ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

THE REAL INTERPOLATION METHOD

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Abstract

This Master's Final Project provides a systematic study of the real interpolation method, tracing its development from the classical theorems of Riesz-Thorin and Marcinkiewicz. The abstract framework is constructed using Peetre's K-functional and the associated J-method, establishing essential structural properties such as the Density, Reiteration, and Wolff's theorems. The project also investigates the behavior of operator compactness under interpolation. To address this, we present the approach relying on the ball measure of non-compactness to demonstrate how compactness is preserved between interpolated spaces. Furthermore, the core theory is extended to the broader category of quasi-normed Abelian groups. This generalization is a necessary mathematical step to provide a proof of the general Marcinkiewicz Interpolation Theorem. Finally, the work concludes by briefly illustrating these abstract concepts through concrete mathematical examples, such as the computation of interpolation spaces for Sobolev couples or the classical endpoints in harmonic analysis H^1 and BMO.

Resum

Aquest Treball Final de Màster proporciona un estudi sistemàtic del mètode d'interpolació real, resseguint el seu desenvolupament des dels teoremes clàssics de Riesz-Thorin i Marcinkiewicz. El marc abstracte es construeix utilitzant el K-funcional de Peetre i el J-mètode associat, establint propietats estructurals essencials com els teoremes de Densitat, Reiteració i Wolff.

El projecte també investiga el comportament de la compacitat dels operadors sota interpolació. Per abordar-ho, presentem l'enfocament basat en la mesura de no-compacitat de bola per demostrar com es preserva la compacitat entre espais interpolats. A més, la teoria central s'estén a la categoria més àmplia dels grups Abelians quasi-normats. Aquesta generalització és un pas matemàtic necessari per proporcionar una demostració del Teorema d'Interpolació de Marcinkiewicz general. Finalment, el treball conclou il·lustrant breument aquests conceptes abstractes a través d'exemples matemàtics concrets, com ara el càlcul d'espais d'interpolació per a parelles de Sobolev o per als típics extrems en anàlisi harmònica H^1 i BMO.

Contents

1 Introduction

The theory of interpolation of operators, now an independent and highly developed branch of functional analysis, traces its origins back to a set of classical theorems from the early 20th century. Specifically, the foundational results were obtained by M. Riesz (1926), S. Thorin (1939), and J. Marcinkiewicz (1939). These early interpolation theorems were initially driven by concrete problems in Fourier analysis and the theory of integral operators. During the 1950s, through the efforts of A. Zygmund and his collaborators and students—including A. P. Calderón, E. M. Stein, and G. Weiss—these classical results were profoundly generalized, laying the analytical foundation for the modern mathematical theory.

The early 1960s marked a dramatic phase transition characterized by a sudden shift toward high-level abstraction. The entire subject was recast into the language of functional analysis and, eventually, category theory. Sparked by a 1958 letter from N. Aronszajn to J.-L. Lions, pioneering abstract work erupted almost simultaneously from Lions, E. Gagliardo, A. P. Calderón, and S. G. Kreĭn. The fundamental pillars of the theory were established during this fertile period: the Riesz-Thorin theorem gave birth to the *complex method*, rigorously introduced by Calderón, while Marcinkiewicz's theorem inspired the *real method*, systematically developed by Lions and J. Peetre [11]. It was also during this time that Peetre invented the elegantly simple yet remarkably powerful K-functional, a tool that fundamentally reshaped how real interpolation spaces were constructed and understood.

The subsequent decade (1965–1975) witnessed a considerable advancement in applying these abstract methods to specific functional Banach couples, such as Lebesgue, Sobolev, and Besov spaces. This era of immense calculation and consolidation culminated in several comprehensive treatises that collected the vast "mosaic" of scattered results into unified frameworks. Most notably, the exhaustive work by J. Bergh and J. Löfström [4] became the definitive standard reference for a generation of analysts. This systematization was later masterfully complemented by the extensive exposition on the real method by C. Bennett and R. Sharpley [3].

This master's final project is mainly based on these two last references, although they are both a vast collection of other results themselves. In particular, it develops and studies the real method of interpolation. The structure of the project is rather akine to the historical development. Chapter 2 introduces the classical interpolation theorems of Riesz-Thorin and Marcinkiewicz. The next two chapters are devoted to the construction of the real method of interpolation and its main properties. The main tool is going to be Peetre's K-functional, inspired by the latter of the two classical theorems. These two chapters constitute the main work of the project. Chapters 6 and 7 give a general overview of two very different problems: compactness of operators and extension of the method to quasi-normed Abelian groups. The second one allows us to finally give a proof of the general Marcinkiewicz Interpolation Theorem, closing the loop. The final chapter provides a general framework for computing interpolation spaces of some specific spaces.

Text in **bold** refers to definitions outside formal definition environments.

By $f(x) \sim g(x)$ we mean the existence of two constants $c_1, c_2 > 0$ such that
$$c_1 f(x) \leq g(x) \leq c_2 f(x) \quad \forall x.$$
Recall that c_1 and c_2 are uniform on the points x. If f and g are functions of more than one variable (potentially in different spaces), the constants are always independent of them.

Throughout this work, C denotes a generic positive constant whose exact value may change from line to line, or even within the same sequence of inequalities.

2 Classical Interpolation Theorems

$\mathbb{K}$ will always denote the scalar field $\mathbb{R}$ or $\mathbb{C}$. Let (U, μ) be a measure space, with μ always being a positive measure. Given $f : U \to \mathbb{K}$ a μ-measurable function, we set for $1 \leq p < +\infty$

$$\|f\|_{L_p} = (\int_U |f(x)|^p d\mu)^{1/p}, \tag{2.1}$$

and

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_U |f(x)| \tag{2.2}$$

for $p = \infty$. The **Lebesgue-space** $L_p(U, d\mu)$ (or simply $L_p(U), L_p(d\mu)$ or L_p) is the set of μ-measurable and scalar-valued functions f such that $\|f\|_{L_p}$ is finite and under the convention that two functions are equal if they are the same except on a null set.

Given f a μ-measurable and scalar-valued function such that it is finite almost everywhere, we define the **distribution function of** f, $m(\sigma, f)$, as

$$m(\sigma, f) = \mu(\{x \in U : |f(x)| > \sigma\}).$$

Since μ is positive, $m(\sigma, f)$ is a real-valued function of σ, defined on the non-negative axis $[0, +\infty)$. Clearly $m(\sigma, f)$ is non-increasing and right-continuous.

Proposition 2.1. *Let f be μ-measurable and scalar function and $m(\sigma, f)$ its distribution function, then*

$$\|f\|_{L_p} = (p \int_0^\infty \sigma^{p-1} m(\sigma, f) d\sigma)^{1/p} \quad \text{if} \quad 1 \leq p < \infty \tag{2.3}$$

and

$$\|f\|_{L_\infty} = \inf\{\sigma : m(\sigma, f) = 0\}. \tag{2.4}$$

Proof. We can prove (2.3) straightforward from the definition,

$$\begin{aligned}
\|f\|_{L_p}^p = \int_U |f(x)|^p d\mu(x) &= \int_U \Big(\int_0^{|f(x)|} p\sigma^{p-1} d\sigma \Big) d\mu(x) \\
= \int_U \Big(\int_0^\infty p\sigma^{p-1} \chi_{\{|f(x)|>\sigma\}} d\sigma \Big) d\mu(x) &= \int_0^\infty p\sigma^{p-1} \Big(\int_U \chi_{\{|f(x)|>\sigma\}} d\mu(x) \Big) d\sigma \\
&= p \int_0^\infty \sigma^{p-1} m(\sigma, f) d\sigma,
\end{aligned}$$

where we only used Fubini. (2.4) is trivial. $\square$

We can now introduce the **weak L_p-spaces**, denoted by $L_p^*(U, d\mu)$ (also $L_p^*(d\mu)$ or L_p^*). For $1 \leq p < \infty$, L_p^* is the set of functions f such that

$$\|f\|_{L_p^*} = \sup_{\sigma>0} \sigma m(\sigma, f)^{1/p} < +\infty. \tag{2.5}$$

In the limiting case $p = \infty$ we simply set $L_\infty^* = L_\infty$. It can be proven that the expressions given by (2.1) and (2.2) are norms on the L_p-spaces. Moreover, with those norms, the Lebesgue spaces become Banach. On the other hand, for $1 \leq p < \infty$, the expression given by (2.5) is not a norm. This follows easily from the fact that

$$m(\sigma, f + g) \leq m(\sigma/2, f) + m(\sigma/2, g). \tag{2.6}$$

Hence, using the inequality $(a + b)^{1/p} \leq a^{1/p} + b^{1/p}$, we conclude that

$$\|f + g\|_{L_p^*} \leq 2(\|f\|_{L_p^*} + \|g\|_{L_p^*}). \tag{2.7}$$

So $\| \cdot \|_{L_p^*}$ does not satisfy the triangular inequality. When $\|f + g\| \leq k(\|f\| + \|g\|)$ is true for some $k \geq 1$, we say that the vector space is a **quasi-normed space**. This is the case of L_p^*. However, for $p > 1$, one can find an equivalent norm to $\| \cdot \|_{L_p^*}$ for which the space becomes normable and Banach. For $p = 1$ one can show that L_1^* is complete but not a normable space.

It follows from the definition of L_p^* that for $1 \leq p < \infty$

$$\|f\|_{L_p} = \sup_{\sigma>0} \Big(\int_{|f(x)|>\sigma} |f(x)|^p d\mu \Big)^{1/p} \geq \sup_{\sigma>0} \sigma m(\sigma, f)^{1/p} = \|f\|_{L_p^*},$$

which proves $L_p \subseteq L_p^*$ for all $p \geq 1$.

If f is a μ-measurable function, we denote by f^* its **decreasing rearrangement**

$$f^*(t) = \inf_{\sigma>0} \{\sigma : m(\sigma, f) \leq t\}. \tag{2.8}$$

This is a non-negative, non-increasing and right-continuous function on $[0, +\infty)$.

Proposition 2.2. *For a μ-measurable function f,*

$$m(\sigma, f) = m(\sigma, f^*), \quad \forall \sigma \geq 0. \tag{2.9}$$

When (2.9) holds we say that f and f^* are **equimeasurable**.

Proof. By definition we have $f^*(m(\sigma, f)) \leq \sigma$. Thus, as f^* is non-increasing, $m(\sigma, f^*) \leq m(\sigma, f)$. Moreover, since f^* is continuous on the right, $f^*(m(\sigma, f^*)) \leq \sigma$. Again by the definition as an infimum of f^* we conclude $m(\sigma, f) \leq m(\sigma, f^*)$. ☐

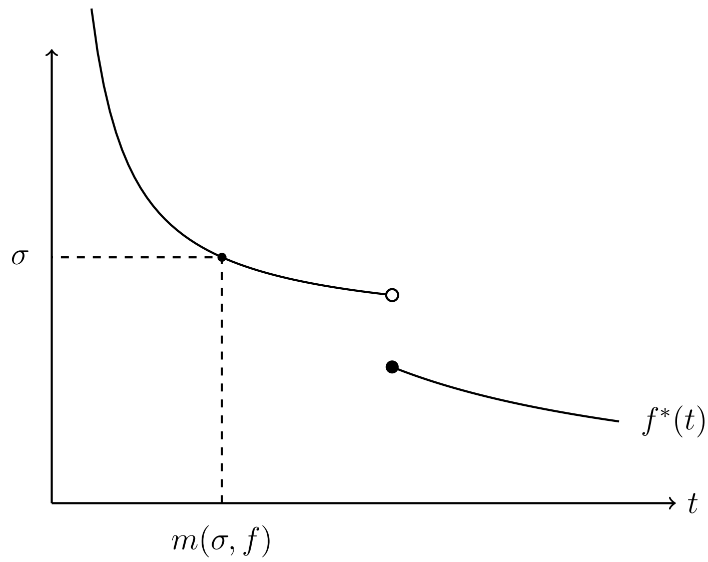

Figure 1: graph of the decreasing rearrangment for a function f.

Notice that for all points t for which $f^*(t) = \sigma$ is continuous, $f^*(m(\sigma, f)) = \sigma$. Figure 1 illustrates this case.

Both the weak L_p-spaces and the Lebesgue spaces L_p are a specific case of the more general **Lorentz spaces**. We say that for $1 \leq p \leq +\infty$, f belongs to the Lorentz space $L_{pr}(U, d\mu)$ (or simply $L_{pr}(d\mu)$ or L_{pr}) if

$$\|f\|_{L_{pr}} = \Big(\int_0^\infty (t^{1/p} f^*(t))^r dt/t \Big)^{1/r} < +\infty \qquad \text{for } 1 \leq r < +\infty, \tag{2.10}$$

$$\|f\|_{L_{p\infty}} = \sup_{t>0} t^{1/p} f^*(t) < +\infty \qquad \text{for } r = \infty. \tag{2.11}$$

Then, we have $L_{pp} = L_p$ and $L_{p\infty} = L_p^*$, for any $p \geq 1$, with equality of norms. For the first equality, simply observe that $\|f\|_{L_{pp}} = \|f^*\|_{L_p}$. Now apply Propositions 2.1 and 2.2 to get $\|f\|_{L_p} = \|f^*\|_{L_p}$. In general, the Lorentz spaces are quasi-normed, but when $p > 1$, one can replace the quasi-norm with an equivalent norm, for which the Lorentz space becomes Banach (see Example 4.1).

Finally, we shall consider linear and bounded mappings T between two Lebesgue spaces L_p and L_q. A mapping T is said to be **linear** if $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ for all $f, g \in L_p$ and $\alpha, \beta \in \mathbb{K}$. We say that T is **bounded** if there exists a real constant $C > 0$ such that $\|Tf\|_{L_q} \leq C\|f\|_{L_p}$ for all $f \in L_p$. The infimum of these constants is called the norm of T. We then write $T : L_p \to L_q$.

For a linear operator T from L_p to L_q^* we will extend the definition of boundedness as expected. That is, T will be bounded (or weakly bounded) if there exists a real constant $C > 0$ such that $\|Tf\|_{L_q^*} \leq C\|f\|_{L_p}$ for all $f \in L_p$. Again, the norm of T will be the infimum of these constants and we will write $T : L_p \to L_q^*$.

Theorem 2.3 (Marcinkiewicz Interpolation Theorem)**.** *Let (U, μ) be a non-atomic measure space. Assume $1 \leq p_0 < p_1 \leq \infty$ and that*

$$T : L_{p_0}(U, d\mu) \to L_{p_0}^*(U, d\mu)$$
$$T : L_{p_1}(U, d\mu) \to L_{p_1}^*(U, d\mu)$$

If $p_0 < p < p_1$, then $T : L_p(U, d\mu) \to L_p(U, d\mu)$.

For the proof we will need Hardy's inequality

Proposition 2.4 (Hardy's Inequality)**.** *Let $p > 1$ and f be a non-negative, measurable function on $(0, \infty)$ such that $f \in L_p(0, \infty)$. Then,*

$$\int_0^\infty (\frac{1}{x} \int_0^x f(t)dt)^p dx \leq (\frac{p}{p-1})^p \int_0^\infty f(x)^p dx.$$

Proof. Using the change of variables $t = xs$, we rewrite the norm:

$$(\int_0^\infty (\frac{1}{x} \int_0^x f(t)dt)^p dx)^{1/p} = (\int_0^\infty (\int_0^1 f(xs)ds)^p dx)^{1/p}.$$

By Minkowski's integral inequality, we can swap the order of integration to bound this by:

$$\leq \int_0^1 (\int_0^\infty f(xs)^p dx)^{1/p} ds.$$

Applying a second change of variables $u = xs$ yields:

$$= \int_0^1 (\int_0^\infty f(u)^p du/s)^{1/p} ds = (\int_0^\infty f(u)^p du)^{1/p} \int_0^1 s^{-1/p} ds$$

And since $p > 1$, $\int_0^1 s^{-1/p} ds = p/(p-1)$. Raising both sides to p yields the desired inequality. $\square$

Proof (Marcinkiewicz Interpolation Theorem). Assume without loss of generality that the weak-type operator norms of T at endpoints p_0 and p_1 are bounded by 1. That is, for $i \in \{0, 1\}$, we have $(Tf)^*(t) \leq t^{-1/p_i}\|f\|_{p_i}$ for $f \in L_{p_i}$.

Now consider $f \in L_p$. We want to see $\|Tf\|_{L_p} \leq C\|f\|_p$ for some constant C. Fix $t > 0$. Because μ is a non-atomic measure, its distribution function is continuous. This guarantees we can choose a measurable set $E \subset \{x : |f(x)| \geq f^*(t)\}$ such that exactly

$$\mu(E) = \min(t, \mu(U)).$$

We split the function as $f = f_0 + f_1$, where:

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

and $f_1(x) = f(x) - f_0(x)$. This construction ensures that $f_0 \in L_{p_0}$ and $f_1 \in L_{p_1}$. Indeed, applying Hölder's inequality with exponents $r = p/p_0 > 1$ and $r' = (p/p_0)'$,

$$\begin{aligned} \|f_0\|_{L_{p_0}}^{p_0} &= \int_E |f(x)|^{p_0} d\mu \leq (\int_E |f(x)|^p d\mu)^{p_0/p} \mu(E)^{(p-p_0)/p} \\ &\leq \|f\|_p^{p_0} t^{(p-p_0)/p} < +\infty. \end{aligned}$$

On the other hand, since $x \in E^c$ implies that $|f(x)| \leq f^*(t)$. This automatically proves $\|f_1\|_{L_\infty} \leq f^*(t) < \infty$ in the case $p_1 = \infty$. For $p_1 < \infty$, since $p_1 - p$ is positive, we get

$$\begin{aligned} \|f_1\|_{L_{p_1}}^{p_1} &= \int_{E^c} |f(x)|^{p_1} d\mu = \int_{E^c} |f(x)|^{p_1 - p} |f(x)|^p d\mu \\ &\leq (f^*(t))^{p_1 - p} \|f\|_{L_p} < +\infty. \end{aligned}$$

By linearity of T, we have $Tf = Tf_0 + Tf_1$. Using (2.6) and the definition of the decreasing rearrangement,

$$(Tf)^*(t) \leq (Tf_0)^*(t/2) + (Tf_1)^*(t/2).$$

Applying the assumed weak-type bounds to f_0 and f_1 gives

$$(Tf_i)^*(t/2) \leq Ct^{-1/p_i} \|f_i\|_{p_i} \quad \text{for } i = 0, 1.$$

Therefore if we compute de L_p norm of Tf and use Minkowski,

$$\|Tf\|_{L_p} = (\int_0^\infty ((Tf)^*(t))^p dt)^{1/p} \leq C(I_0 + I_1),$$

where, for the case $p_1 < \infty$,

$$\begin{aligned} I_0 &= (\int_0^\infty (t^{-1/p_0} \|f_0\|_{p_0})^p dt)^{1/p} = (\int_0^\infty t^{-p/p_0} (\int_0^t f^*(s)^{p_0} ds)^{p/p_0} dt)^{1/p} \\ I_1 &= (\int_0^\infty (t^{-1/p_1} \|f_1\|_{p_1})^p dt)^{1/p} = (\int_0^\infty t^{-p/p_1} (\int_t^\infty f^*(s)^{p_1} ds)^{p/p_1} dt)^{1/p}. \end{aligned}$$

As $p/p_0 > 1$, we can rewrite I_0^p to perfectly match the form of Hardy's inequality

$$I_0^p = \int_0^\infty (\frac{1}{t}\int_0^t f^*(s)^{p_0} ds)^{p/p_0} dt.$$

Therefore it yields,

$$I_0^p \le (\frac{p/p_0}{p/p_0 - 1})^{p/p_0} \int_0^\infty (f^*(t)^{p_0})^{p/p_0} dt = (\frac{p}{p - p_0})^{p/p_0} \|f\|_p^p$$

For the bounding of I_1, let $\theta = p/p_1$. Since $p < p_1$, we have $0 < \theta < 1$. Define then the decreasing function $\phi(s) = f^*(s)^{p_1}$. We must bound

$$I_1^p = \int_0^\infty t^{-\theta} (\int_t^\infty \phi(s) ds)^\theta dt.$$

Since ϕ is decreasing, we can bound the inner integral by a discrete sum over dyadic intervals $[t2^k, t2^{k+1}]$:

$$\int_t^\infty \phi(s) ds \le \sum_{k=0}^\infty t2^k \phi(t2^k).$$

Because $0 < \theta < 1$, the function $x \mapsto x^\theta$ is subadditive. Applying this to our sum:

$$(\int_t^\infty \phi(s) ds)^\theta \le \sum_{k=0}^\infty t^\theta 2^{k\theta} \phi(t2^k)^\theta.$$

Substitute this back into the integral for I_1^p,

$$\begin{aligned} I_1^p &\le \int_0^\infty \sum_{k=0}^\infty 2^{k\theta} \phi(t2^k)^\theta dt = \sum_{k=0}^\infty 2^{k\theta} \int_0^\infty \phi(t2^k)^\theta dt \\ &= \sum_{k=0}^\infty 2^{k\theta} 2^{-k} \int_0^\infty \phi(u)^\theta du = (\sum_{k=0}^\infty 2^{k(\theta-1)}) \int_0^\infty \phi(u)^\theta du. \end{aligned}$$

Since $\theta < 1$, the exponent $(\theta - 1)$ is strictly negative, so the geometric series converges. Replacing $\phi(u)^\theta = (f^*(u)^{p_1})^{p/p_1} = f^*(u)^p$, we get $I_1^p \le C\|f\|_p^p$.

For the case $p_1 = \infty$, the bound for I_1 becomes much easier, since $(Tf_1)^*(t/2) \le \|Tf_1\|_\infty \le \|f_1\|_\infty \le f^*(t)$ implies

$$I_1 = \left(\int_0^\infty (f^*(t))^p dt\right)^{1/p} = \|f\|_{L_p}.$$

Since both I_0 and I_1 are bounded by constant multiples of $\|f\|_p$, we conclude that $T : L_p(U, d\mu) \to L_p(U, d\mu)$. $\square$

The Marcinkiewicz Interpolation Theorem can actually be proved in a much broader setting. In general, the underlying measures do not need to be non-atomic. Furthermore, the theorem also holds for off-diagonal interpolation, where the source and target exponents at each endpoint may differ. Indeed, if we assume that $p_0 \neq p_1$ and that

$$\begin{aligned} T &: L_{p_0}(U, d\mu) \to L^*_{q_0}(V, d\nu) \quad \text{with norm } M_0 \text{ and} \\ T &: L_{p_1}(U, d\mu) \to L^*_{q_1}(V, d\nu) \quad \text{with norm } M_1, \end{aligned}$$

then we have that $T : L_p(U, d\mu) \to L_q(V, d\nu)$ with norm $M \leq C_\theta M_0^{1-\theta} M_1^\theta$ for all $p \leq q$ satisfying

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{2.12}$$

for some $\theta \in (0, 1)$. In Chapter 7 we shall give a complete proof of this version.

The other classical interpolation theorem is the Riesz-Thorin Interpolation Theorem. We now set $\mathbb{K} = \mathbb{C}$. The theorem states the following.

Theorem 2.5 (The Riesz-Thorin Interpolation Theorem)**.** *Assume that $p_0 \neq p_1$ and $q_0 \neq q_1$ and that T is an operator such that*

$$\begin{aligned} T &: L_{p_0}(U, d\mu) \to L_{q_0}(V, d\nu) \quad \textit{with norm } M_0 \\ T &: L_{p_1}(U, d\mu) \to L_{q_1}(V, d\nu) \quad \textit{with norm } M_1. \end{aligned}$$

Then for $0 < \theta < 1$ and any p, q satisfying equation (2.12),

$$T : L_p(U, d\mu) \to L_q(V, d\nu),$$

with norm $M \leq M_0^{1-\theta} M_1^\theta$.

Note this theorem has many differences with respect to the Marcinkiewicz one. On one hand, having an operator bounded from L_{p_i} to L_{q_i} implies that the operator is also weakly bounded from L_{p_i} to $L^*_{q_i}$. Therefore, the result provided by de Riesz-Thorin theorem on the cases $p \leq q$ was already known by the Marcinkiewicz theorem. The converse is not true: having a weak bound for an operator doesn't always imply a "strong" bound. Hence the Marcinkiewicz theorem can be used in cases where the Riesz-Thorin theorem fails. However, in the Marcinkiewicz theorem we cannot get rid of the restriction $p \leq q$.

In the cases where both theorems apply, the bound for the norm of the interpolated T given by the Riesz-Thorin is better than the one given by the

Marcinkiewicz theorem. That is because the Riesz-Thorin proof uses complex function-theoretic methods to get precise bounds. As a fact, most proofs rely on the well-known Hadamard three-lines theorem.

Neverthless it is not the purpose of this project to delve into the complex case. The proof of the Marcinkiewicz Interpolation Theorem is what essentially inspires the K-method or real method of interpolation. The core idea behind both approaches is to be able to "split" an object in a specific space into two other spaces which we already control. Those objects for which the break is sufficiently well-behaved to provide a uniform control will become our interpolated spaces.

3 Interpolation spaces

3.1 Normed Vector Spaces

The Lebesgue spaces form a subcategory[1] of normed vector spaces. A vector space A over a field $\mathbb{K}$ is a **normed vector space** if it can be equipped with a norm $\|\cdot\|_A : A \to \mathbb{R}$ such that for any $a \in A$,

(i) $\|a\|_A \geq 0$, and $\|a\|_A = 0$ if and only if $a = 0$,

(ii) $\|\lambda a\|_A = |\lambda| \|a\|_A \ \forall \lambda \in \mathbb{K}$,

(iii) $\|a + b\|_A \leq \|a\|_A + \|b\|_A$.

We will call the category of normed vector spaces by $\mathcal{N}$. As was done with the Lebesgue spaces, a linear map (or operator) T from two normed vector spaces A and B will be bounded if

$$\|T\|_{A,B} = \sup_{a \neq 0} \frac{\|Ta\|_B}{\|a\|_A} < +\infty.$$

The value $\|T\|_{A,B}$ will be the norm of T. We will write $T : A \to B$ when T is a linear and bounded map. The space of all bounded linear operators between two normed spaces A and B is itself a new normed vector space, with the norm given by $\|\cdot\|_{A,B}$, usually denoted by $\mathcal{L}(A, B)$.

The most studied subcategory of $\mathcal{N}$ is the category of complete normed vector spaces, or **Banach spaces**, $\mathcal{B}$. A complete normed space A is a normed vector space such that any Cauchy sequence in A has a limit in A. Recall that $(a_n)_{n=1}^{\infty}$ was Cauchy if for any $\varepsilon > 0$ there existed $N \geq 1$ such that

$$\|a_n - a_m\|_A < \varepsilon \quad \text{for all } n, m > N,$$

and had a limit in A if there existed $a \in A$ such that

$$\|a_n - a\|_A \to 0 \quad \text{as} \quad n \to \infty.$$

However, instead of using this characterization of Banach spaces, the following lemma is very often used.

Lemma 3.1. *Let A be a normed vector space. Then A is complete if and only if any absolutely convergent series in A is also convergent in A.*

[1]A **category** $\mathcal{C}$ is a set of objects and a set of morphisms. The objects can be related by morphisms following a set of rules (see Mac Lane [12] for further details). In the case of normed vector spaces it is enough to know that morphisms are linear and bounded operators.

Recall that a series $\sum_{n=1}^{\infty} a_n$ was said to be absolutely convergent if the series of real numbers $\sum_{n=1}^{\infty} \|a_n\|_A$ was finite. We said that the same series converged to some element $a \in A$ if the sequence of partial sums $S_N = \sum_{n=1}^{N} a_n \in A$ converged to a.

Proof. Suppose that A is complete and $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. One can see that then the sequence $S_N = \sum_{n=1}^{N} a_n$ is Cauchy, since for $N \geq M$,

$$\|S_N - S_M\|_A = \|\sum_{n=M+1}^{N} a_n\|_A \leq \sum_{n=M+1}^{N} \|a_n\|_A \to 0 \quad \text{as } M \to \infty.$$

Then S_N must converge in A. For the other implication, let $(a_n)_{n=1}^{\infty}$ be any Cauchy sequence in A. As it is Cauchy, one can find a subsequence $(a_{n_j})_{j=1}^{\infty}$ such that $\|a_{n_{j+1}} - a_{n_j}\| < 2^{-j}$ for all $j \geq 1$. If we then define the series $S_N = \sum_{j=1}^{N} b_j$, with $b_1 = a_{n_1}$ and $b_j = (a_{n_j} - a_{n_{j-1}})$ for all $j > 1$, we obtain that

$$\sum_{j=1}^{\infty} \|b_n\|_A \leq \|a_{n_1}\|_A + \sum_{j=1}^{\infty} 2^{-j} < +\infty.$$

By hypothesis, the series must converge to some element in A. But notice now that by construction $S_N = a_{n_N}$. Hence the subsequence $(a_{n_j})_{j=1}^{\infty}$ is convergent in A and as the original sequence is Cauchy it must also converge to the same element of A.

□

Another useful result will be the following lemma.

Lemma 3.2. *Let A be a normed vector space and B a complete normed vector space. Then the set of bounded and linear operators from A to B, $\mathcal{L}(A, B)$ is a Banach space under the norm $\|\cdot\|_{A,B}$.*

Proof. We will use the previous lemma. We must see that any absolutely convergent series in $\mathcal{L}(A, B)$ converges to some element in $\mathcal{L}(A, B)$. Hence, let $\sum_{n=1}^{\infty} T_n$ be a series of operators in $\mathcal{L}(A, B)$ such that $\sum_{n=1}^{\infty} \|T_n\|_{A,B} < \infty$. Then, for any element $a \in A$,

$$\sum_{n=1}^{\infty} \|T_n(a)\|_B \leq \|a\|_A \sum_{n=1}^{\infty} \|T_n\|_{A,B} < \infty, \tag{3.1}$$

so $\sum_{n=1}^{\infty} T_n(a)$ is an absolutely convergent series in B. As B is Banach, the series must converge to some element in B, say $T(a)$. As this can be done for any $a \in A$,we define $T(a) = \sum_{n=1}^{\infty} T_n(a)$. We will see that $T \in \mathcal{L}(A, B)$ and $\lim_{N\to\infty} \sum_{n=1}^{N} T_n = T$ in $\mathcal{L}(A, B)$.

As T is the limit of linear maps it is also linear. For the boundedness, simply use the triangle inequality $\|T(a)\|_B \leq \sum_{n=1}^{\infty} \|T_n(a)\|_B$ and equation (3.1). Finally, observe that for a given $a \in A$,

$$\|(T - \sum_{n=1}^{N} T_n)(a)\|_B = \| \sum_{N+1}^{\infty} T_n(a)\| \leq \|a\|_A \sum_{N+1}^{\infty} \|T_n\|_{A,B}.$$

By the absolute convergence of the original series the right hand side of the inequality goes to zero as N tends to infinity. Taking supremums over $a \in A$ we conclude that $\sum_{n=1}^{N} T_n$ converges to T. $\square$

3.2 Compatible spaces

Any normed vector space A can be equipped with the natural topology induced by the norm. That is the topology generated by the open balls $B_r(a) = \{b \in A : \|b - a\| < r\}$. For this topology, the operations of adding vectors and multiplying by scalars become continuous. We wil say that a vector space is a **topological vector space** if it is equipped with a topology such that these two operations are continuous. Hence, any normed vector space is a topological vector space with the natural topology induced by the norm.

If A is normed vector space, for the natural topology induced given by the norm, a sequence $(a_n)_{n=1}^{\infty} \subseteq A$ tends to $a \in A$ if and only if $\|a_n - a\|_A \xrightarrow{n\to\infty} 0$.

Definition 3.3. *A pair of topological vector spaces A_0 and A_1 are said to be compatible if there is a Hausdorff topological vector space $\mathcal{X}$ such that A_0 and A_1 are continuously embedded in $\mathcal{X}$.*

By A being continuously embedded in $\mathcal{X}$, noted as

$$A \hookrightarrow \mathcal{X},$$

we simply mean that A is a sub-space of $\mathcal{X}$ and that the inclusion map is continuous with respect to their topologies. When A and B are normed vector spaces with topologies induced by their norms, having A continuously embedded in B is equivalent to A being a sub-space of B and there existing a constant $C > 0$ such that

$$\|a\|_B \leq C\|a\|_A \quad \text{for all } a \in A.$$

Here, the element a on the left-hand side is implicitly understood as its image under the inclusion map into B. Any pair (A, B) of normed vector spaces with A continuously embedded in B naturally forms a compatible couple, as we may simply choose $\mathcal{X} = B$ as the common Hausdorff topological vector space.

Example 3.1. As a second example on might consider the couples of Lebesgue spaces $(L_p(U, d\mu), L_q(U, d\mu)$ or weak Lebesgue spaces $(L_p^*(U, d\mu), L_q^*(U, d\mu))$, for $1 \leq p, q \leq \infty$ and some measure space (U, μ). Clearly, all four spaces are continuously embedded in the Hausdorff topological vector space of measurable functions (identified up to μ-almost everywhere equality) that are finite mu-almost everywhere, $\mathcal{M}_0(U, d\mu)$. Indeed, $\mathcal{M}_0(U, d\mu)$ is a Hausdorff topological vector space when equipped with the convergence in measure topology. For a complete explanation see, for instance, Bennet and Sharpley [3].

In the sequel, for any compatible couple (A_0, A_1) contained in a Hausdorff topological vector space $\mathcal{X}$ and any element $a_i \in A_i$, we will also write $a_i \in \mathcal{X}$ for the inclusion of a_i in $\mathcal{X}$. Hence the expressions $a_0 + a_1$ with $a_0 \in A_0$ and $a_1 \in A_1$ make sense in $\mathcal{X}$.

Definition 3.4. *Let A_0 and A_1 be two compatible topological vector spaces contained in the Hausdorff topological vector space $\mathcal{X}$. We define the intersection sub-space $A_0 \cap A_1$ as the set of elements $a \in \mathcal{X}$ such that $a \in A_0$ and $a \in A_1$. We define their sum $A_0 + A_1$ as the set of elements $a \in \mathcal{X}$ such that we can write $a = a_0 + a_1$ for some $a_0 \in A_0$ and $a_1 \in A_1$.*

Notice that $A_0 \cap A_1$ and $A_0 + A_1$ do not depend on the choice of $\mathcal{X}$. Hence,when it is not needed, we will not specify which is the Hausdorff space $\mathcal{X}$ containing both spaces. Instead we will simply speak about elements of $A_0 \cap A_1$ and $A_0 + A_1$ for some compatible couple (A_0, A_1).

Lemma 3.5. *Suppose that A_0 and A_1 are compatible normed vector spaces. Then $A_0 \cap A_1$ and $A_0 + A_1$ are both normed vector spaces with norms given by*

$$\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1}) \tag{3.2}$$

and

$$\|a\|_{A_0 + A_1} = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}). \tag{3.3}$$

Moreover, if A_0 and A_1 are complete, then $A_0 \cap A_1$ and $A_0 + A_1$ are also complete.

Proof. It is trivial to see that $A_0 \cap A_1$ is a vector space and that $\| \cdot \|_{A_0 \cap A_1}$ satisfies the three conditions for being a norm stated before. It is also straightforward to see that $A_0 + A_1$ forms a vector space and that the norm $\| \cdot \|_{A_0+A_1}$ satisfies (ii) and (iii). Since the norm is clearly non-negative we must only see that if $\|a\|_{A_0+A_1} = 0$ for some $a \in A_0 + A_1$, then $a = 0$. By definition, if $\|a\|_{A_0+A_1} = 0$, there exist two sequences $(a_n^0)_{n=1}^\infty \subseteq A_0$ and $(a_n^1)_{n=1}^\infty \subseteq A_1$ such that $a = a_n^0 + a_n^1$ for all $n > 0$ and

$$\|a_n^0\|_{A_0} + \|a_n^1\|_{A_1} \xrightarrow{n\to\infty} 0.$$

Therefore $\|a_n^i\|_{A_i} \to 0$ and $a_n^i \to 0$ in A_i as n tends to infinity. As we have a Hausdorff space $\mathcal{X}$ for which the inclusions maps are continuous we get that $a = a_n^0 + a_n^1 \to 0$ in $\mathcal{X}$ as n tends to infinity. But since $\mathcal{X}$ is Hausdorff, $a = 0$.

For the completeness of $A_0 \cap A_1$, assume $(a_n)_{n=1}^\infty$ is a Cauchy sequence with the norm given by (3.2). Then the same sequence is Cauchy in A_0 and A_1 with their respective norms. But as A_0 and A_1 are complete, there are two elements $a_0 \in A_0$ and $a_1 \in A_1$ such that $a_n \to a_i$ in A_i. Since the inclusions of A_i into some Hausdorff space $\mathcal{X}$ are continuous, $a_n \to a_i$ in $\mathcal{X}$. But $\mathcal{X}$ is Hausdorff, so $a_0 = a_1 = a \in A_0 \cap A_1$. Then

$$\|a_n - a\|_{A_0 \cap A_1} \leq \max(\|a_n - a\|_{A_0}, \|a_n - a\|_{A_1}) \xrightarrow{n\to\infty} 0,$$

which proves $A_0 \cap A_1$ is complete.

To establish the completeness of $A_0 + A_1$ we will use the caracterization of Banach spaces given by Lemma 3.1. Assume that

$$\sum_{n=1}^\infty \|a_n\|_{A_0+A_1} < \infty.$$

Then we can find two sequences $(a_n^0)_{n=1}^\infty \subseteq A_0$ and $(a_n^1)_{n=1}^\infty \subseteq A_1$ such that for all $n > 0$, $a_n = a_n^0 + a_n^1$ and

$$\|a_n^0\|_{A_0} + \|a_n^1\|_{A_1} \leq 2\|a_n\|_{A_0+A_1}.$$

It follows that

$$\sum_{n=1}^\infty \|a_n^0\|_{A_0} < \infty, \qquad \sum_{n=1}^\infty \|a_n^1\|_{A_1} < \infty.$$

As A_0 and A_1 are complete we obtain that $\sum a_n^0$ converges in A_0 and $\sum a_n^1$ converges in A_1. Put $a^0 = \sum a_n^0$ and $a^1 = \sum a_n^1$ and $a = a^0 + a^1$. Then $a \in A_0 + A_1$ and since

$$\|a - \sum_{n=1}^N a_n\|_{A_0+A_1} \leq \|a^0 - \sum_{n=1}^N a_n^0\|_{A_0} + \|a^1 - \sum_{n=1}^N a_n^1\|_{A_1}$$

we conclude that $\sum_n a_n$ converges in $A_0 + A_1$ to a and so the space is complete. $\square$

From now on we will only be considering normed vector spaces in some subcategory $\mathcal{C}$ of $\mathcal{N}$. Observe that having a compatible couple in $\mathcal{C}$ ensures that the intersection and sum spaces are well defined. Nevertheless, it could happen that the sum and intersection spaces might not be in $\mathcal{C}$. Therefore, we will ask the subcategory $\mathcal{C}$ of $\mathcal{N}$ to be closed under the operations of adding and intersecting spaces. A compatible couple (A_0, A_1) will essentially mean a pair of normed vector spaces $A_0, A_1 \in \mathcal{C}$, both continuously embedded in some Hausdorff topological vector space $\mathcal{X}$, where $\mathcal{C}$ is a subcategory of $\mathcal{N}$ closed under the operations of adding and intersecting spaces.

Of course $\mathcal{C} = \mathcal{N}$ and $\mathcal{C} = \mathcal{B}$ work as subcategory by the previous lemma.

Example 3.2. As a second example we can consider the category $\mathcal{C}$ of all spaces $L_{1,w}(\mathbb{R}^n, dx)$ defined by the norms

$$\|f\|_{L_{1,w}} = \int |f(x)|w(x)dx,$$

where w is a non-negative and locally integrable function and dx denotes the Lebesgue measure. One can easily see that $L_{1,w_0} \cap L_{1,w_1} = L_{1,w'}$ and $L_{1,w_0} + L_{1,w_1} = L_{1,w''}$, where

$$w'(x) = \max(w_0(x), w_1(x)) \quad \text{and} \quad w''(x) = \min(w_0(x), w_1(x)).$$

Definition 3.6. *If (A_0, A_1) is a compatible couple, then a normed vector space $A \in \mathcal{C}$ is said to be an intermediate space between A_0 and A_1 (or with respect to (A_0, A_1)) if it is continuously embedded between $A_0 \cap A_1$ and $A_0 + A_1$:*

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1.$$

Clearly, A_0, A_1, $A_0 \cap A_1$ and $A_0 + A_1$ are all intermediate spaces with respect to (A_0, A_1). We still need another assumption over A to have an interpolation space. For two compatible couples (A_0, A_1) and (B_0, B_1), a linear mapping from $A_0 + A_1$ to $B_0 + B_1$ will be said to be **admissible** if

$$T|_{A_0} : A_0 \to B_0 \quad \text{and} \quad T|_{A_1} : A_1 \to B_1. \tag{3.4}$$

That is, the restrictions of T are also bounded from A_i to B_i. The set of all admissible operators from (A_0, A_1) to (B_0, B_1) will be denoted by $\mathcal{A}((A_0, A_1), (B_0 B_1))$. Also, we may write $T : (A_0, A_1) \to (B_0, B_1)$ whenever T is an admissible operator. The set of all compatible couples becomes a category if we consider admissible operators as morphisms. We will denote it by $\mathcal{C}_1$.

From (3.4) it is easy to see that for an admissible operator $T \in \mathcal{A}((A_0, A_1), (B_0 B_1))$ and any decomposition $a = a_0 + a_1$ of $a \in A_0 + A_1$,

$$\begin{aligned} \|T(a)\|_{B_0+B_1} &\leq \|T|_{A_0}(a_0)\|_{B_0} + \|T|_{A_1}(a_1)\|_{B_1} \\ &\leq \|T\|_{A_0,B_0}\|a_0\|_{A_0} + \|T\|_{A_1,B_1}\|a_1\|_{A_1} \end{aligned}$$

and hence

$$\|T\|_{A_0+A_1,B_0+B_1} \leq \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}). \tag{3.5}$$

Also, for $a \in A_0 \cap A_1$

$$\begin{aligned} \|T(a)\|_{B_0 \cap B_1} &= \max(\|T(a)\|_{B_0}, \|T(a)\|_{B_1}) \\ &\leq \max(\|T\|_{A_0,B_0}\|a\|_{A_0}, \|T\|_{A_1,B_1}\|a\|_{A_1)} \end{aligned}$$

and hence

$$\|T\|_{A_0 \cap A_1,B_0 \cap B_1} \leq \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}). \tag{3.6}$$

So for any admissible operator, although we did not assume it in the definition, we eventually have $T : A_0 + A_1 \to B_0 + B_1$ and $T : A_0 \cap A_1 \to B_0 \cap B_1$.

3.3 Interpolation Spaces

We will write $\overline{A} = (A_0, A_1) \in \mathcal{C}_1$ for a compatible couple (A_0, A_1). Given a pair of compatible couples $\overline{A}$, $\overline{B}$ and an admissible operator $T \in \mathcal{A}(\overline{A}, \overline{B})$, we will also write T for the restrictions of T into A_i.

Definition 3.7. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be two compatible couples in $\mathcal{C}_1$. Let $A \in \mathcal{C}$ and $B \in \mathcal{C}$ be two intermediate spaces with respect to $\overline{A}$ and $\overline{B}$, respectively. We will say that the couple (A, B) is an interpolation couple with respect to (or between) $\overline{A}$ and $\overline{B}$ if*

$$T : \overline{A} \to \overline{B} \quad \textit{implies} \quad T : A \to B.$$

Often we will deal with the case $\overline{A} = \overline{B}$. In this case we will simply say that $A \in \mathcal{C}$ is an **interpolation space** with respect $\overline{A}$ (or between A_0 and A_1) if the couple (A, A) is an interpolation couple with respect to $\overline{A}$ and $\overline{A}$.

While this construction may initially appear complex, it becomes more intuitive when framed within the context of the Riesz-Thorin theorem. As previously stated, the pairs (L_{p_0}, L_{p_1}) form compatible couples in $\mathcal{N}$. Furthermore, one can easily verify that when the relations between p_0, p_1, and

p given by equations (2.12) are satisfied, L_p is indeed an intermediate space between L_{p_0} and L_{p_1}. Hence, what the Riesz-Thorin theorem establishes is that (L_p, L_q) acts as interpolation couple between (L_{p_0}, L_{p_1}) and (L_{q_0}, L_{q_1}) whenever relations (2.12) are held.

Still, we can not assert the same for the Marcinkiewicz Interpolation Theorem. Although being normable spaces for $p > 1$, the norms used for the operators in the theorem involve the quasi-norms of L_p^* given by (2.5). Nevertheless, the theory developed up the point is for bounded operators with respect to norms. We will see later in Chapter 7 that this theory is also valid in a more general framework.

Clearly, for a compatible couple $\overline{A} = (A_0, A_1)$, both

$$\Delta(\overline{A}) := A_0 \cap A_1 \quad \text{and} \quad \Sigma(\overline{A}) := A_0 + A_1 \tag{3.7}$$

are interpolation spaces between A_0 and A_1. Given another compatible couple $\overline{B} = (B_0, B_1)$ it is also straightforward form equations (3.5) and (3.6) that the couples $(\Delta(\overline{A}), \Delta(\overline{B}))$ and $(\Sigma(\overline{A}), \Sigma(\overline{B}))$ are both interpolation couples between $\overline{A}$ and $\overline{B}$.

Definition 3.8. *Given two compatible couples $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$, and (A, B) an interpolation couple between them, we say that (A, B) is an exact interpolation couple if for any admissible operator $T = \overline{A} \to \overline{B}$ we have*

$$\|T\|_{A,B} \leq \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}). \tag{3.8}$$

As discussed before the definition, it is clear that both $(\Delta(\overline{A}), \Delta(\overline{B}))$ and $(\Sigma(\overline{A}), \Sigma(\overline{B}))$ are exact interpolation spaces between $\overline{A}$ and $\overline{B}$.

However, not all interpolation spaces act like this. The classical interpolation theorems of Marcinkiewicz and Riesz-Thorin, gave us bounds of the style $\|T\|_{A,B} \leq C\|T\|_{A_0,B_0}^{1-\theta\|}\|T\|_{A_1,B_1}^{\theta}$ for some $0 \leq \theta \leq 1$. This motivates the definition:

Definition 3.9. *In the same hypothesis as in the previous definition, we say that (A, B) is an interpolation couple of exponent θ $(0 \leq \theta \leq 1)$ if for any admissible operator $T = \overline{A} \to \overline{B}$ we have*

$$\|T\|_{A,B} \leq C\|T\|_{A_0,B_0}^{1-\theta}\|T\|_{A_1,B_1}^{\theta}. \tag{3.9}$$

If $C = 1$, we say the interpolation couple (A, B) is of exact exponent θ.

Our last type of interpolation couples will be **uniform interpolation** couples.

Definition 3.10. *In the same hypothesis as in Definition 3.8, we say that* (A, B) *is a uniform interpolation couple if there exists* $C > 0$ *such that for any admissible operator* $T = \overline{A} \to \overline{B}$ *we have*

$$\|T\|_{A,B} \leq C \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}). \tag{3.10}$$

As expected, we say that an interpolation space A between A_0 and A_1 is an exact interpolation space (or interpolation space of exponent θ or a uniform interpolation space) between A_0 and A_1 (or with respect to $\overline{A}$) if (A, A) is an exact interpolation couple (or an interpolation couple of exponent θ or a uniform interpolation couple, respectively) between $\overline{A}$ and $\overline{A}$.

Theorem 3.11. *Consider the category* $\mathcal{B}$. *Suppose that* (A, B) *is an interpolation couple between* $\overline{A}$ *and* $\overline{B}$. *Then it is a uniform interpolation couple.*

When the couples $\overline{A}$ and $\overline{B}$ interpolated are clear, we may simply write $\mathcal{A}$ instead of $\mathcal{A}(\overline{A}, \overline{B})$. For the proof of the theorem we will need the following lemma:

Lemma 3.12. *Let* $\overline{A}$ *and* $\overline{B}$ *be two compatible couples in the category of Banach. Then* $\mathcal{A}$ *equipped with the norm* $\|T\|_{\mathcal{A}} = \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1})$, $\mathcal{A}$ *is also a Banach space.*

Proof. The fact that $\mathcal{A}$ is a vector space and $\|\cdot\|_{\mathcal{A}}$ is a norm is trivial. We will only prove the completeness of the space. Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence of operators in $\mathcal{A}$. By the definition of the norm, the restrictions of T_n to the spaces A_0 and A_1 also form Cauchy sequences in $\mathcal{L}(A_0, B_0)$ and $\mathcal{L}(A_1, B_1)$ respectively. But these are complete spaces with respect to the norms $\|\cdot\|_{A_i,B_i}$, $i = 0, 1$ by Lemma 3.2. Therefore the sequences $\{T_n|_{A_i}\}_{n=1}^{\infty}$ both converge to some operator $T^{(i)}$ in $\mathcal{L}(A_i, B_i)$ for $i = 0, 1$.

We will see that the two operators $T^{(0)}$ and $T^{(1)}$ induce an admissible operator $T \in \mathcal{A}$, which is the limit of our original sequence. First, observe that $T^{(0)}$ and $T^{(1)}$ both coincide on the intersection $A_0 \cap A_1$. Indeed, let $a \in A_0 \cap A_1$. Then $\lim_n T_n(a) = T^{(0)}(a)$ in the topology of B_0 and $\lim_n T_n(a) = T^{(1)}(a)$ in the topology of B_1. But since B_0 and B_1 are continuously embedded in $B_0 + B_1$, these limits are also valid for the topology of $B_0 + B_1$. Finally as B_0 and B_1 are contained in a larger Hausdorff topological vector space, the limits must be the same, $T^{(0)}(a) = T^{(1)}(a)$. We finally set the global operator $T : A_0 + A_1 \to B_0 + B_1$ defined as

$$T(a_0 + a_1) = T^{(0)}(a_0) + T^{(1)}(a_1).$$

We must see that T is independent of the choice of the representation in A_0+A_1. Let $a_0'+a_1' = a_0+a_1$ be two different representations of an element $a \in A_0+A_1$. Then $a_0'-a_0 = a_1-a_1' \in A_0 \cap A_1$. Therefore $T^{(0)}(a_0'-a_0) = T^{(1)}(a_1-a_1')$. By linearity, we obtain $T^{(0)}(a_0')+T^{(1)}(a_1') = T^{(0)}(a_0)+T^{(1)}(a_1)$ as we wanted. By construction, $T|_{A_i} = T^{(i)} : A_i \to B_i$, $i = 0, 1$, so T is an admissible operator. Finally,

$$\|T - T_n\|_{\mathcal{A}} = \max(\|T^{(0)} - T_n\|_{A_0,B_0}, \|T^{(1)} - T_n\|_{A_1,B_1}) \xrightarrow{n\to\infty} 0.$$

□

Proof (Theorem 3.11). Consider the set $\mathcal{A}$ of admissible operators from $\overline{A}$ to $\overline{B}$. As any element in $\mathcal{A}$ also acts as a bounded and linear operator from A to B, we might now consider the norm on $\mathcal{A}$ given by

$$\|T\|_1 = \max(\|T\|_{A_0,B_0}, \|T\|_{A,B}, \|T\|_{A_1,B_1}).$$

Carefully modifying the proof of the previous lemma, one can show that $\mathcal{A}_1 = (\mathcal{A}, \|\cdot\|_1)$ is a Banach space. On the other hand, we already know that $\mathcal{A}_2 = (\mathcal{A}, \|\cdot\|_{\in})$ is a Banach space, where $\|T\|_2 = \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1})$. The identifying map $i : \mathcal{A}_1 \to \mathcal{A}_2$ is clearly linear, bijective and bounded. By the Open Mapping Theorem, i^{-1} is also bounded, which means that $\|T\|_1 \leq C\|T\|_2$ for any admissible $T \in \mathcal{A}$. Therefore,

$$\|T\|_{A,B} \leq \|T\|_1 \leq C\|T\|_2 = C \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}).$$

So we conclude that (A, B) is a uniform interpolation couple between $\overline{A}$ and $\overline{B}$. □

3.4 Interpolation method

The main objective of the interpolation theory is to find interpolation couples or spaces between compatible pairs. The real or K-method is a constructive method. Given a compatible couple $\overline{A}$ we seek an algorithm or method to construct an intermediate space A such that if B is also found from $\overline{B}$ using this method, (A, B) is an interpolation couple between $\overline{A}$ and $\overline{B}$. Formally, an **interpolation method** on $\mathcal{C}$ is a functor[2] F from $\mathcal{C}_1$ to $\mathcal{C}$ such that if $\overline{A}$ and $\overline{B}$ are couples in $\mathcal{C}_1$, then $(F(\overline{A}), F(\overline{B}))$ is an interpolation couple

[2]A **functor** F between two cateories $\mathcal{C}_1$ and $\mathcal{C}_2$ maps elements from $\mathcal{C}_1$ to elements in $\mathcal{C}_2$ and morphisms in $\mathcal{C}_1$ to morphisms in $\mathcal{C}_2$ in such a way that if $T : A \to B$ in $\mathcal{C}_1$, then $F(T) : F(A) \to F(B)$ in $\mathcal{C}_2$.

between them. Moreover, for any admissible operator $T : \overline{A} \to \overline{B}$, it must hold

$$F(T) = T|_{F(\overline{A})} : F(\overline{A}) \to F(\overline{B}).$$

Notice that the functoriality of F ensures that the construction of $F(\overline{A})$ depends only on the couple $\overline{A}$, and not on the target couple $\overline{B}$. That is, if we consider a third compatible couple $\overline{D} \in \mathcal{C}_1$, then $F(\overline{A})$ is an intermediate space with respect to $\overline{A}$ such that both $(F(\overline{A}), F(\overline{B}))$ and $(F(\overline{A}), F(\overline{D}))$ are interpolation couples with respect to $(\overline{A}, \overline{B})$ and $(\overline{A}, \overline{D})$, respectively. Of course, if F is an interpolation method, $F(\overline{A})$ is also an interpolation space with respect to $\overline{A}$.

An interpolation method F is said to be a **uniform (exact) interpolation method** if for any couples $\overline{A}$ and $\overline{B}$ in $\mathcal{C}_1$, then $(F(\overline{A}),F(\overline{B}))$ is a uniform (exact) interpolation couple with respect to $\overline{A}$ and $\overline{B}$. Similarly, we say that the method is an **(exact) interpolation method of exponent** θ if $(F(\overline{A}), F(\overline{B}))$ is exact of exponent θ.

By definition, the method is uniform if for any two couples $\overline{A}$ and $\overline{B}$ in $\mathcal{C}_1$, $F(\overline{A})$ and $F(\overline{B})$ are intermediate spaces with respect to $\overline{A}$ and $\overline{B}$ and for any admissible operator $T : \overline{A} \to \overline{B}$,

$$\|T\|_{F(\overline{A}),F(\overline{B})} \leq C \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}),$$

where C can depend on $\overline{A}$ and $\overline{B}$. When the choice of C can be made independently of $\overline{A}$ and $\overline{B}$, we say that the method is a **bounded interpolation method**. Indeed, an exact interpolation method is a bounded interpolation method with constant $C = 1$.

We can already introduce two simple interpolation methods: Σ and Δ. Defined in equations (3.7), these two can be understood as functors from $\mathcal{C}_1$ to $\mathcal{C}$. As was proven in equations (3.5) and (3.6), these two functors give place to exact interpolation methods.

4 The K-Interpolation Method

This chapter essentially follows C.Bennet and R.Sharpley [3] and J. Bergh and J.Löfström [4], which itself develops the theory in J. Peetre [14]. Unless stated we will always consider couples in $\mathcal{N}_1$, the category of compatible couples in $\mathcal{N}$.

The K-method constructs a family of interpolation methods $K_{\theta,q}$ with parameters θ and q. The way of constructing these functors is deeply linked with the proof of the Marcinkiewicz Interpolation Theorem. Remember that the interpolation space in the theorem was composed of those objects that could be split well enough for any $t > 0$. Our role of t follows from the next definition.

Definition 4.1. *Let $\overline{A}$ a compatible couple in $\mathcal{N}_1$. The K-functional is defined for each $t > 0$ as the map from $\Sigma(\overline{A})$ to $\mathbb{R}$ given by*

$$K(t,a) := K_t(a;\overline{A}) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + t\|a_1\|_{A_1})$$

Proposition 4.2. *For a compatible couple $\overline{A} \in \mathcal{N}_1$ and $a \in \Sigma(\overline{A})$, $K(t,a)$ is a non-negative, non-decreasing and concave function of t. In particular*

$$K(t,a) \leq \max(1,t/s)K(s,a). \tag{4.1}$$

Therefore $\{K(t,\cdot)\}_{t>0}$ is a family of norms in $\Sigma(\overline{A})$ equivalent to $\|\cdot\|_{\Sigma(\overline{A})}$.

Proof. $K(t,a)$ is clearly a non-negative and non-decreasing function of t. For concavity, one can use the following argument from real analysis: for any decomposition $a = a_0 + a_1$, the function $f_{a_0,a_1}(t) = \|a_0\|_{A_0} + t\|a_1\|_{A_1}$ is linear on t and hence concave. Since $K(t,a)$ is the pointwise infimum of these functions, it is also concave.

Equation (4.1) follows from considering

$$K(t,a) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + s\|a_1\|_{A_1}(t/s)).$$

If $(t/s) \geq 1$, we have $K(t,a) \leq (t/s)\inf(\|a_0\|_{A_0} + s\|a_1\|_{A_1}) = (t/s)K(s,a)$. If $(t/s) < 1$, we note that $\|a_0\|_{A_0} + t\|a_1\|_{A_1} \leq \|a_0\|_{A_0} + s\|a_1\|_{A_1}$, which implies $K(t,a) \leq K(s,a)$. Finally, by an argument analogous to Lemma 3.5, for any fixed $t > 0$, $K(t,\cdot)$ defines a norm on $\Sigma(\overline{A})$. Manipulating (4.1) yields, for any $s,t > 0$ and $a \in \Sigma(\overline{A})$

$$\min(1,t/s)K(s,a) \leq K(t,a) \leq \max(1,t/s)K(s,a),$$

which proves that $\{K(t,\cdot)\}_{t>0}$ is a family of equivalent norms. In particular, for $t = 1$, we have $K(1,a) = \|a\|_{\Sigma(\overline{A})}$. $\square$

The next step of the method considers the following idea: motivated by the Marcinkiewicz Interpolation Theorem, for $a \in \Sigma(\overline{A})$ and $t > 0$, we want to decompose a into $a = a_0 + a_1$ in such a way that we can simultaneously control $\|a_0\|_{A_0}$ and $\|a_1\|_{A_1}$. Notice that as t tends to infinity, the definition of $K(t, a)$ places much more weight on the A_1 component. Therefore, by requiring that $K(t, a)$ does not grow too fast, we ensure that a can be decomposed in a manner where $\|a_1\|_{A_1}$ is tightly controlled by t. On the other hand, when t tends to zero, $K(t, a)$ places more weight on the A_0 component. Hence, controlling $\|a_0\|_{A_0}$ can be achieved by ensuring that $K(t, a)$ decreases fast enough when t tends to zero. This idea is done by considering the functional $\phi_{\theta,q}$ defined by

$$\phi_{\theta,q}(\varphi(t)) = \begin{cases} (\int_0^\infty (t^{-\theta}\varphi(t))^q dt/t)^{1/q} & 1 \le q < \infty \\ \sup_{t>0} t^{-\theta}\varphi(t) & q = \infty, \end{cases} \tag{4.2}$$

and establishing:

Definition 4.3. *Let $\overline{A} = (A_0, A_1)$ be a compatible couple in $\mathcal{N}_1$. In the cases $0 < \theta < 1$ for $1 \le q < \infty$ and $0 \le \theta \le 1$ for $q = \infty$, the space $\overline{A}_{\theta,q;K} = K_{\theta,q}(\overline{A})$, consists of all $a \in \Sigma(\overline{A})$ for which*

$$\|a\|_{\overline{A}_{\theta,q;K}} = \phi_{\theta,q}(K(t, a)) \tag{4.3}$$

is finite.

From the inequality (4.1) it is clear that the cases $q < \infty$ and $\theta = 0, 1$ do not make sense, as $\phi_{\theta,q}(K(t, a))$ is never finite. When the couple $\overline{A}$ being interpolated is clear, we may write $\|a\|_{\theta,q;K}$ instead of $\|a\|_{\overline{A}_{\theta,q;K}}$.

Theorem 4.4 (Interpolation Theorem)**.** *$K_{\theta,q}$ is an exact interpolation functor of exponent θ on the category $\mathcal{N}$.*

Proof. We divide the proof in two parts. First, we will prove that $K_{\theta,q}(\overline{A})$ defines an intermediate space with respect to $\overline{A}$. Then, we will show that $K_{\theta,q}$ is an exact interpolation functor of exponent θ.

Before proving the inclusions, we verify that the expression in (4.3) defines a norm. This is straightforward once we know $K(t, \cdot)$ is a norm on $\Sigma(\overline{A})$ for any $t > 0$. Clearly, $\|a\|_{\theta,q;K}$ is non-negative, and equals zero if and only if $K(t, a) = 0$ for all $t > 0$.[3] But since $K(t, \cdot)$ are norms equivalent to $\|\cdot\|_{\Sigma(\overline{A})}$, having $K(t, a) = 0$ for some $t > 0$ means $a = 0$ in $\Sigma(\overline{A})$. Absolute

[3]In the case $q < \infty$ one gets $K(t, a) = 0$ at almost every $t > 0$. But having $K(\cdot, a)$ concave it must also be continuous, and hence $K(t, a) = 0$ at all $t > 0$.

homogeneity is trivial, and the triangle inequality is clear when $q = \infty$. For $q < \infty$, simply use the fact that $K(t, \cdot)$ satisfies the triangle inequality and Minkowski's inequality for integrals.

To establish the continuity of the inclusions, observe first that if $a \in \Delta(\overline{A})$, then $K(t, a) \leq \|a\|_{A_0}$ and $K(t, a) \leq t\|a\|_{A_1}$ simultaneously. Hence,

$$K(t, a) \leq \min(1, t)\|a\|_{\Delta(\overline{A})} \quad \text{for all } a \in \Delta(\overline{A}).$$

A basic calculation shows that $0 < \phi_{\theta,q}(\min(1, t)) = C_{\theta,q} < \infty$ for any permissible combination of θ and q. Therefore, applying $\phi_{\theta,q}$ yields:

$$\|a\|_{\theta,q;K} \leq \|a\|_{\Delta(\overline{A})}\phi_{\theta,q}(\min(1, t)) = \|a\|_{\Delta(\overline{A})}C_{\theta,q}.$$

Now let $a \in \overline{A}_{\theta,q;K}$. Equation (4.1) for $t = 1$ establishes

$$K(s, a) \geq \|a\|_{\Sigma(\overline{A})} \min(1, s) \quad \text{for all } a \in \Sigma(\overline{A}).$$

Similarly to the previous case, applying $\phi_{\theta,q}$ gives

$$\|a\|_{\theta,q;K} \geq \|a\|_{\Sigma(\overline{A})}\phi_{\theta,q}(\min(1, s)) = \|a\|_{\Sigma(\overline{A})}C_{\theta,q},$$

and so $\|a\|_{\Sigma(\overline{A})} \leq C_{\theta,q}^{-1}\|a\|_{\theta,q;K}$. This proves $\Delta(\overline{A}) \hookrightarrow K_{\theta,q}(\overline{A}) \hookrightarrow \Sigma(\overline{A})$.

It remains to prove that $K_{\theta,q}$ is an exact interpolation method of exponent θ. Suppose $T : \overline{A} \to \overline{B}$ is an admissible operator, where $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$. Let

$$M_i = \|T\|_{A_i,B_i}, \quad i = 0, 1.$$

We want to prove that $\|T\|_{\overline{A}_{\theta,q;K},\overline{B}_{\theta,q;K}} \leq M_0^{1-\theta}M_1^{\theta}$. By the definition of the K-functional and the linearity of T,

$$\begin{aligned} K(t, Ta; \overline{B}) &\leq \inf_{a=a_0+a_1} (\|Ta_0\|_{B_0} + t\|Ta_1\|_{B_1}) \\ &\leq \inf_{a=a_0+a_1} (M_0\|a_0\|_{A_0} + tM_1\|a_1\|_{A_1}) = M_0K(tM_1/M_0, a; \overline{A}). \end{aligned} \tag{4.4}$$

Observe now that for a fixed $s > 0$ and $q < \infty$,

$$\begin{aligned} \phi_{\theta,q}(\varphi(t/s)) &= (\int_0^\infty (t^{-\theta}\varphi(t/s))^q dt/t)^{1/q} \\ &= s^{-\theta}(\int_0^\infty (u^{-\theta}\varphi(u))^q \frac{du}{u})^{1/q} = s^{-\theta}\phi_{\theta,q}(\varphi(t)), \end{aligned} \tag{4.5}$$

where we used the change of variables $u = t/s$, for which $du/u = dt/t$. The same change of variables proves that the equality is also valid in the case

$q = \infty$. Considering $s = M_0/M_1$ and computing $\phi_{\theta,q}$ on both sides of the inequality (4.4), we obtain

$$\|Ta\|_{\overline{B}_{\theta,q;K}} \leq M_0(M_0/M_1)^{-\theta}\|a\|_{\overline{A}_{\theta,q;K}} = M_0^{1-\theta}M_1^{\theta}\|a\|_{\overline{A}_{\theta,q;K}}.$$

This completes the proof. □

Corollary 4.5. *$K_{\theta,q}$ is an exact interpolation method of exponent θ on the category of Banach normed spaces $\mathcal{B}$.*

Proof. By the previous theorem and the functoriality of $K_{\theta,q}$, we must only prove that for any compatible couple $\overline{A} = (A_0, A_1)$ with A_0 and A_1 complete, then $\overline{A}_{\theta,q;K}$ is also complete. We will use Lemma 3.1 and see that any convergent series in $\overline{A}_{\theta,q;K}$ has a limit in $\overline{A}_{\theta,q;K}$. Suppose therefore that $(a_n)_{n=1}^{\infty}$ is a sequence in $\overline{A}_{\theta,q;K}$ such that $\sum_n \|a_n\|_{\theta,q;K}$ is finite. Notice that because $\overline{A}_{\theta,q;K}$ is continuously embedded in $\Sigma(\overline{A})$, the original sequence is also absolutely summable in $\Sigma(\overline{A})$. By Lemma 3.5, we know that $\Sigma(\overline{A})$ is complete, so the series $\sum_n a_n$ converges in $\Sigma(\overline{A})$ to some element $a \in \Sigma(\overline{A})$.

Using the fact that $K(t,\cdot)$ is a norm on $\Sigma(\overline{A})$ and $L_q(dt/t)$ is a complete metric space,

$$\begin{aligned}\|\sum_{j>N} a_j\|_{\theta,q;K} &\leq \|\sum_{j>N} t^{-\theta}K(t,a_j)\|_{L^q(dt/t)} \\ &\leq \sum_{j>N}^{\infty} \|t^{-\theta}K(t,a_j)\|_{L^q(dt/t)} = \sum_{j>N}^{\infty} \|a_j\|_{\theta,q;K} \xrightarrow{N\to\infty} 0,\end{aligned}$$

which proves at the same time that $a \in \overline{A}_{\theta,q;K}$ and that $\sum_n a_n$ tends to a in $\overline{A}_{\theta,q;K}$.

□

There are several variants of the K-method. A very useful one is the equivalence with its discrete version.

Definition 4.6. *The space $\lambda^{\theta,q}$ consists of all sequences $(\alpha_n)_{n=-\infty}^{\infty}$ such that*

$$\|(\alpha_n)\|_{\lambda^{\theta,q}} = \begin{cases} (\sum_{n=-\infty}^{\infty}(2^{-\theta n}|\alpha_n|)^q)^{1/q} & \text{for } 1 \leq q < \infty. \\ \sup_{n\in\mathbb{Z}} 2^{-\theta n}|\alpha_n| & \text{for } q = \infty. \end{cases}$$

is finite.

We then have the following equivalence.

Lemma 4.7. *Let $\overline{A}$ be a compatible couple in $\mathcal{N}_1$ and $a \in \Sigma(\overline{A})$. Set $\alpha_n = K(2^n, a; \overline{A})$ for any $n \in \mathbb{Z}$. Then a belongs to $\overline{A}_{\theta,q;K}$ if and only if $(\alpha_n)_{n=-\infty}^{\infty}$ belongs to $\lambda^{\theta,q}$*

Proof. We will prove the following equivalence

$$2^{-\theta}\log(2)\|(\alpha_n)\|_{\lambda^{\theta,q}} \leq \|a\|_{\theta,q;K} \leq 2\log(2)\|(\alpha_n)\|_{\lambda^{\theta,q}}.$$

By the inequality (4.1) we know that for $t \in [2^n, 2^{n+1})$,

$$K(2^n, a) \leq K(t, a) \leq 2K(2^n, a).$$

Multiplying by $t^{-\theta}$ and applying the bounds $2^{-(n+1)\theta} \leq t^{\theta} \leq 2^{-\theta n}$ we get

$$2^{-\theta}2^{-n\theta}\alpha_n \leq t^{-\theta}K(t, a) \leq 2 \cdot 2^{-\theta n}\alpha_n, \quad 2^n \leq t \leq 2^{n+1}. \tag{4.6}$$

In the case $q = \infty$ this immediately proves the equivalence. For $q < \infty$, as the intervals $[2^n, 2^{n+1})$ are disjoint, we have

$$\|a\|_{\theta,q;K} = (\sum_{n=-\infty}^{\infty} \int_{2^n}^{2^{n+1}} (t^{-\theta}K(t, a))^q dt/t)^{1/q}. \tag{4.7}$$

Plugging the inequalities (4.6) into the expression and using $\int dt/t = \log(t) + C$ completes the proof. □

To finish this section, we will see an explicit example of how the K-functional and and the interpolation space $K_{\theta,q}(\overline{A})$ look like for the couple (L_1, L_∞).

Example 4.1. Consider the compatible couple (L_1, L_∞) over a σ-finite measure space (U, μ). For any function $f \in L_1 + L_\infty$, computing the K-functional directly via its infimum definition is non-trivial. However, as was originally proved by J. Peetre [14], it can be evaluated exactly using the decreasing rearrangement of the function, $f^*(s)$ (recall the definition from Chapter 2). See also Bennett and Sharpley [3]:

$$K(t, f; (L_1(d\mu), L_\infty(d\mu))) = \int_0^t f^*(s)ds := tf^{**}(t),$$

where $f^{**}(t) = (\int_0^t f^*)/t$. It is easy to verify that for $p > 1$ and $q \geq 1$,

$$\|f\|_{(p,q)} = \begin{cases} (\int_0^\infty (t^{1/p}f^{**}(t))^q dt/t)^{1/q} & (1 \leq q < \infty) \\ \sup_{t>0} t^{1/p}f^{**}(t) & (q = \infty) \end{cases} \tag{4.8}$$

is an equivalent norm for the Lorentz spaces (recall equations (2.10) and (2.11)). Indeed, one can check that the expression $\|\cdot\|_{(p,q)}$ is a norm and that the space is complete. Inequality $\|f\|_{L_{pq}} \leq \|f\|_{(p,q)}$ is clear because having f^* non-increasing,

$$f^*(t) \leq 1/t \int_0^t f^*(s)ds = f^{**}(t).$$

The other inequality follows directly from Hardy's Inequality 2.4. Therefore, applying $\phi_{\theta,q}$ with $0 < \theta < 1$ and $1 \leq q \leq \infty$ to the K-functional naturally generates the Lorentz spaces :

$$(L_1(d\mu), L_\infty(d\mu))_{\theta,q} = L_{pq}(d\mu), \quad \text{where } \frac{1}{p} = 1 - \theta.$$

In particular, by property (ii) of Proposition 5.1, one finds that for $1 \leq q \leq r \leq \infty$, then $L_{pq} \hookrightarrow L_{pr}$. In the special diagonal case where $q = p$, the Lorentz space collapses back into a classical Lebesgue space, yielding $(L_1, L_\infty)_{1-1/p,p} = L_p$.

4.1 The J-method and the Equivalence Theorem

Replicating the K-functional one finds a similar interpolation method considering now what is called the J-functional. This new functional will act on the space $\Delta(\overline{A})$ instead of $\Sigma(\overline{A})$ as follows:

Definition 4.8. *Let $\overline{A} = (A_0, A_1)$ be a compatible couple in $\mathcal{N}_1$. The J-functional is defined for each $t > 0$ as the map from $\Delta(\overline{A})$ to $\mathbb{R}$ given by*

$$J(t,a) := J_t(a, \overline{A}) = \max(\|a\|_{A_0}, t\|a\|_{A_1}). \tag{4.9}$$

Similarly to the Proposition 4.2 and of immediate proof, we get the following proposition:

Proposition 4.9. *For a compatible couple $\overline{A} \in \mathcal{N}_1$ and $a \in \Delta(\overline{A})$, $J(t,a)$ is a non-negative, non-decreasing and convex function of t. In particular,*

$$J(t,a) \leq \max(1, t/s) J(s,a) \tag{4.10}$$

and

$$K(t,a) \leq \min(1, t/s) J(s,a). \tag{4.11}$$

Again one observes that for $t = 1$, $J(1, \cdot)$ is exactly the regular norm $\|\cdot\|_{\Delta(\overline{A})}$ and so $J(t, \cdot)$ defines a family of equivalent norms in $\Delta(\overline{A})$.

The J-interpolation method consists now of certain elements in $a \in \Sigma(\overline{A})$ that are attainable from $\Delta(\overline{A})$ by means of a Bochner-integral representations. That is

$$a = \int_0^\infty u(t)dt/t \quad \text{(with convergence in } \Sigma(\overline{A})), \tag{4.12}$$

where $u(t)$ is measurable and gets values on $\Delta(\overline{A})$.

Definition 4.10. *Let $\overline{A} = (A_0, A_1)$ be a compatible couple in $\mathcal{N}_1$. In the cases $0 < \theta < 1$ for $1 \leq q \leq \infty$ and $0 \leq \theta \leq 1$ for $q = 1$, the space $J_{\theta,q}(\overline{A}) = \overline{A}_{\theta,q;J}$, consists of all $a \in \Sigma(\overline{A})$ that are representable in the form* (4.12) *and for which*

$$\|a\|_{\theta,q;J} = \inf \phi_{\theta,q}(J(t, u(t)) \tag{4.13}$$

is finite (where the infimum is taken over all representations in the form (4.12)*).*

It is straightforward to see that the expression (4.13) defines a norm on $J_{\theta,q}(\overline{A})$. The next theorem shows that the (θ, q)-spaces generated by the K and J-methods are the same.

Theorem 4.11 (Equivalence theorem)**.** *Let $\overline{A} = (A_0, A_1)$ be a compatible couple in $\mathcal{N}_1$ and suppose $0 < \theta < 1$, $1 \leq q \leq \infty$. Then $J_{\theta,q}(\overline{A}) = K_{\theta,q}(\overline{A})$ with equivalent norms.*

Proof. We shall show the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|a\|_{\theta,q;K} \leq \|a\|_{\theta,q;J} \leq C_2\|a\|_{\theta,q;K}. \tag{4.14}$$

For the first inequality, let a be an element in $J_{\theta,q}(\overline{A})$. Given a representation $a = \int_0^\infty u(t)dt/t$ with $u(t) \in \Delta(\overline{A})$, we can use the subadditivity of the K-functional and inequality (4.11) to obtain:

$$K(t, a) \leq \int_0^\infty K(t, u(s))ds/s \leq \int_0^\infty \min(1, t/s)J(s, u(s))ds/s.$$

Applying $\phi_{\theta,q}$, making de variable change $v = t/s$ and using equation (4.5), we get

$$\begin{aligned} \|a\|_{\theta,q;K} &\leq \int_0^\infty \min(1, v)\phi_{\theta,q}(J(t/v, u(t/v))dv/v \\ &\leq \phi_{\theta,q}(J(t, u(t)) \int_0^\infty \min(1, v)v^{-\theta}dv/v = C\phi_{\theta,q}(J(t, u(t)). \end{aligned}$$

Taking the infimum over all such representations of a gives the first inequality, with $C_1 = 1/C$.

For the second inequality, we can assume that $0 < ||a||_{\theta,q;K} < \infty$ for some $a \in \overline{A}_{\theta,q;K}$. By the definition of the norm and the use of equation (4.1) one verifies that

$$\lim_{t\to 0} K(t,a) = 0 = \lim_{t\to\infty} t^{-1}K(t,a).$$

For each $j \in \mathbb{Z}$, we can choose a decomposition $a = a_0^j + a_1^j$ with $a_i^j \in A_i$ such that:

$$||a_0^j||_{A_0} + 2^j||a_1^j||_{A_1} \leq 2K(2^j,a). \tag{4.15}$$

Let $v_j = a_0^{j+1} - a_0^j$. Notice that $v_j = a_1^j - a_1^{j+1}$, which implies $v_j \in \Delta(\overline{A})$. Furthermore, the sequence of partial sums forms a telescoping series, and one can verify that:

$$\|a - \sum_{j=-M}^{N-1} v_j\|_{\Sigma(\overline{A})} \leq ||a_1^N||_{A_1} + ||a_0^{-M}||_{A_0} \leq 2[2^{-N}K(2^N,a) + K(2^{-M},a)],$$

which tends to 0 as $M, N \to \infty$. Thus, $a = \sum_{j=-\infty}^{\infty} v_j$ in $\Sigma(\overline{A})$. We can now construct the representation of a in the form (4.12) by considering

$$u(s) = (\ln 2)^{-1} \sum_{j=-\infty}^{\infty} v_j \chi_{(2^j,2^{j+1}]}(s).$$

For $2^j < s \leq 2^{j+1}$, we can bound $J(s,u(s))$ as follows:

$$\begin{aligned} J(s,u(s)) &\leq C\max(||v_j||_{A_0}, 2^{j+1}||v_j||_{A_1}) \\ &\leq C'(||a_0^{j+1} - a_0^j||_{A_0} + 2^j||a_1^j - a_1^{j+1}||_{A_1}), \end{aligned}$$

with C' not depending on j. Now applying the hypothesis (4.15) and property (4.1) we end up with

$$\begin{aligned} J(u(s),s) &\leq 2C'(K(2^j,a) + K(2^{j+1},a)) \\ &\leq 6C'K(2^j,a) \leq 6C'K(s,a). \end{aligned}$$

Applying $\phi_{\theta,q}$ to both sides and taking the infimum yields the desired bound, with $C_2 = 6C'$. □

This theorem automatically proves that when $0 < \theta < 1$, the J-functor also defines an interpolation method of exponent θ for the category $\mathcal{N}$ and, by Corollary 4.5, for the subcategory $\mathcal{B}$. However one cannot say that the J-method is an exact interpolation method of exponent θ, as proved for the K-method. To do so, we would require the constants C_1 and C_2 in the proof to be the same. With a little more work, an analogous result to the

Interpolation Theorem can be given for $J_{\theta,q}(\overline{A})$, stating that it is also an exact interpolation method of exponent θ, for all compatible combinations of q and θ in Definitinon 4.10. See, for instance, Bergh and Löfström [4].

Similarly to Lemma 4.7, there is a discrete representation of the space $J_{\theta,q}(\overline{A})$.

Lemma 4.12. *Given a compatible couple $\overline{A} = (A_0, A_1)$ and $a \in \Sigma(\overline{A})$, then the following are equivalent:*

(i) $a \in J_{\theta,q}(\overline{A})$,

(ii) There exists a sequence $(v_n)_{n=-\infty}^{\infty} \subseteq \Delta(\overline{A})$ such that

$$a = \sum_{n=-\infty}^{\infty} v_n \quad \textit{(with convergence in } \Sigma(\overline{A})\textit{)} \tag{4.16}$$

and such that $(J(2^n, v_n))_{n=-\infty}^{\infty} \in \lambda^{\theta,q}$.

Moreover $\|a\|_{\theta,q;J} \sim \inf_{v_n} \|J(2^n, v_n)\|_{\lambda^{\theta,q}}$

Proof. Assume (i) holds. Let $a \in J_{\theta,q}(\overline{A})$ be given by a representation $a = \int_0^\infty u(s)ds/s$ with $u(s) \in \Delta(\overline{A})$ and $\phi_{\theta,q}(J(s, u(s))) < \infty$. We define the sequence v_n by integrating over dyadic intervals

$$v_n = \int_{2^n}^{2^{n+1}} u(s)ds/s,$$

which clearly satisfies (4.16). Using the subadditivity of J, property (4.10) and the fact that $2^{-n\theta} \leq 2^\theta s^{-\theta}$ for $s \leq 2^{n+1}$,

$$\begin{aligned} 2^{-n\theta} J(2^n, v_n) &\leq \int_{2^n}^{2^{n+1}} 2^{-n\theta} J(2^n, u(s))ds/s \\ &\leq 2^\theta \int_{2^n}^{2^{n+1}} s^{-\theta} J(s, u(s))ds/s. \end{aligned} \tag{4.17}$$

The case $q = \infty$ follows now from taking the supremum inside the integral and computing $\int_{2^n}^{2^{n+1}} ds/s = \ln(2)$.

For $q < \infty$, applying Hölder's inequality to expression (4.17) yields:

$$\begin{aligned} \|(J(2^n, v_n))_n\|_{\lambda^{\theta,q}}^q &= \sum_{n=-\infty}^{\infty} (2^{-\theta n} J(2^n, v_n))^q \\ &\leq \sum_{n=-\infty}^{\infty} 2^{\theta q} (\ln 2)^{q-1} \int_{2^n}^{2^{n+1}} (s^{-\theta} J(s, u(s)))^q ds/s \\ &= C_1 \phi_{\theta,q}^q (J(t, u(t))) < \infty. \end{aligned}$$

Conversely, assume (ii) holds for a sequence $(v_n)_n$. We construct a representation $u(s)$ for a defined exactly as in the Equivalence Theorem:

$$u(s) = (\ln 2)^{-1} \sum_{n=-\infty}^{\infty} v_n \chi_{(2^n, 2^{n+1}]}(s).$$

By (4.10), $J(s, u(s)) \leq 2J(2^n, u(s))$ for $s \in [2^n, 2^{n+1})$. Thus, $s^{-\theta} J(s, u(s)) \leq (2/\ln 2)2^{-n\theta} J(2^n, v_n)$ on this interval. This immediately proves the case $q = \infty$.

For $q < \infty$, we compute the continuous norm:

$$\begin{aligned}
\phi^q_{\theta,q}(J(t, u(t))) &= \int_0^\infty (s^{-\theta} J(s, u(s)))^q ds/s \\
&= \sum_{n=-\infty}^{\infty} \int_{2^n}^{2^{n+1}} (s^{-\theta} J(s, u(s)))^q ds/s \\
&\leq (2^q/(\ln 2)^{q-1}) \sum_{n=-\infty}^{\infty} (2^{-n\theta} J(2^n, v_n))^q = C\|(J(2^n, v_n))_n\|^q_{\lambda^{\theta,q}},
\end{aligned}$$

proving (i). □

In the sequel we shall speak of real interpolation method and denote the interpolation space by $\overline{A}_{\theta,q}$. In the cases $0 < \theta < 1$, $\overline{A}_{\theta,q}$ can either be thought as $J_{\theta,q}(\overline{A})$ or $K_{\theta,q}(\overline{A})$, since the interpolation spaces are the same. If $\theta = 0$ or $\theta = 1$, we shall only consider the case $q = \infty$. Then the space $\overline{A}_{\theta,\infty}$ will be $K_{\theta,\infty}(\overline{A})$. If it is not stated, we will always use the norm $\|\cdot\|_{\theta,q;K}$, for which we know that the interpolation property is of exact exponent θ. From now on we will simply write $\|\cdot\|_{\theta,q} = \|\cdot\|_{\theta,q;K}$.

Although it may not seem of special interest *a priori*, the J-method will help us prove some of the properties of the interpolation spaces $\overline{A}_{\theta,q}$.

5 Structure Theorems for the Spaces $\overline{A}_{\theta,q}$

This chapter is devoted to state and prove the most significant structure properties of the interpolation spaces generated by the real method, including the Reiteration and Wolff theorems. It is divided into three sections, each of them needed for the following one.

5.1 Simple Properties and the Density Theorem

In this section we shall see the main basic properties of the interpolation spaces $\overline{A}_{\theta,q}$. The first proposition is concerning inclusions between them.

Proposition 5.1. *Let $\overline{A} = (A_0, A_1)$ be a compatible couple. We have:*

(i) $(A_0, A_1)_{\theta,q} = (A_1, A_0)_{1-\theta,q}$ *with equal norms.*

(ii) $\overline{A}_{\theta,q} \hookrightarrow \overline{A}_{\theta,r}$ *if* $q \le r$*. In particular for* $0 \le \theta \le 1$*, we have*

$$K(t,a) \le C_{\theta,q} t^{\theta} \|a\|_{\theta,q} \quad \text{for } a \in \Sigma(\overline{A}). \tag{5.1}$$

$$\|a\|_{\theta,r} \le C'_{\theta,q} t^{-\theta} J(t,a) \quad \text{for } a \in \Delta(\overline{A}) \ . \tag{5.2}$$

(iii) $\overline{A}_{\theta_0,q_0} \cap \overline{A}_{\theta_1,q_1} \hookrightarrow \overline{A}_{\theta,q}$ *if* $\theta_0 < \theta < \theta_1$*.*

(iv) $A_1 \hookrightarrow A_0$ *implies* $\overline{A}_{\theta_1,q} \hookrightarrow \overline{A}_{\theta_0,q}$ *if* $\theta_0 < \theta_1$*.*

Proof. For (i), simply observe that

$$K(t, a; (A_0, A_1)) = tK(t^{-1}, a; (A_1, A_0)),$$

and by the change of variables $s = t^{-1}$, we also have $\phi_{\theta,q}(\varphi(t)) = \phi_{1-\theta,q}(t\varphi(t^{-1}))$. Combining these two yields (i).

By Lemma 4.7, the continuous norm $\|a\|_{\theta,q}$ is equivalent to the discrete sequence norm $\|(\alpha_n)\|_{\lambda^{\theta,q}}$, which corresponds to the standard ℓ^q-norm of the weighted sequence $x_n = 2^{-\theta n} K(2^n, a)$.

It is a well-known elementary property of sequence spaces that $\ell^q \hookrightarrow \ell^r$ for any $1 \le q \le r \le \infty$. Specifically, for any sequence $x \in \ell^q$, we have $\|x\|_{\ell^r} \le \|x\|_{\ell^q}$. Hence,

$$\|a\|_{\overline{A}_{\theta,r}} \le C\|(\alpha_n)\|_{\lambda^{\theta,r}} \le C\|(\alpha_n)\|_{\lambda^{\theta,q}} \le C'\|a\|_{\overline{A}_{\theta,q}}.$$

Equation (5.1) follows directly from considering this inclusion in the extreme case $r = \infty$. For (5.2) consider now the case $q = 1$. By the Equivalence Theorem, in the case $0 < \theta < 1$,

$$\|a\|_{\theta,r} \le C\|a\|_{\theta,1} \le C'\|a\|_{\theta,1;J}.$$

As $a \in \Delta(\overline{A})$, for any $t > 0$, a can be represented as

$$a = (\ln 2)^{-1} \int_t^{2t} a \; ds/s = \int_0^\infty (\ln 2)^{-1} a\chi_{[t,2t]}(s) ds/s.$$

Hence,

$$\begin{aligned} \|a\|_{\theta,1;J} &\le (\ln 2)^{-1} \int_t^{2t} s^{-\theta} J(s,a) \; ds/s \\ &\le (\ln 2)^{-1} t^{-\theta} J(2t,a) \int_t^{2t} ds/s = t^{-\theta} J(2t,a), \end{aligned}$$

and using equation (4.10) yields the desired bound. For $\theta = 0, 1$ we simply use the definition and property (4.11) to get

$$\begin{aligned} \|a\|_{\theta,\infty} &= \sup_{s>0} s^{-\theta} K(s,a) \\ &\le J(t,a) \sup_{s>0} (s^{-\theta} \min(1, s/t)) = J(t,a) t^{-\theta}. \end{aligned}$$

This proves (ii).

For (iii), note that

$$\phi_{\theta,q}(K(t,a))^q \le \int_0^1 (s^{-\theta} K(s,a))^q ds/s + \int_1^\infty (s^{-\theta} K(s,a))^q ds/s.$$

By equation (5.1), $K(t,a) \le C_i t^{\theta_i} \|a\|_{\theta_i,q_i}$ for $i = 0, 1$. Using the bound for θ_1 on $(0,1]$ and the bound for θ_0 on $[1,\infty)$, we get:

$$\|a\|_{\theta,q}^q \le C_1^q \|a\|_{\theta_1,q_1}^q \int_0^1 t^{(\theta_1-\theta)q} dt/t + C_0^q \|a\|_{\theta_0,q_0}^q \int_1^\infty t^{(\theta_0-\theta)q} dt/t.$$

Since $\theta_0 < \theta < \theta_1$, both integrals converge to finite constants. Therefore, we get $\|a\|_{\theta,q} \le M \max(\|a\|_{\theta_1,q_1}, \|a\|_{\theta_0,q_0})$, which establishes the embedding. For $q = \infty$, simply adapt the proof by splitting the supremum of $t^{-\theta} K(t,a)$ over the intervals $(0,1]$ and $[1,\infty)$ and applying the same argument.

For (iv), assume $A_1 \hookrightarrow A_0$ with embedding constant k. Then, for any $t \ge k$, we have

$$\|a_0\|_{A_0} + t\|a_1\|_{A_1} \ge \|a_0\|_{A_0} + k\|a_1\|_{A_1} \ge \|a_0 + a_1\|_{A_0} = \|a\|_{A_0}.$$

Taking the trivial decomposition $a_0 = a$ and $a_1 = 0$ gives $K(t,a) \le \|a\|_{A_0}$, so we conclude that $K(t,a) = \|a\|_{A_0}$ for all $t \ge k$. Bounding the integral or supremum from below by considering only $t \ge k$, one finds $\|a\|_{A_0} \le C\|a\|_{\theta_1,q}$.

We now split the $\|a\|_{\theta_0,q}$ norm evaluation at $t = k$:

$$\|a\|_{\theta_0,q} \leq (\int_0^k (t^{-\theta_0} K(t,a))^q dt/t)^{1/q} + \|a\|_{A_0} (\int_k^\infty t^{-\theta_0 q - 1} dt)^{1/q}.$$

The second integral converges because $\theta_0 > 0$. For the first, using $t^{-\theta_0} = t^{\theta_1 - \theta_0} t^{-\theta_1}$ and the hypothesis $\theta_1 > \theta_0$, we obtain:

$$\|a\|_{\theta_0,q} \leq C' \|a\|_{\theta_1,q} + C'' \|a\|_{\theta_1,q}.$$

For $q = \infty$, analogously split the supremum over the same two intervals. This completes the proof. □

Theorem 5.2 (Density Theorem)**.** *Let $\overline{A} = (A_0, A_1)$ be a compatible couple. If $0 < \theta < 1$ and $1 \leq q < \infty$, then the intersection $\Delta(\overline{A}) = A_0 \cap A_1$ is dense in $\overline{A}_{\theta,q}$.*

Proof. In view of the Equivalence Theorem, we will prove $\Delta(\overline{A})$ is dense in $J_{\theta,q}(\overline{A})$. From the constructive proof of the Equivalence Theorem, for any $a \in J_{\theta,q}(\overline{A})$, we can find a sequence $(v_j)_{j=-\infty}^{\infty}$ in $\Delta(\overline{A})$ such that $a = \sum_{j=-\infty}^{\infty} v_j$ in $\Sigma(\overline{A})$. Furthermore, $a = \int_0^\infty u(s) ds/s$, with

$$u(s) = (\ln 2)^{-1} \sum_{j=-\infty}^{\infty} v_j \chi_{(2^j, 2^{j+1}]}(s).$$

We also established that $\phi_{\theta,q}(J(s, u(s))) < \infty$.

To approximate a, we define the truncated partial sums

$$a_N = \sum_{j=-N}^{N-1} v_j = \int_{2^{-N}}^{2^N} u(s) ds/s.$$

Since it is a finite sum of elements in the intersection, $a_N \in \Delta(\overline{A})$ for every integer $N > 0$. Therefore, the error $a - a_N$ can be represented as the sum of the two tails of the integral:

$$a - a_N = \int_0^{2^{-N}} u(s) ds/s + \int_{2^N}^\infty u(s) ds/s.$$

This is equivalent to a valid J-representation for $a - a_N$ using the function $u_N(s) = u(s) \chi_{(0, 2^{-N}] \cup (2^N, \infty)}(s)$. Hence

$$||a - a_N||_{\theta,q;J}^q \leq \int_0^{2^{-N}} (s^{-\theta} J(s, u(s)))^q ds/s + \int_{2^N}^\infty (s^{-\theta} J(s, u(s)))^q ds/s.$$

These last two integrals are exactly the tails of the convergent integral $\phi_{\theta,q}(J(s, u(s)))^q$. Since $q < \infty$, the Dominated Convergence Theorem ensures that the tails of a convergent integral must tend to zero as the limits of integration approach 0 and ∞. This completes the proof. □

5.2 θ-Classes and the Reiteration Theorem

Let $\overline{A} = (A_0, A_1)$ be a compatible couple and consider the two interpolation spaces $\overline{A}_{\theta_0,q_0}$ and $\overline{A}_{\theta_1,q_1}$ constructed by the real method, where $0 < \theta_0 < \theta_1 < 1$ and $1 \leq q_0, q_1 \leq \infty$. Then $(\overline{A}_{\theta_0,q_0}, \overline{A}_{\theta_1,q_1})$ is itself a compatible couple and we can repeatedly form the interpolation space between them

$$(\overline{A}_{\theta_0,q_0}, \overline{A}_{\theta_1,q_1})_{\theta,q} \tag{5.3}$$

by means of the real interpolation method, where $0 < \theta < 1$ and $1 \leq q \leq \infty$. A very natural question to ask is if this space can be constructed from the original couple $\overline{A}$ by choosing an adequate θ' and q'. The Reiteration Theorem is going to provide us an affirmative answer to the question. In particular

$$(\overline{A}_{\theta_0,q_0}, \overline{A}_{\theta_1,q_1})_{\theta,q} = \overline{A}_{\theta',q}, \quad \text{with } \theta' = (1-\theta)\theta_0 + \theta\theta_1.$$

Thus there is stability for repeated use of the real interpolation method.

The Reiteration Theorem will be proved in a slightly broader setting. Instead of asking $\overline{A}_{\theta,q_0}$ and $\overline{A}_{\theta_1,q_1}$ to be interpolation spaces, it will suffice to have them both to be **intermediate spaces of class** θ.

Definition 5.3. *Let $\overline{A} = (A_0, A_1)$ be a compatible couple and X an intermediate space with respect to $\overline{A}$. For $0 \leq \theta \leq 1$, we say that*

(i) X is of class $\mathcal{C}_K(\theta; \overline{A})$ if there exists a constant $C > 0$ such that for any $a \in X$ and $t > 0$,
$$K(t, a; \overline{A}) \leq Ct^{\theta} \|a\|_X. \tag{5.4}$$

(ii) X is of class $\mathcal{C}_J(\theta; \overline{A})$ if there exists a constant $C > 0$ such that for all $a \in \Delta(\overline{A})$ and $t > 0$,
$$\|a\|_X \leq Ct^{-\theta} J(t, a; \overline{A}). \tag{5.5}$$

When an intermediate space X is both of class $\mathcal{C}_K(\theta, \overline{A})$ and $\mathcal{C}_K(\theta, \overline{A})$ we say that it is of class $\mathcal{C}(\theta, \overline{A})$ or simply of θ-class.

It is straightforward to verify from Proposition 5.1 (ii) that the real interpolation space $\overline{A}_{\theta,q}$ is of class $\mathcal{C}(\theta; \overline{A})$ for any valid combination of θ's and q's.

Some books, as for instance Bennet and Sharpley [3], usually define the θ-classes as intermediate spaces between $\overline{A}_{\theta,1}$ and $\overline{A}_{\theta,\infty}$. The following shows the relation between them.

Lemma 5.4. *Let X be an intermediate space for the couple $\overline{A} = (A_0, A_1)$.*

(a) X is of class $\mathcal{C}_K(\theta;\overline{A})$ if and only if

$$\Delta(\overline{A}) \hookrightarrow X \hookrightarrow \overline{A}_{\theta,\infty},$$

(b) A Banach space X is of class $\mathcal{C}_J(\theta;\overline{A})$ if and only if

$$\begin{aligned} &\overline{A}_{\theta,1} \hookrightarrow X \hookrightarrow \Sigma(\overline{A}) && \text{for } 0<\theta<1, \\ &A_\theta \hookrightarrow X \hookrightarrow \Sigma(\overline{A}) && \text{for } \theta = 0,1. \end{aligned}$$

Proof. By definition, $X \hookrightarrow \overline{A}_{\theta,\infty}$ if and only if there exists a constant C independent of a such that

$$\sup_{t>0} t^{-\theta} K(t,a;\overline{A}) \le C\|a\|_X,$$

which is clearly equivalent to being of class $\mathcal{C}_K(\theta;\overline{A})$. This proves (a).

For (b), in the cases $0<\theta<1$, assume X is a Banach space of class $\mathcal{C}_J(\theta;\overline{A})$ and let $a \in \overline{A}_{\theta,1}$. Using the Equivalence Theorem and Lemma 4.12, there exists a representation $a = \sum u_n$ in $\Sigma(\overline{A})$ with $(u_n)_n \subseteq \Delta(\overline{A})$. For any such representation, the completeness of X and the definition of $\mathcal{C}_J(\theta;\overline{A})$ classes yields

$$\|a\|_X \le \sum_{n=-\infty}^{\infty} \|u_n\|_X \le C \sum_{n=-\infty}^{\infty} 2^{-\theta n} J(2^n, u_n;\overline{A}).$$

In view of the equivalence of norms given by Lemma 4.12 we get $\|a\|_X \le C\|a\|_{\theta,1}$. Conversely, if $\overline{A}_{\theta,1} \hookrightarrow X$, we use equation (5.2) with $r=1$ to obtain

$$\|a\|_X \le C\|a\|_{\theta,1} \le C' t^{-\theta} J(t,a;\overline{A}),$$

which proves X is of class $\mathcal{C}_J(\theta;\overline{A})$.

Finally, for $\theta = 0$, X is of class $\mathcal{C}_J(0;\overline{A})$ if and only if $\|a\|_X \le CJ(t,a;\overline{A})$ for any $t>0$. Taking the limit as $t \to 0^+$, the term $t\|a\|_{A_1}$ vanishes from the maximum, yielding $\|a\|_X \le C\|a\|_{A_0}$, hence $A_0 \hookrightarrow X$. Conversely, if $A_0 \hookrightarrow X$, then $\|a\|_X \le C\|a\|_{A_0} \le C\max(\|a\|_{A_0}, t\|a\|_{A_1}) = CJ(t,a;\overline{A})$. The case $\theta = 1$ follows symmetrically by bounding with $Ct^{-1}J(t,a;\overline{A})$ and evaluating the limit as $t \to \infty$. □

It follows that A_0 and A_1 are of class $\mathcal{C}(0;\overline{A})$ and $\mathcal{C}(1;\overline{A})$ respectively. We are now ready to prove the Reiteration Theorem, one of the most important results in interpolation theory.

Theorem 5.5 (Reiteration Theorem)**.** *Let $\overline{A} = (A_0, A_1)$ be a compatible couple. Assume that X_0 and X_1 are complete intermediate spaces of class $\mathcal{C}(\theta_0; \overline{A})$ and $\mathcal{C}(\theta_1; \overline{A})$ respectively, where $0 \leq \theta_i \leq 1$ and $\theta_0 \neq \theta_1$. Let $\overline{X} = (X_0, X_1)$.*

For $0 < \eta < 1$ and $1 \leq q \leq \infty$, consider

$$\theta = (1 - \eta)\theta_0 + \eta\theta_1. \tag{5.6}$$

Then,

$$\overline{X}_{\eta,q} = \overline{A}_{\theta,q} \tag{5.7}$$

with equivalent norms. In particular, $(\overline{A}_{\theta_0,q_0}, \overline{A}_{\theta_1,q_1})_{\eta,q} = \overline{A}_{\theta,q}$.

Proof. Without loss of generality, we may assume $\theta_0 < \theta_1$. Consider any decomposition $a = a_0 + a_1$ with $a_i \in X_i$ of an element $a \in \overline{X}_{\eta,q}$. By the subadditivity of the K-functional and the hypothesis that X_i is of class $\mathcal{C}_K(\theta_i; \overline{A})$, we obtain:

$$\begin{aligned} K(t, a; \overline{A}) &\leq K(t, a_0; \overline{A}) + K(t, a_1; \overline{A}) \\ &\leq C(t^{\theta_0} \|a_0\|_{X_0} + t^{\theta_1} \|a_1\|_{X_1}) \\ &= Ct^{\theta_0}(\|a_0\|_{X_0} + t^{\theta_1-\theta_0} \|a_1\|_{X_1}). \end{aligned}$$

Taking the infimum over all such decompositions $a = a_0 + a_1$ in $\Sigma(\overline{X})$, we establish the bound:

$$K(t, a; \overline{A}) \leq Ct^{\theta_0} K(t^{\theta_1-\theta_0}, a; \overline{X}).$$

We now apply the functional $\phi_{\theta,q}$ to both sides with respect to the variable t. Assuming $q < \infty$:

$$\begin{aligned} \|a\|_{\overline{A}_{\theta,q}} &= (\int_0^\infty (t^{-\theta} K(t, a; \overline{A}))^q dt/t)^{1/q} \\ &\leq C(\int_0^\infty (t^{-(\theta-\theta_0)} K(t^{\theta_1-\theta_0}, a; \overline{X}))^q dt/t)^{1/q} = Ct^{\theta_0} \phi_{\theta,q}(K(t^{\theta_1-\theta_0}, a; \overline{X})). \end{aligned} \tag{5.8}$$

Performing the change of variables $s = t^{\theta_1-\theta_0}$, for which $ds/s = (\theta_1 - \theta_0)dt/t$ and noting that $\eta = (\theta - \theta_0)/(\theta_1 - \theta_0)$ yields:

$$\|a\|_{\overline{X}_{\eta,q}} = \phi_{\eta,q}(K(s, a; \overline{X})) = (\theta_1 - \theta_0)t^{\theta_0} \phi_{\theta,q}(K(t^{\theta_1-\theta_0}, a; \overline{X}). \tag{5.9}$$

And so equation (5.8) becomes $\|a|\|_{\overline{A}_{\theta,q}} \leq C\|a\|_{\overline{X}_{\eta,q}}$. The case $q = \infty$ follows analogously by evaluating the supremum. This establishes the continuous embedding $\overline{X}_{\eta,q} \hookrightarrow \overline{A}_{\theta,q}$.

For the other direction, let $a \in \overline{A}_{\theta,q}$. By the Equivalence Theorem, $a \in J_{\theta,q}(\overline{A})$, which means we can find a representation $a = \int_0^\infty u(s)ds/s$ with $u(s) \in \Delta(\overline{A})$ such that $\phi_{\theta,q}(J(s, u(s); \overline{A})) < \infty$.

Using the subadditivity of the K-functional on the integral representation, we have for any $\tau > 0$:

$$K(\tau, a; \overline{X}) \leq \int_0^\infty K(\tau, u(s); \overline{X})ds/s. \tag{5.10}$$

Since $u(s) \in \Delta(\overline{A}) \subset X_i$, the trivial decompositions $(u(s), 0)$ and $(0, u(s))$ in $\Sigma(\overline{X})$ give $K(\tau, u(s); \overline{X}) \leq \|u(s)\|_{X_0}$ and $K(\tau, u(s); \overline{X}) \leq \tau\|u(s)\|_{X_1}$. Using the hypothesis that X_i is of class $\mathcal{C}_J(\theta_i; \overline{A})$, we can bound these norms by the J-functional:

$$\begin{aligned} \|u(s)\|_{X_0} &\leq Cs^{-\theta_0} J(s, u(s); \overline{A}), \\ \tau\|u(s)\|_{X_1} &\leq C\tau s^{-\theta_1} J(s, u(s); \overline{A}). \end{aligned}$$

Taking the minimum of these two bounds provides:

$$K(\tau, u(s); \overline{X}) \leq C \min(s^{-\theta_0}, \tau s^{-\theta_1}) J(s, u(s); \overline{A}).$$

We substitute this into the integral (5.10) and evaluate the K-functional at $\tau = t^{\theta_1 - \theta_0}$. Multiplying both sides by t^{θ_0}, we get:

$$\begin{aligned} t^{\theta_0} K(t^{\theta_1-\theta_0}, a; \overline{X}) &\leq C \int_0^\infty t^{\theta_0} \min(s^{-\theta_0}, t^{\theta_1-\theta_0} s^{-\theta_1}) J(s, u(s); \overline{A}) ds/s \\ &= C \int_0^\infty \min((t/s)^{\theta_0}, (t/s)^{\theta_1}) J(s, u(s); \overline{A}) ds/s. \end{aligned}$$

We now apply $\phi_{\theta,q}$ to both sides and do the variables change $s = vt$ on the integral. By Minkowski inequality we obtain:

$$\phi_{\theta,q}(t^{\theta_0} K(t^{\theta_1-\theta_0}, a; \overline{X})) \leq C \int_0^\infty \min(v^{-\theta_0}, v^{-\theta_1}) \phi_{\theta,q}(J(vt, u(vt); \overline{A}) dv/v.$$

By performing again the change $t = s/v$, we have $\phi_{\theta,q}(J(vt, u(vt); \overline{A}) = v^\theta \phi_{\theta,q}(J(s, u(s))$ and so,

$$\phi_{\theta,q}(t^{\theta_0} K(t^{\theta_1-\theta_0}, a; \overline{X})) \leq C\phi_{\theta,q}(J(s, u(s))) \int_0^\infty v^\theta \min(v^{-\theta_0}, v^{-\theta_1}) dv/v.$$

As the integral is finite, equivalence (5.9) proves $\|a\|_{\overline{X}_{\eta,q}} \leq C\phi_{\theta,q}(J(s, u(s); \overline{A}))$.This completes the proof view of the Equivalence Theorem and Lemma 4.12. □

This theorem provides a wider setting for which Theorem 4.4 is valid. If $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_1, B_2)$ are compatible couples in $\mathcal{N}_1$, Theorem 4.4 certifies that for any admissible operator $T : \overline{A} \to \overline{B}$ then $T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}$ is bounded. However one may not have T to be an admissible operator and still be bounded for a big number of interpolation spaces. Indeed,

Corollary 5.6. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be compatible couples in $\mathcal{N}_1$. Let $0 \leq \theta_i \leq 1$ and $0 \leq \psi_i \leq 1$ for $i = 0, 1$, with $\theta_0 \neq \theta_1$, $\psi_0 \neq \psi_1$. Suppose that A_{θ_i} and B_{ψ_i} are intermediate spaces of class $\mathcal{C}(\theta_i; \overline{A})$ and $\mathcal{C}(\psi_i; \overline{B})$, respectively. If T satisfies*

$$T : (A_{\theta_0}, A_{\theta_1}) \to (B_{\psi_0}, B_{\psi_1}),$$

(that is T is admissible for the couples $(A_{\theta_0}, A_{\theta_1})$ and (B_{ψ_0}, B_{ψ_1})), then

$$T : \overline{A}_{\theta,q} \to \overline{B}_{\psi,q}$$

for any $1 \leq q \leq \infty$ and (θ, ψ) satisfying

$$(\theta, \psi) = (1 - \eta)(\theta_0, \psi_0) + \eta(\theta_1, \psi_1) \tag{5.11}$$

for some $0 < \eta < 1$.

Proof. By the Reiteration Theorem,

$$\overline{A}_{\theta,q} = (A_{\theta_0}, A_{\theta_1})_{\eta,q} \quad \text{and} \quad \overline{B}_{\psi,q} = (B_{\psi_0}, B_{\psi_1})_{\eta,q}$$

with equivalent norms, whenever relations (5.11) are held. Apply now the Interpolation Theorem 4.4 to the compatible couples $(A_{\theta_0}, A_{\theta_1})$ and (B_{ψ_0}, B_{ψ_1}). □

5.3 The Wolff Theorem

We conclude this chapter concerning the structure of the $\overline{A}_{\theta,q}$ spaces with a variant of the reiteration theorem. Before doing so we need a different characterization $\mathcal{C}_J(\theta; \overline{A})$ classes. That is, X is of class $C_J(\theta; \overline{A})$ for a couple (A_0, A_1) if and only if

$$\|a\|_X \leq C \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta} \quad \text{for all } a \in \Delta(\overline{A}). \tag{5.12}$$

Indeed, if X is of class $C_J(\theta; \overline{A})$, then

$$\|a\|_X \leq C t^{-\theta} J(t, a; \overline{A}) = C \max(t^{-\theta} \|a\|_{A_0}, t^{1-\theta} \|a\|_{A_1})$$

for ant $t > 0$. Choosing $t = \|a\|_{A_0} / \|a\|_{A_1}$ yields (5.12). Conversely, if equation (5.12) is satisfied,

$$\|a\|_X \leq C t^{-\theta} \|a\|_{A_0}^{1-\theta} (t \|a\|_{A_1})^{\theta} \leq C t^{-\theta} J(t, a; \overline{A}).$$

Theorem 5.7 (T. H. Wolff)**.** *Let A_1 and A_4 be complete Banach spaces forming a compatible couple $\overline{A} = (A_1, A_4)$. Let A_2 and A_3 be intermediate spaces with respect to $\overline{A}$. Let $0 < \phi, \psi < 1$ and $1 \leq q, r \leq \infty$, and assume that:*

$$A_2 = (A_1, A_3)_{\phi,q}, \quad \text{and} \quad A_3 = (A_2, A_4)_{\psi,r}. \tag{5.13}$$

Then, up to equivalence of norms,

$$A_2 = (A_1, A_4)_{\rho,q}, \quad \text{and} \quad A_3 = (A_1, A_4)_{\theta,r}, \tag{5.14}$$

where the parameters are given by

$$\rho = \frac{\phi\psi}{1 - \phi + \phi\psi}, \quad \text{and} \quad \theta = \frac{\psi}{1 - \phi + \phi\psi}. \tag{5.15}$$

Proof. We will only establish the first identity in (5.14), since the proof for the second one is entirely analogous. By the Reiteration Theorem, it suffices to show that A_3 is of class $\mathcal{C}(\theta; \overline{A})$. Thus, we need to prove that A_3 belongs to both $\mathcal{C}_K(\theta; \overline{A})$ and $\mathcal{C}_J(\theta; \overline{A})$.

To establish the $\mathcal{C}_J$ condition, we use the characterization given by (5.12). From the hypotheses in (5.13), A_2 is of class $\mathcal{C}_J(\phi; (A_1, A_3))$ and A_3 is of class $\mathcal{C}_J(\psi; (A_2, A_4))$. Thus, for any $a \in A_1 \cap A_4$, we have

$$\|a\|_{A_2} \leq C\|a\|_{A_1}^{1-\phi}\|a\|_{A_3}^{\phi}, \quad \text{and} \quad \|a\|_{A_3} \leq C\|a\|_{A_2}^{1-\psi}\|a\|_{A_4}^{\psi}.$$

Substituting the first inequality into the second yields

$$\|a\|_{A_3} \leq C(\|a\|_{A_1}^{1-\phi}\|a\|_{A_3}^{\phi})^{1-\psi}\|a\|_{A_4}^{\psi} = C\|a\|_{A_1}^{(1-\phi)(1-\psi)}\|a\|_{A_3}^{\phi(1-\psi)}\|a\|_{A_4}^{\psi},$$

which is equivalent by (5.15) to

$$\|a\|_{A_3} \leq C\|a\|_{A_1}^{1-\theta}\|a\|_{A_4}^{\theta}.$$

To establish the $\mathcal{C}_K$ condition, let $a \in A_3$ and fix $t > 0$. We claim that there is a decomposition $a = b_0 + c_0 + a_1$ satisfying:

(i) $b_0 \in A_1$, $c_0 \in A_4$, and $a_1 \in A_3$;

(ii) $\|a_1\|_{A_3} \leq \frac{1}{2}\|a\|_{A_3}$;

(iii) $\|b_0\|_{A_1} + t\|c_0\|_{A_4} \leq Ct^{\theta}\|a\|_{A_3}$.

Since A_3 is of class $\mathcal{C}_K(\psi; (A_2, A_4))$, for any $u > 0$, there exists a decomposition $a = b' + c_0$ such that

$$\|b'\|_{A_2} + u\|c_0\|_{A_4} \leq C_1 u^{\psi}\|a\|_{A_3}. \tag{5.16}$$

Similarly, since $b' \in A_2$ and A_2 is of class $\mathcal{C}_K(\phi; (A_1, A_3))$, for any $v > 0$, there is a decomposition $b' = b_0 + a_1$ such that

$$\|b_0\|_{A_1} + v\|a_1\|_{A_3} \leq C_2 v^{\phi} \|b'\|_{A_2}. \tag{5.17}$$

Combining (5.16) and (5.17), we get $a = b_0 + c_0 + a_1$, along with the bounds:

$$\|b_0\|_{A_1} \leq C_1 C_2 u^{\psi} v^{\phi} \|a\|_{A_3}, \tag{5.18}$$

$$t\|c_0\|_{A_4} \leq C_1 t u^{\psi-1} \|a\|_{A_3}, \tag{5.19}$$

$$\|a_1\|_{A_3} \leq C_1 C_2 u^{\psi} v^{\phi-1} \|a\|_{A_3}. \tag{5.20}$$

Choosing $v = (2C_1C_2u^{\psi})^{\frac{1}{1-\phi}}$ ensures that (5.20) yields $\|a_1\|_{A_3} \leq \frac{1}{2}\|a\|_{A_3}$, satisfying (ii). With this choice of v, the bound in (5.18) becomes proportional to $u^{\frac{\psi}{1-\phi}}$. We then equate the exponents of t by setting $u = t^{\frac{1-\phi}{1-\phi+\phi\psi}}$. A direct substitution confirms that both $tu^{\psi-1}$ and $u^{\frac{\psi}{1-\phi}}$ become equal to t^{θ}. Thus, (5.18) and (5.19) provide $\|b_0\|_{A_1} + t\|c_0\|_{A_4} \leq C_3 t^{\theta} \|a\|_{A_3}$, satisfying (iii).

We can apply this procedure inductively to generate sequences $(b_j)_j \subseteq A_1$, $(c_j)_j \subseteq A_4$, and $(a_j)_j \subseteq A_3$, starting with $a_0 = a$, such that $a_j = b_j + c_j + a_{j+1}$, along with:

$$\|a_{j+1}\|_{A_3} \leq 2^{-j-1} \|a\|_{A_3},$$
$$\|b_j\|_{A_1} + t\|c_j\|_{A_4} \leq C 2^{-j} t^{\theta} \|a\|_{A_3}.$$

Summing the decompositions yields $a = \sum_{j=0}^{n} b_j + \sum_{j=0}^{n} c_j + a_{n+1}$. But $a_n \to 0$ in A_3 and hence in $\Sigma(\overline{A})$. Furthermore, since A_1 and A_4 are complete and the series $b = \sum_{j=0}^{\infty} b_j$ and $c = \sum_{j=0}^{\infty} c_j$ satisfy

$$\sum_j \|b_j\|_{A_1} + t \sum_j \|c_j\|_{A_4} \leq 2Ct^{\theta} \|a\|_{A_3},$$

they both converge, say, to b and c in A_1 and A_4, respectively. Therefore, $a = b + c$ is a valid decomposition in $\Sigma(\overline{A})$, and

$$K(t, a; \overline{A}) \leq \|b\|_{A_1} + t\|c\|_{A_4} \leq \sum_{j=0}^{\infty} (\|b_j\|_{A_1} + t\|c_j\|_{A_4}) \leq 2Ct^{\theta} \|a\|_{A_3}.$$

This establishes that A_3 is of class $\mathcal{C}_K(\theta; \overline{A})$, and the proof is complete. □

6 Compactness of the real interpolation method

Recall that a bounded operator $T : A \to B$ between normed spaces A and B is said to be **compact** if the image of any bounded set M in A is relatively compact in B, that is $\overline{T(M)}$ is compact in B. Equivalently, T is compact if the image of the closed unit ball $B_A \subseteq A$ is relatively compact in B. It follows directly from de definition that any compact operator is bounded.

One of the classical problems in interpolation theory is whether compactness of a linear operator is preserved under interpolation. Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be compatible couples in $\mathcal{N}_1$, and let T be an admissible operator from $\overline{A}$ to $\overline{B}$. Remember that then, the restrictions of T to both endpoints are bounded. If one of the endpoint operators is also compact or if both of them are compact, it is natural to ask whether the interpolated operator

$$T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$$

is compact.

This question has a long history. In the foundational work of Lions and Peetre [11], compactness was shown to behave well under interpolation in several important situations. In particular, their results cover certain degenerate cases, for instance when one of the couples is fixed.

Theorem 6.1 (Lions–Peetre Compactness Theorem)**.** *Let $\overline{A} = (A_0, A_1)$ be a compatible couple in the category of Banach compatible couples $\mathcal{B}_1$ and B be a Banach space. For $0 < \theta < 1$,*

1. *If $T : \overline{A} \to (B, B)$ is an admissible operator such that*

$$T : A_0 \to B \quad \textit{is compact,}$$

then $T : X \to B$ is compact for any intermediate space X of class $\mathcal{C}_K(\theta; \overline{A})$.

2. *If $T : (B, B) \to \overline{A}$ is an admissible operator such that*

$$T : B \to A_0 \quad \textit{is compact,}$$

then $T : B \to X$ is compact for any intermediate space X of class $\mathcal{C}_J(\theta; \overline{A})$.

The first general result for the real method, without assuming that one of the couples degenerates, is due to Hayakawa [10]. His theorem shows that compactness at both endpoints is preserved by the real interpolation method.

Theorem 6.2 (Hayakawa)**.** *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be compatible couples in $\mathcal{B}_1$. Let $0 < \theta < 1$ and $1 \leq q < \infty$. Assume that T is an admissible operator from $\overline{A}$ to $\overline{B}$ and that*

$$T : A_0 \to B_0, \qquad T : A_1 \to B_1$$

are compact. Then

$$T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$$

is compact.

Hayakawa's proof is based on a discrete description of the real interpolation spaces and on a careful truncation argument. The interpolated space is represented through sequences, the operator is applied coordinatewise, and compactness is obtained by approximating the operator by finite-coordinate truncations.

A stronger one-sided theorem was later proved by Cwikel [6]. It shows that, for the real K-method, compactness at only one endpoint already implies compactness at every intermediate space, provided the operator is bounded at the other endpoint.

Theorem 6.3 (Cwikel)**.** *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be compatible couples in $\mathcal{B}_1$. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Assume that $T : \overline{A} \to \overline{B}$ is an admissible operator. If*

$$T : A_0 \to B_0$$

is compact, then

$$T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$$

is compact.

Cwikel's argument uses Hayakawa's theorem together with the Reiteration Theorem. In outline, if one also knew that $T : A_1 \to B_1$ were compact, the conclusion would follow directly from Hayakawa's theorem. Thus the main point is to produce an auxiliary compactness result at an intermediate level. Cwikel proves that, for each $0 < \alpha < 1$,

$$T : (A_0, A_1)_{\alpha,1} \longrightarrow (B_0, B_1)_{\alpha,\infty}$$

is compact. Once this is known, Hayakawa's theorem is applied to the couples

$$\big(A_0, (A_0, A_1)_{\alpha,1}\big), \qquad \big(B_0, (B_0, B_1)_{\alpha,\infty}\big),$$

and the conclusion follows from the Reiteration Theorem; see Cwikel [6].

This line of proof is quite long and technically complex. Since our purpose here is not to reproduce the full machinery of the compactness theory for the K-method, we shall follow a different route, based on measures of non-compactness. This approach gives a concise and efficient proof of the compactness result. The key point is that compactness of an operator $T : A \to B$ can be characterized by the vanishing of the ball measure of non-compactness of the set $T(B_A)$, where B_A denotes the unit closed ball in A. Moreover, the ball measure of non-compactness satisfies an interpolation inequality for the real method.

More precisely, Cobos, Fernández-Martínez and Martínez [5] proved that if $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ are compatible couples in $\mathcal{B}_1$ and $T : \overline{A} \to \overline{B}$ and admissible operator, then

$$\beta\big(T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}\big) \leq C_{\theta,q}\, \beta(T : A_0 \to B_0)^{1-\theta} \beta(T : A_1 \to B_1)^{\theta}. \tag{6.1}$$

Here β denotes the ball measure of non-compactness. Therefore, if one of the endpoint mappings is compact, the corresponding factor on the right-hand side is zero, and the interpolated operator is compact. This is the approach we shall use below.

Definition 6.4. *Let A be a Banach space and let $M \subseteq A$ be a bounded set. The ball measure of non-compactness of M in A is defined by*

$$\beta_A(M) = \inf \left\{ r > 0 : \exists x_1, \ldots, x_N \in A \text{ such that } M \subset \cup_{j=1}^{N} (x_j + rB_A) \right\},$$

where B_A denotes the closed unit ball of A.

Equivalently, $\beta_A(M)$ is the infimum of all radius $r > 0$ such that M can be covered by finitely many closed balls of radius r in A.

If $T : A \to B$ is a bounded linear operator between Banach spaces, we write

$$\beta(T : A \to B) = \beta_B(T(B_A)).$$

Thus $\beta(T) = \beta(T : A \to B)$ measures how far the image of the unit ball of A under T is from being relatively compact in B.

Lemma 6.5. *Let A be a Banach space and let $M \subseteq A$ be bounded. Then*

$$\beta_A(M) = 0$$

if and only if M is relatively compact in A.

Before proving so, let us recall a basic result in metric spaces.

Proposition 6.6. *Let A be a metric space. Then $M \subseteq A$ is relatively compact if and only if M is totally bounded and $\overline{M}$ is complete.*

Proof. Assume first that M is relatively compact. Then $\overline{M}$ is compact in A. Hence, for any $\epsilon > 0$, the open cover

$$\overline{M} \subseteq \bigcup_{x \in \overline{M}} B_A(x, \epsilon)$$

has a finite subcover, say $\{B_A(x_i, \epsilon)\}_{i=1}^N$. But then,

$$M \subseteq \overline{M} \subseteq \bigcup_{i=1}^{N} B_A(x_i, \epsilon).$$

Since this can be done for an arbitrary ϵ we get that M is totally bounded. For the completeness assume that $(y_n)_n$ is a Cauchy sequence in $\overline{M}$. As $\overline{M}$ is compact and A is a metric space, $\overline{M}$ is sequentially compact. Hence $(y_n)_n$ has a subsequence converging to some element in $\overline{M}$ and as the sequence is Cauchy, it converges to the same limit.

Conversely, suppose that M is totally bounded and $\overline{M}$ is complete. Let $(y_n)_n$ be a sequence in $\overline{M}$. Since $\overline{M}$ is totally bounded, in can be covered by finitely many balls of radius 1. One of these balls contains an infinite number of terms in the sequence. Choose a subsequence $(y_n^1)_n$ contained in that ball. Inductively, for every $k \in \mathbb{N}$, $\overline{M}$ can be covered in finitely many balls of radius $1/k$. Hence, there must be one of these balls containing an infinite number of elements in the sequence $(y_n^{k-1})_n$. Set $(y_n^k)_n$ a subsequence of $(y_n^{k-1})_n$ contained in that ball.

Define now the diagonal subsequence

$$z_k = y_k^k.$$

We claim that $(z_k)_k$ is a Cauchy sequence. Indeed, for $\varepsilon > 0$ choose k_0 such that $2/k_0 < \varepsilon$. For all $p, q \geq k_0$, the elements z_p and z_q both belong to the same ball of radius $1/k_0$. Hence

$$\|z_p - z_q\|_A \leq 2/k_0 < \varepsilon,$$

which proves the claim. Finally, since $\overline{M}$ is complete, $(z_k)_k$ converges to some element $z \in \overline{M}$. Hence $\overline{M}$ is sequentially compact and therefore compact. $\square$

Notice that if A is a Banach space, then any closed subset of A is also complete. As the closure of a set is always closed, if M is a subset of A, then M is relatively compact if and only if M is totally bounded.

(Proof on Lemma 6.5). Assume first that $\beta_A(M) = 0$. Then, for every $\varepsilon > 0$, there exist $x_1, \ldots, x_N \in A$ such that

$$M \subset \cup_{j=1}^{N}(x_j + \varepsilon B_A) = \cup_{j=1}^{N} B_A(x_j, \varepsilon).$$

Therefore M is totally bounded and by the previous proposition it is relatively compact.

Conversely, assume that M is relatively compact. Then $\overline{M}$ is compact in A. Let $\varepsilon > 0$. Since $\overline{M}$ is compact, the open cover

$$\overline{M} \subset \bigcup_{x \in \overline{M}} B_A(x, \varepsilon)$$

has a finite subcover, say $\{B(x_i, \varepsilon)\}_{i=1}^{N}$. In particular,

$$M \subseteq \overline{M} \subseteq \bigcup_{i=1}^{N} B_A(x_i, \varepsilon) = \bigcup_{i=1}^{N} (x_i + \varepsilon B_A).$$

By the definition of $\beta_A(M)$, this implies

$$\beta_A(M) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\beta_A(M) = 0$.

This proves the equivalence. □

Theorem 6.7 (Cobos–Fernández-Martínez–Martínez)**.** *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be compatible couples in $\mathcal{B}_1$, let $0 < \theta < 1$, and let $1 \leq q \leq \infty$ and suppose that $T : \overline{A} \to \overline{B}$ is an admissible operator.*

Then there exists a constant $C_{\theta,q} > 0$, depending only on θ, q, and on the normalization of the interpolation norms, such that

$$\beta\big(T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}\big) \leq C_{\theta,q}\, \beta(T : A_0 \to B_0)^{1-\theta} \beta(T : A_1 \to B_1)^{\theta}.$$

This is Theorem 1.2 of Cobos, Fernández-Martínez and Martínez, *Interpolation of the measure of non-compactness by the real method*, see [5] for the original statement and proof. In their notation the constant is explicitly estimated. The precise value of the constant is irrelevant for the compactness consequence below; only the fact that it is finite is used.

Theorem 6.8. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be compatible couples in $\mathcal{B}_1$. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Let $T : \overline{A} \to \overline{B}$ be an admissible operator such that*

$$T : A_0 \to B_0$$

is compact. Then

$$T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$$

is compact.

Proof. We apply the interpolation estimate for the ball measure of non-compactness given by the previous theorem,

$$\beta\big(T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}\big) \le C_{\theta,q}\, \beta(T : A_0 \to B_0)^{1-\theta} \beta(T : A_1 \to B_1)^{\theta}.$$

Since $T : A_0 \to B_0$ is compact, the set $T(B_{A_0})$ is relatively compact in B_0. Therefore

$$\beta(T : A_0 \to B_0) = \beta_{B_0}(T(B_{A_0})) = 0.$$

Moreover, since $T : A_1 \to B_1$ is bounded, $\beta(T : A_1 \to B_1) < \infty$. Consequently,

$$\beta_{\overline{B}_{\theta,q}}(T(B_{\overline{A}_{\theta,q}})) = \beta\big(T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}\big) = 0,$$

And by Lemma 6.5, the set $T(B_{\overline{A}_{\theta,q}})$ is relatively compact in $\overline{B}_{\theta,q}$, which by definition means that $T : \overline{A}_{\theta,q} \to \overline{B}_{\theta,q}$ is compact. $\square$

7 Extension to Quasi-Normed Abelian Groups

We couldn't end this project without mentioning its extension to quasi-normed Abelian groups. Notice that, while the whole interpolation theory is developed from the idea behind Marcinkiewicz's Interpolation Theorem, it still does not apply to it. The theory up to this point was constructed for normed spaces and the weak spaces L_p^* are not normed. However, the development of the method did not depend heavily on all the properties of a norm. Indeed, homogeneity $\|\lambda a\| = |\lambda|\|a\|$ is not used.

In this Chapter we shall give the basic results concerning the extension of the real interpolation method to the more general quasi-normed Abelian groups. These are cosntructed to be the most basic spaces for which the real interpolation method works. We essentially follow Bergh and Löfström[4]. Let us first define the notion of quasi-normed Abelian group.

Definition 7.1. *Given an Abelian group $(A, +)$ with neutral element 0, we say that a real-valued function $\|\cdot\|_A : A \to \mathbb{R}$ is a quasi-norm if:*

(i) $\|a\|_A \geq 0$ *and* $\|a\|_A = 0$ *if and only if* $a = 0$.

(ii) $\|-a\|_A = \|a\|_A$,

(iii) $\|a + b\|_A \leq c(\|a\|_A + \|b\|_A)$ *for some* $c \geq 1$.

We call (iii) the c-triangle inequality and $\|\cdot\|_A$ a c-norm.

When A is an Abelian group with a c-norm, we call $(A, \|\cdot\|_A)$, or simply A, a c-normed Abelian group or c-normed space. It is clear that, forgetting the scalars, any normed vector space is a 1-normed space. Also the weak-L_p spaces are 2-normed spaces, with the norm given by (2.5). These are two examples of quasi-normed Abelian groups whose quasi-norm is also absolutely homogeneous, that is $\|\lambda f\| = |\lambda|\|f\|$. This type of quasi-normed Abelian groups that are vector spaces and (ii) is replaced by absolute homogeneity are called quasi-normed vector spaces.

As was done with normed spaces, a natural topology over A is the one generated by the open balls $B_r(a) = \{b \in A : \|b - a\|_A < r\}$. When $c = 1$, $d(a, b) = \|b - a\|_A$ defines a distance and so A is a metrizable topological space. One can prove that for $c > 1$, any quasi-normed Abelian group A is also metrizable (see Bergh and Löfström [4]). Indeed, one finds a 1-norm $\|\cdot\|_A^*$ satisfying

$$\|a\|_A^* \leq \|a\|_A^\rho \leq 2\|a\|_A^*, \tag{7.1}$$

where ρ solves $(2c)^\rho = 2$. Hence, every quasi-normed Abelian group is metrizable and we can use notions of Cauchy sequences and completeness. Analogous now to Lemma 3.1 we get:

Lemma 7.2. *Let A be a c-normed Abelian group and ρ solve $(2c)^\rho = 2$. Then A is complete if and only if for any sequence $(a_n)_n$ such that $\sum_{j=0}^{\infty} \|a_j\|_A^\rho$ is finite, then $\sum_j a_j$ converges to some element $a \in A$. Moreover*

$$\|a\|_A \leq C(\sum_{j=0}^{\infty} \|a_j\|_A^\rho)^{1/\rho}. \tag{7.2}$$

For two given quasi-normed Abelian groups A and B, a mapping $T : A \to B$ is called a **homomorphism** if $T(-a) = T(a)$, $T(a+b) = T(a) + T(b)$. We then say that T is **bounded** if

$$\|T\|_{A,B} = \sup_{a\neq 0} \frac{\|T(a)\|_B}{\|a\|_A}$$

is finite. As with normed spaces, bounded implies continuous. Furthermore, bounded homomorphisms constitute a quasi-normed Abelian group themselves. Quasi-normed Abelian groups, together with homomorphisms become a category, which we will denote by $\mathcal{QA}$. Quasi-normed vector spaces, together with morphisms given by homomorphisms of c-spaces that also satisfy $T(\lambda a) = \lambda T(a)$, become a subcategory of $\mathcal{QA}$, which we will denote by $\mathcal{QN}$.

The notion of compatible spaces remains equal, now having them both continuously embedded in a Hausdorff **topological quasi-normed Abelian group**, that is a c-normed space with a topology for which it is Hausdorff and the sum is continuous. Lemma 3.5 is also true in the new category $\mathcal{QA}$: for two compatible c-normed spaces A_0 and A_1 with c_0 and c_1-norms respectively, $\Delta(\overline{A}) = A_0 \cap A_1$ and $\Sigma(\overline{A}) = A_0 + A_1$ are c-normed vector spaces, when equipped with the norms $\|\cdot\|_{\Delta(\overline{A})}$ and $\|\cdot\|_{\Sigma(\overline{A})}$. These two are c-norms with $c = \max\{c_0, c_1\}$. Moreover, a use of Lemma 7.2 proves that if A_0 and A_1 are complete, so are $\Delta(\overline{A})$ and $\Sigma(\overline{A})$.

If we now consider subcategories $\mathcal{C}$ of $\mathcal{QA}$ for which $\Delta(\cdot)$ and $\Sigma(\cdot)$ are closed, we can again form the category of compatible couples $\mathcal{C}_1$ along with admissible homomorphisms defined in the same way, but with the new notion of boundedness. As a consequence, the definitions of intermediate space, interpolation space and interpolation functor carry over without a change, simply considering now spaces in the subcategory $\mathcal{C}$. Consider for now the category $\mathcal{QA}$ and compatible couples $\mathcal{QA}_1$ on it. The category of complete quasi-normed Abelian groups, or simply quasi-Banach spaces, noted as $\mathcal{QB}$ will play the analogous role to $\mathcal{B}$.

The definitons of the functionals $K(t, a; \overline{A})$ and $J(t, a; \overline{A})$ still remain the same. Propositions 4.2 and 4.9 still hold. The interpolation space $K_{\theta,q}(\overline{A})$ is defined exactly in the same way. Lemma 4.7 also holds in this case. For

$J_{\theta,q}(\overline{A})$, in order to avoid the integration in quasi-normed Abelian groups we will use the discrete definition of the space given by Lemma 4.12.

To prove now that the K-method for c-spaces is also an interpolation functor, we should prove the analogous to the Interpolation Theorem 4.4. The first step was to prove that $K_{\theta,q}(\overline{A})$ gives an intermediate c-space. To do so we must prove that $\|\cdot\|_{\theta,q;K}$ is a c-norm. We rely on the fact that, as expected, $K(t,a;\overline{A})$ is a c-norm on $\Sigma(\overline{A})$. The points (i) and (ii) are immediate and similar to the case for norms. For (iii) we see:

Lemma 7.3. *If $\overline{A}=(A_0,A_1)$ is a compatible couple in $\mathcal{QA}_1$, with c_0 and c_1-norms, respectively, then*

$$K(t,a+b;\overline{A}) \leq c_0(K(c_1t/c_0,a;\overline{A})+K(c_1t/c_0,b;\overline{A})). \tag{7.3}$$

In particular

$$K(t,a+b;\overline{A}) \leq c_0 \max(1,c_1t/c_0)\,(K(t,a;\overline{A})+K(t,b;\overline{A})) \tag{7.4}$$

Proof. If $a=a_0+a_1$ and $b=b_0+b_1$ with $a_i,b_i\in A_i$, we have

$$\begin{aligned} K(t,a+b;\overline{A}) &\leq \|a_0+b_0\|_{A_0}+t\|a_1+b_1\|_{A_1} \\ &\leq c_0(\|a_0\|_{A_0}+\|b\|_{A_0})+tc_1(\|a_1\|_{A_1}+\|b_1\|_{A_1}) \\ &= c_0((\|a_0\|_{A_0}+tc_1/c_0\|a_1\|_{A_1})+(\|b_0\|_{A_0}+tc_1/c_0\|b_1\|_{A_1})). \end{aligned}$$

By taking the infimum over all such representations of a and b yields (7.3). Equation (7.4) now comes directly from applying (4.1) to the expression. □

From (7.3) and equation (4.5) (set as usual $K(t,a)=K(t,a;\overline{A})$),

$$\begin{aligned} \phi_{\theta,q}(K(t,a+b)) &\leq c_0(\phi_{\theta,q}(K(c_1t/c_0,a)+\phi_{\theta,q}(K(c_1t/c_0,b)) \\ &= c_0^{1-\theta}c_1^{\theta}(\phi_{\theta,q}(K(t,a))+\phi_{\theta,q}(K(t,b))). \end{aligned}$$

Hence, for $1\leq q<\infty$ with $0<\theta<1$ and also $\theta=0,1$ in the case $q=\infty$, $K_{\theta,q}(\overline{A})$ is a c-normed space, with $c=c_0^{1-\theta}c_1^{\theta}$. [4]

Using the same arguments, it is verified that $J_{\theta,q}(\overline{A})$ is also a c-normed space, with the same constant c. The rest of the proof of the Interpolation Theorem 4.4 is valid in the category $\mathcal{QA}$, as the triangle inequality is not used. Although the basic Interpolation Theorem translates smoothly, establishing the core equivalence and reiteration results in the category $\mathcal{QA}$ requires a

[4] In the category $\mathcal{QN}$ one can actually extend the values of q to all positive real numbers, and all results remain the same; one finds that for $q<1$, $\|\cdot\|_{\theta,q;K}$ is a $c$norm with $c=c_0^{1-\theta}c_1^{\theta}2^{(1-q)/q}$. See Bergh and Löfstrom [4].

significant change of their proofs. When transitioning from normed spaces to quasi-normed spaces, the fundamental challenge lies in the loss of the strict triangle inequality. Because the quasi-norm introduces a constant $c > 1$, norms of infinite discrete sums cause the constant c to multiply geometrically, leading to exploding error bounds.

To resolve this, the proofs for deeper interpolation results –such as the Equivalence Theorem between the J and K-methods and the Reiteration Theorem– must be routed through Lemma 7.2. This lemma introduces a parameter $\rho \in (0, 1]$ such that $(2c)^\rho = 2$, which guarantees that the sum can be bounded without the geometric explosion of the constant c.

With Lemma 7.2 in hand, we can bypass the failure of the strict triangle inequality. Instead of applying classical bounds directly to infinite discrete sums, one evaluates the terms using the ρ-power, allowing the modified sums to be treated with standard subadditive techniques. This structure naturally leads to the introduction of a shifted parameter $p = q/\rho$. As long as $p > 1$, classical tools like Minkowski's or Hölder's inequalities can safely be applied to these adjusted series. While this approach unifies the proofs, it introduces significant algebraic complexity, requiring careful management of fractional exponents and iterative geometric series across the different spaces. For a proof of the extension of these theorems, see, for instance, Bergh and Löfström [4].

Example 7.1. Let (U, μ) be a σ-finite measure space. Both the Lebesgue and Lorentz spaces can be extended using the same definitions for all values of $p > 0$ and $q > 0$. The Lebesgue spaces $L_p(U, d\mu)$ for $0 < p < 1$ are not normable, taking the form of complete quasi-Banach spaces with c-norm constant $2^{1/p-1}$.

We saw in Example 4.1 that for values of $p > 1$ and $q \geq 1$, the Lorentz spaces $L_{pq}(U, d\mu)$ were normable. However, in the remaining cases (specifically when $0 < p < 1$ with arbitrary q, $0 < q < 1$ with arbitrary p, or $p = 1$ with $q > 1$), the Lorentz spaces are not normable and remain strict quasi-Banach spaces.

Hence, we shall try to interpolate both Lebesgue and Lorentz spaces for arbitrary values of p and q. The following sequence of results leads to the proof of the general Marcinkiewicz Interpolation Theorem. Equivalently to the K-functional for the couple (L_1, L_∞), one gets (see Bennet and Sharpley [3]):

Theorem 7.4. *Let $f \in L_p(U, d\mu) + L_\infty(U, d\mu)$, with $0 < p < \infty$. Then,*

$$K(t, f; (L_p(d\mu), L_\infty(d\mu)) \sim \left(\int_0^{t^p} (f^*(s))^p ds \right)^{1/p}. \tag{7.5}$$

If we now consider $p < q \leq \infty$ and $0 < \theta < 1$, a careful manipulation of integrals or supremums, changes of variables and the Minkowski inequality for $q/p > 1$ proves

$$(L_p(d\mu), L_\infty(d\mu))_{\theta,q} = L_{p'q}(d\mu) \quad (1/p' = (1-\theta)/p). \tag{7.6}$$

By using the Reiteration Theorem 5.5 we get:

Theorem 7.5. *For $0 < p_0 < p_1 \leq \infty$, arbitrary $0 < q \leq \infty$, and $0 < \theta < 1$,*

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{pq} \quad (1/p = (1-\theta)/p_0 + \theta/p_1), \tag{7.7}$$

with equivalent norms. In particular, $(L_{p_0}, L_{p_1})_{\theta,p} = L_p$.

Proof. Let $0 < r < p_0$ and $0 < \theta_0, \theta_1 < 1$ such that $1/p_0 = (1-\theta_0)/r$ and $1/p_1 = (1-\theta_1)/r$. Then by (7.6),

$$(L_{p_0}, L_{p_1})_{\theta,q} = ((L_r, L_\infty)_{\theta_0,p_0}, (L_r, L_\infty)_{\theta_1,p_1})_{\theta,q}.$$

By the Reiteration Theorem, setting $\eta = (1-\theta)\theta_0 + \theta\theta_1$, this simplifies to

$$(L_{p_0}, L_{p_1})_{\theta,q} = (L_r, L_\infty)_{\eta,q} \quad \text{(equivalent norms)}.$$

Finally, applying (7.6) again with exponent $1/p = (1-\eta)/r = (1-\theta)/p_0 + \theta/p_1$, we obtain

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{pq} \quad \text{(equivalent norms)}.$$

$\square$

This theorem provides a first version of the Riesz-Thorin Interpolation Theorem 2.5. Theorem 7.5 states that given an admissible operator $T : (L_{p_0}, L_{p_1}) \to (L_{q_0}, L_{q_1})$, where $0 < p_0 < p_1 \leq \infty$ and $0 < q_0 < q_1 \leq \infty$ with respective norms on the extremes M_0 and M_1, then by the functorial property of real interpolation method:

$$T : L_{pr} \to L_{qr}$$

for any arbitrary $0 < r \leq \infty$, where p and q satisfy equation (7.7).

If we assume that for some $0 < \theta < 1$ we have $p \leq q$, we are perfectly free to pick $r = p$. This yields $T : L_{pp} \to L_{qp}$. Since $p \leq q$, property (ii) of Proposition 5.1 guarantees the continuous embedding $L_{qp} \hookrightarrow L_{qq} = L_q$. Composing the operator with this embedding yields:

$$T : L_p \to L_q,$$

with norm $M \leq C_\theta M_0^{1-\theta} M_1^\theta$. A further investigation on this setup eventually proves that $C_\theta = 1$ (see, for instance, Bergh and Löfström [4]). Notice that this provides an extension of the Riesz-Thorin Theorem to the quasi-Banach cases where $p_i, q_i < 1$, although with the limitation that $p \leq q$.

To end this chapter we will finally give the proof of the general Marcinkiewicz Interpolation Theorem, which itself has provided the idea behind the whole real method. Using again the Reiteration Theorem and Theorem 7.5 we first get the following identification:

Lemma 7.6. *Let p_0, p_1, q_0, q_1 and q be positive numbers, possibly infinite, with $p_0 \neq p_1$. For $0 < \eta < 1$ set $1/p = (1-\eta)/p_0 + \eta/p_1$. Then*

$$(L_{p_0 q_0}, L_{p_1 q_1})_{\eta,q} = L_{pq}.$$

Theorem 7.7 (The general Marcinkiewicz Interpolation Theorem)**.** *Suppose that*

$$\begin{aligned} T : L_{p_0 r_0}(U, d\mu) \to L_{q_0 s_0}(V, d\nu) \quad &\textit{with norm } M_0, \\ T : L_{p_1 r_1}(U, d\mu) \to L_{q_1 s_1}(V, d\nu) \quad &\textit{with norm } M_1, \end{aligned}$$

where $p_0 \neq p_1$ and $q_0 \neq q_1$. By setting

$$1/p = (1-\theta)/p_0 + \theta/p_1, \quad 1/q = (1-\theta)/q_0 + \theta/q_1$$

we get

$$T : L_{pr}(U, d\mu) \to L_{qr}(V, d\nu)$$

with norm $M \leq C_\theta M_0^{1-\theta} M_1^\theta$ for all $0 < r \leq \infty$. In particular we have $T : L_p(U, d\mu) \to L_q(V, d\nu)$, provided $p \leq q$.

Proof. It follows straight from using the real interpolation method and Lemma 7.6. For the particular case, set $r = p$ and use the inclusion $L_{q,p} \hookrightarrow L_{q,q} = L_q$. □

Notice that by picking $r_0 = p_0$, $p_1 = r_1$ and $s_0 = s_1 = \infty$ we recover the Marcinkiewicz Interpolation Theorem stated in Chapter 2.

8 Final remarks and examples

In the preceding chapters we have developed the real interpolation method through the K-functional, established its basic properties, discussed the preservation of compactness under interpolation and extended the main results to the category of quasi-normed Abelian groups. We now close the work with some comments and examples that illustrate how the abstract theory enters concrete problems in analysis.

The main point is that, once the abstract interpolation theorems are available, the essential difficulty in applications is often the identification of the interpolation spaces. In other words, one has to compute, estimate, or otherwise characterize the corresponding K-functional. This problem is a central theme in interpolation theory and is closely related to several areas of analysis, including approximation theory, harmonic analysis, the theory of function spaces and partial differential equations.

This point of view is developed systematically in the monographs of Peetre [14], Bergh and Löfström [4], and Bennett and Sharpley [3]. In particular, the last chapters of the latter one contain many examples showing how abstract interpolation results become effective only after the relevant K-functionals have been identified.

The spirit of this chapter is similar. We shall briefly discuss some classical examples. For simplicity, we will always consider the euclidian space $\mathbb{R}^n$ with the Lebesgue measure. The first one concerns the couple

$$\big(L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)\big),$$

where the K-functional is described in terms of moduli of smoothness and the interpolated spaces are Besov spaces. We then show how the classical Sobolev embedding theorem, combined with interpolation, yields a Besov–Lorentz embedding. The third one concerns Sobolev–Sobolev couples, such as

$$\big(W^{1,1}(\mathbb{R}^n), W^{1,\infty}(\mathbb{R}^n)\big),$$

where the K-functional involves both the function and its weak gradient. Finally, we mention a classical endpoint phenomenon in harmonic analysis involving $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1_{\mathrm{at}}(\mathbb{R}^n)$, which arise as natural substitutes for the failure of $L^\infty(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ to serve as stable endpoints for Calderón–Zygmund operators.

8.1 The couple $(L_p, W^{m,p})$ and Besov spaces

A basic and important example of the real method is obtained from the couple

$$\big(L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)\big).$$

The identification of the corresponding interpolation spaces is based on the computation of the K-functional. This is a central topic in interpolation theory; we refer to Bennett and Sharpley [3] for a general treatment of the K-method and its applications to function spaces. In the particular case of Besov spaces, we use the standard description in terms of moduli of smoothness; see also DeVore and Sharpley [8].

Let $1 \le p \le \infty$ and let $m \in \mathbb{N}$. The **Sobolev space** $W^{m,p}(\mathbb{R}^n)$ is defined by

$$W^{m,p}(\mathbb{R}^n) = \{f \in L_p(\mathbb{R}^n) : D^\alpha f \in L_p(\mathbb{R}^n) \text{ for all } |\alpha| \le m\},$$

where derivatives are understood in the weak sense. We endow it with the norm

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} = \sum_{|\alpha|\le m} \|D^\alpha f\|_{L_p(\mathbb{R}^n)}.$$

For $r \in \mathbb{N}$, the r-th **order difference** of f is

$$\Delta_h^r f(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x+jh), \qquad x, h \in \mathbb{R}^n.$$

The r**-th modulus of smoothness** in $L_p(\mathbb{R}^n)$ is

$$\omega_r(f,t)_p = \sup_{|h|\le t} \|\Delta_h^r f\|_{L_p(\mathbb{R}^n)}, \qquad t > 0.$$

If $0 < s < r$ and $0 < q \le \infty$, the **Besov space** $B^s_{p,q}(\mathbb{R}^n)$ can be described by the norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)} + \left(\int_0^\infty \left[t^{-s}\omega_r(f,t)_p\right]^q \frac{dt}{t} \right)^{1/q},$$

with the usual modification when $q = \infty$. Up to equivalence of norms, this definition is independent of the integer $r > s$.

The fundamental point is the equivalence between the K-functional of the couple $(L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n))$ and the modulus of smoothness:

$$K(t^m, f; L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)) \sim \omega_m(f,t)_p.$$

Equivalently, after the change of variables $u = t^m$,

$$K(u, f; L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)) \sim \omega_m(f, u^{1/m})_p.$$

Consequently,

$$(L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n))_{\theta,q} = B^{\theta m}_{p,q}(\mathbb{R}^n), \qquad 0 < \theta < 1.$$

Thus the real interpolation method naturally produces Besov spaces from the couple formed by L_p and the Sobolev space $W^{m,p}$.

8.2 A Besov embedding obtained by interpolation

The previous section naturally dives into a simple example showing how Besov embeddings can be deduced from Sobolev embeddings by interpolation. Let $1 \le p < n/m$ and define p^* by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}.$$

The classical Sobolev embedding theorem gives

$$W^{m,p}(\mathbb{R}^n) \hookrightarrow L_{p^*}(\mathbb{R}^n), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{n},$$

whenever $1 \le p < n/m$; see, for instance, Adams and Fournier [1] or Maz'ya [13]. Together with the trivial boundedness

$$L_p(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n),$$

we can interpolate the identity operator

$$I : \big(L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n)\big) \longrightarrow \big(L_p(\mathbb{R}^n), L_{p^*}(\mathbb{R}^n)\big).$$

The real interpolation theorem yields

$$I : (L_p, W^{m,p})_{\theta,q} \longrightarrow (L_p, L_{p^*})_{\theta,q}.$$

Using

$$(L_p(\mathbb{R}^n), W^{m,p}(\mathbb{R}^n))_{\theta,q} = B^{\theta m}_{p,q}(\mathbb{R}^n)$$

and the Lorentz space identity proved in Theorem 7.5

$$(L_p(\mathbb{R}^n), L_{p^*}(\mathbb{R}^n))_{\theta,q} = L_{rq}(\mathbb{R}^n), \qquad \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{p^*},$$

we obtain

$$B^{\theta m}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{rq}(\mathbb{R}^n),$$

where

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{p^*} = \frac{1}{p} - \frac{\theta m}{n}.$$

Writing $s = \theta m$, this becomes

$$B^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow L^{r,q}(\mathbb{R}^n), \qquad \frac{1}{r} = \frac{1}{p} - \frac{s}{n}.$$

This is the Besov version of the Sobolev embedding.

8.3 The Sobolev couple $(W^{1,1}, W^{1,\infty})$

The computation of K-functionals becomes more delicate when both spaces in the couple involve derivatives. A basic example is the couple

$$\big(W^{1,1}(\mathbb{R}^n), W^{1,\infty}(\mathbb{R}^n)\big).$$

In this case the K-functional is governed both by the size of f and by the size of its gradient. A fundamental result of DeVore and Scherer [7] gives, in particular, the equivalence

$$\begin{aligned} K\big(t, f; W^{1,1}(\mathbb{R}^n), W^{1,\infty}(\mathbb{R}^n)\big) &\sim K\big(t, f; L_1(\mathbb{R}^n), L_\infty(\mathbb{R}^n)\big) \\ &\quad + K\big(t, |\nabla f|; L_1(\mathbb{R}^n), L_\infty(\mathbb{R}^n)\big). \end{aligned}$$

Here ∇f is understood in the weak sense.

Using the classical formula first described in Example 4.1

$$K(t, g; (L_1(\mathbb{R}^n), L_\infty(\mathbb{R}^n)) = \int_0^t g^*(s)\, ds,$$

we obtain

$$K\big(t, (f; W^{1,1}(\mathbb{R}^n)), W^{1,\infty}(\mathbb{R}^n)\big) \sim \int_0^t f^*(s)\, ds + \int_0^t |\nabla f|^*(s)\, ds.$$

Consequently, for $0 < \theta < 1$ and $1 \leq q \leq \infty$,

$$\big(W^{1,1}(\mathbb{R}^n), W^{1,\infty}(\mathbb{R}^n)\big)_{\theta,q} = W^1 L^{p,q}(\mathbb{R}^n), \qquad \frac{1}{p} = 1 - \theta,$$

with equivalence of norms. Here $W^1 L_{pq}(\mathbb{R}^n)$ denotes the **Sobolev–Lorentz space** of all functions f such that

$$f \in L^{p,q}(\mathbb{R}^n), \qquad \nabla f \in L_{pq}(\mathbb{R}^n).$$

This example illustrates a general phenomenon: computing the K-functional for Sobolev couples requires controlling how the decomposition of f interacts with weak derivatives. The original systematic treatment is due to DeVore and Scherer [7]; see also Bennett and Sharpley [3].

8.4 A comment on harmonic analysis: BMO and $H^1_{\mathbf{at}}$

To end this chapter we shall dive into a classic example in harmonic analysis. For any operator, one would like to establish a boundedness of the form

$$\begin{aligned} T :& L_1(\mathbb{R}^n) \to L_1(\mathbb{R}^n) \\ T :& L_\infty(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n) \end{aligned}$$

and then apply the real method of interpolation to get $T : L_p \to L_p$ for all $1 \leq p \leq \infty$. However, these endpoints are not bounded for a large number of operators, including many Calderon-Zygmung integral operators. This leads to the creation of two other spaces $BMO(\mathbb{R}^n)$ and $H^1_{at}(\mathbb{R}^n)$ (or simply BMO and H^1_{at}) such that $L_\infty \subseteq BMO$ and $H^1_{at} \subseteq L_1$. For these two, the bounds

$$\begin{aligned} T :& H^1_{at}(\mathbb{R}^n) \to L_1(\mathbb{R}^n) \quad \text{and} \\ T :& L_\infty(\mathbb{R}^n) \to BMO(\mathbb{R}^n) \end{aligned}$$

are easier to prove in many setups. As a fact, any operator bounded in $L_2(\mathbb{R}^n)$ satisfies both bounds. These are the cases of the Hilbert or Riesz transform. For a further explanation, see E.M Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* [15].

Hence, these operators suggest seeking for the interpolation spaces produced by the real method between the couples (H^1_{at}, L_∞) and (L_1, BMO). This section is devoted to these two and essentially follows Bennett and Sharpley [3].

8.4.1 $BMO(\mathbb{R}^n)$ and the couple (L_1, BMO)

Recall that, for a locally integrable function f, its **sharp maximal function** is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x, and

$$f_Q = \frac{1}{|Q|} \int_Q f(y)\, dy.$$

The BMO-norm is given by

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy.$$

Equivalently,

$$\|f\|_* = \|f^\#\|_{L_\infty}.$$

Of course $BMO(\mathbb{R}^n)$ is the set of measurable functions f in $\mathbb{R}^n$ for which $\|f\|_*$ is finite. However, $\|\cdot\|_*$ is not a norm, as for any constant function $f = c$, $\|f\|_* = 0$. This is why BMO usually denotes $BMO/\mathbb{R}$, where two functions differing by a constant are the same. It is clear that $L_\infty \subseteq BMO$. Indeed, let $f \in L_\infty$ and $Q \subseteq \mathbb{R}^n$ be any cube. Clearly $|f_Q| \leq \|f\|_{L_\infty}$ and so

$$\frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy \leq \frac{1}{|Q|} \int_Q |f(y)| + |f_Q|\, dy \leq 2\|f\|_{L_\infty}.$$

As the bound is uniform over all cubes $Q \subseteq \mathbb{R}^n$ we get $\|f\|_* \leq 2\|f\|_{L_\infty}$. One can also prove that BMO is a Banch space with such norm.

For the couple (L_1, BMO), the corresponding K-functional is more subtle but can also be described explicitly in terms of oscillation quantities and sharp maximal functions. This description is one of the main results in the work of Bennett, DeVore and Sharpley [2]; see also Bennett and Sharpley [3]. In particular they prove

$$K(t, f; (L_1(\mathbb{R}^n), BMO(\mathbb{R}^n))) \sim t(f^{\#})^*(t),$$

where $(f^{\#})^*$ is the decreasing rearrangement of the sharp maximal function. By definition of Lorentz spaces, we have $\|f\|_{\theta,q} = \|f^{\#}\|_{L_{pq}}$ for $0 < \theta < 1$, $1 \leq q \leq \infty$ and $p = 1/(1-\theta)$. In this regime, the Fefferman-Stein inequality (extended to Lorentz spaces, see Fefferman and Stein [9]) ensures that the L_{pq} norm of a function is equivalent to the L_{pq} norm of its sharp maximal function. Therefore it yields for $0 < \theta < 1$ and $1 \leq q \leq \infty$ that

$$(L_1(\mathbb{R}^n), BMO(\mathbb{R}^n))_{\theta,q} = L_{pq}(\mathbb{R}^n), \quad (p = 1/(1-\theta)).$$

This results suggests that BMO is the right substitute to L_∞ as an endpoint, since we recover the same result as the interpolation of the couple (L_1, L_∞).

8.4.2 $H^1_{at}(\mathbb{R}^n)$ and the couple (H^1_{at}, L_∞)

To define the space $H^1_{at}(\mathbb{R}^n)$ we first need the notion of an atom.

Definition 8.1. *An atom in $\mathbb{R}^n$ is a measurable function $a : \mathbb{R}^n \to \mathbb{R}$ such that there exists a cube $Q \subseteq \mathbb{R}^n$ with*

(a) $supp(a) \subseteq Q$,

(b) $\int_{\mathbb{R}^n} a \, dx = 0$ and

(c) $\|a\|_{L_\infty} \leq |Q|^{-1}$

Then, the **Hardy atomic space** $H^1_{at}(\mathbb{R}^n)$ or H^1 is the set of functions in $L_1(\mathbb{R}^n)$ such that there exist two sequences $(\lambda_n)_n \subseteq \mathbb{R}$ and $(a_n)_n$ of atoms such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{and} \quad \sum_{j=1}^{\infty} |\lambda_j| < \infty, \tag{8.1}$$

with the convergence of the first series in L_1. We then define the atomic norm as

$$\|f\|_{at} = \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all the sequences $(\lambda_n)_n$ satisfying (8.1) for some sequence of atoms $(a_n)_n$. One can prove that $\|\cdot\|_{at}$ defines a norm on H^1 and that H^1 is complete with respect to the norm.

For the characterization of the K-functional we first recall that the **Poisson kernel** $P_t(x)$ in $\mathbb{R}^n$ is the function on $\mathbb{R}^n$ defined as

$$P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{(n+1)/2}} \quad \text{with } c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}. \tag{8.2}$$

One can prove that for $f \in L_p$ for any $p \geq 1$, the function defined on the upper half plane $\mathbb{R}^{n+1}_+$ as $u(t,x) = (P_t * f)(x)$ extends f harmonically into $\mathbb{R}^{n+1}_+$, that is

$$\Delta u = 0 \quad \text{and } \lim_{t\to 0} u(t,x) = f(x) \text{ } a.e.$$

We then define the **non-tangential maximal function of** $f \in L_1(\mathbb{R}^n)$, Nf as

$$Nf(x) = \sup_{(y,t)\in\Gamma_x} |(P_t * f)(y)|,$$

where Γ_x denotes the cone $\Gamma_x = \{(t,y) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$. Nf will play the role of the sharp maximal function in this scenario. In fact, one can prove that the L_{pq} norm of a function is equivalent to the L_{pq} norm of its non-tangential maximal function. The proof relies on a technical sequence of results; first proving that Nf is pointwise bounded by the Maximal Hardy-Littlewood operator and later the Coifman-Latter Theorem, which equates H^1 to the Hardy space $Re(H^1)$ (see Bennett and Sharpley [3]). Similar to the previous case, one has

$$K(f,t;(H^1_{at}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))) \sim \int_0^t (Nf)^*(s)ds = t(Nf)^{**}(t)$$

for all $f \in H^1 + L_\infty$. Hence, using the equivalence of norms in L_{pq} stated in Example 4.1, we get for $0 < \theta < 1$, $1 \leq q \leq \infty$, and $p = 1/(1-\theta)$, $\|f\|_{\theta,q} = \|Nf\|_{L_{pq}}$. As we previously mentioned the latter one is equivalent to the L_{pq} norm of the function. Therefore it yields for $0 < \theta < 1$ and $1 \leq q \leq \infty$ that

$$(H^1_{at}(\mathbb{R}^n), L_\infty(\mathbb{R}^n))_{\theta,q} = L_{pq}(\mathbb{R}^n) \quad (p = 1/(1-\theta))$$

Considering now the both interpolation spaces from the last two sections, we see that for any admissible operator satisfying $T : (H^1, L_\infty) \to (L_1, BMO)$, then given $0 < \theta < 1$ and $1 \leq q \leq \infty$,

$$T : L_{pq}(\mathbb{R}^n) \to L_{pq}(\mathbb{R}^n) \quad (p = 1/(1-\theta)).$$

In particular, by setting $q = 1/(1-\theta)$, one recovers the classical bound $T : L_p \to L_p$ for any $1 < p < \infty$. Some operators satisfying $T : (H^1, L_\infty) \to (L_1, BMO)$ include all integral operators with a Calderón-Zygmund kernel, which can be proven to be bounded in $L_2(\mathbb{R}^n)$.

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