

# Cardinals Beyond Choice and the HOD-Dichotomy

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## Abstract

In the 2019 paper “Large Cardinals Beyond Choice” [1], Bagaria, Koellner and Woodin apply the large cardinal techniques and results developed from Woodin’s work on the HOD-Dichotomy to determine the structural resemblance of HOD to  $V$ . Whereas standard inner model theory attempts to find suitable inner models for large cardinals, this new program is aimed at exploring very large cardinals that “break” the resemblance of HOD to  $V$ . This paper attempts to explain in full detail the tools and arguments required for that body of work.

## 1 Introduction

The program of inner model theory attempts to glean insight into the theory of ZFC by examining the structure of inner models of the set-theoretic universe  $V$ . However, the degree of resemblance between  $V$  and its inner models determines the effectiveness of such a study. In the case of the constructible universe  $L$ , its structure is easy to investigate because it is built level by level in a strict way. So if  $L$  and  $V$  have a high degree of resemblance, then analyzing the structure of  $L$  should yield interesting information about the structure of  $V$ . Conversely, if they are not so correlated, then the inner model theorist ought to abandon  $L$  for a model better resembling  $V$ . Jensen discovered this dichotomy in a strong form.

**Theorem 1** (Jensen’s Covering Lemma [3]). *Exactly one of the following is true:*

- (i) *For every singular cardinal  $\gamma$ ,  $\gamma^+ = (\gamma^+)^L$*
- (ii) *Every uncountable cardinal is inaccessible in  $L$ .*

In the former case,  $L$  is “close” to  $V$  because  $L$  correctly computes much of the cardinal structure of  $V$ . In the latter case,  $L$  is “far” from  $V$  in its cardinal structure. As it happens, the outcome of this dichotomy is

dependent on a single condition:  $0^\sharp$ . This set  $0^\sharp$  is a collection of natural numbers which codes a non-trivial elementary embedding  $j : L \prec L$ . If  $0^\sharp$  does not exist, then the first side of the dichotomy holds where  $L$  is “close” to  $V$ ; if  $0^\sharp$  does exist, then the second side of the dichotomy holds where  $L$  is “far” from  $V$ . Its existence is implied by even relatively weak large cardinal assumptions (e.g., Ramsey or measurable cardinals are more than sufficient)<sup>1</sup> so for this reason,  $0^\sharp$  is treated as a sort of large cardinal axiom. Hence in order to use inner models to provide insight to the structure of  $V$  with the existence of large cardinals beyond  $0^\sharp$ , it is necessary to consider more complex inner models which (1) contain those large cardinals and (2) correctly compute the cardinal structure of  $V$  (in the sense of the “close” side of Theorem 1). For many large cardinal properties, minimal inner models have been found. For example,  $L[U]$  is the minimal inner model containing one measurable cardinal,  $L[\mathcal{U}]$  is the minimal inner model for a proper class of measurable cardinals, and so on. Unfortunately these models suffer the same fate as  $L$ ; despite exhibiting a large cardinal property, there are stronger large cardinal properties which act on  $L[U]$  and  $L[\mathcal{U}]$  in the same manner as  $0^\sharp$  acts on  $L$ . That is, there are large cardinal properties which cause these models to completely fail at computing the cardinal structure of  $V$ .

A natural question is to ask whether there is an inner model which can correctly approximate all of  $V$ ’s large cardinal properties and structure. The unsatisfying trivial answer is *yes*, because  $V$  is of course its own inner model, but  $V$  is obviously not a useful inner model for examining its own structure. One would like an inner model which is sufficiently “simple” so that examining its structure is easy. The benchmark for simplicity is being *L-like*<sup>2</sup>. The non-trivial answer is unsatisfyingly unknown; it seems perfectly plausible that for every *L-like* inner model for a large cardinal property there is a stronger large cardinal property which violates its resemblance to  $V$ . Fortunately, if one expects inner model theory to succeed in finding inner models for ever-stronger large cardinal properties, then there is a ceiling to the kind of failure just described. Woodin has established that this is the level of supercompactness; if an inner model can capture the supercompactness of any single cardinal, then such a model will capture all of the standard large cardinals present in  $V$  and correctly compute much of  $V$ ’s cardinal structure (see Theorem 10).

While the prospects of an inner model for supercompactness are promising, there are still many obstacles to overcome in constructing an *L-like* inner model for a supercompact cardinal. Perhaps as a test, one should first consider larger inner models and their relationship to cardinal structure of  $V$ . In a question, can even very “large” inner models approximate the cardinal and large cardinal structure of  $V$ ? The candidate inner model for this question is HOD, whose structure is determined by a notion of definability in  $V$ , thus it might be expected that HOD’s structure is closely correlated to  $V$ ’s. Yet a proof that HOD is “close” to  $V$  is not

<sup>1</sup>See §9 of [7],

<sup>2</sup>That is, being sufficiently similar to  $L$  in its construction, so that one may analyze its *fine-structure*. See §2 about *L-like* models and [9] for details on fine-structure

forthcoming. Woodin has shown that a similar situation to Theorem 1 plays out for HOD.

**Theorem 2** (The HOD Dichotomy Theorem, Woodin [16]). *If  $\kappa$  is an extendible cardinal, then exactly one of the following hold:*

- (i) *Every singular cardinal  $\gamma$  greater than  $\kappa$  is singular in HOD and  $(\gamma^+)^{\text{HOD}} = \gamma^+$ . (HOD is “close” to  $V$ )*
- (ii) *Every regular cardinal greater than  $\kappa$  is measurable in HOD. (HOD is “far” from  $V$ )*

Analogously to  $L$  in the first theorem, HOD either strongly resembles  $V$  or it very strongly diverges from  $V$ 's cardinal structure. The advantage of the covering lemma for  $L$  is that there is a single condition,  $0^\sharp$ , which determines the outcome of the dichotomy. However, for HOD there is no such condition which is readily implied by large cardinal axioms, so the intermediate goal is to develop new techniques to determine which side of the dichotomy is true.

To this end, Woodin has isolated some hypotheses which entail various degrees of success in proving the “close” half of the dichotomy and finding inner models for strong large cardinal properties. The most relevant hypothesis to our study is the Weak HOD Conjecture,<sup>3</sup> from which it follows that HOD is always an inner model for the standard large cardinals in  $V$ . Alas, proving the conjecture outright is also beyond current technology.

The work of “Large Cardinals Beyond Choice” [1] moves in the opposite direction by introducing the extremely large Berkeley cardinals which entail the “far” part of the dichotomy. Berkeley cardinals are so strong, even more than Reinhardt cardinals, that they contradict ZFC. Despite failing to cohere with the axiom of choice, it is not known if Berkeley cardinals are inconsistent with just ZF. Supposing Berkeley cardinals are indeed consistent with ZF, by adding relatively mild large cardinal assumptions one may obtain the consistency of HOD-Berkeley cardinals with ZFC. These relativized and choice-compatible versions would also cause HOD to be “far” from  $V$ . Hence Berkeley cardinals and HOD-Berkeley cardinals are, in a sense, the HOD-analogues for  $0^\sharp$ . The following main theorems of [1] capture this motto.

**Main Theorem 1** (LCBC Corollary 8.1). *If the Weak HOD Conjecture is true, then the former side of Theorem 2 holds where HOD is “close” to  $V$  and Berkeley Cardinals are inconsistent with ZF.*

**Main Theorem 2** ([1]). *Assume there is a proper class of supercompact cardinals. If Berkeley cardinals are consistent with ZF, then HOD-Berkeley cardinals are consistent with ZFC and the latter side of Theorem 2 holds where HOD is “far” from  $V$ .*

The remaining portion of the paper is devoted to developing the techniques required for these theorems as well as promoting their significance in the context of inner model theory.

<sup>3</sup>The precise statement is given and discussed in Definition 17 and §3.

## 2 Inner Models and Covering

We'll begin by loosely tracing the development of inner model theory with large cardinals, starting with Gödel's constructible universe  $L$  and its variants.

**Definition 3.** For any set or class  $A$ ,  $L[A]$  is the unique minimal transitive model closed under intersections with  $A$ . Formally,  $L[A]$  is constructed recursively as follows:

$$L_0[A] := \emptyset$$

$$L_{\alpha+1}[A] := \{X \subseteq L_\alpha[A] : X \text{ is definable in } L_\alpha[A] \text{ using } A \cap L_\alpha[A] \text{ as a unary predicate}\}$$

$$L_\alpha[A] := \bigcup_{\beta < \alpha} L_\beta[A] \text{ for } \alpha \text{ a limit ordinal, and}$$

$$L[A] := \bigcup_{\alpha \in ORD} L_\alpha[A]$$

Of note about  $L[A]$  is that it is a  $\Sigma_1$  definable class with  $A$  as a predicate,  $L[A] \models ZFC$ , and is absolute between transitive models of ZF. Finally  $L := L[\emptyset]$ , from which it follows that  $L \models GCH$ . Call an inner model *L-like* if it is of the form  $L[A]$ . These kinds of models are extremely important because they admit a *fine-structural analysis*. That is, the inner mechanics of definability are laid bare because of the strict hierarchical progression of constructibility. One may look within the levels constructibility to find important global properties of *L-like* models. This type of analysis is exactly the methodology which Jensen developed to prove Theorem 1 among other important results.

Despite the definable paradise that it may appear to be,  $L$  is entirely incompatible with most large cardinals. The most famous and easiest example is its inability to contain a measurable cardinal.

**Definition 4.** An uncountable cardinal  $\kappa$  is *measurable* if there is a  $\kappa$ -complete non-principal ultrafilter  $U$  on  $\kappa$ .

Suppose a measurable cardinal exists, and let  $\kappa$  be the least such cardinal. There is a  $\kappa$ -complete non-principal ultrafilter  $U$  on  $\kappa$ , so using an ultrapower construction,<sup>4</sup> one may derive a non-trivial elementary embedding  $j : V \prec \mathcal{M}$  with critical point  $\kappa$ . Because  $L$  is the minimal inner model it must also be that  $j \upharpoonright L : L \prec L$ . Now  $V$  cannot be identical to  $L$ , for otherwise the elementarity of  $j$  would cause  $L$  to compute that  $\kappa$  and  $j(\kappa)$  are simultaneously the least measurable cardinal.

**Theorem 5** (Scott [10]). *If there is a measurable cardinal, then  $V \neq L$ .*

In fact, the given characterization of  $0^\sharp$  shows that measurable cardinals directly imply the existence of

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<sup>4</sup>See [7]

$0^\sharp$ . Mentioned earlier, the existence of such a set causes a strong structural discordance between  $L$  and  $V$  as in Theorem 1. This promotes a certain criterion for structural similarity of an inner model.

**Definition 6.** An inner model  $M$  of  $V$  has the *covering* property in  $V$  if for every uncountable set of ordinals  $X \in V$ , there is a  $Y \in M$  such that  $X \subseteq Y$  and  $|X| = |Y|$ .  $M$  has the *weak covering* property in  $V$  if it correctly computes the successors of singular cardinals, i.e.  $(\lambda^+)^M = \lambda^+$  for any singular cardinal  $\lambda$ .

While the version of Theorem 1 given above only makes reference to weak covering, Jensen actually had shown that if  $0^\sharp$  does not exist, then  $L$  has the covering property in  $V$ . In the case of  $L$  and any inner model which satisfies choice, one may easily derive the weak covering property from the covering property.

**Proposition 7.** *If an inner model  $M$  has the covering property in  $V$  then  $cf\lambda = |\lambda|$  for every  $\lambda$  which is regular in  $M$ , and every singular cardinal in  $V$  is singular in  $M$ . Moreover, if  $M$  also satisfies the axiom of choice, then  $M$  has the weak covering property in  $V$ .*

*Proof.* For any  $\lambda$  which is regular in  $M$ , take  $X \in V$  to be a cofinal sequence in  $\lambda$  of minimal size (i.e.  $|X| = cf\lambda$ ). By the covering property, there is a  $Y \supseteq X$  for which  $|Y| = |X| = cf\lambda$  and  $Y \in M$ . Then  $Y \cap \lambda$  is cofinal in  $\lambda$ , so  $cf\lambda = |Y| = |\lambda|$  by the regularity of  $\lambda$ . Contrapositively one obtains that every singular cardinal must be singular in  $M$ .

Let  $\lambda$  be a singular cardinal in  $V$ , and suppose towards a contradiction that  $(\lambda^+)^M \neq \lambda^+$ . Necessarily  $(\lambda^+)^M < \lambda^+$  so we have that  $|(\lambda^+)^M| = \lambda$ . But then  $cf(\lambda^+)^M < |\lambda|$ . Because choice holds in  $M$ ,  $(\lambda^+)^M$  is a regular cardinal. This is a direct contradiction to the first claim of this proposition.  $\square$

As mentioned earlier, inner model theory takes interest in those inner models with large cardinal properties which at a minimum satisfy weak covering. The first inner model beyond  $L$  to capture a large cardinal was  $L[U]$ , containing one measurable cardinal by adding a normal ultrafilter  $U$  witnessing that some  $\kappa$  is measurable. Unfortunately this model cannot even contain 2 measurable cardinals, as the second measurable induces a non-trivial elementary embedding  $j : L[U] \prec L[U]$ . The existence of such an embedding is equated with a set of natural numbers  $0^\dagger$ , and it is the  $L[U]$  analogue of  $0^\sharp$ . If  $0^\dagger$  exists, then  $L[U]$  also fails the weak covering lemma.

The pattern continues.  $L$ -like inner models capturing one cardinal property are violated by a stronger property which adds a *sharp* (a subset of  $\omega$  coding a non-trivial elementary embedding of a model into itself, like  $0^\sharp$  or  $0^\dagger$ ) thereby failing the covering lemma. In the cases of some famous inner models, the construction of  $0^\sharp$  can be generalized to obtain: a sharp for the inner model for a proper class of measurable cardinals  $L[\mathcal{U}]$ ; the sharp  $0^\blacksquare$  for the inner model for a strong cardinal  $L[\mathcal{E}]$ ; the sharp  $M_1^\sharp$  for the inner model for a Woodin cardinal  $M_1$ .

On the one hand, it is easy to conjecture that this pattern continues for all large cardinal properties. If in some possible future this were to happen, then a major goal of inner model theory would fail: no inner model could successfully capture the entire large cardinal structure of  $V$ . On the other hand, generalizing the argument for the existence of sharps is increasingly hard according to the complexity of the inner model. So it is not clear which way the future will unfold.

Fortunately, if we expect inner model theory to succeed in building an  $L$ -like inner models for large cardinals, there is a light at the end of the tunnel. By a result of Woodin, *supercompactness* is really the only large cardinal property we need to find an  $L$ -like inner model for.

**Definition 8.** A cardinal  $\kappa$  is supercompact if for every  $\gamma > \kappa$  there is a normal fine ultrafilter over  $\mathcal{P}_\kappa(\gamma)$ .

What Woodin has shown is summarized as follows.

**Definition 9.**  $M$  is a *weak extender model for the supercompactness of  $\delta$* <sup>5</sup> if  $M \models ZFC$  and for every  $\gamma > \delta$ , there is a  $\delta$ -complete normal fine ultrafilter  $U$  over  $\mathcal{P}_\delta(\gamma)$  where;

- (i)  $M \cap \mathcal{P}_\delta(\gamma) \in U$  and
- (ii)  $U \cap M \in M$ .

Even though the name is quite specific it is quite a mouthful. So in the rest of paper I will say “ $M$  is a weak extender model” and specify a supercompact  $\delta$  only when needed.

**Theorem 10** (Woodin [14]). *If  $M$  is a weak extender model for the supercompactness of some  $\delta$ , then*

- (i)  $M$  satisfies the weak covering lemma.
- (ii) If  $\kappa \geq \delta$  is (supercompact, extendible, huge,  $n$ -huge, etc..) then  $\kappa$  is (supercompact, extendible, huge,  $n$ -huge, etc..) in  $M$ .

Thus if one can find an  $L$ -like weak extender model, then the whole large cardinal structure of  $V$  is reflected in that model. The theory of sharps would end at the level of supercompactness and such models would become of primary interest to inner model theory. I will not digress further on the topic of weak extender models except for motivational purposes. For more on this topic and its relation to the HOD-Dichotomy theorem see [14].

### 3 HOD

Changing focus from small  $L$ -like inner models, consider now the very large inner model HOD. Instead of being constructed level-by-level according to a strict notion of absolute definability, HOD’s definition is

<sup>5</sup>The original formulation and many of the results surrounding weak extender models are given in section 5 of [15]. However, the results and presentation are streamlined in [14]

founded on the definability of objects in  $V$ . Naturally, the complexity of  $V$  may induce HOD to be highly complex as well.

**Definition 11.** OD is the class of sets which are definable using only ordinals as parameters. HOD is the collection of *hereditarily ordinal definable* sets, i.e.

$$\text{HOD} = \{x : TC(\{x\}) \subseteq OD\}$$

Occasionally, an augmented version of HOD will be useful,  $\text{HOD}_A$  which is the class of sets which are hereditarily definable using only ordinals and  $A$  as parameters. One can see that this is the HOD-analogue of  $L[A]$  by comparing the definitions. HOD can be quickly characterized as follows:  $\text{HOD} \models ZFC$ , is a  $\Delta_2$  definable class and is the largest inner model containing all ordinals and a global definable well-ordering. Moreover, HOD has a strong property in relation to its possible sets of ordinals. That is, every set of ordinals is set-generic over HOD.

**Lemma 12** ([13]). *To every ordinal  $\kappa$  there is an associated poset  $\mathbb{B}_\kappa \in \text{HOD}$  such that for any  $A \subseteq \kappa$  there is a  $\mathbb{B}_\kappa$ -generic filter  $G$  over HOD where  $A \in \text{HOD}[G]$ .*

*Proof.* Fix  $\kappa$  and  $A \subseteq \kappa$  and consider  $\mathbb{P}$  to be the partial order of  $\mathcal{P}(\mathcal{P}(\kappa)) \cap \text{OD}$  ordered by inclusion ( $\subseteq$ ). Set  $G := \{X \in \mathbb{P} : A \in X\}$ , and observe that  $G$  is an ultrafilter on  $\mathbb{P}$ . To see that  $G$  is generic over OD, take  $D \in \text{OD}$  to be a dense open subset of  $\mathbb{P}$ . If  $G$  does not intersect  $D$ , then  $\bar{D} := \{\mathcal{P}(\mathcal{P}(\kappa)) \setminus X : X \in D\} \subseteq G$  (i.e. the  $\bar{D}$  is the collection of complements of sets in  $D$ ). Hence each  $X \in \bar{D}$  contains  $A$ , guaranteeing that  $\bigcap \bar{D} \in G$ . But  $D$  is dense, so there is a  $X \in D$  such that  $X \subseteq \bigcap \bar{D}$ , but this is impossible by the definition of  $\bar{D}$ . Thus  $G$  must intersect  $D$  and  $G$  is  $\mathbb{P}$ -generic over OD.

OD has a definable well-ordering  $\pi$ , so  $\pi[\mathbb{P}] \in \text{HOD}$  is a poset of ordinals. Define  $\mathbb{B}_\kappa = \pi[\mathbb{P}]$  to obtain the desired poset:  $\pi[G]$  is  $\mathbb{B}_\kappa$ -generic over HOD. We can recover  $A$  in  $\text{HOD}[\pi[G]]$  by considering the function  $f : \kappa \rightarrow \mathbb{B}_\kappa$  defined by  $\alpha \mapsto \pi(\{X \subseteq \kappa : \alpha \in X\})$ .  $f$  is OD and hence belongs to HOD. Finally,  $\alpha \in A$  if and only if  $f(\alpha) \in \mathbb{B}_\kappa$ , so  $A \in \text{HOD}[\pi[G]]$ .  $\square$

Instead of immediately trying to isolate a weak extender model for the supercompactness of some  $\delta$ , it's relevant to first ask about the limitations of such models. For example, if there is no  $L$ -like weak extender model, or no weak extender model admits a fine-structural analysis, then weak extender models fail the needs of the inner model theorist. Noting that HOD is the broadest of inner models, it is pertinent to ask whether or not HOD is or contains a weak extender model. If not, it is hard to imagine that a weak extender model would fulfill the needs of the inner model theorist, insofar as admitting a fine-structural analysis. This is where the HOD Dichotomy Theorem comes into play. It turns out that the “close” outcome of the

dichotomy implies that HOD itself is a weak extender model (see the full statement of the HOD Dichotomy, Theorem 16).

Thus, the inner model theorist hopes (believes) that the “close” side of the dichotomy holds, but the current toolbox for proving it is primitive. So Woodin adopts some reasonable-sounding conjectures to aid the mission. To this end, the full version of the HOD Dichotomy Theorem is needed, which requires a refined notion of measurability in HOD.

**Definition 13.** For  $\gamma$  a regular uncountable cardinal,  $\gamma$  is  $\omega$ -strongly measurable in HOD if there is a  $\kappa < \gamma$  such that  $(2^\kappa)^{\text{HOD}} < \gamma$  and there is no partition  $\langle S_\alpha : \alpha < \kappa \rangle \in \text{HOD}$  of  $E_\omega^\gamma := \{\alpha < \gamma : \text{cof}(\alpha) = \omega\}$  into stationary sets.

The next lemma justifies the *measurable* part of the name.

**Lemma 14** (Woodin [15]). *If  $\gamma$  is  $\omega$ -strongly-measurable in HOD, then  $\text{HOD} \models$  “ $\gamma$  is measurable”.*

*Proof.* This version of the proof appears in [14], which offers a strategy different from the original given in [15]. The following claim is the key to showing the result.

**Claim 14.1.** *If  $\gamma$  is  $\omega$ -strongly measurable in HOD, then there is a subset  $S \in \text{HOD}$  of  $E_\omega^\gamma$  which is stationary in  $\gamma$  and has no stationary partitions in HOD.*

*Proof of claim:* Suppose not. Build a binary tree  $T$  of stationary subsets of  $E_\omega^\gamma$  labelled by functions  $\sigma \in {}^\beta 2$ .

$S_\emptyset = E_\omega^\gamma$  is the root of the tree. If  $\sigma$  is a label on the tree for a stationary subset  $S_\sigma$  of  $E_\omega^\gamma$ , then  $S_{\sigma \frown 0}$  and  $S_{\sigma \frown 1}$  are obtained by the failure of the claim.

At limit stages (i.e for  $\text{length}(\sigma) = \alpha$  a limit ordinal),  $\sigma$  is a label on the tree if and only if  $\sigma \upharpoonright \beta \in T$  for all  $\beta < \alpha$  and  $S_\sigma := \bigcap_{\beta < \alpha} S_{\sigma \upharpoonright \beta}$  is stationary. If  $|2^\alpha|^{\text{HOD}} < \gamma$ , then, since we are working in HOD where choice holds, the club filter must be  $\gamma$ -complete. So there must be at least one  $\sigma \in {}^\alpha 2$  where  $\sigma \in T$  and  $S_\sigma$  is stationary.

Thus the tree has height at least  $\alpha$  for every ordinal such that  $|2^\alpha|^{\text{HOD}} < \gamma$ . Using this fact, we may take  $\sigma \in {}^\alpha 2$  to be a label on the tree. Then  $S_\beta^* := S_{\sigma \upharpoonright \beta+1} \setminus S_{\sigma \upharpoonright \beta}$  (i.e  $S_\beta^*$  is the “other” half of the stationary partition of  $S_{\sigma \upharpoonright \beta}$ ) forms a stationary partition  $\langle S_\beta^* : \beta < \alpha \rangle$  of the stationary set  $\bigcup_{\beta < \alpha} S_\beta^* \subseteq E_\omega^\gamma$ . Moreover, by using the definable well-ordering in HOD for choices at successor steps we can guarantee that  $\langle S_\beta^* : \beta < \alpha \rangle \in \text{HOD}$ . This contradicts the supposition that  $\gamma$  is  $\omega$ -strongly-measurable in HOD. ■

So let  $S$  be as in the claim. We can define a  $\gamma$ -complete ultrafilter  $G$  on  $\mathcal{P}(S)$  as the club filter of  $\gamma$  in HOD but restricted to subsets of  $S$ . Precisely,

$$G := \{X \in \text{HOD} : X \subseteq S \text{ and there is a club } C \text{ such that } C \cap X \subseteq S\}.$$

The  $\gamma$ -completeness of  $G$  follows from the  $\gamma$ -completeness of the club filter on  $\gamma$  in HOD. If neither  $X \subseteq S$  nor its complement belongs to the filter, then both sets must be stationary, contradicting the choice of  $S$ , thus  $G$  is an ultrafilter. A further filter  $G'$  on  $\mathcal{P}(\gamma)^{\text{HOD}}$  is generated by  $G$  which is easily checked to retain the ultrafilter property,  $\gamma$ -completeness, and non-principality hence witnessing the measurability of  $\gamma$ .  $\square$

Contrast the definition of  $\omega$ -strongly-measurable with Solovay's Theorem, a standard fact about partitions of stationary sets.

**Theorem 15** (Solovay [11]). *(ZFC) If  $\kappa$  is a regular uncountable cardinal, then every stationary subset of  $\kappa$  has a partition into  $\kappa$ -many stationary subsets.*

One may find a proof of this fact at Theorem 8.10 of [6]. It might seem that a direct application of this theorem rules out the existence of cardinals which are  $\omega$ -strongly-measurable in HOD, but it in no way guarantees that the partition obtained will belong to HOD. Conversely, trying to apply the theorem inside HOD also does not necessarily contradict  $\omega$ -strong-measurability in HOD. Consider what  $E_\omega^\gamma$  is in HOD:  $E_\omega^\gamma$  certainly belongs to HOD, but it might not be the same as  $(E_\omega^\gamma)^{\text{HOD}}$ . In particular, it could be that some club sets of  $V$  are not in HOD, so Theorem 15 inside HOD could yield a partition which is stationary in HOD, but fails to be stationary in  $V$ .

We now state the full version of the HOD dichotomy.

**Theorem 16** (Woodin [16]). *If  $\kappa$  is an extendible cardinal, then exactly one of the following hold:*

- (i) *No regular cardinal  $\gamma \geq \kappa$  is  $\omega$ -strongly measurable in HOD. Moreover,*
  - (i.a) *Every singular cardinal  $\gamma$  greater than  $\kappa$  is singular in HOD and  $(\gamma^+)^{\text{HOD}} = \gamma^+$ .*
  - (i.b) *HOD is a weak extender model.*
- (ii) *Every regular cardinal  $\gamma \geq \kappa$  is  $\omega$ -strongly measurable in HOD. Moreover,*
  - (ii.a) *HOD is not a weak extender model.*
  - (ii.b) *There is no weak extender model  $N$  for any supercompact cardinal such that  $N \subseteq \text{HOD}$ .*

As stated earlier, there aren't many tools available to show or refute the existence of even a few  $\omega$ -strongly-measurable in HOD cardinals. A stepping stone to showing the former side of the dichotomy is to weaken the premises by working under the Weak HOD Conjecture.

**Definition 17.** [1] The *HOD Hypothesis* is the statement “there is a proper class of regular cardinals which are not  $\omega$ -strongly measurable in HOD.” The *Weak HOD Conjecture* states “ZFC + ‘there exists an extendible cardinal  $\delta$  and a huge cardinal  $\kappa > \delta$ ’ proves the HOD Hypothesis”.<sup>67</sup>

Assuming an extendible cardinal, the HOD hypothesis just says the former (“close”) side of the HOD Dichotomy holds. Whereas the Weak HOD conjecture makes the strong claim that this fact is *provable* in ZFC under large cardinal assumptions. With that said, proving the Weak HOD Conjecture is still out of reach from current techniques. So we will shelve the topic to instead search for sufficient conditions for the “far” side of the HOD Dichotomy.

Having reviewed covering lemmas and sharps in the previous section, the obvious suggestion is to look for a HOD-analogue of  $0^\sharp$  for HOD by analyzing large cardinal properties. In order to do so, one must go beyond the traditional large cardinal axioms because all of the standard large cardinal axioms appear to be consistent within HOD.

Recall that most strong large cardinal properties (measurable, strong, supercompact, huge, etc..) have an equivalent formulation positing the existence of some non-trivial elementary embedding  $j : V \prec M$ , with the strength of the property increasing by requiring  $M$  to more strongly resemble  $V$ . Yet, choice forbids us from developing stronger axioms than those traditional ones.

**Theorem 18** (Kunen [8]). (ZFC) *There is no non-trivial elementary embedding  $j : V \prec V$ . In particular, there is no non-trivial elementary embedding  $j : V_{\lambda+2} \prec V_{\lambda+2}$ .*

*Proof.* (Woodin) First it is to be shown that there is no  $j : V \prec V$ , and then that the relevant details of the proof may be coded into any  $\mathcal{P}(\mathcal{P}(\lambda))$  for which  $j : V_{\lambda+2} \prec V_{\lambda+2}$ .

Suppose towards a contradiction that there is some  $j : V \prec V$ . Let  $\kappa$  be the critical point of  $j$  and  $\lambda := \sup_n j^n(\kappa)$ . Then  $\lambda^+$  is a regular cardinal, so apply Solovay’s Theorem 15 to get a partition  $\langle S_\alpha : \alpha < \kappa \rangle$  of  $E_\omega^{\lambda^+}$  into stationary sets. Consider the image of the stationary sequence:  $\langle \bar{S}_\alpha : \alpha < j(\kappa) \rangle := j(\langle S_\alpha : \alpha < \kappa \rangle)$ .  $\lambda^+$  is fixed (i.e.  $j(\lambda^+) = \lambda^+$ ) by the elementarity of  $j$ , so the sequence of  $\bar{S}_\alpha$  is a partition of  $E_\omega^{\lambda^+}$  into  $j(\kappa)$ -many stationary sets. Fix  $C \subseteq E_\omega^{\lambda^+}$  as the  $\omega$ -club of fixed points of  $j$ , i.e.:

$$C := \{\beta \in E_\omega^{\lambda^+} : j(\beta) = \beta\}$$

Since  $C$  is  $\omega$ -club, then  $C \cap \bar{S}_\alpha$  is non-empty for any  $\alpha < j(\kappa)$ , for example  $\kappa$  itself. That is, there is some  $\beta \in C \cap \bar{S}_\kappa$ . But since the  $S_\alpha$  are a stationary partition, it must be that  $\beta \in S_\alpha$  for some  $\alpha < \kappa$ . This leads

<sup>6</sup>Choiceless but equivalent definitions of *huge* and *extendible* cardinals are given in §4

<sup>7</sup>In [15] and [14], “The HOD Conjecture” is used to denote what I’ve defined as *The HOD Hypothesis*, but I depart from this usage to maintain consistency with [1].

to a contradiction as

$$\beta = j(\beta) \in S_\alpha \cap \bar{S}_\kappa$$

which violates that the  $\bar{S}_\alpha$  are a stationary partition of  $E_\omega^{\lambda^+}$ .

To see that this proof can be carried out in  $V_{\lambda+2}$ , observe that the “largest” sets used in this proof are sequences of subsets of  $\lambda^+$  of length  $< \lambda$ . Such sets are fully codeable as subsets of  $\mathcal{P}(\lambda)$  inside  $V_{\lambda+2}$  where the proof can be carried out on the codes as if they were the actual subsets of  $\lambda^+$ .

An example of such a coding: for  $R$  any well-order of  $\lambda$ , denote  $\alpha_R$  its associated order-type, and  $e_R$  as the isomorphism between  $\langle \lambda, R \rangle$  and  $\langle \alpha_R, \in \rangle$ . Fix a definable bijection  $p : \lambda^3 \rightarrow \lambda$ . Then the code  $C_X$  for any  $X \subseteq \lambda^+$  is given by

$$C_X := \{p[R \times e_R[X \cap \alpha_R]] : R \text{ well-orders } \lambda\}.$$

Since the  $\alpha_R$  are unbounded in  $\lambda^+$ , this coding loses no information about ordinals in  $X$ . Finally  $C_X \in \mathcal{P}(\mathcal{P}(\lambda)) \subseteq V_{\lambda+2}$ . In a similar way, by the regularity of  $\lambda^+$  one can string together sequences of subsets of  $\lambda^+$  of short length, which codes objects such as  $\langle S_\alpha : \alpha < \kappa \rangle$ . Lastly, properties like *clubness* or *stationarity* are now expressible as properties of codes. Therefore the contradiction obtained earlier can be obtained “virtually” inside  $V_{\lambda+2}$ .  $\square$

The proof offers a strategy that is fundamental for the rest of this work in the choiceless setting, so let’s momentarily digress to review the proof’s argument. Given a non-trivial elementary embedding  $j$ , the contradiction requires:

- (I) a regular cardinal  $\lambda > \text{crit}(j)$  which is fixed by  $j$ ,
- (II) the club filter on  $\lambda$  is  $\lambda$ -complete,
- (III) and a stationary partition  $\langle S_\alpha : \alpha < \kappa \rangle$  of  $E_\omega^\lambda$  where  $\kappa \neq j(\kappa)$ .

All three features are easily obtained from choice with  $\lambda = (\kappa_\omega(j))^+$ . Although, these facts are results of seemingly superficial uses of choices. For this reason one might be convinced that  $j : V \prec V$  is inconsistent with  $ZF$ . It has long been suspected that this is the case, but a proof has continually evaded discovery. Those who are convinced of such an inconsistency will see the project of cardinals beyond choice as an exercise in futility, but there is perhaps merit to the investigation even in this situation. By exploring cardinals that contradict choice, one might find a structural barrier of ZF to the existence of very strong embeddings, which could forge a path to proving Kunen’s theorem outright without choice.

The investigation continues by trying to import into ZF as much of the proof of Kunen’s theorem as possible. The requirements (I) and (II) are more tractable, so we will later isolate certain cardinals which have these properties. For any non-trivial elementary embedding  $j$ , if we have a  $\lambda$  which satisfies (I) and

(II), the argument above shows that (III) fails, i.e., there can be no stationary partition  $\langle S_\alpha : \alpha < \text{crit}(j) \rangle$  of  $E_\omega^\lambda$ . Obviously this also implies that there is no stationary partition  $\langle S_\alpha : \alpha < \text{crit}(j) \rangle \in \text{HOD}$  of  $E_\omega^\lambda$ . Thus in the presence of a non-trivial elementary embedding  $j$ , such a  $\lambda$  which satisfies (I) and (II) is necessarily  $\omega$ -strongly-measurable in HOD.

With respect to the HOD Dichotomy Theorem, the goal of finding a  $0^\sharp$ -analogue which places HOD “far” from  $V$  is framed in terms of finding a large cardinal property which causes many cardinals to be  $\omega$ -strongly measurable. At a minimum, if one assumes the existence of an extendible cardinal  $\gamma$ , a sharp for HOD should require at least one cardinal  $\lambda \geq \gamma$  to be  $\omega$ -strongly measurable in HOD. If this happens, then Theorem 16 ensures that all cardinals above  $\gamma$  are  $\omega$ -strongly measurable in HOD. This kind of large cardinal property would be a sort of *weak* sharp for HOD. Without making reference to extendible cardinals nor the HOD dichotomy, a *strong* sharp for HOD should necessitate the “far” side of the dichotomy directly by requiring that a tail of regular cardinals are  $\omega$ -strongly measurable in HOD.

The analysis of Woodin’s proof gives sufficient criteria for large cardinal properties to be weak or strong sharps for HOD. Specifically, if there is an extendible cardinal  $\gamma$ , a large cardinal property  $\Phi$  is a *weak sharp for* HOD if it adds a non-trivial elementary embedding  $j$  with critical point  $\text{crit}(j) > \gamma$  and a cardinal  $\lambda$  satisfying (I) and (II). On the other hand a large cardinal property  $\Phi$  is a *strong sharp for* HOD if for every sufficiently large regular  $\lambda$ , there is an elementary embedding  $j$  where (I) and (II) hold. What will be shown (and what is claimed by the Main Theorem 2) is that a Reinhardt cardinal is a weak sharp for HOD and a Berkeley cardinal is a strong sharp for HOD.

## 4 Cardinals Beyond Choice

### 4.1 Standard Large Cardinals

In order to approach Reinhardt and Berkeley cardinals, one must work without choice, therefore the rest of this paper takes place in ZF unless explicitly stated. Before developing those choiceless cardinals, we remark that the standard definitions of traditional large cardinals need refinement. This is because the standard definitions and equivalences fragment into non-equivalent statements without choice.

**Definition 19.** An uncountable cardinal  $\kappa$  is *inaccessible* if for every  $\alpha < \kappa$  there is no  $f : V_\alpha \rightarrow \kappa$  with range unbounded in  $\kappa$ .

For larger cardinal axioms we need a technical notion related to  $\Sigma_1$ -reflection.

**Definition 20.** We write  $V_\gamma \prec_{\Sigma_1^*} V$  if  $V_\gamma \prec_{\Sigma_1} V$  and for every  $\alpha < \gamma$ ,  $a \in V_\alpha$ , and  $\Sigma_0$  formula  $\phi(x, y)$ :

$$\text{if } \exists b \in V [V^\alpha b \subseteq b \text{ and } V \models \phi(a, b)], \text{ then } \exists b \in V_\gamma [V^\alpha b \subseteq b \text{ and } V \models \phi(a, b)].$$

A definition for supercompactness utilizing ultrafilters was given in §2, yet this characterization is only relevant in the presence of choice. Ultrafilters like those in the Definition 8 can be used with the axiom of choice to obtain ultrapowers  $M$  of  $V$  and embeddings  $j$  from  $M$  to  $V$ . The embeddings  $j$  are the true objects of interest because of the reflection principles they cause at the level of the supercompact cardinal. Hence the next definition of supercompactness more directly captures this phenomenon, and it is the definition we will take to be official in the choiceless setting.

**Definition 21.** A cardinal  $\kappa$  is *supercompact* if for every  $\gamma > \kappa$  with  $V_\gamma \prec_{\Sigma_1^*} V$  and  $a \in V_\gamma$ , there is a  $\bar{\gamma} < \kappa$ , and  $\bar{a} \in V_{\bar{\gamma}}$  with a non-trivial elementary embedding

$$j : V_{\bar{\gamma}+1} \prec V_{\gamma+1}$$

such that  $\text{crit}(j) = \bar{\kappa}$ ,  $j(\bar{\kappa}) = \kappa$ ,  $j(\bar{a}) = a$  and  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$ .

It's not hard to verify that this version is equivalent modulo choice to the ultrafilter characterization. Although, a key difference is that the statement “ $\kappa$  is supercompact” is not necessarily a  $\Pi_2$  statement as it is in Definition 8, but rather a  $\Pi_3$  statement in this form.

Alternatively, Woodin gives yet another definition of supercompactness that is more closely related to extendibility. The next lemma states this characterization and introduces the key idea of using supercompact embeddings to make reflection arguments.

**Lemma 22** ([15]). *A cardinal  $\kappa$  is supercompact if and only if for every  $\alpha > \kappa$  there is a  $\beta > \alpha$  and an elementary embedding  $j$  of  $V_\beta$  into a transitive set  $N$  with  $\text{crit}(j) = \kappa$ ,  $j(\kappa) > \alpha$  and  $N^{V_\alpha} \subseteq N$ .*

*Proof.* Suppose  $\kappa$  is supercompact. Define  $\Phi(\delta, \alpha)$  to declare “There are transitive sets  $M$  and  $N$  satisfying  $Z$ ,<sup>8</sup> both  $M$  and  $N$  are closed under  $V_\alpha$ -sequences and there is an embedding  $j : M \prec N$  such that  $\text{crit}(j) = \delta$  and  $j(\delta) > \alpha$ .” Then  $\Phi(\kappa, \alpha)$  holds for all  $\alpha > \kappa$  by the following reflection argument.

Take some  $\gamma > \alpha$  such that  $V_\gamma \prec_{\Sigma_1^*} V$ . By supercompactness of  $\kappa$ , there is a  $j : V_{\bar{\gamma}+1} \prec V_{\gamma+1}$  with  $\text{crit}(j) = \bar{\kappa} < \kappa$ ,  $j(\bar{\alpha}) = \alpha$  and  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$ . In this scenario  $V_\kappa^{V_{\bar{\alpha}}} \subseteq V_\kappa$  because  $\kappa$  is inaccessible. But using the fact that  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$ , there is some  $\bar{\beta} < \bar{\gamma}$  where  $V_{\bar{\beta}}^{V_{\bar{\alpha}}} \subseteq V_{\bar{\beta}}$ . Letting  $\beta = j(\bar{\beta})$  we have  $V_\beta^{V_\alpha} \subseteq V_\alpha$ . Hence

<sup>8</sup>Hereafter,  $Z$  denotes the axioms of ZF excluding replacement.

$\Phi(\bar{\kappa}, \bar{\alpha})$  is witnessed by  $j$ ,  $M = V_{\bar{\beta}}$  and  $N = V_{\beta}$ . Again using that  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$  there are some  $\bar{j}$ ,  $\bar{M}$ , and  $\bar{N}$  contained in  $V_{\bar{\gamma}}$  satisfying  $\Phi(\bar{\kappa}, \bar{\alpha})$ . Using the elementarity of  $j$ ,  $\Phi(\kappa, \alpha)$  is witnessed by  $j(\bar{j})$ ,  $j(\bar{M})$  and  $j(\bar{N})$ .

Now, fix  $\alpha > \kappa$ . We've shown that  $\Phi(\kappa, \alpha)$  holds so define  $j$ ,  $M$ , and  $N$  to be an embedding and transitive sets which witness this. Let  $\beta$  be defined as  $j(\kappa)$ . Then  $\Phi(\kappa, \beta)$  holds as well so take  $k, M', N'$  to be the elementary embedding and transitive sets witnessing this. Then  $(M')^{V_{\beta}} \subseteq M'$  and  $(N')^{V_{\beta}} \subseteq N'$ . By transitivity of  $M'$  and separation,  $V_{\beta}$  is a set in  $M'$ . Also,  $\beta$  is inaccessible so  $V_{\beta}^{V_{\alpha}} \subseteq V_{\beta}$ . Finally by elementarity of  $k$  this means that  $k(V_{\beta})^{V_{\alpha}} \subseteq k(V_{\beta})^{V_{k(\alpha)}} \subseteq k(V_{\beta})$ . Therefore we can take  $k \upharpoonright V_{\beta}$  as the embedding from  $V_{\beta}$  to  $k(V_{\beta})$  to satisfy the lemma for  $\kappa$  and  $\alpha$ .

For the other direction, suppose  $\gamma > \kappa$ ,  $V_{\gamma} \prec_{\Sigma_1^*} V$  and  $a \in V_{\gamma}$ . Now if there is a  $\beta > \gamma + \omega$  and  $j : V_{\beta} \prec N$  where  $N$  is closed under  $V_{\gamma+1}$ -sequences, then the important part of our embedding,  $j \upharpoonright V_{\gamma+1}$ , also belongs as a set to  $N$ . So  $N$  satisfies the following: “there is a  $\bar{\gamma} < j(\gamma)$ ,  $\bar{\delta} < \bar{\gamma}$ ,  $\bar{a} \in V_{\bar{\gamma}}$  and a  $\bar{j} : V_{\bar{\gamma}+1} \prec V_{j(\bar{\gamma})+1}$  such that  $\text{crit}(\bar{j}) = \bar{\kappa}$ ,  $\bar{j}(\bar{\kappa}) = \kappa$ ,  $\bar{j}(\bar{a}) = j(a)$  and  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V_{\bar{j}(\bar{\gamma})}$ .” By elementarity of  $j$  we can reflect back to  $V$  to find a  $\bar{\kappa}$ ,  $\bar{\gamma}$ , and  $\bar{a} \in V_{\bar{\gamma}}$  with embedding

$$\bar{j} : V_{\bar{\gamma}+1} \prec V_{\bar{j}(\bar{\gamma})+1}$$

such that  $\text{crit}(\bar{j}) = \bar{\kappa}$ ,  $\bar{j}(\bar{\kappa}) = \kappa$ ,  $\bar{j}(\bar{a}) = a$  and  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V_{\bar{j}(\bar{\gamma})}$ . Lastly,  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V_{\gamma}$  and  $V_{\gamma} \prec_{\Sigma_1^*} V$  imply  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$ . Hence  $\bar{j}$  is our desired supercompactness embedding.  $\square$

Framed in the manner of the lemma, the requirement and definition of  $\prec_{\Sigma_1^*}$  no longer seems so contrived. In fact, it's merely a closure property to ensure that  $V_{\alpha}$  has sufficient reflection properties. A consequence of this argument yields a further refinement of supercompactness.

**Proposition 23.** *A cardinal  $\kappa$  is supercompact if and only if for every  $\gamma > \kappa$  and  $\eta < \kappa$  with  $V_{\gamma} \prec_{\Sigma_1^*} V$  and  $a \in V_{\gamma}$ , there is a  $\bar{\gamma} < \kappa$ , and  $\bar{a} \in V_{\bar{\gamma}}$  with a non-trivial elementary embedding*

$$j : V_{\bar{\gamma}+1} \prec V_{\bar{j}(\bar{\gamma})+1}$$

such that  $\text{crit}(j) = \bar{\kappa}$ ,  $j(\bar{\kappa}) = \kappa$ ,  $j(\bar{a}) = a$  and  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$  and moreover that  $\text{crit}(j) > \eta$ .

*Proof.* All that is needed to be shown here is that we can move the critical point of a supercompact embedding arbitrarily high below  $\kappa$ . Fix some  $\eta < \kappa$ . Since  $\kappa$  is supercompact we can use the alternative characterization given in Lemma 22. There is a  $\beta > \gamma + 1$  and  $j : V_{\beta} \prec N$  with  $\text{crit}(j) = \kappa$ ,  $j(\kappa) > \gamma$ , where  $N$  is closed under  $V_{\gamma+1}$  sequences. Hence  $k := j \upharpoonright V_{\gamma+1}$  belongs to  $N$  and witnesses (in  $N$ ): “There is a  $\bar{\gamma} < j(\kappa)$ , and

$\bar{a} \in V_{\bar{\gamma}}$  with a non-trivial

$$k : V_{\bar{\gamma}+1} \prec V_{j(\bar{\gamma})+1}$$

such that  $\eta < \text{crit}(k) < j(\kappa)$ ,  $k(\text{crit}(k)) = j(\kappa)$ ,  $k(\bar{a}) = j(a)$  and  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V_{j(\bar{\gamma})}$ .

Reflecting back to  $V$  via the elementarity of  $j$  we have that there is a  $\bar{\gamma} < \kappa$ , and  $\bar{a} \in V_{\bar{\gamma}}$  with a non-trivial elementary embedding

$$j : V_{\bar{\gamma}+1} \prec V_{\gamma+1}$$

such that  $\eta < \text{crit}(j) < \kappa$ ,  $j(\text{crit}(j)) = \kappa$ ,  $j(\bar{a}) = a$  and  $V_{\bar{\gamma}} \prec_{\Sigma_1^*} V$  and moreover that  $\text{crit}(j) > \eta$ .  $\square$

Essentially, this just states that the elementary embeddings obtained from a supercompact cardinal  $\kappa$  can have their critical point arbitrarily high below  $\kappa$ , thereby fixing arbitrarily large sets below  $\kappa$ . We'll see shortly that supercompacts resemble Berkeley cardinals in this respect.

**Definition 24.** Let  $A$  be the class of  $\gamma$  such that  $V_\gamma \prec_{\Sigma_1^*} V$ . A cardinal  $\kappa$  is *extendible* if for all  $\beta > \kappa$  there is a  $\beta'$  and a non-trivial elementary embedding

$$j : V_\beta \prec V_{\beta'}$$

such that  $\text{crit}(j) = \kappa$ ,  $j(\kappa) > \beta$ , and  $j(A \cap V_\alpha) = A \cap V_{j(\alpha)}$  for all  $\kappa < \alpha < \beta$ .

When working in ZFC, the given definitions of supercompactness and extendibility are equivalent to the same statements *without* the stipulations about  $\prec_{\Sigma_1^*}$ .<sup>9</sup>

**Definition 25.** A cardinal  $\kappa$  is *n-huge* if there is a elementary embedding  $j : V \prec M$  such that  $\text{crit}(j) = \kappa$  and  $j^{n(\kappa)}M \subseteq M$ .

**Lemma 26** ([12] Theorem 7.4). (*ZFC*)  $\kappa$  is *n-huge* iff there is a sequence of cardinals  $\kappa = \kappa_0 < \kappa_1 < \dots < \kappa_n$  and  $\kappa$ -complete normal ultrafilter  $\mathcal{U}$  over  $\mathcal{P}(\kappa_n)$  such that for all  $i < n$ :  $\{x \in \mathcal{P}(\kappa_n) : \text{ot}(x \cap \kappa_{i+1}) = \kappa_i\} \in \mathcal{U}$ .

This characterization also readily yields that “ $\kappa$  is *n-huge*” is just a  $\Sigma_2$  statement in the presence of choice. Yet, the *n-huge* cardinals are not directly formalizable in ZF, as we do not have recourse to characterizations via ultrafilters. Yet, they are important to this program because they are implied by stronger large cardinal notions and because of their role in the Weak HOD Conjecture.

**Definition 27.** We say  $I3(\lambda, \kappa)$  holds if there is a non-trivial elementary embedding  $j : V_\lambda \prec V_\lambda$  with critical point  $\kappa$ .

<sup>9</sup>Precisely, the equivalent statements modulo choice are as follows: A cardinal  $\kappa$  is *supercompact* if for every  $\gamma > \kappa$  with  $V_\gamma \prec_{\Sigma_1^*} V$  and  $a \in V_\gamma$ , there is a  $\bar{\gamma} < \kappa$ , and  $\bar{a} \in V_{\bar{\gamma}}$  with a non-trivial elementary embedding  $j : V_{\bar{\gamma}+1} \prec V_{\gamma+1}$  such that  $\text{crit}(j) = \bar{\kappa}$ ,  $j(\bar{\kappa}) = \kappa$  and  $j(\bar{a}) = a$ . A cardinal  $\kappa$  is *extendible* if for all  $\beta > \kappa$  there is a  $\beta'$  and a non-trivial elementary embedding  $j : V_\beta \prec V_{\beta'}$  such that  $\text{crit}(j) = \kappa$  and  $j(\kappa) > \beta$ .

Suppose there is a non-trivial elementary embedding witnessing  $\text{I3}(\lambda, \kappa)$ , i.e.,

$$j : V_\lambda \prec V_\lambda \text{ and } \text{crit}(j) = \kappa.$$

The *critical sequence*  $\langle \kappa_i : i < \omega \rangle$  of  $j$  is the sequence of  $j^n(\kappa)$  for  $n < \omega$ . Write  $\kappa_\omega(j)$  for  $\sup_n j^n(\kappa)$ .  $\kappa_\omega(j)$  is the least fixed point of  $j$  above  $\text{crit}(j)$ . In ZFC, Kunen's theorem shows that

$$\kappa_\omega(j) \leq \lambda < \kappa_\omega(j) + 2.$$

In the choiceless setting, it seems possible that if  $\lambda \geq \kappa_\omega(j) + 2$ , but the restriction of  $j$  to  $V_{\kappa_\omega(j)}$  witnesses that  $\text{I3}(\kappa_\omega(j), \kappa)$  holds. Hence we will assume hereafter that  $\text{I3}(\lambda, \kappa)$  indicates that  $\lambda$  is the supremum of the critical sequence. The *application* of  $j$  is a function  $j^+ : V_{\lambda+1} \rightarrow V_{\lambda+1}$  defined as

$$j^+(A) := \bigcup_{\alpha < \lambda} j(A \cap V_\alpha)$$

Moreover  $j^+(j)$  results in a full non-trivial elementary embedding with critical point  $j(\kappa)$ . By iterating the use of application and composition, we can justify the existence of an elementary embedding where  $\text{crit}(j) = \kappa_m$  and  $j(\kappa_m) = \kappa_n$  for any  $\kappa_m < \kappa_n$  on the critical sequence. So if  $\text{I3}(\lambda, \kappa)$  holds, then there is a multitude of  $\kappa'$  where  $\text{I3}(\lambda, \kappa')$  hold and by elementarity have identical properties. Despite having countable cofinality,  $\lambda$  is actually the large cardinal of interest, rather than the points of its critical sequences. For this reason I will abbreviate  $\text{I3}(\lambda, \kappa)$  as  $\text{I3}(\lambda)$ , and isolate elements of the critical sequence when needed. In natural language I will say “ $\lambda$  is an I3 cardinal.” I3 cardinals and similar rank-into-rank embeddings are well covered in the survey [4], which establishes the previous facts. The next observations follow easily.

**Observation 28.** If  $\lambda$  is an I3 cardinal, then

- (i) in  $V_\lambda$  every element of its critical sequence is supercompact and extendible,
- (ii)  $V_\kappa \prec V_\lambda$  for all  $\kappa$  in the critical sequence of  $\lambda$ ,
- (iii)  $V_\lambda \models ZF$ .

It is worth mentioning that some of the results appearing in [1] are formulated in terms of  $\omega$ -huge cardinals. The name “ $\omega$ -huge” is given to non-equivalent large cardinal properties in various sources, so I have omitted this terminology. It is very easy to check that the theorems of [1] which require  $\omega$ -huge cardinals can be formulated equivalently with I3 cardinals.

## 4.2 Beyond Choice

Continuing our foray upwards into extremely large cardinals, we really are forced to work without choice by Kunen's theorem.

**Definition 29.** A cardinal  $\kappa$  is *Reinhardt* if there is a non-trivial elementary embedding  $j : V \prec V$  with  $\text{crit}(j) = \kappa$ .

As with  $n$ -huge cardinals in the choiceless context, one may not directly formulate “ $j$  is a non-trivial elementary embedding with domain  $V$ ” in the first order language of set theory. One way to avoid this problem is to suppose any claim about such embeddings is actually formulated in the conservative extension NBG (without choice) where one may assert “ $j$  is a cofinal  $\Delta_0$ -elementary embedding” ([1], p. 8).

**Definition 30.** [1] A cardinal  $\delta$  is *proto-Berkeley* if for every transitive set  $M$  containing  $\delta$ , there is a non-trivial elementary embedding  $j : M \prec M$  with  $\text{crit}(j) < \delta$ . Furthermore,  $\delta$  is *Berkeley* if for every transitive set  $M$  containing  $\delta$ , and every  $\eta < \delta$ , there is a non-trivial elementary embedding  $j : M \prec M$  with  $\eta < \text{crit}(j) < \delta$ .

While Berkeley cardinals are the cardinals of interest, proto-Berkeley cardinals simplify arguments by using the next facts.

**Proposition 31.** [1] For any set  $a$  there is a transitive set  $M$  such that  $a$  is definable without parameters in  $M$ .

*Proof.* Fix  $a$  and let  $a \in V_\lambda$ . Define

$$M := V_\lambda \cup \{(a, x) : x \in V_\lambda\}.$$

$M$  is transitive and  $a$  is now definable as “the first coordinate of every set in the set of highest rank.”  $\square$

**Proposition 32.** [1]  $\delta$  is proto-Berkeley if and only if for every transitive set  $M$  containing  $\delta$  and any  $a \in M$ , there is a  $j : M \prec M$  with  $\text{crit}(j) < \delta$  and  $j(a) = a$ .

*Proof.* One direction is trivial. For the other, let  $M$  be transitive,  $\delta \in M$  and fix some  $a \in M$ . By the previous proposition, choose a transitive  $M'$  such that  $M$  and  $a$  are definable in  $M'$ . By the proto-Berkeleyness of  $\delta$ , there is a  $j : M' \prec M'$  with  $\text{crit}(j) < \delta$ . The elementarity of  $j$  ensures that  $j(M) = M$  and  $j(a) = a$ . Hence  $j \upharpoonright M$  is the desired embedding.  $\square$

Finally, we have a unification of the notions of being Berkeley and proto-Berkeley for the least cardinals of each type.

**Proposition 33.** [1] *The least proto-Berkeley cardinal is also the least Berkeley cardinal.*

*Proof.* Clearly the least Berkeley cardinal is proto-Berkeley, so fix  $\delta$  as the least proto-Berkeley cardinal, and suppose towards a contradiction that  $\delta$  is not Berkeley. This amounts to saying there is an  $\eta < \delta$  and transitive  $M$  such that there is no elementary  $j : M \prec M$  with  $\eta < \text{crit}(j) < \delta$ . Among all such counterexamples, define  $\eta_0$  to be the least with  $M_0$  a corresponding transitive set.

Now if  $M$  is any transitive set containing  $\eta_0$ , then we consider a larger transitive set  $M'$  such that  $M, M_0$ , and  $\eta_0$  are definable without parameters in  $M'$ . By transitivity,  $\delta \in M'$  so we can find an embedding  $j : M' \prec M'$  with  $\text{crit}(j) < \delta$ . Such a  $j$  fixes  $M, M_0$ , and  $\eta_0$ , which induces  $j \upharpoonright M : M \prec M$  and  $j \upharpoonright M_0 : M_0 \prec M_0$ . But the second embedding ensures that  $\text{crit}(j) < \eta_0$  by our definition of  $\eta_0$ . Thus we have just shown that for every transitive  $M$  containing  $\eta_0$  there is an embedding  $j : M \prec M$  with  $\text{crit}(j) < \eta_0$ . This means  $\eta_0$  is proto-Berkeley, contradicting the assumption that  $\delta$  is the least proto-Berkeley cardinal.  $\square$

**Lemma 34.** [1] *If  $\delta_0$  is the least Berkeley cardinal, then for a tail of limit ordinals  $\gamma$ : if  $j : V_\gamma \prec V_\gamma$  with  $\text{crit}(j) < \delta_0$ , then  $j(\delta_0) = \delta_0$  and the fixed points of  $j$  are cofinal in  $\delta_0$ .*

*Proof.* By the previous proposition, we will interchangeably use that  $\delta_0$  is both Berkeley and proto-Berkeley. A first observation is that for any limit ordinal  $\lambda$  greater than  $\delta_0$ ,  $V_\lambda$  contains all of the (proto-)Berkeley embeddings for its transitive sets containing  $\delta_0$ , i.e. it recognizes that  $\delta_0$  is Berkeley.

Conversely, every  $\delta < \delta_0$  is not proto-Berkeley, so there is a transitive set  $M_\delta$  witnessing this fact. That is, there is no

$$j : M_\delta \prec M_\delta$$

with  $\text{crit}(j) < \delta$ . Any one of these counterexamples  $M_\delta$  to the proto-Berkeleyness of  $\delta$  is recognized as such by  $V_{\gamma+2}$  for  $\gamma = \text{rank}(M_\delta)$ . So let  $\beta_\delta$  be the least ordinal such that  $V_{\beta_\delta}$  contains a counterexample to the proto-Berkeleyness of  $\delta$ . Then for any limit ordinal  $\lambda \geq \bigcup_{\delta < \delta_0} \beta_\delta$ ,  $V_\lambda$  recognizes that every  $\delta < \delta_0$  is not proto-Berkeley.

This shows  $\delta_0$  is definable as the least (proto)-Berkeley cardinal in sufficiently large limit  $V_\lambda$ , hence if  $j : V_\lambda \prec V_\lambda$  then  $j(\delta_0) = \delta_0$ .

For the other part, let  $\lambda$  be as above and suppose towards a contradiction that the fixed points of the Berkeley embedding  $j : V_\lambda \prec V_\lambda$  have  $\eta_0 < \delta_0$  as a least upper bound. Define  $\eta_{i+1} := j(\eta_i)$  for all  $i < \omega$ . Then  $\delta_0 = \sup_{i < \omega} \eta_i$ , otherwise if  $\sup_{i < \omega} \eta_i$  were below  $\delta_0$ , then it would be a definable fixed point of  $j$ , which would imply  $\sup_{i < \omega} \eta_i < \sup_{i < \omega} \eta_i$ , a contradiction.

Since  $\eta_0$  is not proto-Berkeley, take a transitive  $M_0$  as a counterexample. Define  $M_{i+1} := j(M_i)$ , so that by elementarity of  $j$ , each  $M_i$  is a counterexample to the proto-Berkeleyness of  $\eta_i$ . Choose a (perhaps larger)

limit  $\lambda'$  such that  $V_{\lambda'}$  contains  $\langle M_i : i < \omega \rangle$ . With Proposition 32, we can ensure that there is an elementary embedding  $k : V_{\lambda'} \prec V_{\lambda'}$  with  $\text{crit}(k) < \delta$  which fixes  $\langle M_i : i < \omega \rangle$ . The critical point of  $k$  must be below some  $\eta_i$ , as remarked earlier. But then  $k \upharpoonright M_i : M_i \prec M_i$ , which contradicts that  $M_i$  is a counterexample to  $\eta_i$  being proto-Berkeley.  $\square$

The last theorem of this section confirms that the Berkeley cardinals are indeed much stronger (consistency-wise) than Reinhardt cardinals.

**Theorem 35** ([1] 3.7). *If  $\delta_0$  is the least Berkeley cardinal, then there is a cardinal  $\gamma < \delta_0$  where*

$$\langle V_\gamma, V_{\gamma+1} \rangle \models \text{ZF}_2 + \text{“there is a Reinhardt cardinal witnessed by } j \text{ and there is an I3 cardinal above } \kappa_\omega(j)\text{”}$$

*Proof.* Fix a limit  $\beta$  as in the previous lemma and take  $j : V_\beta \prec V_\beta$  with  $\text{crit}(j) < \delta_0$ . Set  $\kappa = \text{crit}(j)$  and  $\lambda = \kappa_\omega(j)$ . Because  $\delta_0$  is Berkeley, there are many embeddings  $k : V_\beta \prec V_\beta$  with  $\lambda < \text{crit}(k) < \delta_0$ , hence there are many I3 cardinals (formed as  $\kappa_\omega(k)$ ) sandwiched between  $\lambda$  and  $\delta_0$ .

Let  $\lambda'$  be the least I3 cardinal above  $\lambda$  and  $\gamma$  the least inaccessible cardinal above  $\lambda'$ . Then both  $\lambda'$  and  $\gamma$  are definable from  $\lambda = j(\lambda)$ . Hence  $j(\lambda') = \lambda'$  and  $j(\gamma) = \gamma$ . Finally  $V_\gamma \models \text{ZF}_2$  by inaccessibility,  $V_\gamma$  thinks that  $j \upharpoonright V_\gamma : V_\gamma \prec V_\gamma$  is a Reinhardt embedding and contains all embeddings which witness  $\lambda'$  as an I3 cardinal.  $\square$

Before continuing, I will reiterate here some remarks in [1] about the program of cardinals beyond choice. It may be true that Kunen’s theorem is provable in ZF alone, eliminating all Reinhardt and Berkeley cardinals. Despite this possibility, we don’t yet have any proof of such an inconsistency. Moreover, if there is indeed a proof in ZF of Kunen’s inconsistency, then it is very likely to be much more sophisticated than any of the usual proofs of Theorem 18. (Many proofs are less than a page in length, are mostly combinatorial in nature, and do not require much reasoning beyond elementary set theory.) Assuming that the choiceless cardinals are actually inconsistent, developing their theory will likely uncover the “deeper” structural constraints that negate their existence. In this scenario, the study of cardinals beyond choice still has utility in its ability to probe the structure of  $V$  and lead to a choiceless proof of Kunen’s theorem.

On the other hand, if no such inconsistency is to be found, it is likely that the sophistication of the theory of large cardinals beyond choice will be evidence for their consistency. Establishing confidence in this possibility could happen in the same way one establishes confidence in the standard large cardinals. It is the sophistication of the theory of strong large cardinals that leads one to believe that measurable cardinals are consistent, for example. Therefore a standard interpretation of results like Theorem 5, is that axioms similar to  $V = L$  are merely limiting principles on the structure of  $V$ . If the theory of cardinals beyond

choice becomes sufficiently advanced without encountering an inconsistency, then set theorists might regard the axiom of choice as a limiting principle in the same vein as  $V = L$ . Although, this possibility is quite farfetched, as the axiom of choice is a near-universal tool for most of mathematics. For this scenario to play out, it would require an extreme shift in our beliefs about the axiom of choice.

## 5 Woodin's Forcing <sup>10</sup>

This section's objective is to introduce the forcing notion  $\mathbb{Q}_\delta$  needed for the main theorems of this paper. The importance of  $\mathbb{Q}_\delta$  is its ability to take large cardinals in a model of ZF and produce a model of ZFC where those cardinals have more or less the same features, but in a manner which is compatible with choice.

### 5.1 Dependent Choices and Forcing

Before giving the full forcing notion, I account for some interactions between forcing and fragments of the axiom of choice. Moreover, one can use these developments to lift the proof of Theorem 18 to certain cardinals in the choiceless setting.

**Definition 36.** For cardinals  $\kappa < \delta$ ,  $\mathbb{P}_\kappa^\delta$  is the following poset of partial functions, ordered by reverse inclusion:

$$\{f : \kappa \times \delta \rightarrow V_\delta : |\text{dom}f| < \kappa \wedge \forall(\alpha, \eta) \in \text{dom}f [f(\alpha, \eta) \in V_{1+\eta}]\}.$$

Essentially, in the generic extension, every set of rank  $< \delta$  is collapsed to cardinality  $\leq \kappa$ . A few important properties of this forcing notion which will be used without mention: (1)  $\mathbb{P}_\kappa^\delta$  is  $(< \kappa)$ -closed, (2) If  $\delta$  is inaccessible, then  $\mathbb{P}_\kappa^\delta$  is  $\delta^+$ -c.c., (3)  $\mathbb{P}_\kappa^\delta$  is *homogeneous* (i.e.  $(\mathbb{P} \upharpoonright p, \supseteq) \cong (\mathbb{P} \upharpoonright q, \supseteq)$  for any  $p, q \in \mathbb{P}$ ), and (4)  $\mathbb{P}_\kappa^\delta$  is  $\Sigma_1$ -definable from  $\delta$  and  $\kappa$ . It also follows that if  $\lambda > \delta$  is an (inaccessible, supercompact, extendible, etc..) then  $\lambda$  remains (inaccessible, supercompact, extendible, etc..) after forcing with  $\mathbb{P}_\kappa^\delta$ .

The homogeneity of this forcing is especially important in later theorems because of the next fact.

**Proposition 37.** *If  $\mathbb{P}$  is an ordinal definable homogeneous forcing notion, then for any  $\mathbb{P}$ -generic filter  $G$  over  $V$ ,  $\text{HOD}^{V[G]}$  is a subclass of  $\text{HOD}$  which is definable in  $V$ .*

*Proof.* If  $\phi$  is any formula of the forcing language involving only standard names (i.e. those with a  $\check{\phantom{x}}$ ), then homogeneity ensures that  $p \Vdash_{\mathbb{P}} \phi$  if and only if  $\mathbf{1} \Vdash_{\mathbb{P}} \phi$ , where  $\mathbf{1}$  is the “top” element of  $\mathbb{P}$ . Suppose inductively that  $A \subseteq V$  and  $A$  belongs to  $\text{HOD}^{V[G]}$ . Then there is some sequence of ordinals  $\bar{\beta}$  and formula

<sup>10</sup>I take for granted knowledge of forcing and its mechanics. [6] is a standard reference.

$\phi$  such that  $\phi(x, \bar{\beta})$  is a definition for  $A$ . Then

$$x \in A \iff \mathbf{1} \Vdash_{\mathbb{P}} \phi(\check{x}, \check{\beta})$$

is a definition of  $A$  in  $V$ . □

An immediate consequence is that  $\text{HOD}^{V[G]} \subseteq \text{HOD}$  for any  $\mathbb{P}_\kappa^\delta$ -generic  $G$  over  $V$ . The real power of the forcing notion  $\mathbb{P}_\kappa^\delta$  comes as a result of its connection with fragments of the axiom of choice, so called *Principles of Dependent Choices*.

**Definition 38.** For an ordinal  $\gamma$ ,  $\gamma$ -DC is the statement that if  $X \neq \emptyset$  and  $F : <^\gamma X \rightarrow (\mathcal{P}(X) \setminus \{\emptyset\})$ , then there is a function  $f : \gamma \rightarrow X$  such that  $f(\alpha) \in F(f \upharpoonright \alpha)$  for all  $\alpha < \gamma$ .

Equivalently, for every  $\gamma$ -closed partial order  $(X, R)$ ,  $X$  has an  $R$ -descending chain of length  $\gamma$ . Of note is that the axiom of choice is equivalent to saying  $\gamma$ -DC holds for all  $\gamma$ . The motivation of the collapsing notion  $\mathbb{P}_\kappa^\delta$  is that its combination with supercompact cardinals boosts DC to a higher level in the extension.

In order to say anything about large cardinals and forcing simultaneously, it is pertinent to mention how one may obtain transference of a large cardinal property from the ground to the extension. Silver completely characterized when this happens for elementary embeddings, which can be applied to the embeddings obtained from supercompactness (see [2] for an easy proof of this result and more on the theme of forcing with elementary embeddings).

**Lemma 39** (Silver's Lifting Criterion). *Suppose  $j : M \prec N$  is an elementary embedding,  $\mathbb{P}$  is a forcing notion in  $M$ ,  $G$  is  $\mathbb{P}$ -generic over  $M$ , and  $G^*$  is  $j(\mathbb{P})$ -generic over  $N$ . Define  $j_G(x) := i_{G^*}(j(\dot{x}))$  where  $\dot{x}$  is any  $\mathbb{P}$ -name for  $x$ . Then  $j_G : M[G] \prec N[G^*]$  is a well-defined elementary embedding if and only if  $j[G] \subseteq G^*$ . In this situation we say  $j$  "lifts" to  $j_G$ .*

**Lemma 40** ([15] Theorem 225). *If  $\kappa$  is an infinite regular cardinal where  $(< \kappa)$ -DC holds, then for any supercompact  $\delta > \kappa$ :*

$$V^{\mathbb{P}_\kappa^\delta} \models (< \delta)\text{-DC}$$

*Proof.* Let  $G$  be  $\mathbb{P}_\kappa^\delta$ -generic over  $V$  and  $(X, R)$  a non-empty  $(< \delta)$ -closed partial order in  $V[G]$ . Fixing an arbitrary  $\eta < \delta$ , we will find an  $R$ -descending sequence of length  $\eta$  in  $X$ . Working in  $V$ , we take a limit ordinal  $\gamma > \delta$  and  $\tau \in V_\gamma$  to be a  $\mathbb{P}_\kappa^\delta$ -name for  $(X, R)$ . Let  $p \in G$  be a condition which forces that  $\tau$  is a non-empty poset which is  $\eta$ -closed. Define  $\xi := \max\{\eta, \kappa, \text{rank}(p)\}$ .

By the supercompactness of  $j$  and Proposition 23 there is  $\bar{\gamma} < \delta$ ,  $\bar{\tau} \in \bar{\gamma}$  and an embedding  $j : V_{\bar{\gamma}+1} \prec V_{\bar{\gamma}+1}$  where  $\xi < \text{crit}(j)$ ,  $j(\text{crit}(j)) = \delta$ , and  $j(\bar{\tau}) = \tau$ . By choosing  $\text{crit}(j) > \xi$  we have also ensured that  $j(p) = p$ ,

$j(\kappa) = \kappa$  and  $j(\eta) = \eta$ . We finally let  $\bar{G} := G \cap V_{\bar{\delta}}$ . Using Silver's Criterion (Proposition 39), and that  $j[\bar{G}] \subseteq G$ ,  $j$  lifts to

$$j_{\bar{G}} : V_{\bar{\gamma}+1}[\bar{G}] \prec V_{\gamma+1}[G].$$

By elementarity, we can see  $p$  forces that  $\bar{\tau}$  is a  $\mathbb{P}_{\bar{\kappa}}^{\bar{\delta}}$ -name for an  $\eta$ -closed partial order  $(\bar{X}, \bar{R})$ . In  $V[\bar{G}]$  it remains true that  $(< \kappa)$ -DC holds, hence in  $V[G]$ :  $V[\bar{G}]$  is closed under  $(< \kappa)$ -sequences and  $(\bar{X}, \bar{R})$  is  $(< \kappa)$ -closed. Recall that our forcing collapses  $V_{\bar{\gamma}}$  to cardinality  $\leq \kappa$  in the extension. By  $(< \kappa)$ -DC and that  $\eta < \bar{\gamma}$ , there is a  $\bar{R}$ -decreasing sequence  $\bar{Y} \subseteq \bar{X}$  in  $V[G]$  of length  $\eta$ . But  $\gamma$  was chosen to be a limit ordinal, hence  $\bar{Y} \in V_{\bar{\gamma}}[\bar{G}]$ . Finally  $j_G[\bar{Y}] \subseteq X$  is a  $R$ -descending sequence of length  $\eta$  in  $V[G]$ .  $\square$

These lemmas have immediate consequences with respect to the goal of lifting the proof of Theorem 18 to the choiceless setting. In particular, singular limits of supercompact cardinals fulfill the conditions (I) and (II) described on page 11.

**Theorem 41** ([15] Theorem 227). *If  $\lambda$  is a singular limit of supercompact cardinals then  $\lambda^+$  is a regular cardinal, and the club filter  $\mathcal{F}$  on  $\lambda^+$  is  $\lambda^+$ -complete.*

*Proof.* The following claim is key to the demonstrating the theorem, so it shall be proved first:

**Claim 41.1.** *If  $\mathbb{P} \in V_{\lambda}$  is a partial order and  $G$  is  $\mathbb{P}$ -generic over  $V$ , then for any set  $A \in V[G]$  of ordinals, there is a finite set  $Z \in V$  where  $\text{HOD}_Z^V[A]$  is a  $\kappa$ -cc generic extension of  $\text{HOD}_Z^V$  for some  $\kappa < \lambda$ .*

*Proof of Claim.* In  $V$ , let  $\dot{A}$  be a  $\mathbb{P}$ -name for  $A$ . Define  $Z := \{(\mathbb{P}, <), \dot{A}\}$ .  $\text{HOD}_Z$  then satisfies ZFC because  $Z$  is finite (observe that a well-ordering of  $\text{HOD}_Z$  is contingent upon a well-ordering of  $Z$ ).  $\mathbb{P}$  belongs to some  $\delta < \lambda$ , so consider the collection  $C$  of ordinals  $\alpha$  for which there is a map from  $V_{\delta+1}$  onto  $\alpha$ .  $C$  must be bounded below  $\lambda$  because  $\lambda$  is a limit of supercompact cardinals, so define  $\kappa := \sup C$ .

Now, for all  $n \in \omega$  let  $\mathbb{Q}_n := \{X \subseteq (V_{\delta})^n : X \neq \emptyset \wedge X \in OD_Z\}$ . Given any  $m \leq n$ , define the pointwise projection map  $\pi_{n,m} : \mathbb{Q}_n \rightarrow \mathbb{Q}_m$  as  $X \mapsto \{x \upharpoonright m : x \in X\}$ . Finally taking  $\mathbb{Q}$  to be the union of all  $\mathbb{Q}_n$  for  $n \in \omega$  we have a poset with ordering given as follows: if  $X \in \mathbb{Q}_m$ ,  $Y \in \mathbb{Q}_n$  then

$$X \leq Y \iff m \geq n \wedge \pi_{m,n}(X) \subseteq Y.$$

Take  $g$  to be a  $\text{Coll}(\omega, V_{\delta})$ -generic filter over  $V$ , and in  $V[g]$ , let  $g'$  be the added surjection from

$\omega$  onto  $V_\delta$ . We use this surjection to define

$$H := \{X \in \mathbb{Q} : g' \upharpoonright n \in X \text{ for } X \in \mathbb{Q}_n\}.$$

$H$  is easily verified to be  $\mathbb{Q}$ -generic over  $(\text{HOD}_Z)^V$ . As argued in Theorem 12, there is a poset  $\mathbb{Q}^* = \langle \alpha, E \rangle \in \text{HOD}$  and isomorphism  $e : \mathbb{Q} \rightarrow \mathbb{Q}^*$  where  $e \in \text{OD}_Z$ . Thus taking  $H^* := e[H]$ ,  $H^*$  is then also  $\mathbb{Q}^*$ -generic over  $(\text{HOD}_Z)^V$ . Moreover, we have that  $(\text{HOD}_{Z \cup V_\delta})^V[g] = (\text{HOD}_Z)^V[H^*]$ . The inclusion from left to right is clear. Conversely, from  $e$  and  $H^*$  one can recover  $g$ , by which all of  $V_\delta$  becomes ordinal definable.

By the assumption on  $\delta$ , we have that there is a surjection from  $V_{\delta+1}$  onto  $\mathbb{Q}^*$ , ensuring that  $\mathbb{Q}^*$  is  $\kappa$ -cc in  $(\text{HOD}_Z)^V$ .  $V_\delta$  belongs to  $(\text{HOD}_Z)^V[H^*]$ , hence  $\mathbb{P}$  does as well. But  $V_\delta$  is countable in this model (which satisfies ZFC) so  $\mathbb{P}$  is countable too.

Finally, we can assume that  $G$  is  $(\text{HOD}_Z)^V[H^*]$ -generic by choosing  $g$  to be generic over  $V[G]$ . It follows that  $A \in (\text{HOD}_{Z \cup V_\delta})^V[H^*][G] = (\text{HOD}_Z)^V[H^*][G]$ . So  $\text{HOD}_Z^V[A]$  is an extension of  $\text{HOD}_Z^V$  via the  $\kappa$ -cc iteration  $\mathbb{Q}^* * \mathbb{P}$ . ■

Now for any  $\delta < \lambda$  a supercompact cardinal, recall Definition 36 and take  $G$  to be  $\mathbb{P}_\omega^\delta$ -generic over  $V$ . Lemma 40 then gives us that  $\omega$ -DC holds in  $V[G]$ .

By the claim we can now assure that  $(\lambda^+)^V = (\lambda^+)^{V[G]}$  and  $cf(\lambda^+) \geq \delta$ . As we've left  $\delta$  to be arbitrary,  $cf(\lambda^+) > \delta$  for every  $\delta < \lambda$ . This, together with the fact that  $\lambda$  is a limit of supercompact cardinals shows that  $\lambda^+$  is regular in  $V[G]$ .

We now check the  $\lambda^+$  completeness of  $\mathcal{F}$ . Note by the forcing  $\mathbb{P}_\omega^\delta$  we have

$$\delta = (\omega_1)^{V[G]} \text{ and } \lambda^+ = (\lambda^+)^{V[G]}.$$

Moreover,  $V[G] \models DC$ , so  $\mathcal{F}^{V[G]}$  is  $\delta$ -complete. Lastly, for any club  $C \in \mathcal{F}^{V[G]}$ , by the claim there is a finite set  $Z$  for which  $\text{HOD}_Z^V[C]$  is a generic extension of  $\text{HOD}_Z^V$  by a  $\kappa$ -cc poset, with  $\kappa < \lambda$ . By this fact there is some club  $D \subseteq C$  belonging to  $\text{HOD}_Z^V$  (see [6] 22.25). Thus  $\mathcal{F}$  is also  $\delta$ -closed in  $V$ .

This argument holds for all supercompact cardinals below  $\lambda$ , which we've decided to be a limit of supercompact cardinals, so  $\mathcal{F}$  is  $< \lambda$ -complete. Hence by the singularity of  $\lambda$ ,  $\mathcal{F}$  must be  $\lambda^+$ -complete. □

With another assumption about the computation of  $\lambda$  in  $L$ -like models, we have a full lifting of Kunen's theorem to the choiceless context.

**Theorem 42** ([15] Theorem 228). *If  $\lambda$  is a limit of supercompact cardinals with countable cofinality and for*

some set of ordinals  $A$ ,  $\lambda^+ = (\lambda^+)^{L[A]}$ , then there is no non-trivial elementary embedding  $j : V_{\lambda+2} \prec V_{\lambda+2}$ .

*Proof.* First note that  $\mathcal{P}(\lambda^+)$  can be coded into  $\mathcal{P}(\mathcal{P}(\lambda))$  in such a way that the relevant properties, such as being stationary, of the proof of Theorem 18 are expressible in  $V_{\lambda+2}$  on the codes. Thus, in the remainder of the proof we work exclusively with  $\mathcal{P}(\lambda^+)$ , but in fact the argument is translatable to work with the codes.

Suppose towards a contradiction that there is a non-trivial elementary embedding  $j : V_{\lambda+2} \prec V_{\lambda+2}$ . The coding guarantees that  $j(\lambda^+) = \lambda^+$ . Of course we also have that  $\text{crit}(j) < \lambda = j(\lambda)$ .

By Theorem 41,  $\lambda^+$  is regular and the club filter  $\mathcal{F}$  is  $\lambda^+$  complete. The same facts carry into  $\text{HOD}_A$ . Since  $E_\omega^{\lambda^+}$  belongs to  $\text{HOD}_A$ , I claim there must be a partition  $\langle S_\alpha : \alpha < \lambda \rangle \in \text{HOD}_A$  of  $E_\omega^{\lambda^+}$ . If not, then by the proof of Lemma 14,  $\lambda^+$  must be measurable. But this is impossible, since  $\lambda^+ = (\lambda^+)^{L[A]} = (\lambda^+)^{\text{HOD}_A}$ . In other words,  $\text{HOD}_A$  already recognizes  $\lambda^+$  as a successor cardinal and therefore not as measurable.

With such a partition the proof of Theorem 18 functions inside  $\text{HOD}_A$ , yielding a contradiction.  $\square$

## 5.2 The Forcing

To obtain models of large fragments of choice as forcing extensions, Lemma 40 encourages us to iterate the forcing  $\mathbb{P}_\kappa^\delta$  to raise the level of DC as high as any supercompact we like. For the purposes of building Woodin's forcing notion, assume a proper class of supercompact cardinals. The Axiom of Choice is equivalent to  $\gamma$ -DC for all  $\gamma$  thus Lemma 40 along with a proper class iteration of  $\mathbb{P}_\kappa^\delta$  will result in a model of ZFC. Although, care is needed in the construction of Woodin's forcing iteration [[15] pp. 323-4]:

**Definition 43.** For any ordinal  $\alpha$  Woodin's Forcing  $\mathbb{Q}_\alpha$  is the iteration  $\langle \mathbb{Q}_\beta : \beta \leq \alpha \rangle$  and a sequence of cardinals  $\langle \kappa_\beta : \beta \leq \alpha \rangle$  where:

- (i)  $\mathbb{Q}_0 = \emptyset$  and  $\kappa_0$  is the least regular cardinal for which  $(< \kappa_0)$ -DC holds and  $\kappa_0$ -DC fails.
- (ii)  $\mathbb{Q}_{\beta+1} = \mathbb{P}_{\kappa_\beta}^{\kappa_{\beta+1}}$  where  $\kappa_{\beta+1}$  is the least inaccessible cardinal greater than  $\kappa_\beta$  such that if  $V^{\mathbb{Q}_\beta} \models (< \kappa_\beta)$ -DC, then  $V^{\mathbb{Q}_\beta * \mathbb{Q}_{\beta+1}} \models (< \kappa_{\beta+1})$ -DC.
- (iii) For a limit ordinal  $\beta$ ,
  - (a) if  $\sup\{\kappa_\gamma : \gamma < \beta\}$  is an inaccessible cardinal then  $\mathbb{Q}_\beta$  is the direct limit of  $\langle \mathbb{Q}_\gamma : \gamma < \beta \rangle$  and  $\kappa_\beta$  is  $\sup\{\kappa_\gamma : \gamma < \beta\}$ ,
  - (b) otherwise  $\mathbb{Q}_\beta$  is the inverse limit of  $\langle \mathbb{Q}_\gamma : \gamma < \beta \rangle$  and  $\kappa_\beta$  is  $(\sup\{\kappa_\gamma : \gamma < \beta\})^+$ .

If the axiom of choice holds or if there is no inaccessible cardinal after the least failure of DC, then Woodin's Forcing is empty. There are many situations where the iteration is either empty or short (i.e., has length less than a supercompact cardinal). In order to make the forcing useful for our purposes, we want the iteration to be as long as a supercompact cardinal. To indicate the length of the forcing, I will call the

iteration *well-defined up to*  $\delta$  whenever  $\mathbb{Q}_\alpha$  is non-empty for all  $\alpha < \delta$ . To guarantee that the iteration is well-defined beyond short lengths one needs only to assume that choice fails and that there is a supercompact cardinal above the least failure of DC. In this case, the resultant model has some spectacular properties.

**Theorem 44** (Woodin [15], pp. 323-4). *Assume the axiom of choice fails and  $\kappa$  is the least cardinal where  $\kappa$ -DC fails. For any supercompact cardinal  $\delta > \kappa$ , Woodin's Forcing  $\langle \mathbb{Q}_\alpha : \alpha \leq \delta \rangle$  and corresponding sequence of cardinals  $\langle \kappa_\alpha : \alpha \leq \delta \rangle$  is well-defined up to  $\delta$ . Moreover, if  $G$  is  $\mathbb{Q}_\delta$ -generic over  $V$ , then*

- (i)  $V[G]_\delta \models ZFC$ .
- (ii) If  $\gamma \leq \delta$  is supercompact, then  $\kappa_\gamma = \gamma$ .
- (iii) If  $\gamma = \kappa_\gamma$ , then for every  $\alpha < \gamma$ ,
  - (a)  $\mathbb{Q}_\gamma \cong \mathbb{Q}_{\kappa_\alpha} * \mathbb{Q}$  where  $\mathbb{Q} = (\mathbb{Q}_\gamma)^{V^{\mathbb{Q}_{\kappa_\alpha}}}$
  - (b)  $(\mathbb{Q}_\gamma)^{V^{\mathbb{Q}_{\kappa_\alpha}}}$  is homogeneous and  $(< \kappa_\alpha)$ -directed-closed in  $V^{\mathbb{Q}_{\kappa_\alpha}}$
  - (c)  $V_\gamma[G \upharpoonright \gamma] = (V[G])_\gamma$
- (iv)  $\mathbb{Q}_\delta$  is homogeneous
- (v)  $\mathbb{Q}_\delta$  is  $\Sigma_3$  definable in  $V_\delta$ .

*Proof.* The least failure of DC is at the level of  $\kappa$ , which is below  $\delta_0$ , so  $\kappa_0 = \kappa$ . At limit stages the construction of  $\kappa_\alpha$  and  $\mathbb{Q}_\alpha$  is straightforward by the definition. It is left to be shown that the successor stages of Woodin's Forcing are well defined. Suppose  $\mathbb{Q}_\alpha$  and  $\kappa_\alpha$  have already been defined. An application of Theorem 40 shows that if  $G \subseteq \mathbb{P}_{\kappa_\alpha}^\delta$  is generic over  $V$  then  $(< \delta)$ -DC holds in  $V[G]$ . By the supercompactness of  $\delta$ , there is an inaccessible cardinal  $\bar{\delta} > \kappa_\alpha$  such that  $V_{\bar{\delta}+\omega} \prec V_{\delta+\omega}$ . Thus, if  $G \subseteq \mathbb{P}_{\kappa_\alpha}^{\bar{\delta}}$  is generic over  $V$ , then  $(< \bar{\delta})$ -DC holds in  $V[G]$ . This guarantees that there is a least inaccessible cardinal satisfying the successor condition (ii) of Definition 43. Therefore the construction of Woodin's Forcing is well defined up to the  $\delta^{\text{th}}$  stage.

We now inductively check that  $(< \kappa_\alpha)$ -DC holds in  $V[G]$  for all  $\alpha < \delta$ . By definition  $(< \kappa_0)$ -DC must hold. If  $V^{\mathbb{Q}_\alpha} \models (< \kappa_\alpha)$ -DC, then  $V^{\mathbb{Q}_\alpha * \mathbb{Q}_{\alpha+1}} \models (< \kappa_{\alpha+1})$ -DC is guaranteed by the successor step of the construction of  $\mathbb{Q}_{\alpha+1}$ . At limit stages  $\alpha$ , if  $\gamma = \sup\{\kappa_\beta : \beta < \alpha\}$  is an inaccessible cardinal, then  $\kappa_\alpha = \gamma$  and the direct limit  $\mathbb{Q}_\alpha$  of  $\langle \mathbb{Q}_\beta : \beta < \alpha \rangle$  ensures that  $V^{\mathbb{Q}_\alpha} \models (< \kappa_\beta)$ -DC for all  $\beta < \alpha$ . Since the  $\kappa_\beta$  are unbounded in  $\kappa_\alpha$ , it follows that  $V^{\mathbb{Q}_\alpha} \models (< \kappa_\alpha)$ -DC. If  $\gamma$  is not an inaccessible cardinal, then it must be a singular cardinal, so the inverse limit  $\mathbb{Q}_\alpha$  of  $\langle \mathbb{Q}_\beta : \beta < \alpha \rangle$  ensures that  $V^{\mathbb{Q}_\alpha} \models (< \kappa_\beta)$ -DC for all  $\beta < \alpha$ . From this it not only follows that  $V^{\mathbb{Q}_\alpha} \models (< \gamma)$ -DC, but because  $\gamma$  is singular we also have that  $V^{\mathbb{Q}_\alpha} \models (\gamma)$ -DC. Hence  $\kappa_\alpha = \gamma^+$  and  $V^{\mathbb{Q}_\alpha} \models (< \kappa_\alpha^+)$ -DC.

It's now been shown that if  $G$  is a  $\mathbb{Q}_\delta$ -generic filter over  $V$ , then  $(< \delta)$ -DC holds in the extension  $V[G]$ . Every supercompact cardinal is inaccessible and inaccessibility is not destroyed by any of the forcing notions  $\mathbb{Q}_\alpha$ , so  $\delta$  remains inaccessible in  $V[G]$ . Therefore  $V[G]_\delta \models ZF \wedge (< \delta)$ -DC. As remarked after Definition 38,

the axiom of choice is equivalent to  $\alpha$ -DC for all  $\alpha$ , modulo the remaining axioms of ZF. Finally we have (i)  $V[G]_\delta \models ZFC$ .

The facts (ii) and (iii) are immediate by construction, and (iv) follows from the previous two.

(v) Since  $\mathbb{P}_\kappa^\delta$  is  $\Sigma_1$ -definable from  $\delta$  and  $\kappa$ , we need only compute correctly the sequence  $\langle \kappa_\alpha : \alpha < \delta \rangle$ . The cardinal  $\kappa_0$  is both  $\Sigma_2$  and  $\Pi_2$  definable. Computing  $\kappa_{\alpha+1}$  from  $\kappa_\alpha$  amounts to stating that  $\kappa_{\alpha+1}$  is the least inaccessible cardinal above  $\kappa_\alpha$  such that if (a)  $(< \kappa_\alpha)$ -DC holds and  $\kappa_\alpha$ -DC fails then (b)  $(< \kappa_{\alpha+1})$ -DC holds and  $\kappa_{\alpha+1}$ -DC fails in the extension. (a) is the conjunction of a  $\Sigma_2$  statement and a  $\Pi_2$  statement and (b) is the conjunction of a  $\Sigma_3$  statement and a  $\Pi_3$  statement (the increase in complexity results from having to quantify over all  $\mathbb{P}_{\kappa_\alpha}^{\kappa_{\alpha+1}}$ -names). Note for any  $\eta < \delta$  and sufficiently large inaccessible  $\gamma$  below  $\delta$ , that

$$\langle \kappa_\alpha : \alpha < \eta \rangle = (\langle \kappa_\alpha : \alpha < \eta \rangle)^{V_\gamma}.$$

So checking the construction of  $\mathbb{Q}_\alpha$  can be formulated as a statement about a tail of inaccessible cardinals in  $V_\delta$ , adding just a single existential quantifier to a  $\Sigma_3$  statement, hence it is  $\Sigma_3$ .  $\square$

In summary  $\mathbb{Q}_\delta$  is a forcing iteration of length  $\delta$  with direct limits taken at inaccessible limit stages. In the extension  $(< \delta)$ -DC holds and  $\kappa_\alpha$  is the  $\alpha^{\text{th}}$  regular cardinal. It's easy to see that this forcing yields some very nice features.

In addition to raising the level of DC in the extension,  $\mathbb{Q}_\delta$  has the advantage of preserving large cardinal properties under certain circumstances. In particular, embeddings will lift to the extension in the following sense:

**Lemma 45.** [15] *Suppose there is a non-trivial elementary embedding  $j : V_{\bar{\gamma}+\omega} \prec V_{\gamma+\omega}$  with critical point  $\delta$ , and that Woodin's forcing is well-defined up to some  $\eta \geq \gamma$  and  $G$  is a  $\mathbb{Q}_\eta$ -generic filter over  $V$  for some  $\eta \geq \gamma$ . If (1)  $j(\delta) > \bar{\gamma}$ , (2)  $\kappa_\gamma = \gamma$  and (3)  $j(\mathbb{Q}_{\bar{\gamma}}) = \mathbb{Q}_\gamma$  then  $j$  lifts to an elementary embedding*

$$j_{G \upharpoonright \bar{\gamma}} : (V[G])_{\bar{\gamma}+1} \prec (V[G])_{\gamma+1}.$$

*Proof.* Let  $j_{G \upharpoonright \delta}(x)$  be the lifting of  $j$  as defined in Lemma 45.  $j[(G \upharpoonright \delta)] \subseteq G \upharpoonright j(\delta)$  because  $j$  is the identity below  $\delta$ , thereby ensuring

$$j_{G \upharpoonright \delta} : V_{\bar{\gamma}+1}[G \upharpoonright \delta] \prec V_{\gamma+1}[G \upharpoonright j(\delta)].$$

Similarly we define the elementary embedding  $j_{G \upharpoonright \bar{\gamma}}$ . But we do not necessarily have that  $j[(G \upharpoonright \bar{\gamma})] \subseteq G \upharpoonright \gamma$ . To remedy the situation, we use the essential properties of our forcing.

By (ii) of Theorem 44,  $\mathbb{Q}_\gamma = \mathbb{Q}_{j(\delta)} * \mathbb{Q}$  for some  $\mathbb{Q} \in V_{\gamma+\omega}[G \upharpoonright j(\delta)]$ . With the elementarity of  $j$

and (3),  $\mathbb{Q}_{\bar{\gamma}} = \mathbb{Q}_{\delta} * \bar{\mathbb{Q}}$  for some homogeneous,  $(< j(\delta))$ -directed-closed  $\bar{\mathbb{Q}}$  such that  $j(\bar{\mathbb{Q}}) = \mathbb{Q}$ . Therefore  $V[G \upharpoonright \bar{\gamma}] = V[G \upharpoonright \delta][H]$  where  $H$  is  $\bar{\mathbb{Q}}$ -generic over  $V[G \upharpoonright \delta]$ . Since  $\bar{\gamma} < j(\delta)$ , then  $H$  must also have size less than  $j(\delta)$ . Using this fact and the  $(< j(\delta))$ -directed-closure of  $\mathbb{Q}$ , we can find a  $p \in \mathbb{Q}$  which is pointwise beneath all  $j(q)$  for  $q \in H$ . Now, by the homogeneity of  $\mathbb{Q}$ , we can assume  $p \in G \upharpoonright \gamma$ ; for if not, we can take a  $\bar{G}$  such that  $G \upharpoonright j(\delta) = \bar{G} \upharpoonright j(\delta)$ ,  $V[G] = V[\bar{G}]$  and  $p \in \bar{G}$ .

So if  $p \in G \upharpoonright \bar{\gamma}$  we can freely assume that there is some  $q \leq j(p)$  in  $G \upharpoonright \gamma$ . By upward closure of the filter,  $j[(G \upharpoonright \bar{\gamma})] \subseteq G \upharpoonright \gamma$ . Hence the embedding lifts.

$$j_{G \upharpoonright \bar{\gamma}} : V_{\bar{\gamma}+1}[G \upharpoonright \bar{\gamma}] \prec V_{\gamma+1}[G \upharpoonright \gamma].$$

Noting that  $\kappa_{\bar{\gamma}} = \bar{\gamma}$  and  $\kappa_{\gamma} = \gamma$ , (ii) again gives  $V_{\bar{\gamma}}[G \upharpoonright \bar{\gamma}] = (V[G])_{\bar{\gamma}}$  and  $V_{\gamma}[G \upharpoonright \gamma] = (V[G])_{\gamma}$ . We have our desired conclusion:

$$j_{G \upharpoonright \bar{\gamma}} : (V[G])_{\bar{\gamma}+1} \prec (V[G])_{\gamma+1}$$

□

A direct application of the lemma shows that if  $\delta$  is supercompact (or extendible) in  $V$  and Woodin's forcing is well-defined up to some  $\eta > \delta$ , then in the extension  $\delta$  retains all of its supercompact (or extendible) embeddings whose range is contained in  $V_{\eta}$ . In other words,  $\delta$  is  $(< \eta)$ -supercompact (or  $(< \eta)$ -extendible) in the extension  $V[G]$ . With the given lemmas we easily obtain the next theorem, which is another variation of 226 of [15].

**Theorem 46.** [1] *Suppose I3( $\lambda$ ) holds. There is a homogeneous forcing notion  $\mathbb{Q}_{\lambda}$  which is  $\Sigma_3$ -definable in  $V_{\lambda}$  such that for any  $\mathbb{Q}_{\lambda}$ -generic  $G$  over  $V_{\lambda}$ :*

- (1)  $V_{\lambda}[G] \models ZFC$
- (2) *There is a proper class of huge cardinals and a proper class of extendible cardinals in  $V_{\lambda}[G]$ .*

*Proof.* If I3( $\lambda$ ) holds, let  $\langle \delta_i : i < \omega \rangle$  be its critical sequence. Then each  $\delta_i$  is supercompact. So each  $\mathbb{Q}_{\delta_i}$  is well-define and its construction can be carried out in  $V_{\lambda}$ . The forcing  $\mathbb{Q}_{\lambda}$  is defined as the direct limit of  $\langle \mathbb{Q}_{\delta_i} : i < \omega \rangle$ . Suppose  $G$  is  $\mathbb{Q}_{\lambda}$  generic over  $V_{\lambda}$ . Then  $V_{\lambda}[G] \models ZFC$  because  $V_{\lambda} \models ZF$  and  $G$  ensures  $(< \delta_i)$ -DC for all  $\kappa_i$  which are unbounded in  $\lambda$ . Theorem 44 has already guaranteed  $\Sigma_3$ -definability. (1) has been thus been verified.

(2) The critical sequence  $\langle \delta_i : i < \omega \rangle$  of  $\lambda$  is such a class of extendible and n-huge cardinals: Extendibility follow immediately from Lemma 45 and the remarks which follow it. One may obtain the n-hugeness of  $\delta_0$

by lifting  $j : V_{\delta_{n+1}} \prec V_{\delta_{n+2}}$  with  $\text{crit}(j) = \delta_0$ , and  $j(\delta_0) = \delta_1$  to the extension and defining

$$\mathcal{U} := \{X \subseteq \mathcal{P}(\delta_n) : j[\delta_n] \in j(X)\}$$

This is routinely checked to be an ultrafilter in  $V_\lambda[G]$ . Because AC holds in  $V_\lambda[G]$  we can apply the ultrafilter characterization of  $n$ -hugeness as given in Theorem 26 to verify that  $\mathcal{U}$  witnesses the  $n$ -hugeness of  $\delta_0$ . One can argue similarly for any other element of the critical sequence by lifting the appropriate elementary embeddings.  $\square$

Unfortunately, the argument of Lemma 45 cannot lift  $j$  to a full elementary embedding  $\bar{j} : V_\lambda[G] \prec V_\lambda[G]$  to preserve the I3 property of  $\lambda$ , but the final chapter will show how the homogeneity of this forcing can reproduce stronger cardinal properties “from the outside”.

## 6 An inner model from Woodin's forcing

We next show that Woodin's forcing up to an I3 cardinal allows one to isolate a very useful inner model of ZFC. The relevant theorem appears as Theorem 8.4 of [1] which is a variation on Theorem 229 of [15].

**Theorem 47.** *If the Weak HOD Conjecture Holds and  $I3(\lambda)$  holds, then there is a transitive class  $N \subseteq V_\lambda$  and parameter  $X \in V_\lambda$  such that:*

- (i)  $N \models ZFC$
- (ii)  $N$  is  $\Sigma_2$ -definable in  $V_\lambda$  from  $X$
- (iii) *There is a partial order  $\mathbb{P} \in N$  such that for every set of ordinals  $A \in V_\lambda$  there is an  $\mathbb{P}$ -generic filter  $G$  over  $N$  where  $A \in N[G$ ].*

Before the proof, we will treat some basic facts of ground model definability in forcing extensions, which happens to be intimately related with the covering properties of Definition 6. These arguments are essential to satisfying properties (i) and (ii) of the previous theorem. The next definitions and results are due to Hamkins [5], but Theorem 50 was also discovered independently by Woodin and Laver.

**Definition 48.** For  $M \subseteq N$  both transitive models of  $ZC + \Sigma_1$ -Replacement and a regular uncountable cardinal  $\delta$ ,

- (i)  $M$  has the  $\delta$ -covering property in  $N$  if for every  $A \in N$  such that  $A \subseteq M$  and  $|A| < \delta$ , there is some  $B \in M$  with  $|B| < \delta$  and  $A \subseteq B$
- (ii)  $M$  has the  $\delta$ -approximation property in  $N$  if for every  $A \in N$  with  $A \subseteq M$ , if  $A \cap B$  belongs to  $M$  for every  $B \in M$  with  $|B|^M < \delta$ , then  $A \in M$ .

In the context of these properties, we will often just consider  $A$  and  $B$  to be sets of ordinals, as choice applies here so all sets are coded by sets of ordinals. Immediately we find that these properties crop up in small forcing extensions.

**Lemma 49** (Hamkins [5]). *Let  $M \models ZC + \Sigma_1$ -Replacement and  $\delta > \omega$  a regular cardinal. Forcing with any  $\mathbb{P}$  with  $|\mathbb{P}| < \delta$  ensures that  $M$  has the  $\delta$ -covering and  $\delta$ -approximation properties in  $M[G]$ .*

*Proof.* Suppose  $\dot{A}$  is a  $\mathbb{P}$ -name for  $A \subseteq M$  and  $p \Vdash |\dot{A}| < \delta$ . Then the possible values of  $|\dot{A}|$  are bounded below  $\delta$ , because  $\mathbb{P}$  satisfies the  $\delta$ -c.c. Let  $\gamma$  be the supremum of all such values and  $\dot{f}$  a name for a surjection from  $\gamma$  onto  $A$ . Then  $B := \{\beta : \exists q \leq p \exists \alpha < \gamma (q \Vdash \dot{f}(\check{\alpha}) = \check{\beta})\}$  (i.e. the possible members of  $A$ ) is a set in  $M$  and  $|B| < \delta$ . Thus  $M$  has the  $\delta$ -covering property in  $M[G]$ .

To see that  $M$  has the  $\delta$ -approximation property, suppose that  $A$  is a subset of some ordinal  $\lambda$  and that all of its approximations of size  $< \delta$  belong to  $M$ . Note that all such approximations are subsets of  $\lambda$ . So let  $\dot{A}$  be a name for  $A$  and  $p \in G$  force those facts about  $\dot{A}$ . Precisely:

$$(*) \quad p \Vdash \text{“}\dot{A} \subseteq \lambda \wedge \forall X \in \mathcal{P}(\check{\lambda})^M [ |X| < \delta \rightarrow X \cap \dot{A} \in \mathcal{P}(\check{\lambda})^M ] \text{”}$$

Now suppose that  $q \nVdash \dot{A} \in \mathcal{P}(\lambda)^M$  for every  $q \leq p$ . This translates to the fact that there is some  $\alpha < \lambda$  and conditions  $q^0, q^1 \leq q$  that disagree on  $\alpha$ , i.e.  $q^0 \Vdash \check{\alpha} \in \dot{A}$  and  $q^1 \Vdash \check{\alpha} \notin \dot{A}$ . Since  $|\mathbb{P} \restriction p| = \gamma < \delta$ , take an enumeration  $\langle p_\beta : \beta < \gamma \rangle$  of  $\mathbb{P} \restriction p$ . Then for every  $\beta < \gamma$  there must be some  $\alpha_\beta$  and conditions  $q_\beta^0, q_\beta^1 \leq q_\beta$  such that  $q_\beta^0 \Vdash \check{\alpha}_\beta \in \dot{A}$  and  $q_\beta^1 \Vdash \check{\alpha}_\beta \notin \dot{A}$  by the observation made at the beginning of this paragraph.

Consider the set  $C = \{\alpha_\beta : \beta < \gamma\}$ .  $C$  is definable in  $M$  from  $\mathbb{P}, \dot{A}$  and the enumeration  $\langle p_\beta : \beta < \gamma \rangle$ , hence it belongs to  $M$ . Notably  $|C| < \delta$  so the approximation  $C \cap A$  belongs to  $M$ , i.e.  $p \Vdash \dot{A} \cap \check{C} = \check{B}$  for some  $B \in M$  and  $r \in G$ . This means that  $r$  must decide every element of the set  $B$ , and without loss of generality  $r \leq p$  so  $r = q_\beta$ . But then  $r$  fails to decide whether  $\alpha_\beta \in \dot{A}$ , which demonstrates the contradiction. Therefore the conditions which force  $\dot{A} \in \mathcal{P}(\lambda)^M$  are dense below all  $p$  which satisfy  $(*)$ , from which it follows that  $A \in \mathcal{P}(\lambda)^M$ .  $\square$

**Theorem 50.** *If  $\delta$  is a regular uncountable cardinal,  $M, N, V$  are transitive models of  $Z + AC + \Sigma_1$ -Replacement and  $M, N \subseteq V$  satisfy the  $\delta$ -covering and  $\delta$ -approximation properties in  $V$ ,  $(\delta^+)^V = (\delta^+)^M = (\delta^+)^N$  and  $\mathcal{P}(\delta)^M = \mathcal{P}(\delta)^N$ . Then*

- (i)  $M=N$  and
- (ii)  $M$  is  $\Sigma_2$  definable from  $M \cap \mathcal{P}(\delta)$ .

*Proof.* (i) It suffices to show that  $M$  and  $N$  have the same sets of ordinals. The key observation is that for any set of ordinals  $A \in V$  there is a covering set  $B \supseteq A$  where  $|B| \leq \delta$  and  $B$  belongs to both  $M$  and  $N$ .

Fix  $A \subseteq \gamma$  of size  $< \delta$ . To find such a  $B$ , we build a sequence of sets  $\langle (B_\alpha^M, B_\alpha^N) : \alpha < \delta \rangle$  such that for all  $\alpha$ :

- (1)  $A \subseteq B_\alpha^M \subseteq B_\alpha^N \subseteq \gamma$ ,
- (2)  $|B_\alpha^M|, |B_\alpha^N| < \delta$ ,
- (3)  $B_\alpha^M \in M$  and  $B_\alpha^N \in N$ ,
- (4) and if  $\alpha$  is a limit ordinal, then  $\bigcup_{\beta < \alpha} B_\beta^N \subseteq B_\alpha^M$ .

For any alpha we obtain  $B_\alpha^N$  by applying the  $\delta$ -covering property to  $B_\alpha^M$  and similarly obtain  $B_{\alpha+1}^M$  from  $B_\alpha^N$ . Finally we have that

$$\bigcup_{\alpha < \delta} B_\alpha^M = \bigcup_{\alpha < \delta} B_\alpha^N.$$

Call this union  $B$ , and observe that  $B \in V$  by  $\Sigma_1$ -Replacement. To see that  $B$  is a set in  $M$ , use the  $\delta$ -approximation property: consider a set  $C \in M$  with  $|C| < \delta$ . By the regularity of  $\delta$  there is some  $\alpha < \delta$  for which  $B \cap C = B_\alpha^M \cap C$ . But this is now an intersection of sets in  $M$ , whereby it must belong to  $M$ . Having established that  $M$  contains all its approximations of  $B$ , the  $\delta$ -approximation property yields that  $B \in M$ . Arguing symmetrically, we find that  $B \in N$ .

Suppose  $A$  is a set of ordinals in  $V$  of size  $< \delta$ . Taking  $|B| \leq \delta$  to be a covering set of  $A$  in  $M \cap N$ , then  $\text{otp}(B) = \xi < \delta^+$  and the Mostowski collapse  $\pi : B \rightarrow \xi$  also belongs to  $M \cap N$ . Because  $(\delta^+)^V = (\delta^+)^M = (\delta^+)^N$ ,  $\pi[A]$  corresponds to a subset  $X \in V$  of  $\delta$ . Therefore,

$$A \in M \iff X \in \mathcal{P}(\delta)^M \iff X \in \mathcal{P}(\delta)^N \iff A \in N.$$

From the previous statement any set of ordinals  $A \in M$  can be shown to be in  $N$  (and vice versa) by the  $\delta$ -approximation property. If  $B \in N$  is a set of size  $< \delta$ , then  $B \in M$  and  $A \cap B \in M$ . But  $|A \cap B| < \delta$  too, so  $A \cap B \in N$ . Hence all of  $N$ 's  $\delta$ -approximations of  $A$  belong to  $N$ , thereby showing  $A \in N$  and  $M \subseteq N$ . The symmetric argument follows, from which we have  $M = N$ .

(ii) Finally,  $M$  is definable by the  $\Sigma_2$  formula which states: “ $X \in M$  if and only if there is a regular cardinal  $\xi$  and  $N \subseteq V_\xi$  where both  $N, V_\xi \models ZC + \Sigma_1$ -Replacement,  $N$  satisfies the  $\delta$ -properties in  $V_\xi$ ,  $(\delta^+)^N = (\delta^+)^{V_\xi}$ ,  $\mathcal{P}(\delta)^N = \mathcal{P}(\delta)^{V_\xi}$  and  $X \in N$ .” Certainly  $M \cap V_\xi$  will also satisfy these properties, thus the argument of (i) follows in this context and verifies that the definition is correct.  $\square$

Lastly, a variant of the next theorem is instrumental in the proof of Theorem 47. Essentially this lemma entitles us to a definable set that belongs to any ultrafilter witnessing that a cardinal  $\delta$  is supercompact.

**Lemma 51** ([12] Theorem 3.4). (ZFC) *Suppose  $\delta$  is supercompact and  $\lambda > \delta$  is a regular cardinal. If  $E_\omega^\lambda = \{\alpha < \lambda : \text{cof}(\alpha) = \omega\}$  and  $\langle S_\alpha : \alpha < \lambda \rangle$  is a partition of  $E_\omega^\lambda$  into stationary sets, then there is a set  $X$*

definable from  $\langle S_\alpha : \alpha < \lambda \rangle$  such that

$$\forall \sigma, \tau \in X [\text{sup}(\sigma) = \text{sup}(\tau) \implies \sigma = \tau],$$

and  $X$  belongs to every normal fine ultrafilter on  $\mathcal{P}_\delta(\lambda)$ .

*Proof.* Fix  $\langle S_\alpha : \alpha < \lambda \rangle$  as a sequence of stationary subsets of  $E_\omega^\lambda$  as above.  $X$  merely depends on this sequence and not on  $\mathcal{U}$ . Take  $Z_\eta$  to be  $\{\alpha < \eta : S_\alpha \cap \eta \text{ is stationary in } \eta\}$ . We can define

$$X := \{Z_\eta : \eta < \lambda \wedge \omega < \text{cof}(\eta) < \delta \wedge \text{sup } Z_\eta = \eta\}$$

Clearly  $X$  satisfies the property we want, so we just need to show that  $X \in \mathcal{U}$ .  $\mathcal{U}$  induces a non-trivial elementary embedding  $j : V \prec M$  such that (1)  $\text{crit}(j) = \delta$ , (2)  $j(\delta) > \lambda$ , (3)  $j''\lambda \in M$  and (4)  $Y \in \mathcal{U}$  if and only if  $j[\lambda] \in j(Y)$ .<sup>11</sup>

Working in  $M$ , fix  $\beta_\lambda := \text{sup } j[\lambda]$ . Letting  $\langle \bar{S}_\alpha : \alpha < j(\lambda) \rangle = j(\langle S_\alpha : \alpha < \lambda \rangle)$  we have that  $Z_\eta^M = \{\alpha < \eta : M \models \bar{S}_\alpha \cap \eta \text{ is stationary in } \eta\}$ . The idea is to show  $j[\lambda] = Z_{\beta_\lambda}^M$  so that  $X \in \mathcal{U}$  by (4).

Suppose  $\gamma < \lambda$  and  $C$  is a club of  $\beta_\lambda$  in  $M$ . Then consider the preimage of  $C$  under  $j$ :

$$C' = \{\alpha < \lambda : j(\alpha) \in C\}.$$

By the elementarity of  $j$ ,  $C'$  must be  $\omega$ -closed. But  $S_\gamma$  being a stationary subset of  $E_\omega^\lambda$  intersects all  $\omega$ -clubs of  $\lambda$ . Thus  $C' \cap S_\gamma$  is non-empty hence  $C \cap \bar{S}_{j(\gamma)}$  is also non-empty. So  $j[\lambda] \subseteq Z_{\beta_\lambda}^M$ .

Conversely, suppose  $M \models \bar{S}_\alpha \cap \eta_\lambda$  is stationary in  $\beta_\lambda$  for some  $\alpha < \beta_\lambda$ . Then  $\bar{S}_\alpha \cap j[\lambda]$  is non-empty since  $j[\lambda]$  is an  $\omega$ -club in  $\beta_\lambda$  and  $\bar{S}_\alpha$  is a stationary subset of  $E_\omega^{\beta_\lambda}$ . Fix some  $\beta$  with countable cofinality such that  $j(\beta) \in \bar{S}_\alpha$ . Now  $\beta$  must occur in some  $S_{\alpha'}$ , hence  $j(\beta) \in \bar{S}_{j(\alpha')}$ , but the  $\bar{S}_\alpha$  form a partition of  $j[\lambda]$ , so  $\bar{S}_{j(\alpha')} = \bar{S}_\alpha$ . Hence  $\alpha \in j[\lambda]$  and  $Z_{\beta_\lambda}^M \subseteq j[\lambda]$ .  $\square$

Now we have all the tools to prove the main theorem of this section.

*Proof of Theorem 47.* Suppose I3( $\lambda$ ) holds and  $\langle \delta_i : i < \omega \rangle$  is the critical sequence. Let  $\mathbb{Q}_\lambda$  be Woodin's forcing from Theorem 46 so that if  $G$  is a  $\mathbb{Q}$ -generic filter over  $V_\lambda$ , then every  $\delta_i$  is supercompact, extendible, and huge in the extension. Particularly we have

$$V_\lambda[G] \models ZFC + \text{“}\delta_0 \text{ is extendible”} + \text{“}\delta_1 \text{ is huge”}.$$

<sup>11</sup>(1)-(4) are the standard features of the elementary embedding obtained from the ultrapower of  $V$  by  $\mathcal{U}$ . See section 2 of [12] for details.

It is a well known fact of ZFC that if  $\kappa$  is extendible, then  $V_\kappa \prec_{\Sigma_3} V$ . Because adding a universal quantifier is downward absolute,  $V_\kappa$  is also correct about  $\Pi_4$  statements ([12], Theorem 5.11). Because  $\delta_2$  is extendible in  $V_\lambda[G]$ , it must be the case that  $(V[G])_{\delta_2}$  is a  $\Pi_4$ -elementary substructure of  $V_\lambda[G]$ . In ZFC, “ $\delta_0$  is extendible” is a  $\Pi_3$ -statement and “ $\delta_1$  is huge” is a  $\Sigma_2$  statement, so these cardinals reflect downward to the level of  $\delta_2$ :

$$(V[G])_{\delta_2} \models ZFC + \text{“}\delta_0 \text{ is extendible”} + \text{“}\delta_1 \text{ is huge”}$$

By the Weak Hod Conjecture  $(V[G])_{\delta_2} \models$  “The HOD Hypothesis”. That is, there is a proper class of cardinals which are not  $\omega$ -strongly measurable in HOD.

Let  $\Phi(M, \kappa, \delta)$  describe the following properties of a transitive set  $M \in V[G]$ :

- (i)  $M \models ZFC$
- (ii)  $M \in V$
- (iii)  $M \cap OR = \kappa$
- (iv) There is a partial order  $\mathbb{P}$  such that for every bounded  $A \subseteq \kappa$  in  $V$  there is some  $\mathbb{P}$ -generic  $G \in V$  over  $M$  such that  $A$  belongs to  $M[G]$  and
- (v) If  $\gamma$  is the least inaccessible in  $V[G]$  above  $\delta$  and  $\xi < \gamma$ , then any  $g$  which is  $\text{Coll}(\omega, \mathcal{P}^V(\xi))$ -generic over  $V$  is set-generic over  $M \cap V_\gamma$ .

**Claim 51.1.** *If both  $\Phi(M, \delta, \kappa)$  and  $\Phi(N, \delta, \kappa)$  hold in  $V[G]$  and furthermore  $N \cap V_\gamma = M \cap V_\gamma$  for  $\gamma$  the least inaccessible in  $V[G]$  above  $\delta$ , then  $M = N$ .*

*Proof of Claim.* Suppose  $\mathbb{P}^M, \mathbb{P}^N$  witnesses the property (iv) for  $M$  and  $N$ , respectively. Set

$$\xi = \max\{|\mathbb{P}^M|^M, |\mathbb{P}^N|^N\}$$

and let  $g$  be  $\text{Coll}(\omega, \mathcal{P}^V(\xi))$ -generic over  $V$ . Using property (v),  $g$  is set-generic over  $M \cap V_\gamma$  and  $N \cap V_\gamma$ . So suppose  $B \in N$  is a bounded subset of  $\kappa$ . By (iv), there is a  $\mathbb{P}^M$ -generic  $G$  over  $M$  such that  $G \in V$  and  $B \in M[G]$ . Because  $|\mathbb{P}^M|^M < \xi$  we have that  $G$  is coded by  $g$ . Hence  $G \in M[g]$  and thus  $B \in M[g]$  as well. So it is established that  $N \subseteq M[g]$ ; a symmetric argument shows that  $M \subseteq N[g]$ . Noting that a model of ZFC is uniquely identifiable by its sets of ordinals, it has been shown that  $M[g] = N[g]$ . Since  $g$  is generic for some forcing notion of size  $< \gamma$ , Lemma 49 declares that both  $M$  and  $N$  have the  $\gamma$ -covering and  $\gamma$ -approximation properties in  $M[g]$ . Finally, applying Theorem 50, we have that  $M = N$ .  $\blacksquare$

Let  $\delta$  be the least supercompact cardinal and define

$$I := \{\kappa \geq \delta : V[G]_\kappa \prec V[G]_{\delta_2}\}.$$

The embeddings obtained from the supercompactness of  $\delta_2$  ensure that  $I$  is non-empty and unbounded in  $\delta_2$ . Fix some  $\kappa \in I$ . By the supercompactness of  $\delta$ , there is a  $\bar{\kappa} < \delta$  and  $j : V_{\bar{\kappa}+1} \prec V_{\kappa+1}$  such that  $\text{crit}(j) = \bar{\delta}$  and  $j(\bar{\delta}) = \delta$ . Using Lemma 45 we can see that  $j$  lifts to an elementary embedding  $j_{G \upharpoonright \bar{\kappa}}$  in the extension  $V[G]$  and therefore it one may further restrict its domain to  $\text{HOD}^{V[G]}$ .

To avoid the inevitable typographical mess, I will use  $j$  to refer to both the original embedding  $j$  and the liftings  $j_G, j_G \upharpoonright \text{HOD}^{V[G]}$  to  $V[G]$  and  $\text{HOD}^{V[G]}$  which are obtained from  $j$ .

**Claim 51.2.** *For every  $\xi < \bar{\kappa}$  and  $A \in \mathcal{P}^V(\xi)$  there is a  $B \subseteq \bar{\delta}$  such that  $A \in L[j(B)]$ .*

*Proof of Claim.* Fix  $\xi < \bar{\kappa}$  and define  $\lambda$  to be the least cardinal in  $V[G]_{\delta_2}$  for which  $j(2^\xi) < \lambda$  and  $\lambda$  is not  $\omega$ -strongly measurable in  $\text{HOD}$ . It must be that  $\lambda < \kappa$  because  $\kappa \in I$ , and  $\lambda$  is definable from  $j(\xi)$ , so there is some  $\bar{\lambda} < \bar{\kappa}$  such that  $j(\bar{\lambda}) = \lambda$ . Using the elementarity of  $j$ ,  $\bar{\lambda}$  remains greater than  $2^\xi$  and not  $\omega$ -strongly measurable in  $\text{HOD}$  (in  $V[G]_{\bar{\kappa}}$ ). Thus we take  $\langle S_\alpha : \alpha < \xi \rangle$  as a partition of  $E_\omega^{\bar{\lambda}}$  into stationary sets which belongs to  $\text{HOD}^{V[G]_{\bar{\kappa}}}$ .

For ease of notation, write  $\langle \bar{S}_\alpha : \alpha < j(\xi) \rangle$  for  $j(\langle S_\alpha : \alpha < \xi \rangle)$ ,  $\beta_\xi = \sup j[\xi]$ , and  $\beta_{\bar{\lambda}} = \sup j[\bar{\lambda}]$ .

Because  $j : (\text{HOD}^{V[G]})_{\bar{\kappa}+1} \prec (\text{HOD}^{V[G]})_{\kappa+1}$  and working in the context of  $V[G]_{\bar{\kappa}}$  where choice holds, the elementary embedding  $j$  induces a normal fine ultrafilter  $\mathcal{U} \in \text{HOD}^{V[G]}$  on  $\mathcal{P}_{\bar{\delta}}(\xi)$ . An easy modification of the proof from Lemma 51 yields an  $X \in \mathcal{U}$  definable from  $\langle S_\alpha : \alpha < \gamma \rangle$ :

Set

$$Z_{\eta_0, \eta_1} := \{\alpha < \eta_0 : S_\alpha \cap \eta_1 \text{ is stationary in } \eta_1\}$$

and

$$X := \{Z_{\eta_0, \eta_1} : \eta_0 < \xi \wedge \omega < \text{cof}(\eta_1) < \bar{\delta}\}.$$

The point here is that  $j[\xi] = Z_{\beta_\xi, \beta_{\bar{\lambda}}}$  by a nearly identical argument as before thus showing that  $j[\xi] \in j(X)$  whereby  $X \in \mathcal{U}$ .

Because we've accomplished all this working in  $\text{HOD}^{V[G]}$ , we find by the homogeneity of  $\mathbb{Q}$  that  $X$  belongs to  $V$ .

Let  $B \subseteq \bar{\delta}$  be such that  $B \in L[A, X]$  and  $L[B] \cap V_{\bar{\delta}} = L[A, X] \cap V_{\bar{\delta}}$  (i.e.  $B$  is a coding of all

bounded subsets of  $L[A, X] \cap V_{\bar{\delta}}$ ; such a coding exists because  $\bar{\delta}$  is inaccessible, and we have a definable well ordering in  $L[A, X]$ .

Using the elementarity of  $j$ :  $L[j(B)] \cap V_{\delta} = L[j(A), j(X)] \cap V_{\delta}$ . Because  $X \in \mathcal{U}$ , we have  $j[\xi] \in j(X)$ . So we can recover  $j \upharpoonright \xi$  and therefore  $A$  as well. Hence  $A \in L[j(A), j(X)]$  and also  $A \in L[j(B)]$  since  $A \subseteq \xi < \delta$ .  $\blacksquare$

Define  $\bar{N}$  to be HOD computed in  $V[G]_{\delta_2}$  with parameter  $j \upharpoonright \mathcal{P}^V(\bar{\delta})$ . It's clear that  $\bar{N} \in V$  because it is a definable subclass of  $V$ , seen by the homogeneity of  $\mathbb{Q}$  and Proposition 37.

**Claim 51.3.** *For every  $\kappa \in I$  there is a transitive set  $M$  such that  $\Phi(M, \kappa, \delta)$  holds.*

*Proof of Claim.* Define  $\bar{M} := \bar{N} \cap V_{\bar{\kappa}}$ . Then  $\bar{M} \in V$ ,  $\bar{M} \cap OR = \bar{\kappa}$  and  $\bar{\kappa}$  is inaccessible so  $\bar{M} \models ZFC$ .

The partial order relevant to (iv) is the Vopěnka Algebra from Theorem 12:  $\mathbb{B}_{\bar{\delta}}$ . If  $A$  is a bounded subset of  $\bar{\kappa}$ , then by the previous claim there is some  $B \subseteq \bar{\delta}$  such that  $A \in L[j(B)]$ . By Vopenka's theorem, there is a  $\mathbb{B}_{\bar{\delta}}$ -generic filter  $G$  over  $\bar{N}$  which adds  $B$  to  $\bar{N}[G]$ . But then  $j(B) \in \bar{N}$ , hence  $A \in \bar{N}[G]$ .  $A$  is bounded below  $\bar{\kappa}$  so this shows  $A$  is a set in  $\bar{N}[G] \cap V_{\bar{\kappa}} = \bar{M}[G]$ .

(v) Fix  $\bar{\gamma}$  as the least inaccessible cardinal in  $V[G]$  above  $\bar{\delta}$ . If  $\xi < \bar{\gamma}$  and  $g$  is  $\text{Coll}(\omega, \mathcal{P}^V(\xi))$ -generic over  $V$ , then extending  $\bar{M}$  by  $g$  adds a well-order of  $\mathcal{P}^V(\xi)$  to  $\bar{M}$ . Hence  $(\bar{M} \cap V_{\bar{\gamma}})[g] \models ZFC$ . By choice we can encode  $(TC\{g\}, \in)$  as some  $(\alpha, E)$ , where  $E$  is a binary relation on  $\alpha$ , i.e.  $E \subseteq \alpha \times \alpha$ . Using a standard pairing function on ordinals,  $E$  itself can be coded by some set of ordinals  $X \subseteq \alpha$ . By Theorem 12, we find there is a filter  $G$  of the Vopenka algebra  $\mathbb{B}_{\alpha}$  which is generic over  $\bar{M} \cap V_{\bar{\gamma}}$  and adds  $X$ . In  $(\bar{M} \cap V_{\bar{\gamma}})[G]$  recover  $E$  from  $X$  using the pairing function and apply the transitive collapse to  $(\alpha, E)$  to yield  $(TC\{g\}, \in)$ . Thus  $g$  is set-generic over  $\bar{M} \cap V_{\bar{\gamma}}$ .

We've just shown  $\Phi(\bar{M}, \bar{\kappa}, \bar{\delta})$  to hold, so by the elementarity of  $j$  and defining  $M = j(\bar{M})$ , it has been shown that  $\Phi(M, \kappa, \delta)$  holds.  $\blacksquare$

By the third claim, and because  $G$  adds a well ordering of  $V[G]_{\delta_2}$ , we can suppose there is a function  $F : I \rightarrow V[G]_{\delta_2}$  mapping each  $\kappa$  to a transitive set  $M \subseteq V_{\kappa}$  belonging to  $V$  such that  $\Phi(M, \kappa, \delta)$  holds.  $I$  is cofinal in  $\delta_2$ , which is inaccessible, so there must be a subcollection  $I_0 \subseteq I$  and  $\bar{N} \subseteq V_{\bar{\gamma}}$  for which

$$\forall \kappa_1, \kappa_2 \in I_0 [F(\kappa_1) \cap V_{\bar{\gamma}} = F(\kappa_2) \cap V_{\bar{\gamma}} = \bar{N}].$$

But then we also have by uniqueness

$$\forall \kappa_1, \kappa_2 \in I_0 [\kappa_1 < \kappa_2 \implies F(\kappa_2) \cap V_{\kappa_1} = F(\kappa_1)].$$

Thus we can define  $N := \bigcup_{\kappa \in I_0} F(\kappa)$ . With this definition in hand we apply the uniqueness argument from the first claim to get a similar observation about  $N$ : if  $\kappa > \gamma$  satisfies  $Z + \Sigma_1$ -Replacement, then  $M = N \cap V_\kappa$  iff  $M \models ZC + \Sigma_1$ -Replacement,  $M \cap V_\gamma = \bar{N}$  and there is a partial order  $\mathbb{P} \in M \cap V_\gamma$  such that for every bounded  $A \subseteq \kappa$  there is some  $\mathbb{P}$ -generic  $G$  over  $M$  such that  $A$  belongs to  $M[G]$ . The uniqueness condition along with the set  $\bar{N}$  allows the use of the I3 embeddings of  $V_\lambda$  to bring  $N$  to a  $\Sigma_2$ -definable class of  $V_\lambda$ , completing the theorem.  $\square$

The interest in this inner model will shortly be clear, as it is important in the relationship between the Weak HOD Conjecture (a statement about provability in ZFC) and Berkeley cardinals (a cardinal notion only possibly consistent with ZF).

## 7 Determining the Dichotomy

First, we begin by assuming the Weak HOD Conjecture. In this scenario inner model theory does not yet fail, as HOD acts as an inner model for all standard large cardinals. Indeed the success of the Weak HOD conjecture, which is a statement about ZFC, leaks into the choiceless setting as well, prohibiting the choiceless cardinals.

**Theorem 52.** *Suppose the Weak HOD Conjecture Holds. There is no non-trivial elementary embedding  $j : V \prec V$  with an I3 cardinal  $\lambda > \kappa_\omega(j)$ .*

*Proof.* Suppose towards a contradiction that there is a Reinhardt embedding  $j : V \prec V$  with  $\kappa = \text{crit}(j)$  and  $\lambda_0 = \kappa_\omega(j)$  with  $\lambda_1$  the least I3 cardinal above  $\lambda_0$ . Fix  $\kappa_1$  as the least element of the critical sequence of  $\lambda_1$  which lies above  $\lambda_0$ .  $\kappa_1$  and  $\lambda_1$  are definable from  $\lambda_0$ , hence  $j(\lambda_0) = \lambda_0$ ,  $j(\kappa_1) = \kappa_1$ , and  $j(\lambda_1) = \lambda_1$ .

Thus we can apply Theorem 47 to  $\lambda_1$ , although we choose the  $N \subseteq V_\lambda$  and  $X$  to be minimal in the following sense: Let  $\alpha \geq \lambda_0$  be the least ordinal for which there is an  $X \in V_\lambda$  and  $\mathbb{P} \in V_\alpha$  which satisfy the theorem. In other words,  $\alpha$  is the least ordinal greater than  $\lambda_0$  for which there is a transitive  $N \subseteq V_{\lambda_1}$  where

- (i)  $N \models ZFC$
- (ii)  $N$  is  $\Sigma_2$ -definable in  $V_{\lambda_1}$  from parameter  $X \in V_\alpha$
- (iii) There is a partial order  $\mathbb{P} \in N \cap V_\alpha$  such that for every set of ordinals  $A \in V_{\lambda_1}$  there is an  $\mathbb{P}$ -generic filter  $G$  over  $N$  where  $A \in N[G]$ .

$\alpha$  is definable from  $\lambda_0$  so  $j(\alpha) = \alpha$ . Fix such an  $N$  with  $X, \mathbb{P} \in V_\alpha$ . In  $V_{\lambda_1}$  there is a proper class of supercompact cardinals, namely those of the critical sequence of  $\lambda_1$ . In  $V_{\lambda_1}$ , let  $\lambda$  be the limit of the first  $\omega$ -many supercompact cardinals above  $\alpha$ . Since we are computing this limit inside  $V_{\lambda_1}$  (which satisfies ZF),  $\lambda$  must be an ordinal less than  $\lambda_1$ . Again,  $\lambda$  is definable from  $\alpha$  and  $\lambda_1$ , so  $j(\lambda) = \lambda$ . By our choice of  $\mathbb{P} \in V_\alpha$ , we have guaranteed that  $(\lambda^+)^{V_{\lambda_1}} = (\lambda^+)^N$ . But  $N \models ZFC$ , so we can code  $(V_{\lambda+1})^N$  as a set of ordinals  $A \in N$ . (i.e.  $(V_{\lambda+1})^N \subseteq L[A]$ ). Thus it follows that

$$(\lambda^+)^{V_{\lambda_1}} = (\lambda^+)^N = (\lambda^+)^{L[A]^{V_{\lambda_1}}}.$$

We can now find a contradiction: since  $j(\lambda) = \lambda$ , it must be the case that  $j : V_{\lambda+2} \prec V_{\lambda+2}$ , whereas Theorem 42 applied inside  $V_{\lambda_1}$  declares that no such embedding is possible.  $\square$

Thus the first of the two main theorems of this paper is obtained as a direct corollary of the previous theorem and Theorem 35:

**Corollary 53** (Main Theorem 1). *If the Weak HOD Conjecture holds, then Berkeley Cardinals are inconsistent with ZF.*

This is strong evidence that Berkeley Cardinals and the Weak HOD conjecture are good indicators for the outcome of the HOD dichotomy.

Conversely we drop the Weak HOD conjecture as an assumption and ask what happens if Reinhardt and Berkeley cardinals are actually consistent. This possibility produces relativized versions of those same cardinals.

**Definition 54.**

- (i) A cardinal  $\kappa$  is *HOD-Reinhardt* if there is a non-trivial  $j : \text{HOD} \prec \text{HOD}$  with  $\text{crit}(j) = \kappa$ .
- (ii) A cardinal  $\delta$  is *HOD-Berkeley* if for all transitive sets  $M \in \text{HOD}$  containing  $\delta$  and every  $\eta < \delta$ , there is a  $j : M \prec M$  with  $\eta < \text{crit}(j) < \delta$ .

In the next results,  $S$  is shorthand for “there is a proper class of supercompact cardinals”.

**Proposition 55.** [1] *If  $ZF + S +$  “there is a Reinhardt cardinal” is consistent then  $ZFC + S +$  “there is a HOD-Reinhardt cardinal” is also consistent. Similarly if  $ZF + S +$  “there is a Berkeley cardinal” is consistent then  $ZFC + S +$  “there is a HOD-Berkeley cardinal” is consistent.*

*Proof.* Since there is a proper class of supercompacts, we can use the class forcing  $\mathbb{Q}_\infty$  defined to be the direct limit of all the supercompact forcings  $\mathbb{Q}_\kappa$  as introduced in §5. If  $G$  is  $\mathbb{Q}_\infty$ -generic over  $V$  then it has already

been shown that  $V[G] \models ZFC$  and preserves all the supercompact embeddings. Since  $\mathbb{Q}_\infty$  is homogenous, then we also have that  $\text{HOD}^{V[G]}$  is a definable class in  $V$ . So if  $j$  is a Reinhardt embedding, then its restriction to  $\text{HOD}^{V[G]}$  is a HOD-Reinhardt embedding. Likewise, if  $\delta$  is Berkeley, and  $M \in \text{HOD}^{V[G]}$  contains  $\delta$ , then this situation is of course true in  $V$ . So  $V[G]$  inherits all the Berkeley embeddings on  $\text{HOD}^{V[G]}$  by virtue of the fact that  $\text{HOD}^{V[G]} \subseteq V$ .  $\square$

Consider the effect that these cardinals have on the HOD-Dichotomy in ZFC. If there is an extendible cardinal  $\gamma$  and a HOD-Reinhardt cardinal above it, then there is a regular cardinal  $\lambda > \gamma$  which is  $\omega$ -strongly measurable in HOD. Namely  $\lambda = (\kappa_\omega(j))^+$ , which is a regular cardinal with a  $\lambda$ -complete club filter by choice. If  $\lambda$  were not  $\omega$ -strongly-measurable in HOD, then there would be a stationary partition  $\langle S_\alpha : \alpha < \kappa \rangle \in \text{HOD}$  of  $E_\omega^\lambda$ , to which one may apply Woodin's argument in the proof of Theorem 18 to obtain a contradiction. Yet, the HOD-dichotomy states that this is equivalent to *all* regular cardinals greater than  $\gamma$  being  $\omega$ -strongly measurable in HOD.

**Proposition 56.** (ZFC) *If there is an extendible cardinal  $\gamma$  and a HOD-Reinhardt cardinal  $\kappa > \gamma$ , then every regular cardinal is  $\omega$ -strongly measurable in HOD and HOD is “far” from  $V$ .*

A similar result holds for HOD-Berkeley cardinals, but an important difference here is that it requires no mention of the HOD-dichotomy nor extendible cardinals to show that HOD “far” from  $V$ .

**Theorem 57.** [1] (ZFC) *If there is a HOD-Berkeley cardinal  $\delta$ , then every regular cardinal  $\gamma > \delta$  is  $\omega$ -strongly measurable in HOD.*

*Proof.* If  $\gamma > \delta$  is regular, then we can take  $j : \text{HOD}_{\gamma^+} \prec \text{HOD}_{\gamma^+}$  with  $\text{crit}(j) = \eta < \delta$ . By choice we also have that the club filter on  $\gamma$  is  $\gamma$ -complete. It follows that  $(2^\eta)^{\text{HOD}} < \delta < \gamma$ , so suppose there is a partition  $\langle S_\alpha : \alpha < \eta \rangle \in \text{HOD}$  of  $E_\omega^\gamma$  into stationary sets. Apply the proof of Kunen's Theorem 18 to obtain a contradiction.  $\square$

The conjunction of the previous theorem and Proposition 55 demonstrates the Main Theorem 2.

Let's take a moment to review what has been shown and the effects on inner models and large cardinals. In one possible future, the Weak HOD Conjecture holds where is it *provable* from standard large cardinal assumptions that HOD is “close” to  $V$ , i.e. it satisfies the weak covering property. Moreover, HOD is a weak extender model, thereby reflecting all the standard large cardinals into HOD. Lastly, the closeness of HOD to  $V$  acts as a structural barrier to the consistency of large cardinals beyond choice with ZF alone. Inner model theory edges closer to finding an “ultimate”  $L$ -like inner model which contains nearly all of  $V$ 's cardinal structure.<sup>12</sup>

<sup>12</sup>[1] gives a more detailed discussion of the possible futures of inner models and large cardinals, especially relating to Woodin's Ultimate-L program.

Alternatively, it could be true that Berkeley cardinals are consistent with ZF. Even if this is true, there will likely never be a proof of this fact, since it will have to appeal to even stronger principles which would require a radical shift in views on the role of the axiom of choice. If it is indeed the case that they are consistent, then Berkeley cardinals and HOD-Berkeley cardinals act as strong sharps for HOD. The Weak HOD Conjecture fails and HOD is not a weak extender model. Inner model theory only captures weak portions of  $V$ 's large cardinal hierarchy and fails to give a fine-structural insight to the whole of  $V$ . The program of inner models for large cardinals fails at the level of supercompactness.

Whether or not we prove the Weak Hod Conjecture or that Berkeley cardinals are inconsistent, it is certain that the development of the theory of cardinals beyond choice will have a profound effect on inner models and set theory as a whole. While a solution to any of these problems is distant, one should keep an eye on the theory of cardinals beyond choice and its relation to inner models; these programs have the potential to fundamentally shift our understanding of set theory.

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