Then, the individual with the highest income may want to buy the big an expensive car. The individual with mid income probably prefers the medium-priced one. Finally, the individual with low income may want to buy the low-priced one.

This type of preferences are what Black (1948) [3] and Arrow (1951, 1963) [1] [2] call *single-peaked preferences*. On those preferences, there is a peak, the point of highest desirability. Later, on either side of that peak (or just one side if the peak is at one extreme), the individual desirability declines.

Now, we want to show that the simple majority rule turns out to be a social welfare function in the Arrovian sense for:

i) Any number of alternatives.

ii) The number of individuals is odd.

iii) The property of *single-peakedness* is fulfilled for every triple of alternatives.

Strict Ordering

To be able to define that property, the *single-peakedness* we have to introduce a strict ordering between the set of alternatives. This strict ordering assumes that the alternatives can be ordered and therefore we can see between three distinct alternatives which one is between the others.

Definition 3.5. (Strict ordering). *We say that S is a* strict ordering *if for $x, y \in X, x \neq y$ then either xSy or ySx. And for any $x, y, z \in X$, all distinct from each other, if xSy and ySz then xSz.*

Now, we will define the notion of "betweenness" with respect to S.

Definition 3.6. (Between). *Given $x, y, z \in X$, we say that y is* between *x and z if either xSy and ySz, or zSy and ySx. We will write $B(x, y, z)$.*

So, given $x, y, z \in X$ all distinct from each other, one of the following holds: $B(x, y, z)$, $B(x, z, y)$ or $B(y, x, z)$. It's now that given these notions, we can define what Black calls *single-peaked preferences*.

Condition of single-peakedness. A profile $(R_1, \ldots, R_n)$ of individual preferences satisfies the property of *single-peakedness* if there exists a strict ordering S such that for all $i \in N$, xR_iy and $B(x, y, z)$ imply yP_iz, where $B(x, y, z)$ is the betweenness relation derived from S.

Possibility Theorem for Single-Peaked Preferences

It turns out that whenever we are in the domain of single-peakedness we obtain a possibility result. This was given by Black (1948) [3] and Arrow (1951, 1963) [1] [2].

Theorem 3.7. (A possibility theorem for single-peaked preferences). *Provided the number of concerned voters is odd, the majority decision rule is a social welfare function for any number of alternatives if the individual orderings satisfy the property of single-peakedness over each triplet of alternatives.*

Proof. We have to prove that given the *single-peakedness preference* property, the social relation generated by the method of majority decision is an ordering. More precisely, we have to prove that is complete and transitive.

Complete. Given $x, y, z \in X$, we have either $N(xR_iy) \geq N(yR_ix)$ or $N(xR_iy) \leq N(yR_ix)$. Under the method of majority decision, this means that we have either xRy or yRx for any $x, y \in X$, which is indeed the definition of completeness of a binary relation.

Transitive. Here we want to prove that R is transitive. Given $x, y, z \in X$ all distinct, under the majority rule, zRy and yRx correspond to $N(zP_iy) \geq N(yP_iz)$ and $N(yP_ix) \geq N(xP_iy)$. Now, with respect to betweenness, we have to consider three cases: (1) $B(x, y, z)$, (2) $B(x, z, y)$ and (3) $B(y, x, z)$.

Case 1: $B(x, y, z)$.

Using single-peakedness property, we have that zR_iy implies yP_ix. Using the transitivity of individual preference relation we get zP_ix, and therefore zR_iy implies zP_ix. Now, $N(zP_ix) \geq N(zR_iy) \geq N(zP_iy)$. Also $N(zP_iy) \geq N(yP_iz)$.

From the fact that zR_iy implies zP_ix, we get that (not zP_ix) implies (not zR_iy), which is indeed yP_iz. Then, (not zP_ix), which is xR_iz, implies yP_iz. So, $N(yP_iz) \geq N(xR_iz) \geq N(xP_iz)$. With these three steps we taken, we obtain $N(zP_ix) \geq N(zP_iy) \geq N(xP_iz)$. So zRx under the majority rule has been shown.

Case 2: $B(x, z, y)$.

Using single-peakedness property we get that xR_iz implies zP_iy. Together with transitivity, we get that xR_iz implies xP_iy. And so, $N(xP_iy) \geq N(xR_iz)$. According to what we have supposed, $N(yP_ix) \geq N(xP_iy)$, $\frac{1}{2} \cdot |N| \geq N(xP_iy)$. Therefore, $\frac{1}{2} \cdot |N| \geq N(xR_iz)$. Then, since $N(xR_iz) \geq N(xP_iz)$, then it is clear that $\frac{1}{2} \cdot |N| \geq N(xP_iz)$.

$N(zP_ix) = |N| - N(xR_iz)$. Then, as $N(xR_iz) \leq \frac{1}{2} \cdot |N|$, then we have $|N| - N(xR_iz) \geq |N| - \frac{1}{2} \cdot |N| = \frac{1}{2} \cdot |N|$. So, $N(zP_ix) \geq \frac{1}{2} \cdot |N|$. If we combine the last two steps, we get $N(zP_ix) \geq N(xP_iz)$. Then, we have again that zRx under the majority rule.

Case 3: $B(y, x, z)$.

We want to prove that this case can not occur under the supposition that zRy and yRx and that the number of voters is odd.

By single-peakedness, we have that yR_ix implies that xP_iz. Using transitivity we get yR_ix implies yP_iz. Then, $N(yP_iz) \geq N(yR_ix)$. But zRy from our supposition so that $N(yP_iz) \leq \frac{1}{2} \cdot |N|$. Then, $N(yR_ix) \leq \frac{1}{2} \cdot |N|$. With our hypothesis yRx so that $N(yR_ix) \geq \frac{1}{2} \cdot |N|$. Taking the last two results, we get $N(yR_ix) = \frac{1}{2} \cdot |N|$. But that implies that $\frac{1}{2} \cdot |N|$ must be an integer value. And so, $|N|$ has to be even, which is contrary to the hypothesis of the theorem.

Therefore, we have proved that in all cases in which it was possible to have zRy and yRx, we were able to prove that zRx. This proves the transitivity property we wanted and the proof is finally complete. □

Chapter 4

The Gibbard-Satterthwaite theorem

In this chapter we will state and prove another impossibility result, the Gibbard-Satterthwaite theorem. For that, we will follow closely the explanation of Gaertner (2006) [6]. The result is that strategy-proof and non-dictatorial allocation mechanisms (such as a social choice function) do not exist in general.

4.1 Introduction

The original proof of Gibbard (1973), gives a close relation between strategy-proof social choice functions and Arrovian social welfare functions under the assumption of unrestricted domain. The idea behind is that from every strategy-proof social choice function h one can construct via binary relation on the alternatives that is shown to be transitive, a social welfare function. Then, that function will hold the Pareto condition and independence of irrelevant alternatives over the range of h. Now, if the range has more than two alternatives, the social welfare function will be dictatorial, and the same holds for the social choice function from which the welfare function was derived.

We will move forward taking some of the ideas on the proof from Reny (2001) [9]. He follows closely the proof of Arrow's impossibility theorem which is already presented. Remember here what we just introduced when presenting the social choice function on the previous pages.

A function $h : \sum \mapsto X$ will be called a social choice function. Where X denotes the finite set of alternatives. $\sum$ denotes the set of all strict linear orderings on

X. n is the number of individuals. A member of $\sum^n$ is called a profile of linear orderings with its ith component being individual i's strict ranking that we shall denote by P_i.

Definition 4.1. (Onto). *We say that the function h is* onto *(or surjective), if every element of X will be chosen for some profile.*

Now, some of the conditions on the social choice function that we will look closely on our next lines are if h is: Pareto efficient, monotonic or Dictatorial. To have a detailed definition of those, the reader can go to Definition 1.14.

4.2 G-S Theorem and Proof

Theorem 4.2. (Gibbard-Satterthwaite theorem). *Let be h a social choice function defined on an unrestricted domain of strict linear preferences. If the range of h contains at least three alternatives and* $h : \sum^n \mapsto X$ *is onto and strategy-proof, then h is dictatorial.*

We will split the proof into two parts. The first one will be similar to the one given when proving the Arrow's impossibility theorem.

First part of the proof

Result (a): If there are at least three alternatives and if the social choice function $h : \sum^n \mapsto X$ is Pareto efficient and monotonic, then h is dictatorial.

First Step: Let's consider alternatives $a, b \in S$ and a profile of strict ranking in which a is ranked highest and b is ranked lowest by every individual $i \in \{1, \ldots, n\}$. Then, Pareto efficiency implies that the social choice for this profile is a.

Now, on individual 1's, we change it's order by raising b one step at the time. Due to monotonicity, as long as b is below a, the social choice function is still equal to a. Eventually, b will finally rise above a and so, will be on the top. Due to monotonicity, the social choice function will switch to b or either a will stay. If the latter holds, we iterate this process for the 2nd individual, 3rd, and so on. Eventually, when we have done this process for an individual m, the social choice function will change to b, because of Pareto efficiency. The Tables 4.1 and 4.2 show the situation before the social choice function switches.

P_1	...	P_{m-1}	P_m	P_{m+1}	...	P_n	Social choice
b	...	b	a	a	...	a	a
a		a	b	.		.	
.		.	.	.		.	
.		.	.	.		.	
.		.	.	b		b	

Table 4.1

P_1	...	P_{m-1}	P_m	P_{m+1}	...	P_n	Social choice
b	...	b	b	a	...	a	b
a		a	a	.		.	
.		.	.	.		.	
.		.	.	.		.	
.		.	.	b		b	

Table 4.2

Second Step: Here we change the situation we just presenting. Particularly, we move the alternative a to the lowest position of individual i's ranking for $i < m$ and move a to the second lowest position in the ranking of persons $i > m$. We can see how the situation is on the Tables 4.3 and 4.4.

P_1	...	P_{m-1}	P_m	P_{m+1}	...	P_n	Social choice
b	...	b	a	.	...	.	a
.		.	b	.		.	
.		.	.	.		.	
.		.	.	a		a	
a		a	.	b		b	

Table 4.3

P_1	...	P_{m-1}	P_m	P_{m+1}	...	P_n	Social choice
b	...	b	b	.	...	.	b
.		.	a	.		.	
.		.	.	.		.	
.		.	.	a		a	
a		a	.	b		b	

Table 4.4

But now, which is the effect on the social choice by these changes made on the second step? There is no change at all. Why is that?

First note that the social choice in Figure 4.4 must be b since in Figure 4.2 the social choice is b. This is due to monotonicity and that no agent's ordering of b against any other alternative changes between the move between Tables 4.2 and 4.4.

Second note that profiles of Tables 4.3 and 4.4 are the same except for the ordering of agent m. Then, due to monotonicity and that Figure 4.4 is b, Figure 4.3 must be either a or b. But suppose that in Figure 4.3 the result were b, then due to monotonicity, the same result would have to occur in Figure 4.1. And this will lead to a contradiction, since the social choice in Figure 4.1 is a. Then we can say that the social choice in Figure 4.3 is a.

Third Step: Let's suppose we add another alternative $c \in S$, which is distinct from a and b. In Figure 4.5 we construct a now profile in which the ranking of a is not changed versus any other alternative in any individual's ordering compared to the Figure 4.3. As in Figure 4.3 the result is a, as argued, now by monotonicity the result of Figure 4.5 is also a.

P_1	...	P_{m-1}	P_m	P_{m+1}	...	P_n	Social choice
.	...	.	a	.	...	.	a
.		.	c	.		.	
c		c	b	c		c	
b		b	.	a		a	
a		a	.	b		b	

Table 4.5

Fourth Step: Here we do only one change on profile of Figure 4.5. For individuals $i > m$ we reverse the ordering of alternatives a and b. Then, we want to see that because of the outcome of Figure 4.5 is a, then the outcome of Figure 4.6 is a too, so can not be either b, c or any other alternative d. Let's do it by cases:

- Assume the outcome is c. Because of monotonicity, then the outcome of Figure 4.5 would also had to be c, but we have proven that is a.

- Assume the outcome is b. Recall that in Figure 4.6, c is ranked above b in all individual orderings. Now, this result will continue to hold due to monotonicity, even if c were raised to the top of every individual ranking. Then, however, we will get a contradiction with Pareto efficiency.

- Assume the outcome is d. Going from Figure 4.6 to 4.5, d does not fall under anyone's ordering below a or b or c. Then, by monotonicity, d would have to be chosen in Figure 4.5, a contradiction.

Therefore, the social choice in Figure 4.6 is a.

P_1	...	P_{m-1}	P_m	P_{m+1}	...	P_n	Social choice
.	...	.	a	.	...	.	a
.		.	c	.		.	
c		c	b	c		c	
b		b	.	b		b	
a		a	.	a		a	

Table 4.6

Fifth Step: We take it from the situation of Figure 4.6. Then we construct an arbitrary profile of orderings, with a at the top of person m's ordering. Recall that all this profiles have the ranking of a versus any other alternative in any individual's ordering is nowhere reduced. Now, again due to monotonicity, the social choice must be a whenever the person m's ranking has a at the top. So, person m turns to be a dictator for the alternative a. As a was chosen randomly, it appears that there is a dictator for every alternative $a \in S$.

Now the question arises, can there be different dictators for different alternatives? But this does not happen, as our social choice mapping give us a unique outcome for every set of alternatives. Then, there is a single dictator for all alternatives.

Second part of the proof

Before we begin with the proof, let us give the definitions of the terms mentioned in the present context.

We say that the social choice function $h : \sum^n \mapsto X$ is *onto* (or surjective) if for each singleton set $S' \subseteq S$, there exists a profile $(P_1, \dots, P_n)$ such that $h(P_1, \dots, P_n) = S'$. This property was referred previously as the *voters' sovereignty*, and disallows that some $S' \subseteq S$ will never be chosen via h.

We say that the social choice function $h : \sum^n \mapsto X$ is *strategy-proof* if for every individual i, every profile $(P_1, \dots, P_n) \in \sum^n$, and every $P_i' \in \sum$, $h(P_i', P_{-i}) \neq$

$h(P_1, \ldots, P_n)$ implies that $h(P_1, \ldots, P_n)$ is ranked above $h(P'_i, P_{-i})$ according to P_i, the honest strict preference of person i.

Clearly, if $h(P_1, \ldots, P_n)$ were ranked below $h(P'_i, P_{-i})$, there would be room for successful manipulation, given preference P_i. Of course, strategy-proofness requires that $h(P'_i, P_{-i})$ is tanked above $h(P_1, \ldots, P_n)$ according to P'_i.

Result (b): If $h : \sum^n \mapsto X$ is strategy-proof and onto, then h is Pareto efficient and monotonic.

Let's assume that $h(P_1, \ldots, P_n) = a$ and that for every alternative b, the ordering P'_i ranks a above b when P_i does. We want to show that $h(P'_i, P_{-i}) = a$. Let's suppose the contrary, so that $h(P'_i, P_{-i}) = b \neq a$. Then, the strategy-proofness implies that $h(P_1, \ldots, P_n) = a$ is ranked above $h(P'_i, P_{-i}) = b$ according to P_i (if not, there would be a case for profitable manipulation). But the ranking of a does not fall in the move to P'_i, so it follows that $a = h(P_1, \ldots, P_n)$ must also be ranked above $h(P'_i, P_{-i}) = b$ according to P'_i. However, this would lead to a contradiction with strategy-proofness (as it would be profitable for person i to switch from P'_i to P_i and then reach a). Therefore, we have seen that $h(P'_i, P_{-i}) = a = h(P_1, \ldots, P_n)$.

Now suppose that $h(P_1, \ldots, P_n) = a$ and that for every person i and every alternative b, the ordering P'_i ranks a above b whenever P_i does. When we nos move from $(P_1, \ldots, P_n)$ to $(P'_1, \ldots, P'_n)$ by changing the ordering of each person i from P_i to P'_i one at a time. Then, we must obtain $h(P'_1, \ldots, P'_n) = h(P_1, \ldots, P_n)$. Because on the previous step it was shown that the social outcome must remain unchanged for every such change. Therefore, h is monotonic.

Let's take $a \in S$. As we know that the mapping h is onto or surjective, there is some profile $(P'_1, \ldots, P'_n) \in \sum^n$ such that $h(P'_1, \ldots, P'_n) = a$. Thanks to the monotonicity, the social choice remains equal to the alternative a when a is raised to the top of every individual ranking. Again due to the monotonicity, the social choice function must remain a independently of how the alternatives below a are ranked by each individual. This means that every time that the alternative a is at the top of every individual ordering, the social outcome is a. Since a was randomly chosen, it is proven that h is Pareto efficient.

Let's bring now everything we shown together. Result (b) says that if h is strategy-proof and onto, then h satisfies Pareto-efficiency and monotonicity. Result (a) has shown that with the latter two properties, h is dictatorial. So then, results (a) and (b) together prove Theorem 4.2, the stated variant of the Gibbard-Satterthwaite impossibility result.

Conclusions

Personally, my studies (at this moment, still to be finished) are in mathematics and economics.

- As a maths student, I like to get abstract ideas and frameworks of systems to be able to understand them better. However, although most of the time you get to understand better the system, but the system alone and really far from any concept in reality.
- As an economics student, I like to understand how the people behave in terms of production, distribution and consumption of goods and services. However, even understanding the system we are studying, most of the time we don't get to create such robust frameworks. And normally, the cases observed are not general but specific situations.

Keeping that in mind, I tried to make a student thesis which takes the maximum benefits from both sides of the coin. In that direction, I really liked working in a social science topic, the social choice theory, while making an effort to add and understand as much as possible the mathematical framework behind.

In my opinion, I found quite shocking the result and consequences that Arrow and Gibbard-Satterthawite give us on their theorems. That's why I wanted to add to the thesis some positive results like the simple majority rule and the single-peaked preferences.

Also, we have been able to see that that all the framework of analysis we described with the social choice theory is clearly important outside the mathematical boundaries.

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