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Extended running vacuum model: cosmological equations and thermodynamical implications

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Abstract In this work, we consider an extension of a Running Vacuum Model (RVM), a particular subclass of dynamical vacuum model based on a renormalisation group equation that establishes the time-dependence of the vacuum energy density ρ_Λ as a function of the Hubble parameter H. We aim to inspect the thermodynamical implications of considering a RVM that presumes two different inflation scales. In particular, we want to analyse the behaviour of the entropy of the universe that arises from considering a dynamical vacuum energy density with such particular behaviour. To this end, we introduce the Generalised Second Law (GSL) of Thermodynamics together with the concept of cosmological horizons, emphasising on the apparent horizon. Moreover, we review a deduction of the Einstein field equations and a derivation of the Friedmann equations in arbitrary dimensions, both from thermodynamical principles, following our objectives of maintaining this work as thermodynamically-focused as possible and to wit the possible deep connection between Thermodynamics and General Relativity. Finally, we study the RVM in the early universe and the entropy production within, reviewing recent work on the matter considering a RVM and introducing our extended RVM with two different higher powers of the Hubble parameter H, which poses the possible existence of two different epochs of inflation.

Contents

1

Introduction

More than two decades have passed by after the accelerated expansion of the universe was first observed [38] and much more evidence supporting this discovery has been provided by independent measurements ever since [27; 36; 51]. Nowadays, it is well-established that the universe is undergoing an accelerated expansion, yet a successful explanation for this phenomenon is still missing. Several attempts have taken place in the last twenty years and numerous models have been proposed, such as quintessence fields of phantom fields, among others, but none of them has been able to bring a conclusive answer to the observations [43].

From a theoretical point of view, the aim is to explain what kind of mechanism lies behind the cosmological constant term, or Λ term for short, in Einstein field equations [16]. Einstein introduced the Λ term in 1917, just two years after proposing his field equations [15], so that his equations would contain a solution for a static universe, as this was the belief at the time. However, in 1929, Hubble published his observations showing the expansion of the universe, contradicting the conception of the universe of the epoch [18]. Moreover, Eddington proved in 1931 that the static solution presented by Einstein was, in fact, unstable, hence it represented an unrealistic scenario [14]. That same year, Einstein published another article concerning his field equations, but this time without the Λ term [17].

Nevertheless, a few years later, an unexpected turn of events took place, when Lemaître discussed for the first time the idea of vacuum energy being the cause of the cosmological constant [29], associating a vacuum energy density to Λ through an expression of the form $\rho_\Lambda = \Lambda/(8\pi G)$. This offered a new way to explain the expansion of the universe, which has remained till our days. Besides, either we like it or not, there is no good reason to erase the Λ term from Einstein field equations since it is allowed by general covariance. Nowadays, we write the Einstein field equations in the following form [11]:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $g_{\mu\nu}$ is the metric tensor and

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{1.2}$$

is the Einstein tensor, with $R_{\mu\nu}$ being the Ricci tensor and R the Ricci scalar. As it is conventional, we shall always consider a connection in which the covariant derivative of the metric is zero, that is $\nabla^\mu g_{\mu\nu} = 0$. From the Bianchi identity we know that $\nabla^\mu G_{\mu\nu} = 0$. Assuming that G is a fundamental constant and that matter is conserved, i.e. $\nabla^\mu T_{\mu\nu} = 0$, if we perform a covariant derivative on equation (1.1) we are left with the expression

$$\nabla_\mu \Lambda = \partial_\mu \Lambda = 0, \tag{1.3}$$

which implies that Λ must be constant, hence the name "cosmological constant". At this point we should mention that, in this work, we may use Planck units [11], which means that we will normalise to 1 the speed of light in the vacuum, the Newton's gravitational constant, the reduced Planck constant, and the Boltzmann constant: $c = G = \hbar = k = 1$. Thus, eventually, we shall rewrite equation (1.1) as

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.4}$$

We have just seen how the Λ term is allowed within the structure of General Relativity and mentioned that was postulated as an explanation for the expansion of the universe if we associate it with vacuum energy. Unfortunately, this promising perspective was frustrated when theoretical results were compared to measurements. As it turns out when we try to obtain a prediction for Λ from quantum theory, the value that we get is several orders of magnitude greater than the one that we acquire from observations. On the one hand, the present current value for the vacuum energy is $\rho_\Lambda^0 \approx 2.5 \times 10^{-47}$ GeV^4 [36]. On the other hand, the "induced cosmological constant" due to the electroweak phase transition caused by the Higgs potential is $\rho_{\Lambda\mathrm{ind}} \approx -1.2 \times 10^8$ GeV^4 [43]. As we can see, the vacuum energy density predicted by electroweak theory is 55 orders of magnitude bigger than the present value of vacuum energy density. This discrepancy is called the *cosmological constant problem* or *CC problem* for short; first formulated by Zeldovich [55], it is one of the most famous and challenging open problems in cosmology and in physics.

Furthermore, we notice that ρ_Λ^0 is very close to the value for the current matter density ρ_m^0. Actually, both values are of the same order of magnitude, as we see from [36] that $\rho_\Lambda^0/\rho_m^0 \approx 2.2 \sim \mathcal{O}(1)$. As it is well known, the matter density is inversely proportional to the cube of the scale factor, i.e. $\rho_m \propto a^{-3}$, since we assume that matter is conserved and, consequently, it becomes more and more diluted as the universe expands. In contrast, we consider the cosmological constant to be indeed constant, thus the vacuum energy density is also constant. So it seems to be that we find ourselves in an epoch of the history of the universe in which an evolving and a constant magnitudes, which apparently do not hold any relation between them, meet. Are we extremely lucky or are we wrong about (at least) one of the assumptions we have made in our way here? Most likely, the latter is the actual situation, which leads to the problematic known as the *cosmic coincidence problem*.

An interesting hypothesis is to assume that the cosmological constant is, in fact, not constant [43]. From this perspective, we think of Λ as of a spatially homogeneous function of the cosmic time, i.e. $\Lambda = \Lambda(t)$, since we must respect the Cosmological Principle. At first sight, one may think that this violates general covariance in General Relativity, but this is not the case. Recall the general covariance of equation (1.1) relied on the assumptions that G is constant and that matter is conserved. Postulating that the Newton's gravitational constant is a varying parameter is not as senseless as it sounds since measurements carried on in the past decade suggest that basic quantities of the Standard Model may not be preserved during cosmological evolution [43], so it could be the same for G. In addition, there could be the possibility of matter and vacuum interacting and interchanging energy, so that matter density and vacuum energy density would evolve together through time. Either one of these two phenomena or a combination of both may allow for a dynamical cosmological constant, preserving the general covariance of Einstein field equations.

Contemporary works have considered a time-evolving gravitational constant G in an attempt to better fit the observed cosmological data [33; 45; 46]. In this work, in contrast, we shall take G as constant and consider that matter is not covariantly conserved and that the vacuum energy density is dynamical. In other words, we will work in the context of dynamical vacuum models (DVMs), in which we regard Λ and, consequently, ρ_Λ as quantities that depend on the cosmic time, that is $\Lambda = \Lambda(t)$ and $\rho_\Lambda = \rho_\Lambda(t)$. In particular, we will work with a specific subclass of dynamical vacuum models called Running Vacuum Models (RVMs) [43; 44; 50].

Opposite to old models [34], which are essentially phenomenological and hardly ever based on any fundamental theory, RVMs are built upon theoretical principles within QFT in curved spacetime [40; 41; 42]. As we shall see, the renormalization group (RG) in curved spacetime is invoked to construct an expression for the time-dependent vacuum energy density $\rho_\Lambda(t)$. This confers to the dynamical vacuum energy density its time dependence through the Hubble rate (and its time derivatives)[4], that is $\rho_\Lambda(t) = \rho_\Lambda(H(t), \dot{H}(t), \dots)$, in the context of RVMs. Within RVMs, we can build unified models of cosmological evolution which can encompass from the inflation epoch to the present time [30; 35; 44; 50]. As a matter of fact, RMVs appear to be quite consistent with the current observational data [45; 46].

In this models, vacuum plays a dynamical role with matter, in the sense that vacuum energy decays into particles and that particles annihilate into the vacuum. This gives rise to certain energy exchange between the vacuum and particles which takes place through the creation and annihilation of particles at a macroscopic scale, i.e., the entire universe [50]. We are interested in the thermodynamical aspects of this phenomenon, hence the study of the entropy of the universe will be paramount in this work. Needless to say, Thermodynamics has become more and more prominent in the field of cosmology since Bekenstein and Hawking discovered that the event horizon of a black-hole has entropy and temperature [5; 19; 20], which lead to the formulation of black-hole thermodynamics by Hawking, Carter and Bardeen [3]. In turn, black-hole thermodynamics served as inspiration for the formulation of the holographic principle [24; 52], in particular, for a possible profound relation between Thermodynamics and General Relativity [7; 8; 25], which we shall review.

In Section 2 we introduce the concept of the Generalised Second Law (GSL) of Thermodynamics, together with the definitions of event horizon, particle horizon, and apparent horizon. In Section 3 we discuss Jacobson's [25] original deduction of the Einstein field equations from classical entropy, showing that the Einstein equations can be deemed as an equation of state. In Section 4, we discuss how the Friedmann equations can be obtained from the unified first law, both considering and not considering an apparent horizon, closing with a derivation of the aforementioned equations in arbitrary dimension. Finally, in Section 5, we introduce Running Vacuum Models (RVMs) and we examine its thermodynamical implications within the early universe. We review the study performed in [50] and we introduce an extended RVM which explores the possibility of the existence of two different inflation epochs.

2

Cosmological horizons and the Generalised Second Law

In this section, we introduce the concept of the Generalised Second Law of Thermodynamics, that offers a new study perspective of entropy which will be the one we shall consider in this work. To be capable of applying the Generalised Second Law in our discussion of entropy, we are in need of a certain cosmological horizon, namely the apparent horizon. Here we describe this particular horizon together with two other crucial cosmological horizons.

In this day and age, the thermodynamic study of the expanding universe is performed from the point of view of the Generalised Second Law (GSL) of Thermodynamics. The GSL was first formulated by Bekenstein, who conjectured that: "The common entropy in the black-hole exterior plus the black-hole entropy never decreases." [5], where the black-hole entropy refers to the entropy associated to the area of the event horizon of the black-hole and the common entropy to the entropy of what resides outside the event horizon. Nowadays, this postulate has been extended to cosmological horizons, so that we consider that the sum of the entropy of radiation and matter inside the horizon - which is called the *volume entropy* $S_{\mathcal{V}}$ - and the entropy of the horizon itself - which we refer to as the *horizon entropy* $S_{\mathcal{A}}$ - is the magnitude that must never decrease.

Specifically, the total entropy $S_T \equiv S_{\mathcal{V}} + S_{\mathcal{A}}$ must fulfil two thermodynamic conditions. First, it must increase with the expansion of the universe. Second, its increase must diminish over time until the system the universe eventually reaches a final state of thermodynamic equilibrium. That is to say that

$$S'_T = S'_{\mathcal{V}} + S'_{\mathcal{A}} \geq 0, \quad S''_T = S''_{\mathcal{V}} + S''_{\mathcal{A}} < 0, \tag{2.1}$$

where the differentiation will be (in our case) with respect to the scale factor, for it is directly related to the expansion of the universe. Observe that the first condition is what we commonly refer to as the Second Law of Thermodynamics. The second one is called the Law of Thermodynamic Equilibrium (LTE), which is necessary in order to prevent entropy from increasing in a rampant manner. Then, the equilibrium situation is attained when the first derivative of the total entropy is zero and the second one is negative, that is $S'_T = 0$ and $S''_T < 0$, which corresponds to a state with maximum entropy, which conforms to what we understand as thermodynamic equilibrium.

Let us now explain what we mean when we talk about cosmological horizons and to which horizon we refer to when we speak of the horizon entropy. Rindler adequately treated the concept of cosmological horizons for the first time [39], distinguishing between event horizons and particle horizons. The *particle horizon* is defined as the boundary of the volume of space from which an observer can receive information at a given time t [32]. Then, the physical distance from the observer to the particle horizon is given by

$$d_p(t) = a(t) \int_{t_i}^{t} \frac{dt'}{a(t')} = a(t) \int_{0}^{a(t)} \frac{da'}{a'^2 H(a')}, \tag{2.2}$$

where t_i is the initial instant when the scale factor vanishes, i.e., $a(t_i) = 0$. One can see that if the universe has always been in decelerated expansion (as happens in radiation and matter dominated epochs), then $d_p(t) < \infty$, $\forall t$, that is the universe indeed has a particle horizon. In contrast, if the universe begins with an accelerated expansion, then $d_p(t) = \infty$, $\forall t$, which means that there is no particle horizon.

The *event horizon* is the complement of the particle horizon: it is the boundary of the set of points from which signals sent at a given time t will never be received by an observer in the future [32], hence the physical size of the event horizon is

$$d_e(t) = a(t) \int_{t}^{\infty} \frac{dt'}{a(t')} = a(t) \int_{a(t)}^{\infty} \frac{da'}{a'^2 H(a')}. \tag{2.3}$$

For flat or open Friedmann-Lemaître-Robertson-Walker (FLRW) universes with no cosmological constant, one has that $d_p(t) = \infty$, $\forall t$, so there is no event horizon for such universes, which means that all future events are accessible to the present observer. Conversely, the standard ΛCDM model does have an event horizon since $d_p(t) < \infty$, $\forall t$. It can also be seen that closed FLRW universes with no cosmological constant have an event horizon.

For our purposes, we ought to define another cosmological horizon, the apparent horizon. We consider the $(n+1)$-dimensional FLRW metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2 \right) = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.4}$$

where k is the spatial curvature and $d\Omega_{n-1}^2$ is the metric of the unit (n-1)-sphere. Assuming that all components present in the universe can be considered as perfect fluids, the energy-momentum tensor in the FLRW metric takes the form [37]

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}, \tag{2.5}$$

where u_μ is the four-velocity of a comoving observer.

In our choice of metric signature, the four-velocity satisfies $u_\mu u^\mu = -c^2 = -1$, for we are working in Planck units. Then, we immediately have that $u_\mu = (1, 0, \cdots, 0)$ in the $(n+1)$-dimensional FLRW metric (2.4). Thus, in this metric, the expansion scalar θ of the perfect fluid takes the form

$$\begin{aligned} \theta \equiv \nabla_\mu u^\mu &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} u^\mu \right) = \frac{1 - kr^2}{a^n(t) r^{n-1}} \partial_\mu \left(\frac{a^n(t) r^{n-1}}{1 - kr^2} u^\mu \right) \\ &= \frac{1 - kr^2}{a^n(t) r^{n-1}} \partial_0 \left(\frac{a^n(t) r^{n-1}}{1 - kr^2} \right) = \frac{1 - kr^2}{a^n(t) r^{n-1}} \frac{n a^{n-1}(t) \dot{a}(t) r^{n-1}}{1 - kr^2} = nH, \end{aligned} \tag{2.6}$$

where g is the determinant of the matrix formed by the components of the metric and $H \equiv \dot{a}/a$ is the Hubble parameter, where the dot denotes the derivative with respect to the cosmic time t. We know that the energy-momentum tensor fulfils the conservation law $\nabla_\mu T^{\mu\nu} = 0$, whose $\nu = 0$ component reads

$$\partial_0 \rho + \nabla_\mu u^\mu (\rho + p) = \dot{\rho} + nH(\rho + p) = 0, \tag{2.7}$$

where we have applied equation (2.6).

If we define $x^0 = t$ and $x^1 = r$, the radial coordinate $\tilde{r} \equiv a(t)r$ and the 2-dimensional metric h_{ab} as the projection of the FLRW metric $g_{\mu\nu}$ in the normal direction of the $(n-1)$-sphere, we can write equation (2.4) as

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega^2_{n-1}, \tag{2.8}$$

In these coordinates, the *apparent horizon* is defined by $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$ [2; 9; 12], for it is a marginally trapped surface where the expansion θ becomes 0 (recall equation (2.6), see also [37]). In the FLRW metric, the expression $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r}$ becomes

$$h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = -(\dot{a}r)^2 + \frac{1-kr^2}{a^2}a^2 = -H^2\tilde{r}^2 + 1 - \frac{k}{a^2}\tilde{r}^2 = 1 - \left(H^2 + \frac{k}{a^2}\right)\tilde{r}^2, \tag{2.9}$$

thus the radius of the apparent horizon is given by the expression

$$\tilde{r}_{AH}(t) = a(t)r_{AH}(t) = \frac{1}{\sqrt{H^2(t) + \frac{k}{a^2(t)}}}. \tag{2.10}$$

The apparent horizon is more suitable for the thermodynamic discussion of the expansion of the universe (see [8] for a review). For starters, we do not need to know the causal structure of spacetime, as happens with the particle and event horizons. Secondly, the holography based on it obeys the First Law of Thermodynamics, in contrast to the particle horizon [2] - as we will discuss in Section 4 and Section 3, respectively-. Moreover, the apparent horizon has a natural gravitational entropy associated with it, which is nothing less but the generalisation for cosmological horizons of the black-hole entropy found by Bekenstein and Hawking. For all these reasons, the apparent horizon is much more appropriate for thermodynamic discussions than the event or particle horizons.

As we have already explained, in this work we will consider the GSL of Thermodynamics, by which we must take into account both the volume entropy $S_\mathcal{V}$ and the horizon entropy $S_\mathcal{A}$. Regarding the apparent horizon, the volume entropy is just the entropy of all the matter and radiation contained by the sphere with radius $\tilde{r}_{AH}$. We know that the entropy density is defined as [28]

$$s \equiv \frac{S}{V} = \frac{\rho + p}{T} = \frac{1+w}{T}\rho, \tag{2.11}$$

where S is the entropy, V is the volume of the sphere, and ρ and p are, respectively, the density and the pressure of the matter and radiation contained by the sphere.

In practice, we assume that the universe is geometrically flat, i.e., that its spatial curvature is $k = 0$ since the current observational data appears to indicate so [36], so the radius of the apparent horizon reduces to $\tilde{r}_{AH}(t) = 1/H(t)$. Hence, we have that the volume of the sphere contained by the apparent horizon is $V_h(t) = (4\pi/3)r_{AH}^3(t) = 4\pi/(3H^3(t))$, thus the volume entropy is given by

$$S_{\mathcal{V}} = \frac{4\pi}{3}\frac{(1+w)\rho}{TH^3}. \tag{2.12}$$

The horizon entropy is given by the Bekenstein-Hawking expression [5; 19; 20]

$$S_{\mathcal{A}} = \frac{kA}{4l_P^2} = \frac{1}{4}\frac{A}{l_P^2}, \tag{2.13}$$

where k is the Boltzmann constant (the natural unit of entropy), $A = 4\pi\tilde{r}_{AH}^2$ is the area of the apparent horizon, and $l_P = \sqrt{\hbar G/c^3} = \sqrt{G}$ is the Planck length, where $\hbar$ is the reduced Planck constant and c is the speed of light in the vacuum. Notice that we have set $k = \hbar = c = 1$ since we are working in natural units, in which entropy is a dimensionless magnitude, as we realise from the second equality in equation (2.13). From this equation, we also see that the horizon entropy $S_{\mathcal{A}}$ is just one fourth of the area of the horizon in units of the fundamental area l_P^2. Since in natural units one has that $l_P = \sqrt{G} = 1/M_P$, where M_P is the Planck mass, we can rewrite the previous equation for the horizon entropy as

$$S_{\mathcal{A}} = \frac{A}{4G} = \pi\tilde{r}_{AH}^2 M_P^2 = \pi\frac{M_P^2}{H^2}. \tag{2.14}$$

We should keep in mind expressions (2.12) and (2.14), for we will employ them in our entropy discussion in Section 5.

3

Einstein field equations from classical entropy

In this section, we aim to provide a sensible deduction of the Einstein field equations. Our intention is to show how Einstein equations can be regarded as an equation of state when we consider them from a thermodynamic point of view, more precisely, when we think of entropy from a holographic perspective, taking into account its connection with information within the universe. To this end, we will closely follow [25] and [26].

In Classical Thermodynamics, entropy is defined as $dS = \delta Q/T$, where δQ is the heat flow and T is the temperature. Let us contemplate the region of spacetime that is causally connected from the viewpoint of a certain observer, e.g., for us, that would be the observable universe. Let us think of the boundary of this region, namely the particle horizon for the observer. The particle horizon is a null hypersurface composed of generators which are null geodesic segments that travel back in time from the observer [25]. Then, we can think of the causally connected region as our thermodynamic system, of the particle horizon as its boundary and of the universe beyond the horizon as its surroundings. Thus, the energy flowing across the particle horizon can be regarded as the heat flow δQ.

We observe that the boundary of our thermodynamic system is a causality barrier, in the sense that it prevents our causally connected region of sending information to or receiving any from the rest of the universe. The fact that there is information which is inaccessible from our region leads us to think that the particle horizon must be related to entropy [5]. This unavailable information is contained within correlations between vacuum fluctuations inside and outside the horizon [6]. If there is no lower limit to the wavelength of the quantum fields in the vicinity of the particle horizon, the entanglement entropy associated with it diverges in continuum quantum field theory. On the contrary, if there is a fundamental cutoff length l_c, then the entanglement entropy is finite and proportional to the area of the particle horizon in units of l_c^2, provided the radius of curvature of spacetime R satisfies $l_c \ll R$ [25]. Therefore, we assume that the entropy of our thermodynamic system is proportional to the particle horizon area. Notice that this assumption is coherent with the extensive nature of entropy, for the area also is an extensive characteristic of a horizon.

We have discussed how the vacuum fluctuations of quantum fields give rise to the entropy associated with the particle horizon. They also are responsible for the effective temperature experienced by a uniformly accelerated observer, according to the Unruh effect [11; 54]. Hence, we assume that the temperature of our system is that measured by such an observer within the particle horizon because of the Unruh effect, that is, the Unruh temperature, which is given by [1; 11; 53; 54]

$$T_U = \frac{\hbar a}{2\pi kc} = \frac{a}{2\pi}, \tag{3.1}$$

where a is the acceleration in the instantaneous rest frame of the observer, $\hbar$ is the reduced Planck constant, k is the Boltzmann constant, and c is the speed of light. Notice that we have set $\hbar = k = c = 1$ since we are working in Planck units. It is not surprising that the Unruh temperature has the same form as the Hawking temperature [1; 11; 19; 20]

$$T_H = \frac{\hbar g}{2\pi kc} = \frac{g}{2\pi}, \tag{3.2}$$

where g is the surface gravity.

We have finally managed to define the heat flow δQ, the entropy S and the temperature T_U for our thermodynamic system. At this point we realise that, in order to be consistent, the uniformly accelerated observer that measures the Unruh temperature must be also the one that measures the heat flow. However, this implies that different accelerated observers will measure different values, and that is a serious problem. Moreover, the observer acceleration diverges as the accelerated worldline approaches the particle horizon [25]. Then, we see from equation (3.1) that this means that the Unruh temperature diverges, as so does the heat flow. Yet, their ratio tends to a finite limit [25], hence we will study our thermodynamic system in this limit. The interest in this limit relies on the fact that it allows us to proceed with the discussion in local terms, which is what we want.

Having solved the problem that arises from the divergence of the observer acceleration, there is still another concern with our system. Our thermodynamic system is delimited by a boundary which is not stationary. The particle horizon evolves together with the expansion of the universe, so our system is not in equilibrium, hence we cannot apply Equilibrium Thermodynamics. To overcome this problem, we introduce the concept of *local Rindler horizon* (LRH), which is also another step towards making our argument local.

Let us consider a spacetime point x and a spacelike 2-surface element Σ such that $x \in \Sigma$. We choose Σ so that its past directed null normal congruence to one side - which we call the *inside* - has vanishing expansion and shear [37] at x (we can always choose Σ in a way that the expansion and shear vanish at first order in the surroundings of x [25]). Then the LRH of Σ is defined as the boundary to the past (on the inside) of Σ [26]. Hence, the LRH reaches equilibrium at x. Now we can apply the equilibrium thermodynamic relation $\delta Q = TdS$. However, since we have introduced a new notion of horizon, we need to adapt the previous definitions of heat flow, temperature, and entropy to our new system.

By the equivalence principle, we know that in the vicinity of any spacelike 2-surface element Σ the spacetime can be regarded as Minkowski spacetime and, consequently, with the Poincaré symmetry. Let us examine then an approximate Killing vector field χ^μ generating boosts orthogonal to and vanishing at Σ. The

Unruh effect tells [11; 54] us an observer moving with uniform acceleration through the Minkowski vacuum along χ^μ will observe a thermal spectrum of particles with temperature

$$T_U = \frac{\kappa}{2\pi}, \tag{3.3}$$

where κ is the acceleration of the Killing orbit, which can be identified with the surface gravity [37].

Having found an expression for the temperature, let us discuss now the heat flow. Assuming that all the heat flow across the LRH is energy carried by matter and radiation, we can define [26]

$$\delta Q = \int_{\mathcal{H}} T_{\mu\nu}\chi^\mu dx^\nu, \tag{3.4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter and radiation and $\mathcal{H}$ is the horizon generating the Killing vector field χ^μ. Let k^μ be the tangent vector to the horizon generators for an affine parameter λ that vanishes at Σ and is negative to its past. Then, one has that $\chi^\mu = -\kappa\lambda k^\mu$ and that $dx^\nu = k^\nu d\lambda dA$, where dA is the area element on a horizon cross section [25], so the heat flow can be rewritten as

$$\delta Q = -\kappa \int_{\mathcal{H}} \lambda T_{\mu\nu} k^\mu k^\nu d\lambda dA. \tag{3.5}$$

As for the entropy, we assume that is proportional to the horizon area, as we did before, so we have that $dS = C\delta A$, where C is a proportionality constant and δA is the area variation of a cross section of a pencil of generators of $\mathcal{H}$, which is given by

$$\delta A = \int_{\mathcal{H}} \theta d\lambda dA, \tag{3.6}$$

where θ is the expansion of the null congruence of horizon generators [25; 26] (in [26] we find a bit different but equivalent expression for the area variation δA). It is known that the Raychaudhuri equation reads [37]

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma^2 + \omega^2 - R_{\mu\nu}k^\mu k^\nu, \tag{3.7}$$

where $\sigma^2 = \sigma^{\mu\nu}\sigma_{\mu\nu}$ is the square of the shear, $\omega^2 = \omega^{\mu\nu}\omega_{\mu\nu}$ is the square of the vorticity, and $R_{\mu\nu}$ is the Ricci tensor. Since we are studying the expansion θ of the null geodesic congruence generating the LRH, we know from [37] that the vorticity becomes zero, that is, $\omega_{\mu\nu} = 0$. Moreover, we recall from the definition of the LRH that the congruence has vanishing expansion θ and shear σ at x (in fact, θ and σ vanish at first order in Σ). Thus, θ^2 and σ^2 are higher order terms which can be neglected when evaluating equation (3.7) in the neighbourhood of x. Therefore, integrating equation (3.7) under these conditions we obtain that

$$\theta = -\lambda R_{\mu\nu}k^\mu k^\nu, \tag{3.8}$$

when λ is small enough. Replacing the expression we have found for the expansion of the null congruence in equation (3.6), one has that

$$\delta A = -\int_{\mathcal{H}} \lambda R_{\mu\nu}k^\mu k^\nu d\lambda dA. \tag{3.9}$$

Finally, from equations (3.3), (3.5) and (3.9), we see that thermodynamic relation $\delta Q = TdS$ holds if, and only if, the expression

$$\begin{aligned} -\kappa \int_{\mathcal{H}} \lambda T_{\mu\nu} k^\mu k^\nu d\lambda dA &= -\frac{\kappa}{2\pi} C \int_{\mathcal{H}} \lambda R_{\mu\nu} k^\mu k^\nu d\lambda dA \\ T_{\mu\nu} k^\mu k^\nu &= \frac{C}{2\pi} R_{\mu\nu} k^\mu k^\nu \end{aligned} \tag{3.10}$$

is true $\forall\, k^\mu$ null, which is equivalent to

$$\frac{2\pi}{C} T_{\mu\nu} = R_{\mu\nu} + f g_{\mu\nu}, \tag{3.11}$$

where f is an arbitrary function and $g_{\mu\nu}$ is the metric tensor [25; 26]. It is known that the energy-momentum tensor fulfils the conservation law $\nabla_\mu T^{\mu\nu} = 0$ and that the contracted Bianchi identity reads [11]

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} \nabla^\nu R, \tag{3.12}$$

where R is the Ricci scalar. Therefore, since we have chosen a connection so that $\nabla_\mu g^{\mu\nu} = 0$, the covariant differentiation of equation (3.11) yields

$$0 = \frac{1}{2} \nabla^\nu R + g^{\mu\nu} \nabla_\mu f = \frac{1}{2} \nabla^\nu R + \nabla^\nu f, \tag{3.13}$$

which means that $f = -R/2 + \Lambda$, where Λ is a constant. Thus, equation (3.11) can be rewritten as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{2\pi}{C} T_{\mu\nu}. \tag{3.14}$$

All it remains is to specify the proportionality constant C. As we know from [19], Hawking proved that $C = 1/4$, which is something that we had already foreshadowed in equation (3.2), since the Hawking temperature was was first predicted to be of the form $T_H = g/8\pi$ [3]. At last, we obtain the **Einstein field equations**

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{3.15}$$

where Λ is regarded as the cosmological constant. Bear in mind that we are employing Planck units, so the Newton's gravitational constant G is set to 1.

4

Friedmann equations from the unified first law

In this section, we derive Friedmann equations from the unified first law of Thermodynamics. Our purpose, as it also was in Section 3, is to show how cosmological laws can be obtained from basic thermodynamic principles together with entropic concepts coming from the black-hole mechanics and holographic theory. For the usual derivation of the Friedmann equations in $n = 3$ spatial dimensions from the Einstein field equations, see Appendix A.

Before introducing the unified first law, we need to adequately define the concept of energy in our context. We want to consider a a spherical system geometrically defined by its radius $\tilde{r}$ (recall Section 2). The matter contained in this spherical surface gives rise to a certain gravitational field, which has an associated energy. Nevertheless, it is troublesome to properly define energy for a gravitational field in General Relativity, that is why quasi-local definitions of energy can be found in the literature [10]. These different conceptions of energy are defined on the boundary of a spacetime region, as is our case. One among them is the *Misner-Sharp energy* [31], particularly useful for our purposes.

The Misner-Sharp energy E is defined in a spherically symmetric spacetime and possesses some interesting properties, of which we mention a few here. In the small-sphere limit, the leading term in E is the product of volume and the energy density of the matter. In the vacuum, E reduces to the Schwarzschild energy. In $n = 3$ spatial dimensions, a sphere is trapped (see [37]) if $E > R/2$, marginally trapped if $E = R/2$, and untrapped if $E < R/2$, where R is the areal radius. A central singularity is spatial and trapped if $E > 0$, and temporal and untrapped if $E < 0$. For a detail discussion on these and more properties of the Misner-Sharp energy, we refer the curios reader to [21].

In our work, we consider a generalised definition of the Misner-Sharp energy, by which the total energy inside a sphere of radius $\tilde{r}$ is given by [2; 9; 12]

$$E \equiv \frac{n(n-1)V_n}{16\pi G}\tilde{r}^{n-2}\left(1 - h^{ab}\partial_a\tilde{r}\partial_b\tilde{r}\right), \tag{4.1}$$

where $V_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit n-sphere. Using equation (2.9), the last expression can be rewritten as

$$E = \frac{n(n-1)V_n}{16\pi G}\tilde{r}^{n}\left(H^2 + \frac{k}{a^2}\right). \tag{4.2}$$

If we now consider the system described by an apparent horizon, we see that the total energy inside it is given by the expression

$$E = \frac{n(n-1)V_n}{16\pi G}\tilde{r}_{AH}^{n-2}, \tag{4.3}$$

which coincides with the expression for the mass of the Schwarzschild black-hole if we consider the event horizon of the black-hole instead of the apparent horizon [2].

Let us now define [2; 9; 12] the work density W as

$$W \equiv -\frac{1}{2}T^{ab}h_{ab}, \tag{4.4}$$

and the energy-supply vector Ψ_a as

$$\Psi_a \equiv T_a^b \partial_b \tilde{r} + W\partial_a \tilde{r}, \tag{4.5}$$

where

$$T_{ab} = (\rho + p)\, u_a u_b + p h_{ab} \tag{4.6}$$

is the projection of the $(n+1)$-dimensional energy-momentum tensor $T_{\mu\nu}$ in the normal direction of the $(n-1)$-sphere (recall equation (2.5)). Recall that in our choice of metric signature we have that $u_\mu = (1, 0, \cdots, 0)$, so it is obvious that in the projected metric h_{ab} one has that $u_a = (1, 0)$. Therefore, the projected $(1+1)$-dimensional energy-momentum tensor T_{ab} reads

$$T_{ab} = (\rho + p)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + p\begin{pmatrix} -1 & 0 \\ 0 & \frac{a^2}{1-kr^2} \end{pmatrix} = \mathrm{diag}\left(\rho, \frac{a^2}{1-kr^2}p\right). \tag{4.7}$$

Then, we can compute the work density

$$W = \frac{1}{2}(\rho - p), \tag{4.8}$$

and the energy-supply vector

$$\Psi_a = \frac{1}{2}(\rho + p)(-H\tilde{r}, a). \tag{4.9}$$

As noted in [23], the work density at the apparent horizon should be deemed as the work done by a change of the apparent horizon and the energy-supply as the total energy flow through the apparent horizon. There it is shown that these three magnitudes are related by

$$\nabla E = A\Psi + W\nabla V, \tag{4.10}$$

where $A = nV_n\tilde{r}^{n-1}$ and $V = V_n\tilde{r}^n$ respectively are the area and the volume of the n-sphere with radius $\tilde{r}$. We shall refer to equation (4.10) as the **unified first law** [2; 9; 12; 22; 23]. We notice that there is a certain resemblance among this equation and the First Law of Thermodynamics - which reads

$$dU = \delta Q + \delta W, \tag{4.11}$$

where U the internal energy, Q is the heat flow and W is the work - if we regard $A\Psi$ as a certain heat flow and $W\nabla V$ as a form of differential work.

Classical Thermodynamics tells us that the heat flow is related to the entropy by $\delta Q = TdS$. Since the heat flow is connected to the change in energy, the entropy is related to the energy-supply term in equation (4.10), which we can rewrite as [2; 9]

$$A\Psi = \frac{\kappa}{8\pi G}\nabla A + \tilde{r}^{n-2}\nabla\left(\frac{E}{\tilde{r}^{n-2}}\right), \tag{4.12}$$

where the surface gravity κ is given by

$$\kappa \equiv \frac{1}{2\sqrt{-h}}\partial_a\left(\sqrt{-h}h^{ab}\partial_b\tilde{r}\right). \tag{4.13}$$

From equation (4.3) we acknowledge that the last term in equation (4.12) vanishes on the apparent horizon. If we assume that we can associate the Hawking temperature $T_H = \kappa/2\pi$ [19; 20] to the apparent horizon, then equation (4.12) becomes

$$A\Psi = \frac{T_H}{4G}\nabla A = T_H\nabla\left(\frac{A}{4G}\right) \tag{4.14}$$

at the apparent horizon, which suggest that we can associate the Bekenstein-Hawking entropy $S_{BH} = A/4G$ to the apparent horizon [9; 19; 20; 23].

Now that we have reached expressions for both the temperature and the entropy of the apparent horizon, let us calculate the heat flow through it. As we have previously stated, heat is just a form of energy, so the heat flow δQ through the apparent horizon during an infinitesimal interval of time dt is just the change of energy inside the apparent horizon that takes place in that same interval dt. In other words, we have that $\delta Q = -dE$, where the minus signs accounts for the fact that an outward heat flow from the apparent horizon implies a loss of energy.

At this point, we have discussed all the elements intervening in the expression $\delta Q = TdS$. As we see in [9], a couple of more steps would lead us to the deduction of the Friedmann equations from the classical definition of entropy. However, we observe that this derivation relies on the thermodynamic characteristics of the apparent horizon. Nevertheless, we can dispense with the latter and also obtain the Friedmann equations from thermodynamic principles if we consider the unified first law instead of classical entropy. As it is shown in [12] for the $n = 3$ case, this derivation makes us realise that the unified first law and the Friedmann equations are equivalent on any spherical surface. Therefore, in this way, we obtain the Friedmann equations in more general grounds, without being compelled to employ a specific surface.

We begin with differentiating equation (4.2), which yields

$$\begin{aligned} dE &= \frac{n(n-1)V_n}{16\pi G}\left[n\tilde{r}^{n-1}\left(H\tilde{r}dt + adr\right)\left(H^2 + \frac{k}{a^2}\right) + 2H\tilde{r}^n\left(\dot{H} - \frac{k}{a^2}\right)dt\right] \\ &= \frac{n(n-1)A}{16\pi G}\left\{\left[\left(H^2 + \frac{k}{a^2}\right) + \frac{2}{n}\left(\dot{H} - \frac{k}{a^2}\right)\right]H\tilde{r}dt + \left(H^2 + \frac{k}{a^2}\right)adr\right\}, \end{aligned} \tag{4.15}$$

where we have used that $A = nV_n\tilde{r}^{n-1}$. From equations (4.8) and (4.9), one has that

$$\begin{aligned} A\Psi + WdV &= A\frac{1}{2}\left(\rho + p\right)\left(-H\tilde{r}dt + adr\right) + \frac{1}{2}\left(\rho - p\right)V_n n\tilde{r}^{n-1}\left(H\tilde{r}dt + adr\right) \\ &= A\left(-pH\tilde{r}dt + a\rho dr\right). \end{aligned} \tag{4.16}$$

The unified first law (4.10) tells us that the two above expressions are equivalent. Therefore, matching the terms carrying dr in equation (4.15) with those in equation (4.16), we obtain that

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho \equiv \frac{16\pi G}{n(n-1)}\left(\rho_m + \rho_r + \rho_\Lambda\right), \tag{4.17}$$

which is nothing but the **first Friedmann equation** for an $(n+1)$-dimensional FLRW universe with spatial curvature k, where we have relabelled $\rho \equiv \rho_m + \rho_r + \rho_\Lambda$ to indicate that ρ stands for the total density of the different components of the universe, that is matter, radiation and vacuum energy. Analogously, the unified first law allows us to identify the terms carrying dt in equations (4.15) and (4.16), which leads us to the result

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}\left(\rho + p\right) \equiv -\frac{8\pi G}{n-1}\left(\rho_m + \rho_r + \rho_\Lambda + p_m + p_r + p_\Lambda\right), \tag{4.18}$$

where we have also redefined $p \equiv p_m + p_r$. Considering the equation of state (EoS) parameters for each component of the universe, namely $w_i = p_i/\rho_i$ for $i = m, r, \Lambda$, we can rewrite the previous expression as

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}\left[\left(1+w_m\right)\rho_m + \left(1+w_r\right)\rho_r + \left(1+w_\Lambda\right)\rho_\Lambda\right]. \tag{4.19}$$

Then, using that

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2, \tag{4.20}$$

equation (4.19) becomes

$$\frac{\ddot{a}}{a} - H^2 - \frac{k}{a^2} = -\frac{8\pi G}{n-1}\left[\left(1+w_m\right)\rho_m + \left(1+w_r\right)\rho_r + \left(1+w_\Lambda\right)\rho_\Lambda\right]. \tag{4.21}$$

Finally, if we substitute the Hubble parameter for the expression that equation (4.17) provides us, we obtain that

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{n(n-1)}\left[\left(n-2+nw_m\right)\rho_m + \left(n-2+nw_r\right)\rho_r - 2\rho_\Lambda\right], \tag{4.22}$$

which is the **second Friedmann equation** for an $(n+1)$-dimensional FLRW universe with spatial curvature k. Notice that we have used that the EoS parameter for the cosmological constant Λ is $w_\Lambda = -1$. As we should expect, the found Friedmann equations (4.17) and (4.22) coincide with those derived in [12] when we set $n = 3$.

5

Running Vacuum Models in the early universe

In this section, we study Running Vacuum Models (RVMs) from the perspective of the GSL of Thermodynamics. We examine whether or not the total entropy of the universe arising from dynamical vacuum energy density evolving under the RVM description satisfies or violates the GSL. As we have aforementioned, RVMs provide a good fit to the current cosmological observations as compared to the standard ΛCDM model [43; 44; 50]. Besides, the RVM can also be extended so that it includes the inflationary epoch and its transition to the radiation dominated epoch [30; 35; 44; 50]. This will be our object of study, the RVM in the early universe, when inflation took place and when radiation was dominant, before the matter-dominated epoch of the universe expansion. In particular, we postulate the existence of two different inflationary epochs and inspect whether or not this assumption leads us to an entropy that fulfils the GSL of Thermodynamics.

The RVM is well described in the aforementioned references; we encourage the interested reader to see [43; 44; 47; 48; 49] for review and further analysis on RVM. In our work, we limit ourselves to the expressions which are essential for our interest. The RVM is based on a renormalization group equation that determines the evolution of the vacuum energy density ρ_Λ as a function of the Hubble parameter H, which up to $\mathcal{O}(H^6)$ reads [4]

$$\frac{d\rho_\Lambda(H)}{d\ln H^2} = \frac{1}{(4\pi)^2}\sum_i \left[a_i M_i^2 H^2 + b_i H^4 + c_i \frac{H^6}{M_i^2}\right], \tag{5.1}$$

where the coefficients $a_i, b_i, c_i \in \mathbb{R}$ are dimensionless and receive contributions from loop corrections arising from both boson and fermion matter fields with different masses M_i. Using that

$$d\ln H^2 = \frac{1}{H^2} 2H dH = \frac{2}{H} dH \tag{5.2}$$

we integrate the differential equation (5.1) and obtain that

$$\begin{aligned} \rho_\Lambda(H) &= \frac{1}{8\pi^2}\sum_i \left[\frac{1}{2}a_i M_i^2 H^2 + \frac{1}{4}b_i H^4 + \frac{1}{6}c_i \frac{H^6}{M_i^2}\right] + C \\ &= \frac{3c_0}{8\pi G} + \frac{3}{8\pi G}\sum_i \left[\frac{G}{6\pi}a_i M_i^2 H^2 + \frac{G}{12\pi}b_i H^4 + \frac{G}{18\pi}c_i \frac{H^6}{M_i^2}\right], \end{aligned} \tag{5.3}$$

In natural units $\hbar = c = 1$, the reduced Planck mass reads $M_P = \sqrt{\hbar c/8\pi G} = M_P = 1/\sqrt{8\pi G}$. Let us then define the following dimensionless parameters

$$\nu \equiv \frac{1}{48\pi^2}\sum_i a_i \frac{M_i^2}{M_P^2}, \quad \alpha \equiv \frac{1}{96\pi^2}\frac{H_I^2}{M_P^2}\sum_i b_i, \quad \beta \equiv \frac{1}{144\pi^2}\frac{H_J^4}{M_P^2 M_i^2}\sum_i c_i, \tag{5.4}$$

where H_I and H_J are the Hubble rate at two different scales of inflation. Defining $\kappa^2 \equiv 8\pi G$, equation (5.3) can be rewritten as

$$\rho_\Lambda(H) = \frac{\Lambda(H)}{\kappa^2} = \frac{3}{\kappa^2}\left(c_0 + \nu H^2 + \alpha\frac{H^4}{H_I^2} + \beta\frac{H^6}{H_J^4}\right). \tag{5.5}$$

Obviously, for $\nu = \alpha = \beta = 0$, we recover the standard ΛCDM model with no dynamical evolution of the vacuum energy density. For the sake of argument, we define three more dimensionless parameters

$$\chi \equiv \frac{H}{H_I}, \quad \eta \equiv \frac{H_I}{H_J}, \quad \delta \equiv \beta\eta^4, \tag{5.6}$$

so that we can express the vacuum energy density in the early universe as

$$\rho_\Lambda(H) = \frac{3}{\kappa^2}\left(\nu H^2 + \alpha\chi^2 H^2 + \delta\chi^4 H^2\right) = \frac{3H^2}{\kappa^2}\left(\nu + \alpha\chi^2 + \delta\chi^4\right), \tag{5.7}$$

where we have omitted the constant term c_0, for at early stages of the universe it can be neglected in front of the other terms. In the early universe, since the Hubble rate H has an extremely large value, c_0 is much smaller then the higher order terms in equation (5.5). Then, the constant term c_0 acquires relevance only in the late time universe, when H is sufficiently small so that the higher order terms are irrelevant (see [50]). Therefore, we will not consider the c_0 term throughout this section.

Let us discuss the two different scales of inflation that we have introduced above. First, we suppose that $H_I \ll H_J$. Then, if α and β are of the same order of magnitude, i.e., $\alpha \sim \beta$, we see that $\delta \ll \alpha$ since $\delta/\alpha \sim \eta^4 \ll 1$. Hence, when $H_I \ll H_J$, the δ term can be neglected, which means that inflation is dominated by the H_I Hubble rate. In contrast, if $H_I \gg H_J$, then $\delta \gg \alpha$ by the same reasoning that we followed before, and consequently inflation is dominated by the H_J Hubble rate.

Surprisingly, we have just realised that inflation is governed by the lowest inflationary scale among H_I and H_J. For instance, if the highest Hubble rate power (in our case, H^6) is associated with the highest inflationary scale (that is, H_J, so that $H_I \ll H_J$), it is the lower inflationary Hubble rate power (in our case, H^4) the one that controls inflation. This behaviour can be understood as follows. In the early universe, we start from very high values for the Hubble rate $H(a)$, which decreases along with the expansion of the universe, in other words, decreases with the scale factor a. At some point $a = a_J$, the Hubble rate reaches the value $H(a_J) = H_J$, so that the H^6 term in equation (5.7) (which is the χ^4 term) is of order 1, while the H^4 (χ^2) term happens to be much greater than one, for at $a = a_J$ one has that $H(a_J)^4/H_I^2 = H_J^4/H_I^2 \gg 1$. From the point $a = a_J$ forward, the H^6 term falls below order 1, while the H^4 remains much greater than 1 for a considerable long period, therefore it ends up dominating inflation.

Regarding the dimensionless parameters ν and α, they are expected to be naturally small. First, we know that $M_i^2 \ll M_P^2$ for all particles, even for the heavy fields within a typical Grand Unified Theory (GUT) since $M_P \approx 2.43 \times 10^{18}$ GeV and in GUTs we expect $M_i \sim 10^{16}$ GeV [42], so we can assure that $|\nu| \ll 1$. Second, we expect the Hubble rate at inflationary scale to positively be below the Planck scale, hence we have that $|\alpha| \ll 1$. The δ parameter concerns the two inflationary scales. In this work, we shall assume that $H_I \ll H_J$, for it is the usual scenario, so we are certain that $|\delta| \ll 1$.

To calculate the vacuum energy density and, afterwards, the radiation density, we should employ the first (4.17) and second (4.22) Friedmann equations for $n = 3$ and flat spacetime, which read as follows:

$$3H^2 = \kappa^2 \left(\rho_m + \rho_r + \rho_\Lambda\right), \tag{5.8}$$

and

$$3\frac{\ddot{a}}{a} = -\frac{\kappa^2}{2}\left[\left(1 + 3w_m\right)\rho_m + \left(1 + 3w_r\right)\rho_r - 2\rho_\Lambda\right], \tag{5.9}$$

Using (4.20) and (4.21), one can rewrite equation (5.9) as

$$3H^2 + 2\dot{H} = -\kappa^2 \left(w_m\rho_m + w_r\rho_r - \rho_\Lambda\right). \tag{5.10}$$

Due to the very high temperature, the early universe is dominated by radiation, so the mass density ρ_m can be neglected at this stage. Then, using that the EoS parameter for radiation is $w_r = 1/3$, equations (5.8) and (5.10) become

$$3H^2 = \kappa^2 \left(\rho_r + \rho_\Lambda\right), \tag{5.11}$$

and

$$3H^2 + 2\dot{H} = \kappa^2 \left(\rho_\Lambda - \frac{1}{3}\rho_r\right). \tag{5.12}$$

From equation (5.11) we obtain an expression for ρ_r that, replaced in equation (5.12), leads to the equation

$$\dot{H} + 2H^2 = \frac{2}{3}\kappa^2\rho_\Lambda, \tag{5.13}$$

Considering equation (5.7), the previous expression reads

$$H' = \frac{2}{a}H\left(\nu - 1 + \alpha\chi^2 + \delta\chi^4\right), \tag{5.14}$$

where we have used that $\dot{H} = aHH'$, with prime denoting d/da, hence finally obtaining an ordinary differential equation for the Hubble parameter H as a function of the scale factor a.

5.1 RVM up to $\mathcal{O}(H^4)$

In this subsection, we shall review the results obtained in [50] in their study of RVM in the early universe. To this end, we consider the vacuum energy density up to fourth order, i.e., we set $\beta = 0$, so that equation (5.14) becomes

$$H' = \frac{2}{a} H \left(\nu - 1 + \alpha \chi^2\right) = \frac{2}{a} \left[(\nu - 1) H + \alpha \frac{H^3}{H_I^2} \right], \tag{5.15}$$

where we have employed definition (5.6). Integrating this differential equation we obtain that

$$H(a) = \sqrt{\frac{1-\nu}{\alpha}} \frac{H_I}{\sqrt{1 + D a^{4(1-\nu)}}}, \tag{5.16}$$

where $D > 0$ is a positive constant since H must decrease with the expansion. From equation (5.7), we obtain the vacuum energy density

$$\rho_\Lambda(a) = \frac{3}{\kappa^2} \left(\nu H^2 + \alpha \frac{H^4}{H_I^2} \right) = \frac{3 H_I^2 (1-\nu) \left(1 + \nu D a^{4(1-\nu)}\right)}{\kappa^2 \alpha \left(1 + D a^{4(1-\nu)}\right)^2}, \tag{5.17}$$

and from equation (5.11), we obtain the radiation density

$$\rho_r(a) = \frac{3H^2}{\kappa^2} - \rho_\Lambda(a) = \frac{3 H_I^2 (1-\nu)^2 D a^{4(1-\nu)}}{\kappa^2 \alpha \left(1 + D a^{4(1-\nu)}\right)^2}. \tag{5.18}$$

The vacuum energy density is enormous at the early stages of the universe, but as vacuum decays into radiation the radiation density rises. Eventually, both densities, ρ_Λ and ρ_r, meet at the so-called *equality point* $a = a_{eq}$, at which one has that $\rho_\Lambda(a_{eq}) = \rho_r(a_{eq})$. We can use this fact to rescalate the scale factor a at the equality point and, consequently, eliminate the constant D from our equations:

$$D = \frac{1}{1 - 2\nu} a_{eq}^{-4(1-\nu)} \equiv a_*^{-4(1-\nu)}. \tag{5.19}$$

Therefore, we can rewrite equation (5.16) as

$$H(\hat{a}) = \frac{\tilde{H}_I}{\sqrt{1 + \hat{a}^{4(1-\nu)}}}, \tag{5.20}$$

where we have defined

$$\hat{a} \equiv \frac{a}{a_*}, \quad \tilde{H}_I \equiv \sqrt{\frac{1-\nu}{\alpha}} H_I, \tag{5.21}$$

Correspondingly, we can rewrite equations (5.17) and (5.18) as

$$\rho_\Lambda(\hat{a}) = \tilde{\rho}_I \frac{1 + \nu \hat{a}^{4(1-\nu)}}{\left[1 + \hat{a}^{4(1-\nu)}\right]^2}, \quad \rho_r(\hat{a}) = \tilde{\rho}_I (1-\nu) \frac{\hat{a}^{4(1-\nu)}}{\left[1 + \hat{a}^{4(1-\nu)}\right]^2}, \tag{5.22}$$

where we have defined

$$\tilde{\rho}_I \equiv \frac{3}{\kappa^2} \tilde{H}_I^2. \tag{5.23}$$

It is well known that the energy density for relativistic particles reads

$$\rho_r = \frac{\pi^2}{30} g_* T_r^4, \tag{5.24}$$

where g_* is the total number of effectively massless degrees of freedom [28]. The radiation temperature associated to the radiation density is given by

$$T_r = \tilde{T}_I(1-\nu)^{1/4}\frac{\hat{a}^{(1-\nu)}}{\sqrt{1+\hat{a}^{4(1-\nu)}}}, \tag{5.25}$$

where we have associated a temperature $\tilde{T}_I$ to the characteristic inflation density $\tilde{\rho}_I$, as in equation (5.24)

$$\tilde{\rho}_I \equiv \frac{\pi^2}{30}g_*\tilde{T}_I^4. \tag{5.26}$$

On the one hand, the horizon entropy of the universe is given by equation (2.14), so by equation (5.20) we have that

$$S_{\mathcal{A}}(\hat{a}) = \pi\frac{M_P^2}{H^2(\hat{a})} = \frac{\pi M_P^2\left(1+\hat{a}^{4(1-\nu)}\right)}{\tilde{H}_I^2}. \tag{5.27}$$

On the other hand, the volume entropy of the universe is given by equation (2.12), so by equations (5.20), (5.22) and (5.25), we have that

$$S_{\mathcal{V}}(\hat{a}) = \frac{4\pi}{3}\frac{(1+w_r)\rho_r}{TH^3(\hat{a})} = \frac{8\pi^3}{135}g_*\left(\frac{\tilde{T}_I}{\tilde{H}_I}\right)^3(1-\nu)^{3/4}\hat{a}^{3(1-\nu)}, \tag{5.28}$$

where we have used that the EoS parameter for radiation is $w_r = 1/3$. Therefore, we have that the total entropy reads

$$S_T(\hat{a}) = \frac{\pi M_P^2\left(1+\hat{a}^{4(1-\nu)}\right)}{\tilde{H}_I^2} + \frac{8\pi^3}{135}g_*\left(\frac{\tilde{T}_I}{\tilde{H}_I}\right)^3(1-\nu)^{3/4}\hat{a}^{3(1-\nu)}. \tag{5.29}$$

We calculate the first and second derivatives, which read

$$S_T'(\hat{a}) = \frac{4\pi(1-\nu)\hat{a}^{2-3\nu}}{45\tilde{H}_I^3}\left[45\tilde{H}_I M_P^2\hat{a}^{1-\nu} + 2\pi^2 g_*\tilde{T}_I^3(1-\nu)^{3/4}\right], \tag{5.30}$$

and

$$S_T''(\hat{a}) = \frac{4\pi(1-\nu)\hat{a}^{1-3\nu}}{45\tilde{H}_I^3}\left[45\tilde{H}_I M_P^2(3-4\nu)\hat{a}^{1-\nu} + 2\pi^2 g_*\tilde{T}_I^3(2-3\nu)(1-\nu)^{3/4}\right]. \tag{5.31}$$

If we take a look at Figure 5.1, we observe that the horizon entropy $S_{\mathcal{A}}$ increases more rapidly than the volume entropy $S_{\mathcal{V}}$. For instance, in the two first rows of plots, we notice that the value for the entropy is greater in the horizon contribution. In addition, regarding the other two rows, one can appreciate that the first and second derivatives of $S_{\mathcal{A}}$ are steeper than those of $S_{\mathcal{V}}$. All plots in Figure 5.1 do not just indicate that entropy grows with the scale factor, but that it grows exponentially, something that violates the GSL of Thermodynamics. This violation of the GSL in the early universe is, in fact, welcomed. Such explosive behaviour at the beginning of the universe is useful for solving the entropy/horizon problems [50]. Also, we must keep in mind that there is a positive cosmological constant c_0 related to the vacuum energy, which becomes relevant in the late times of the universe and leads the entropy to behaviour that fulfils the GSL at the present time (see [50]). In other words, the fact that entropy violates the GSL in the early universe is not troublesome as long as it satisfies it after this exponential increase. It is worth mentioning the initial plateau that $S_{\mathcal{A}}$ displays for $\hat{a} \in [0, 1)$, whose value is approximately $S_{\mathcal{A}}(\hat{a} \ll 1) \approx \pi M_P^2/\tilde{H}_I^2$.

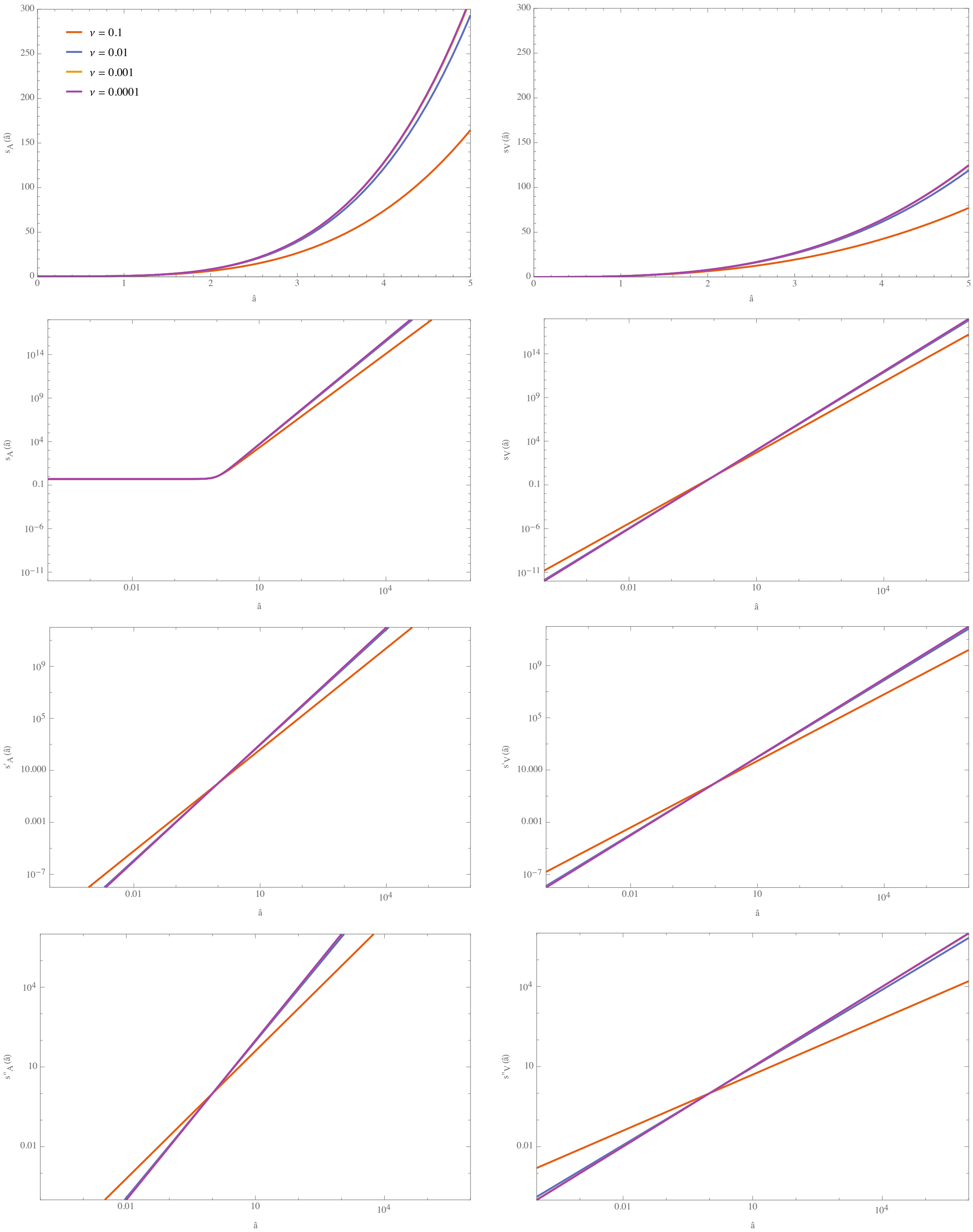

Figure 5.1: Here we present the entropy plots arising from the RVM up to $\mathcal{O}(H^4)$. On the left side, we display the horizon entropy $S_{\mathcal{A}}(\hat{a})$, equation (5.27), both in a linear scale and in a bilogarithmic scale, and its corresponding first and second derivatives in bilogarithmic scale, which we find within equations (5.30) and (5.31). On the right side, we display the corresponding plots for the volume entropy $S_{\mathcal{V}}(\hat{a})$, that is equation (5.28), and its first and second derivatives, which we there again find in equations (5.30) and (5.31). In all plots we have normalised the magnitudes with respect to the corresponding value at the equality point $\hat{a} = 1$, hence the lower case s in the axis label to indicate the normalised value, e.g. $s_{\mathcal{A}}(\hat{a}) \equiv S_{\mathcal{A}}(\hat{a})/S_{\mathcal{A}}(1)$. Four different values for the ν parameter are displayed: $\nu = 0.1$ (red), $\nu = 0.01$ (blue), $\nu = 0.001$ (orange), and $\nu = 0.0001$ (purple).

5.2 RVM up to $\mathcal{O}(H^6)$

In this subsection, we consider the vacuum energy density up to sixth order, that is we recover equation (5.14)

$$H' = \frac{2}{a} H \left(\nu - 1 + \alpha\chi^2 + \delta\chi^4\right) = \frac{2}{a}\left[(\nu-1)H + \alpha\frac{H^3}{H_I^2} + \delta\frac{H^5}{H_I^4}\right]. \tag{5.32}$$

Integrating this differential equation, we obtain the following expression for the Hubble parameter H as a function of the scale factor a:

$$Aa^{8(\nu-1)\gamma} = H^{4\gamma}\frac{\left[(\alpha+\gamma)H_I^2 + 2\delta H^2\right]^{\alpha-\gamma}}{\left[(\alpha-\gamma)H_I^2 + 2\delta H^2\right]^{\alpha+\gamma}}, \tag{5.33}$$

where $\gamma \equiv \sqrt{\alpha^2 - 4\delta(\nu-1)}$, and where $A > 0$ is a positive constant. Unfortunately, we have not found any analytical solution to the equation above. Because of that, and because we want to study the behaviour of the terms related to inflation, we neglect the first order term in equation (5.32). As consequence, we consider the differential equation

$$H' = \frac{2}{a}\left(\alpha\frac{H^3}{H_I^2} + \delta\frac{H^5}{H_I^4}\right). \tag{5.34}$$

This differential equation yields

$$\begin{aligned} 2\int\frac{da}{a} &= \int\frac{dH}{\alpha\frac{H^3}{H_I^2} + \delta\frac{H^5}{H_I^4}}, \\ 2\log(a) + B_2 &= H_I^4\int\frac{dH}{\alpha H_I^2 H^3 + \delta H^5}, \end{aligned} \tag{5.35}$$

where $B_2 > 0$ is an integration constant. Let us solve the integral on the right hand side of the equation above:

$$\begin{aligned} \int\frac{dH}{\alpha H_I^2 H^3 + \delta H^5} &= \int\frac{dH}{H^3\left(\alpha H_I^2 + \delta H^2\right)} \\ &= \int\left(\frac{\delta^2 H}{\alpha^2 H_I^4\left(\delta H^2 + \alpha H_I^2\right)} - \frac{\delta}{\alpha^2 H_I^4 H} + \frac{1}{\alpha H_I^2 H^3}\right)dH \\ &= \frac{\delta^2}{\alpha^2 H_I^4}\int\frac{H dH}{\delta H^2 + \alpha H_I^2} - \frac{\delta}{\alpha^2 H_I^4}\int\frac{dH}{H} + \frac{1}{\alpha H_I^2}\int\frac{dH}{H^3} \\ &= \frac{\delta}{2\alpha^2 H_I^4}\log\left(\delta H^2 + \alpha H_I^2\right) - \frac{\delta}{\alpha^2 H_I^4}\log(H) - \frac{1}{2\alpha H_I^2 H^2}, \end{aligned} \tag{5.36}$$

thus equation (5.35) reads

$$B_1 + \frac{2\log(a)}{H_I^4} = -\frac{\delta\log(H)}{\alpha^2 H_I^4} + \frac{\delta\log\left(\alpha H_I^2 + \delta H^2\right)}{2\alpha^2 H_I^4} - \frac{1}{2\alpha H_I^2 H^2}, \tag{5.37}$$

where we have defined $B_1 \equiv B_2/H_I^4$. If we define $\log(B) \equiv 2\alpha^2 H_I^4 B_1$, the last expression turns into

$$\begin{aligned} \log(B) + 4\alpha^2\log(a) &= -2\delta\log(H) + \delta\log\left(\alpha H_I^2 + \delta H^2\right) - \alpha\frac{H_I^2}{H^2}, \\ \log\left(Ba^{4\alpha^2}\right) &= \delta\log\left(\alpha\frac{H_I^2}{H^2} + \delta\right) - \alpha\frac{H_I^2}{H^2}, \\ \log\left(B^{1/\delta}a^{4\alpha^2/\delta}\right) + \frac{\alpha}{\delta}\frac{H_I^2}{H^2} &= \log\left(\alpha\frac{H_I^2}{H^2} + \delta\right). \end{aligned} \tag{5.38}$$

Since the exponential function is a bijective function, we know that equation (5.38) yields

$$\begin{aligned} B^{1/\delta}a^{4\alpha^2/\delta}e^{\frac{\alpha}{\delta}\frac{H_I^2}{H^2}} &= \alpha\frac{H_I^2}{H^2}+\delta, \\ B^{1/\delta}a^{4\alpha^2/\delta} &= \left(\alpha\frac{H_I^2}{H^2}+\delta\right)e^{-\frac{\alpha}{\delta}\frac{H_I^2}{H^2}}, \\ -\frac{B^{1/\delta}a^{4\alpha^2/\delta}}{\delta e} &= \left(-\frac{\alpha}{\delta}\frac{H_I^2}{H^2}-1\right)e^{-\frac{\alpha}{\delta}\frac{H_I^2}{H^2}-1}, \end{aligned} \tag{5.39}$$

which, by definition, leads us to

$$H(a) = \sqrt{\frac{\alpha}{\delta}}\frac{H_I}{\sqrt{-W\left(-\frac{B^{1/\delta}a^{4\alpha^2/\delta}}{\delta e}\right)-1}}, \tag{5.40}$$

where $B>0$ is a positive constant and $W(x)$ is called the *Lambert W function* or the *product logarithm* (see Appendix B). Thus, using equation (5.7), we can write the vacuum energy density ρ_Λ as a function of the scale factor a

$$\rho_\Lambda(a) = \frac{3}{\kappa^2}\left(\alpha\frac{H^4}{H_I^2}+\delta\frac{H^6}{H_I^4}\right) = \frac{3\alpha^3}{\kappa^2\delta^2}H_I^2\frac{W\left(-\frac{D^{1/\delta}a^{4\alpha^2/\delta}}{\delta e}\right)}{\left[W\left(-\frac{B^{1/\delta}a^{4\alpha^2/\delta}}{\delta e}\right)+1\right]^3}. \tag{5.41}$$

Likewise, from equation (5.11), we can write the radiation density ρ_r as a function of the scale factor a

$$\rho_r(a) = \frac{3\alpha^3}{\kappa^2\delta^2}H_I^2\left[-\frac{\delta}{\alpha^2}\frac{1}{W\left(-\frac{B^{1/\delta}a^{4\alpha^2/\delta}}{\delta e}\right)+1}-\frac{W\left(-\frac{B^{1/\delta}a^{4\alpha^2/\delta}}{\delta e}\right)}{\left[W\left(-\frac{B^{1/\delta}a^{4\alpha^2/\delta}}{\delta e}\right)+1\right]^3}\right]. \tag{5.42}$$

Just as we have done in the previous subsection, we can eliminate the constant B by use of the equality condition, namely $\rho_\Lambda(a_{eq}) = \rho_r(a_{eq})$, for it implies that

$$B = \left(\mp\Delta+\alpha^2+\delta\right)^\delta e^{\pm\Delta-\alpha^2}a_{eq}^{-4\alpha^2} \equiv a_*^{-4\alpha^2}, \tag{5.43}$$

where $\Delta \equiv \sqrt{\alpha^4+2\alpha^2\delta}$. Hence, we can rewrite equation (5.40) as

$$H(\hat{a}) = \frac{\tilde{H}_I}{\sqrt{-W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right)-1}}, \tag{5.44}$$

where we have redefined $\tilde{H}_I$

$$\tilde{H}_I \equiv \sqrt{\frac{\alpha}{\delta}}H_I. \tag{5.45}$$

Then, the radiation and vacuum energy densities take the forms

$$\rho_\Lambda(\hat{a}) = \tilde{\rho}_I\frac{W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right)}{\left[W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right)+1\right]^3}, \tag{5.46}$$

and

$$\rho_r(\hat{a}) = \tilde{\rho}_I \left[-\frac{\delta}{\alpha^2} \frac{1}{W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right) + 1} - \frac{W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right)}{\left[W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right) + 1\right]^3} \right], \tag{5.47}$$

where we have redefined

$$\tilde{\rho}_I \equiv \frac{3\alpha^3}{\kappa^2\delta^2} \tilde{H}_I^2. \tag{5.48}$$

From equation (5.24) we know that, in this case, the radiation temperature associated to the radiation density is given by

$$T_r = \tilde{T}_I \left[-\frac{\delta}{\alpha^2} \frac{1}{W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right) + 1} - \frac{W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right)}{\left[W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right) + 1\right]^3} \right]^{1/4}, \tag{5.49}$$

where we have associated a temperature $\tilde{T}_I$ to the characteristic inflation density $\tilde{\rho}_I$, as in equation (5.26). As before, the horizon entropy of the universe is given by equation (2.14), so by equation (5.44) we have that

$$S_{\mathcal{A}}(\hat{a}) = \pi \frac{M_P^2}{H^2(\hat{a})} = \pi \frac{M_P^2}{H_I^2} \left[-W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e} \right) - 1 \right], \tag{5.50}$$

Likewise, the volume entropy of the universe is given by equation (2.12), so by equations (5.44), (5.47) and (5.49), we have that

$$S_{\mathcal{V}}(\hat{a}) = \frac{4\pi}{3} \frac{(1+w_r)\rho_r}{TH^3(\hat{a})} = \frac{8\pi^3}{135} g_* \frac{\tilde{T}_I^3}{\tilde{H}_I^3} f(\hat{a}), \tag{5.51}$$

where we have defined the function

$$f(\hat{a}) \equiv \left[-\frac{\delta}{\alpha^2} \frac{1}{W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right) + 1} - \frac{W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right)}{\left[W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e}\right) + 1\right]^3} \right]^{3/4} \left[-W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e} \right) - 1 \right]^{3/2}, \tag{5.52}$$

Therefore total entropy inside the horizon reads

$$S_T(\hat{a}) = \pi \frac{M_P^2}{H_I^2} \left[-W\left(-\frac{\hat{a}^{4\alpha^2/\delta}}{\delta e} \right) - 1 \right] + \frac{8\pi^3}{135} g_* \frac{\tilde{T}_I^3}{\tilde{H}_I^3} f(\hat{a}). \tag{5.53}$$

Both in Figure 5.2 and in Figure 5.3 we observe that the horizon entropy $S_{\mathcal{A}}$ grows faster that the volume entropy $S_{\mathcal{V}}$, as it happened in Figure 5.1. Nonetheless, in Figure 5.1, the entropy increases exponentially, opposite to the current situation, in which we observe that both horizon and volume entropy increase slower as the universe expands. The plots we observe remind us of asymptotic behaviour, approaching an upper bound. Actually, this is what the GSL of Thermodynamics requires, an entropy that constantly increases towards a saturation point, a maximum. Then, we see that, in this situation, both the horizon and the volume entropy

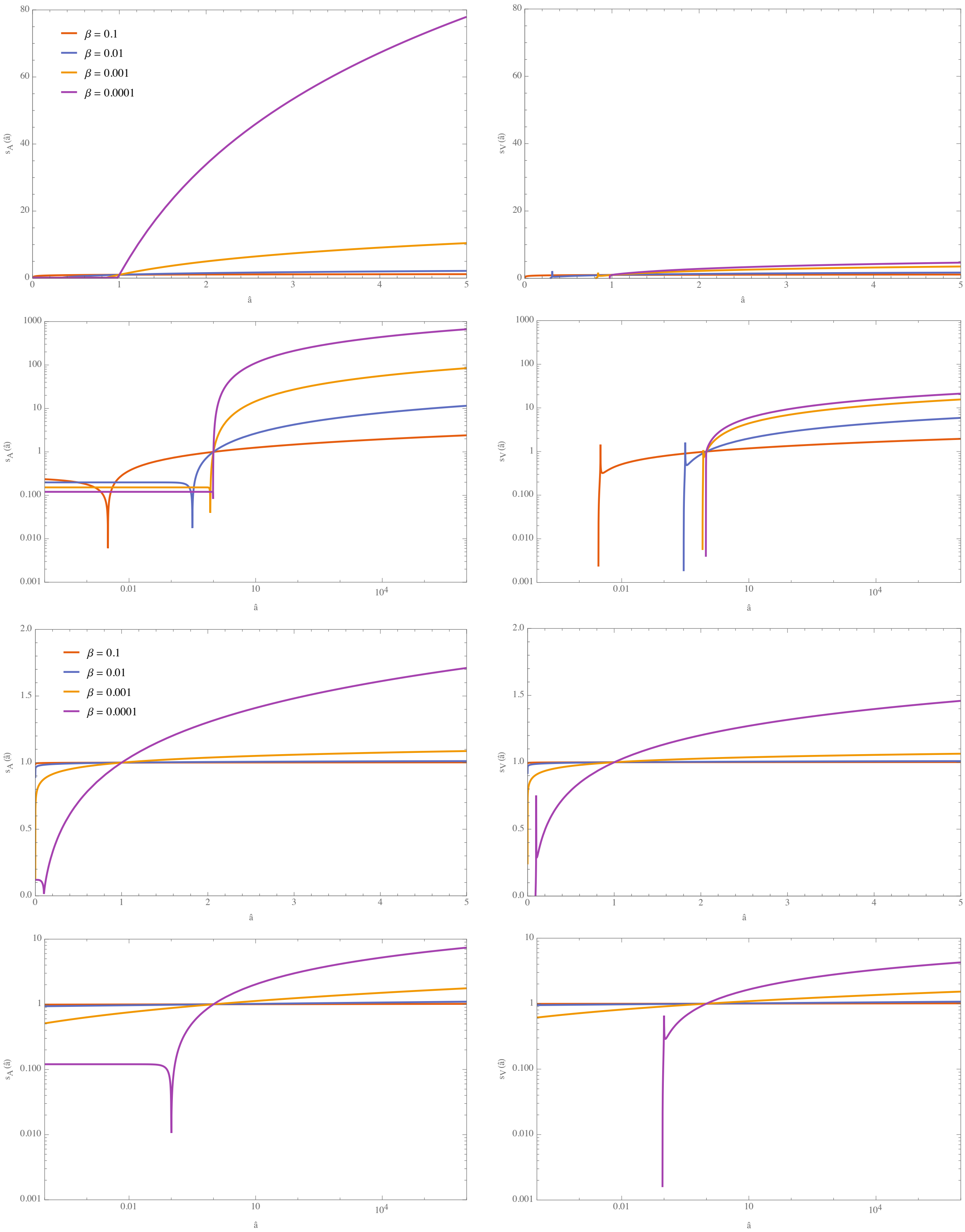

Figure 5.2: Here we present the entropy plots arising from the RVM up to $\mathcal{O}(H^6)$ for α and $\eta = 1$ fixed. On the left side, we display the horizon entropy $S_{\mathcal{A}}(\hat{a})$, equation (5.50), both in a linear scale and in a bilogarithmic scale. On the right side, we display the corresponding plots for the volume entropy $S_{\mathcal{V}}(\hat{a})$, that is equation (5.51). In all plots we have normalised the magnitudes with respect to the corresponding value at the equality point $\hat{a} = 1$, hence the lower case s in the axis label to indicate the normalised value, e.g. $s_{\mathcal{A}}(\hat{a}) \equiv S_{\mathcal{A}}(\hat{a})/S_{\mathcal{A}}(1)$, as in Figure 5.1. Four different values for the β parameter are displayed: $\beta = 0.1$ (red), $\beta = 0.01$ (blue), $= 0.001$ (orange), and $\beta = 0.0001$ (purple). The four top plots correspond to $\alpha = 0.1$, and the four bottom ones to $\alpha = 0.01$.

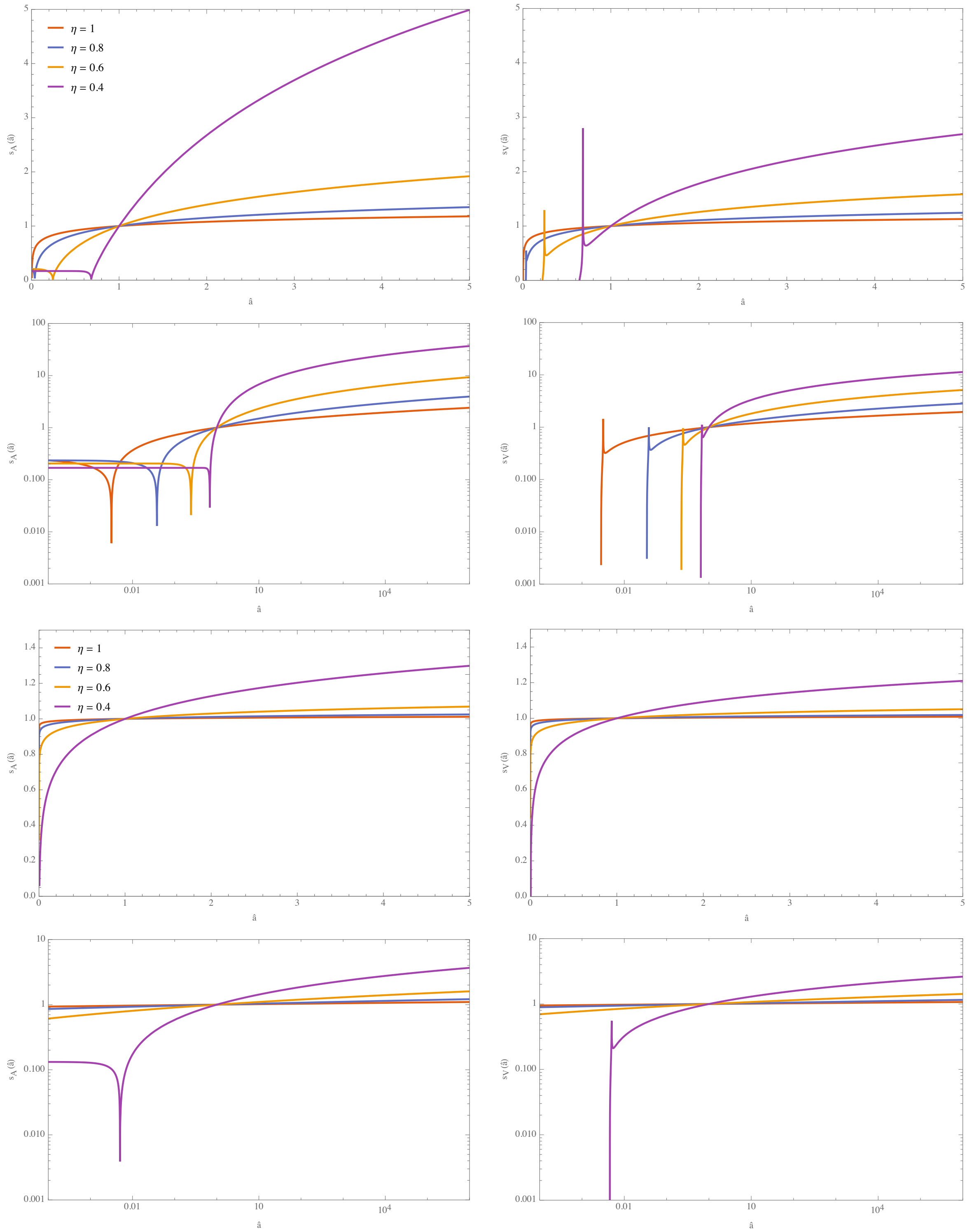

Figure 5.3: Here we present the entropy plots arising from the RVM up to $\mathcal{O}(H^6)$ for $\alpha = \beta$ fixed. On the left side, we display the horizon entropy $S_{\mathcal{A}}(\hat{a})$, equation (5.50), both in a linear scale and in a bilogarithmic scale. On the right side, we display the corresponding plots for the volume entropy $S_{\mathcal{V}}(\hat{a})$, that is equation (5.51). In all plots we have normalised the magnitudes with respect to the corresponding value at the equality point $\hat{a} = 1$, hence the lower case s in the axis label to indicate the normalised value, e.g. $s_{\mathcal{A}}(\hat{a}) \equiv S_{\mathcal{A}}(\hat{a})/S_{\mathcal{A}}(1)$, as in Figure 5.1. Four different values for the η parameter are displayed: $\eta = 1$ (red), $\eta = 0.8$ (blue), $\eta = 0.6$ (orange), and $\eta = 0.4$ (purple). The four top plots correspond to $\alpha = \beta = 0.1$, and the four bottom ones to $\alpha = \beta = 0.01$.

seem to fulfil the GSL. However, this might leave us without a solution for the entropy/horizon problem since we notice that the entropy in Figures 5.2 and 5.3 does not reach the high values we encountered in Figure 5.1.

However, we do observe an interesting feature. In the same way that smaller values of ν originated faster entropy growths in Figure 5.1, in Figures 5.2 and 5.3 we observe that smaller values of β and η also give birth to entropy that increases faster, yet with a more remarkable difference between the different curves we observe in the same plot. Regarding the definition of δ, which is proportional to β and η^4, we wonder if a decrease in δ, which can be related to a greater difference between H_I and H_J through η, is what triggers a faster increase of entropy. In other words, we ask ourselves if considering two different inflationary epochs, characterised by the Hubble rate scales H_I and H_J, could be responsible for the generation of more entropy. That, perhaps, could be a path towards solving the entropy/horizon problem without violating the GSL; that is something which indeed deserves further dedication.

6

Conclusions and future prospects

This work aims to bring out the relevance that Thermodynamics processes within the field of cosmology. In the last half a century, with the discovery of the temperature and entropy of black-holes, together with the formulation of the holographic principle, this classical field has gone deep within cutting-edge physics and offered us a new way of understanding and describing cosmos.

We have reviewed how the Einstein field equations can be inferred on fundamental grounds from entropic principles, a result that gives us priceless insight into General Relativity and, consequently, gravity.

We have also provided a derivation of the Friedmann equations in arbitrary dimension from a generalisation of the First Law of Thermodynamics. This does not only constitute another witness in favour of the deep connection between Thermodynamics and Cosmology, but also a powerful way of obtaining those equations, without being forced to undergo numerous calculations and without dimensional restrictions.

We have connected the study of the Running Vacuum Model (RVM) with the aforementioned results, showing that a thermodynamical analysis of this particular model can be carried out employing results that have been obtained also from thermodynamical principles, conferring compactness to the discussion.

We have extended the RVM from one high-order power to two of them, examining the possibility of two inflationary epochs characterised by two different higher powers of the Hubble parameter. We have reviewed a recent entropic analysis on the RVM and compared it to our extended model, showing its similarities and, even more important, its differences. For instance, how the former violates the Generalised Second Law (GSL) of Thermodynamics but leads to a solution to the entropy/horizon problem, and how the latter might behave in the opposite manner.

We have performed an analytical study of the extended RVM deep in the inflationary epoch in order to show the possibility to further enhance the production of entropy. This study, however, must be completed in order to connect in a smoother way the inflationary period of the extended RVM with the radiation dominated epoch. Such connection is fully analytical for the RVM model with a single higher power of the Hubble function, but is more difficult for the extended one, and therefore it requires further investigations.

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I would like to express my very great appreciation to my advisor, Dr Joan Solà Peracaula, for his teachings, his guidance and his valuable advice, and for his patience with me during the realisation of this work.

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Appendix A

Friedmann equations from the Einstein field equations

In this appendix, we derive the Friedmann equations from the Einstein field equations. Apart from providing a detailed deduction of the former, this appendix shows that deriving Friedmann equations from General Relativity suppose a much greater deal of calculations than deriving them from Thermodynamics, as we have shown in Section 4. Moreover, the derivation carried on in the aforesaid section allowed an $(n+1)$-dimensional treatment, while the one we calculate here is performed in $(3+1)$ dimensions. The advantage of this derivation is that it is just straightforward calculation, needles of delicate assumptions or introducing any concept we have not previously seen in our work. Moreover, it may be helpful for the reader who is becoming acquainted with the underlying mathematics of General Relativity.

The Einstein field equations can be written as [11]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{A.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R$ is the Einstein tensor, defined by the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R, and $T_{\mu\nu}$ is the energy-momentum tensor. Besides, Λ is the cosmological constant, which we can associate to a vacuum energy density through the expression $\Lambda \equiv 8\pi G \rho_\Lambda$. Hence, we can rewrite equation (A.1) as

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \rho_\Lambda g_{\mu\nu} \right). \tag{A.2}$$

We know that the Einstein tensor satisfies the Bianchi identity, that is $\nabla^\mu G_{\mu\nu} = 0$. Therefore, the energy-momentum tensor must fulfil the following condition:

$$\nabla^\mu T_{\mu\nu} = g_{\mu\nu} \nabla^\mu \rho_\Lambda, \tag{A.3}$$

since $\nabla^\mu g_{\mu\nu} = 0$, as we have aforementioned. At this point we notice that the energy-momentum tensor is not conserved if $\rho_\Lambda = \rho_\Lambda(t)$ is not a constant but a variable which evolves with time.

We consider the $(3+1)$-dimensional FLRW metric

$$\begin{aligned} ds^2 = g_{\mu\nu} dx^\mu dx^\nu &= -dt^2 + a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2 \right) \\ &= -dt^2 + a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right), \end{aligned} \tag{A.4}$$

where θ is the polar angle and ϕ is the azimuthal angle. If we consider the $\nu = 0$ component of equation (A.3) in the FLRW metric just given, we obtain the following conservation law

$$\dot{\rho}_\Lambda + \dot{\rho} + 3H(\rho + p) = 0, \tag{A.5}$$

where we have assumed that the energy-momentum tensor is that for a perfect fluid, as in equation (2.5). Notice that this expression is the $n = 3$ case for equation (2.7) but with a dynamical cosmological constant, i.e., $\rho_\Lambda = \rho_\Lambda(t)$.

Let us calculate the Christoffel symbols for the given $(3+1)$-dimensional FLRW metric. We shall use the Greek alphabet for spacetime components and we will reserve the Latin alphabet for the spatial components. The Christoffel symbols are given by the expression

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right), \tag{A.6}$$

which are symmetric with respect to their lower indices, i.e., $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$, $\forall \mu \neq \nu$. Since the FLRW metric is diagonal, the previous equation reduces to

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\rho} \left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right). \tag{A.7}$$

Then, the calculation of the Christoffel symbols is considerably simplified. Let us begin with those containing at least one time index, for the time component of the metric is the only one independent of the scale factor:

$$\begin{aligned}
&\Gamma^0_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} = 0, \quad \Gamma^0_{0i} = \frac{1}{2} g^{00} \partial_i g_{00} = 0, \quad \forall i, \quad \Gamma^0_{ij} = -\frac{1}{2} g^{00} \partial_0 g_{ij} = 0, \quad \forall i \neq j,\\
&\Gamma^i_{00} = -\frac{1}{2} g^{ii} \partial_i g_{00} = 0, \quad \forall i, \quad \Gamma^i_{0j} = \frac{1}{2} g^{ii} \partial_0 g_{ij} = 0, \quad \forall i \neq j,\\
&\Gamma^0_{11} = -\frac{1}{2} g^{00} \partial_0 g_{11} = \frac{1}{2} \partial_0 \left(\frac{a^2(t)}{1-kr^2}\right) = \frac{1}{2} \frac{2a(t)\dot{a}(t)}{1-kr^2} = \frac{a\dot{a}}{1-kr^2},\\
&\Gamma^0_{22} = -\frac{1}{2} g^{00} \partial_0 g_{22} = \frac{1}{2} \partial_0 \left(a^2(t) r^2\right) = \frac{1}{2} 2a(t)\dot{a}(t) r^2 = a\dot{a} r^2,\\
&\Gamma^0_{33} = -\frac{1}{2} g^{00} \partial_0 g_{33} = \frac{1}{2} \partial_0 \left(a^2(t) r^2 \sin^2\theta\right) = \frac{1}{2} 2a(t)\dot{a}(t) r^2 \sin^2\theta = a\dot{a} r^2 \sin^2\theta,\\
&\Gamma^1_{01} = \frac{1}{2} g^{11} \partial_0 g_{11} = \frac{1}{2} \frac{1-kr^2}{a^2(t)} \partial_0 \left(\frac{a^2(t)}{1-kr^2}\right) = \frac{1}{2} \frac{1-kr^2}{a^2(t)} \frac{2a(t)\dot{a}(t)}{1-kr^2} = \frac{\dot{a}(t)}{a(t)} = H,\\
&\Gamma^2_{02} = \frac{1}{2} g^{22} \partial_0 g_{22} = \frac{1}{2} \frac{\partial_0 \left(a^2(t) r^2\right)}{a^2(t) r^2} = \frac{1}{2} \frac{2a(t)\dot{a}(t) r^2}{a^2(t) r^2} = \frac{\dot{a}(t)}{a(t)} = H,\\
&\Gamma^3_{03} = \frac{1}{2} g^{33} \partial_0 g_{33} = \frac{1}{2} \frac{\partial_0 \left(a^2(t) r^2 \sin^2\theta\right)}{a^2(t) r^2 \sin^2\theta} = \frac{1}{2} \frac{2a(t)\dot{a}(t) r^2 \sin^2\theta}{a^2(t) r^2 \sin^2\theta} = \frac{\dot{a}(t)}{a(t)} = H,
\end{aligned} \tag{A.8}$$

where $i, j \in \{1, 2, 3\}$ for they are spatial indices. We draw the reader's attention to the fact that $\Gamma^i_{0i} = H$, $\forall i$, since it will be useful afterwards. Let us now compute the Christoffel symbols whose upper index corresponds to the radial component:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}g^{11}\partial_1 g_{11} = \frac{1}{2}\frac{1-kr^2}{a^2(t)}\partial_1\left(\frac{a^2(t)}{1-kr^2}\right) = \frac{1}{2}\frac{1-kr^2}{a^2(t)}\frac{2a^2(t)kr}{\left(1-kr^2\right)^2} = \frac{kr}{1-kr^2},\\
\Gamma^1_{12} &= \frac{1}{2}g^{11}\partial_2 g_{11} = 0, \quad \Gamma^1_{13} = \frac{1}{2}g^{11}\partial_3 g_{11} = 0,\\
\Gamma^1_{22} &= -\frac{1}{2}g^{11}\partial_1 g_{22} = -\frac{1}{2}\frac{1-kr^2}{a^2(t)}\partial_1\left(a^2(t)r^2\right) = -r\left(1-kr^2\right),\\
\Gamma^1_{23} &= \frac{1}{2}g^{11}\left(\partial_3 g_{12} + \partial_2 g_{13} - \partial_1 g_{23}\right) = 0,\\
\Gamma^1_{33} &= -\frac{1}{2}g^{11}\partial_1 g_{33} = -\frac{1}{2}\frac{1-kr^2}{a^2(t)}\partial_1\left(a^2(t)r^2\mathrm{sin}^2\theta\right) = -r\left(1-kr^2\right)\mathrm{sin}^2\theta.
\end{aligned} \tag{A.9}$$

We continue with the Christoffel symbols with polar angle upper index:

$$\begin{aligned}
&\Gamma^2_{11} = -\frac{1}{2}g^{22}\partial_2 g_{11} = 0, \quad \Gamma^2_{12} = \frac{1}{2}g^{22}\partial_1 g_{22} = \frac{1}{2}\frac{\partial_1\left(a^2(t)r^2\right)}{a^2(t)r^2} = \frac{1}{r},\\
&\Gamma^2_{13} = \frac{1}{2}g^{22}\left(\partial_3 g_{21} + \partial_1 g_{23} - \partial_2 g_{13}\right) = 0, \quad \Gamma^2_{22} = \frac{1}{2}g^{22}\partial_2 g_{22} = 0,\\
&\Gamma^2_{23} = \frac{1}{2}g^{22}\partial_3 g_{22} = 0, \quad \Gamma^2_{33} = -\frac{1}{2}g^{22}\partial_2 g_{33} = -\frac{1}{2}\frac{\partial_2\left(a^2(t)r^2\mathrm{sin}^2\theta\right)}{a^2(t)r^2} = -\mathrm{sin}\theta\mathrm{cos}\theta.
\end{aligned} \tag{A.10}$$

Finally, we calculate the Christoffel symbols with azimuthal angle upper index:

$$\begin{aligned}
&\Gamma^3_{11} = -\frac{1}{2}g^{33}\partial_3 g_{11} = 0, \quad \Gamma^3_{12} = \frac{1}{2}g^{33}\left(\partial_2 g_{31} + \partial_1 g_{32} - \partial_3 g_{12}\right) = 0,\\
&\Gamma^3_{13} = \frac{1}{2}g^{33}\partial_1 g_{33} = \frac{1}{2}\frac{\partial_1\left(a^2(t)r^2\mathrm{sin}^2\theta\right)}{a^2(t)r^2\mathrm{sin}^2\theta} = \frac{1}{r}, \quad \Gamma^3_{22} = -\frac{1}{2}g^{33}\partial_3 g_{22} = 0,\\
&\Gamma^3_{23} = \frac{1}{2}g^{33}\partial_2 g_{33} = \frac{1}{2}\frac{\partial_2\left(a^2(t)r^2\mathrm{sin}^2\theta\right)}{a^2(t)r^2\mathrm{sin}^2\theta} = \frac{\mathrm{cos}\theta}{\mathrm{sin}\theta} = \mathrm{cot}\theta, \quad \Gamma^3_{33} = \frac{1}{2}g^{33}\partial_3 g_{33} = 0.
\end{aligned} \tag{A.11}$$

It is well known that the Ricci tensor can be obtained from the Christoffel symbols via the expression

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\mu\Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}. \tag{A.12}$$

As we did for the Christoffel symbols, we shall begin with the components of the Ricci tensor that contain at least one time index:

$$\begin{aligned}
R_{00} =& -\partial_0\Gamma^\rho_{\rho 0} - \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0} = -\partial_0\Gamma^1_{10} - \partial_0\Gamma^2_{20} - \partial_0\Gamma^3_{30} - \Gamma^1_{01}\Gamma^1_{10} - \Gamma^2_{02}\Gamma^2_{20} - \Gamma^3_{03}\Gamma^3_{30}\\
=& -3\partial_0 H - 3H^2 = -3\frac{\ddot{a}a - \dot{a}^2}{a^2} - 3H^2 = -3\frac{\ddot{a}}{a} + 3H^2 - 3H^2 = -3\frac{\ddot{a}}{a},\\
R_{0i} =& \partial_i\Gamma^i_{0i} - \partial_0\Gamma^j_{ji} + \Gamma^j_{ji}\Gamma^i_{0i} - \Gamma^j_{0j}\Gamma^j_{ji} = \partial_i\Gamma^i_{0i} - \partial_0\Gamma^1_{1i} - \partial_0\Gamma^2_{2i} - \partial_0\Gamma^3_{3i}\\
&+ \Gamma^1_{1i}\Gamma^i_{0i} + \Gamma^2_{2i}\Gamma^i_{0i} + \Gamma^3_{3i}\Gamma^i_{0i} - \Gamma^1_{01}\Gamma^1_{1i} - \Gamma^2_{02}\Gamma^2_{2i} - \Gamma^3_{03}\Gamma^3_{3i} = -\partial_0\left(\Gamma^1_{1i} + \Gamma^2_{2i} + \Gamma^3_{3i}\right),\\
R_{01} =& -\partial_0\left(\frac{kr}{1-kr^2} + \frac{1}{r} + \frac{1}{r}\right) = 0, \quad R_{02} = -\partial_0\mathrm{cot}\theta = 0, \quad R_{03} = 0.
\end{aligned} \tag{A.13}$$

Let us proceed to the spatial components of the Ricci tensor:

$$
\begin{aligned}
R_{11} =& \partial_0 \Gamma^0_{11} - \partial_1 \Gamma^2_{21} - \partial_1 \Gamma^3_{31} + \Gamma^2_{20}\Gamma^0_{11} + \Gamma^3_{30}\Gamma^0_{11} \\
&+ \Gamma^2_{21}\Gamma^1_{11} + \Gamma^3_{31}\Gamma^1_{11} - \Gamma^0_{11}\Gamma^1_{01} - \Gamma^2_{12}\Gamma^2_{21} - \Gamma^3_{13}\Gamma^3_{31} \\
=& \partial_0 \left(\frac{a\dot{a}}{1-kr^2} \right) - 2\partial_1 \left(\frac{1}{r} \right) + \frac{2Ha\dot{a}}{1-kr^2} + \frac{2}{r}\frac{kr}{1-kr^2} - \frac{a\dot{a}H}{1-kr^2} - \frac{2}{r^2} \\
=& \frac{a\ddot{a}+\dot{a}^2}{1-kr^2} + \frac{2}{r^2} + \frac{\dot{a}^2}{1-kr^2} + \frac{2k}{1-kr^2} - \frac{2}{r^2} = \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2}, \\
R_{22} =& \partial_0 \Gamma^0_{22} + \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^3_{32} + \Gamma^1_{10}\Gamma^0_{22} + \Gamma^3_{30}\Gamma^0_{22} \\
&+ \Gamma^1_{11}\Gamma^1_{22} + \Gamma^3_{31}\Gamma^1_{22} - \Gamma^0_{22}\Gamma^2_{02} - \Gamma^1_{22}\Gamma^2_{12} - \Gamma^3_{23}\Gamma^3_{32} \\
=& \partial_0 \left(a\dot{a}r^2 \right) + \partial_1 \left[-r\left(1-kr^2\right) \right] - \partial_2 \left(\cot\theta \right) + 2Ha\dot{a}r^2 - \frac{kr}{1-kr^2} r\left(1-kr^2\right) \\
&- \frac{1}{r} r\left(1-kr^2\right) - a\dot{a}r^2 H + r\left(1-kr^2\right)\frac{1}{r} - \cot^2\theta \\
=& a\ddot{a}r^2 + \dot{a}^2 r^2 - \left(1-kr^2\right) - r\left(-2kr\right) + \csc^2\theta + \dot{a}^2 r^2 - kr^2 - \cot^2\theta \\
=& \left(a\ddot{a}+2\dot{a}^2+2k\right) r^2 - 1 + \csc^2\theta - \cot^2\theta = \left(a\ddot{a}+2\dot{a}^2+2k\right) r^2, \\
R_{33} =& \partial_0 \Gamma^0_{33} + \partial_1 \Gamma^1_{33} + \partial_2 \Gamma^2_{33} + \Gamma^1_{10}\Gamma^0_{33} + \Gamma^2_{20}\Gamma^0_{33} \\
&+ \Gamma^1_{11}\Gamma^1_{33} + \Gamma^2_{21}\Gamma^1_{33} - \Gamma^3_{30}\Gamma^0_{33} - \Gamma^3_{31}\Gamma^1_{33} - \Gamma^3_{32}\Gamma^2_{33} \\
=& \partial_0 \left(a\dot{a}r^2 \sin^2\theta \right) + \partial_1 \left[-r\left(1-kr^2\right)\sin^2\theta \right] + \partial_2 \left(-\sin\theta\cos\theta \right) + Ha\dot{a}r^2 \sin^2\theta \\
&- \frac{kr}{1-kr^2} r\left(1-kr^2\right)\sin^2\theta + \cot\theta\sin\theta\cos\theta \\
=& a\ddot{a}r^2\sin^2\theta + \dot{a}^2 r^2 \sin^2\theta - \left(1-kr^2\right)\sin^2\theta - r\left(-2kr\right)\sin^2\theta - \cos^2\theta + \sin^2\theta \\
&+ \dot{a}^2 r^2 \sin^2\theta - kr^2\sin^2\theta + \cos^2\theta = \left(a\ddot{a}+2\dot{a}^2+2k\right) r^2 \sin^2\theta, \\
R_{12} =& \partial_2 \Gamma^2_{12} - \partial_1 \Gamma^3_{32} + \Gamma^3_{32}\Gamma^2_{12} - \Gamma^3_{13}\Gamma^3_{32} = \partial_2 \left(\frac{1}{r} \right) - \partial_1 \left(\cot\theta \right) + \cot\theta \frac{1}{r} - \frac{1}{r}\cot\theta = 0, \\
R_{13} =& \partial_3 \Gamma^3_{13} = \partial_3 \left(\frac{1}{r} \right) = 0, \quad R_{23} = \partial_3 \Gamma^3_{23} = \partial_3 \left(\cot\theta \right) = 0.
\end{aligned}
\tag{A.14}
$$

We observe that the non-zero components of the Ricci tensor are R_{00}, R_{11}, R_{22} and R_{33}, in other words, the Ricci tensor is diagonal. Therefore, the Ricci scalar R is

$$
\begin{aligned}
R =& g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} \\
=& 3\frac{\ddot{a}}{a} + \frac{1-kr^2}{a^2} \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} + \frac{\left(a\ddot{a}+2\dot{a}^2+2k\right) r^2}{a^2 r^2} + \frac{\left(a\ddot{a}+2\dot{a}^2+2k\right) r^2 \sin^2\theta}{a^2 r^2 \sin^2\theta} \\
=& 3\frac{\ddot{a}}{a} + 3\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} \right] = 6\left[\frac{\ddot{a}}{a} + H^2 + \frac{k}{a^2} \right].
\end{aligned}
\tag{A.15}
$$

Let us consider the time-time component of the Einstein field equations (A.2). Considering the energy-momentum tensor for a perfect fluid and the previously calculated terms of the Ricci tensor and the Ricci scalar, one has that

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \left(\rho_m + \rho_r + \rho_\Lambda\right), \tag{A.16}$$

which is the first Friedmann equation for a $(3+1)$-dimensional FLRW universe with spatial curvature k, where we have relabelled $\rho \equiv \rho_m + \rho_r$, for we consider that there are both matter and radiation in the universe. Notice that the above expression coincides with equation (4.17) when $n = 3$, as we should expect. We can obtain the second Friedmann equation by deriving the first equation with respect to time

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho_m + 3p_m + \rho_r + 3p_r - 2\rho_\Lambda\right), \tag{A.17}$$

where we have used (4.20), (A.5) and (A.16), and accordingly redefined $p \equiv p_m + p_r$ (recall equation (2.5)). Observe that, once again, this equality is the same that we obtain from equation (4.22) in the $n = 3$ case, bearing in mind that $p_m = w_m \rho_m$ and $p_r = w_r \rho_r$.

Appendix B

Lambert W function

In this appendix, we briefly discuss the Lambert W function, also known as the omega function or product logarithm. To this end, we have followed [13], to which we refer the reader eager for further details and a deeper analysis of this particular function.

The Lambert W function is defined as the multivalued inverse of the function $f(w) = we^w$, where $w \in \mathbb{C}$ and e^w is the exponential function. In other words, $W(z)$ is defined as the function satisfying

$$W(z)e^{W(z)} = z, \tag{B.1}$$

where $z \in \mathbb{C}$. More precisely, being the Lambert W function the inverse of the complex function $f(w)$, we know it is split into branches, which we denote as $W_k(z)$, where $k \in \mathbb{Z}$.

For our pur poses, we are interested in the W function when its argument is real, that is $W(x)$ with $x \in \mathbb{R}$. We see that the $W(x)$ function is real if $x \geq -1/e$. However, if $x \in [-1/e, 0)$, there are two possible real values of $W(x)$ (see Figure B.1). The branch that satisfies $W(x) \geq -1$ is denoted by $W_0(x)$, and the one satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$. The $W_0(x)$ branch is called the *principal branch* of the W function, thus it may be designated as $W(x)$. In our work we need to evaluate $W(x)$ with $x < -1/e$, when the W function is not completely real. In that case, we will consider the real part of the $W(x)$ complex function (see Figure B.2).

Let us now calculate the derivative of the function $W(x)$ when $x \in \mathbb{R}$, considering it as a real function. Differentiating both sides of equation (B.1), one has that

$$W(x)W'(x)e^{W(x)} + W'(x)e^{W(x)} = W'(x)\left(x + e^{W(x)}\right) = 1. \tag{B.2}$$

If we multiply both sides of the equation by $W(x)$ and isolate $W'(x)$, we have that

$$W'(x) = \frac{W(x)}{x\left(W(x)+1\right)} = \frac{1}{x + e^{W(x)}}. \tag{B.3}$$

We observe that the first expression for the derivative of the function $W(x)$ is not defined if $x \in \{-1/e, 0\}$. Nevertheless, the second expression is not defined only for $x = -1/e$, which tells us that the W function is not differentiable only for $x = -1/e$, despite what it may seem when looking at the first equality. As we see in Figure B.2, the Lambert W function is discontinuous at $x = -1/e$.

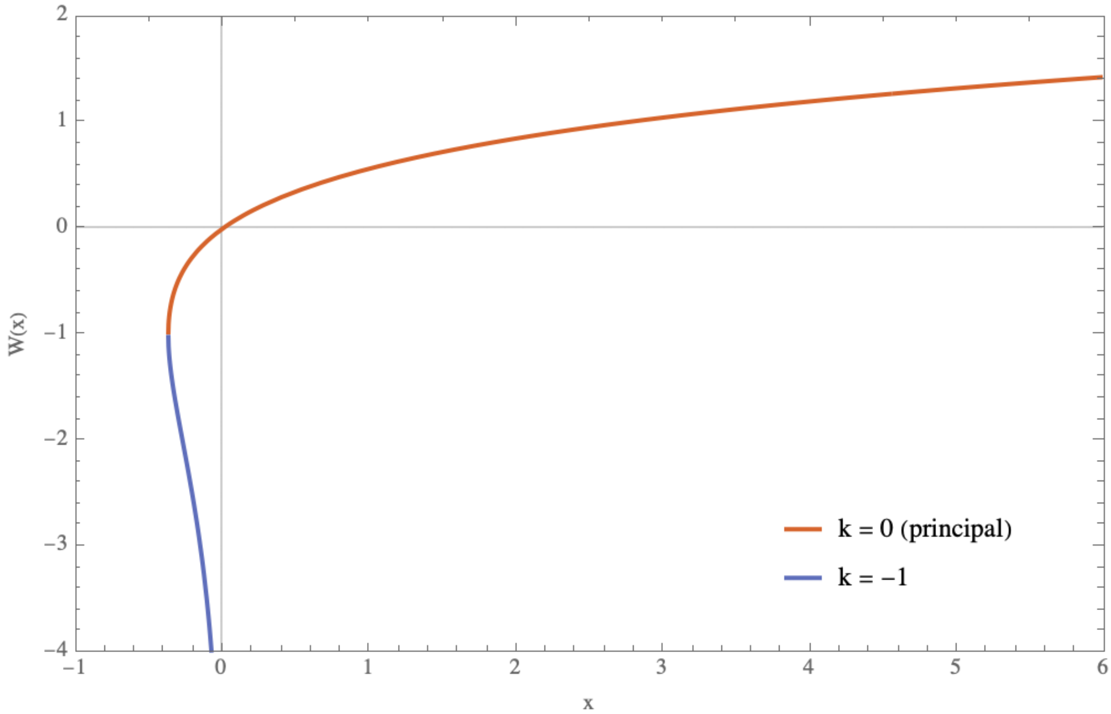

Figure B.1: The two real branches of the Lambert W function: the principal branch $W_0(x)$, in orange, and $W_{-1}(x)$, in blue.

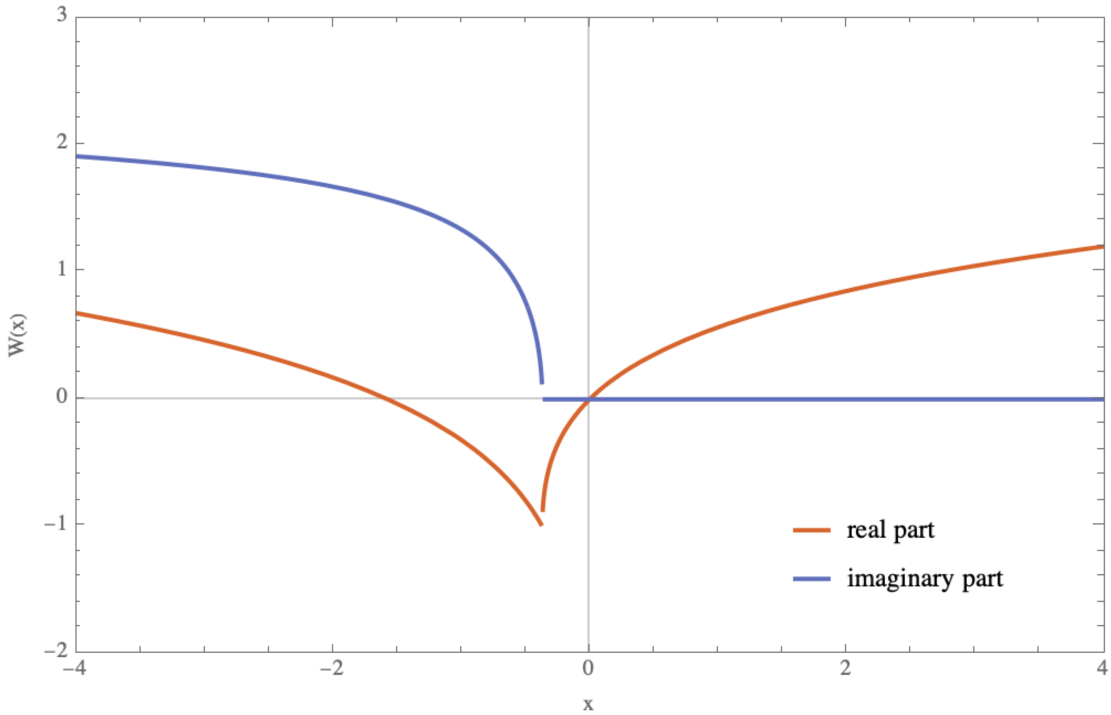

Figure B.2: Real (orange) and imaginary (blue) parts of the principal branch $W(x)$ of the Lambert W function. Notice that for $x \geq -1/$ the imaginary part of the W function is zero, and we recover the real principal branch displayed in Figure B.1.

Bibliography

[1] P. M. Alsing and P. W. Milonni, Am. J. Phys. **72** (2004) 1524-1529.

[2] D. Bak and S. J. Rey, Class. Quant. Grav. **17** (2000) L83.

[3] J. M. Bardeen, B. Carter, S. W. Hawking, Comm. Math. Phys. **31** (1973) 161.

[4] S. Basilakos, N. E. Mavromatos, J. Solà, Phys. Rev. D **101** (2020) 045001.

[5] J. D. Bekenstein, Phys. Rev. D **7** (1973) 2333.

[6] L. Bombelli *et al.*, Phys. Rev. D **34** (1986) 373.

[7] R. Bousso, JHEP **9907** (1999) 004.

[8] R. Bousso, Rev. Mod. Phys. **74** (2002) 825.

[9] R-G. Cai and S. P. Kim, JHEP **0502** (2005) 050.

[10] R-G. Cai *et al.*, Phys. Rev. D **80** (2009) 104016.

[11] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, London, UK, Pearson Education Limited (1985).

[12] S. Chakraborty, R. Biswas, N. Mazumder, Nuovo Cim. B **125** (2011) 1209.

[13] R. M. Corless *et al.*, Adv Comput Math **5** (1996) 329.

[14] A. S. Eddington, Mon. Not. Roy. Astron. Soc. **91** (1931) 490.

[15] A. Einstein, Sitz. Preuss. Akad. Wiss. Berlin (1915) 844.

[16] A. Einstein, Sitz. Königl. Preuss. Akad. Wiss. phys.-math. **K. VI** (1917) 142.

[17] A. Einstein, Sitz. Königl. Preuss. Akad. Wiss. phys.-math. **K. XII** (1931) 235.

[18] E. P. Hubble, Proc. Natl. Acad. Sci. **15** (1929) 168.

[19] S. W. Hawking, Nature **248** (1974) 30.

[20] S. W. Hawking, Comm. Math. Phys. **43** (1975) 199.

[21] S. A. Hayward, Phys. Rev. D **53** (1996) 1938.

[22] S. A. Hayward, Class. Quant. Grav. **15** (1998) 3147.

[23] S. A. Hayward *et al.*, Phys. Lett. A **256** (1999) 347.

[24] G. 't Hooft, Conf. Proc. C **930308** (1993) 284.

[25] T. Jacobson, Phys. Rev. Lett. **75** (1995) 1260.

[26] T. Jacobson and R. Parentani, Found. Phys. **33** (2003) 323.

[27] R. Knop *et al.*, Astrophys. J. **598** (2003) 102.

[28] E. W. Kolb and M. S. Turner, *The Early Universe*, Boca Raton, USA, CRC Press (2018).

[29] G. Lemaître, Proc. Natl. Acad. Sci. **20** (1934) 12.

[30] J. A. S. Lima, S. Basilakos, J. Solà, MNRAS **431** (2013) 923.

[31] C. W. Misner and D. H. Sharp, Phys. Rev. **136** (1964) B571.

[32] V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge, UK, Cambridge University Press (2005).

[33] S. Nesseris, G. Pantazis, L. Perivolaropoulos, Phys. Rev. D **96** (2017) 023542.

[34] J. M. Overduin and F. I. Cooperstock, Phys. Rev. D**D58** (1998) 043506.

[35] E. L. D. Perico, J. A. S. Lima, S. Basilakos and J. Solà, Phys. Rev. D **88** (2013) 063531.

[36] Planck Collaboration, *Planck 2018 results. VI.* [arXiv:1807.06209v2].

[37] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*, Cambridge, UK, Cambridge University Press (2004).

[38] A. G. Riess *et al.*, Astronom. J. **116** (1998) 1009.

[39] W. Rindler, MNRAS **116** (1956) 662.

[40] I. L. Shapiro and J. Solà, JHEP **0202** (2002) 006.

[41] I. L. Shapiro and J. Solà, Phys. Lett. B **682** (2009) 105.

[42] J. Solà, J. Phys. Math. Theor. A **41** (2008) 164066.

[43] J. Solà, J. Phys. Conf. Ser. **453** (2013) 012015.

[44] J. Solà and A. Gómez-Valent, Int. J. Mod. Phys. D **24** (2015) 1541003.

[45] J. Solà, A. Gómez-Valent, J. de Cruz Pérez, Astrophys. J. **811** (2015) L14.

[46] J. Solà, A. Gómez-Valent, J. de Cruz Pérez, Astrophys. J. **836** (2017) 43.

[47] . Solà, A. Gómez-Valent, J. de Cruz Pérez, Phys. Lett. B **774** (2017) 317.

[48] J. Solà, J. de Cruz Pérez, A. Gómez-Valent, MNRAS **478** (2018) 4357.

[49] J. Solà, J. de Cruz Pérez, A. Gómez-Valent, EPL **121** (2018) 39001.

[50] J. Solà and H. Yu, Gen. Rel. Grav. **52** (2020) no.2, 17.

[51] D. N. Spergel *et al.*, Astrophys. J. Suplem. **170** (2007) 377.

[52] L. Susskind, J. Math. Phys. **36** (1995) 6377.

[53] S. Takagi, Prog. Theor. Phys. **88** (1986) 1.

[54] W.G. Unruh, Phys. Rev. D **14** (1976) 870.

[55] Y. B. Zeldovich, Sov. Phys. JETP Lett **6** (1967) 3167.