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GRAU DE MATEMÀTIQUES

Treball final de grau

Can one hear the area of a drum?

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Abstract

Timbre is one of the most important treats in music as it enables the distinction of different instruments even when each one plays the same note. For more general vibrating smooth bounded domains, timbre can be related with the set of eigenvalues of the Laplacian. Having this considered, the aim of this work is to explore this vibrational problem, by studying the existence of solutions of the Laplacian problem with different boundary conditions and explore the inverse spectral problem with Weyl's law, in order to see how the spectrum determines the area or volume of the domain. In addition, nodal domains will be defined and Courant's theorem proved and sharpened .

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Chapter 1

Introduction

When a musical instrument is played, independently of the form, colors and materials, there is always a unique requirement: the vibration of an element in the instrument in a oscillatory way. The string of a guitar, the membrane of a snare drum, and even the metallic surface of a trumpet, they all need to vibrate in order to produce musical notes, and music, eventually.

When a membrane is struck, it undergoes complex oscillations that propagate across its surface, producing audible sound waves. These vibrations can be mathematically described using the wave equation, a partial differential equation that models the propagation of waves in elastic media. The solutions to the wave equation gives us the displacement of each point a vibrating surface and are expressed, normally in physics, as normal modes, which are specific patterns of vibration associated with discrete frequencies. These frequencies, also called as harmonics, are determined by the geometry, tension, and material properties of the surface, as well as conditions in its edges. The combination of these frequencies forms the spectrum of frequencies, which includes the fundamental frequency and a series of overtones.

In this context, any musical instrument is characterized by an important aspect: its timbre. The Acoustical Society of America (ASA) Acoustical Terminology defines timbre as "that attribute of auditory sensation which enables a listener to judge that two nonidentical sounds, similarly presented and having the same loudness and pitch, are dissimilar". It also adds, "Timbre depends primarily upon the frequency spectrum, although it also depends upon the sound pressure and the temporal characteristics of the sound".

Therefore, timbre allows us to differentiate between a violin and a trumpet even when they play the same note, with the same frequency. In other words, what allows to differentiate between musical instruments is completely related to the collection of harmonics or spectrum.

The main doubt that appears when talking about timbre is the following: if a membrane or a vibrating surface characterizes and defines the collection of frequencies of the instrument, given the complete set of harmonics which properties of the membrane could be predicted?

1.1 Mathematical set up of the physical problem

Let us put mathematical context to the vibrational description given above. If we have a membrane Ω that is fixed along its boundary $\partial\Omega$ and is set in motion in the direction perpendicular to the plane, the vertical displacement of any point of Ω at any time can be described by a function $F(x, y, t)$ (smooth) that obeys the wave equation:

$$\partial_{tt}F - c^2\Delta F = 0$$

where c is a constant that depends on the tension and density of the surface.

Of special interest (both to the mathematician and to the musician) are solutions of the form

$$F(x, y, t) = U(x, y)T(t)$$

for being harmonic in time, they represent the pure tones the membrane is capable of producing. These special solutions are also known as normal modes. Substituting this function inside the wave equation we get that

$$\frac{T''(t)}{T(t)} = \frac{\Delta U(x, y)}{U(x, y)} \quad \text{in } \Omega$$

As this equals expressions that do depend on different variables, each side must be the same constant value $-\lambda < 0$. The reason behind the negative sign has a more physical meaning, since the force that governs a vibrational domain must be restoring and the force is proportional to $\partial_{tt}F$ by Newton's Second Law.

Looking into the temporal part, we have directly the equation:

$$T''(t) = -\lambda T(t)$$

which has the general solution

$$T(t) = A\sin(\sqrt{\lambda}t) + B\cos(\sqrt{\lambda}t)$$

where A, B are constants. For the spacial part, however, we obtain:

$$\begin{cases} \Delta U + \lambda U = 0 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

As we can see, U will be a eigenfunction of the Laplacian and λ the eigenvalue and associated to the harmonics. Then, in order to understand the vibration of a membrane, as the temporal part is well-described (constants are to be found), we need to focus on the spacial part.

Therefore, this problem does not only have implications and interest in acoustic physics and musical circles, it can be studied in a more general mathematical sense. And this is the way we will study this problem. Given a bounded smooth domain $\Omega \subset \mathbb{R}^n$, how are the functions that suffixes (1.1), if they exist? Given the spectrum, can we excel some properties of the bounded domain, for example its area?

1.2 Overview

All things considered, our main objective with this paper is to answer these last questions. In order to do it, the structure of this text is as follows:

Firstly, we are defining all the basic concepts related with EDPs and their solution's spaces in what is left of **Chapter 1**.

In **Chapter 2**, we are going to show in detail the existence of eigenfunctions and eigenvalues of (1.1) for a bounded smooth domain $\Omega \subset \mathbb{R}^n$. The main result is:

Theorem 1.1. *For any $\Omega \subset \mathbb{R}^n$, there existst a sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ and functions $\phi_k \in H_0^1(\Omega)$ that are distinct weak solutions of:*

$$\left.\begin{aligned} -\Delta\phi = \lambda\phi \quad in \quad \Omega \\ \phi = 0 \quad on \quad \partial\Omega \end{aligned}\right\}$$

Moreover, these functions form an orthonormal basis for $L^2(\Omega)$.

Chapter 3 explores the asymptotics of the eigenvalues of (1.1), connecting it with the area or volume of the domain Ω, eventually proving Weyl's law. In other words, proving that the dimensions of a domain can be predicted or listened (if we see it physically) with the spectrum of eigenvalues.

Theorem 1.2. *(Weyl's law) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Let $\rho > 0$ and denote $D(\rho) = \#\{\lambda_k \in Spectrum(Q) : \lambda_k \leqslant \rho)\}$, then:*

$$\lim_{\rho\to\infty} \frac{D(\rho)}{\rho^{n/2}} = \frac{\omega_n}{2^n}\frac{|\Omega|}{\pi^n}$$

where ω_n is the hypervolume of the unitary ball in $\mathbb{R}^n$. Equivallently,

$$\lim_{k\to\infty} \lambda_k = 4\pi^2 \frac{k^{2/n}}{(\omega_n|\Omega|)^{2/n}}$$

Eventually, in **Chapter 4** we will define nodal domains for completeness and using all the important results proved, we will try to obtain new results about nodal domains. Pleijel's nodal domain theorem is the one we will be proving.

Theorem 1.3. *(Pleijel's nodal domain theorem)*

$$\limsup_{k\to\infty} \frac{N}{k} \leqslant \left(\frac{2}{\beta_{1,0}}\right)^2 \approx 0.691...$$

where N is the number of nodal domains of ϕ_k and $\beta_{1,0}$ is the first zero of the Bessel function J_0 and its value is approximately $\beta_{1,0} \approx 2.4048...$

1.3 Preliminars

We will first give a quick review on integration by parts, L^p, Sobolev and Hölder spaces, stating the results that will be used later in the text.

Firstly, a **partial derivative equation**, or also called a PDE , is any equation where the unknown is a function f of several variables (usually $x \in \mathbb{R}^n$ and maybe also $t \in \mathbb{R}$), and which their **partial derivatives** are involved (i.e, $\partial_{x_i} f$ or $\partial_{x_i x_j} f$ or even $\partial_t f$). Generally in this text, f will be a smooth function defined in a bounded domain $\Omega \subset \mathbb{R}^n$. In this context, we define the laplacian of a function f as :

$$\Delta f = \nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + ... + \frac{\partial^2 f}{\partial x_n^2} \tag{1.2}$$

The most famous EDPs associated with the laplacian are:

$$\Delta f = 0 \quad \text{Laplace equation} \tag{1.3}$$

$$\Delta f = u \quad \text{Poisson equation} \tag{1.4}$$

where $u : \mathbb{R}^n \to \mathbb{R}$ is a function.

Integration by parts

A fundamental identity in the study of PDEs is the following.

Theorem 1.4. *(Integration by parts) For any bounded (C^1) domain $\Omega \subset \mathbb{R}^n$, and any pair of functions $f, g \in C^1(\overline{\Omega})$, we have:*

$$\int_\Omega f \partial_{x_i} g = -\int_\Omega g \partial_{x_i} f + \int_{\partial\Omega} f g \nu_i \quad (i = 1, ..., n) \tag{1.5}$$

where ν is the unit (outward) normal vector to $\partial\Omega$, and $i = 1, 2, ..., n$. In particular, for any functions $u, w \in C^2(\overline{\Omega})$ we have:

$$\int_\Omega \nabla u \cdot \nabla w = -\int_\Omega u \Delta w + \int_{\partial\Omega} u \nabla w \cdot \nu \tag{1.6}$$

L^p spaces

Given $\Omega \subset \mathbb{R}^n$ and $1 \leqslant p < \infty$, the space $L^p(\Omega)$ is the set:

$$L^p(\Omega) := \left\{ u \textit{ measurable in } \Omega : \int_\Omega |u|^p dx < \infty \right\}$$

It is a complete normed space, with the norm: $\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p\right)^{1/p}$. When $p = \infty$, the integral is replaced by the essential supremum, $\|u\|_{L^\infty(\Omega)} := \sup\limits_{x \in \Omega} |u(x)|$, and $L^\infty(\Omega)$ is the set of bounded functions (up to sets of measure zero).

Moreover, C^∞ functions are dense in $L^p(\Omega)$ for $1 \leqslant p < \infty$. of functions u such that $\int_\Omega |u|^p < \infty$. Eventually, it is crucial the next result.

Lemma 1.5. ***(Hölder Inequality)****. Let f, g be measurable functions on $\mathbb{R}^n$. For $p, q \geqslant 1$ with $1/p + 1/q = 1$,*

$$\|fg\|_{L^1(\Omega)} \leqslant \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^q(\Omega)}$$

Sobolev spaces

As we will see soon in order to prove existence theorems for PDEs Sobolev spaces are crucial. The idea behind them is to consider the space of functions $u \in L^2(\Omega)$ whose derivatives are also living in $L^2(\Omega)$:

$$H^1(\Omega) = \{u \in L^2(\Omega) \mid \partial_{x_i} u \in L^2(\Omega) \textit{ for i=1, ..., n}\} \tag{1.7}$$

However, one needs to be careful with this definition (we will see it is a good definition, though) as L^2 functions are not differential in general. Then, it can be provided, a more general definition.

Definition 1.6. *Let $u \in C^\infty(\overline{\Omega})$ a function. We consider ther norm:*

$$\|u\|^2_{H_1(\Omega)} = \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2$$

Then, the space $H^1(\Omega)$ is defined as the clousure of $C^\infty(\overline{\Omega})$ underneath the norm $\|u\|_{H_1(\Omega)}$.

It remains to understand in which sense (1.7) is a good definition.

Proposition 1.7. *For any $u \in H^1(\Omega)$, in $i = 1, ..., n$, satisfying:*

$$\int_\Omega u \partial_{x_i} \varphi = -\int_\Omega u_i \varphi \quad \forall \varphi \in C^\infty_C(\overline{\Omega}) \tag{1.8}$$

*Such functions u_i are called **weak deriavtives** of u. Morevover, if $u \in C^1(\Omega)$ then we simply have $u_i = \partial_{x_i} u$.*

Proof. By definition, a function $u \in L^2\Omega$ belongs to $H^1(\Omega)$ if and only if we have a sequence $v_k \in C^\infty(\overline{\Omega})$ such that $v_k \to u$ in $L^2\Omega$ and v_k is a Cauchy sequence with the $H^1(\Omega)$ norm.

Since $v_k \in C^\infty$, we know the integration by parts formula:

$$\int_\Omega v_k \partial_{x_i} \varphi = -\int_\Omega \partial_{x_i} v_k \varphi$$

$\forall \varphi \in C^\infty_C(\overline{\Omega})$. We want to see the asymtotic behavior of the identity, in other words, we want to take $k \to \infty$. Since:

$$\|v_k\|^2_{H_1(\Omega)} = \int_\Omega |v_k|^2 + \int_\Omega |\nabla v_k|^2 \geqslant \int_\Omega |\nabla v_k|^2 \geqslant \|\partial_{x_i} v_k\|^2_{L_2(\Omega)}$$

and v_k is a Cauchy sequence with the $H^1(\Omega)$ norm, which means that $\partial_{x_i} v_k$ is a Cauchy sequence with $L^2(\Omega)$ norm. As $L^2(\Omega)$ is complete then there exists $u_i \in L^2(\Omega)$ such that $\partial_{x_i} v_k \to u_i$ in $L^2(\Omega)$.But then, we are able to pass to the limit the integration by parts identity, in order to get:

$$\int_\Omega v_k \partial_{x_i} \varphi = -\int_\Omega \partial_{x_i} v_k \varphi \xrightarrow[k\to\infty]{} \int_\Omega u \partial_{x_i} \varphi = -\int_\Omega u_i \varphi$$

and we are done. $\square$

Remark 1.8. In particular, we have shown that for any $u \in H^1(\Omega)$ we can always find $v_k \in C^\infty(\overline{\Omega})$ such that:

$$\begin{aligned} v_k \to u \quad & in \quad L^2(\Omega) \\ \partial_{x_i} v_k \to u_i \quad & in \quad L^2(\Omega) \end{aligned}$$

We will also need the following:

Definition 1.9. *We denote by $H_0^1(\Omega)$ the clousure of $C_C^\infty(\overline{\Omega})$ in $H^1(\Omega)$. This is the set of functions $u \in H^1$ that, in some sense, satisfy $u = 0$ on $\partial\Omega$.*

As it is compacted in [FR22], we can subtract the basic properties of these spaces:

- $H_0^1(\Omega)$ and $H^1(\Omega)$ are complete spaces. In addition, H_0^1 and H^1 functions can be approximated by C_C^∞ and C^∞, respectively.
- $C^\infty(\overline{\Omega})$ functions are dense in $H^1(\Omega)$.
- We have the inclusions $C^\infty(\overline{\Omega}) \subset H^1(\Omega) \subset L^2(\Omega)$
- Integration by parts holds for any $u \in H^1(\Omega)$.
- $H_0^1(\Omega)$ is the set of functions $u = 0$ on $\partial\Omega$.

Remark 1.10. Density and compactness properties of both L^p and Sobolev spaces are well proved in [Kal23].

Theorem 1.11. ***(Poincaré inequality, 1890)*** *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. Then, there exists a constant R, depending only on Ω, such that:*

$$\int_\Omega |u|^2 \leqslant R \int_\Omega |\nabla u|^2 \qquad \forall u \in H_0^1(\Omega)$$

Proof. **Step 1:** It can be easily proved that in dimensions $n = 1$ we have that $\forall u \in C_C^\infty((a,b))$:

$$\int_a^b |u|^2 \leqslant R \int_a^b |u'|^2$$

where $u' = du/dx$.

Step 2: We prove the inequality for $u \in C_C^\infty(\Omega)$, as follows. If we describe a point in $\mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we can write $\int_\Omega |u|^2$ (since the function is nule in $\mathbb{R}^n \setminus \Omega$ and as the domain is bounded, its also bounded in its nth-component):

$$\int_{\mathbb{R}^{n-1}\times(a,b)} |u|^2 = \int_{\mathbb{R}^{n-1}} \left(\int_a^b |u(x', x_n)|^2 dx_n \right) dx' \underset{\text{ST 1}}{\leqslant} \int_{\mathbb{R}^{n-1}} \left(R \int_a^b |\partial_{x_n} u(x', x_n)|^2 dx_n \right) dx'$$

Taking this last term:

$$\int_{\mathbb{R}^{n-1}} \left(R \int_a^b |\partial_{x_n} u(x', x_n)|^2 dx_n \right) dx' = R \int_\Omega |\partial_{x_n} u|^2 \leqslant R \int_\Omega |\nabla u|^2$$

And we get the result that we wanted.

Step 3: Suppose we have $u \in H_0^1(\Omega)$. We have learned that we can always find $u_k \in C_C^\infty(\Omega)$ such that $u_k \to u$ in $L^2(\Omega)$ and $\partial_{x_i} u_k \to \partial_{x_i} u$ in $L^2(\Omega)$. But then we have that by the Step 2, that the inequality hols for the succession, then:

$$\left(\int_\Omega |u_k|^2 \leqslant R \int_\Omega |\nabla u_k|^2 \right) \xrightarrow[k \to \infty]{} \int_\Omega |u|^2 \leqslant R \int_\Omega |\nabla u|^2$$

and the result follows. □

Remark 1.12. It is crucial that the constant R does not depend on the function u. Moreover, the inequality holds in $H_0^1(\Omega)$, but not in $H^1(\Omega)$.

We can finally state the compactness result for bounded sequences in H^1.

Theorem 1.13. ***(Rellich, 1930)*** *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, and let $f_k \in H^1(\Omega)$ be any sequence satisfying*

$$\int_\Omega |u|^2 \leqslant M_1 \quad \textit{and} \quad \int_\Omega |\nabla u|^2 \leqslant M_2$$

Then, there exists $f \in H^1(\Omega)$ and a sequence such that

$$f_{k_m} \to f \quad \textit{in} \quad L^2(\Omega)$$
$$\int_\Omega |\nabla f|^2 \leqslant \liminf_{k_m \to \infty} \int_\Omega |\nabla f_{k_m}|^2$$

Morever, if $f_k \in H_0^1(\Omega)$ for all k, then $f \in H_0^1(\Omega)$, too.

Chapter 2

Laplace Equation and Eigenfunctions

In order to prove the existence of eigenvalues and eigenfunctions of the Laplacian, we will first start the study of harmonic functions:

$$\Delta u = 0 \quad \text{in} \quad \Omega$$

and more generally of solutions of

$$\Delta u = f \quad \text{in} \quad \Omega$$

(where $f(x)$ is given, and $u(x)$ is the unknown).

This is one of the simplest and at the same time most ubiquitous and central PDEs. Usually one wants to construct harmonic functions in Ω, given its boundary values on $\partial\Omega$. This is called the Dirichlet problem:

$$\left.\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \partial\Omega \end{aligned}\right\} \tag{$\star$}$$

where $g(x)$ is given. Our first goal will be to prove the existence of solutions to the Dirichlet problem for any smooth domain Ω and any smooth function g.

2.1 Existence and Uniqueness

The existence of solutions to the Dirichlet problem turned out to be fundamental in many areas of Mathematics and Physics, and the approach to solve it was originally suggested around 1847 by William Thomson and Gustav Lejeune-Dirichlet. It is the following:

Proposition 2.1. ***(Dirichlet principle):*** *Let $u \in C^2(\overline{\Omega})$, with $u = g$ on $\partial\Omega$. Then the following are equivalent:*

(i) u is harmonic in Ω, i.e., $\Delta u = 0$ in Ω.

(ii) u minimizes the Dirichlet energy:

$$E(u) = \int_\Omega |\nabla u|^2 \, dx$$

among all functions w satisfying $w = g$ on $\partial\Omega$.

Proof. **(i)⇒(ii)** Assume $\Delta u = 0$ in Ω, and let us prove that u minimizes the Dirichlet energy for all $w \in C^2(\overline{\Omega})$ with $w = g$ on $\partial\Omega$, in other words, that

$$\int_\Omega |\nabla u|^2 \leqslant \int_\Omega |\nabla w|^2$$

Let us denote $v = w - u$, which satisfies $v = 0$ on $\partial\Omega$. Then:

$$E(w) = \int_\Omega |\nabla u + \nabla v|^2 = \int_\Omega |\nabla u|^2 + 2\int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla v|^2 \ \geqslant \int_\Omega |\nabla u|^2$$

where we used that:

$$\int_\Omega \nabla u \cdot \nabla v \ = 0 \quad \text{(integration by parts expression)}$$

This proves the first part: if u is harmonic, then it minimizes the Dirichlet energy.

(ii)⇒(i) Now assume that u minimizes the Dirichlet energy, and let us prove that $\Delta u = 0$ in Ω. Consider $w = u + \varepsilon v$, for any $v \in C_C^\infty(\Omega)$ and any $\varepsilon \in \mathbb{R}$. Then:

$$E(u + \varepsilon v) \geq E(u).$$

Expanding this expression:

$$E(u + \varepsilon v) = \int_\Omega |\nabla u|^2 + 2\varepsilon \int_\Omega \nabla u \cdot \nabla v + \varepsilon^2 \int_\Omega |\nabla v|^2$$

At $\varepsilon = 0$, the left-hand side has a minimum. Thus, as a function of ε we have:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} E(u + \varepsilon v) = 0 \quad \forall v \in C_C^\infty(\Omega)$$

Computing this we obtain:

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left\{ \int_\Omega |\nabla u|^2 + 2\varepsilon \int_\Omega \nabla u \cdot \nabla v + \varepsilon^2 \int_\Omega |\nabla v|^2 \right\} = 2\int_\Omega \nabla u \cdot \nabla v = -2\int_\Omega \Delta u \cdot v$$

which implies that $\Delta u = 0$ in Ω (This is because if we supposed $\Delta u > 0$ in a Disk Q contained in Ω and we took a $v \in C_C^\infty(\Omega)$ positive in the disk and nule in the rest of the domain, then we would get that: $\int_\Omega \Delta u \cdot v = \int_Q \Delta u \cdot v > 0$, which contradicts the result obtained). □

Remark 2.2. This transforms the existence problem for a PDE into a minimization problem (in finite dimension).

In a more general Dirichlet problem,

$$\left.\begin{aligned} \Delta u = f \quad &\text{in} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}\right\} \tag{$\star\star$}$$

it exists an equivalent result:

Proposition 2.3. *Let $u \in C^2(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$. Then the following are equivalent:*

(i) u solves $\Delta u = f$ in Ω.

(ii) u minimizes the energy functional:

$$E(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx + \int_\Omega f w \, dx$$

among all functions $w \in C^2(\overline{\Omega})$ with $w = 0$ on $\partial\Omega$.

Remark 2.4. If we can solve $(\star\star)$ for any f, then we can solve $(\star)$ for any smooth boundary condition g. Indeed, for any smooth domain Ω and any $g \in C^\infty(\partial\Omega)$, we can extend g to a function $g \in C^\infty(\overline{\Omega})$. Just by denoting $\overline{u} := u - g$

$$\left.\begin{array}{lll} \Delta u = 0 & \text{in} & \Omega \\ u = g & \text{on} & \partial\Omega \end{array}\right\} \quad \Leftrightarrow \quad \left.\begin{array}{lll} \Delta \overline{u} = -\Delta g & \text{in} & \Omega \\ \overline{u} = 0 & \text{on} & \partial\Omega \end{array}\right\} \quad \Leftrightarrow \quad \left.\begin{array}{lll} \Delta \overline{u} = f & \text{in} & \Omega \\ \overline{u} = 0 & \text{on} & \partial\Omega \end{array}\right\}$$

where $f := -\Delta g$. Thus, it suffices to prove the existence of solutions to $(\star\star)$.

2.2 Existence of Minimizers and Regularity

We will next prove the existence of minimizers:

Theorem 2.5. *For any $f \in L^2(\Omega)$, there exists a function $u \in H_0^1(\Omega)$ such that:*

$$E(u) \leqslant E(w) \quad \forall w \in H_0^1(\Omega).$$

where $E(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx + \int_\Omega f w \, dx$

Moreover, u satisfies the equation:

$$\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

in the weak sense. Notice that $u \in H_0^1(\Omega)$ already means that $u = 0$ on $\partial\Omega$ (in the sense of Sobolev spaces).

Proof. **Step 1:** *Boundedness from below.* First, we show that the functional $E(w)$ is bounded from below. Using Poincaré's inequality, we have:

$$\frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega f \cdot w \underset{\text{P.I}}{\geqslant} \frac{c_n}{2}\int_\Omega w^2 - \int_\Omega |f|\cdot|w| \underset{\text{H.I}}{\geqslant} \frac{c_n}{2}\int_\Omega w^2 - \left(\int_\Omega f^2\right)^{1/2} \cdot \left(\int_\Omega w^2\right)^{1/2}$$

Definying as $\lambda = c_n$, $A = \left(\int_\Omega w^2\right)^{1/2}$ and $B = \left(\int_\Omega f^2\right)^{1/2}$, we have that $\lambda A^2 + 1/\lambda B^2 \geqslant 2AB$ as always if $\sqrt{\lambda}A - 1/\sqrt{\lambda}B)^2 \geqslant 0$, then $\lambda A^2 + 1/\lambda B^2 \geqslant 2AB$. Thus,

$$\frac{c_n}{2}\int_\Omega w^2 - \left(\int_\Omega f^2\right)^{1/2} \cdot \left(\int_\Omega w^2\right)^{1/2} \geqslant -\frac{1}{2c_n}\int_\Omega f^2 = -C_o$$

Therefore, the functional $E(w)$ is bounded from below, and we may consider

$$\theta_0 := \inf_{w \in H_0^1(\Omega)} \{E(w)\}$$

and minimizing sequence $\{u_k\} \subset H_0^1(\Omega)$ such that:

$$E(u_k) \to \theta_0 \quad \text{as} \quad k \to \infty.$$

Step 2: We will next prove that the sequence u_l is bounded in $H_0^1(\Omega)$. Indeed, we have that $E(u_k) \leqslant M$, then

$$M \geqslant \frac{1}{2}\int_\Omega |\nabla u_k|^2 + \int_\Omega f \cdot u_k \underset{\text{H.I}}{\geqslant} \frac{1}{2}\int_\Omega |\nabla u_k|^2 - \left(\int_\Omega u_k^2\right)^{1/2} \cdot \left(\int_\Omega f^2\right)^{1/2} = T$$

We will use again that $\lambda A^2 + 1/\lambda B^2 \geqslant 2AB$ and the Poincare inequality,

$$T \geqslant \frac{1}{2}\int_\Omega |\nabla u_k|^2 - \lambda \int_\Omega u_k^2 - \frac{1}{\lambda}\int_\Omega f^2 \underset{\text{P.I}}{\geqslant} \frac{1}{2}\int_\Omega |\nabla u_k|^2 - \lambda \cdot c_n \int_\Omega |\nabla u_k|^2 - \frac{1}{\lambda}\int_\Omega f^2$$

If choosing $\lambda \cdot c_n = 1/4$ then we get that

$$\frac{1}{2}\int_\Omega |\nabla u_k|^2 - \lambda \cdot c_n \int_\Omega |\nabla u_k|^2 - \frac{1}{\lambda}\int_\Omega f^2 = \frac{1}{4}\int_\Omega |\nabla u_k|^2 - 4c_n \int_\Omega f^2$$

We deduce that,

$$\int_\Omega |\nabla u_k|^2 \leqslant 4M + 16c_n \int_\Omega f^2 = M_1$$

By Poincaré inequality we have,

$$\int_\Omega u_k^2 \leqslant c_n \int_\Omega |\nabla u_k|^2 \leqslant c_n M_1 = M_2$$

This means that the sequence $\{u_k\}$ is bounded in $H_0^1(\Omega)$

Step 3: Let's finally see the existence of the minimizer. Since the sequence $\{u_k\}$ is bounded in $H_0^1(\Omega)$, by the Rellich-Kondrachov compactness theorem, there exists a function $u \in H^1(\Omega)$ and a subsequence $\{u_{k_m}\}$ such that

$$u_{k_m} \to u \quad in \quad L^2(\Omega) \qquad and \qquad \int_\Omega |\nabla u|^2 \leqslant \liminf_{k_m \to \infty} \int_\Omega |\nabla u_{k_m}|^2$$

In particular, $\int_\Omega u_{k_m} \cdot f \to \int_\Omega u \cdot f \quad \text{as} \quad k \to \infty$. Hence, by $E(u)$'s definition:

$$\frac{1}{2}\int_\Omega |\nabla w|^2 + \int_\Omega f \cdot w \leqslant \liminf_{k_m \to \infty} \frac{1}{2}\int_\Omega |\nabla u_{k_m}|^2 + \lim_{k_m \to \infty} \int_\Omega u_{k_m} \cdot f = \liminf_{k_m \to \infty} E(u_{k_m}) = \theta_0$$

In conclusion,

$$E(u) \leqslant E(w) \quad \forall w \in H_0^1(\Omega).$$

and thus u is the minimizer of the functional $E(u)$. □

We have proved that if $u \in C^2$ then, u solves $(\star\star)$ if and only if u minimizes $E(u)$. Moreover, there always a minimizer of the functional, which belongs to $H_0^1(\Omega)$. However, in which sense does $u \in H_0^1(\Omega)$ solve $\Delta u = f$ in Ω? We will first answer the last question:

Proposition 2.6. *Let $f \in L^2(\Omega)$, and let with $u \in H_0^1(\Omega)$ be the minimizer of*

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu$$

in the sense that

$$E(u) \leqslant E(w) \quad \forall w \in H_0^1(\Omega).$$

Then, we have

$$-\int_\Omega \nabla u \cdot \nabla \eta = \int_\Omega f\eta \quad \forall\, \eta \in H_0^1(\Omega), \tag{W}$$

Proof. For any $\eta \in H_0^1(\Omega)$ and any $\varepsilon \in \mathbb{R}$ we have $E(u) \leqslant E(u + \varepsilon\eta)$ and this means that $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(u+\varepsilon\eta) = 0$:

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 + \varepsilon \int_\Omega \nabla u \cdot \nabla \eta + \frac{\varepsilon^2}{2}\int_\Omega |\nabla \eta|^2 + \int_\Omega fu + \varepsilon \int_\Omega f\eta \right\}$$

Computing the derivative correctly:

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\eta = 0$$

and we are done □

Corollary 2.7. *There is a unique minimizer $u \in H_0^1(\Omega)$ of $E(u)$.*

Proof. Assume $u_1, u_2 \in H_0^1(\Omega)$ are two minimizers of $E(u)$. Then,

$$-\int_\Omega \nabla u_i \cdot \nabla \eta = \int_\Omega f\eta \quad \forall\, \eta \in H_0^1(\Omega), \quad (i = 1, 2)$$

Subtracting these equations, we deduce:

$$-\int_\Omega (\nabla u_1 - \nabla u_2) \cdot \nabla \eta = 0 \quad \forall\, \eta \in H_0^1(\Omega)$$

Choosing $\eta = u_1 - u_2$, we deduce:

$$\int_\Omega |\nabla(u_1 - u_2)|^2 = 0,$$

and therefore $u_1 = u_2$, as required. □

Definition 2.8. *We say that $u \in H_0^1(\Omega)$ solves $\Delta u = f$ in Ω in the weak sense whenever (**W**) holds.*

Notice that if $u \in C^2(\Omega)$ then this is equivalent to $\Delta u = f$ in Ω:

$$W \underset{\text{Int by parts}}{\Leftrightarrow} -\int_\Omega \Delta u \cdot \eta = \int_\Omega f\eta \quad \forall\, \eta \in C_C^\infty(\Omega) \Leftrightarrow -\Delta u = f \text{ in } \Omega$$

Thus, we have proved that there exists a unique minimizer $u \in H_0^1(\Omega)$ of $E(u)$, and such minimizer solves $-\Delta u = f$ in Ω in the weak sense. Therefore, it is a weak solution of the Dirichlet problem.

2.3 Regularity and further properties

Recall the first simple Dirichlet problem:

$$\Delta u = 0 \quad \text{in}\, \Omega, \quad u = g \quad \text{on}\, \partial\Omega.$$

We have proved that, given any $f \in L^2(\Omega)$, there exists a unique weak solution of the Dirichlet problem ($\star\star$). Moreover, as a consequence, given any $g \in C^\infty(\partial\Omega)$, there exists a unique weak solution of ($\star$).

The first natural question concerns the regularity of solutions: are these solutions actually more regular?

Theorem 2.9. *Given any $f \in C^\infty(\overline{\Omega})$, the weak solution of ($\star\star$) is $C^\infty(\overline{\Omega})$. In particular, for any $g \in C^\infty(\partial\Omega)$, the weak solution of ($\star$) is $C^\infty(\overline{\Omega})$.*

Remark 2.10. We will not prove such a regularity result here. It is somewhat difficult to prove, and our main focus is on the existence of solutions. We refer to [Owe03, FR22] for further reading in this direction.

Definition 2.11. *For $C^\infty(\overline{\Omega})$ functions and $k \geqslant 1$, we define the norm:*

$$\|u\|^2_{H^k(\Omega)} = \left(\sum_{|\alpha| \le k} \|\partial^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2} = \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 + ... + \int_\Omega |\nabla^k u|^2$$

Then, we define the space $H^k(\Omega))$ as the closure of $C^\infty(\overline{\Omega})$ functions under the norm $H^k(\Omega)$.

$H^k(\Omega)$ is a complete normed space, and $C^\infty(\overline{\Omega})$ functions are dense in $H^k(\Omega)$. For all $k \geqslant 0$, an important property is the following:

Theorem 2.12. *Let $f \in H^k$, and u be the week solution of ($\star\star$). Then, $u \in H^{k+2}$ and*

$$\|u\|_{H^{k+2}(\Omega)} \leqslant C \|f\|_{H^k(\Omega)}$$

with C depending only on k and Ω. In particular, if $f \in H^k(\Omega)$ for all k, then $u \in C^k(\overline{\Omega})$

The powerful idea behind the result is that: " u is two derivatives more regular than f".

2.4 Eigenvalues of the Laplacian

We are now in position to start studying the main element of this text: the eigenvalues and eigenfunctions of the Laplacian, i.e., nontrivial solutions of

$$\left.\begin{aligned} -\Delta\phi = \lambda\phi \quad \text{in} \quad \Omega \\ \phi = 0 \quad \text{on} \quad \partial\Omega \end{aligned}\right\} \tag{2.1}$$

for some $\lambda \in \mathbb{R}$.

The number λ is the eigenvalue, and ϕ is the eigenfunction. Our main goal will be to prove the existence of a sequence of eigenvalues of the Laplacian for any bounded smooth domain of $\Omega \subset \mathbb{R}^n$:

Theorem 2.13. *For any $\Omega \subset \mathbb{R}^n$, there existst a sequence $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$ and functions $\phi_k \in H_0^1(\Omega)$ that are distinct weak solutions of* (2.1). *Moreover, these functions form an orthonormal basis for $L^2(\Omega)$.*

Definition 2.14. *Let $\Omega \subset \mathbb{R}^n$ be any bounded smooth domain. The spectrum of Ω is the set of $\lambda \in \mathbb{R}$ eigenvalues for which exist non-trivial solutions of* (2.1) *(eigenfunctions). It can be denoted as $Spectrum(\Omega)$.*

Remark 2.15. Orthonormal Basis of $L^2(\Omega)$ The functions ϕ_k are said to form an orthonormal basis of $L^2(\Omega)$when

1. $\int_\Omega \phi_k^2 = 1 \quad$ and $\quad \int_\Omega \phi_i \cdot \phi_j = 0 \quad \forall i \neq j$
2. Any function $f \in L^2(\Omega)$ can be written as an infinite sum $f(x) = \sum_{i=1}^{\infty} a_k \phi_k(x)$, for some constants $a_k \in \mathbb{R}$. Thanks to orthonormality, the constants are given by $a_k = \int_\Omega f \cdot \phi_k$.

A key observation to prove the theorem is the following:

Lemma 2.16. *Let $\phi \in H_0^1(\Omega)$ be the function that minimizes the Rayleigh quotient*

$$J(w) := \frac{\int_\Omega |\nabla w|^2}{\int_\Omega w^2}$$

among all functions $w \in H_0^1(\Omega), u \neq 0$. Then, ϕ is a weak solution of $-\Delta\phi = \lambda\phi$ in Ω, for some $\lambda > 0$.

Proof. Let $\phi \in H_0^1(\Omega)$ be a minimizer of Rayleigh quotient and let $v = \phi + \varepsilon\varphi$ for some $\varphi \in H_0^1(\Omega)$ and $\varepsilon \in \mathbb{R}$.Then, we have that $J(\phi) \leqslant J(w) \quad \forall \in H_0^1(\Omega), u \neq 0$. In particular,

$$J(\phi) \leqslant J(\phi + \varepsilon\varphi)$$

Then, we have a minimum at $\epsilon = 0$, and

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J(\phi + \varepsilon\varphi) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left\{ \frac{\int_\Omega |\nabla u|^2 + 2\varepsilon \int_\Omega \nabla u \cdot \nabla v + \varepsilon^2 \int_\Omega |\nabla v|^2}{\int_\Omega \phi^2 + 2\varepsilon \int_\Omega \phi\varphi + \varepsilon^2 \int_\Omega \varphi^2} \right\} =$$

which gives

$$0 = \frac{2\left(\int_\Omega \nabla\phi \cdot \nabla\varphi\right)\left(\int_\Omega \phi^2\right) - \left(\int_\Omega |\nabla\phi|^2\right)\left(2\int_\Omega \phi \cdot \varphi\right)}{\left(\int_\Omega \phi^2\right)^2} = 2\frac{\int_\Omega \nabla\phi \cdot \nabla\varphi - J(\phi)\int_\Omega \phi \cdot \varphi}{\int_\Omega \phi^2}$$

As $J(\phi)$ is the minimum value it is constant that depends only on Ω, which we denote $\lambda = J(\phi)$. We then have

$$\int_\Omega \nabla\phi \cdot \nabla\varphi = \int_\Omega \lambda\phi \cdot \varphi \quad \forall\, \varphi \in H_0^1(\Omega)$$

Thus, ϕ is a weak solution of $-\Delta\phi = \lambda\phi$ in Ω, as wanted. □

This lemma gives us a way of looking at eigenfunctions as minimizers of an appropriate functional. We can try to prove their existence similarly to what we did for harmonic functions.

2.5 Existence of Eigenvalues

Lemma 2.17. *Given $\Omega \subset \mathbb{R}^n$, there exists a function $\phi \in H_0^1(\Omega)$ that minimizes the functional*

$$J(w) = \frac{\int_\Omega |\nabla w|^2}{\int_\Omega w}$$

among all functions $w \in H_0^1(\Omega) \quad w \neq 0$.

Proof. First, notice that for any constant $M > 0$ and any function $w \in H_0^1(\Omega)$, we have $J(Mw) = J(w)$. Therefore, given any function not nule$w \in H_0^1(\Omega)$, we can multiply it by a constant so that $\int_\Omega w^2 = 1$, and the value of $J(w)$ does not change. Thus, the problem is equivalent to minimizing

$$\int_\Omega |\nabla w|^2$$

among all functions $w \in H_0^1(\Omega)$ satisfying $\int_\Omega w^2 = 1$. Let us denote $\lambda = \inf\{\int_\Omega |\nabla w \nabla^2 : w \in H_0^1(\Omega), \int_\Omega w^2 = 1\}$. Thanks to the Poincaré inequality, we know that

$$\int_\Omega |\nabla w|^2 \geqslant C_n \int_\Omega w^2$$

for some constant $C_n > 0$, and hence λ is strictly positive. By definition of infimum, there exists a sequence of functions $u_j \in H_0^1(\Omega)$ with $\int_\Omega u_j^2 = 1$ and with $J(u_j) \to \lambda_1$ as $j \to \infty$. This sequence of functions is bounded in $H^1(\Omega)$, and hence by the Rellich compactness theorem (Chapter 1), there exists a subsequence u_{j_k} that satosfies $u_{j_k} \to \phi$ in $L^2(\Omega)$ and some $\phi \in H_0^1(\Omega)$, and moreover the limit ϕ satisfies

$$\int_\Omega |\nabla \phi|^2 \leqslant \liminf_{j_k \to \infty} \int_\Omega |\nabla u_{j_k}|^2 \quad \textit{and} \quad \int_\Omega \phi^2 = \lim_{j_k \to \infty} \int_\Omega u_{j_k}$$

This means that

$$\int_\Omega |\nabla \phi|^2 \leqslant \lambda \quad \textit{and} \quad \int_\Omega \phi^2 = 1$$

Since $\phi \in H_0^1(\Omega)$ and $\int_\Omega \phi^2 = 1$, then by definitions of λ we must have that

$$\int_\Omega |\nabla \phi|^2 = \lambda$$

and the infimum is attained. we conclude that ϕ is the minimizer of J(w) among all functions $w \in H_0^1(\Omega) \quad w \neq 0$, as desired. □

Combining the previous two lemmas, we have shown that there exists one eigenvalue of the Laplacian, let us call it λ_1

$$\lambda_1 := \min_{w \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla w|^2}{\int_\Omega w}$$

Our next goal is to construct the full sequence $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \to \infty$. Let us start the proof of the theorem.

Proof. **Step 1.**

Let us denote

$$X_0 = \{w \in H_0^1(\Omega) : \int_\Omega w \neq 0\}$$

and

$$\lambda_1 = \min_{w \in X_0} J(w).$$

We already proved that the minimum is attained, and let us denote by $\phi_1 \in H_0^1(\Omega)$ a function satisfying $\int_\Omega \phi_1^2 = 1$ and $J(\phi_1) = \lambda_1$. We also proved that ϕ_1 is a weak solution of

$$-\Delta\phi_1 = \lambda_1\phi_1 \text{ in } \Omega.$$

Step 2.

We now want to repeat the same procedure, but with the space

$$X_1 = \{w \in X_0 : \int_\Omega w\phi_1 = 1\}.$$

Namely, we define

$$\lambda_2 = \inf_{w \in X_1} J(w).$$

Since $X_1 \subset X_0$, then $\lambda_2 \geqslant \lambda_1 > 0$.

To show that the infimum is attained, we argue as before. Take a sequence $w_j \in X_1$ such that $J(w_j) \to \lambda_2$ as $j \to \infty$ and $\int_\Omega w_j^2 = 1$. Then, $\int_\Omega |\nabla w_j|^2 \leqslant$ and hence by Rellich's compactness theorem, there is a subsequence $w_{j_k} \to \phi_2$ in $L^2(\Omega)$ for some $\phi_2 \in H_0^1(\Omega)$ satisfying $\int_\Omega |\nabla\phi_2|^2 \leqslant \liminf \int_\Omega |w_{j_k}|^2 = \lambda_2$

Moreover, since $w_{j_k} \to \phi_2$ in $L^2(\Omega)$ then

$$\int_\Omega \phi_2^2 = \lim \int_\Omega w_{j_k}^2 = 1 \quad \text{and} \quad \int_\Omega \phi_2 \cdot \phi_1 = \lim \int_\Omega w_{j_k} \cdot \phi_1 = 0$$

Thus, $\phi_2 \in H_0^1(\Omega)$ satisfies

$$J(\phi_2) \leqslant J(w) \quad \forall w \in X_2$$

and with the same computation developed in the lemma,

$$J(\phi_2) \leqslant J(\phi_2 + \varepsilon\eta) \quad \forall \eta \in X_1 \quad \forall \varepsilon \in \mathbb{R}$$

implies

$$\int_\Omega \nabla\phi_2 \cdot \nabla\eta = \int_\Omega \lambda_2\phi_2 \cdot \eta \quad \forall\, \eta \in X_1$$

We require the same identity to hold $\forall \eta \in H_0^1(\Omega)$. For this, we descompose any $\eta \in H_0^1(\Omega)$ as $\eta = \overline{\eta} + c\phi_1$ with $c = \int_\Omega \eta\phi_1$. Then,

$$\overline{\eta} \in X_1 \quad \text{since} \quad \int_\Omega \overline{\eta}\phi_1 = \int_\Omega \eta\phi_1 - c\int_\Omega \phi_1^2 = c - c = 0$$

Therefore,

$$\int_\Omega \nabla\phi_2 \cdot \nabla\eta = \int_\Omega \nabla\phi_2 \cdot \nabla\overline{\eta} + c\int_\Omega \nabla\phi_2 \cdot \nabla\phi_1 = \int_\Omega \lambda_2\phi_2 \cdot \overline{\eta} + c\int_\Omega \nabla\phi_2 \cdot \nabla\phi_1 =$$

$$= \int_\Omega \lambda_2\phi_2 \cdot \overline{\eta} + c\int_\Omega \lambda_1\phi_2 \cdot \phi_1 = -\int_\Omega c\lambda_2\phi_2 \cdot \phi_1 + \int_\Omega \lambda_2\phi_2 \cdot \eta = 0 + \int_\Omega \lambda_2\phi_2 \cdot \eta$$

We have proved that $\int_\Omega \nabla\phi_2 \cdot \nabla\eta = \int_\Omega \lambda_2\phi_2 \cdot \eta$, $\forall \eta \in H_0^1(\Omega)$, which means that ϕ_2 is a weak solution of $-\Delta\phi_2 = \lambda_2\phi_2$ in Ω.

Step 3.

Iterating this procedure, we get a sequence

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots$$

and a sequence $\phi_k \in H_0^1(\Omega)$ with $\int_\Omega u_k^2 = 1$ and $\int_\Omega \phi_i \cdot \phi_j = 0 \quad \forall i \neq j$, satisfying

$$\lambda_k = \min_{w \in X_k} J(w) = J(\phi_k),$$

where

$$X_k = \{w \in X_{k-1} : \int_\Omega w \cdot \phi_k = 0\}.$$

Moreover, the ϕ_k are weak solutions of

$$-\Delta\phi_k = \lambda_k\phi_k \text{ in } \Omega.$$

Let us now prove that $\lambda_k \to \infty$ as $k \to \infty$.

Assume this was not true. Then, $\lambda_k \to \overline{\lambda}$ as $k \to \infty$, and hence $J(u_k) = \lambda_k$ is bounded. In other words,

$$\int_\Omega |\nabla\phi_k|^2 = J(\phi_k) = \lambda_k \leqslant \overline{\lambda}$$

By Rellich's compactness theorem, we can extract a subsequence $\phi_{k_j} \to u$ in $L^2(\Omega)$ for some $\overline{\phi} \in H_0^1(\Omega)$. However, this is a contradiction, since

$$\int_\Omega |\phi_{k_j} - \phi_{k_i}|^2 = \int_\Omega \phi_{k_j}^2 + \int_\Omega \phi_{k_i}^2 - 2\int_\Omega \phi_{k_j} \cdot \phi_{k_i} = 1 + 1 - 0 = 2 \quad \forall i \neq j$$

so this sequence does not converge in $L^2(\Omega)$

Therefore, we have proved that $\lambda_k \to \infty$ as $k \to \infty$. Furthermore, we notice that this implies that

$$\bigcap_{k=1}^{\infty} X_k = \{0\}.$$

Indeed, if $f \in \bigcap_{k=1}^{\infty} X_k$, then in particular $f \in X_j \quad \forall j$. But by the definition of X_j, if $f \neq 0$, we have

$$J(f) \geqslant \min_{w \in X_j} J(w) = \lambda_j \to \infty \quad \text{as} \quad j \to \infty,$$

which is impossible, since $J(f)$ is a fixed number.

Step 4.

It only remains to prove that the functions ϕ_k functions form an orthonormal basis of $L^2(\Omega)$.

In other words, any function $f \in L^2(\Omega)$ can be written as

$$f = \sum_{k=1}^{\infty} c_k \cdot \phi_k \quad \text{where} \quad c_k = \int_\Omega f \cdot \phi_k$$

Since smooth functions $C_C^\infty(\Omega)$ are dense in $L^2(\Omega)$, it suffices to show this for $f \in C_C^\infty(\Omega)$. In particular, we may assume $f \in H_0^1(\Omega)$.

Then, we have

$$f - c_1\phi_1 \in X_1 \quad \text{as} \quad \left.\begin{aligned} f \in X_0 \\ \int_\Omega (f - c_1\phi_1)\phi_1 = 0 \\ \dots \\ f - \sum_{i=1}^{k} c_i \cdot \phi_i \in X_k \end{aligned}\right\}$$

and then

$$\int_\Omega |f - \sum_{i=1}^{k} c_i \cdot \phi_i|^2 = \int_\Omega f^2 + \sum_{i=1}^{k} c_i^2 \cdot \int_\Omega \phi_i^2 - 2c_i \int_\Omega f \cdot \phi_i = \int_\Omega f^2 - \sum_{i=1}^{k} c_i^2$$

$$\int_\Omega |\nabla f - \sum_{i=1}^{k} c_i \cdot \nabla\phi_i|^2 = \int_\Omega \nabla f^2 + \sum_{i=1}^{k} c_i^2 \cdot \int_\Omega \nabla\phi_i^2 - 2c_i \int_\Omega \nabla f \cdot \nabla\phi_i = \int_\Omega \nabla f^2 - \sum_{i=1}^{k} \lambda_i c_i^2$$

Therefore, the function $\overline{f} := f - \sum_{k=1}^{\infty} c_k \cdot \phi_k$ belongs to $H_0^1(\Omega)$ and is orthonormal to all ϕ_j, so

$$\overline{f} \in \bigcap_{k=1}^{\infty} X_k = \{0\}$$

This means that $\overline{f} = 0$, and hence $f = \sum_{k=1}^{\infty} c_k \cdot \phi_k$, as wanted. $\square$

During the proof of this theorem, we proved, indeed, a simple method to find the n-th eigenvalue for the Dirichlet problem (same description as [Lev03], Theorem 5).

Proposition 2.18. *(Minimum Principle for the nth Eigenvalue of the Dirichlet problem) Fix an integer $n \geqslant 1$. Let v_i, $i = 1, ..., n-1$ be the first $n-1$ eigenfunctions of the Dirichlet problem. Without loss of generality, these eigenfunctions can be chosen orthogonal. Let*

$$X_n \equiv \left\{ w : w \in H_0^1(\Omega), w \neq 0, \int_\Omega w^2 = 1, \int_\Omega w v_i = 0 \quad \forall i = 1, ..., n-1 \right\}$$

Suppose there exists a function $u_n \in X_n$ which minimizes the Rayleigh quotient over all functions $w \in X_n$. This means that:

$$m_n = ||\nabla u_n||^2 = \min_{w \in X_n} ||\nabla w||^2$$

Then m_n is the nth eigenvalue of the Dirichlet problem. This is, $\lambda_n = m_n$ and u_n is the corresponding eigenfunction.

For any bounded smooth domain $\Omega \subset \mathbb{R}^n$, we have shown the existence of orthonormal eigenfunctions $\phi_k \in H_0^1(\Omega)$, which are weak solutions of

$$-\Delta \phi_k = \lambda_k \phi_k \text{ in } \Omega, \quad \phi_k = 0 \text{ on } \partial\Omega.$$

The last step concerns their regularity through this theorem:

Theorem 2.19. *The eigenfunctions ϕ_k are actually $C^\infty(\Omega)$ and they can not vanish in an open set. Moreover, for all $m \geqslant 0$, we have*

$$\|\phi_k\|_{C^m(\overline{\Omega})} \leqslant C_m \cdot \lambda_k^{m/2+n/2},$$

with constants C_m independent of k.

The boundary condition in our Dirichlet problem can be changed, when instead of looking for solutions that are null in the $\partial\Omega$, we search for solutions that their normal component to the boundary $\partial\Omega$ is null. In other words:

$$\begin{cases} -\Delta \varphi = \mu \varphi & \text{in } \Omega \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

where ν denotes the (typically exterior) normal to the boundary $\partial\Omega$. This new eigenvalue problem is called the Neumann boundary condition problem, or in some cases *free condition*, as distinguished from the "fixed" Dirichlet condition. We will see that its understanding can truly be helpful in the asymptotics study, when we combine both Neumann's and Dirichlet's problems. Moreover, the Neumann problem gives us analog results to the Dirichlet condition, here being the most important ones.

Theorem 2.20. *For any $\Omega \subset \mathbb{R}^n$ bounded smooth and connected domain, it exists a sequence $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \to \infty$ of eigenvalues for the Laplacian with Neumann boundary condition, with eigenfunctions $\varphi_k \in H^1(\Omega)$ solving in the weak sense:*

$$\begin{cases} -\Delta\varphi_k = \mu\varphi_k & \text{in } \Omega \\ \partial_\nu\varphi_k = 0 & \text{on } \partial\Omega \end{cases}$$

Moreover, these functions form an orthonormal basis for $L^2(\Omega)$.

Proposition 2.21. *(Minimum Principle for the nth Eigenvalue with Neumann's boundary condition) Fix an integer $n \geqslant 1$. Let φ_i, $i = 1, ..., n-1$ be the first $n-1$ eigenfunctions of the Neumann problem. Without loss of generality, these eigenfunctions can be chosen orthogonal. Let*

$$Y_n \equiv \left\{ w : w \in H^1(\Omega), \int_\Omega w^2 = 1, \int_\Omega w v_i = 0 \quad \forall i = 1, ..., n-1 \right\}$$

Suppose there exists a function $u_n \in Y_n$ which minimizes the Rayleigh quotient over all functions $w \in X_n$. This means that:

$$b_n = ||\nabla u_n||^2 = \min_{w \in Y_n} ||\nabla w||^2$$

Then b_n is the nth eigenvalue of the Neumann problem. This is, $\mu_n = b_n$ and u_n is the corresponding eigenfunction.

2.6 Examples

By having proved Theorem 2.13, we have the existence of eigenvalues and eigenfunctions for any bounded domain $\Omega \subset \mathbb{R}^n$ (for the Dirichlet problem and even the Neumann- although I have not talked about it yet) guaranteed. In other words, any bounded domain Ω characterizes its own spectrum of eigenvalues and eigenfunctions of the Dirichlet problem. Nevertheless, there exists, actually, a short list of few domains that, due to their symmetries, allow an explicit calculation for its spectrum and eigenfunctions. For instance, we have the cases of the rectangle and the disk, that will be studied carefully (the results correspond to the space modes obtained in [Mor48]).

Indeed, we will see, later, the importance of understanding the spectrum of a rectangle, in order to describe the asymptotic behavior of the eigenvalues of an arbitrary bounded domain.

Rectangle

Let $Q = (0, a) \times (0, b)$ be a rectangle in $\mathbb{R}^2$. Our main goal is to find, by separation of variables, all eigenfunctions of the form $\phi(x_1, x_2) = \xi(x_1)\eta(x_2)$, as well as their corresponding eigenvalues λ.

Suppose $\phi(x_1, x_2)$ is an eigenfunction of the Laplacian and it suffices:

$$\left.\begin{aligned} -\Delta\phi = \lambda\phi \quad &\text{in} \quad Q \\ \phi = 0 \quad &\text{on} \quad \partial Q \end{aligned}\right\}$$

In other words $\phi_k \equiv \phi(x_1, x_2) = \xi_k(x_1)\eta_k(x_2)$, but to simplify calculus, we fix $\xi_k = \xi, \quad \eta_k = \eta, \quad \lambda_k = \lambda$. Next, we substitute in the Dirichlet system above, and we get (using $f'' = d^2 f/dx^2$):

$$-\xi''(x_1)\eta(x_2) - \xi(x_1)\eta''(x_2) = \lambda\xi(x_1)\eta(x_2)$$

Dividing firstly by ξ and the η:

$$-\frac{\xi''(x_1)}{\xi(x_1)} - \frac{\eta''(x_2)}{\eta(x_2)} = \lambda(= f_1 + f_2)$$

As we have two functions that depend on different variables equal to a constant, each one would be constant. Let α and β be positive constants that do satisfy $\alpha + \beta = \lambda$. Then, it appears, naturally, two new eigenvalue problem in 1 dimension.

$$\left.\begin{aligned} \xi(x_1) = -\alpha\xi(x_1) \quad \text{in} \quad (0, a) \\ \xi(x_1) = 0 \quad \text{if} \quad x_1 = \{0, a\} \end{aligned}\right\} \qquad \left.\begin{aligned} \eta(x_2) = -\beta\eta(x_2) \quad \text{in} \quad (0, b) \\ \eta(x_2) = 0 \quad \text{if} \quad x_2 = \{0, b\} \end{aligned}\right\}$$

The general solutions for these two Dirichlet problems are:

$$\xi(x_1) = A\sin(\sqrt{\alpha}\cdot x_1) + B\cos(\sqrt{\alpha}\cdot x_1) \tag{2.3}$$
$$\eta(x_2) = C\sin(\sqrt{\beta}\cdot x_2) + D\cos(\sqrt{\beta}\cdot x_2) \tag{2.4}$$

Applying the boundary conditions when $x_1 = x_2 = 0$ we get $B = D = 0$. In order to avoid trivial solutions we apply the second boundary condition; $\xi(a) = 0 = \eta(b)$, which allows us to affirm that:

$$\alpha = \frac{n^2\cdot\pi^2}{a^2} \quad \text{and} \quad \beta = \frac{m^2\cdot\pi^2}{b^2}$$

where $m, n \in \mathbb{N}$ are positive constants in order to be well defined. Recapping, the eigenvalues $\lambda = \alpha + \beta$ depend on two variables and they can be generally written as:

$$\lambda_{n,m} = \pi^2 \cdot \left\{ \frac{n^2}{a^2} + \frac{m^2}{b^2} \right\}$$

Moreover, the eigenfunctions will have the final version:

$$\phi(x_1, x_2) = A\cdot C \sin\left(\frac{n\pi}{a}x_1\right)\sin\left(\frac{m\pi}{b}x_2\right)$$

Using the normalizing factor of eigenvalues of the Dirichlet problem ($||\phi_k||_{L^2} = 1$), we could find the normalizing constant. The calculation appears just bellow:

$$A^2\cdot C^2 \int_0^a \sin^2\left(\frac{n\pi}{a}x_1\right)dx_1 \int_0^b \sin^2\left(\frac{m\pi}{b}x_2\right)dx_2 = A^2\cdot C^2\cdot\frac{ab}{4} = 1 \Leftrightarrow A\cdot C = \frac{2}{\sqrt{ab}}$$

where we have used that $\int_0^a \sin^2\left(\frac{n\pi}{a}x_1\right)dx_1 = \frac{1}{2}\int_0^a 1 - \cos\left(\frac{n\pi}{a}x_1\right)dx_1 = \frac{a}{2}$. In conclusion, the Dirichlet eigenfunctions and collection of eigenvalues for the rectangle domain have the form:

$$\phi_{n,m}(x_1, x_2) = \frac{2}{\sqrt{ab}}\cdot\sin\left(\frac{n\pi}{a}x_1\right)\sin\left(\frac{m\pi}{b}x_2\right) \quad \text{and} \quad \lambda_{n,m} = \pi^2\cdot\left\{\frac{n^2}{a^2} + \frac{m^2}{b^2}\right\}$$

for $n, m \in \mathbb{N}$.

Using the Theorem 2.13 the set $\{\phi_{n,m}\}$ of eigenfunctions form a $L^2(Q)$ orthogonal basis.

If we proceed in the exact way, but we change the Dirichlet boundary condition for the Neumann one, we will end up deriving (2.3) and $A = C = 0$, which will change a bit the form of the eigenfucntions, but the eigenvalues remain exactly the same. Then, the Neumann eigenfunctions and collection of eigenvalues for the rectangle domain can be written:

$$\varphi_{k,l}(x_1, x_2) = \frac{2}{\sqrt{ab}} \cdot \cos\left(\frac{k\pi}{a}x_1\right)\cos\left(\frac{l\pi}{b}x_2\right) \quad \text{and} \quad \mu_{k,l} = \pi \cdot \left\{\frac{k^2}{a^2} + \frac{l^2}{b^2}\right\}$$

for $k, l \in \mathbb{N} \cup \{0\}$.

Parallelepiped

Let us denote $\Omega = (0, a_1) \times \cdots \times (0, a_n)$ a n-dimensional parallelepiped in $\mathbb{R}^n$. If we follow the exact same steps that were required for the rectangular domain in $\mathbb{R}^2$, we eventually come up with the following spectrum (for both Dirichlet and Neumann problems):

$$\lambda_{m_1,\ldots,m_k} = \pi^2 \cdot \left\{\frac{m_1^2}{a_1^2} + \cdots + \frac{m_k^2}{a_n^2}\right\} \tag{2.5}$$

where $m_1, ..., m_n \in \mathbb{N}$. For the Neumann boundary condition, these n parameters could also be 0.

Remark 2.22. When $n = 1$, $\Omega = (0, L)$ the domain becomes a line and the spectrum and the collection of eigenfunctions only depend on one parameter $n \in \mathbb{N}$:

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \quad \text{and} \quad \lambda_n = \frac{n^2 \cdot \pi^2}{L^2}$$

where L represents the longitude of the domain and the normalizing constant is computed. The main difference with respect to the last three domains is that a property of the domain appears explicitly in the eigenvalue expression, its length L. Evermore, eigenvalues do only depend on the length of the domain, which means that if two lines have the same spectrum, it means they will have the same length.

Disk

Continuing with the two-dimensional domains, suppose that we have a disk or a circular membrane Ω with the same area as the open bounden rectangle we have constructed above. In order to simplify the computations, let us use the polar coordinates (r, θ) since $\Omega = [0, R) \times [0, 2\pi)$, $R > 0$. Similarly as we have proceeded before, we need to apply the Dirichlet boundary conditions in order to find the spectrum and the list of eigenfunctions.

We will be using separation of variables in order to separate the radial and the angular part of the domain. Thus, let $\gamma(r,\theta) = U(r)V(\theta)$ be a Dirichlet eigenvalue. In polar coordinates, the Laplacian has the form:

$$\Delta\gamma = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\gamma}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\gamma}{\partial\theta^2} = \frac{1}{r}\frac{\partial\gamma}{\partial r} + \frac{\partial^2\gamma}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\gamma}{\partial\theta^2} \tag{2.6}$$

Dividing by $U(r)V(\theta)$, multiplying each side by r^2 and reordering, we obtain:

$$r\frac{V'(r)}{V(r)} + r^2\frac{V''(r)}{V(r)} + \lambda r^2 = -\frac{U''(\theta)}{U(\theta)}$$

Once again, we can see both sides depend on different variables, so they must be constant. Let us call it α our positive constant, $\alpha \geqslant 0$. For the right side, we follow the same steps as in the one-dimensional case, as we have $U''(\theta) = -\alpha U(\theta)$ with $\theta \in [0, 2\pi)$ and its general solution:

$$U(\theta) = A\sin(\sqrt{\alpha}\theta) + B\cos(\sqrt{\alpha}\theta)$$

Since $U(\theta)$ is 2π-periodic, $\sqrt{\alpha} = n$ where $\mathbb{N} \cup \{0\}$. Considering the left-hand sided equation, we then have:

$$r^2V''(r) + rV'(r) + (r^2\lambda - n^2)V(r) = 0$$

where we have already substituted $\alpha = n^2$. This equation is called the Bessel differential equation which is a linear second-order ordinary differential equation. The general solution is given by

$$V(r) = CJ_n(\sqrt{\lambda}r) + DY_n(\sqrt{\lambda}r)$$

where J_n and Y_n are the first and second kind Bessel functions. Because of the fact that Y_m diverges when you approach to 0, we impose that D=0. Moreover, by applying the boundary conditions and avoiding trivial solutions we obtain that

$$V(R) = J_n(\sqrt{\lambda}R) = 0 \Leftrightarrow \beta_{k,n} = \sqrt{\lambda}R$$

where $\beta_{k,n}$ represents the k-th zero of the Bessel function J_n. In other words $\sqrt{\lambda}R$ must be a zero of the first kind Bessel function J_n. For instance, the first eigenvalue λ_1 will be:

$$\beta_{1,0} = \sqrt{\lambda_1}R \Leftrightarrow \lambda_1 = \frac{\beta_{1,0}^2}{R^2}$$

With everything said, setting φ as a constant given by the initial conditions, the general solution will be given as:

$$\gamma(r,\theta) = \sum_{n\geqslant 0} A_n J_n(\sqrt{\lambda}r) sin(n\theta + \varphi)$$

2.7 Linear evolutionary PDE

We have seen so far the existence and uniqueness of eigenvalues of the Dirichlet and Neumann Laplacian. However, we were interested in vibrating domains (from strings in one dimension, drums in two dimensions, or even n-dimensional membranes), which require in all of the cases to suffix the wave equation $\partial_{tt}u - \Delta u = 0$ inside the domain and be in repose in the border. In this cases, we will be dealing with time-dependent PDEs, which existence and regularity of solutions should be proved. Haven said that, the eigenfunctions of the Laplacian will be of crucial use.

Theorem 2.23. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Suppose there are two initial conditions $f_0 \in L^2(\Omega)$ and $g_0 \in L^2(\Omega)$. Then, it exists a solution of the wave equation:*

$$\left.\begin{aligned} \partial_{tt}u - \Delta u \quad in \quad \Omega \times (0, \infty) \\ u \quad on \quad \partial\Omega \times (0, T) \\ u(x,0) = f_0(x) \quad in \quad \Omega \quad for\ t = 0 \\ \partial_t u(x,0) = g_0(x) \quad in \quad \Omega \quad for\ t = 0 \end{aligned}\right\} \tag{2.7}$$

Proof. The strategy of the proof is to look for solutions of the form:

$$u(t,x) = T(t) \cdot \phi(x)$$

that suffices the wave equation and its boundary condition. Thus, by separation of variables and dividing by u, we end up with:

$$\frac{T''(t)}{T(t)} = \frac{\Delta\phi(x)}{\phi(x)} \quad \text{in } \Omega$$

As this equals expressions that do depend on different variables, each side must be the same constant value $-\lambda < 0$. The reason behind the negative sign has a more physical meaning, since the force that governs a vibrational domain must be restoring. Apart from that, this choice also helps connect with the Laplacian eigenvalues because for the spacial part we get:

$$\Delta\phi(x) = -\lambda\phi(x)$$

when x is in Ω and respecting the boundary condition. In other words, the spacial part will be eigenvalues $\phi_k(x)$ of the Dirichlet Laplacian and then $\lambda = \lambda_k$ for $k \geqslant 1$.

Secondly, we have directly the equation $T''(t) = -\lambda_k T(t)$, which has the general solution

$$T(t) = A_k \sin(\sqrt{\lambda_k} t) + B_k \cos(\sqrt{\lambda_k} t)$$

Both constants A_k, B_k can be related with $\frac{1}{\sqrt{\lambda}}\frac{d}{dt}|_{t=0}T(t)$ and $T(t=0)$, respectively. Taking both solutions, we have that any linear combination

$$u(x,t) = \sum_{k=1}^{\infty}[A_k \sin(\sqrt{\lambda_k}t) + B_k \cos(\sqrt{\lambda_k}t)]\phi_k(x) \tag{2.8}$$

solves the wave equation:

$$\left.\begin{aligned} \partial_{tt}u - \Delta u \quad \text{in} \quad \Omega \times (0,\infty) \\ u \quad \text{on} \quad \partial\Omega \times (0,T) \end{aligned}\right\}$$

We just need to impose the rest of boundary conditions, $u(x,0) = f_0(x)$ and $\partial_t u(x,0) = g_0(x)$. Since the set of eigenfunctions of the Laplacian form an orthogonal basis in $L^2(\Omega)$, we choose:

$$f_0 = \sum_{k=1}^{\infty} A_k\phi_k(x) \quad \text{with} \quad A_k = \int_\Omega f_0\phi_k \tag{2.9}$$

$$g_0 = \sum_{k=1}^{\infty} B_k\phi_k(x) \quad \text{with} \quad B_k = \int_\Omega g_0\phi_k \tag{2.10}$$

in order that the general function (2.8) is a solution of the wave equation as we wanted. □

Remark 2.24. As the time part is completely solved, solutions for the wave equation only depend on the eigenfunctions and its respectively eigenvalues of the Laplacian. Rephrasing it, solving a Dirichlet problem is equivalent to computing solutions for the wave equation.

Remark 2.25. When considering a physical membrane or a string, we define $c^2 = T/\epsilon$ where T is the tension and ϵ the density of the surface. Thus, if the wave equation looks like $\partial_{tt}u - c^2\Delta u = 0$ and we consider the tension and density be homogeneous among all the surface, the eigenvalues will be $\gamma_k = \lambda_k c^2 = \lambda_k T/\epsilon$

Lastly, the regularity of the wave equation's solutions can be precised with the following proposition (that we do not intend to prove it).

Proposition 2.26. *Suppose there are two initial conditions $f_0, g_0 \in C^\infty(\overline{\Omega})$. Then, if $u(x,t)$ is a solution of the wave equation it is $C^\infty(\overline{\Omega} \times [0,\infty))$.*

Chapter 3

Asymptotics of the spectrum

All the results presented, so far, have been destined to prove in which way a bounded domain Ω in $\mathbb{R}^n$ characterizes a set of eigenvalues and their corresponding eigenfunctions. Apart from that, we have learned their guaranteed existence for both Dirichlet and Neumann boundary conditions problems, and how their corresponding eigenfunctions are actually $C^\infty(\Omega)$ and form an orthogonal $L^2(\Omega)$ basis.

Although, having also proved that the eigenvalues for the Dirichlet problem tend to infinity, the main goal from now on will be understanding the asymptotic growth of the eigenvalues of our Laplacian problems, indeed, how fast the eigenvalues go to infinity. Eventually, we want to ask the opposite question that motivated the existence of eigenfunctions, and the chain of eigenvalues: in which way does a spectrum characterizes a bounded domain? For the one-dimensional case, we have seen that this question is affirmatively answered, since the spectrum depends directly of the dimensions of the domain. However, can we confirm that for a general bounded domain Ω there are properties or characteristics that might be inferred just by the set of eigenvalues?

In order to answer this question, the section has the goal to prove the result known as Weyl's law that affirms that:

Theorem 3.1. *(Weyl's law) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Let $\rho > 0$ and denote $D(\rho) = \#\{\lambda_k \in Spectrum(Q) : \lambda_k \leqslant \rho)\}$, then:*

$$\lim_{\rho\to\infty} \frac{D(\rho)}{\rho^{n/2}} = \frac{\omega_n}{2^n}\frac{|\Omega|}{\pi^n}$$

where ω_n is the hypervolume of the unitary ball in $\mathbb{R}^n$. Equivalently,

$$\lim_{k\to\infty} \lambda_k = 4\pi^2 \frac{k^{2/n}}{(\omega_n|\Omega|)^{2/n}}$$

There are different paths to prove it, but the approach will be similar to the ones discussed in [CH89,Gar64].

3.1 Rectangular domain asymptotics

We might start studying the asymptotics of the spectrum for a rectangular domain. If $Q = (0, a) \times (0, b)$ is a rectangle in $\mathbb{R}^2$ with $area(Q) = |Q| = ab$, we have computed explicitly its spectrum (for both Dirichlet and Neumann conditions):

$$\lambda_k = \lambda_{n,m} = \pi \cdot \left\{ \frac{n^2}{a^2} + \frac{m^2}{b^2} \right\}$$

for $n, m \in \mathbb{N}$. Because each eigenvalue depends directly on two integer indices, it is unlikely to see how fast does λ_k (taking the k-th eigenvalue) grow. Therefore, we must need to rethink the way to visualize this eigenvalues. In order to do this, let $\rho > 0$ and denote:

$$D(\rho) = \# \{ \lambda_k \in Spectrum(Q) : \lambda_k \leqslant \rho) \}$$

which denotes the number of eigenvalues not exceeding a fix value $\rho > 0$.

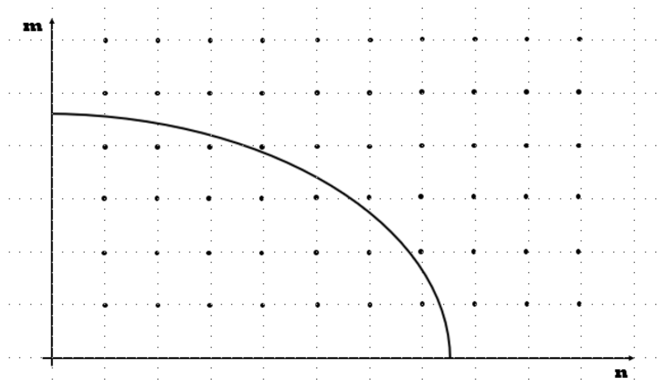

Figure 3.1: Points of the form $(n, m) \in \mathbb{N} \times \mathbb{N}$ lying inside the quarter of ellipse of semi axes a $\frac{a\sqrt{\rho}}{\pi}$ and $\frac{b\sqrt{\rho}}{\pi}$.

If an eigenvalue for this domain suffices the condition of being smaller or equal to ρ, then it means that:

$$\pi^2 \cdot \left\{ \frac{n^2}{a^2} + \frac{m^2}{b^2} \right\} \leqslant \rho \Leftrightarrow \frac{n^2}{(\frac{a\sqrt{\rho}}{\pi})^2} + \frac{m^2}{(\frac{b\sqrt{\rho}}{\pi})^2} \leqslant 1$$

Taking into consideration this last inequality, it is clear that $D(\rho)$ can be understood as the number of lattice points (n,m) that are contained within the quarter ellipse (only the first quadrant as $n, m \in \mathbb{N}$) in the (n,m) plane (see Figure 3.1). Each point in this plane is the upper-right corner of a unit square; its area is 1. In other words, all the eigenvalues that suffix the condition given by ρ can be represented as full squares that lay inside of a quarter ellipse with semi axis $\frac{a\sqrt{\rho}}{\pi}$ and $\frac{b\sqrt{\rho}}{\pi}$.

Therefore, $D(\rho)$ is at most the area of this quarter ellipse:

$$D(\rho) \leqslant \frac{ab\rho}{4\pi}$$

Furthermore, the difference between the area of this quarter of ellipse and the number $D(\rho)$ has the order of magnitude $\sqrt{\rho}$ of the perimeter of the ellipse, as it coincides with an area between the quarter of the ellipse and a system of unit squares exhausting that region form within.

If we denote E_ρ the first quadrant of the ellipse described above. We can associate each point $(n, m) \in E_\rho$ with the square:

$$SQ_{n,m} = [n-1, n] \times [m-1, m]$$

Apart from that, in order to give a lower bound to the number of eigenvalues not exceeding ρ, recalling the idea developed in [Lau12] Proposition 2.1, we see that the union of the squares covers a copy $\widetilde{E_\rho}$ of E_ρ translated by (-1,-1), which allows:

$$\left(\widetilde{E_\rho} \cap \{(n,m) : n, m \in \mathbb{N}\}\right) \subset \bigcup_{(n,m)\in E_\rho} SQ_{n,m}$$

Using this fact, by comparison of areas:

$$D(\rho) \geqslant \frac{ab\rho}{4\pi} - \frac{\sqrt{\rho}n}{\pi} - \frac{\sqrt{\rho}m}{\pi} = \frac{ab\rho}{4\pi} - \frac{n+m}{\pi}\sqrt{\rho}$$

Writing everything in the same chain of inequalities:

$$\frac{|Q|}{4\pi}\rho - \frac{perimeter(Q)}{2\pi}\sqrt{\rho} \leqslant D(\rho) \leqslant \frac{|Q|}{4\pi}\rho$$

Dividing by ρ, and taking to infinity, we can apply the sandwich and then obtain:

$$\lim_{\rho\to\infty} \frac{D(\rho)}{\rho} = \frac{|Q|}{4\pi} \tag{3.1}$$

This expression can be rewritten in terms of the k-th eigenvalue, because for large ρ, this can be approximated to $\rho = \lambda_k$. Furthermore, as the eigenvalues are written in increasing order (with multiplicity), $N(\lambda_k) = k$. This gives us:

$$\lim_{k\to\infty} \lambda_k = \frac{4\pi k}{|Q|} \tag{3.2}$$

Remark 3.2. As the Neumann eigenvalues for a rectangular domain Q are $\mu_k = \mu_{q,l} = \pi \cdot \left\{ \frac{q^2}{a^2} + \frac{l^2}{b^2} \right\}$ where, now, $q, l \in \mathbb{N} \cup \{0\}$. The only difference, is that both q, l are allowed to be 0.This represents that the lower bound of $D(\rho)$ includes contributions given by the area of strips of unit width along the axis of the ellipse that correspond to extra lattice points situated there (see Figure 3.2).

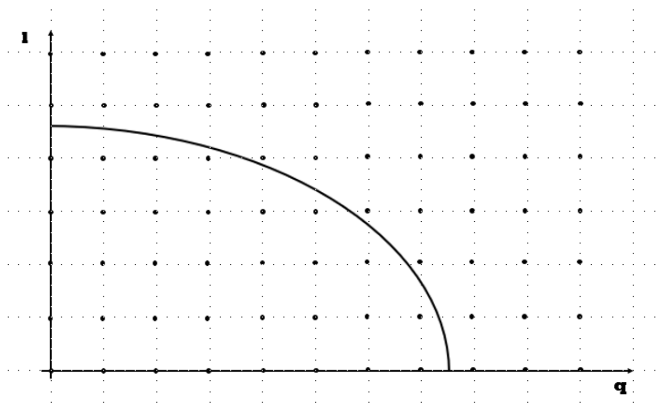

Figure 3.2: Points of the form $(l, q) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ lying inside the quarter of ellipse of semi axes a $\frac{a\sqrt{\rho}}{\pi}$ and $\frac{b\sqrt{\rho}}{\pi}$.

However, if we proceed the exact same way, by describing:

$$N(\rho) = \# \{ \mu_k \in Spectrum(Q) : \mu_k \leqslant \rho) \}$$

we obtain an analog result:

$$\lim_{\rho \to \infty} \frac{N(\rho)}{\rho} = \frac{|Q|}{4\pi} \tag{3.3}$$

or, equally, it can be rewritten as:

$$\lim_{k \to \infty} \mu_k = \frac{4\pi k}{|Q|} \tag{3.4}$$

Remark 3.3. In case we are treating with bigger dimensions, we in position to compute the asymptotic growth in case our domain is a parallelepiped. Recall the spectrum (2.5) for $\Omega = (0, a_1) \times \cdots \times (0, a_n)$, with n-volume $vol(\Omega) = |\Omega| = a_1 \cdots a_n$. If we keep the steps done with the rectangle in two dimensions, let $\rho > 0$ and denote:

$$D(\rho) = \# \{ \lambda_k \in Spectrum(\Omega) : \lambda_k \leqslant \rho \}$$

which denotes the number of eigenvalues not exceeding a fix value $\rho > 0$. In this bigger dimensional case, geometrically, this will denote the number of unitary hypercubes contained in the first orthant (we will need to divide by 2^n) of an ellipsoid with axis $\frac{a_1\sqrt{\rho}}{\pi}, ..., \frac{a_n\sqrt{\rho}}{\pi}$. The idea behind the bounding of $D(\rho)$ is the exact same as for the two-dimensional case. For this reason we need the volume of n-dimensional ellipsoid:

$$V_n = \frac{2}{n}\frac{\pi^{n/2}}{\Gamma(n/2)}(a_1 \dots a_n)$$

where $a_1, ..., a_n$ are the length of the axis. Therefore,

$$\lim_{\rho\to\infty} \frac{D(\rho)}{\rho} = \frac{1}{2^n}\frac{2}{n}\frac{\pi^{n/2}}{\Gamma(n/2)}\left(\frac{a_1\sqrt{\rho}}{\pi} \dots \frac{a_n\sqrt{\rho}}{\pi}\right) = \frac{1}{2^{n-1}n}\frac{\rho^{n/2}}{\pi^{n/2}\Gamma(n/2)}|\Omega|$$

If we considerate the volume ω of the unitary n-dimensional ball :

$$\omega_n = \frac{2}{n}\frac{\pi^{n/2}}{\Gamma(n/2)}$$

The growth expression can be rewritten as:

$$\lim_{\rho\to\infty} \frac{D(\rho)}{\rho^{n/2}} = \frac{\omega_n}{2^n}\frac{|\Omega|}{\pi^n} \tag{3.5}$$

and with the same last reasoning, for large ρ,it can be approximated to λ_k, and D(λ_k)=k and finally:

$$\lim_{k\to\infty} \lambda_k = 4\pi^2\frac{k^{2/n}}{(\omega_n|\Omega|)^{2/n}} \tag{3.6}$$

For the free condition Neumman problem, as in the case of the rectangular domain in two dimensions, the results are completely analogous:

$$\lim_{\rho\to\infty} \frac{N(\rho)}{\rho^{n/2}} = \frac{\omega_n}{2^n}\frac{|\Omega|}{\pi^n} \tag{3.7}$$

or, equally, it can be rewritten as:

$$\lim_{k\to\infty} \mu_k = 4\pi^2\frac{k^{2/n}}{(\omega_n|\Omega|)^{2/n}} \tag{3.8}$$

If we are in $\mathbb{R}^2$, $w_2 = \pi$, and this last expression turns (3.2), as we could expected.

3.2 Maximin principle and monotonicity of eigenvalues

An important step to Weyl's law proof is to show the monotonicity of eigenvalues for subdomains. Let us start with the Maximin principle.

Proposition 3.4. *(Maximin Principle) Let $0 < \lambda_1 < \lambda_2 \leqslant \ldots$ be the eigenvalues of the Laplacian in Ω with Dirichlet boundary conditions. Then,*

$$\lambda_k = \max_{f_1,\ldots,f_{k-1 \in L^2(\Omega)}} \min_{\substack{u \in H_0^1(\Omega) \\ \int_\Omega u f_i = 0 \\ ||u||_{L^2(\Omega)} = 1}} ||\nabla u||_{L^2(\Omega)} \tag{3.9}$$

Proof. Let us take $f_1, \ldots, f_{k-1} \in L^2(\Omega)$ and let $\phi_1, \ldots \phi_k$ be the first normalized eigenfunctions of the Dirichlet problem with Ω as the bounded domain. If we construct, $w(x) = \sum_{j=1}^{n} c_j \phi_j(x)$ a linear combination of the first k eigenfunctions which is chosen to be orthogonal to $f_1, \ldots, f_{k-1}$. This means that

$$0 = \int_\Omega w f_i = \sum_{j=1}^{k} \int_\Omega \phi_j f_i \quad (for \quad i = 1, \ldots, k-1)$$

As it is a system of only $k-1$ equations and k unknowns, it has a solution $c_1, \ldots, c_k$, which not all constants are zero. Moreover, we impose that $||w||_{L^2(\Omega)} = 1 \Leftrightarrow \sum_i c_i^2 = 1$.In other words, these constants $c_1, \ldots, c_n$ are well chosen to satisfy the linear system above and the quadratic sum is equal to 1. In order the ease the reading we will be writing

$$\lambda_k^*(f_1, \ldots, f_{k_1}) = \min_{\substack{u \in H_0^1(\Omega) \\ \int_\Omega u f_i = 0 \\ ||u||_{L^2(\Omega)=1}}} ||\nabla u||_{L^2(\Omega)}$$

Then we can start a chain of inequalities:

$$\lambda_k^*(f_1, \ldots, f_{k_1}) \leqslant ||\nabla w||_{L^2(\Omega)} = \sum_{j=1}^{k} \sum_{l=1}^{k} c_j c_l \int_\Omega \nabla \phi_j \cdot \nabla \phi_l = \sum_{j=1}^{k} \sum_{l=1}^{k} c_j c_l \int_\Omega (-\Delta \phi_j) \cdot \phi_l =$$

$$= \sum_{j=1}^{k} \sum_{l=1}^{k} c_j c_l \int_\Omega (\lambda_j \phi_j) \cdot \phi_l = \sum_{l=1}^{k} c_j^2 \lambda_j \leqslant \lambda_k \sum_{j=1}^{k} c_j^2 = \lambda_k$$

where the integration by parts' formula has been used, and the fact that the set of eigenvalues is crescent in value. The inequality $\lambda_k^*(f_1, \ldots, f_{k_1}) \leqslant \lambda_k$ is true $\forall f_1, \ldots, f_{k-1}$ chosen. Hence, $\max \lambda_k^*(f_1, \ldots, f_{k_1}) \leqslant \lambda_k$. In order to prove the exact equality of the proposition, we just need a special choice of these $f_1, ..., f_{k-1} \in L^2(\Omega)$ functions. Indeed, only by selecting, $f_i = \phi_i, \quad i = 1, ..., k-1$, we can then apply the Minimum Principle for the nth eigenvalue, proved in Proposition (2.18), that allows us to say that, by this choice of functions, $\lambda_k^*(\phi_1, ..., \phi_{k_1}) = \lambda_k$. Therefore, we have the chain of inequalities:

$$\lambda_k = \lambda_k^*(\phi_1, ..., \phi_{k_1}) \leqslant \max \lambda_k^*(f_1, ..., f_{k_1}) \leqslant \lambda_k$$

which means, as desired, that $\lambda_k = \lambda_k^*(f_1, ..., f_{k_1})$. □

The same Maximin Principle has also its own version, valid for the Neumann boundary condition. In other words,

Proposition 3.5. *(Neumann's Maximin Principle) Let $\mu_1 < \mu_2 \leqslant ...$ be the eigenvalues of the Laplacian in Ω with Neumann boundary conditions. Then,*

$$\mu_k = \max_{f_1, ..., f_{k-1 \in L^2(\Omega)}} \min_{\substack{u \in H^1(\Omega) \\ \int_\Omega u f_i = 0 \\ ||u||_{L^2(\Omega)} = 1}} ||\nabla u||_{L^2(\Omega)} \tag{3.10}$$

Now, if we consider both boundary Laplacian problems, we have the next result: the k-th Dirichlet eigenvalue is never smaller than the k-th Neumann one.

Proposition 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If λ_n and μ_n the nth eigenvalue for the Dirichlet and Neumann problem, respectively, then $\mu_n \leqslant \lambda_n$.*

Proof. We start proving the inequality for the first eigenvalue. By Proposition (2.18) and Proposition (2.21), both first eigenvalues λ_1 and μ_1 are found searching the same minimum of the Rayleigh quotient. However, the trial functions for the Dirichlet first eigenvalue λ_1 satisfy an extra condition; the nullity in the boundary of Ω. As the computation of μ_1 requires one less constraint, the first Neumann eigenvalue μ_1 will be possibly smaller. Then $\mu_1 \leqslant \lambda_1$.

Suppose now that $k \geqslant 2$. Let us take $f_1, ..., f_{k-1} \in L^2(\Omega)$ and due to the presence of an extra constraint, we have:

$$\mu_k^*(\phi_1, ..., \phi_{k_1}) \leqslant \lambda_k^*(f_1, ..., f_{k_1})$$

Just by taking the maximum of both sides and applying the Maximin Principle for each type of boundary eigenvalue problem (Proposition 3.4 and the Neumann analog) we have

$$\mu_k = \max \mu_k^*(\phi_1, ..., \phi_{k_1}) \leqslant \max \lambda_k^*(f_1, ..., f_{k_1}) = \lambda_k$$

and the result is proved. □

The general principle illustrated by Proposition 3.6 is that any additional constraint will increase the value of the Maximin Principle. In the path of learning about the asymptotics, it is crucial to comprehend the relation between eigenvalues of the bounded domain and a subdomain that is contained (for each Laplacian boundary problem). In particular, we can take advantage of the Maximin principle to prove the monotonicity of the eigenvalues with respect to the domain.

Theorem 3.7. *(Monotonicity of eigenvalues). Let $\Omega \subset \mathbb{R}^n$ a bounded domain and $\Omega' \subset \Omega$ a subdomain. If $\lambda_k(\Omega)$ and $\lambda'_k(\Omega')$ represent the k-th Dirichlet eigenvalue for the domains Ω and Ω', respectively. Then,*

$$\lambda'_k(\Omega') \geqslant \lambda_k(\Omega)$$

Proof. For the proof it is need to be suffixed that the functions $f_1, ..., f_{k-1} \in L^2(\Omega)$ that allows us to construct the Maximin Principle for the subdomain Ω' are well defined not only in Ω' but also in ω. Observe that for any trial function $w' \in H^1_0(\Omega')$ different to 0 and orthogonal to $f_i \quad \forall i = 1, ..., k-1$ could be extended to a function $w \in H^1_0(\Omega)$ with the same two properties as w'. Just by defining,

$$w(x) = \begin{cases} w'(x) & \text{if} \quad x \in \Omega' \\ 0 & \text{if} \quad x \in \Omega \setminus \Omega' \end{cases}$$

With this definition it is clear that every trial function in Ω' represents a trial function in Ω. Apart from that, $||\nabla w||^2_\Omega = ||\nabla w'||^2_{\Omega'}$ and $||w||^2_\Omega = ||w'||^2_{\Omega'}$.

As the trial functions for Ω' compared to those in Ω have the extra condition of vanishing in the rest of the domain Ω, it means that

$$\lambda_k{}^*(f_1, ..., f_{k_1}) \leqslant \lambda'^*_k(f_1, ..., f_{k_1})$$

which directly applying the Maximin Principle over all f_i chosen we directly see that

$$\lambda'_k(\Omega') \geqslant \lambda_k(\Omega)$$

We should beware that we are avoiding the difficulty that by extending the function to be zero, it might stop being a trial function, in the sense of its regularity. Despite this, the extended function $w(x)$ is that it still is continuous. For a rigorous justification of this point, see [CH89, Gar64]. □

Remark 3.8. For the Neumann case, there exists a counterexample for the monotonicity of eigenvalues. Considering a 2D rectangle of sides a and b with $a > b$ with

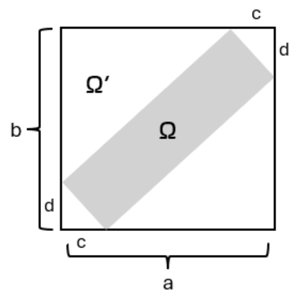

Figure 3.3: Neumann's counterexample

an inscribed thin rectangle as shown in Figure 3.3, $\Omega \subset \Omega'$. As it is precised [Sai07] and recalling the Neumann's eigenvalues for a rectangular domain, you can have $\mu_2 < \mu_2'$, even if $\Omega \subset \Omega'$.

Having proved the monotonicity of the eigenvalues for subdomains and recalling the asymptotic growth of the Dirichlet or Neumann eigenvalues for a rectangle:

$$\lim_{k\to\infty} \lambda_k = \lim_{k\to\infty} \mu_k = \frac{4\pi k}{|Q|}$$

we are in position to prove the next theorem,

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^n$ a bounded smooth domain and $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots$ be the Dirichlet eigenvalues. Then,there are positive constants C_1 and C_2 independent of k, such that*

$$C_1 k^{2/n} \leqslant \lambda_k \leqslant C_2 k^{2/n} \text{ for all k large enough}$$

Proof. As Ω is bounded, we can always find domains $\Omega(R)$ and $\Omega(r)$ in $\mathbb{R}^n$ which are contained or can entirely cover Ω, such that, $\Omega(r) \subset \Omega \subset \Omega(R)$. For proper sets of $a_1, ..., a_n$ and $b_1, ..., b_n$ lengths for two n-dimensional parallelepipeds, such us the first is completely contained and the second covers all Ω , such that, $\Omega(r) \subset \Omega \subset \Omega(R)$ (look figure 3.4). Using the monotonicity of Dirichlet eigenvalues, we have that $\lambda_k(\Omega(r)) \leqslant \lambda_k(\Omega) \leqslant \lambda_k(\Omega(R))$ This means that the k-th eigenvalue of Ω will be bounded by the k-th eigenvalues of both parallelepipeds. By equation (3.6),

$$4\pi^2 \frac{k^{2/n}}{(\omega_n|\Omega(r)|)^{2/n}} \leqslant \lambda_k \leqslant 4\pi^2 \frac{k^{2/n}}{(\omega_n|\Omega(R)|)^{2/n}}$$

which means there are positive constants C_1 and C_2 that do not depend on k such that for k large enough

$$C_1 k^{2/n} \leq \lambda_k \leqslant C_2 k^{2/n} \text{ for all k large enough}$$

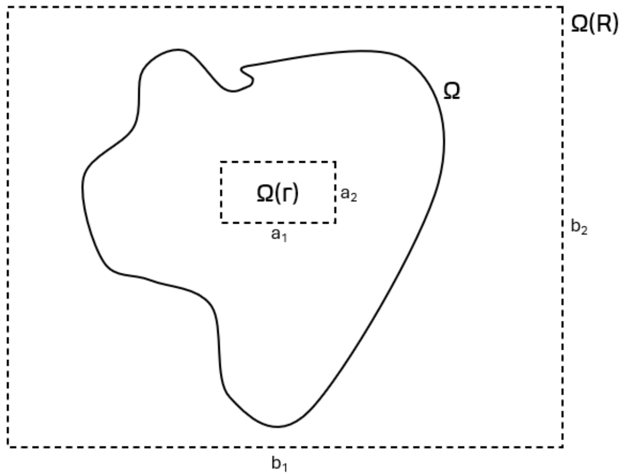

Figure 3.4: An arbitrary bounded domain Ω that contains a little rectangle and which is also contained by a bigger one.

□

3.3 Subdomains

Our next step is to divide the general domain Ω into a finite number of subdomains $\Omega_1, \ldots, \Omega_n$ by introducing inside Ω a system of smooth surfaces $S_1, S_2, \ldots$. Let Ω have Dirichlet eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ and Neumann eigenvalues $\mu_1 \leqslant \mu_2 \leqslant \ldots$. Each of the subdomains $\Omega_1, \ldots, \Omega_n$ has its own collection of eigenvalues. We combine all of the Dirichlet eigenvalues of all of the subdomains $\Omega_1, \ldots, \Omega_n$ into a single increasing sequence $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots$. Similarly, we combine all of their Neumann eigenvalues into another single increasing sequence $\tilde{\mu}_1 \leqslant \tilde{\mu}_2 \leqslant \ldots$.

By the Maximin Principle (for both conditions), each of these numbers can be obtained as the maximum over piecewise continuous functions $(L^2(\Omega))$ $f_1, \ldots, f_{n-1}$ of the minimum over trial functions orthogonal to $f_1, \ldots, f_{n-1}$. As discussed above, although each $\tilde{\lambda}_n$ is a Dirichlet eigenvalue of a single one of the subdomains, the trial functions can be defined in all of Ω simply by making them vanish in the other subdomains. They will be continuous but not C^2 in the whole domain Ω. Thus, each of the competing trial functions for $\tilde{\mu}_n$ has the extra restriction, compared with the trial functions for λ_n for Ω, of vanishing on the internal boundaries. For this reason:

$$\lambda_n \leq \tilde{\lambda}_n \quad \text{for each } n = 1, 2, \ldots. \tag{3.11}$$

On the other hand, the trial functions defining μ_n for the Neumann problem in Ω are arbitrary C^2 functions. As above, we can characterize $\tilde{\mu}_n$ as

$$\tilde{\mu}_n = \max \tilde{\mu}_n^*(f_1, ..., f_{n-1}) \tag{3.12}$$

where the competing trial functions are arbitrary on each subdomain and orthogonal to each f_i. But these trial functions are allowed to be discontinuous on the internal boundaries, so they comprise a significantly more extensive class than the trial functions for λ_n, which are required to be continuous in Ω. Therefore, we have $\tilde{\mu}_n \leq \mu_n$ for each n. Combining this fact with Proposition 3.6 and (3.11), we have proven the following inequalities:

$$\tilde{\mu}_n \leqslant \mu_n \leqslant \lambda_n \leqslant \tilde{\lambda}_n \tag{3.13}$$

From this development, we have already proved the next result:

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and consider a partition of subdomains $\Omega_1, ..., \Omega_n$. Thus, for all $k \geqslant 1$,*

$$\mu_k(\Omega_1 \cup ... \cup \Omega_n) \leqslant \mu_k(\Omega) \leqslant \lambda_k(\Omega) \leqslant \lambda_k(\Omega_1 \cup ... \cup \Omega_n)$$

This last Theorem can be rewritten in the form of the next Corollary.

Corollary 3.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and consider a partition of subdomains $\Omega_1, ..., \Omega_n$. If we deine for any $\rho > 0$*

$$\begin{cases} D(\rho) = \#\{\lambda_k \in Spectrum(\Omega) : \lambda_k \leqslant \rho)\} \\ N(\rho) = \#\{\mu_k \in Spectrum(\Omega) : \mu_k \leqslant \rho)\} \end{cases}$$

Thus,

$$D_{\Omega_1}(\rho) + \cdots + D_{\Omega_n}(\rho) \leqslant D_{\Omega}(\rho) \leqslant N_{\Omega}(\rho) \leqslant N_{\Omega_1}(\rho) + \cdots + N_{\Omega_n}(\rho) \tag{3.14}$$

Remark 3.12. Let $\Omega \subset \mathbb{R}^n$ is be the union of a finite number of parallelepipeds (rectangles in two dimensions) $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$. It is clear that the $|\Omega| = |\Omega_1| + \cdots + |\Omega_n|$. We want to take advantage of knowing the eigenvalues for the Dirichlet and Neumann conditions for a n-dimensional parallelepiped (rectangular domain) domain. Taking a $\rho > 0$ and dividing the equation (3.14) of Corollary 3.11 by it:

$$\frac{D_{\Omega_1}(\rho) + \cdots + D_{\Omega_n}(\rho)}{\rho} \leqslant \frac{D_\Omega(\rho)}{\rho} \leqslant \frac{N_{\Omega_1}(\rho) + \cdots + N_{\Omega_n}(\rho)}{\rho}$$

Taking ρ to infinity and using equations (3.5) and (3.7) for each parallelepiped we have:

$$\lim_{\rho\to\infty} \sum_{i=1}^{n} \frac{D_{\Omega_i}(\rho)}{\rho} = \frac{\omega_n}{2^n} \frac{\rho^{n/2-1}}{\pi^n} \sum_{i=1}^{n} |\Omega_i| = \frac{\omega_n}{2^n} \frac{\rho^{n/2-1}}{\pi^n} |\Omega| = \lim_{\rho\to\infty} \sum_{i=1}^{n} \frac{N_{\Omega_i}(\rho)}{\rho}$$

Thus, using the sandwich theorem, we obtain:

$$\lim_{\rho\to\infty} \frac{D(\rho)}{\rho^{n/2}} = \frac{\omega_n}{2^n} \frac{|\Omega|}{\pi^n} \tag{3.15}$$

which for n=2, it is the (Weyl's law for grids).

3.4 Weyl's law for bounded smooth domains

We are in position to prove Theorem 3.1 that it is what we firstly have aimed for: proving Weyl's law for any bounded domain.

From the monotonicity of eigenvalues to Weyl's law for grids, seems pretty reasonable to consider that in order to evaluate a general case, we just need to encapsulate our domain between two set of finite union of rectangles, that in one case contains the domain, and in the other is contained by it.

Proof. Let a $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. As it is bounded, given any $\varepsilon > 0$, it exists a finite union of rectangles Ω_1 and Ω_2 that suffix

$$\Omega_1 \subset \Omega \subset \Omega_2, \quad \text{with} \quad |\Omega_1| \leqslant |\Omega| \leqslant |\Omega_2| \qquad \text{and} \qquad |\Omega_2| - |\Omega_1| < \varepsilon$$

Since $\Omega_1 \subset \Omega \subset \Omega_2$, by Theorem 3.7 of monotonicity of eigenvalues, the eigenvalues of Ω, Ω_1 and Ω_2 are well-ordered. For all $k \geqslant 1$,

$$\lambda_k(\Omega_2) \leqslant \lambda_k(\Omega) \leqslant \lambda_k(\Omega_1)$$

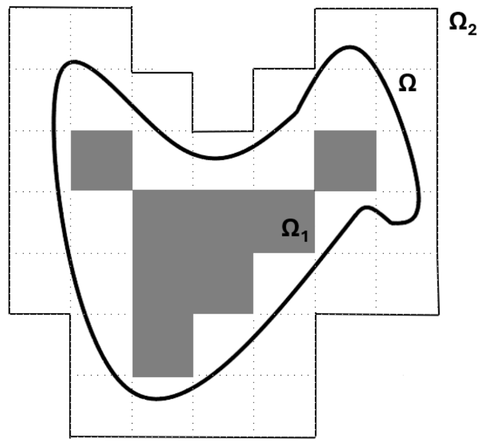

Figure 3.5: A bounded domain Ω encapsulated between two sets of finite rectangles.

which means that for a given $\rho > 0$

$$\frac{D_{\Omega_1}(\rho)}{\rho^{k/2}} \leqslant \frac{D_{\Omega}(\rho)}{\rho^{k/2}} \leqslant \frac{D_{\Omega_2}(\rho)}{\rho^{k/2}}$$

As we know the expression of the Weyl's law for grids and Ω_1 and Ω_2 are a collection of finite rectangles these inequalities hold:

$$\liminf_{\rho\to\infty} \frac{D_{\Omega}(\rho)}{\rho^{k/2}} \geqslant \lim_{\rho\to\infty} \frac{D_{\Omega_1}(\rho)}{\rho^{k/2}} = \frac{\omega_n}{2^n}\frac{|\Omega_1|}{\pi^n}$$

$$\limsup_{\rho\to\infty} \frac{D_{\Omega}(\rho)}{\rho^{k/2}} \leqslant \lim_{\rho\to\infty} \frac{D_{\Omega_2}(\rho)}{\rho^{k/2}} = \frac{\omega_n}{2^n}\frac{|\Omega_2|}{\pi^n} \leqslant \frac{\omega_n}{2^n}\frac{\varepsilon + |\Omega_1|}{\pi^n}$$

Eventually, as we first defined $|\Omega_1| \leqslant |\Omega| < |\Omega_1| + \varepsilon$, and ε is arbitrary, in the limit $\varepsilon \to 0^+$, by the sandwich theorem we get that:

$$\lim_{\rho\to\infty} \frac{D_{\Omega}(\rho)}{\rho^{k/2}} = \frac{\omega_n}{2^n}\frac{|\Omega|}{\pi^n}$$

for all $k \geqslant 1$. In addition, as we have done before, for large ρ,this value can be approximated to λ_k, and D(λ_k)=k which means that finally:

$$\lim_{k\to\infty} \lambda_k = 4\pi^2 \frac{k^{2/n}}{(\omega_n|\Omega|)^{2/n}}$$

□

Finally, for completeness, an important exploration is related to how the bounds of $D_\Omega(\rho)$ are approximated for large enough ρ. Although we do not intend to prove results like this, Weyl conjectured in [Wey11] the existence of a second asymptotic term would give the perimeter of Ω for the planar case.

$$D_\Omega(\rho) = \frac{|\Omega|}{4\pi}\rho \mp \frac{|\partial\Omega|}{4\pi}\sqrt{\rho} + O(\sqrt{\rho}) \quad \text{as} \quad \rho \to \infty \tag{3.16}$$

where $|\partial\Omega|$ denotes the length of the boundary or its perimeter and the (–) sign refers to the Dirichlet boundary condition and the (+) sign to the Neumann boundary problem.

However, until de 1980 the most general expression for the Weyl's law was not proved. For a smooth boundary, this was proved by Russian, Canadian mathematician Victor Ivrii in 1980. In [Ivr80] he presented the precise formula:

$$D_\Omega(\rho) = \frac{\omega_n}{2^n}\frac{|\Omega|}{\pi^n}\rho^{n/2} \mp \frac{\omega_{n-1}}{4}\frac{|\partial\Omega|}{(2\pi)^{n-1}}\rho^{(n-1)/2} + O(\rho^{(n-1)/2}) \quad \text{as} \quad \rho \to \infty \tag{3.17}$$

where $|\partial\Omega|$ is the (n-1)-dimensional volume of the boundary $\partial\Omega$.

3.5 "Can one hear the shape of a drum?" and the inverse spectral problem

Independently of how sharp are the bounds, Weyl's law answers a question we asked at the beginning of the the Introduction: there exists properties, such as the volume, that can be inferred from the spectrum of a bounded smooth domain Ω. For this reason, Weyl's law provides the following insight: if $\Omega \subset \mathbb{R}^N$ is a bounded open set and all the eigenvalues of the Dirichlet Laplacian are known, then the volume of Ω can also be deduced. In other words, the spectrum of the Dirichlet Laplacian determines the volume. However, this question can be pushed further: are there additional properties or features of Ω that can be inferred from its spectrum? Such inquiries fall under the category of *inverse spectral problems*. We can define the following:

Definition 3.13. *Consider two bounded smooth open sets Ω_1 and Ω_2 in $\mathbb{R}^N$. These sets are called isospectral if Ω_1 and Ω_2 share the same Dirichlet Laplacian sequence of eigenvalues, or what it is the same if $Spectrum(\Omega_1) = Spectrum(\Omega_2)$.*

Then, by Weyl's law, a direct result is:

Proposition 3.14. *Suppose Ω_1 and Ω_2 in $\mathbb{R}^N$ are two bounded smooth isospectral domains. Thus, Ω_1 and Ω_2 have the same volume.*

After fifty years of Weyl's law, the inverse spectral problem gained a lot of popularity after Polish-American mathematician Mark Kac popularized a zany but mathematically deep question in his 1966 paper, [Kac66] "Can One Hear the Shape of a Drum?". In other words, if you hear someone beat a drum, and you know the frequencies of the sounds it makes, can you work backward to figure out the shape of the drum that created those sounds? Or can more than one drum shape create the exact same set of frequencies? As we showed in the Introduction, as the wave equation problem is completely dependent to the Dirichlet Laplacian problem, clearly this paper motivated a more general spectral problem. With Kac's language and Weyl's law, the volume of a bounded smooth domain can be listened. Is the domain's shape listenable, too? In other words, can the shape be inferred by the spectrum of the set?

Formulating it more precisely, let us define:

Definition 3.15. *Two bounded sets Ω_1, Ω_2 are called congruent in the Euclidean sense if there exists an orthogonal matrix B and a vector b in $\mathbb{R}^n$ such that $\Omega_2 = \{Bx + b : x \in \Omega_1\}$. In the plane, two figures are called congruent exactly if one can be transformed into the other by translation, rotation, and reflection.*

The critical question that Kac asked was: *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two bounded smooth domains. If they are isospectral, are they necessarily congruent?*

Although we will not go much deeper, Kac was not the first person to pose this or related questions. Indeed, before Kac posed the question for plane domains, Milnor [Mil64] had shown in 1964, the existence of two isospectral, but non-congruent, 16-dimensional tori. Thus, for compact manifolds (at least in 16 dimensions!), the question had been answered negatively. What it is more, for a more restricted boundaries such as Lipschitz domains, a first counterexample was given by Urakawa [Ura82] in 1982 who constructed two isospectral Lipschitz domains in $\mathbb{R}^4$ which are not congruent. Ten years later, Gordon, Webb and Wolpert [GWW92] found a two-dimensional example. By putting together seven triangles they obtained two polygons in $\mathbb{R}^2$ which are isospectral but not congruent. These two polygons are not convex, though. It is an open question whether convex isospectral polygons in $\mathbb{R}^2$ are congruent. However, in four dimensions convexity alone does not help. There are convex isospectal sets which are not congruent. In fact, by modifying Urakawa's example, Gordon and Webb [GW96] obtained two truncated convex cones in $\mathbb{R}^4$ which are isospectral but not congruent.

So far, a counterexample for a smooth boundary is unknown in any dimension. For this reason, the question raised by Kac is completely open today, although more related results have been found.

There exists, despite all this, some positive results. One of them appears if we impose that one of both sets is a ball. Before showing this, let us present as a lemma, the well-known Faber-Krahn inequality:

Lemma 3.16. *(Faber-Krahn inequality) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Suppose $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet problem on Ω and Ω^* the open centered ball whose volume coincides with Ω. Then,*

$$\lambda_1(\Omega) \geqslant \lambda_1(\Omega^*) \tag{3.18}$$

with equality if and only if Ω is a ball.

Taking into consideration this important theorem (We refer the reader to [Bur09] in order to have a complete understanding of the proof), the promised conclusion can be easily proved.

Proposition 3.17. *Let Ω_1, $\Omega_2 \subset \mathbb{R}^n$ be two bounded smooth domains. Suppose Ω_1 is a n-dimensional ball. If they are isospectral, then they are congruent.*

Proof. For a smooth bounded domain Ω, Weyl's law shows that its volume $|\Omega|$ can be determined by the spectrum. Moreover, if the Faber-Krahn inequality is used:

$$\lambda_1(\Omega) \geqslant c_n |\Omega|^{-2/n}, \tag{3.19}$$

which holds for all such domains. Here, $\lambda_1(\Omega)$ represents the first eigenvalue of the Dirichlet Laplacian on Ω, and c_n is a constant that depends only on the dimension n. Additionally, equality only hold if and only if Ω is a ball. Having said that, as $|\Omega_1| = |\Omega_2|$ for being isospectral, then

$$\lambda_1(\Omega_1) = c_n |\Omega_1|^{-2/n} = c_n |\Omega_2|^{-2/n} = \lambda_1(\Omega_2),$$

which means that Ω_2 is also a ball. Therefore, clearly they are congruent in the Euclidean sense. $\square$

This means that one can hear if a smooth bounded domain in $\mathbb{R}^n$ is a ball.

Chapter 4

Nodal domains of eigenfunctions

In our derivation of Weyl's law we have been able to use a surprisingly successful use of estimates of the eigenvalue λ_k by its monotonicity properties when dividing our bounded domain Ω into rectangles. The final element to analyze around the asymptotic behavior of the eigenvalues is the examination of nodal domains.

Definition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. A nodal domain of a function $u \in H_0^1(\Omega)$ is a connected component of the set $\{u \neq 0\} \subset \Omega$.*

In particular, if ϕ_k is the k-th eigenfunction of the Dirichlet problem, nodal lines are the set of points in the domain that suffix $\phi_k(x) = 0$. These level curves separate Ω in nodal domains which, while they are not compulsory rectangular, they need to diminish in size as $k \to \infty$, because if the eigenvalue λ_k associated to ϕ_k grows without limit, then it will not remain the principal eigenvalue of any fixed proportion region. Therefore, although nodal domains become smaller or narrower as $k \to \infty$ we need to see that they are not more than k in number.

Theorem 4.2. *(Courant's Theorem) Any eigenfunction with λ_k as its eigenvalue has at most k nodal domains.*

Proof. Assume that u is an eigenfunction with λ_k as its eigenvalue with at least $k+1$ nodal domains, and let $\Omega_1, ..., \Omega_{k+1}$ be distinct nodal domains of u. For each $i = 1, ..., k$, we define the following functions:

$$\varepsilon_i = \begin{cases} u & \text{on} \quad \Omega_i \\ 0 & \text{elsewhere} \end{cases}$$

By Theorem 3.7 we saw that for any subdomain $\Omega' \subset \Omega$, a function $w' \in H_0^1(\Omega')$ can be extended to a function in $w \in H_0^1(\Omega)$ just the way we have done above. Apart from that, $||\nabla w||_\Omega^2 = ||\nabla w'||_{\Omega'}^2$ and $||w||_\Omega^2 = ||w'||_{\Omega'}^2$. Thus, $||u||_\Omega^2 = ||\varepsilon_i||_{\Omega_i}^2$ and

$||\nabla u||^2_\Omega = ||\nabla \varepsilon_i||^2_{\Omega_i}$. Notice that ε_i and $\nabla \varepsilon_i$ are clearly pairwise orthogonal, as they are defined in distinct subdomains.

For any linear combination $v = \beta_1 \varepsilon_1 + ... + \beta_k \varepsilon_k$, we get:

$$||\nabla v||^2 = \sum_{i=1}^{k} \beta_i^2 \int_{\Omega_i} |\nabla \varepsilon_i|^2 = -\sum_{i=1}^{k} \beta_i^2 \int_{\Omega_i} \varepsilon_i \Delta \varepsilon_i = \sum_{i=1}^{k} \beta_i^2 \lambda_k \int_{\Omega_i} \varepsilon_i^2 = \lambda_k ||v||^2$$

then recalling the Rayleigh quotient, $J(v) = \lambda_k$. Suppose ϕ_j be the first k-1 eigenfunctions of the Laplacian. If we introduce the linear conditions:

$$\int_\Omega v\phi_j = 0 \quad \text{for each} \quad 1 \leqslant j \leqslant k-1$$

Since there are more unknown coefficients than constraints, we can affirm that it does exist a $i \in \{1, ..., k\}$ such as $\beta_i \neq 0$ such that v nontrivial linear combination.

Apart from that, as $J(v) = \lambda_k$ by Proposition 2.18, v must be an eigenfunction.

However, as eigenfunctions do not vanish on open sets and v by definition vanishes on Ω_{k+1} we end up with a contradiction. □

From this result it follows an immediate corollary.

Corollary 4.3. *Let $\Omega' \subset \Omega$ be a nodal domain of an eigenfunction with λ_k as its eigenvalue. Then, the first eigenvalue of the Dirichlet problem on Ω' is λ_k.*

4.1 Sharpening of the Nodal Line Theorem

Let $\Omega \subset \mathbb{R}^n$ be a planar bounded smooth domain, $0 < \lambda_1 < \lambda 2 < ...$ the collection of Dirichlet eigenvalues and ϕ_1, ϕ_2... its Dirichlet eigenfunctions. By Courant's Theorem (Theorem 4.2) we know that the number of nodal domains of a given eigenvalue ϕ_k is at most k. Firstly presented by A. Plejel in [Ple56], this Theorem can be sharpened using Faber-Krahn inequality and Weyl's law as you can bound the number of nodal domains of ϕ_k by the zeros of Bessel functions.

Theorem 4.4. *(Pleijel's nodal domain theorem)*

$$\limsup_{k\to\infty} \frac{N}{k} \leqslant \left(\frac{2}{\beta_{1,0}}\right)^2 \approx 0.691...$$

where N is the number of nodal domains of ϕ_k and $\beta_{1,0}$ is the first zero of the Bessel function J_0 (as we defined in section 2.6) and its value is approximately $\beta_{1,0} \approx 2.4048...$

Proof. Let $\Omega' \subset \Omega$ be some nodal domain of ϕ_k. Let $|\Omega'|$ and $|\Omega|$ be their areas, respectively. By Courant's nodal domain corollary (Corollary 4.3) we have:

$$\lambda_1(\Omega') = \lambda_k \tag{4.1}$$

Using the Rayleigh–Faber–Krahn inequality for the membrane (i.e., n=2) we can compare the first eigenvalues of the nodal domain with the first one of a disk of area $|\Omega'|$, B :

$$\lambda_1(\Omega') \geqslant \lambda_1(B) = \frac{\pi\beta_{1,0}^2}{|\Omega'|}, \tag{4.2}$$

Combining the expression (4.1) and (4.2) we get

$$|\Omega'| \geqslant \frac{\pi\beta_{1,0}^2}{\lambda_k}$$

Let $\Omega_1, ..., \Omega_N$ be the nodal domains of domains of ϕ_k with areas $|\Omega_1|, ..., |\Omega_N|$, respectively. Thus,

$$|\Omega| = \sum_{i=1}^{N} |\Omega_i| \geqslant \frac{N\pi\beta_{1,0}^2}{\lambda_k}$$

Dividing by k and rearranging

$$\frac{N}{k} \leqslant \left(\frac{2}{\beta_{1,0}}\right)^2 \frac{\lambda_k|\Omega|}{4\pi k}$$

Finally, we know that by Weyl's asymptotic law when $n = 2$

$$\lim_{k\to\infty} \frac{\lambda_k}{k} = \frac{4\pi}{|\Omega|}$$

Therefore, combining this two last results

$$\limsup_{k\to\infty} \frac{N}{k} \leqslant \left(\frac{2}{\beta_{1,0}}\right)^2$$

as we wanted. □

Remark 4.5. Generalizing it to an n-dimensional domain we just need to change the formulas. Using equation (4.2) and recalling Weyl's law for a n-dimensional domain in Theorem 3.1:

$$\limsup_{k\to\infty} \frac{N}{k} \leqslant \left(\frac{2}{\beta_{1,x}}\right)^n \frac{\pi^n}{\omega_n^{n/2+1}}$$

where ω_n is the hypervolume of the unitary ball in $\mathbb{R}^n$ and $x = n/2 - 1$ if n is even or $x = n/2 - 3/2$ if n is odd.

4.2 Example of Maximal Subdivision by Nodal Lines

The maximal case $N = k$ in Courant's theorem holds in a trivial way when $k = 1, 2$. That it can also occur for larger values of n is usually exemplified by reference to the case of the square membrane with fixed boundary. If Ω is the square $0 \le x \le \pi$, $0 \le y \le \pi$, a complete system of eigenfunctions vanishing on the boundary is given by:

$$\phi_{m,n}(x, y) = A_{m,n} \sin mx \sin ny \tag{4.3}$$

where k and l are positive integers. The corresponding eigenvalues are $\lambda = k^2 + l^2$, the first ones being $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = \lambda_6 = 10$, ... The number of nodal domains of (4.3) is $N = kl$, which for $\lambda = \lambda_4$ equals 4. The general eigenfunction with eigenvalue λ is:

$$\phi(x, y) = \sum_{m^2+n^2=\lambda} A_{mn} \sin mx \sin ny, \tag{4.4}$$

where the C_{mn} are constants. It should be observed that even if the number of nodal domains for each term in (4.4) is less than a certain value, it is quite possible that the sum itself has a larger number of nodal domains. This, for instance, occurs when $\lambda = 10$.

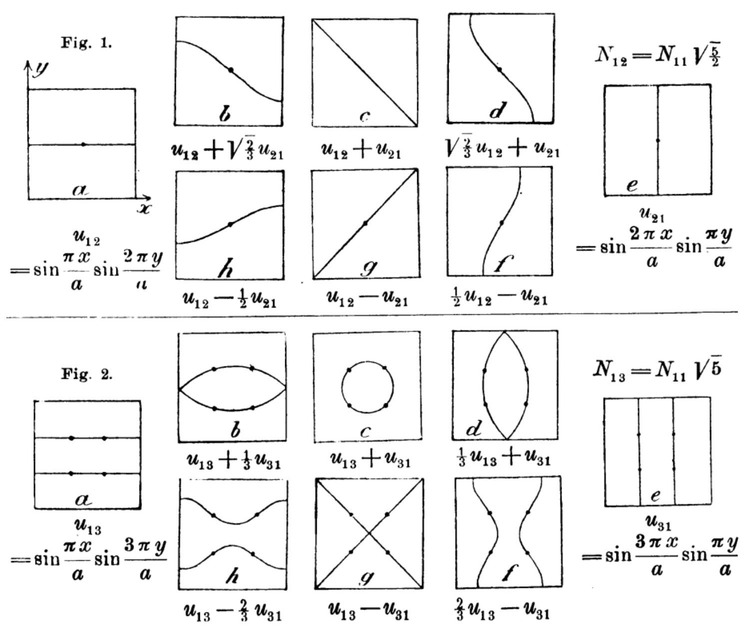

Figure 4.1: Nodal sets, Dirichlet eigenvalues λ_2 and λ_5 (Pockels, [Poc91])

Chapter 5

Conclusions

At the Introduction, there were asked some questions about timbre that have been, eventually, successfully answered through the evolution of this text.

The existence of a list eigenvalues and smooth eigenfunctions of the Laplacian for any vibrating boundary condition problem (for instance a drum) allows us to see that these eigenvalues (strictly connected with the harmonics of an instrument) asymptotically converge to an expression that depend on the volume (or area) of the vibrating domain. In other words, the area of a vibrating membrane could be deciphered by its spectrum. For this reason, using Kac's choice of words, one can listen the area of a drum (or the volume of a vibrating membrane). Moreover, the asymptotics of the spectrum induces the definition of nodal domains, which for the k-th eigenvalue of the spectrum the number of nodal domains is at most k.

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