Treball final de grau

GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques
Universitat de Barcelona

AVATAR ANIMATION USING GREEN COORDINATES

Anna Isabel Bellido Rivas

Directors: Xavier Jarque i Xavier Massaneda
Realitzat a: Departament de Matemàtica Aplicada i Anàlisi. UB
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Contents

Chapter 1

Introduction

1.1 Brief avatar animation introduction

Previous to the very core introduction of this work should come an abridged description of the location this work will occupy in the field of avatar animation, just for the sake of informing the reader about where we are going. I will start with a description of how an avatar's mesh is built and how it can be animated.

1.1.1 An avatar's mesh

There are many kinds of meshes that can be used to build an avatar. In this case, we will focus on closed triangular meshes.

These meshes are formed of a set of vertices, edges and faces. The way these items are connected and positioned in space is what sculpts the avatar we want to create.

Closed triangular meshes restrict the faces to be triangles, meaning that each face is formed of three vertices connected by three edges and all the edges are included in exactly two triangles. This way, the mesh is entirely formed of triangles and all of them are connected to another triangle through each edge. Moreover, each face is assigned an outward normal vector which we will also use in this work. An example can be seen in Figure 1.1.

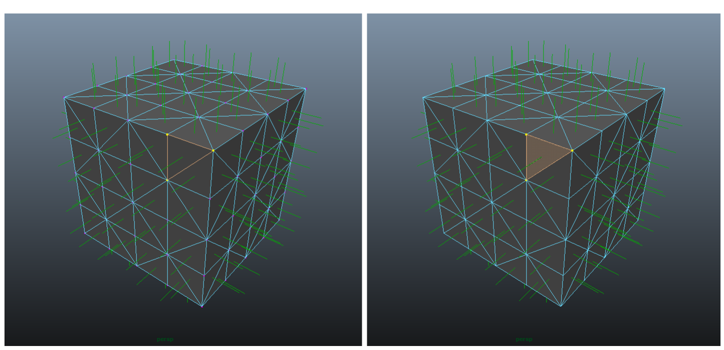


Figure 1.1: Three vertices and three edges are selected on the left. On the right, the resulting face surrounded by these items is shown. The exterior facial normals can be seen in both pictures in green.

As one can easily imagine, if we only had this mesh, the way to animate it would be to move all the vertices of the mesh adequately until it adopted the desired pose, and we would have to do this for each keyframe of the animation. When the mesh is very coarse, this simple methodology could be feasible; however, this is not normally the case, especially nowadays when computational power increases so drastically every year, making it possible to create and control meshes of millions of vertices. Following this need, many structures aimed to control the mesh and animate it have been built. In the following subsection I will briefly introduce the skeletons and the free-form deformations (FFD). Only these two will be mentioned, because the FFD's will be used in this work and the skeleton is the most commonly used and it's important to point out the advantages of the method proposed in the paper *Green Coordinates.*

1.1.2 Two methods to animate a mesh

A skeleton is a hierarchical structure formed of bones. Additionally, a map defining the influence every bone has on each vertex is built, namely a weight map. These influences are specified in a vector stored per vertex. This vector's dimension is the number of bones in the skeleton, and on each coordinate the influence level is indicated. The influence is a value ranging from 0 to 1, 0 meaning that the bone has no influence on the vertex, and 1 means that the vertex moves with the bone.

These two things together (and some other possible control substructures that don't apply to this work) form a rig. An example of a 3-boned rig controlling a cylinder can be seen in Figure 1.2.

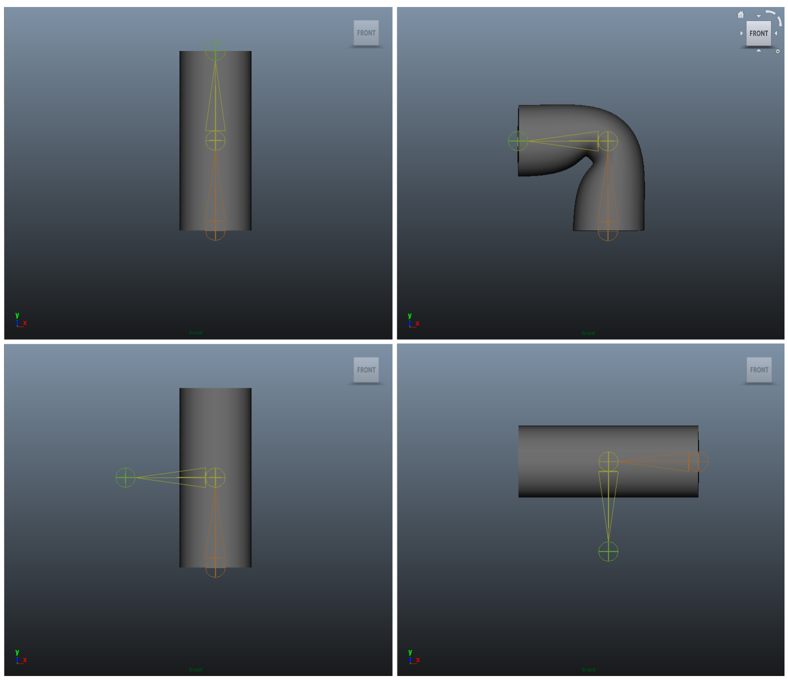


Figure 1.2: At the top left corner, the cylinder and the 3-boned rig can be seen (the root bone is the one at the bottom, namely bone0). At the top right corner we can see how the vertices move adequately when we rotate the second bone in the hierarchy; thus, the coordinates of the vertices which moved are (b0,b1,b2) where b1>0. At the bottom we can see the same skeleton and mesh but with a different weight map, as the vertices are not moving at all with bone1; thus, all the vertices coordinates are (b0,0,b2). We can also see that all the influence comes from bone0 as the mesh moves when we rotate it; then, all the vertices influence vectors are (b0,0,0).

This structure is currently widely used in computer graphics because of its simplicity, its ease of use, its adaptability to different meshes and the possibility of adding extra controls very straight forward to improve the mesh movements. However, it has some limitations that may be too difficult to control by these extras in the rig. These are due to the fact that the influences in the weight map can't guarantee conformal deformations, what makes the mesh deform inadequately sometimes (see Figure 1.3).

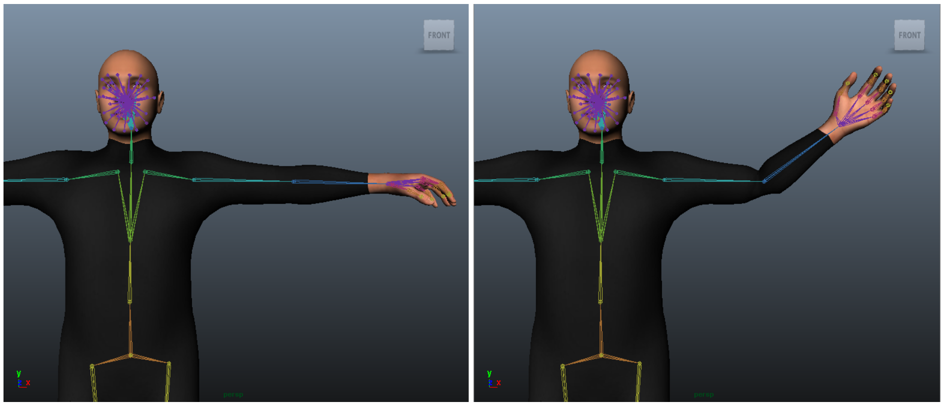


Figure 1.3: The picture on the left shows the mesh without any deformation and the picture on the right shows how the shape of the muscles around the elbow is not preserved when only a slight rotation to start a greeting animation is done

The other structure is a free-form deformation (FFD). This is basically a coarser mesh that encloses the mesh we want to animate. We will call this latter mesh a *cage*. A simple example can be seen in Figure 1.4.

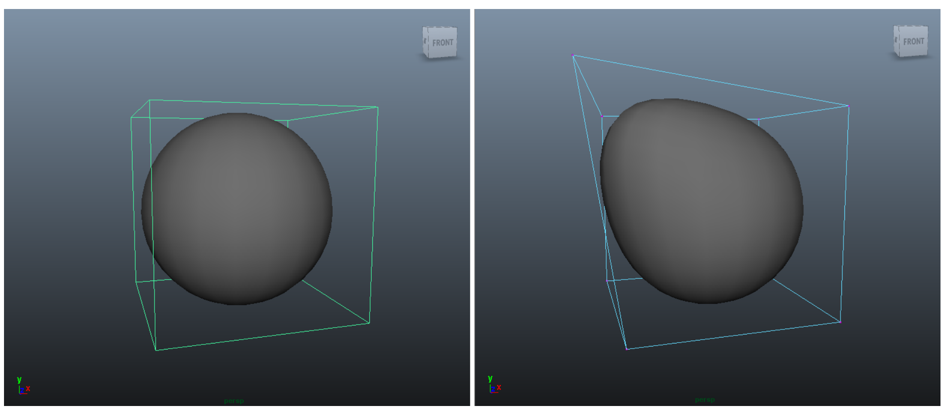


Figure 1.4: Free-form deformation controlling a sphere. Original deformation on the left, and result after moving one of the cage's vertices on the right

This also needs an influence vector per vertex in the original mesh determining how much it moves with each vertex in the coarser mesh or *cage*. There are different methodologies for assigning the influences to the vertices, but none of these deal with the limitation that the former structure has. The main goal of the paper *Green Coordinates* [1] is to set a map that does guarantee a deformation keeping the original shape of the mesh as much as possible. This deformation won't exactly be a conformal map; however, the authors argue that an holomorphic map in the case of a 2D mesh and an adequate map in 3D have been empirically shown to be sufficient to minimize the shape deformation (only drastic deformations of the cage won't work). In fact this makes sense, as the difference between an holomorphic map and a conformal one is that there are no zeros in its derivative, but due to the analytic continuation principle, the set of zeros of the derivative of the holomorphic map is discrete, with no accumulation points in the domain. Then, it's rare to find one of these points, so we can argue that the map will be generally conformal.

The paper *Green Coordinates* was published in 2008. This paper presented a method to represent a point using the vertices and the faces normals of a simplicial surface (the coarser mesh mentioned earlier could be an example of a simplicial surface), which can either be in 2D or 3D. This representation will be linear in nature and the coordinates assigned to each vertex or normal will be derived mainly using Green's identities and the fundamental solution of the Laplace equation which will be denoted as $G(\xi, \eta)$. A more detailed description of these concepts is given in *Annex I*.

Another paper titled *Derivation and Analysis of Green Coordinates* [2] was published in 2010. This latter paper was aimed to describe the proposed coordinates in a bit more mathematical detail. This is what I used to understand the ins and outs of the original paper.

I have to say that this paper required extra effort. It was very common to find errors in almost every proof given. Then, the work done was not merely based on understanding the paper, but also on correcting the inadequacies, what happened very often. Nevertheless, the results were correct, as we corroborated in the end.

Regarding the structure followed to organize this work, chapter 2 will be focused on finding formulas to get the influence vectors (called coefficients as we will see in short) of the exterior cage over the finer interior mesh. These formulas will be first derivated for interior points, meaning those points of the finer mesh which are totally enclosed by the cage, and they will be later extended to points in the exterior of the cage. At the end of each, we will get close-form formulas for the influences, which will be programmable. These formulas will vary depending on whether we are in 2D or 3D. Chapter 3 will aim to describe the piece of software done to calculate these influences (the algorithms in pseudocode can be seen at the end of *Annex II*). This software will provide the implementation for the 2D case. Chapter 4 will provide the results obtained after programming it. And Chapter 6 will include the conclusions after having done this work.

Chapter 2

Derivation of the Green Coordinates

This chapter will start with the definitions and the goal we would like to achieve, and it will end with a closed-form formula determining the exact coordinates each point will have with respect to the vertices and normals of the given simplicial surface (or cage). All the calculations will depend on whether we are working with a 2D mesh or a 3D one, what will directly imply that we have also a closed piecewise curve or a closed piecewise surface as the exterior *cage*, respectively.

Let's denote by $P = (\mathbb{V}, \mathbb{T})$ the cage we will use, where $\mathbb{V} = \{v_i\}_{i \in I_{\mathbb{V}}} \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) are the vertices in the cage and $\mathbb{T} = \{t_j\}_{j \in I_{\mathbb{T}}}$ are its faces (edges (2D) or triangles (3D)). Thus, each element t_j in the set $\mathbb{T}$ will be formed of d vertices: $t_j = (v_{j_1}, ..., v_{j_d})$. We will also denote the outward unitary normal of each face t_j by $n(t_j)$.

Now we can reformulate the main goal of the paper as finding a closed-form formula for the coefficients in a linear combination of $\{v_i\}$ and $\{n(t_j)\}$, which we will denote by $F(\eta; P)$. For every point $\eta \in \mathbb{R}^d$ and a given cage $P = (\mathbb{V}, \mathbb{T})$, we would like to have

$$\eta \mapsto F(\eta; P) = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) \tag{2.1}$$

The coefficients we are seeking for are $\phi_i(\eta)$ and $\psi_j(\eta)$, named *Green Coordinates* in the paper.

Once we have these coefficients, the expression above will allow us to control the finer mesh (the way it was described in the section including FFD's) just by moving the vertices of the cage (we will denote the deformed cage by $P' = (\mathbb{V}', \mathbb{T}')$) and recalculating the normals accordingly:

$$\eta \mapsto F(\eta; P') = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j) \tag{2.2}$$

The variable s_j will control the deformation so that the map is least-distorting. By map I mean the change that one point η makes due to movements of the vertices in the cage to its new position $F(\eta; P')$.

We will see later in this chapter the properties these coefficients have, which will be of great use in the control of the deformation of the finer mesh, because holomorphic maps will be achieved in 2D and least-distorting deformations will be obtained in 3D.

We will start by considering points enclosed in the cage, and in later subsections we will find an extension of the already defined formula to the exterior points.

2.1 Deriving the Green Coordinates for interior points

Let u be a harmonic function[1] in a certain domain $D \subset \mathbb{R}^d$ enclosed by a piecewise continuous boundary ∂D. Let's also introduce the fundamental solution to the Laplace equation with pole η, which is denoted by $G(\xi, \eta)$ for $\xi, \eta \in \mathbb{R}^d$:

$$G(\xi,\eta) = \begin{cases} \frac{1}{(2-d)\omega_d} \|\xi - \eta\|^{2-d} & d \geq 3 \\ \\ \frac{1}{2\pi} \log \|\xi - \eta\| & d = 2 \end{cases} \tag{2.3}$$

We know by definition that $\Delta_\xi G(\xi, \eta) = \delta_\eta$, and in particular $G(\cdot, \eta)$ is harmonic in $\mathbb{R}^d \backslash \eta$. See more details in *Annex II*.

We will derive the coordinates we need for the linear map mentioned before from Green's third identity. This derivation is the result of making the right substitution for one of the functions involved in Green's second identity.

Theorem 2.1. Green's Third Identity:
Be D a bounded domain in $\mathbb{R}^n$, with ∂D a regular hypersurface oriented with the unit outward normal n. Let $u \in \mathcal{C}^2(\bar{D}) \cap Har(D)$. Now, $\forall \eta \in D^{in} = interior(D)$ we can express

$$u(\eta) = \int_{\partial D} \left(u(\xi) \frac{\partial_\xi G(\xi,\eta)}{\partial n} - G(\xi,\eta) \frac{\partial u(\xi)}{\partial n} \right) d\sigma_\xi \tag{2.4}$$

where $d\sigma_\xi$ is the surface element in ∂D.

Proof. Given a point $\eta \in supp(u)$[2], let $U = D \backslash B(\eta, \epsilon)$ where $\epsilon > 0$ is small. Notice that $u, G \in Har(U)$[3], because U does not include its pole η.
By Green's Second Identity (see *Annex I*), we obtain (no parameters will be included to simplify the expression):

$$\int_U (u \Delta G - G \Delta u)\, dA = \int_{\partial U} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\sigma \tag{2.5}$$

Since $u, G \in Har(U)$, the left-hand side vanishes. Consequently, we can rearrange the non-zero terms in (2.5) and split the integrals with respect to the integration domain ($\partial U = \partial D \cup \partial B(\eta, \epsilon)$).

$$\int_{\partial D} \left(u(\xi) \frac{\partial G(\xi,\eta)}{\partial n} - G(\xi,\eta) \frac{\partial u(\xi)}{\partial n} \right) d\sigma = \int_{\partial B(\eta,\epsilon)} u(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma + \int_{\partial B(\eta,\epsilon)} G(\xi,\eta) \frac{\partial u(\xi)}{\partial n} d\sigma \tag{2.6}$$

From now until the end of these calculations we will assume we are in 2D, then, d=2. The 3D case is analogous to this one changing the parametrization of $\partial B(\eta, \epsilon)$. This gives us a concrete expression for $G(\xi, \eta) = \frac{1}{2\pi} \log \|\xi - \eta\| = \frac{1}{4\pi} \log \|\xi - \eta\|^2$.
As we are in 2D, we can parametrize every point $\xi \in \partial B(\eta, \epsilon)$ as $\xi(\theta) = \epsilon e^{i\theta} + \eta$, where $\theta \in [0, 2\pi]$. We can also calculate the outward normal accordingly: $n = \frac{\xi - \eta}{\|\xi - \eta\|} = \frac{\epsilon e^{i\theta}}{\epsilon} = e^{i\theta}$.

[1] u is a harmonic function if it is a solution to the Laplace equation: $\Delta u = \nabla \cdot \nabla u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} = 0$

[2] supp(u) stands for the support of the function u, which is the closure of the set $\{\xi : u(\xi) \neq 0\}$

[3] $u \in Har(D)$ if u is a harmonic function in D, then, $\forall \xi \in D$ $\Delta u(\xi) = 0$

We will compute the first integral on the right side of (2.6) first:

$$\int_{\partial B(\eta,\epsilon)} u(\xi)\frac{\partial G(\xi,\eta)}{\partial n}d\sigma = \int_{\partial B(\eta,\epsilon)} u(\xi)\frac{\partial(\log\|\xi-\eta\|^2)}{\partial n}d\sigma$$

We calculate the normal derivative $\frac{\partial G}{\partial n}(\xi,\eta) = \nabla G(\xi,\eta)\cdot n$ using that

$$\nabla G(\xi,\eta) = \frac{1}{4\pi}\left(\frac{\partial\left(\log\|\xi-\eta\|^2\right)}{\partial\xi_1}, \frac{\partial\left(\log\|\xi-\eta\|^2\right)}{\partial\xi_2}\right) = \frac{1}{2\pi}\frac{\xi-\eta}{\|\xi-\eta\|}$$

Therefore, in terms of the parameterization,

$$\frac{\partial G}{\partial n}(\xi,\eta) = \nabla G(\xi,\eta)\cdot n = \frac{1}{2\pi}\frac{\xi-\eta}{\|\xi-\eta\|^2}\frac{\xi-\eta}{\|\xi-\eta\|} = \frac{1}{2\pi}\frac{1}{\|\xi-\eta\|} = \frac{1}{2\pi\epsilon}$$

Since $d\sigma = \|\xi'\|\,d\theta = \epsilon d\theta$, we get then

$$\int_{\partial B(\eta,\epsilon)} u(\xi)\frac{\partial(\log\|\xi-\eta\|^2)}{\partial n}d\sigma = \int_0^{2\pi} u(\eta+\epsilon e^{i\theta})\frac{1}{2\pi\epsilon}\epsilon d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(\eta+\epsilon e^{i\theta})d\theta$$

Letting the radius ϵ tend to 0:

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} u(\eta+\epsilon e^{i\theta})d\theta = u(\eta) \tag{2.7}$$

This result is due to the following (and the dominant convergence theorem at the final step) :

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\left(u(\eta+\epsilon e^{i\theta})-u(\eta)\right)d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|u(\eta+\epsilon e^{i\theta})-u(\eta)\right|d\theta \xrightarrow{\epsilon\to 0} 0$$

Now we can calculate the second integral in equation 2.6.
Let us show that

$$\lim_{\epsilon\to 0}\int_{\partial B(\eta,\epsilon)} G(\xi,\eta)\frac{\partial u(\xi)}{\partial n}d\sigma = 0$$

The previous parameterization gives

$$G(\xi,\eta) = G(\eta+\epsilon e^{i\theta},\eta) = \frac{1}{4\pi}\log\left\|\epsilon e^{i\theta}\right\|^2 = \frac{1}{4\pi}\log\epsilon^2$$

and therefore

$$\int_{\partial B(\eta,\epsilon)} G(\xi,\eta)\frac{\partial u(\xi)}{\partial n}d\sigma = \int_0^{2\pi}\frac{1}{4\pi}(\log\epsilon^2)\epsilon\frac{\partial u(\eta+\epsilon e^{i\theta})}{\partial n}d\theta = \frac{\epsilon(\log\epsilon^2)}{4\pi}\int_0^{2\pi}\frac{\partial u(\eta+e^{i\theta})}{\partial n}d\theta$$

Again, if we make ϵ go to 0 we find what we expected ($u\in\mathcal{C}^2(D)$):

$$\left|\frac{\epsilon(\log\epsilon^2)}{4\pi}\int_0^{2\pi}\frac{\partial u(\eta+\epsilon e^{i\theta})}{\partial n}d\theta\right| \le \frac{\epsilon(\log\epsilon^2)}{4\pi}2\pi\max|\nabla u| \xrightarrow{\epsilon\to 0} 0$$

Finally, let $\epsilon \to 0$ in 2.6 and find:

$$\int_{\partial D} \left(u(\xi) \frac{\partial G(\xi,\eta)}{\partial n} - G(\xi,\eta) \frac{\partial u(\xi)}{\partial n} \right) d\sigma = u(\eta) + 0 = u(\eta)$$

what is exactly equation (2.4), as we wanted to show.

The purpose until the end of this section is to transform the formula we have just shown (2.4) into the linear expression we need (expression (2.1)) for controlling the finer interior mesh. The subsequent step after having found the appropriate coefficients expression is to iteratively place the pole of the fundamental solution to the Laplace equation, denoted by η in this piece of work, in every coordinate of every vertex of the finer interior mesh and get the coordinates for that vertex. After that, the expression $F(\eta; P')$ will let us control the finer mesh using the set of coefficients we just built.

Adapting Green's Third Identity to be our linear expression

We apply the formula (2.4) to the region D limited by the cage. Thus, ∂D is made of all the faces in our cage (edges in 2D, triangles in 3D). Moreover, the function u will be the coordinate function $u_k(\eta) = \xi_k$ (being ξ_k the k-th component of the point η and the function $u(\eta)$), so $\frac{\partial u_k}{\partial n} = \nabla u_k \cdot n = e_1 \cdot n = n_k$.

$$\eta_k = \sum_{j \in I_{\mathbb{T}}} \left(\int_{t_j} \xi_k \frac{\partial G(\xi,\eta)}{\partial n} d\sigma - \int_{t_j} G(\xi,\eta) n_k(t_j) d\sigma \right)$$

Overall, this means that we have to use the former expression of Green's Third Identity (2.4) for every coordinate, and its vectorial form $u = \xi : \mathbb{R}^r \to \mathbb{R}$ results into:

$$\eta = \sum_{j \in I_{\mathbb{T}}} \left(\int_{t_j} \xi \frac{\partial G(\xi,\eta)}{\partial n} d\sigma - \int_{t_j} G(\xi,\eta) n(t_j) d\sigma \right)$$

The next step is to make this last equation turn into the final linear equation shown in (2.1). In order to do so, we have to introduce the vertices in the cage as well. We will do this through writing ξ in barycentric coordinates regarding the set of vertices $\{v_1, ..., v_d\}$ in each face t_j:

$$\xi = \sum_{k=1}^{d} \Gamma_k(\xi) v_k$$

Let's rewrite the previous formula with the barycentric coordinates bearing in mind that the normal is constant for every face, we get:

$$\eta = \sum_{j \in I_{\mathbb{T}}} \left(\int_{t_j} \left(\sum_{k=1}^{d} \Gamma_k(\xi) v_k \right) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma \right) - \sum_{j \in I_{\mathbb{T}}} n(t_j) \int_{t_j} G(\xi,\eta) d\sigma$$

Grouping the same vertices together and denoting by $N\{v_i\}$ the set of all the faces that include v_i as one of its vertices ($i \in I_{\mathbb{V}}$, the set of indexes of the vertices in the cage):

$$\begin{aligned} \eta &= \sum_{i \in I_{\mathbb{V}}} \left(\int_{\xi \in N\{v_i\}} \Gamma_i(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma \right) v_i - \sum_{j \in I_{\mathbb{T}}} \left(\int_{t_j} G(\xi,\eta) d\sigma \right) n(t_j) = \\ &= \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) \end{aligned}$$

where we finally give shape to the coefficients we were seeking for:

$$\phi_i(\eta) = \int_{\xi \in N\{v_i\}} \Gamma_i(\xi) \frac{\partial G(\xi, \eta)}{\partial n} d\sigma, \quad i \in I_{\mathbb{V}} \tag{2.8}$$

$$\psi_j(\eta) = -\int_{\xi \in t_j} G(\xi, \eta) d\sigma, \quad j \in I_{\mathbb{T}} \tag{2.9}$$

Before stating the properties of the map $F(\eta; P)$, let's summarize what we have up to now. We started with a figure or mesh that can either be in 2D or 3D, which is enclosed by a coarser mesh (or *cage*); this last mesh is what we will use to control the finer mesh inside of it. The method to control it is divided into two steps: in the first step there is no movement of the exterior cage (namely P) and what we do is building a set of coordinates for every point inside the coarser mesh. Hence, for each point $\eta \in P^{in}$ the coordinates vector will represent the components of the vertices and the normals of the cage; thus, the coordinates space will have dimension $\#\{\mathbb{V}\} + \#\{\mathbb{T}\}$. These calculations are done only once, as once we have the coefficients we can move the vertices in the cage, what will make the enclosed mesh vary following the linear expression

$$\eta \mapsto F(\eta; P') = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j)$$

which will tell us where every point $\eta \in P^{in}$ should go. The following properties will refer to the deformation of the interior mesh every time we modify the cage.

2.2 Properties of the Green Coordinates expression

The expression $F(\eta; P)$ should fulfill with a list of properties in order to be of good use for our purposes. We must also take into account that we have not defined the variable s_j yet; these properties will help us do so. Let's analyze them one by one:

1. *Linear reproduction:* This property refers to being able to reconstruct every η-point coordinates using a linear expression: $\eta = F(\eta; P)$, for $\eta \in P^{in}$. In our case, the expression $F(\eta; P)$ is linear by nature and if we choose $s_j = 1$ by construction we get exactly a linear reproduction of the point η.

2. *Translation invariance*: The translation invariance comes from the following:
Let $P = (\mathbb{V}, \mathbb{T})$ and $P' = (\mathbb{V}', \mathbb{T}')$, where $\forall v_i \in \mathbb{V}\ v'_i = v_i + t$, and $t \in \mathbb{R}^d$ an arbitrary constant vector. Observe that $n(t_j) = n(t'_j)$. Then, given an $\eta \in P^{in}$:

$$\eta = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j)$$

and therefore

$$\eta' = \eta + t = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t'_j) = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta)(v_i + t) + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j)$$

If we substract the first equation to the second one we get:

$$t = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) t = t \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta)$$

Since we want this to be possible for all possible translations $t \in \mathbb{R}^d$, we see that the translation invariance property is equivalent to

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) = 1, \quad \forall \eta \in P^{in} \tag{2.10}$$

3. *Rotation and scale invariance*: given an affine transformation consisting of a rotation and/or an isotropic scale U, we should have that $F(\eta; UP) = U\eta$.

$$\eta = F(\eta; UP) = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) U v_i + \sum_{j \in I\mathbb{T}} \psi_j(\eta) U n(t_j)$$

If we choose $s_j = \|U\|$ we get $Un(t_j) = s_j n(t'_j)$.

Let's build two simplicial surfaces: the first one (named S) will be formed of the face t_j and the vertex $v_{j_1} + n(t_j)$, and the second one (named S') will be formed of t'_j and the vertex $v'_{j_1} + n(t'_j)$, what at the same time is the result of applying the affine transformation U to the first simplicial surface ($S'_j = U(S_j)$). We would like to get least-distorting deformations, so we should choose an appropriate s_j. It should represent the stretch applied to the face and compensate for that stretch. In 2D, this is very straight-forward, because the faces t_j are edges, so the stretch of the face is exacly $s_j = \frac{\|t'_j\|}{\|t_j\|}$. However, in 3D it's not so obvious. Be σ_1 and σ_2 the eigenvalues of the linear map taking the face t_j to t'_j. Then, some average value between these two is what was finally chosen, as it empirically took to least-distorting deformations:

$$s_j = \sqrt{\frac{\sigma_1 + \sigma_2}{2}}$$

However, this expression implies that we have to find the eigenvalues of every deformation made to each face. We then look for a more straight-forward formula depending only on the faces themselves. Let u, v be the vectors defining the face t_j and u', v' the images of those vectors with respect to the affine transformation U, a computation shows that:

$$s_j = \frac{\|u'\|^2 \|v\|^2 + \|u\|^2 \|v'\|^2 - \|u'\|^2 \left(\frac{u}{\|u\|} v\right)^2 - \|u\|^2 \left(\frac{u'}{\|u'\|} v'\right)^2}{8area^2(t_j)} \tag{2.11}$$

This formula differs from the one in the paper, which does not correspond to the value s_j above.

4. *Smoothness*: Observing the expression of the coordinates $\phi_i(\eta)$ and $\psi_j(\eta)$ we can see that they are infinitely differentiable when $\eta \in P^{in}$. Then, $F(\eta; P)$ must also be so.

5. *Conformality*: the authors argue that they only show that the map $F(\eta; P)$ is holomorphic because they empirically found that the deformation was mostly conformal, and only when the cage was drastically modified conformality was not achieved. This proof will only be done for the case *d=2*, because in 3D we can't build a conformal map. The argumentation for this last statement can be found in the paper [3]. Hence, until the end of the proof of this property, we will assume that *d=2*.

Theorem 2.2. *For d=2 the deformation $\eta \mapsto F(\eta; P')$ defined by equation* (2.1)*, with the coordinates defined in* (2.9)*, is holomorphic in P^{in} for all P'.*

Proof. We will assume that the vertices in the cage are indexed clockwise, and we will also name the edges in the same manner: $t_j = v_{j+1} - v_j$. We will also need to introduce the linear operator $\perp: \mathbb{R}^2 \to \mathbb{R}^2$ making a counter-clockwise rotation of $\frac{\pi}{2}$ radiants. In the complex plane this is:

$$\perp(x,y) = i(x+iy) = (-y,x)$$

Using the expression $s_j = \frac{\|t'_j\|}{\|t_j\|}$ we can find an equivalent expression for the linear map we will use to deform the cage, which will be more useful for the proof of this theorem (and its lemmas):

$$\begin{aligned}\eta \mapsto F(\eta; P') &= \sum_{i\in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j\in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j) = \sum_{i\in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j\in I_{\mathbb{T}}} \psi_j(\eta) \frac{\|t'_j\|}{\|t_j\|} \frac{(t'_j)^{\perp}}{\|t'_j\|} = \\ &= \sum_{i\in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j\in I_{\mathbb{T}}} \tilde{\psi}_j(\eta)(t'_j)^{\perp} \end{aligned} \tag{2.12}$$

where

$$\tilde{\psi}_j(\eta) = \frac{1}{\|t_j\|}\psi_j(\eta)$$

The proof of the theorem is based on the following three lemmas:

Lemma 2.2. *Be $u \in Har(D)$, where $D \subset \mathbb{R}^2$; then $f = u_y + iu_x$ is holomorphic.*

Proof. As u is a harmonic function:

$$0 = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} \Rightarrow u_{xx} = -u_{yy}$$

The first equation comes from the Schwarz theorem, which says that we can reorder the variables with respect to we are differentiating if the function is $\mathcal{C}^2(D)$; which is the case because $\boldsymbol{u}$ is a harmonic function so it's $\mathcal{C}^\infty(D)$. ⊠

Lemma 2.3. *Be $v \in \mathbb{C}$, and be $h, r : \mathbb{C} \to \mathbb{R}$. If the map $h(z) + ir(z)$ is holomorphic then $ivh(z) - vr(z)$ is also holomorphic.*

Proof. This proof is immediate as the second map is the result of multiplying the map $h(z)$ by iv.

Corollary 2.4. *Let $v \in \mathbb{R}^2$ and let $h(x,y)$ and $r(x,y)$ be conjugate harmonic functions in $D \subset \mathbb{R}^2$. Then the mapping $f : D \to \mathbb{R}^2$ defined by $f(x,y) = v^{\perp}h(x,y) - vr(x,y)$ is holomorphic.*

Proof. This proof is also immediate because applying the operator $\perp$ is the same as rotating $\frac{\pi}{2}$ what, at the same time, is equivalent to multiplying by i, and it makes this map to be exactly the same as the resulting map from the previous lemma.

Lemma 2.5. *Let $v_i \in \mathbb{V}$ be an arbitrary vertex of P. Denote by t_{i-1} and t_i the faces (edges in this case) $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{v_iv_{i+1}}$, respectively. Then ϕ_i and $(\tilde{\psi}_i - \tilde{\psi}_{i-1})$ are conjugate harmonic. In other words,*

$$(\tilde{\psi}_i - \tilde{\psi}_{i-1}) + i\phi_i$$

is holomorphic.

Before proving this lemma let us see that it implies theorem 2.2, in conjunction with the current status.

We can observe that moving the whole cage and applying the change to the interior mesh is equivalent to moving all the vertices one by one and applying the change to the mesh each time. Consequently, we can show that the change of the interior mesh when we modify only one vertex in the cage is holomorphic, and later apply the proof to the rest of the vertices.

Proof (Theorem 2.2). Let's denote by P' a cage configuration which differs in only one vertex v_i from the original cage P. Be t_{i-1} and t_i the two consecutive edges that include the vertex v_i, which are also the only two edges that will differ from the original set of edges $\mathbb{T}$. For every $\eta \in P^{in}$ the change done corresponds to the difference between the final position of η induced by the map $F(\eta, P')$ and the initial position, which due to the linear reproduction property, correponds to the result of the initial map $F(\eta, P)$:

$$H(\eta) = F(\eta, P') - F(\eta, P) = \phi_i(\eta)(v_i' - v_i) + \sum_{j=i-1,i} \tilde{\psi}_j(\eta)(t_j'^{\perp} - t_j^{\perp})$$

Leaving the coordinate $\phi_i(\eta)$ aside for one moment, let's calculate what is this summation, bearing in mind that $t_i' = v_{i+1} - v_i'$ and $t_{i-1}' = v_i' - v_{i-1}$, because $v_k = v_k'$, $\forall k \neq i, k \in I_{\mathbb{V}}$:

$$\begin{aligned}\sum_{j=i-1,i} \tilde{\psi}_j(\eta)\left(t_j'^{\perp} - t_j^{\perp}\right) &= \tilde{\psi}_{i-1}(\eta)\left(t_{i-1}'^{\perp} - t_{i-1}^{\perp}\right) + \tilde{\psi}_i(\eta)\left(t_i'^{\perp} - t_i^{\perp}\right) \\ &= \tilde{\psi}_{i-1}(\eta)\left[(v_i' - v_{i-1})^{\perp} - (v_i - v_{i-1})^{\perp}\right] + \tilde{\psi}_i(\eta)\left[(v_{i+1} - v_i')^{\perp} - (v_{i+1} - v_i)^{\perp}\right]\end{aligned}$$

Now we can use the fact that $\perp$ is a linear operator:

$$\begin{aligned}\sum_{j=i-1,i} \tilde{\psi}_j(\eta)\left(t_j'^{\perp} - t_j^{\perp}\right) &= \tilde{\psi}_{i-1}(\eta)\left(v_i' - v_{i-1} - v_i + v_{i-1}\right)^{\perp} + \tilde{\psi}_i(\eta)\left(v_{i+1} - v_i' - v_{i+1} + v_i\right)^{\perp} = \\ &= \left(\tilde{\psi}_{i-1}(\eta) - \tilde{\psi}_i(\eta)\right)(v_i' - v_i)^{\perp}\end{aligned}$$

Let's bring all the calculations together and get:

$$H(\eta) = F(\eta; P') - F(\eta; P) = (v_i' - v_i)\phi_i(\eta) + (v_i' - v_i')^{\perp}(\tilde{\psi}_{i-1}(\eta) - \tilde{\psi}_i(\eta))$$

Conclusively, from Corollary 2.4 and Lemma 2.5 (where the vector $v = -(v_i' - v_i)^{\perp}$) $H(\eta)$ is holomorphic. ⊠

Proof (Lemma 2.5). In order to prove this lemma, for each set of three consecutive vertices $\overrightarrow{v_{i-1}v_iv_{i+1}}$ (indexed clockwise as it was indicated earlier) we build a triangle T; thus, this triangle is formed of the edges $t_{i-1} = \overrightarrow{v_{i-1}v_i}$, $t_i = \overrightarrow{v_iv_{i+1}}$ and an extra edge $e = \overrightarrow{v_{i+1}v_{i-1}}$, as it can be seen in Figure 2.1.

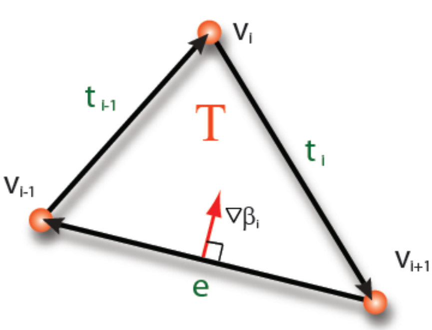


Figure 2.1: Triangle used in the proof

The proof will be split into two cases: $\eta \notin T$ and $\eta \in T$.

Let's denote by β_i the linear function defined over the triangle T that has the value 1 in the vertex v_i and the value 0 in the vertices v_{i-1} and v_{i+1}. Then, we can rewrite the expression found for determining ϕ_i in (2.9):

$$\phi_i(\eta) = \int_{t_{i-1}} \beta_i \frac{\partial G}{\partial n} d\sigma + \int_{t_i} \beta_i \frac{\partial G}{\partial n} d\sigma + \int_e \beta_i \frac{\partial G}{\partial n} d\sigma = \int_{\partial T} \beta_i \frac{\partial G}{\partial n} d\sigma$$

We can add the edge e to the integration domain (in this case it's $N\{v_i\} = t_{i-1} \cup t_i$) because $\int_e \beta_i \frac{\partial G}{\partial n} d\sigma = 0$, as β_i is 0 along that edge. Thus, we can complete the integral on the whole boundary of the triangle T ($t_{i-1} \cup t_i \cup e = \partial T$). Now we can apply Green's First Identity (see *Annex I*) and get:

$$\phi_i(\eta) = \int_{\partial T} \beta_i \frac{\partial G}{\partial n} d\sigma = \int_T (\beta_i \Delta G + (\nabla \beta_i \cdot \nabla G)) dV$$

One of the conditions of Green's Theorem is that the boundary must be positively oriented. As we didn't force any parameterization, we can suppose we are orienting the boundary positively and get the integral above.

Since $\eta \notin T$ the function G is harmonic in T and

$$\phi_i(\eta) = \int_{\partial T} \beta_i \frac{\partial G}{\partial n} d\sigma = \int_T (\nabla_\xi \beta_i(\xi) \cdot \nabla_\xi G(\eta, \xi)) dV$$

Let us compute the coordinates $\psi_j(\eta)$. We will need an explicit parameterization of the edge we are integrating in. We parameterize the edge t_{i-1} as:

$$\xi(t) = v_{i-1} + t(v_i - v_{i-1}), \quad t \in [0, 1]$$

Therefore, the function β_i which is 1 in the vertex v_i and linearly goes to 0 at v_{i-1} is $\beta_i(\xi(t)) = t$. Thus,

$$\nabla \beta_i(\xi(t)) \xi'(t) = \beta_i(\xi(t))(v_i - v_{i-1}) = 1 \tag{2.13}$$

On the other hand, since on the edge t_{i-1} $d\sigma = \|\xi'(t)\|\, dt = \|(v_i - v_{i-1})\|\, dt = \|t_{i-1}\|\, dt$, we obtain:

$$\tilde{\psi}_{i-1}(\eta) = \frac{1}{\|t_{i-1}\|} \psi_{i-1}(\eta) = -\frac{1}{\|t_{i-1}\|} \int_{t_{i-1}} G(\xi, \eta) d\sigma = \frac{1}{\|t_{i-1}\|} \int_0^1 G(\xi(t), \eta)\, \|t_{i-1}\|\, dt = \int_0^1 G(\xi(t), \eta) dt$$

By (2.13) we have

$$\tilde{\psi}_{i-1}(\eta) = \int_0^1 G(\xi(t), \eta) \nabla \beta_i(\xi(t))(v_i - v_{i-1}) dt$$

Finally, we note that $d\overrightarrow{\sigma} = \xi'(t) dt = (v_i - v_{i-1}) dt$. Therefore:

$$\tilde{\psi}_{i-1}(\eta) = \int_0^1 G(\xi(t), \eta) \nabla \beta_i(\xi(t))(v_i - v_{i-1}) dt = -\int_{t_{i-1}} G(\xi, \eta) \nabla \beta_i(\xi) d\overrightarrow{\sigma}$$

Analogously, we can find the equivalent expression for $\psi_i(\eta)$. We only have to bear in mind that $\beta_i(\xi(t)) = 1 - t$ in t_i. This last adjustment changes the sign of the integral, while the rest remains analogous, hence

$$\tilde{\psi}_i(\eta) = \int_{t_i} G(\xi,\eta)\nabla\beta_i(\xi)d\overrightarrow{\sigma}$$

Now bringing back the coefficient ψ_{i-1} we get:

$$\tilde{\psi}_{i-1}(\eta) - \tilde{\psi}_i(\eta) = -\int_{t_{i-1}\bigcup t_i} G(\xi,\eta)\nabla\beta_i(\xi)d\overrightarrow{\sigma}$$

Similarly to what we have done previously, we can add the edge e to this integral because $\beta_i = 0$ all along that edge; therefore

$$\tilde{\psi}_{i-1}(\eta) - \tilde{\psi}_i(\eta) = -\int_{t_{i-1}\cup t_i\cup e} G(\xi,\eta)\nabla\beta_i(\xi)d\overrightarrow{\sigma} = \int_{\partial T} G(\xi,\eta)\nabla\beta_i(\xi)d\overrightarrow{\sigma}$$

In order to compute the value of this integral, we use Green's Theorem (see *Annex I*).
Let $Q = G\nabla\beta_i = (A,B)$, and let's parameterize the boundary of the triangle T: $\xi(t) = (\xi_1(t),\xi_2(t)) = (x(t),y(t))$.
Then,

$$\int_{\partial T} Qd\overrightarrow{\sigma} = \int_{\partial T}(A,B)(\xi_1'(t),\xi_2'(t))dt = \int_{\partial T}(A\xi_1'(t)+B\xi_2'(t))dt = \int_{\partial T} Adx + Bdy$$

By Green's Theorem we get:

$$\int_{\partial T} Adx + Bdy = \int_T \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy = -\int_T \nabla\cdot Q^{\perp} dV$$

where the last equivalence is due to the fact that $Q^{\perp} = (-B,A)$. The negative sign comes from the fact that we oriented the boundary of T negatively, thus, we get the negative resulting integral.
Since $Q = G\nabla\beta_i = \left(G\frac{\partial\beta_i}{\partial x}, G\frac{\partial\beta_i}{\partial y}\right)$ we have that $Q^{\perp} = \left(-G\frac{\partial\beta_i}{\partial y}, G\frac{\partial\beta_i}{\partial x}\right)$. Then,

$$\nabla\cdot Q^{\perp} = -\frac{\partial}{\partial x}\left(G\frac{\partial\beta_i}{\partial y}\right) + \frac{\partial}{\partial y}\left(G\frac{\partial\beta_i}{\partial x}\right) = -\frac{\partial G}{\partial x}\frac{\partial\beta_i}{\partial y} - G\frac{\partial^2\beta_i}{\partial y\partial x} + \frac{\partial G}{\partial y}\frac{\partial\beta_i}{\partial x} + G\frac{\partial^2\beta_i}{\partial x\partial y}$$

But β_i is a linear function, so $\frac{\partial^2\beta_i}{\partial y\partial x} = \frac{\partial^2\beta_i}{\partial x\partial y} = 0$, and

$$\nabla\cdot Q^{\perp} = -\frac{\partial G}{\partial x}\frac{\partial\beta_i}{\partial y} + \frac{\partial G}{\partial y}\frac{\partial\beta_i}{\partial x} = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right)\cdot\left(-\frac{\partial\beta_i}{\partial y}, \frac{\partial\beta_i}{\partial x}\right) = \nabla G\cdot(\nabla\beta_i)^{\perp}$$

If we add this to the integral, we finally get:

$$\tilde{\psi}_i(\eta) - \tilde{\psi}_{i-1}(\eta) = \int_T \nabla_\xi G(\eta,\xi)\cdot(\nabla_\xi\beta_i(\xi))^{\perp} dV$$

Let's bring together the results we found:

$$\phi_i(\eta) = \int_T \nabla\beta_i\nabla_\xi G\ dV(\xi) = \int_T (\beta_x G_x + \beta_y G_y)\,dxdy$$
$$\tilde{\psi}_i(\eta) - \tilde{\psi}_{i-1}(\eta) = \int_T (\nabla\beta_i)^{\perp}\nabla_\xi G\ dV(\xi) = \int_T (-\beta_y G_x + \beta_x G_y)\,dxdy$$

Then,

$$\begin{aligned}(\tilde{\psi}_i(\eta) - \tilde{\psi}_{i-1}(\eta)) + i\phi_i(\eta) &= \int_T [(-\beta_y G_x + \beta_x G_y) + i(\beta_x G_x + \beta_y G_y)]\, dx\ dy = \\ &= \int_T [(G_y + iG_x)\beta_x + (-G_x + iG_y)\beta_y]\, dx\ dy = \\ &= \int_T [(G_y + iG_x)\beta_x + i(iG_x + G_y)\beta_y]\, dx\ dy = \\ &= \int_T (G_y + iG_x)(\beta_x + i\beta_y) dx\ dy\end{aligned}$$

Observe that $(\beta_x + i\beta_y)$ is constant, because β_i is a linear function, therefore, all that matters is the factor $(G_y + iG_x)$. We have to note that the variable we are interested in is η while the derivative of G are taken with respect to $\xi = x + iy$. However, one of the properties of the fundamental solution to the Laplace equation is that it is symmetrical (see its expression in (2.3), or see *Annex I*). Then,

$$\nabla_\xi G(\xi, \eta) = \nabla_\eta G(\eta, \xi)$$

Hence, we can now apply Lemma 2.2 and get what we were looking for. This completes the case $\eta \notin T$.

Let's assume now that $\eta \in T$; what we will do is building the same proof as before excluding the point η from the triangle, splitting all the calculations in two. Therefore, for each of these triangles we will do the same process we went through in the case where $\eta \notin T$.

Let's denote by w the middle point between v_i and η: $w = \frac{\eta + v_i}{2}$. Also consider the following vectors $e_0 = \overrightarrow{wv_{i-1}}$, $e_1 = \overrightarrow{v_{i+1}w}$ and $e = \overrightarrow{v_i w}$. The triangles that can be formed with these edges are denoted $T_0 = \overrightarrow{v_{i-1}v_i w}$ and $T_1 = \overrightarrow{v_i v_{i+1} w}$. All these new variables can be seen in Figure 2.2.

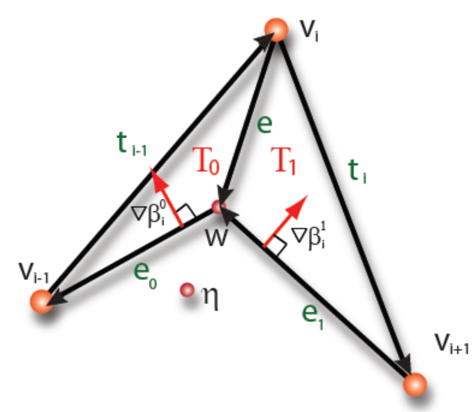


Figure 2.2: Triangle in case $\eta \in T$

Moreover, we will also define two more linear functions β_i^0 and β_i^1, where:

$$\begin{cases} \beta_i^0(\xi) = \beta_i(\xi), & \forall \xi \in t_{i-1} \\ \\ \beta_i^1(\xi) = \beta_i(\xi), & \forall \xi \in t_i \end{cases} \tag{2.14}$$

Similarly to β_i, these two new functions will vanish in the edges e_0 and e_1, respectively:

$$\begin{cases} \beta_i^0(\xi) = 0, & \forall \xi \in e_0 \\ \beta_i^1(\xi) = 0, & \forall \xi \in e_1 \end{cases} \tag{2.15}$$

Under these notations, let's compute again the coefficients:

$$\begin{aligned}
\phi_i(\eta) &= \int_{t_{i-1}} \beta_i(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma + \int_{t_i} \beta_i(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma \overset{(2.14)}{=} \\
&= \int_{t_{i-1}} \beta_i^0(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma + \int_{t_i} \beta_i^1(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma \overset{(2.15)}{=} \\
&= \int_{t_{i-1}\bigcup e_0} \beta_i^0(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma + \int_{t_i \bigcup e_1} \beta_i^1(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma
\end{aligned}$$

Note also the fact that $\beta_i^0(\xi) = \beta_i^1(\xi)$ on e, and that e has opposite sense in T_0 and T_1, hence:

$$\int_e \beta_i^0 \frac{\partial G}{\partial n} d\sigma = \int_{-e} \beta_i^1 \frac{\partial G}{\partial n} d\sigma$$

Therefore, we can complete the boundaries of both triangles T_0 and T_1 as integration domains of the integrals:

$$\begin{aligned}
\phi_i(\eta) &= \int_{t_{i-1}\bigcup e_0} \beta_i^0(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma + \int_{t_i \bigcup e_1} \beta_i^1(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma = \\
&= \int_{t_{i-1}\bigcup e_0 \bigcup e} \beta_i^0(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma + \int_{t_i \bigcup e_1 \bigcup -e} \beta_i^1(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma = \\
&= \int_{\partial T_0} \beta_i^0(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma + \int_{\partial T_1} \beta_i^1(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma
\end{aligned}$$

Analogously to the case where $\eta \notin T$ we can apply Green's first identity bearing also in mind that we excluded η from the integration area, so G is harmonic in both triangles T_0 and T_1. We finally get the following expression for the coefficient $\phi_i(\eta)$:

$$\phi_i(\eta) = \int_{T_0} \nabla \beta_i^0 \cdot \nabla G \; dV + \int_{T_1} \nabla \beta_i^1 \cdot \nabla G \; dV$$

We can do a similar process to find the coefficients $\tilde{\psi}_{i-1}(\eta)$ and $\tilde{\psi}_i(\eta)$, skipping the steps already made in the calculation where $\eta \notin T$ so as not to be too repetitive:

$$\begin{aligned}
\tilde{\psi}_{i-1}(\eta) &= -\frac{1}{\|t_{i-1}\|} \int_{t_{i-1}} G(\xi,\eta) \; d\sigma = -\int_{t_{i-1}} G(\xi,\eta) \nabla \beta_i(\xi) \; d\overrightarrow{\sigma} \overset{(2.14)}{=} \\
&= -\int_{t_{i-1}} G(\xi,\eta) \nabla \beta_i^0(\xi) \; d\overrightarrow{\sigma} \overset{(2.15)}{=} -\int_{t_{i-1}\bigcup e_0} G(\xi,\eta) \nabla \beta_i^0(\xi) \; d\overrightarrow{\sigma}
\end{aligned}$$

$$\begin{aligned}
\tilde{\psi}_i(\eta) &= -\frac{1}{\|t_i\|} \int_{t_i} G(\xi,\eta) \; d\sigma = -\int_{t_i} G(\xi,\eta) \nabla \beta_i(\xi) \; d\overrightarrow{\sigma} \overset{(2.14)}{=} \\
&= -\int_{t_i} G(\xi,\eta) \nabla \beta_i^1(\xi) \; d\overrightarrow{\sigma} \overset{(2.15)}{=} -\int_{t_i \bigcup e_1} G(\xi,\eta) \nabla \beta_i^1(\xi) \; d\overrightarrow{\sigma}
\end{aligned}$$

Finally, we can put it all together and get the other function we need (applying also here the fact that the integral on the edge e is compensated on both integrals):

$$\begin{aligned}\tilde{\psi}_i(\eta) - \tilde{\psi}_{i-1}(\eta) &= \int_{t_{i-1}\bigcup e_0} G(\xi,\eta)\nabla\beta_i^0(\xi)\ d\overrightarrow{\sigma} + \int_{t_i\bigcup e_1} G(\xi,\eta)\nabla\beta_i^1(\xi)\ d\overrightarrow{\sigma} = \\ &= \int_{t_{i-1}\bigcup e_0\bigcup e} G(\xi,\eta)\nabla\beta_i^0(\xi)\ d\overrightarrow{\sigma} + \int_{t_i\bigcup e_1\bigcup -e} G(\xi,\eta)\nabla\beta_i^1(\xi)\ d\overrightarrow{\sigma} = \\ &= \int_{\partial T_0} G(\xi,\eta)\nabla\beta_i^0(\xi)\ d\overrightarrow{\sigma} + \int_{\partial T_1} G(\xi,\eta)\nabla\beta_i^1(\xi)\ d\overrightarrow{\sigma}\end{aligned}$$

Apply Green's first identity once more:

$$\psi_i(\eta) - \psi_{i-1}(\eta) = -\int_{T_0} \nabla G\cdot(\nabla\beta_i^0)^\perp\ dV - \int_{T_1} \nabla G\cdot(\nabla\beta_i^1)^\perp\ dV.$$

We can observe now that we are under the same conditions as in the case $\eta \notin T$. Therefore, we can use the same reasoning and thus conclude the proof. ⊠

2.3 Closed-form formulas for 2D and 3D

In this section we will find explicit formulas for the coefficients for 2D and 3D. Their expression will vary depending on the dimension, so we will regard them separately. Throughout this section we will suppose that η is fixed.

2.3.1 A two-dimensional cage (d=2)

As it was exlicitly indicated in (2.3), the fundamental solution to the Laplace equation with pole η, in the case where d=2 is:

$$G(\xi,\eta) = \frac{1}{2\pi}\log\|\xi-\eta\|$$

We will start establishing the formula for $\psi_i(\eta)$ starting from the definition given in (2.9). We need a bit of notation beforehand. Let's denote by $v_i, v_{i+1} \in \mathbb{V}$ the vertices that form the edge t_i. Moreover, let's also denote $a_i = v_{i+1} - v_i$ and $b_i = v_i - \eta$. The parameterization we will use for the edge t_i will be

$$\xi(t) = v_i + t(v_{i+1} - v_i) = v_i + ta_i,\ t\in[0,1]$$

Hence, as $d\sigma = \|\xi'(t)\|\,dt = \|a_i\|\,dt$:

$$\begin{aligned}\psi_i(\eta) &= -\int_{\xi\in t_j} G(\xi,\eta)d\sigma = \frac{1}{2\pi}\int_{t=0}^{1}\log\|v_i + ta_i - \eta\|\,\|a_i\|\,dt \\ &= \frac{\|a_i\|}{2\pi}\int_{t=0}^{1}\log\|b_i + ta_i\|dt = \\ &= \frac{\|a_i\|}{4\pi}\int_{0}^{1}\log\left(\|b_i\|^2 + 2(a_i\cdot b_i)t + \|a_i\|^2 t^2\right)dt\end{aligned}$$

We can now use the primitive (see the Integration 1 in *Annex II* for detailed integration):

$$\int \log(qt^2+rt+s)dt =$$

$$= \left(t+\frac{r}{2q}\right)\log\left(qt^2+rt+s\right) - 2\left(t+\frac{r}{2q}\right) + \frac{\sqrt{4qs-r^2}}{q}\ \arctan\left(\frac{2qt+r}{\sqrt{4qs-r^2}}\right)$$

Making the substitutions:

$$q = \|a_i\|^2, \quad r = 2(a_i\cdot b_i), \quad s = \|b_i\|^2$$

and integrating in the interval [0,1] we will obtain an explicit value for $\psi_i(\eta)$.
Next, we will calculate the closed-form formula for the coefficient $\phi_i(\eta)$, starting from the definition given in (2.9). Here

$$N\{v_i\} = t_{i-1}\bigcup t_i = \overrightarrow{v_{i-1}v_i}\cup\overrightarrow{v_iv_{i+1}}$$

and therefore,

$$\phi_i(\eta) = \sum_{k=i-1,i}\int_{t_k}\Gamma_i(\xi)\frac{\partial G(\xi,\eta)}{\partial n}d\sigma$$

We will go a bit more in depth into the integrals separately:
For t_{i-1} we will use a parameterization almost identical as the one used above, although refering to the previous edge (in the sense of the clockwise order defined on the edges):

$$\xi(t) = v_{i-1}+a_{i-1}t, \quad t\in[0,1]$$

Moreover, we need to compute

$$\frac{\partial G(\xi,\eta)}{\partial n} = \nabla G(\xi,\eta)\cdot n$$

From the definition of G we have:

$$\nabla G(\xi,\eta) = \frac{1}{4\pi}\frac{1}{\|\xi-\eta\|^2}(2(\xi_1-\eta_1),2(\xi_2-\eta_2)) = \frac{1}{2\pi}\frac{\xi-\eta}{\|\xi-\eta\|^2}$$

And we also need the outward normal of the edge, which is $n(a_{i-1}) = \dfrac{a_{i-1}^{\perp}}{\|a_{i-1}\|}$.
Everything together results into:

$$\nabla G(\xi,\eta)\cdot n(a_{i-1}) = \frac{1}{2\pi}\frac{a_{i-1}t+v_{i-1}-\eta}{\|a_{i-1}t+v_{i-1}-\eta\|^2}\cdot\left(\frac{a_{i-1}^{\perp}}{\|a_{i-1}\|}\right) = \frac{1}{2\pi}\frac{a_{i-1}t+b_{i-1}}{\|a_{i-1}t+b_{i-1}\|^2}\cdot\left(\frac{a_{i-1}^{\perp}}{\|a_{i-1}\|}\right) =$$
$$= \frac{1}{2\pi}\frac{b_{i-1}\cdot a_{i-1}^{\perp}}{\|a_{i-1}t+b_{i-1}\|^2}\cdot\frac{1}{\|a_{i-1}\|}$$

Let's add everything to the integral, including $\Gamma_i(\xi(t)) = t$:

$$\int_{t_{i-1}} \Gamma_i(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma = -\int_0^1 t \left(\frac{b_{i-1} \cdot a_{i-1}^{\perp}}{2\pi \left\| a_{i-1}t + b_{i-1} \right\|^2} \cdot \frac{1}{\left\| a_{i-1} \right\|} \right) \left\| a_{i-1} \right\| dt =$$

$$= -\frac{b_{i-1} \cdot a_{i-1}^{\perp}}{2\pi} \int_0^1 \frac{t}{\left\| a_{i-1} \right\|^2 t^2 + 2(a_{i-1} \cdot b_{i-1})t + \left\| b_{i-1} \right\|^2} dt$$

Using Wolfram Alpha (online version of Mathematica) [8] we can obtain the primitive:

$$\int \frac{t}{qt^2 + rt + s} dt =$$

$$= \frac{1}{2q} \log \left(qt^2 + rt + s \right) - \left(\frac{r}{q\sqrt{4sq - r^2}} \right) \arctan \left(\frac{2qt + r}{\sqrt{4sq - r^2}} \right)$$

We have to substitute

$$q = \left\| a_{i-1} \right\|^2, \quad r = 2(a_{i-1} \cdot b_{i-1}), \quad s = \left\| b_{i-1} \right\|^2$$

and integrate on the interval [0,1].

We can do a similar process to calculate the integral in the edge t_i:

$$\xi(t) = v_i + a_i t, \qquad t \in [0,1]$$

Taking into account this parameterization and the fact that in this case the function Γ_i goes the other way around ($\Gamma_i(\xi(t)) = 1 - t$) we readily get:

$$\int_{t_i} \Gamma_i(\xi) \frac{\partial G(\xi,\eta)}{\partial n} d\sigma = -\int_0^1 (1-t) \left(\frac{b_i \cdot a_i^{\perp}}{2\pi \left\| a_i t + b_i \right\|^2} \cdot \frac{1}{\left\| a_i \right\|} \right) \left\| a_i \right\| dt =$$

$$= -\frac{b_i \cdot a_i^{\perp}}{2\pi} \int_0^1 \frac{1-t}{\left\| a_i \right\|^2 t^2 + 2(a_i \cdot b_i)t + \left\| b_i \right\|^2} dt$$

In this case, the primitive for the integral was also computed using Wolfram Alpha, which returned:

$$\int \frac{t-1}{qt^2 + rt + s} dt =$$

$$= \frac{1}{2q} \log \left(qt^2 + rt + s \right) - \left(\frac{2q + r}{q\sqrt{4sq - r^2}} \right) \arctan \left(\frac{2qt + r}{\sqrt{4sq - r^2}} \right)$$

Again, we have to substitute:

$$q = \left\| a_i \right\|^2, \quad r = 2(a_i \cdot b_i), \quad s = \left\| b_i \right\|^2$$

and integrate on [0,1].

The algorithm to compute these values can be seen in Algorithm 1, in *Annex II*.

2.3.2 A three-dimensional cage (d=3)

In this section the cage provided will be three-dimensional, therefore, all the formulas need to be adapted accordingly, starting with the expression of the fundamental solution to the Laplace equation (see eq. (2.3)), which in this case is:

$$G(\xi,\eta) = -\frac{1}{4\pi\|\xi-\eta\|}$$

Now the faces are triangles instead of edges, so we need three vertices to determine a face.

We will start with the coefficient ψ_i, which according to (2.9) is:

$$\psi_i(\eta) = -\int_{\xi\in t_i} G(\xi,\eta)d\sigma$$

Denote by v_i, v_{i+1}, v_{i+2} the vertices that form the face t_i, and let p be the orthogonal projection of the point η onto the plane defined by t_i. With this new point we can split t_i into three triangles, denoted by $\Delta_1 = (p, v_i, v_{i+1})$, $\Delta_2 = (p, v_{i+1}, v_{i+2})$ and $\Delta_3 = (p, v_{i+2}, v_i)$.

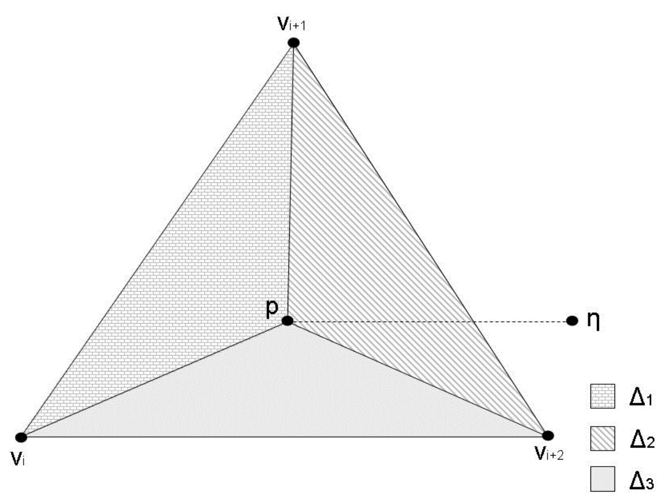


Figure 2.3: Triangle t_i formed of the vertices v_i, v_{i+1}, v_{i+2}

As p is the orthogonal projection of η, Pitagoras Theorem yields, for $\xi \in t_i$:

$$\|\xi-\eta\| = \sqrt{\|\eta-p\|^2 + \|p-\xi\|^2}$$

Since $\|\eta-p\|^2$ is constant, we can denote it by c.

We will assume that $p \in t_i$ (as in Figure 2.3) and later on we will argue that it doesn't really matter where the projection of η lies.

We will calculate the integral of ψ_i by splitting the area of integration into the three triangles $\Delta_1, \Delta_2, \Delta_3$. We will start with triangle Δ_1, and once we have an expression for it, we will be able to apply it to the other two triangles.

Let's denote two angles of Δ_1 by $\alpha = \measuredangle(pv_iv_{i+1})$ and $\beta = \measuredangle(v_{i+1}pv_i)$ and change to polar coordinates with origin at p. A picture can be seen in Figure 2.4.

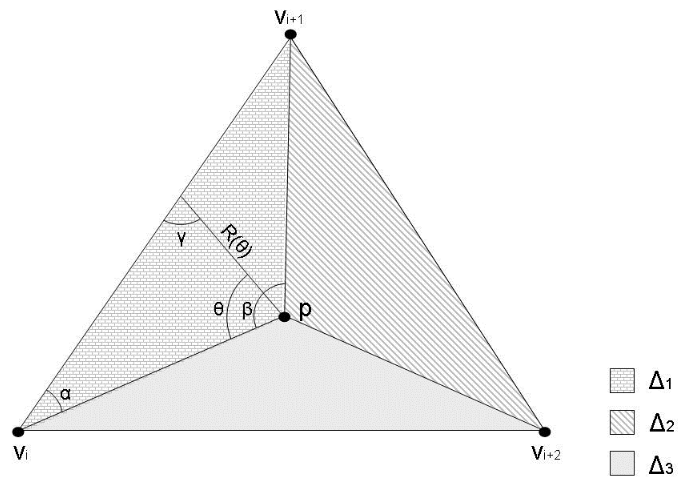


Figure 2.4: Denoted angles in Δ_1 and the change to polar coordinates

The integral on Δ_1 is

$$\int_{\xi\in\Delta_1} G(\xi,\eta)d\sigma = -\frac{1}{4\pi}\int_{\xi\in\Delta_1}\frac{1}{\sqrt{c+\|p-\xi\|^2}}d\sigma =$$
$$= -\frac{1}{4\pi}\int_{\theta=0}^{\beta}\int_{r=0}^{R(\theta)}\frac{r}{\sqrt{c+r^2}}drd\theta = -\frac{1}{4\pi}\left(\int_{\theta=0}^{\beta}\sqrt{c+R(\theta)^2}d\theta\right)+\frac{\sqrt{c}\beta}{4\pi}$$

From the law of sines we can find $R(\theta)$ explicitly:

$$\frac{R(\theta)}{\sin(\alpha)} = \frac{\|pv_i\|}{\sin(\gamma)} = \frac{\|pv_i\|}{\sin(\pi-\alpha-\theta)} \Rightarrow R(\theta) = \frac{\|\overrightarrow{pv_i}\|\sin(\alpha)}{\sin(\pi-\alpha-\theta)}$$

Before substituting $R(\theta)$ in the integral above, let's add some notation to make it clearer:
Denote by $\lambda = \|\overrightarrow{pv_i}\|^2\sin^2(\alpha)$ and $\delta=\pi-\alpha$, and apply a translation over the parameter θ ($\varphi=\delta-\beta+\theta$). Then:

$$\int_{\theta=0}^{\beta}\sqrt{c+R(\theta)^2}d\theta = \int_{\varphi=\delta-\beta}^{\delta}\sqrt{c+\frac{\lambda}{\sin^2(\varphi)}}d\varphi$$

Using Wolfram Alpha we get the primitive for this last integral:

$$\int\sqrt{c+\frac{a}{\sin^2(t)}}dt =$$
$$= \frac{\sqrt{2}\sin(t)\sqrt{c+a\csc^2(t)}}{\sqrt{-2a-c+c\cos(2t)}}\left[\sqrt{a}\arctan\left(\frac{\sqrt{2}\sqrt{a}\cos(t)}{\sqrt{-2a-c+c\cos(2t)}}\right)+\sqrt{c}\log\left(\sqrt{2}\sqrt{c}\cos(t)+\sqrt{-2a-c+c\cos(2t)}\right)\right]$$

The same procedure yields the integrals over Δ_2 and Δ_3, finishing thus the whole integral.

In case the projection of η was to lie outside of the triangle t_i notice that, if the triangles are oriented the same way (in this case, we will suppose it's a clockwise order), the integral on the areas out of the triangle t_i cancel, as they will be covered once on each orientation. An example can be seen in Figure 2.5.

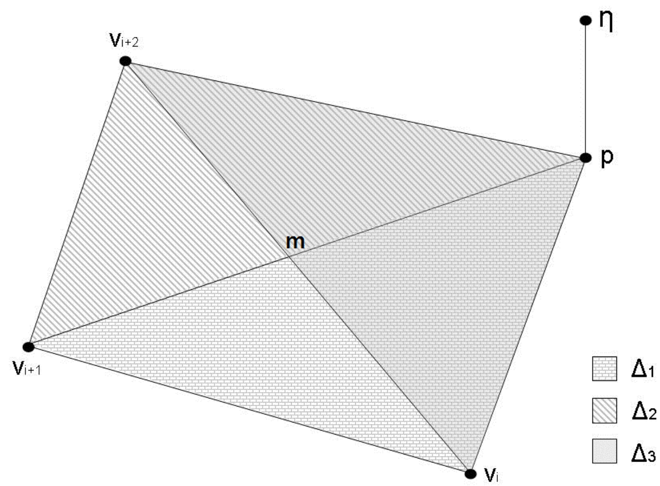


Figure 2.5: Case where $p \notin t_i$. The triangle $\Delta_1 = (p, v_i, v_{i+1})$ is oriented negatively, and so is the triangle $\Delta_2 = (p, v_{i+1}, v_{i+2})$; however, the triangle $\Delta_3 = (p, v_{i+2}, v_i)$ is oriented positively. This fact makes the integral over the triangle outside t_i to be exactly 0, because we integrate twice over that area with opposite orientations. Moreover, the edge (m, v_{i-1}) is also covered twice with opposite orientations. Therefore, all that is left, is the integral over the triangle t_i.

This makes us notice that it's very important to keep these triangles in the right orientation. We will emphasize this in the resulting final formula by adding the sign of the orientation ($sign(\Delta_k)$):

$$\int_{\xi \in t_i} G(\xi, \eta) d\sigma = \sum_{k=1}^{3} sign(\Delta_k) \int_{\xi \in \Delta_k} G(\xi, \eta) d\sigma$$

Now we can derive formulas for the coefficients $\phi_i(\eta)$. We do so by reusing the formulas for $\psi_i(\eta)$. For this purpose, we will need to define some notation. First, we will define the set of vertices $A\{v_i\}$ consisting of those which are connected by an edge to the vertex v_i. As the mesh of the cage is closed and every triangle in the mesh has exactly one neighbour triangle through each edge, we can form a set of continuous triangles ordered in sequence and linked by the consecutive edges. For each of these triangles we form a tetrahedron by adding the point η and building the three extra faces from the triangle to η. An example can be seen in Figure 2.6.

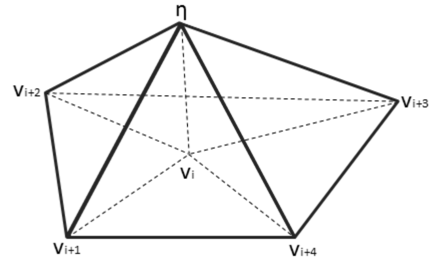


Figure 2.6: Example of polyhedron formed of the tetrahedrons that can be created from each of the triangles and the point η. In this case, $A\{v_i\} = \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$, so there are four triangles on the base of the polyhedron and four more tetrahedrons. Each tetrahedron shares a face with the neighbour tetrahedrons.

Denote $m = \#A\{v_i\}$ and $\Lambda = \bigcup\limits_{\Upsilon \in \aleph} \partial\Upsilon$, where $\aleph$ is the set of tetrahedrons. It will be convenient to name the tetrahedrons. Let's denote by $\Upsilon_{t_{i+k}}$ the tetrahedron that has $t_{i+k} = (v_i, v_{i+k+1}, v_{i+k+2})$ as its base ($k \in \{0, \ldots, m-1\}$). In the example shown in Figure 2.6, the four tetrahedrons would be denoted and formed as:

$$\begin{aligned}
\Upsilon_{t_i} &= (v_i, v_{i+1}, v_{i+2}, \eta), \quad \text{with base } t_i \\
\Upsilon_{t_{i+1}} &= (v_i, v_{i+2}, v_{i+3}, \eta), \quad \text{with base } t_{i+1} \\
\Upsilon_{t_{i+2}} &= (v_i, v_{i+3}, v_{i+4}, \eta), \quad \text{with base } t_{i+2} \\
\Upsilon_{t_{i+3}} &= (v_i, v_{i+4}, v_{i+1}, \eta), \quad \text{with base } t_{i+3}
\end{aligned}$$

We use Λ as area of integration for Green's Third Identity, using the coordinates functions $u_j(\xi) = \xi_j$, $j \in \{1, 2, 3\}$:

$$\eta_j = \int_\Lambda \xi_j \frac{\partial G}{\partial n} d\sigma - \int_\Lambda G n_j d\sigma,$$

and in vectorial form:

$$\eta = \int_\Lambda \xi \frac{\partial G}{\partial n} d\sigma - \int_\Lambda G n d\sigma$$

There is no restriction in assuming that $\eta = 0$. This will let us abuse the notation and identify the vertices v_i with the vectors $\vec{v_i}$.

From the expression above we get:

$$\int_\Lambda \xi \frac{\partial G}{\partial n} d\sigma = \int_\Lambda G n d\sigma = \sum_{k=0}^{m-1} \int_{\partial \Upsilon_{t_{i+k}}} \xi \frac{\partial G}{\partial n} d\sigma \tag{2.16}$$

All the interior triangles (oriented clockwise) will lead to cancelled integrals. In order to check this, we will split the set Λ in sets of two faces:

$$\begin{cases} \delta_k = \{triangle\ \Delta \in \Lambda \mid v_{i+k}, \eta \in \Delta\}, k \in \{1, \ldots, m\} \\ \bar{\delta}_k = \{triangle\ \Delta \in \Lambda \mid \overrightarrow{v_{i+k}v_{i+k+1}} \in \Delta\}, k \in \{1, \ldots, m\} \end{cases}$$

The integral can now be rewritten as:

$$\sum_{r=0}^{m-1}\int_{\Upsilon_{t_{i+r}}}\xi\frac{\partial G}{\partial n}d\sigma=\sum_{k=1}^{m}\int_{\Delta\in\delta_k}\xi\frac{\partial G}{\partial n}d\sigma+\sum_{k=1}^{m}\int_{\Delta\in\bar{\delta}_k}\xi\frac{\partial G}{\partial n}d\sigma$$

Each δ_k includes two equal triangles with identic normals pointing in opposite directions; let's denote these triangles by $\delta_k=\{\Delta_k^{in},\bar{\Delta}_k^{in}\}$. Consequently:

$$\begin{aligned}\forall k\in\{1\ldots m\},\quad \int_{\Delta\in\delta_k}\xi\frac{\partial G}{\partial n}d\sigma &= \int_{\Delta_k^{in}}\xi\frac{\partial G}{\partial n}d\sigma+\int_{\bar{\Delta}_k^{in}}\xi\frac{\partial G}{\partial n}d\sigma=\\ &=\int_{\Delta_k^{in}}\xi\nabla G\ nd\sigma+\int_{\Delta_k^{in}}\xi\nabla G\ (-n)d\sigma=0\end{aligned}$$

Moreover, each $\bar{\delta}_k$ is formed of a triangle t_{i+k-1} from the base ($k\in\{1,\ldots,m\}$), that is, a triangle from $N\{v_i\}$, and another triangle that we will denote by $\bar{\Delta}_{i+k}$, which shares with t_{i+k-1} the two vertices that are not v_i (and includes η as well). Each set of these two triangles will be indexed differently; in fact, the triangles $\bar{\Delta}_{i+k}$ have the index of the base triangle t_{i+k-1} plus 1, as one can see. We do this because these indexes will we repeated many times in the explanation below and writing $i+k-1$ would make the expressions to be longer.

If we consider every $\bar{\delta}_k$ separately and multiply the whole integral on $\bar{\delta}_k$ by $\dfrac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k})\cdot v_i}$, we get that $\forall k\in\{1,\ldots,m\}$:

$$\int_{\Delta\in\bar{\delta}_k}\xi\frac{\partial G}{\partial n}d\sigma=\int_{t_{i+k-1}}\xi\frac{\partial G}{\partial n}d\sigma+\int_{\bar{\Delta}_{i+k}}\xi\frac{\partial G}{\partial n}d\sigma \tag{2.17}$$

$$\frac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k})\cdot v_i}\int_{\Delta\in\bar{\delta}_k}\xi\frac{\partial G}{\partial n}d\sigma=\frac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k})\cdot v_i}\cdot\int_{t_{i+k-1}}\xi\frac{\partial G}{\partial n}d\sigma+\frac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k})\cdot v_i}\cdot\int_{\bar{\Delta}_{i+k}}\xi\frac{\partial G}{\partial n}d\sigma \tag{2.18}$$

Since $\eta=0$, all $\xi\in\bar{\Delta}_{i+k}$ are actually vectors in the plane defined by the triangle. For this reason, $\xi\cdot n(\bar{\Delta}_{i+k})=0$ and therefore

$$\frac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k})\cdot v_i}\int_{\bar{\Delta}_{i+k}}\xi\frac{\partial G}{\partial n}d\sigma=\int_{\bar{\Delta}_{i+k}}\left(\frac{n(\bar{\Delta}_{i+k})\cdot\xi}{n(\bar{\Delta}_{i+k})\cdot v_i}\right)\frac{\partial G}{\partial n}d\sigma=0$$

So all that's remaining is the first integral on the right hand side of (2.18). We now express the points ξ in barycentric coordinates: $\xi=\sum_{r=1}^{3}\Gamma_{i_r}(\xi)v_{i_r}$ where i_r are the ordered indexes $\{i,i+k,i+k+1\}$ (as $\{v_i,v_{i+k},v_{i+k+1}\}$ are the vertices of the corresponding face t_{i+k-1} from the base of the polyhedron):

$$\begin{aligned}&\frac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k})\cdot v_i}\cdot\int_{t_{i+k-1}}\xi\frac{\partial G}{\partial n}d\sigma=\\ &=\frac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k})\cdot v_i}\cdot\left(v_i\int_{t_{i+k-1}}\Gamma_i(\xi)\frac{\partial G}{\partial n}d\sigma+v_{i+k}\int_{t_{i+k-1}}\Gamma_{i+k}(\xi)\frac{\partial G}{\partial n}d\sigma+v_{i+k+1}\int_{t_{i+k-1}}\Gamma_{i+k+1}(\xi)\frac{\partial G}{\partial n}d\sigma\right)\end{aligned} \tag{2.19}$$

As before, since $\eta=0$, the vertices v_{i+k},v_{i+k+1} are in the plane defined by $\bar{\Delta}_{i+k}$; hence, the dot-product by the normal of the face they are in vanishes:

$$n(\bar{\Delta}_{i+k}) \cdot v_{i+k} = 0, \quad n(\bar{\Delta}_{i+k}) \cdot v_{i+k+1} = 0$$

We have to do a remark to make things clearer. The set of vertices connected by an edge to v_i and the edges connecting the vertices in this set form a 1-ring. There will be a moment when $k+1$ is bigger than the number of vertices. In this case, we will have to close the circle grabbing the first vertex again.

Finally, we get:

$$\frac{n(\bar{\Delta}_{i+k}) \cdot v_i}{n(\bar{\Delta}_{i+k}) \cdot v_i} \int_{t_{i+k-1}} \Gamma_i(\xi) \frac{\partial G}{\partial n} d\sigma = \int_{t_{i+k-1}} \Gamma_i(\xi) \frac{\partial G}{\partial n} d\sigma = \frac{n(\bar{\Delta}_{i+k}) \cdot \left(\int_{t_{i+k-1}} \xi \frac{\partial G}{\partial n} d\sigma \right)}{n(\bar{\Delta}_{i+k}) \cdot v_i}$$

And we can substitute the inner integral by (2.16), denoting by $\Delta_1^k, \Delta_2^k, \Delta_3^k$ the triangles in each tetrahedron which are not t_{i+k-1}, thus, those triangles that share the vertex η as one of its vertices ($\Delta_r^k \in \Upsilon_{t_{i+k}}$, $\forall r \in \{1, 2, 3\}$):

$$\int_{t_{i+k-1}} \Gamma_i(\xi) \frac{\partial G}{\partial n} d\sigma = \frac{n(\bar{\Delta}_{i+k})}{n(\bar{\Delta}_{i+k}) \cdot v_i} \cdot \left(\sum_{r=1}^{3} n(\Delta_r^k) \int_{\Delta_r^k} G d\sigma + n(t_{i+k-1}) \int_{t_{i+k-1}} G d\sigma \right)$$

Looking at these integrals closely we see that they actually represent the same calculation that we did to find the coefficient $\psi_i(\eta)$, so we already have a procedure to extract a final closed-form formula for $\phi_i(\eta)$. Then, all we have to do is substituting and ordering the vertices of the triangles adequately.

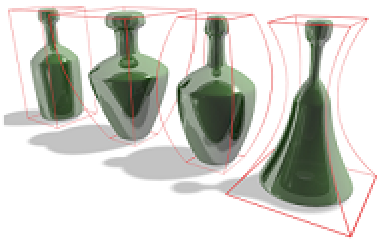
A pseudocode algorithm for computing these coefficients can be found in Algorithm 2 in *Annex II.*

Final remark: When we were building the polyhedron used to obtain a formula for $\phi_i(\eta)$ we implicitly assumed that η was not in the plane defined by t_{i+k-1}. In case it was, the formula would have suffered a drastic simplification, as $\dfrac{\partial G}{\partial n} = 0$, because the Green function is radial with its center at the pole, so each radial direction is orthogonal to the normal. Therefore:

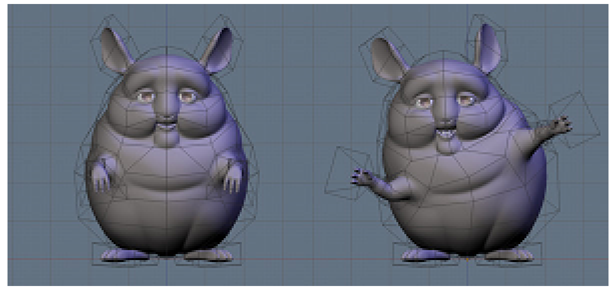
$$\int_{t_{i+k-1}} \Gamma_{i+k} \frac{\partial G}{\partial n} d\sigma = 0, \quad \forall k = 0, 1, 2$$

2.4 Extension to the exterior of the cage

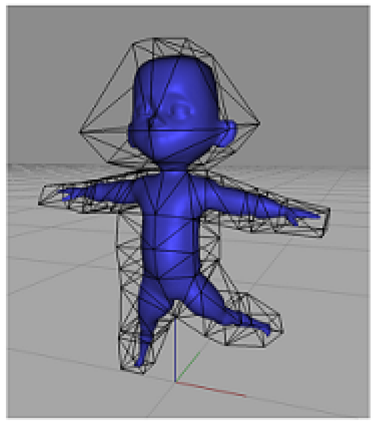
Up to now, all the points considered were located in the interior of the cage P. This is normally all what is necessary to calculate the coordinates due to the fact that the cage is most often built to enclose the finer mesh, as it can be seen in the following examples.



(a) Image of FFD taken from [5]



(b) Image of FFD taken from [6]



(c) Image of FFD taken from [7]

Figure 2.7: Examples of cages enclosing the mesh to be controlled/animated

However, under some circumstances, the artist (the person who generated the interior mesh and the one who will build the animation afterwards) may be interested in setting a cage that is closer to the mesh in some areas, even intersecting with it. In this situation, there will be points outside the cage and we would like to find coordinates for those points as well. In this chapter, we will exlain a strategy to assign coordinates to exterior points by extending the coordinates in the interior.

By looking at the definition of the coordinates in (2.9), and bearing in mind that to calculate the coordinates of a point η we place the pole of the Green function in that very point, we can see that if $\eta \in \partial P = \mathbb{T}$, the coordinates are going to have jump discontinuities along the edge, including the pole. Therefore, we have to find another way to determine the coordinates for the boundary points as well.

If we proofread the properties in the Properties section in this chapter, we see that the linear reproduction is not fulfilled, in fact, we have the following:

Lemma 2.6. *For $\eta \in P^{ext}$,*

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) = 0 \tag{2.20}$$

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) = 0 \tag{2.21}$$

Proof. As we saw when deriving Green's Third Identity earlier in this chapter,

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) = u(\eta) = \int_{\partial P} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\sigma$$

where ∂P is the boundary of the cage (the piecewise linear curve/surface), and $u(\xi) = \xi$.

Bringing back Green's Second Identity, which was used to derive the third identity; and using the fact that in this case $G(\xi, \eta) \in Har(P)$ because $\eta \notin P$, we get (u was already defined to be harmonic in P):

$$\int_{\partial P} \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) d\sigma = \int_{P^{in}} (u \Delta G - G \Delta u) \, dV = 0$$

Therefore, equation (2.20) follows.

For the second statement we will use the translation invariance property. We will translate the origin by a constant vector $c \in \mathbb{R}^d$ making sure that $\eta + c \notin P^{in}$. Using equation (2.20) again with the translated set of points we get:

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta + c)(v_i + c) + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta + c) n(t_j) = 0$$

where we have to note that translating the vertices of the face leaves the normal vector invariable. Now, due to the translation invariance property, $\phi_i(\eta) = \phi_i(\eta + c)$ and $\psi_j(\eta) = \psi_j(\eta + c)$, what produces

$$\sum_{i \in I_{\mathbb{V}}} \phi_i(\eta)(v_i + c) + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) = 0$$

By substracting equation (2.20) to the latter expression we get (2.21), which is the second statement we wanted to show.

Extension through a face

In this section we will describe how each coordinate can be extended through a face t_l (where $l \in I_{\mathbb{T}}$); that is, how we can continuously determine the coordinates of the boundary or the exterior points as η approaches ∂P. In order to do so, we introduce some notation: let $i_1, \ldots, i_d$ be the indexes of the vertices forming the face t_l.

Our starting point is the linear expression of the map $F(\eta; P)$ (2.1). By isolating the coordinates of the vertices and the normal of t_l, which are $\phi_{i_1}(\eta), \ldots, \phi_{i_d}(\eta), \psi_l(\eta)$, we obtain:

$$\eta - \sum_{\substack{i \neq i_1, \ldots, i_d \\ i \in I_{\mathbb{V}}}} \phi_i(\eta) v_i - \sum_{\substack{j \neq l \\ j \in I_{\mathbb{T}}}} \psi_j(\eta) n(t_j) = \sum_{k=1}^{d} \phi_{i_k}(\eta) v_{i_k} + \psi_l(\eta) n(t_l) \tag{2.22}$$

And isolating also the coordinates of the vertices of t_l from the equation of the translation invariance property (see (2.10)), we get:

$$1 - \sum_{\substack{i \neq i_1, \ldots, i_d \\ i \in I_{\mathbb{V}}}} \phi_i(\eta) = \sum_{k=1}^{d} \phi_{i_k}(\eta) \tag{2.23}$$

Equations (2.22) and (2.23) can be seen as a linear system with unknowns $\phi_{i_1}(\eta), \ldots, \phi_{i_d}(\eta), \psi_l(\eta)$. If the system is invertible, it allows to define the values (in a unique way).

Lemma 2.7. *The linear system (2.22), (2.23) with unknowns $\phi_{i_1}(\eta), \ldots, \phi_{i_d}(\eta)$ and $\psi_l(\eta)$ is non-singular.*

Proof. If there existed a non-zero vector $w = (w_1, \ldots, w_{d+1})$ in the kernel of the system, it would fulfill that $\sum_{k=1}^{d} w_k = 0$ because of equation (2.23). Here we can isolate w_1:

$$w_1 = -\sum_{k=2}^{d} w_k \tag{2.24}$$

From equation (2.22) we get:

$$0 = \sum_{k=1}^{d} w_k v_{i_k} + w_{d+1} n(t_l)$$

If we substitute the expression (2.24), we finally obtain:

$$0 = \sum_{k=2}^{d} w_k (v_{i_k} - v_{i_1}) + w_{d+1} n(t_l)$$

As $v_{i_k}, k = 2, \ldots, d$ are vertices of the face t_l, the straight lines $\{v_{i_1} + \lambda_k (v_{i_k} - v_{i_1})\}$ (where $\lambda_k \in \mathbb{R}$) are in the hyperplane defined by the face t_l. Hence, the vectors $(v_{i_k} - v_{i_1})$ are orthogonal to the normal of the face t_l. Conclusively, they are linearly independent and therefore

$$w_2 = \ldots = w_d = w_{d+1} = 0 \overset{(2.24)}{\Rightarrow} w_1 = 0$$

Then, the kernel of the system is solely formed of the vector 0, and thus the system must be invertible. ☒

As we previously noted, this system provides a way to reproduce the coordinates of the normal of a face t_l and those of its vertices when the pole η is located in the interior of the cage. However, we can observe that if we make the point η get close to the boundary of the cage and even go through the face t_l, this definition also provides a smooth extension of the components related to the face t_l (the vertices and the normal) when η goes through that face. The other components of the coordinates vector can be calculated using the method we already had for interior points. A picture can be seen in Figure 2.8.

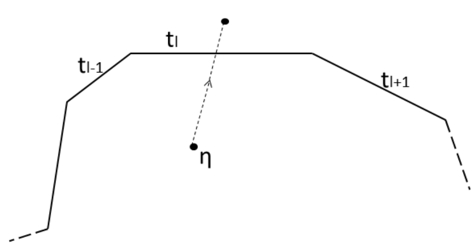


Figure 2.8: The point η goes to the exterior of the 2D cage through the face t_l, so the coordinates related to that face are obtained using the system of equations previously proposed

In order to distinguish between the coordinates given as the solution to the linear system from the previous ones, from now on we will denote them by $\tilde{*}$. Notice that $\tilde{\phi}_i(\eta) = \phi_i(\eta)$ and $\tilde{\psi}_j(\eta) = \psi_j(\eta)$ when $\eta \in P^{in}$.
Let's now try to find a simplified expression for the system (2.22), (2.23).
By Lemma 2.6, if $\eta \in P^{ext}$, and we isolate all what refers to the face t_l (vertices forming it and normal) we get:

$$\sum_{\substack{i\neq i_1,\ldots,i_d\\ i\in I_{\mathbb{V}}}} \phi_i(\eta)v_i + \sum_{\substack{j\neq l\\ j\in I_{\mathbb{T}}}} \psi_j(\eta)n(t_j) = -\sum_{k=1}^{d}\phi_{i_k}(\eta)v_{i_k} - \psi_l(\eta)n(t_l)$$

$$\sum_{\substack{i\neq i_1,\ldots,i_d\\ i\in I_{\mathbb{V}}}} \phi_i(\eta) = -\sum_{k=1}^{d}\phi_{i_k}(\eta)$$

As the components of the other faces remain the same, we can substitute this into the equations of the system (2.22) and (2.23). We get the following:

$$\eta + \sum_{k=1}^{d}\phi_{i_k}(\eta)v_{i_k} + \psi_l(\eta)n(t_l) = \sum_{k=1}^{d}\tilde{\phi}_{i_k}(\eta)v_{i_k} + \tilde{\psi}_l(\eta)n(t_l)$$

$$1 + \sum_{k=1}^{d}\phi_{i_k}(\eta) = \sum_{k=1}^{d}\tilde{\phi}_{i_k}(\eta)$$

If we group by vertices and normals:

$$\eta = \sum_{k=1}^{d}\left(\tilde{\phi}_{i_k}(\eta) - \phi_{i_k}(\eta)\right)v_{i_k} + \left(\tilde{\psi}_l(\eta) - \psi_l(\eta)\right)n(t_l) \tag{2.25}$$

$$1 = \sum_{k=1}^{d}\left(\tilde{\phi}_{i_k}(\eta) - \phi_{i_k}(\eta)\right) \tag{2.26}$$

Let's denote

$$\alpha_k = \tilde{\phi}_{i_k}(\eta) - \phi_{i_k}(\eta)$$
$$\beta = \tilde{\psi}_l(\eta) - \psi_l(\eta)$$

And if we substitute these into (2.25) and (2.26) we finally get:

$$\begin{aligned} \eta &= \textstyle\sum_{k=1}^{d}\alpha_k v_{i_k} + \beta n(t_l) \\ 1 &= \textstyle\sum_{k=1}^{d}\alpha_k \end{aligned} \tag{2.27}$$

If we consider α_k and β to be unknown and we solved the last system to find them, we would only have to substitute here

$$\tilde{\phi}_{i_k}(\eta) = \phi_{i_k}(\eta) + \alpha_k, \quad k = 1\ldots d \tag{2.28}$$

$$\tilde{\psi}_l(\eta) = \psi_l(\eta) + \beta \tag{2.29}$$

and get the values for the new coordinates components. This way, we split the calculations in two, making it easier to compute.

From the first equation in (2.27) we see that η is expressed as an affine sum of the vertices in the face t_l in barycentric coordinates (the second equation guarantees that $\sum_{k=1}^{d} \alpha_k = 1$) and the normal of the face. We can also interpret this as the unique affine coordinates of the point η in the simplex defined by the points $v_{i_1}, \dots, v_{i_d}, v_{i_1} + n(t_l)$. We can reformulate the expression for the coordinates related to t_l by adding and substracting βv_{i_1} to the first equation in (2.27), and denote this expression by $L_l(\eta; P)$:

$$L_l(\eta; P) = (\alpha_1 - \beta)v_{i_1} + \sum_{k=2}^{d} \alpha_k v_{i_k} + \beta(v_{i_1} + n(t_l)) \tag{2.30}$$

Bringing back (2.1) and adding (2.30) we get:

$$\eta = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) n(t_j) + \sum_{k=1}^{d} \alpha_k v_{i_k} + \beta n(t_l) = F(\eta; P) + L_l(\eta; P)$$

The resulting map when we animate the cage will be:

$$\eta \mapsto \tilde{F}(\eta; P') = \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j) + \sum_{k=1}^{d} \alpha_k v'_{i_k} + \beta s_l n(t'_l) \tag{2.31}$$

Likewise the case of the expression for interior points, we have to proofread the properties this new map fulfills. We will also do it depending on the dimension the cage is enclosed in.

Properties of the case d=2

Before checking whether the map is still holomorphic we have to find a concrete expression for it. In this case, we see that the system of equations (2.27) is equivalent to:

$$\eta = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \beta n(t_l) \tag{2.32}$$

$$1 = \alpha_1 + \alpha_2 \tag{2.33}$$

From the second equation, we see that $\alpha_1 = 1 - \alpha_2$, and thus simplify the linear map to:

$$\begin{aligned}\tilde{F}(\eta; P') &= \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j) + (1-\alpha_2) v'_{i_1} + \alpha_2 v'_{i_2} + \beta s_l n(t'_l) = \\ &= \sum_{i \in I_{\mathbb{V}}} \phi_i(\eta) v'_i + \sum_{j \in I_{\mathbb{T}}} \psi_j(\eta) s_j n(t'_j) + v'_{i_1} + \alpha_2 (v'_{i_2} - v'_{i_1}) + \beta s_l n(t'_l)\end{aligned}$$

The values for α_2 and β can be found through the first equation in (2.32) and (2.33):

$$\eta = (1 - \alpha_2) v_{i_1} + \alpha_2 v_{i_2} + \beta n(t_l) = v_{i_1} + \alpha_2 (v_{i_2} - v_{i_1}) + \beta n(t_l) \Rightarrow$$

$$\Rightarrow \eta - v_{i_1} = \alpha_2 (v_{i_2} - v_{i_1}) + \beta n(t_l) \tag{2.34}$$

Since $(v_{i_2} - v_{i_1}) \cdot n(t_l) = 0$, equation (2.34) gives the value of α_2:

$$(\eta - v_{i_1}) \cdot (v_{i_2} - v_{i_1}) = \alpha_2 \|v_{i_2} - v_{i_1}\|^2 \Rightarrow \alpha_2 = \frac{(\eta - v_{i_1}) \cdot (v_{i_2} - v_{i_1})}{\|v_{i_2} - v_{i_1}\|^2}$$

Similarly,

$$(\eta - v_{i_1}) \cdot n(t_l) = \beta \|n(t_l)\|^2 \overset{\|n(t_l)\|=1}{=} \beta$$

We already know by Theorem 2.2 that (2.2) is holomorphic when $\eta \in P^{in}$. Nevertheless, this theorem can also be used when $\eta \in P^{ext}$. Therefore, that this part of the mapping is holomorphic. Now, we have to check that the new term is holomorphic too.

Lemma 2.8. *$L_l(\eta; P)$, for $\eta \in P^{ext}$, is the unique linear holomorphic mapping taking the edge $\overrightarrow{v_{i_1}v_{i_2}}$ to the edge $\overrightarrow{v'_{i_1}v'_{i_2}}$.*

Proof. Let's check first that $L_l(\eta; P')$ takes the edge $\overrightarrow{v_{i_1}v_{i_2}}$ to $\overrightarrow{v'_{i_1}v'_{i_2}}$. We will do this by calculating the images of the points v_{i_1} and v_{i_2}.
Let $\eta = v_{i_1}$. Then,

$$\begin{aligned} \alpha_2 &= \frac{(v_{i_1} - v_{i_1}) \cdot (v_{i_2} - v_{i_1})}{\|v_{i_2} - v_{i_1}\|^2} &&= 0 \\ \beta &= (v_{i_1} - v_{i_1}) \cdot n(t_l) &&= 0 \end{aligned}$$

and therefore $L_l(v_{i_1}; P') = v'_{i_1}$.
Let $\eta = v_{i_2}$:

$$\begin{aligned} \alpha_2 &= \frac{(v_{i_2} - v_{i_1}) \cdot (v_{i_2} - v_{i_1})}{\|v_{i_2} - v_{i_1}\|^2} &&= 1 \\ \beta &= (v_{i_2} - v_{i_1}) \cdot n(t_l) &&= 0 \end{aligned}$$

Then, the maps results into $L_l(v_{i_1}; P') = v'_{i_1} + (v'_{i_2} - v'_{i_1}) = v'_{i_2}$.
Moreover, if we substitute the actual values of α_2 and β we see that

$$L_l(\eta; P') = v'_{i_1} + \frac{(\eta - v_{i_1}) \cdot (v_{i_2} - v_{i_1})}{\|v_{i_2} - v_{i_1}\|^2}(v'_{i_2} - v'_{i_1}) + (\eta - v_{i_1}) \cdot n(t_l)\frac{\|v'_{i_2} - v'_{i_1}\|}{\|v_{i_2} - v_{i_1}\|}n(t'_l) = \tag{2.35}$$

$$= v'_{i_1} + \frac{\|v'_{i_2} - v'_{i_1}\|}{\|v_{i_2} - v_{i_1}\|}\left[\frac{(\eta - v_{i_1}) \cdot (v_{i_2} - v_{i_1})}{\|v_{i_2} - v_{i_1}\|}\frac{(v'_{i_2} - v'_{i_1})}{\|v'_{i_2} - v'_{i_1}\|} + (\eta - v_{i_1}) \cdot n(t_l)n(t'_l)\right] \tag{2.36}$$

The previous map is holomorphic because it is linear (it depends on η and we can set the natural bijection between $\mathbb{R}^2$ and $\mathbb{C}$). It has to be unique as it has 2 degrees of freedom in $\mathbb{R}^2$, and so has a linear conformal mapping in 2D.

Now, we have to show that we actually built an analytic continuation of the linear mapping $F(\eta; P')$ through the face t_l.

Theorem 2.3. *In the case d=2, fixing an edge t_l and defining the coordinates $\tilde{\phi}_{i_k}(\eta)$, $k = 1, 2$ and $\tilde{\psi}_l(\eta)$ by (2.22) and (2.23), we get that for $\eta \in P^{ext}$ $F(\eta; P') + L_l(\eta; P')$ is the unique analytic continuation of the holomorphic mapping $F(\eta; P')$ through the face t_l.*

Proof. Given a point $\eta \in P^{ext}$ and fixing an edge t_l we know by lemma 2.7 and the system of equations (2.22), (2.23) that $\tilde{F}(\eta; P)$ is continuous through t_l. Moreover,

$$\forall \eta \in P^{in} \quad \tilde{F}(\eta; P) = F(\eta; P)$$

Therefore, due to the analytic continuation principle, the map $\tilde{F}(\eta; P)$ must be the unique analytic continuation of the map $F(\eta; P)$.

Limitations on the region of conformality:

In this last subsection we will see that we can't expect to have an analytic continuation of the coordinates map to the whole space, when *P'* is an arbitrary cage. That is, there is no entire function $\tilde{F}(\eta; P')$ such that $\tilde{F}(\eta; P') = F(\eta; P')$ for $\eta \in P^{in}$ for a general *P'*. What the authors argue is that we can place the singularities in a flexible way so that we can have a reasonable region of conformality. The following theorem will do so for the case d=2, although a similar result can be obtained for d>2.

Theorem 2.4. *1) There is no entire function $\bar{F}$ such that $\bar{F}(\eta; P') = F(\eta; P')$ for $\eta \in P^{in}$ for a general P'.*

2) Let P^{ext} be subdivided into disjoint domains O_k, $k \in K$, $P^{ext} = \bigcup_{k \in K} \bar{O}_k$ ($\bar{O}_k$ is the closure of O_k), such that for every $j \in I_{\mathbb{T}}$, t_j is contained in some $\bar{O}_k$, that is $t_j \subset \bar{O}_k$. Assume that for each $k \in K$ one extends F to O_k through a specific face $t_k \in O_k$. Then $\tilde{F}$ is analytic in $\bigcup_{k \in K} O_k$ in exception of all the faces $t_j \in O_k$ which do not satisfy $t'_j = L_k(t_j; P')$.

Proof. In order to give the proof of 1) we will assume the opposite. The, let's assume that there exists such analytic continuation $\bar{F}$.

If we look back to Theorem 2.3, we can assure that the unique analytic continuation to the exterior of the cage through the face t_j must be $\bar{F}(\eta; P') = F(\eta; P') + L_j(\eta; P')$. However, the map $\eta \mapsto F(\eta; P')$ is holomorphic (practically conformal as we argued before) due to Theorem 2.2 everywhere outside the cage ($\eta \in P^{ext}$), and since $L_j(\eta; P')$ are entire (linear) functions, we can assure by the uniquess of the analytic continuation that:

$$L(\cdot; P') \equiv L_2(\cdot; P') \equiv \ldots \equiv L(\cdot; P')$$

Consequently, all the linear transformations $L_j(\eta; P')$ coincide, what is impossible if we consider an arbitrary *P'* as we can build different linear mapping for different cages as one can see in (2.36).

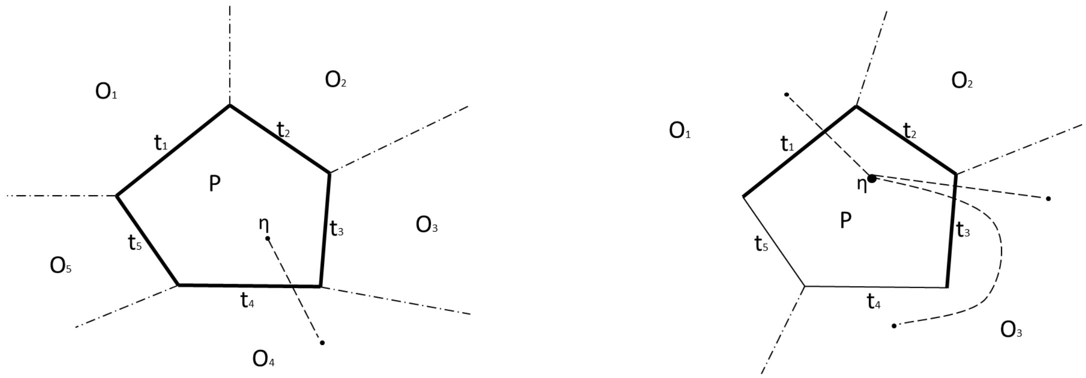
For the proof of the point 2, we will show an image of the possible subdivisions of the space so that is easier to picture.

By Theorem 2.3 we know that $\tilde{F}$ is analytic through all the faces $t_k \in O_k$, and the extensions are $\tilde{F}(\eta; P') = F(\eta; P') + L_k(\eta; P')$. Then, for all the $t_l \in O_k$ which satisfy that $t'_j = L_k(t_j; P')$, we know that the extension in O_k is analytic through t_j (due to Theorem 2.3 and Lemma 2.8).

In case d=3, we already know that we can't have conformality as it is argued in [3], therefore, we can't have continuation in the sense of real analiticity.

Theorem 2.5. *The extended coordinate functions $\tilde{\phi}_i$, $\tilde{\psi}_j$ through a face t_l are harmonic in their domain of definition.*

Proof. We saw at the beginning of this chapter that the functions ϕ_i, ψ_j are harmonic in P^{in}. The same reasoning can be used to see that they are harmonic outside the cage too. On the other hand, the functions $\tilde{\phi}_{i_k}(\eta) = \phi_{i_k}(\eta)$, $\tilde{\psi}_l(\eta) = \psi_l(\eta)$ (where $k = 1 \ldots d$) when $\eta \in P^{in}$, then, they are also harmonic in P^{in}. Moreover, if we bring back the equations for the coefficients $\tilde{\phi}_{i_k}$, $\tilde{\psi}_l$ from (2.29) (the simpler form of the



(a) Example 1 **(b)** Example 2

Figure 2.9: In a) we can see how we subdivide the exterior in as many regions as the number of edges the cage has. These edges can't intersect and the extension of the coordinates regarding every zone is done through the edge included in the area O_k. However, in b) we can see that there are less regions than the number of edges. Therefore, for every region O_k we choose the edge from which we will define the extension (thicker edges). From η crossing the chosen edge of the region O_k we can define a path towards the exterior point we want to find the coordinates of. Building a straight line crossing non-chosen edges does not guarantee continuity of the coordinates through that edge.

system of equations introduced initially), we see they are the coefficients ϕ_{i_k}, ψ_l plus the additional terms $\alpha_k = \alpha_k(\eta)$ and $\beta = \beta(\eta)$. These last terms are linear functions of the coordinates of η as it can be induced from (2.27); hence, they are also harmonic outside the cage. Since every extension is done through a face, the different coordinates (with respect to the ones we defined for the interior of the cage earlier in this chapter) are those of that face, and the rest remain the same. Therefore, the rest of the coordinates $\tilde{\phi}_i$ and $\tilde{\psi}_j$ are equal to ϕ_i and ψ_j respectively, so they are harmonic as well.

Finally, we have to note that the system of equations (see (2.22) and (2.23)) used to define the coordinates $\tilde{\phi}_i$, $\tilde{\psi}_j$, the fact that this system is invertible (Lemma 2.7) and also the fact that the coefficients expressions included in the system are C^∞ functions. Conclusively, by continuity from both sides of the face t_l we get that the coordinates functions $\tilde{\phi}_{i_k}$, $\tilde{\psi}_l$ (where $k = 1 \dots d$) are harmonic outside the cage through the face t_l.

We will get the uniqueness for these coefficient functions in the last theorem:

Theorem 2.6. *Fixing a face t_l and defining the coordinates $\tilde{\phi}_{i_k}(\eta)$, where $k = 1 \dots d$ and $\tilde{\psi}_l(\eta)$ by the system of equations* (2.22), (2.23) *results in the unique real analytic continuation of the harmonic coordinate functions ϕ_i,ϕ_j through the face t_l.*

Proof. We have seen in Theorem 2.5 that the extended coordinates $\tilde{\phi}_i$,$\tilde{\psi}_j$ are harmonic in their domain of definition. We know that being harmonic implies that they are also real analytic in that domain (d=2), because a harmonic function is locally the real part of an analytic function. The analytic continuation principle guarantees the unicity because if there was any other continuation $\bar{\phi}_i$,$\bar{\psi}_j$ it would have to fulfill that $\bar{\phi}_i(\eta) = \phi_i(\eta)$ and $\bar{\psi}_j(\eta) = \psi_j(\eta)$ for every $\eta \in P^{in}$. Therefore, $f_i(\eta) = \tilde{\phi}_i(\eta) - \bar{\phi}_i(\eta) = 0$, $g_j(\eta) = \tilde{\psi}_l(\eta) - \bar{\psi}_j(\eta) = 0$, $\forall\ \eta \in P^{in}$. However, we can only have a discrete number of zeros in the domain (no series of zeros with accumulation points); thus, $f_i = 0$ and $g_j = 0$, what implies that the two coordinates extensions should be the same, ending thus the proof.

Chapter 3

Implementation

The implementation that will be described below was done using Maya 2012. This choice was due to the fact that generating meshes with it is very simple, and the vertices, edges, faces and normals of those meshes can be accessed quite straightforward. One can gather these items either through the user interface or through scripting. Maya offers the possibility of scripting in two languages: Python and Maya Embedded Language (or MEL). I opted for the second one, as I was already familiar with it.

The case chose for the implementation was the 2D.

Before the scripting, given an initial mesh, we have to create the cage we will use to control that mesh. In Maya, all the vertices, edges and faces are assigned an index. There is a set of indexes for each item ranging from 0 to the number of items -1. If we create a general object in Maya, those indexes may not be ordered the desired way, and they cannot be changed. In order to tackle this issue, one can either store the desired index order separately or try to find a way to generate polygons (we are in 2D) forcing a certain index order. I used the option "Create Polygon Tool", because it was the only way I found to generate a polygon (the cage) where the indexes of consecutive vertices were also consecutive. This saved the work of searching for neighbour vertices each time or generating the order structure.

I created a GUI (Graphical User Interface) to call the script easily:

Figure 3.1: GUI to call the script

This window lets the user select the interior mesh or the cage (if they are named adequately: "intMesh" and "cage"). And if the user presses the button "Generar Coordenades", the algorithm to generate the coordinates in 2D will be triggered. It is based on the pseudocode given in the article, which is available in Algorithm 1 in *Annex II*.

As it was explained in chapter 2, the algorithm will generate the coordinates vector for every η given. As the mesh is formed of vertices, and moving them suffices to move everything, we only have to place the pole η in every vertex of the interior mesh.

The script will store two matrices: the first one will store the ϕ_i and the second one will store the ψ_j. There's an important limitation that MEL has. It doesn't allow to create variable-sized matrices. This means that every time that we want to control a new mesh or create a new cage, is the number or vertices changes, the code will need to be changed accordingly. It is not a severe limitation as it's a local problem of MEL, translating the code into a programming language that allows to deal with dynamic memory would solve it.

In addition to this, another vector storing the lengths of the edges of the original cage needs to be stored, as we will use them to calculate s_j.

Once the coefficients are generated, the button "Actualitzar Malla Interior" enables. Pressing it would trigger the calculations related to the linear expression $F(\eta; P')$, and thus get the resulting mesh deformation.

Results can be seen in the following chapter.

The actual script and the example used in the pictures of the Results chapter will also be uploaded.

Chapter 4

Results

The mesh used to test the algorithm can be seen below:

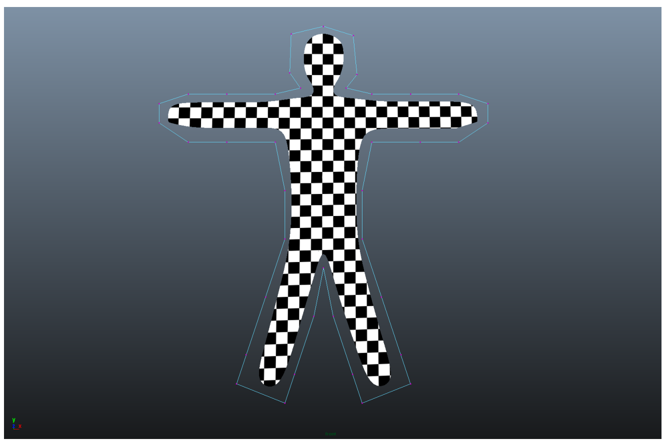


Figure 4.1: Original mesh with cage enclosing it

I textured it with a checkerboard because this will show the deformation in the interior of the mesh, thus, we will see something more than just the contour changing. The wireframe of the interior mesh (the vertices and the edges) can be seen below. In fact, this is what will de deformed, and the area inside every triangle will move accordingly.

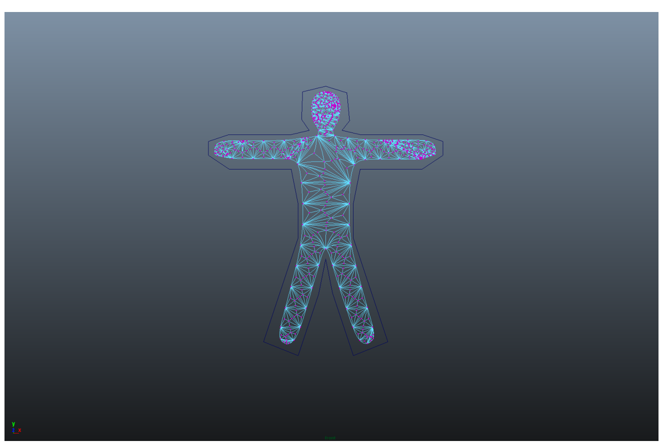
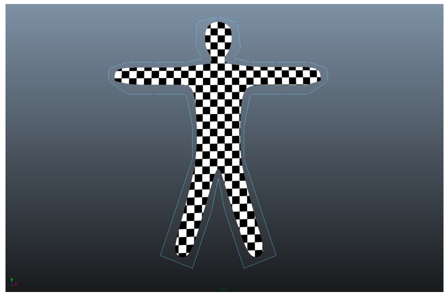
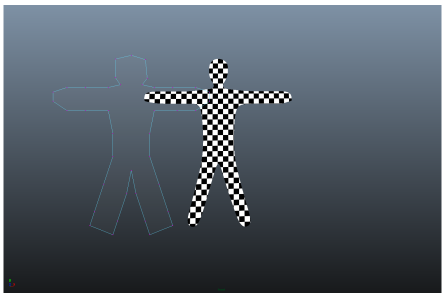


Figure 4.2: Wireframe of the interior mesh and the cage

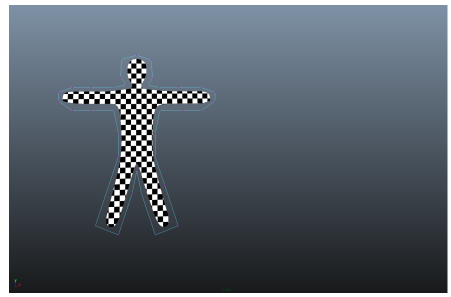
We will start testing the properties outlined and showed in chapter 2:



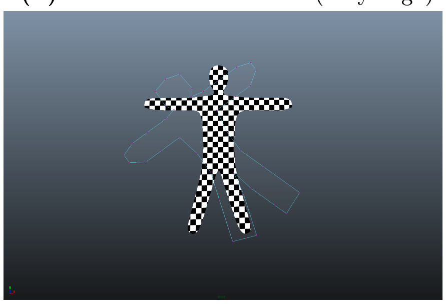
(a) Linear reproduction



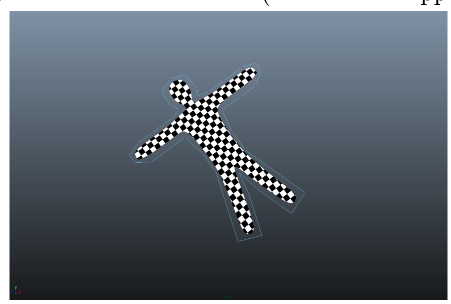
(b) Translation invariance (only cage)



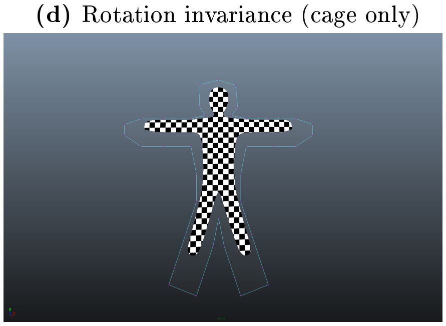
(c) Translation invariance (deformation applied)



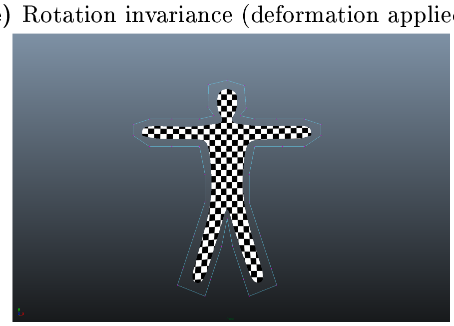
(d) Rotation invariance (cage only)



(e) Rotation invariance (deformation applied)



(f) Uniform scale (cage only)



(g) Uniform scale (deformation applied)

Figure 4.3: First four properties of the deformation. The smoothness can be seen in the checkerboard.

We can see better the deformation algorithm the paper suggests by looking at how the mesh looks when we change the mesh differently:

(a) Only the cage **(b)** Deformation applied

Figure 4.4: Non-trivial deformation

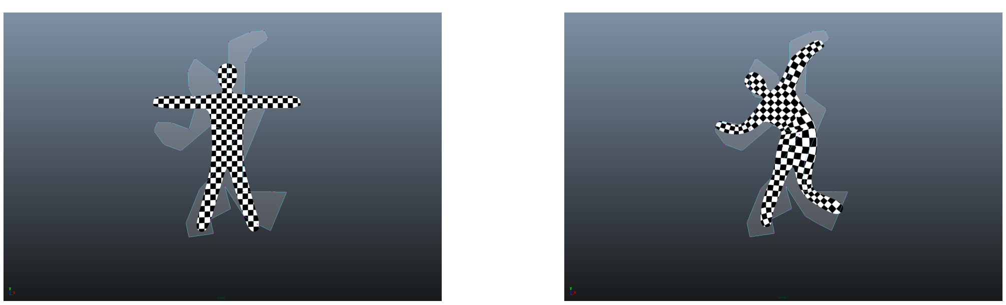
By looking at the checkerboard we can see that the shape is kept as much as it can while adapting the interior figure to the deformation of the cage.

Next, a more drastic deformation was applied to the cage, and the resulting deformation is the following:

(a) Cage only **(b)** Deformation applied

Figure 4.5: More drastic deformation

Finally, and even more drastic deformation was applied to the cage. In this case we can start detecting issues in the checkerboard as the authors foresaw.



(a) Cage only **(b)** Deformation applied

Figure 4.6: Even more drastic deformation

Chapter 5

Conclusions

The authors argue that the proposed method to deform the mesh guarantees having a quasi-conformal map. This means that the distortion carried through is bounded. However, even though they constantly say that the mapping is least-distorting, no explicit quantification of the distortion is made. The only comment on this topic is that they empirically checked that it was least-distorting. Bearing in mind that this paper has been widely spread, those empirical tests are bound to be numerous.

As it can be seen in the Results chapter, the method to control a 2D mesh deforms the interior mesh very smoothly and keeping the shape reasonably well. This fact can be observed more easily if one textures the mesh with a checkerboard, as it was done in the example.
A possible improvement over the current method could be done in this direction. The main goal would be to calculate the quasi-conformality constant[1], and thus quantify the distortion maximal bound.

This constant could also be useful to calculate the distortion of the mesh after every deformation. If we consider a maximal distortion that we can tolerate, the method could recalculate the coefficients once this is exceeded.

Reading this paper, trying to understand it and correcting the numerous errors that could be found very often was a test to patience. Nevertheless, seeing the results once the algorithm worked made it worthwhile. The process I went through has been very different from the one in the whole degree. Even though the leap needed to understand the theoretical parts was reasonable, working with a paper with so many mistakes makes one doubt about everything. It has nothing to do with studying your own notes or a book.

[1]The quasi-conformality constant is the maximum excentricity of an ellipse that is the image of a sphere. The excentricity is the ratio between the length of the semi-minor axis and the length of the semi-major axis

Chapter 6

References

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Annexes

I Green's Theorem, Green's Identities and the Fundamental solution to the Laplace equation

In this section we gather the basic results used in the article. We follow [4].

Green's Theorem

Be U a bounded domain in $\mathbb{R}^2$ with a piecewise regular boundary ∂U positively oriented, and be $\overrightarrow{T}$ the unit tangent vector to ∂U. Be $\overrightarrow{X} = (P, Q)$ a vectorial field, where the components P, Q are differentiable functions in a neighbourhood of $\bar{U}$, so that the function $rot(\overrightarrow{X}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous in $\bar{U}$. Then,

$$\int_{\partial U} < \overrightarrow{X}, \overrightarrow{T} > ds = \int_{\partial U} P dx + Q dy = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

As the Green's First Identity needs the divergence theorem, we will outline it first.

Divergence Theorem

Be U a bounded domain in $\mathbb{R}^n$, and be ∂U a regular hypersurface oriented with the unit outward normal $\overrightarrow{N}$. Be $\overrightarrow{X}$ a continuously differentiable vector field in $\bar{U}$. Then,

$$\int_{\partial U} < \overrightarrow{X}, \overrightarrow{N} > dA = \int_U (div \overrightarrow{X}) dV,$$

where $div \overrightarrow{X} = \nabla \cdot \overrightarrow{X}$.
This theorem is also valid in case ∂U is a piecewise regular hypersurface.

Green's First Identity

Be U a bounded domain in $\mathbb{R}^n$, and be ∂U a regular hypersurface oriented with the unit outward normal $\overrightarrow{N}$.
Let $u, v \in \mathcal{C}^2(\bar{U})$. Now we apply the divergenge theorem to the field $\overrightarrow{X} = u \nabla v$.
Here,

$$div \overrightarrow{X} = \nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla \cdot (\nabla v) =< \nabla u, \nabla v > +u$$

where the last step is because of the definition of the Laplace operator, so: $\nabla \cdot (\nabla v)$.
Therefore, by the divergence theorem we obtain Green's First Identity:

$$\int_{\partial U} < u \nabla v, \overrightarrow{N} > dA = \int_{\partial U} u \frac{\partial v}{\partial \overrightarrow{N}} dA = \int_U < \nabla u, \nabla v > dV + \int_U u \Delta v \; dV$$

Green's Second Identity

We will derive this identity from Green's First Identity; therefore, let's assume we are under the same conditions.

We permute the functions u and v and consider the field $\overrightarrow{Y} = v\nabla u$. Following the same calculations as before we get:

$$\int_{\partial U} < v\nabla u, \overrightarrow{N} > dA = \int_{\partial U} v\frac{\partial u}{\partial \overrightarrow{N}} dA = \int_U < \nabla u, \nabla v > dV + \int_U v\Delta u \; dV$$

By substracting this to Green's First Identity we get Green's Second Identity:

$$\int_{\partial U} \left(u\frac{\partial v}{\partial \overrightarrow{N}} - v\frac{\partial u}{\partial \overrightarrow{N}} \right) dA = \int_U (u\Delta v - v\Delta u) dV$$

The fundamental solution to the Laplace equation

In [4] we can see that this function is derived from imposing a radial function to be harmonic outside the pole. I will also follow this procedure, as it shows its radial nature and due to its simplicity as well .

A radial function has the property that the image of every point in the domain only depends on the distance from the point to the origin, that is, there exists a single variable real function $\varphi : [0, \infty) \to \mathbb{R}$ such that

$$f(x) = \varphi(\|x\|)$$

Now we force f to be harmonic:

$$\Delta f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0$$

As one can readily see, we need to calculate the terms $\dfrac{\partial^2 f}{\partial x_i^2}$, we will do it incrementally:

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}; \quad \frac{\partial f}{\partial x_i} = \varphi'(r)\frac{\partial r}{\partial x_i} = \varphi'(r)\frac{x_i}{r}$$
$$\frac{\partial^2 f}{\partial x_i^2} = \varphi''(r)\frac{x_i^2}{r^2} + \varphi\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

Let's add this to get an expression for Δf:

$$\Delta f = \varphi''(r)\left(\frac{\sum\limits_{i=1}^{n} x_i^2}{r^2}\right) + \varphi'(r)\left(\frac{\sum\limits_{i=1}^{n} 1}{r}\right) - \varphi'(r)\left(\frac{\sum\limits_{i=1}^{n} x_i^2}{r^3}\right) = \varphi''(r) + \varphi'(r)\frac{n}{r} - \varphi'(r)\frac{1}{r} = \varphi''(r) + \frac{n-1}{r}\varphi'(r)$$

Thus, the harmonicity is given by $\varphi''(r) + \frac{n-1}{r}\varphi'(r) = 0$. Then, $\varphi'(r) = Cr^{1-n}$, where C is constant, and integrating once more:

$$\varphi(r) = k_1\frac{r^{2-n}}{2-n} + k_2, \quad if \; n > 2$$
$$\varphi(r) = k_1 \log \; r + k_2, \quad if \; n = 2$$

where k_1, k_2 are constants. Both functions have a singularity in the origin $(r = 0)$.

The function

$$G(x) = \begin{cases} d_n \|x\|^{2-n} & if\ n > 2 \\ \\ d_n \log \|x\| & if\ n = 2 \end{cases}$$

is called the fundamental solution to the Laplace equation with a pole at the origin. If we translate the pole to a point $\eta \in \mathbb{R}^n$, the function $G(x - \eta)$ is called the fundamental solution to the Laplace equation with a pole in η, what we will also denote by $G(x, \eta)$. Then, this function is harmonic everywhere except for domains that include its pole.

In order to determine the value of the constant d_n, we go to the proof given for the Riesz decomposition theorem. We can see that d_n corresponds to

$$d_n = \begin{cases} \frac{1}{2\pi} & if\ n = 2 \\ \frac{1}{(2-n)c_n} & if\ n > 2 \end{cases}$$

where c_n is the (n-1)-dimensional measure of the unit sphere.

In fact, the formula we showed in the beginning of Chapter 2 (equation 2.4) is exactly a Riesz decomposition where the function u is harmonic. I will not go into the proof for the theorem as it's not the goal of this work.

We can also observe from the definition that this function has to be symmetric, because if we exchange the variable and the pole we get exacly the same function:

$$G(x, \eta) = \left\{ \begin{array}{ll} \frac{1}{(2-n)c_n} \|x\|^{2-n}, & if\ n > 2 \\ \\ \frac{1}{2\pi} \log \|x\|, & if\ n = 2 \end{array} \right\} = G(\eta, x)$$

What is most important about the fundamental solution $G(\xi, \eta)$ is that $\Delta_\xi G(\xi, \eta) = \delta_\eta$ in the sense of distributions. This means by definition that

$$\forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}^d) \quad < \Delta G, \psi >= \int G(\xi, \eta)\ \Delta\psi(\xi)\ dV(\xi) = \psi(\eta) =< \delta_\eta, \psi >$$

II Explicit Integrals and Algorithms in pseudocode

Integral 1

The overall strategy will be integrating by parts and reducing the integral to one of a rational function. Previous to this integral we will do some calculations with the polynomial involved in it:

$$qt^2 + rt + s = q\left(t^2 + \frac{r}{q}t + \frac{s}{q}\right) = q\left[\left(t + \frac{r}{2q}\right)^2 + \left(\frac{s}{q} - \frac{r^2}{4q^2}\right)^2\right] = q\left(s^2 + A^2\right)$$

where

$$s = t + \frac{r}{2q} \quad and \quad A = \frac{\sqrt{4qs - r^2}}{2q}$$

We can begin integrating now. As long as we get intermediate results the non-solved integrals will be treated aside (this way we will avoid repetitions):

$$\int \log(qt^2 + rt + s)dt \overset{(\text{ds=dt})}{=} \int \left[\log q + \log(s^2 + A^2)\right] ds = s\log q + \int \log\left(s^2 + A^2\right) ds$$

We will solve this last integral by parts, where:

$$\begin{cases} u = \log\left(s^2 + A^2\right) \\ v' = 1 \end{cases} \Rightarrow \begin{cases} u' = \dfrac{2s}{s^2 + A^2} \\ v = s \end{cases}$$

Then,

$$\int \log\left(s^2 + A^2\right) ds = s\log\left(s^2 + A^2\right) - \int \frac{2s^2}{s^2 + A^2} ds,$$

where

$$\int \frac{2s^2}{s^2 + A^2} ds = \int \frac{2s^2 + 2A^2}{s^2 + A^2} ds - 2\int \frac{A^2}{s^2 + A^2} ds = 2s - 2\int \frac{1}{1 + \left(\frac{s}{A}\right)^2} ds = 2s - 2A\ \arctan\left(\frac{s}{A}\right) + C$$

If we bring all the integrals together and undo the changes:

$$\int \log\left(qt^2 + rt + s\right) dt = s\log q + s\log\left(s^2 + A^2\right) - 2s + 2A\ \arctan\left(\frac{s}{A}\right) + C =$$

$$= s\log\left[q\left(s^2 + A^2\right)\right] - 2s + 2A\ \arctan\left(\frac{s}{A}\right) + C =$$

$$= \left(t + \frac{r}{2q}\right)\log\left(qt^2 + rt + s\right) - 2\left(t + \frac{r}{2q}\right) + \frac{\sqrt{4qs - r^2}}{q}\ \arctan\left(\frac{2qt + r}{\sqrt{4qs - r^2}}\right) + C$$

where the only omitted calculation is:

$$\frac{s}{A} = \frac{t + \frac{r}{2q}}{\frac{\sqrt{4qs - r^2}}{2q}} = \frac{2qt + r}{\sqrt{4qs - r^2}}.$$

Algorithm 1

This algorithm computes the coordinates $\psi_j(\eta)$ and $\phi_i(\eta)$ given a cage $P = (\mathbb{V}, \mathbb{T})$ in 2D. It supposes that the vertices and the edges are ordered clockwise.

```
Input  : cage P=(V,T), set of points Λ = η
Output: 2D GC φ_i(η),ψ_j(η), i ∈ I_V, j ∈ I_T, η ∈ Λ
// Initialization:
set all φ_i = 0 and ψ_j = 0
// Coordinate computation:
foreach point η ∈ Λ do
    foreach face j ∈ I_T, with vertices v_j1, v_j2 do
        a := v_j2 − v_j1
        b := v_j1 − η
        Q := a · a
        S := b · b
        R := 2(a · b)

        BA := b · ||a|| n(t_j)
        SRT := sqrt(4SQ − R^2)

        L0 := log(S)
        L1 := log(Q + R + S)
        L10 := L1 − L0

        A0 := arctan(R/SRT) / SRT

        A1 := arctan((2Q+R)/SRT) / SRT

        A10 := A1 − A0

        ψ_j(η) := −(||a||/(4π)) [ (4S − R^2/Q) A10 + (R/(2Q)) L10 + L1 − 2 ]

        φ_j2(η) := φ_j2(η) − (BA/(2π)) [ L10/(2Q) − A10 (R/Q) ]

        φ_j1(η) := φ_j1(η) + (BA/(2π)) [ L10/(2Q) − A10 (2 + R/Q) ]
    end
end
```

Where, in the notation of the algorithm:

$$SRT := \sqrt{4SQ - R^2}$$

$$A0 := \frac{\arctan(R/SRT)}{SRT}, \qquad A1 := \frac{\arctan((2Q+R)/SRT)}{SRT}$$

$$\psi_j(\eta) := -\frac{\|a\|}{(4\pi)}\left[\left(4S - \frac{R^2}{Q}\right)A10 + \frac{R}{2Q}L10 + L1 - 2\right]$$

$$\phi_{j_2}(\eta) := \phi_{j_2}(\eta) - \frac{BA}{2\pi}\left[\frac{L10}{2Q} - A10\frac{R}{Q}\right]$$

$$\phi_{j_1}(\eta) := \phi_{j_1}(\eta) + \frac{BA}{2\pi}\left[\frac{L10}{2Q} - A10\left(2 + \frac{R}{Q}\right)\right]$$

Algorithm 2

Here we compute the coordinates $\phi_i(\eta)$ and $\psi_j(\eta)$ in 3D.

Input : cage P=(V,T), set of points $\Lambda = \eta$
Output: 3D GC $\phi_i(\eta)$,$\psi_j(\eta)$, $i \in I_V, j \in I_T$, $\eta \in \Lambda$
`// Initialization:`
set all $\phi_i = 0$ and $\psi_j = 0$
`// Coordinate computation:`
foreach *point* $\eta \in \Lambda$ **do**
  foreach *face* $j \in I_T$*, with vertices* $v_{j_1}, v_{j_2}, v_{j_3}$ **do**
    foreach *l=1,2,3* **do**
      $v_{j_l} = v_{j_l} - \eta$
    end
    $p := (v_{j_l} \cdot n(t_j))n(t_j)$
    foreach *l=1,2,3* **do**
      $s_l := sign(((v_{j_l} - p) \times (v_{j_{l+1}} - p)) \cdot n(t_j))$
      $I_l := \mathbf{GCTriInt}(\mathbf{p}, \mathbf{v_{j_l}}, \mathbf{v_{j_l}}, \mathbf{0})$
      $II_l := \mathbf{GCTriInt}(\mathbf{0}, \mathbf{v_{j_{l+1}}}, \mathbf{v_{j_l}}, \mathbf{0})$
      $q_l := v_{j_{l+1}} \times v_{j_l}$
      $N_l := {}^{q_l}/_{\|q_l\|}$
    end
    $I := -|\sum_{k=1}^{3} s_k I_k|; \quad \psi_j(\eta) := -I; \quad w := n(t_j)I + \sum_{k=1}^{3} N_k II_k$
    if $\|w\| > \epsilon$ **then**
      foreach *l=1,2,3* **do**
        $\phi_{j_l}(\eta) := \phi_{j_l}(\eta) + \frac{N_{l+1} \cdot w}{N_{l+1} \cdot v_{j_l}}$
      end
    end
  end
end
Procedure **GCTriInt**(p, v_1, v_2, η)
$\alpha := \arccos\left(\frac{(v_2 - v_1) \cdot (p - v_1)}{\|v_2 - v_1\| \, \|p - v_1\|}\right); \qquad \beta := \arccos\left(\frac{(v_1 - p) \cdot (v_2 - p)}{\|v_1 - p\| \, \|v_2 - p\|}\right)$
$\lambda := \|p - v_1\|^2 \sin(\alpha)^2; \quad c := \|p - \eta\|^2$
foreach $\theta = \pi - \alpha, \pi - \alpha - \beta$ **do**
  $S := \sin(\theta); \quad C := \cos(\theta)$
  $I_\theta := \frac{-sign(S)}{2}\left[2\sqrt{c}\arctan\left(\frac{\sqrt{c}C}{\sqrt{\lambda + S^2 c}}\right) + \sqrt{\lambda}\log\left(\frac{2\sqrt{\lambda}S^2}{(1-C)^2}\left(1 - \frac{2cC}{c(1+C)+\lambda+\sqrt{\lambda^2+\lambda c S^2}}\right)\right)\right]$
end
return $-\frac{1}{4\pi}|I_{\pi-\alpha} - I_{\pi-\alpha-\beta} - \sqrt{c}\beta|$