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Uncertainty and Policy Ineffectiveness: A Macroeconomic Model with Markov Chains

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Abstract

The main objective of this thesis is the study of a new economic relationship between uncertainty (of agents in an economy) and the effectiveness of policy (either fiscal or monetary). To do so, this work develops a theoretical macroeconomic model based on Markov chains, which illustrates that raising uncertainty perceived by economic agents can lower the effectiveness of monetary and fiscal policies, defined as its ability to modify the Gross Domestic Product (GDP). Next, an econometric approach is taken in order to assess whether these conclusions translate to an empirical framework, and results support the model's conclusions with Spanish data from 1995-2018.

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Contents

Introduction

Macroeconomic policymaking has long rested on the premise that governments and central banks can steer aggregate output and employment through well-targeted fiscal or monetary interventions. However, throughout modern economic history, from the Great Recession of 2008 to the Spanish sovereign debt crisis in the early 2010s, policymakers have repeatedly faced unexpected setbacks in which traditional tools failed to deliver the desired outcomes. During the Great Recession, massive fiscal stimulus packages and unconventional monetary policies were deployed across developed economies, yet output and employment recovered only sluggishly [13]. Likewise, in Spain's sovereign debt crisis, despite austerity measures and European Central Bank interventions, the country endured years of high unemployment and subdued growth. These episodes raise fundamental questions about the conditions under which standard policy frameworks can be effective.

One prevalent explanation for these "policy failures" invokes the limiting assumptions of standard macroeconomic models: most notably, the reliance on a representative agent framework and the presumption of fully rational, forward-looking behavior. By ignoring the heterogeneous, stochastic interactions among millions of individual agents, conventional models (such as the IS-LM) may omit crucial sources of aggregate uncertainty and inertia. As Aoki and Yoshikawa [3] demonstrate, when an economy is understood as a vast network of micro-units (households, firms, and financial intermediaries) that interact and revise their decisions in response to idiosyncratic shocks, macroeconomic outcomes emerge not from a single "representative" behavior, but from an evolving distribution over a multitude of states. In such a framework, "equilibrium" is best interpreted not as a fixed point in phase-space but as a probability distribution over states-an insight borrowed from statistical physics and combinatorial stochastic processes.

Rather than imposing a priori that all agents share identical optimization problems and face the same constraints, we endogenize heterogeneity by allowing distinct agent "types" to transition randomly among behavioral regimes. Each micro-unit decision follows probabilistic rules that depend on its current state and on aggregate variables (for instance, interest rates, output gaps, or credit spreads).

Building directly on these ideas, this thesis develops a continuous-time Markov chain model to explore a novel relationship in economic analysis: the interplay between uncertainty and policy ineffectiveness. We will discuss this relationship both at the theoretical level and at the empirical level, using panel data from 1995 to 2018 for Spain. This will allow us to evaluate if the austerity measures imposed during the sovereign debt crisis in Spain were "worth" the short-run discontent of citizens to boost the output level or, on the contrary, were totally "useless" since

high uncertainty constrained policy effectiveness.
Concretely, this thesis proceeds as follows:

- **Chapter 1 (Markov Processes)** introduces the mathematical foundations needed for the development of the model, essentially continuous-time Markov chain, with an emphasis on jump processes that are suited to model discrete changes in agents' states. The main result of this section are the Master equations, which characterize the time-evolution of a Jump Markov Process with discrete states.

- **Chapter 2 (Economic Foundations)** situates our approach within standard micro- and macroeconomic paradigms. Section 2.1 reviews a prototypical micro-foundation consumer-choice model, criticizing its approach based on the representative and rational agent. Section 2.2 revisits the classical IS-LM framework, emphasizing its inability to accommodate persistent heterogeneity. Finally, Section 2.3 develops the basic econometric toolkit which will be used in the empirical analysis conducted.

- **Chapter 3 (A Macroeconomic Model of Uncertainty)** Chapter 3 introduces a continuous-time jump-Markov framework where heterogeneous agents, subject to idiosyncratic shocks, transition probabilistically across micro-states. By embedding fiscal or monetary policy into these transition rates, the chapter derives master equations for the aggregate distribution and characterizes its steady state. It then shows that under high uncertainty, policy is constrained.

- **Chapter 4 (Empirical Evidence on Policy Effectiveness under Uncertainty: The case of Spain)** tests the theoretical predictions using Spanish macro-time-series data. First, it is described the data in Section 4.2. Section 4.3 specifies an econometric regression that maps deviations in government spending and monetary variables to subsequent changes in output growth, while controlling for the evolving "uncertainty index". Then, it is compared OLS estimates of policy multipliers across high and low uncertainty regimes, illustrating that, consistent with the Markov chain model, fiscal multipliers shrink precisely when uncertainty measures spike.

Chapter 1

Markov Processes

1.1 Introduction to Markov Processes

The new approach to macroeconomics presented in the previous section requires mathematical methods and concepts that are not quite common in economics. In particular, tools will be needed to take into account complex features that classical models do not contemplate, such as:

- A finite but large number of micro units interacting
- Different types of agents
- New and unknown types of agents, which can appear and will not be known in advance

Stochastic process will allow us to formally reason on this concepts. Moreover, we will dive deep into Markov process theory and focus particularly on Continuous-time Markov chains, also known as Jump Markov Processes with Discrete States. The reason behind choosing Markov process theory to model macroeconomics was not random. It is because it can be understood as a generalization of ordinary calculus to accommodate mathematical functions that are imbued with a certain kind of unpredictability or "randomness". This is exactly what we need, ordinary calculus has been extensively used in macroeconomics and, while it has provided Economists with meaningful results, it has limitations that are now being accounted for, such as the imposition of the representative agent. Intuitively, we will be able to include different types of agents and their behavior by introducing some randomness in their actions, tilted by their preferences. A more formal discussion will proceed. This chapter will follow [7] closely.

Definition 1.1. *Given a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, a stochastic process is a family of random variables* $\{X_t\}_{t\in T}$*. Moreover:*

- *If* $T = \mathbb{N}$*, the process is in discrete time*
- *If* $T = \mathbb{R}_+$*, then the process is in continuous time*

Note that, on an abuse of notations, we can refer to a stochastic process as $X(t) = X_t(\omega)$, that specifies the instantaneous state of some real or hypothetical system. For each $t \in \mathbb{R}_+$, $X(t) = X_t$ is a random variable.
For the purpose of this study, we will assume that for an initial timestamp t_0, the value of the process if fixed

$$X(t_0) = x_0$$

Let's introduce some notation and consider now the k-variate joint density function with j conditionings, $P_k^{(j)}$.

Definition 1.2. *Given a stochastic process X(t),* $P_k^{(j)}$ *is defined as:*

$$\begin{aligned} &P_k^{(j)}(x_{r_k}, t_{r_k}; ...; x_{r_1}, t_{r_1} | x_{s_j}, t_{s_j}; ...; x_{s_1}, t_{s_1}) \\ &\equiv \mathbb{P}(X(t_{r_z}) = x_{r_z} \forall z = 1, ..., k | X(t_{s_y}) = x_{s_y} \;\; \forall y = 1, ..., j \end{aligned} \tag{1.1}$$

Note that the subscript refers to the number of (x, t) pairs to the left of the "conditional bar", and the superscript to the number of pairs to the right of the conditional bar. With this notation, we can now easily define Markov processes:

Definition 1.3. *A stochastic process* $X(t)$ *is called a Markov process if, and only if for all* $j \geq 2$ *and* $t_{j-1} < t_j$*:*

$$P_1^{(j)}(x_j, t_j | x_{j-1}, t_{j-1}; ...; x_1, t_1, x_0, t_0) = P_1^{(1)}(x_j, t_j | x_{j-1}, t_{j-1}) \tag{1.2}$$

The last equation is called the Markov property, and it states that the process is "memoryless": Given that $X(t') = x'$, then our ability to predict $X(t)$ for any $t > t'$ will not be enhanced by the knowledge of any values of the process earlier than t'.

1.2 General properties of Markov Processes

In this section, concepts that will be further useful regarding Markov processes will be displayed and proven. We shall begin with the Markov state density function.

1.2.1 Markov State density function

Definition 1.4. *The Markov State Density function is* $P \equiv P_1^{(1)}$*, whenever it is a probability density function.*

Its definition is relevant since for Markov processes, every conditioned state density function $P_n^{(1)}$ can be written solely in terms of the particular conditioned state density function. This is shown in the next proposition.

Proposition 1.5.

$$P_n^{(1)}(x_n, t_n; ...; x_1, t_1 | x_0, t_0) = \prod_{i=1}^{n} P(x_i, t_i | x_{i-1}, t_{i-1}) \tag{1.3}$$

Proof. Induction on n is used for this proof. The initial case will be $n = 2$. Then:

$$P_2^{(1)}(x_2, t_2; x_1, t_1 | x_0, t_0) = P_1^{(1)}(x_1, t_1 | x_0, t_0) P_1^{(2)}(x_2, t_2 | x_1, t_1; x_0, t_0)$$

$$= P_1^{(1)}(x_1, t_1 | x_0, t_0) P_1^{(1)}(x_2, t_2 | x_1, t_1) = P(x_1, t_1 | x_0, t_0) P(x_2, t_2 | x_1, t_1)$$

The first equality is the result of the definition of conditional probability, while the second one is given because of the Markov Property. For the induction step, assume the result for $n - 1$.

$$P_n^{(1)}(x_n, t_n; ...; x_1, t_1 | x_0, t_0) = P_{n-1}^{(1)}(x_{n-1}, t_{n-1}; ...; x_1, t_1 | x_0, t_0) P_1^{(1)}(x_n, t_n | x_{n-1}, t_{n-1}; ...; x_0, t_0)$$

$$= \prod_{i=1}^{n-1} P(x_i, t_i | x_{i-1}, t_{i-1}) P(x_n, t_n | x_{n-1}, t_{n-1}) = \prod_{i=1}^{n} P(x_i, t_i | x_{i-1}, t_{i-1})$$

□

This proposition states that for a Markov process, if continuous, every conditioned state density function $P_n^{(1)}$ can be written in terms of the particular conditioned state density function P. P, the Markov state density function, becomes the principle focus of our study.

1.2.2 The Chapman-Kolmogorov equation

In the previous section, we stated that a Markov process is fully characterized by the Markov state density function. However, assuming the process is composed of continuous random variables, this density function is not guaranteed to be a probability density function (pdf). In particular, it must satisfy its basic probability axioms. Moreover, to be a probability density function representing a Markov process, it must verify the Markov property. With this reasoning, the following characterization of a Markov state density function (and thus, a Markov process) arises:

Proposition 1.6. *$P(x_i, t_i|x_{i-1}, t_{i-1})$ is a Markov state density function if, and only if:*

$$P(x_i, t_i|x_{i-1}, t_{i-1}) \geq 0 \tag{1.4}$$

$$\int_{-\infty}^{\infty} P(x_i, t_i|x_{i-1}, t_{i-1}) dx_i = 1 \tag{1.5}$$

$$P(x_i, t_i = t_{i-1}|x_{i-1}, t_{i-1}) = \delta(x_i - x_{i-1}) \tag{1.6}$$

$$P(x_{i+1}, t_{i+1}|x_{i-1}, t_{i-1}) = \int_{-\infty}^{\infty} P(x_{i+1}, t_{i+1}|x_i, t_i) P(x_i, t_i|x_{i-1}, t_{i-1}) dx_i \tag{1.7}$$

where $t_{i-1} \leq t_i \leq t_{i+1}$.

Proof. Conditions (1.4) and (1.5) make sure the function is a PDF. (1.6) assures the function verifies the definition of conditional probability. (1.7) is the condition that makes the function represent a Markov process. Let's show it:

$$P(x_{i+1}, t_{i+1}|x_{i-1}, t_{i-1}) = \int_{-\infty}^{\infty} P_2^{(1)}(x_{i+1}, t_{i+1}; x_i, t_i|x_{i-1}, t_{i-1}) dx_i$$

$$= \int_{-\infty}^{\infty} P_1^{(1)}(x_i, t_i|x_{i-1}, t_{i-1}) P_1^{(2)}(x_{i+1}, t_{i+1}|x_i, t_i; x_{i-1}, t_{i-1}) dx_i$$

$$= \int_{-\infty}^{\infty} P(x_i, t_i|x_{i-1}, t_{i-1}) P(x_{i+1}, t_{i+1}|x_i, t_i)$$

The last equality was achieved with the Markov property, thus, it is embedded in this condition. □

The last equation is the Chapman-Kolmogorov equation. It is important in the characterization of a Markov process, as it is shown in this proposition, but also in the determination of time-evolution equations for $P(x, t|x_0, t_0)$. This will be shown later on.

There is another version of the Chapman-Kolmogorov equation that is useful to calculate $P(x, t|x_0, t_0)$ is:

Proposition 1.7. *For $t_0 < t_1 < ... < t_n$:*

$$P(x_n, t_n|x_0, t_0) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \prod_{i=1}^{n} P(x_i, t_i|x_{i-1}, t_{i-1}) \tag{1.8}$$

Proof. The following lemma is useful to easily prove this

Lemma 1.8.

$$P(x_i, x_k) = \int_{-\infty}^{\infty} dx_j P(x_i, x_j | x_k) \tag{1.9}$$

Proof. Take the definition of conditional probability:

$$\mathbb{P}(x_i, x_j) = \mathbb{P}(x_i | x_k) \mathbb{P}(x_j | x_i, x_k)$$

Now, integrating over x_j and imposing the normalization condition:

$$\int_{-\infty}^{\infty} \mathbb{P}(x_i, x_j | x_k) dx_j = \mathbb{P}(x_i | x_k) \int_{-\infty}^{\infty} \mathbb{P}(x_j | x_i, x_k) dx_j = \mathbb{P}(x_i | x_k)$$

□

Using this lemma recursively:

$$P(x_n, t_n | x_0, t_0) = \int_{-\infty}^{\infty} dx_{n-1} \cdots \int_{-\infty}^{\infty} dx_1 P_n^{(1)}(x_n, t_n; ...; x_1, t_1 | x_0, t_0)$$

Now, because of Proposition 1.5, which is a consequence of the Markov property:

$$P(x_n, t_n | x_0, t_0) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \prod_{i=1}^{n} P(x_i, t_i | x_{i-1}, t_{i-1})$$

where $t_0 < t_1 < ... < t_n$. □

1.2.3 The Markov propagator

The concept of propagator in Markov processes is crucial, since its density function characterizes the Markov process.

Definition 1.9. *Consider a Markov process $X(t)$. Given an infinitesimal time interval dt and $X(t) = x$, the Markov propagator is defined as:*

$$\Xi(dt, x, t) \equiv X(t + dt) - X(t) \tag{1.10}$$

It measures the *displacement* of the system in an infinitesimal time interval, somewhat like a derivative would do in calculus. Note the generalization process this work is trying to achieve with respect to calculus.
As the propagator is clearly a random variable, if continuous, it is completely characterized by its density function.

Definition 1.10. *Consider a Markov process $X(t)$. The propagator density function is:*

$$\Pi(\xi | dt, x, t) d\xi \equiv \mathbb{P}(\Xi(dt, x, t) \in [\xi, \xi + d\xi)) \tag{1.11}$$

There is an important connection between the propagator density function Π and the Markov state density function P, since both characterize the Markov process:

Proposition 1.11.

$$\Pi(\xi|dt,x,t) = P(x+\xi,t+dt|x,t) \tag{1.12}$$

Proof.

$$P(x+\xi,t+dt|x,t) = \mathbb{P}(X(t+dt) = x+\xi|X(t) = x)$$
$$= \mathbb{P}(X(t+dt) - x = \xi|X(t) = x) = \mathbb{P}(X(t+dt) - X(t) = \xi|X(t) = x)$$
$$= \mathbb{P}(\Xi(dt,x,t) = \xi) = \Pi(\xi|dt,x,t)$$

☐

Therefore, a Markov process is fully characterized by specifying the Markov propagator. Now the same argument as with the Markov state density function follows:

Proposition 1.12. *A function $\Pi(\xi|dt,x,t)$ is a Markov propagator representing a Markov process if, and only if:*

$$\Pi(\xi|dt,x,t) \geq 0 \tag{1.13}$$

$$\int_{-\infty}^{\infty} \Pi(\xi|dt,x,t)d\xi = 1 \tag{1.14}$$

$$\Pi(\xi|dt=0,x,t) = \delta(\xi) \tag{1.15}$$

$$\Pi(\xi|dt,x,t) = \int_{-\infty}^{\infty} \Pi(\xi-\xi_1|(1-a)dt,x+\xi_1,t+adt)\Pi(\xi_1|adt,x,t)d\xi_1 \tag{1.16}$$

for any value $a \in (0,1)$.

Proof. Equations (1.13), (1.14) are a consequence of $\Pi(dt,x,t)$ being a probability density function. Now, by the definition of Markov propagator:

$$\Xi(dt=0,x,t) \equiv X(t+0) - X(t) = 0$$

Therefore, its density function will be given by (1.15). Finally:

$$P(x+\xi,t+\Delta t)|x,t) = \int_{-\infty}^{\infty} P(x+\xi,t+\Delta t|x+\xi_1,t+a\Delta t)P(x+\xi_1,t+a\Delta t|x,t)d\xi_1$$

$$= \int_{-\infty}^{\infty} P(x+\xi_1+\xi-\xi_1,t+a\Delta t+(1-a)\Delta t|x+\xi_1,t+a\Delta t)P(x+\xi_1,t+a\Delta t|x,t)$$

Taking Δt to be infinitesimal dt, using Proposition 1.9, (1.16) is obtained. ☐

As stated above, the importance of the propagator density function Π lies in the fact that it completely determines the Markov state density function P, which allows to compute anything about the Markov process $X(t)$. However, this fundamental fact cannot strictly be interpreted from Proposition 1.9, which states that a knowledge of P implies a knowledge of Π. The following theorem proves this:

Theorem 1.13.

$$P(x,t|x_0,t_0) = \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_{n-1} \prod_{i=1}^{n} \Pi(\xi_i|dt, x_{i-1}, t_{i-1}) \tag{1.17}$$

Proof. Start from the compounded Chapman-Kolmogorov equation Proposition 1.7, and consider interval $[t_0, t]$, where $t = t_n > t_0$. Divide the interval into n subintervals of equal length $(t - t_0)/n$.
Also, change integration variables:

$$x_i \to \xi_i \equiv x_i - x_{i-1} \quad (i = 1, ..., n-1)$$

The Jacobian determinant of this transformation has zeros everywhere on one side of its main diagonal, and ones everywhere on its main diagonal, so $dx_1 \cdots dx_{n-1} = d\xi_1 \cdots d\xi_{n-1}$. Moreover, the transformation implies:

$$x_i = x_{i-1} + \xi_i = (x_{i-2} + \xi_{i-1}) + \xi_i = ... = x_0 + \xi_1 + ... + \xi_i$$

Finally, relabel $x_n = x$. Applying this changes to the compounded Chapman-Kolmogorov equation:

$$P(x,t|x_0,t_0) = \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_{n-1} \prod_{i=1}^{n} P(x_{i-1} + \xi_i, t_{i-1} + (t - t_0)/n | x_{i-1}, t_{i-1})$$

Now, take n large enough so that $(t - t_0)/n = dt$ an infinitesimal. Then, because of the definition of the propagator density function, Definition 1.10:

$$P(x,t|x_0,t_0) = \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_{n-1} \prod_{i=1}^{n} \Pi(\xi_i|dt, x_{i-1}, t_{i-1})$$

□

Therefore, if the propagator density function $\Pi(\xi|dt, x', t')$ is specified as a function of ξ for all x', all $t' \in [t_0, t)$ and all infinitesimal dt, then the Markov density function $P(x, t|x_0, t_0)$ is uniquely determined for all x.
This section defined a Markov process and developed some of its general concepts

and formulas. The main takeaway of the section is given by Theorem 1.13, which states that a Markov process can be fully specified with its propagator density function Π. The following section is devoted to studying in depth Jump Markov Processes, a subclass of Markov processes obtained with a particular propagator density function. They are of special interest since will be extensively used in our macroeconomic model.

1.3 Jump Markov Processes

Roughly speaking, a jump Markov process is one for which the propagator $\Xi(dt, x, t)$ defined in Definition 1.9, is usually zero but occasionally finitely different from 0. Therefore, a jump Markov process does not change its state $X(t)$ in a gradual manner, but rather moves in sudden "jumps". In particular, if the process is in state x at time t, i.e $X(t) = x$, then it will remain there until at some time $t' > t$, where it instantaneously jumps to a new state $x' \neq x$. The times $t^{(1)}, t^{(2)}, ...$ at which those jumps occur, and the states $x^{(1)}, x^{(2)}, ...$ reached in these jumps, both form countable sets.
A formal definition of jump Markov process follows, and next it will be proved that this definition is equivalent to specifying a particular propagator density function.
As for a jump Markov process it makes sense to talk about when the system will leave its current state and what will the next state be, we can define the following probabilities:

Definition 1.14.

$$q(x, t; \tau) \equiv \mathbb{P}(\exists t' \in [t, t+\tau] : X(t') \neq x | X(t) = x) \tag{1.18}$$

$$w(\xi|x, t)d\xi \equiv \mathbb{P}\left(X(t+dt) \in [x+\xi, x+\xi+d\xi] | \Xi(dt, x, t) \neq 0\right) \tag{1.19}$$

More intuitively, $q(x, t; \tau)$ is the probability, given $X(t) = x$, that the process will jump away from state x at some instant between t and $t + \tau$. $w(\xi|x, t)$ is the probability that the process, upon jumping away from state x at time t, will land in some state lying in the infinitesimal interval $[x + \xi, x + \xi + d\xi)$. With these functions, it is possible to give a formal definition to a jump Markov process:

Definition 1.15. *A jump Markov process is a Markov process for which the two functions defined in 1.14 exist, and have the following properties:*

- $q(x, t; \tau)$ *is a smooth function of* t *and* τ, *and satisfies* $q(x, t; 0) = 0$
- $w(\xi|x, t)$ *is a smooth function of* t

In this case, by smooth function we refer to a continuous function that has continuous derivatives of any order. Note also that we could have made this definition by simply considering $X(t)$ as a process, since the Markov property is implicit in the definition of functions $q(x,t;\tau)$ and $w(\xi|x,t)$ (the fate of the process $X(t)$ in state x at time t as dictated by this definitions depends only on x and t, and not on the history of the process prior to time t.
Since this functions will be essential for the treatment of jump Markov processes, it is useful to study their properties:

Proposition 1.16.

$$w(\xi|x,t) \geq 0 \qquad \int_{-\infty}^{\infty} d\xi w(\xi|x,t) = 1$$

$$0 \leq q(x,t;\tau) \leq 1 \qquad q(x,t;dt) = a(x,t)dt$$

where $a(x,t)$ is a nonnegative and smooth function of t.

Proof. The properties of function $w(\xi|x,t)$ and the boundaries of function $q(x,t;\tau)$ are given by the definition of probability. The remaining property is a consequence of the process $X(t)$ being Markovian:
Consider the infinitesimal interval $[t, t+dt)$ and divide it into $n \geq 2$ subintervals of equal length dt/n by defining:

$$t_i = t_{i-1} + dt/n \qquad (i = 1, ..., n-1)$$

where we identify $t_0 = t$. Since we are dividing an infinitesimal interval, the probability $[1 - q(x,t;\tau)]$ that the system in the state x at time t will not jump away from that state in $[t, t+dt)$ is equal to the probability that the system does not jump away from state x in $[t, t_1), [t_1, t_2), ..., [t_{n-1}, t+dt)$. However, since the probability for not jumping away from state x in $[t_{i-1}, t_i)$ is $[1 - q(x, t_{i-1}; dt/n]$, by the multiplication law of probabilities:

$$1 - q(x,t;dt) = \prod_{i=1}^{n} [1 - q[x, t_{i-1}; dt/n]$$

Since t_i is infinitesimal close to t and function $q(x,t;dt)$ is smooth in t, then:

$$q(x, t_{i-1}; dt/n) = \lim_{t' \to t} q(x, t'; dt/n) = q(x,t;dt/n) \qquad \forall i = 1, ..., n$$

Therefore:

$$[1 - q(x,t;dt)] = [1 - q(x,t;dt/n]^n$$

Using again Definition 1.15, $q(x,t;dt/n)$ is smooth in τ. In particular, it's continuous, and since dt/n is infinitesimally close to 0, we can use a Taylor series expansion:

$$[1-q(x,t;dt/n)]^n = \lim_{\tau\to 0}[1-q(x,t;\tau)]^n = 1-n\cdot q(x,t;dt/n)$$

The last step is an equality as second order infinitesimals are negligible to first order infinitesimals. Putting everything together:

$$[1-q(x,t;dt)] = 1-n\cdot q(x,t;dt/n) \Rightarrow q(x,t;dt) = n\cdot q(x,t;dt/n)$$

For the final step of the proof, a lemma is needed:

Lemma 1.17. *If $h(z):\mathbb{R}\longrightarrow\mathbb{R}$ is a smooth function of z such that $h(z)=n\cdot h(z/n)$ for all positive integers n, then $h(z)=Cz$, where C is independent of z.*

Proof. Differentiate $h(z)=nh(z/n)$ with respect to z to obtain: $h'(z)=nh'(z/n)(1/n)=h'(z/n)$. Since $h(z)$ is smooth w.r.t z, then:

$$h'(z)=\lim_{n\to\infty}h'(z)=\lim_{n\to\infty}h'(z/n)=h'\left(\lim_{n\to\infty}z/n\right)=h'(0)\equiv C$$

Note that C is independent from z. Moreover, as $h(z/n)=h(z)/n$, then

$$h(0)=\lim_{n\to\infty}h(z/n)=\lim_{n\to\infty}h(z)/n=0$$

Integrating is easily found that the only function $h(z)$ satisfying both $h(0)=0$ and $h'(z)=C$ is $h(z)=Cz$ ☐

Using this lemma, we obtain $q(x,t;dt)=a(x,t)dt$, the desired result. ☐

With this result, function $a(x,t)$ has an intuitive interpretation:

Proposition 1.18. *For any jump Markov process there exists a function $a(x,t)$, smooth on t, such that*

$$a(x,t)dt=\mathbb{P}(\exists t'\in[t,t+dt):X(t')\neq x|X(t)=x) \tag{1.20}$$

That is, $a(x,t)dt$ is the probability, given $X(t)=x$, that the process will jump away from state x in the next infinitesimal time interval $[t,t+dt)$.

Proof. The result follows easily by combining Definition 1.14 and Proposition 1.16. $a(x,t)$ is smooth on t as $q(x,t;dt)$ is smooth on t. ☐

The next result formally proves the intuitive vision about jump Markov processes that was given at the beginning of section 1.3.

Proposition 1.19. *A jump Markov process $X(t)$ will either jump once or not jump at all in any infinitesimal time interval $[t, t+dt)$*

Proof. Take $a \in (0,1)$ and divide the infinitesimal time interval into two pieces $[t, t+a \cdot dt)$ and $[t+a \cdot dt, t+dt)$. The probability of the process jumping in those two subintervals is, by the multiplication law of probabilities:

$$a(x,t)dt \cdot a(x, t+a \cdot dt)(1-a)dt \propto (dt)^2$$

Since second order infinitesimals are negligible, the probability is 0. □

With these results, we can now deduce the form of the propagator density function $\Pi(\xi|dt, x, t)$ for a jump Markov process:

Theorem 1.20. *Given a jump Markov process $X(t)$, its propagator density function satisfies:*

$$\Pi(\xi|dt, x, t) = a(x,t)dt \cdot w(\xi|x,t) + [1 - a(x,t)dt]\delta(\xi) \tag{1.21}$$

Proof. Given $X(t) = x$, by time $t + dt$, with dt infinitesimal, the process either will have jumped once, with probability $a(x,t)dt$, or it will not have jumped at all, with probability $1 - a(x,t)dt$.
If a jump does occur, then the probability that the state change vector $X(t+dt) - X(t) \in [\xi, \xi + d\xi)$ will be $w(\xi|x,t')d\xi$, where $t' \in [t, t+dt)$ the instant where the jump occurred.
If a jump does not occur, then the probability that the state change vector $X(t+dt) - X(t) \in [\xi, \xi + d\xi)$ will be $\delta(\xi, 0)d\xi$, being $\delta(x,y)$ the Kronecker delta function. Therefore, by the propagator density function Definition 1.10 and the addition and multiplication laws of probability:

$$\Pi(\xi|dt, x, t)d\xi = [a(x,t)dt][w(\xi|x,t')d\xi] + [1 - a(x,t)dt][\delta(\xi)d\xi]$$

Now, since $w(\xi|x,t)$ is smooth on t, we can replace t' by t, infinitesimally close, and:

$$\Pi(\xi|dt, x, t)d\xi = a(x,t) \cdot dt \cdot w(\xi|x,t') + [1 - a(x,t)dt]\delta(\xi)$$

□

1.4 Jump Markov process with discrete states

It is preferable, for the chcharacterization of the present macroeconomic model, to dive deep into the study of Jump Markov Processes with discrete states. This section will be devoted to the finding of the forward and backward Chapman-Kolmogorov equations, with define the time evolution of a system described by a Markov process.

Definition 1.21. *A jump Markov process with discrete states is a jump Markov process $X(t)$ such that there exists a function $\phi : \Omega \longrightarrow \mathbb{N}$ such that $\phi(X(t))$ is bijective.*

This means that $X(t)$ takes values in a countable set. More intuitively, that $X(t)$ has only integer-valued states. In the following subsections, our definitions will be adapted for the discrete state case, and this will allow us ultimately to get to the time-evolution equations for a Markov process.
By simplicity, it will assumed that our Markov process takes only integer values, that is:

$$\forall t \in \mathbb{R}_+ \qquad X(t) = X_t : \Omega \longrightarrow \mathbb{N}$$

The following sections will be dedicated to adapt the theoretical concepts that characterize Markov processes for the particular case of jump Markov processes with discrete states.

1.4.1 The Chapman-Kolmogorov equation

First and foremost, note that, in this context, the hierarchy of state density functions take the following form

Definition 1.22. *Given a jump Markov process with discrete states $X(t)$, $P_k^{(j)}$ is defined as:*

$$P_k^{(j)}(n_{r_k}, t_{r_k}; ...; n_{r_1}, t_{r_1} | n_{s_j}, t_{s_j}; ...; n_{s_1}, t_{s_1})$$

$$\equiv \mathbb{P}(X(t_{r_z}) = n_{r_z} \;\; \forall z = 1, ..., k | X(t_{s_y}) = n_{s_y} \;\; \forall y = 1, ..., j) \tag{1.22}$$

And the Markov property:

Definition 1.23. *A stochastic process with integer valued states $X(t)$ is Markovian if, and only if, for all $k > 1$ and all $t_0 \leq t_1 \leq ... \leq t_k$:*

$$P_1^{(k)}(n_k, t_k | n_{k-1}, t_{k-1}; ...; n_0, t_0) = P(n_k, t_k | n_{k-1}, t_{k-1}) \tag{1.23}$$

Again, this equation limits the ability to predict the value of $X(t_k)$ given the value of $X(t_{k-1})$ cannot be enhanced by knowing the values of $X(t)$ at any times prior to t_{k-1}. The process has no memory of the past.
Again, because of Proposition 1.5, which has an identical proof for this discrete state case, it is worthy to define the Markov state density function as it was done in Proposition 1.4.

Definition 1.24. *For a jump Markov process with discrete states $X(t)$, it is Markov state density function is*

$$P(n,t|n_0,t_0) \equiv \mathbb{P}(X(t)=n|X(t_0)=n_0) \tag{1.24}$$

Following Proposition 1.6, the study of the properties of the Markov state density function lead to the Chapman-Kolmogorov equation:

Proposition 1.25. *$P(n,t|n_0,t_0)$ is a Markov state density function if, and only if:*

$$P(n,t|n_0,t_0) \geq 0 \qquad (t_0 \leq t) \tag{1.25}$$

$$P(n,t|n_0,t_0) \geq 0 \qquad (t_0 \leq t) \tag{1.26}$$

$$P(n,t|n_0,t_0) = \delta(n,n_0) \tag{1.27}$$

$$P(n_{i+1},t_{i+1}|n_{i-1},t_{i-1}) = \sum_{-\infty}^{\infty} P(n_{i+1},t_{i+1}|n_i,t_i)P(n_i,t_i|n_{i-1},t_{i-1}) \tag{1.28}$$

where $t_{i-1} \leq t_i \leq t_{+1}$ and $\delta(x,y)$ is the Kronecker delta function.

Proof. The first and second condition make sure the function is a PDF. The third condition assures the function verifies the definition of conditional probability. The last condition is the one that makes the process Markovian. Let us show this:

$$P(n_{i+1},t_{i+1}|n_{i-1},t_{i-1}) = \sum_{n_i=-\infty}^{\infty} P_2^{(1)}(n_{i+1},t_{i+1};n_i,t_i|n_{i-1},t_{i-1})$$

However, since $X(t)$ is Markovian:

$$P(n_{i+1},t_{i+1}|n_0,t_0) = \sum_{n_i=\infty}^{\infty} P(n_{i+1},t_{i+1}|n_i,t_i)P(n_i,t_i|n_{i-1},t_{i-1})$$

□

The last equation is the discrete state version of the Chapman-Kolmogorov equation. Finally, the compounded Chapman-Kolmogorov equation:

Proposition 1.26. *For $k \geq 2$, $t_0 \leq t_1 \leq ... \leq t_k$:*

$$P(n_k, t_k | n_0, t_0) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_{k-1}=-\infty}^{\infty} \prod_{i=1}^{k} P(n_i, t_i | n_{i-1}, t_{i-1}) \tag{1.29}$$

Proof. As in Proposition 1.7, the following lemma is useful.

Lemma 1.27.

$$P(n_i, n_k) = \sum_{n_j=-\infty}^{\infty} P(n_i, n_j | n_k) \tag{1.30}$$

Proof. By the definition of conditional probability

$$\mathbb{P}(n_i, n_j) = \mathbb{P}(n_i | n_k) \mathbb{P}(n_j | n_i, n_k)$$

Summing over all n_j, and imposing the normalization condition:

$$\sum_{n_j=-\infty}^{\infty} \mathbb{P}(n_i, n_j | n_k) = \mathbb{P}(n_i | n_k) \sum_{n_j=-\infty}^{\infty} \mathbb{P}(n_j | n_i, n_k) = \mathbb{P}(n_i, n_k)$$

□

This lemma can be used recursively:

$$P(n_k, t_k | n_0, t_0) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_{k-1}=-\infty}^{\infty} P_k^{(1)}(n_k, t_k; ...; n_1, t_1 | n_0, t_0)$$

Using equation 1.28, consequence of the Markov property

$$P(n_k, t_k | n_0, t_0) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_{k-1}=-\infty}^{\infty} \prod_{i=1}^{k} P(n_i, t_i | n_{i-1}, t_{i-1})$$

□

In the following section, the concept of the propagator and propagator density function will be adapted for the particular case of jump Markov processes with discrete states, continuing with the same logic.

1.4.2 The Propagator

Adapting the definitions of propagator and propagator density function:

Definition 1.28. *Let $X(t)$ be a jump Markov process with discrete state. It is propagator and propagator density function are defined such that, given $X(t) = n$:*

$$\Xi(dt; n, t) \equiv X(t + dt) - X(t) \qquad ; \qquad \Pi(\nu | dt; n, t) \equiv \mathbb{P}(\Xi(dt; n, t) = \nu)$$

The following results will prove, as we did in the general case, that the state density function P uniquely determines the propagator density function Π and vice versa.

Proposition 1.29. *Given $X(t)$ a jump Markov process with discrete states, then:*

$$\Pi(\nu|dt;n,t) = P(n+\nu, t+dt|n,t) \tag{1.31}$$

Proof.

$$\begin{aligned}\Pi(\nu|dt;n,t) &= \mathbb{P}(\Xi(dt;n,t) = \nu|X(t) = n) = \mathbb{P}(X(t+dt) = n+\nu|X(t) = n)\\ &= P(n+\nu, t+dt|n,t)\end{aligned}$$

□

Proposition 1.30. *Given $X(t)$ a jump Markov process with discrete states, then:*

$$P(n,t|n_0,t_0) = \sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_{k-1}=-\infty}^{\infty} \prod_{i=1}^{k} \Pi(\nu_i|dt;n_{i-1},t_{i-1}) \tag{1.32}$$

Proof. Consider the compounded Chapman-Kolmogorov equation (1.29) with $n_k = n$ and $t_k = t$. Let the points $t_1 < t_2 < ... < t_{k-1}$ divide the interval $[t_0, t]$ into k subintervals of equal length $(t-t_0)/k$. Change the summation variables according to:

$$n_i \to \nu_i \equiv n_i - n_{i-1} \qquad i = 1, ..., k-1$$

And define $\nu_k \equiv n - n_{k-1}$. Then, the compounded Chapman-Kolmogorov equation becomes:

$$P(n,t|n_0,t_0) = \sum_{\nu_1=-\infty}^{\infty} \cdots \sum_{\nu_{k-1}=-\infty}^{\infty} \prod_{i=1}^{k} P(n_{i-1}+\nu_i, t_{i-1}+(t-t_0)/k|n_{i-1},t_{i-1})$$

where

$$t_i = t_{i-1} + (t-t_0)/k \qquad i = 1, ..., k-1$$

$$n_i = n_0 + \nu_1 + ... + \nu_i \qquad i = 1, ..., k-1$$

$$\nu_k \equiv n - n_0 - \nu_1 - ... - \nu_{k-1}$$

Let k so large that $(t-t_0)/k = dt$ and infinitesimal. Then the P-factors on the right-hand side of the preceding equation for $P(n,t|n_0,t_0)$ become, because of Proposition 1.29, Π-factors. Finally, using the previous relationships, the final result is obtained. □

Therefore, it is equivalent to provide the Markov state density function or the propagator.

Now that the basic concepts are adapted for discrete state Markov processes, it is time to work on the particular form of the propagator density function for a jump Markov process with discrete states.

We shall start with some definitions. Note that, as $X(t)$ only takes integer values, the only way for the process to change with time is to make instantaneous jumps from one integer to another. Thus, it can be defined:

Definition 1.31. *Let $X(t)$ be a jump Markov process with discrete states. Then:*

$$q(n,t;\tau) \equiv \mathbb{P}(\exists t' \in (t, t+\tau) : X(t') \neq n | X(t) = n) \tag{1.33}$$

$$w(\nu|n,t) \equiv \mathbb{P}(X(t+dt) = n+\nu | X(t) = n) \tag{1.34}$$

Intuitively, $q(n,t;\tau)$ is the probability, given $X(t) = n$, that the process will jump away from state n at some instant between t and $t+\tau$. $w(\nu|n,t)$ is the probability that the process, upon jumping away from state n at time t, will land in state $n+\nu$.

Following Definition 1.15, a formal definition of a discrete state jump Markov process can be provided:

Definition 1.32. *A discrete state jump Markov process is any integer stare process $X(t)$ for with functions q and w exist and have the following properties*

- *$q(n,t;\tau)$ is a smooth function of t and τ, and satisfies $q(n,t;0) = 0$*
- *$w(\nu|n,t)$ is a smooth function of t.*

As in the previous section, the expected properties for these functions can be enounced.

Proposition 1.33. *Let $X(t)$ be a jump Markov process with discrete states. Then:*

$$w(\nu|n,t) \geq 0 \qquad \sum_{\nu=-\infty}^{\infty} w(\nu|n,t) = 1$$

$$0 \leq q(n,t;dt) \leq 1 \qquad q(n,t;dt) = a(n,t)dt$$

with $a(n,t)$ a nonnegative, smooth function of t.

Proof. Analogous to the proof of Proposition 1.16 □

To give some intuition, this result states that $a(n,t)dt$ is the probability, given $X(t) = n$, that the process will jump away from state n in the next infinitesimal time interval $[t, t + dt)$.
With this information, the equivalent result for Proposition 1.19 can be proved:

Proposition 1.34. *A jump Markov process with discrete states $X(t)$ will either jump once or else not at all in the infinitesimal time interval $[t, t + dt)$.*

Proof. Analogous to the proof of 1.19 □

With these concepts, we are in position to deduce an explicit formula for the propagator density function $\Pi(\nu|dt;n,t)$ in terms of the two functions $a(n,t)$ and $w(\nu|n,t)$.

Theorem 1.35. *Let $X(t)$ be a jump Markov process with discrete states. Then:*

$$\Pi(\nu|dt;n,t) = a(n,t)dtw(\nu|n,t) + [1 - a(n,t)dt]\delta(\nu,0) \tag{1.35}$$

Proof. Given $X(t) = n$, then by time $t + dt$ the system either will have jumped once, with probability $a(n,t)dt$ or it will not have jumped at all, with probability $1 - a(n,t)dt$.
If a jump does happen, by the definition of $w(\nu|n,t)$, the probability that the state change vector $X(t + dt) - n$ will be equal to ν will be $w(\nu|n,t')$ where $t' \in [t, t + dt)$ is the precise instant when the jump occurred.
If a jump does not occur, then the probability that the state change vector $X(t + dt) - n$ will be equal ν will be $\delta(\nu,0)$, since it is equal to unity if $\nu = 0$ and zero if $\nu \neq 0$.
Finally, because of the definition of the propagator density function, Definition 1.28, and the multiplication and addition laws of probability, it is obtained:

$$\Pi(\nu|dt;n,t) = [a(n,t)dt]w(\nu|n,t') + [1 - a(n,t)dt]\delta(\nu,0)$$

Finally, since $t' \in [t, t+dt)$, then the smooth dependence of $w(\nu|n,t)$ on t assumed in Definition 1.32 means t' can be replaced by any infinitesimally close value t while maintaining the relationship. The desired result is obtained.

$$\Pi(\nu|dt;n,t) = a(n,t)dtw(\nu|n,t) + [1 - a(n,t)dt]\delta(\nu,0)$$

□

Note that, because of Theorem 1.35, the propagator density function $\Pi(\nu|dt;n,t)$ is completely determined by the two functions $a(n,t)$ and $w(\nu|n,t)$. Therefore, the following definition makes sense.

Definition 1.36. *Take as given a discrete state Markov process $X(t)$. Then, it is defined:*

- *Functions $a(n,t)$ and $w(\nu|n,t)$ as defined in Proposition 1.33 and Definition 1.32, respectively, are called the characterizing functions of the process $X(t)$.*
- *$W(\nu|n,t) \equiv a(n,t)w(\nu|n,t)$ the consolidated characterizing function of the process $X(t)$.*

In this context, note that $W(\nu|n,t)dt$ is the probability, given $X(t) = n$ that the process will in the time interval $[t, t+dt)$ jump from state n to state $n+\nu$.
The following proposition justifies the definition of $W(\nu|n,t)$ as the consolidated characterizing function and will be useful for a later result:

Proposition 1.37. *Consider a discrete state Markov process $X(t)$. Then:*

$$a(n,t) = \sum_{\nu=-\infty}^{\infty} W(\nu|n,t) \tag{1.36}$$

$$w(\nu|n,t) = \frac{W(\nu|n,t)}{\sum_{\nu'=-\infty}^{\infty} W(\nu'|n,t)} \tag{1.37}$$

Proof. To obtain the relationship for $a(n,t)$, start by Definition 1.36 and sum over all ν.

$$\sum_{\nu=-\infty}^{\infty} W(\nu|n,t) = \sum_{\nu=-\infty}^{\infty} a(n,t)w(\nu|n,t) = a(n,t)\sum_{\nu=-\infty}^{\infty} w(\nu|n,t) = a(n,t)$$

Since $\sum_{\nu=-\infty}^{\infty} w(\nu|n,t) = 1$ by Proposition 1.33. Now, substituting back this result into Definition 1.36:

$$W(\nu|n,t) = \sum_{\nu'=-\infty}^{\infty} W(\nu'|n,t)w(\nu|n,t) \Rightarrow w(\nu|n,t) = \frac{W(\nu|n,t)}{\sum_{\nu'=-\infty}^{\infty} W(\nu'|n,t)}$$

□

With this result, it can be proved that the propagator density function can be expressed solely in terms of the consolidated characterizing function:

Proposition 1.38. *Let $X(t)$ be a jump Markov process with discrete states. Then, its propagator density function takes the following form:*

$$\Pi(\nu|dt;n,t) = W(\nu|n,t)dt + \left(1 - \sum_{\nu'=-\infty}^{\infty} W(\nu'|n,t)dt\right)\delta(\nu,0) \tag{1.38}$$

Proof. It is immediate by departing from Theorem 1.35 and using Proposition 1.37 □

To this point, we have presented the basic pillars of Markov processes, and particularized our explanations for jump Markov processes with discrete states, which is the analytical framework our model will use. One last step is missing, and that is the description of the time-evolution of the Markov state density function. In the following (sub)section, it will be proven that the time evolution of a jump Markov process with discrete states will be completely determined by the forward and backward master equations, two differential equations.

1.4.3 The Master equations

As stated, this section is dedicated to the time-evolution of a jump Markov process with discrete states. Throughout this thesis, it has been repeatedly stated that Markov processes are fully characterized by a Markov state density function $P(n,t|n_0,t_0)$ or, equivalently, by the propagator density function $\Pi(\nu|dt;n,t)$. Thus, studying the time-evolution of a Markov process means studying the time evolution of the Markov state density function.

This section is dedicated to providing differential equations that characterize the evolution of $P(n,t|n_0,t_0)$. This will be the so-called Master equations: the backwards master equation provides the time evolution of a (jump) Markov process (with discrete states) for changes in the initial time t_0. On the other hand, the forward master equation provides the evolution of the Markov process with a fixed initial condition.

Theorem 1.39. ***(Forward master equations)*** *Let $X(t)$ be a jump Markov process with discrete states. Then:*

$$\frac{\partial}{\partial t}P(n,t|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} [a(n-\nu,t)w(\nu|n-\nu,t)P(n-\nu,t|n_0,t_0)] - a(n,t)P(n,t|n_0,t_0) \tag{1.39}$$

$$\frac{\partial}{\partial t}P(n,t|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} W(\nu|n-\nu,t)P(n-\nu,t|n_0,t_0) - P(n,t|n_0,t_0)\sum_{\nu=-\infty}^{\infty} W(-\nu|n,t) \tag{1.40}$$

Proof. Start with the Chapman-Kolmogorov equation, Proposition 1.25, but written in the following form:

$$P(n,t+dt|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} P(n,t+dt|n-\nu,t)P(n-\nu,t|n_0,t_0)$$

Now, using the equivalence between the Markov state density function and the propagator density function, Proposition 1.29, and the formula for the propagator density function for a jump Markov process with discrete states, Theorem 1.35:

$$P(n,t+dt|n-\nu,t) = P(n-\nu+\nu,t+dt|n-\nu,t) = \Pi(\nu|dt;n,t)$$

$$= a(n-\nu,t)dtw(\nu|n-\nu,t) + [1-a(n-\nu,t)dt]\delta(\nu,0)$$

Substituting this last expression into the Chapman-Kolmogorov equation:

$$P(n,t+dt|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} [a(n-\nu,t)dtw(\nu|n-\nu,t)]\, P(n-\nu,t|n_0,t_0)$$

$$+\sum_{\nu=-\infty}^{\infty} ([1-a(n-\nu,t)dt]\delta(\nu,0))\, P(n-\nu,t|n_0,t_0)$$

Now, carrying out the summation using the Kronecker delta function:

$$P(n,t+dt|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} \Big(a(n-\nu,t)dtw(\nu|n-\nu,t)\Big)P(n-\nu,t|n_0,t_0)$$

$$+[1-a(n,t)dt]P(n,t|n_0,t_0)$$

Now, subtracting $P(n,t|n_0,t_0)$ from both sides of this last expression and dividing by dt, and taking the limit $dt \to 0$, the definition of derivative is obtained:

$$\frac{\partial}{\partial t}P(n,t|n_0,t_0) = \lim_{dt\to 0}\frac{P(n,t+dt|n_0,t_0) - P(n,t|n_0,t_0)}{dt}$$

$$= \lim_{dt\to 0}\sum_{\nu=-\infty}^{\infty}\Big(a(n-\nu,t)w(\nu|n-\nu,t)\Big)P(n-\nu,t|n_0,t_0) + [1-a(n,t)dt]P(n,t|n_0,t_0)$$

$$\Rightarrow \frac{\partial}{\partial t}P(n,t|n_0,t_0) = \sum_{\nu=-\infty}^{\infty}[a(n-\nu,t)w(\nu|n-\nu,t)P(n-\nu,t|n_0,t_0)] - a(n,t)P(n,t|n_0,t_0)$$

This is the first form of the forward master equation for a jump Markov process $X(t)$ defined by the characterizing functions $a(n,t)$ and $w(\nu|n,t)$. Let us deduce the forward equation in terms of the consolidated characterizing function $W(\nu|n,t)$. Departing from the forward equation in terms of a, w, multiply the second term on the right by unity in the form of $\sum_\nu w(-\nu|n,t)$, which is given by Proposition 1.33 with the summation variable change $\nu \to -\nu$:

$$\frac{\partial}{\partial t}P(n,t|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} [a(n-\nu,t)w(\nu|n-\nu,t)P(n-\nu,t|n_0,t_0)]$$

$$-a(n,t)\sum_{\nu=-\infty}^{\infty} w(-\nu|n,t)P(n,t|n_0,t_0)$$

$$\Rightarrow \frac{\partial}{\partial t}P(n,t|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} [W(\nu|n-\nu,t)P(n-\nu,t|n_0,t_0) - W(-\nu|n,t)P(n,t|n_0,t_0)]$$

In this last step, the definition of the consolidated characterizing function, Definition 1.36 is used. □

See that clearly these equations describe the time evolution of the Markov state density function (which uniquely characterizes the Markov process) as it is a differential equation for $P(n,t|n_0,t_0)$ for fixed n_0 and t_0. They are solved subject to the initial condition $P(n,t=t_0) = \delta(n,n_0)$.
Now, the backward master equations is presented. However, they will not be discussed in such detail since they will not be used in our model. They are presented for completeness.

Theorem 1.40. ***(Backward master equations)*** *Let $X(t)$ be a jump Markov process with discrete states. Then:*

$$-\frac{\partial}{\partial t_0}P(n,t|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} [a(n_0,t_0)w(\nu|n_0,t_0)P(n,t|n_0+\nu,t_0)] - a(n_0,t_0)P(n,t|n_0,t_0) \tag{1.41}$$

$$-\frac{\partial}{\partial t_0}P(n,t|n_0,t_0) = \sum_{\nu=-\infty}^{\infty} W(\nu|n_0,t_0)\,[P(n,t|n_0,t_0) - P(n,t|n_0,t_0)] \tag{1.42}$$

Proof. Analogous to Theorem 1.39. □

This chapter provided the essential mathematical modeling tools needed to understand the formalisms of the model in chapter 3. Some other additional explanations were added into the Annex. The following section will recompile all the economic concepts and intuitions necessary for the comprehension of the model.

Chapter 2

Economic foundations

This chapter provides the proper economic foundations for understanding the model. The objective is not so much as to give a quantitative analysis such as in Chapter 1, but to grasp key economic ideas for modeling the macroeconomy. Economics has traditionally been composed by two disciplines: microeconomics and macroeconomics. As Mas Colell defines [12], Microeconomic Theory aims to model economic activity as an interaction of individual agents pursuing their private interests. Thus, microeconomics studies the agent's behavior and extracts economic relations from that study. According to Wickens [14], Macroeconomic Theory seeks to explain the behavior of the aggregate economy using theories based on strong microeconomic foundations. This is, however, a more modern definition. A more traditional Keynesian approach is based on ad hoc theorizing about relations between aggregate economic variables, without being based on Microeconomic Theory. Following this logic, section 2.1 provides basic notions to understand classical microeconomic analysis, while section 2.2 does so with classical macroeconomic modeling. Finally, section 2.3 provides the basic econometric tools to conduct empirical research.

2.1 Microeconomic foundations

The objective of our study is the macroeconomy and, motivated by Wickens definition of modern macroeconomics, our macroeconomic model will be based on strong micro-foundations. However, our microeconomic model will depart from the traditional modeling techniques. In this section, we will briefly analyze a classical Microeconomic model. The objective is to grasp the basics of the modeling techniques used, to inspire our own model.

2.1.1 The Consumer Choice Model

This model analyses consumption choices a rational agent would make.

Definition 2.1. *The consumer choice set is $S \subset \mathbb{R}^n_+$ containing the physical feasible combinations of goods an economic agent can make.*

This set describes all the available possibilities of consumption. However, not all combinations are economically feasible by the consumer. This motivates the following definition:

Definition 2.2. *The consumer budget set is the collection of all baskets affordable by the consumer. Let $(p_1, ..., p_n) \in \mathbb{R}^n_+$ be the vector of prices, and $m \in \mathbb{R}_+$ the consumer's available income. The consumer budget set is:*

$$\{(x_1, ..., x_n) \in S|\ p_1 x_1 + ... + p_n x_n \leq m\} \tag{2.1}$$

The consumer also has some preferences regarding the different consumption bundle. As proven in [12], under certain assumptions, these preferences can be represented by a utility function.

Definition 2.3. *Let a consumer be characterized by having a preference relation in the consumption set, where $x \succ y$ means that bundle $x \in S$ is strictly preferred to $y \in S$. Under this situation, a utility function for this consumer is a function*

$$U : S \subset \mathbb{R}^n_+ \longrightarrow \mathbb{R}$$

Such that $\forall x, y \in S : x \succ y \Leftrightarrow u(x) > u(y)$.

The question to analyze is what will the preferred and affordable consumption bundle chose by the consumer. Thus, the consumer choice problem is the following:

$$\begin{aligned} \max_{x_1, x_2} \quad & U(x_1, x_2) \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 = m \\ & (x_1, x_2) \in S \end{aligned} \tag{2.2}$$

To solve it, the Lagrangian is used (more details in [12]).

$$\mathcal{L} = U(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m) \tag{2.3}$$

The First Order Conditions (FOC) are:

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow \frac{\partial U(x_1, x_2)}{\partial x_1} - \lambda p_1 = 0 \tag{2.4}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow \frac{\partial U(x_1, x_2)}{\partial x_2} - \lambda p_2 = 0 \tag{2.5}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow p_1 x_1 + p_2 x_2 - m = 0 \tag{2.6}$$

Combining the first two equations:

$$\frac{\frac{\partial U(x_1,x_2)}{\partial x_1}}{\frac{\partial U(x_1,x_2)}{\partial x_2}} = -\frac{p_1}{p_2} \tag{2.7}$$

And combining this equation with the third FOC, the individual demand functions are obtained $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$. Note that this demand functions depend on the prices and income of the consumer.

Notice that, up to this point, the model can be adapted to different individuals, assuming each one has a different utility function. However, yet with this general framework we already assumed that agents are rational: This is, its preferences are complete, transitive and continuous so that a utility function can exist (see [12]). And maybe sometimes an agent's choices are not rational, or even stochastic. However, usually a Cobb-Douglas utility function is assumed:

$$U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha} \tag{2.8}$$

And solving the maximization problem, the individual demand functions are:

$$x_1^* = \frac{m\alpha}{p_1} \quad ; \quad x_2^* = \frac{m(1-\alpha)}{p_2} \tag{2.9}$$

This equilibrium solution can be represented by a point in $\mathbb{R}^2_+$.

Usually, these results are taken and aggregated to construct the aggregate demand for each good.

The idea we want to stress with this model is that mainstream Microeconomics (and thus, the grounds for macroeconomic modeling) is based on the aggregation of behaviors of a series of rational representative agents. Taking this approach, which ignores microeconomic fluctuations, only needs of differential calculus to model, and the outcome of the maximization problem is a "point" in some set or space, as seen in Figure 2.1.

This is not the approach we want to follow to design our foundations for macroeconomic modeling. Some economists try introducing small variations within the framework of the representative agent. For example, Caselli and Ventura [4] make the assumption that consumers differ only in their preferences for public goods to accommodate such "heterogeneity". However, this is not the type of heterogeneity

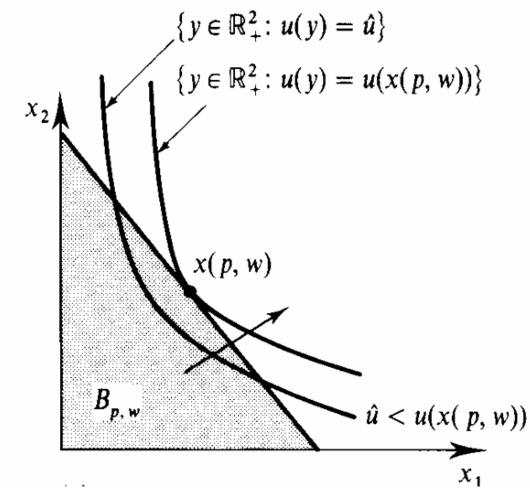


Figure 2.1: Consumer choice model outcome, taken from [12]

we have in mind: It is very limited and artificial. In our microeconomic model, it is assumed that economic agents differ in every aspect, and their behavior is subject to stochastic fluctuations. It is impossible and pointless to model it if our object is to model the macroeconomy. Thus, our microeconomic foundations will be grounded on a stochastic model, a binary choice model.

2.2 Macroeconomic foundations

Originally, Economics as a science was born with the study of the microeconomy. Macroeconomic modeling as we understand it was not born until the General Theory, published by Keynes in 1936. This book was transcendental since it offered a new (and macro) approach to addressing Macroeconomic problems, and it successfully managed to explain the causes of the Great Depression, as well as provided an effective solution (see [11]). It managed to overcome the limitations of the microeconomic approach of the Neoclassical School.

Keynes took the entrepreneur as the starting point, and stated that they demanded employers based on their expected demand for their products. Thus, aggregate demand, determined the employment level. This, in turn, determined the GDP of the economy. His theory was extensively analyzed, and post-Keynesian authors developed Keynes' ideas until developing the IS-LM model, which became the main macroeconomic tool after the Second World War and until the Oil crisis. In the construction of our model, the IS-LM conclusions will be incorporated, so it is important to provide a review.

2.2.1 The IS-LM model

The IS-LM model is a short-run macroeconomic model. It is composed by two curves that represent two markets. To have a more detailed view, please refer to [5].

The model

Definition 2.4. *The IS (Investments-Savings) curve represents the combinations of interest rates and GDP levels such that the goods market is in equilibrium.*

Assuming a closed economy, supply in the goods market is given by the production of the economy, namely Y. Demand is given by private consumption (C), investment (I), and public demand for expenditure (G). Thus, in equilibrium, markets clear and:

$$Y = C + I + G \tag{2.10}$$

For simplicity, let us assume that investment is only sensible to interest rates with a linear dependency:

$$I = \bar{I} - bi \tag{2.11}$$

where $b > 0$ since logically, less interest rate favors investment. Assuming that agents in the economy consume a constant proportion of their net income:

$$C(Y) = \bar{C} + c(Y - tY + Tr) \tag{2.12}$$

where $T = tY$ refers to taxes and Tr to transfers, which we take as exogenous. Public consumption is also assumed to be exogenous. Operating, it is easy to find the IS curve:

$$i(Y) = \frac{\bar{A}}{b} - \frac{1}{b\alpha}Y \tag{2.13}$$

where $\alpha := 1/(1 - c + ct)$.

Note how the micro-foundations of this macroeconomic model are based on the representative agent theory. In particular, in this model, all agents are assumed to consume a constant proportion of their income (see equation 2.12). This approach will be criticized throughout this work, and it will be seen how our more sophisticated model can get to much richer conclusions than the models microfounded with the representative agent. The other market that is considered in this model is the money market, and it is represented by the Liquidity-Money curve (LM curve).

Definition 2.5. *The LM curve is the combination of interest rates and GDP levels such that the money market clears.*

In the monetary market, the supply is given by the real money supply:

$$M^s = \frac{\bar{M}}{P} \tag{2.14}$$

where $\bar{M}$ is the amount of money issued by the Monetary Authority (exogenously given) and P is the price level, which in this short term model is taken as given. The demand of money is defined following the 3 functions of money proposed by Keynes (see [11]):

$$L^d = L(Y, i) = KY - hi \tag{2.15}$$

with $K > 0$ and $h > 0$. Imposing market clearing conditions, the LM curve is:

$$i(Y) = \frac{K}{h}Y - \frac{1}{h}\frac{\bar{M}}{P} \tag{2.16}$$

The equilibrium in this model is the interest rate and output level such that both the goods and money market are in equilibrium. Since the IS curve is continuous and decreasing and the LM curve is continuous and increasing, then there exists en equilibrium point, which is provided by:

$$Y_E = \frac{h\alpha}{h + kb\alpha}\bar{A} + \frac{b\alpha}{h + kb\alpha}\frac{\bar{M}}{P} \tag{2.17}$$

Graphically, this equilibrium can be presented by the following manner:

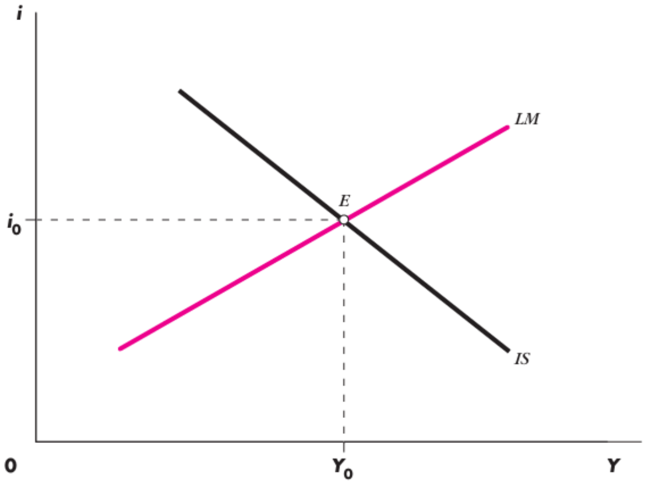


Figure 2.2: Equilibrium in the IS-LM model, extracted from [5]

with $\bar{A}$ an aggregation of constant parameters of the IS and LM curves.

Policy in the IS-LM model

This model features two authorities: Fiscal and Monetary authorities. Both agencies can act over different inputs of the IS and LM function in order to modify GDP. The most common objective of these policies is to respond to crisis, this is, sudden drops of the GDP level. This section shows a brief introduction to the different policies available.

The central bank can directly decide the amount of money supply of the economy. This modifies the LM curve, and ultimately displaces the equilibrium level of output.

If the Monetary Authority considers necessary to raise GDP, raising the monetary supply ($\bar{M}$) will shift the LM curve to the right (see equation 2.16), and this will increase the equilibrium level of output (see Figure 2.3). Analogously, decreasing money supply reduces GDP.

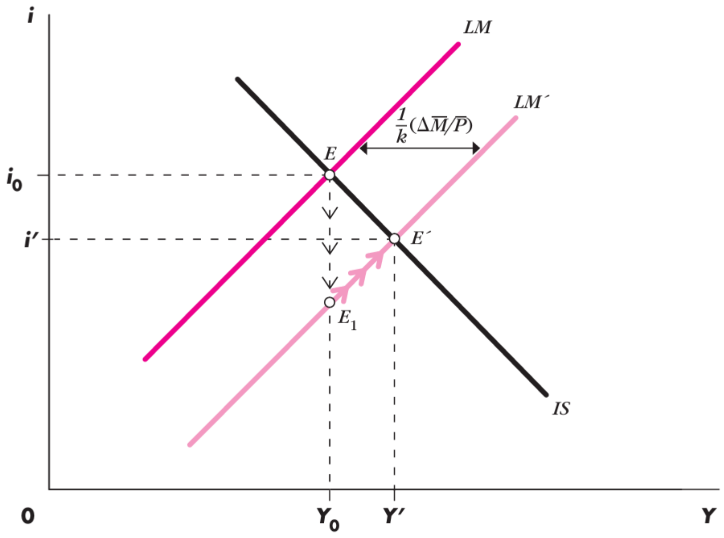


Figure 2.3: Monetary policy in the IS-LM model, extracted from [5]

On the other side, fiscal authorities can modify equilibrium GDP by changing their level of expenditure, taxes, and transfers. This is, because of equation 2.12, they can act on the IS curve, via $\bar{A}$ to modify the GDP level, with the same reasoning as the monetary authorities.

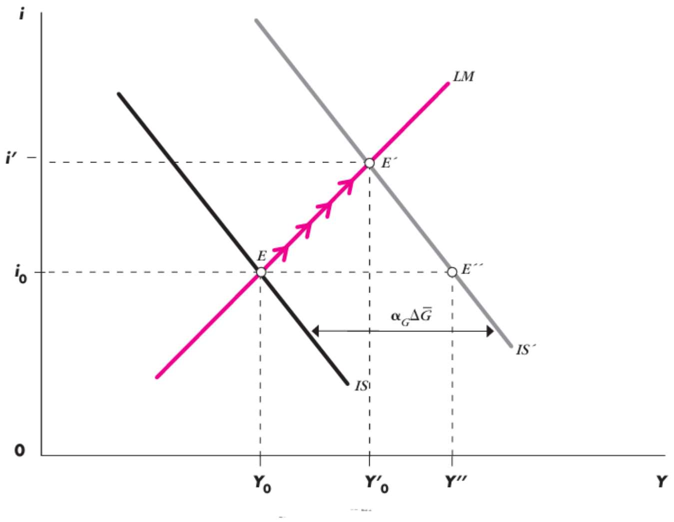


Figure 2.4: Fiscal policy in the IS-LM model, extracted from [5]

This macroeconomic model, although very basic, is very illustrative of a great proportion of the macroeconomic models available. They all are poorly micro-founded, based on the assumption that all agents in the economy are rational and behave as the representative agent. In this framework, only Calculus is needed, and the equilibrium in the economy is given by a "point" in a diagram, with a deterministic level of GDP associated. However, this is not what it is observed in our day to day: GDP fluctuates, and agents make irrational decisions, and have heterogeneous behaviors.

2.3 Econometric foundations

Econometrics is the branch of economics that applies statistical and mathematical methods to analyze economic data. Its main goal is to give empirical content to economic theories and to test hypotheses or forecast future trends. We will use econometrics in this study to test if the real economy is subject to the main conclusions of our model. In this section, a basis of the Econometrics practice is provided in order to understand Chapter 4. This explanations are based on [15].
In our analysis, Econometrics is based on the Multiple Linear Regression Model (MLRM), which is studied in the following section.

2.3.1 The Multiple Linear Regression Model

The objective of the model is to study the dependance of the dependent variable y in terms of $k \in \mathbb{N}$ explanatory variables. It is grounded on the following assumptions.

Assumptions about the model

Conjecture 2.6. *The MLRM is stochastic.*

This assumption makes the model more realistic, since typically a non-countable number of shocks can affect the dependent variable y.

Conjecture 2.7. *The MLRM is linear (in parameters) or can be linearized.*

Conjecture 2.8. *There is enough statistical information, this is, $T \geq k$, with $T \in \mathbb{N}$ the number of observations.*

With these assumptions, it is possible to obtain a functional dependence for y.

$$y = f(x_1, x_2, x_3, ...x_k) \Rightarrow y = \beta_1 + \beta_2 x_2 + ... + \beta_k x_k + u_i$$

Since our empirical study will be based in a time series dataset:

$$y_t = \beta_1 + \beta_2 x_{2,t} + ... + \beta_k x_{k,t} + u_t \tag{2.18}$$

Without loss of generality, we can assume $x_1 = 1 \forall i = 1, ...T$. This provides the model with a constant term, which makes computations easier.
It is useful to formulate the model using a matrix notation:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} 1 & x_{21} & x_{31} & \cdots & x_{k1} \\ 1 & x_{22} & x_{32} & \cdots & x_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2T} & x_{3T} & \cdots & x_{kT} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}$$

$$\Rightarrow Y_{T\times 1} = X_{T\times k} \beta_{k\times 1} + U_{T\times 1} \tag{2.19}$$

It is important to restrict the behavior of the disturbance term u_i, which is now any random variable.

Assumptions about the error term

Conjecture 2.9. *The expected value of the error term is equal to 0.*

Mathematically, this translates to:

$$\mathbb{E}(u_i) = 0 \quad \forall i \Rightarrow \mathbb{E}(U) = 0_{N\times 1} \Rightarrow \mathbb{E}(U) = \begin{bmatrix} \mathbb{E}(u_1) \\ \mathbb{E}(u_2) \\ \vdots \\ \mathbb{E}(u_N) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Conjecture 2.10. *(Homoskedasticity assumption) The variance of the error term is constant.*

Mathematically,

$$\text{VAR}(u_i) = \sigma_u^2 \qquad \forall i = 1, ..., T \tag{2.20}$$

Conjecture 2.11. *The error terms are uncorrelated.*

$$COV(u_i, u_j) := E\left[(u_i - E(u_i))(u_j - E(u_j))\right] = E[u_i u_j] = 0 \quad \forall i, j = 1, ..., T \quad i \neq j \tag{2.21}$$

Conjecture 2.12. *The error term is normally distributed*

Conjectures 2.9, 2.10, 2.11 and 2.12 imply that

$$u_i \sim N(0, \sigma_u^2) \tag{2.22}$$

Assumptions about the explanatory variables

Conjecture 2.13. *(Exogeneity assumption) Explanatory variables are uncorrelated with the error term.*

$$E(x_{j,i} u_i) = 0 \quad \forall i = 1, ..., T \quad \forall j = 1, ..., k$$

This assumption means that there are no relevant omitted explanatory variables.

Conjecture 2.14. *(Lack of perfect multicollinearity). Columns of matrix $X_{k\times T}$ are independent, and it has rank k.*

Conjecture 2.15. *No measurement errors in the explanatory variables.*

Assumptions about the parameters

Conjecture 2.16. *Parameters (β_j) are constant across sample observations.*

These assumptions are essential for the model to have certain properties that will guarantee that our estimations can reflect reality. In the following section, the Ordinary Least Square Estimation (OLS) method is presented, which provides values for parameters, $\hat{\beta}_j$, given a sample of observations $X_{k\times T}$, $Y_{T\times 1}$.

2.3.2 Ordinary Least Square Estimation

As stated, the OLS estimation departs from a sample, and derives values for parameters according to some criteria. In this case, it chooses the parameters that minimize the sum of squared residuals.

Definition 2.17. *Consider an MLRM model with the corresponding assumptions. The Sum of Squared Residuals (SSR) is:*

$$SSR := \sum_{i}^{T}(y_i - \hat{y}_i)^2 \tag{2.23}$$

with $\hat{y}_i := \hat{\beta}_1 + \hat{\beta}_2 x_{2,i} + ... + \hat{\beta}_k x_{k,i}$

Theorem 2.18. *Consider a MLRM model with the usual assumptions. Then, OLS estimation provides:*

$$\hat{\beta}_{OLS} = (X'X)^{-1}(X'Y) \tag{2.24}$$

Proof.

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} SSR = (Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta})$$

The first order condition:

$$\frac{\partial SSR}{\partial \hat{\beta}} = \frac{\partial\left(Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}\right)}{\partial \hat{\beta}} = -2X'Y + 2X'X\hat{\beta} = 0$$

$$\Rightarrow -2X'Y + 2X'X\hat{\beta} = 0 \Rightarrow \hat{\beta}_{OLS} = (X'X)^{-1}(X'Y)$$

The second order condition:

$$\frac{\partial^2 SSR}{\partial^2 \hat{\beta}} = \frac{\partial^2 e'e}{\partial^2 \hat{\beta}} = \frac{\partial^2\left(Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}\right)}{\partial^2 \hat{\beta}}$$

$$= \frac{\partial\left(-2X'Y + 2X'X\hat{\beta}\right)}{\partial \hat{\beta}} = 2X'X$$

□

2.3.3 Statistical properties of the OLS estimator

The properties shown in this section illustrate why the OLS was the chosen method of estimation.

Proposition 2.19. *Consider a MLRM with the usual assumptions. Then, the OLS estimator has the following properties.*

- ***Linearity****: The OLS estimator is a linear function of the population parameters β, explanatory variables and the error term.*

$$\hat{\beta} = (X'X)^{-1}X'Y = (X'X)^{-1}X'(X\beta + U) \tag{2.25}$$

$$= (X'X)^{-1}X'X\beta + (X'X)^{-1}X'U = \beta + (X'X)^{-1}X'U$$

- ***Unbiasedness****: The OLS estimator is an unbiased estimator of the population parameters.*

$$\begin{aligned} \mathbb{E}(\hat{\beta}) &= \mathbb{E}\left((X'X)^{-1}X'Y\right) \\ &= \mathbb{E}\left(\beta + (X'X)^{-1}X'U\right) \\ &= \beta + (X'X)^{-1}X'\mathbb{E}(U) \\ &= \beta \end{aligned} \tag{2.26}$$

- ***Efficiency****: The OLS estimator is the linear unbiased estimator with the smallest variance. This is The Gauss-Markov Theorem, see [15] for details.*

$$\begin{aligned} Var(\hat{\beta}_{OLS}) &= \mathbb{E}\left[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'\right] \\ &= \mathbb{E}\left[((X'X)^{-1}X'U)((X'X)^{-1}X'U)'\right] \\ &= (X'X)^{-1}X'\,\mathbb{E}(UU')\,X(X'X)^{-1} \\ &= (X'X)^{-1}X'\sigma_u^2 I_N X(X'X)^{-1} \\ &= \sigma_u^2(X'X)^{-1}X'X(X'X)^{-1} \\ &= \sigma_u^2(X'X)^{-1} \end{aligned} \tag{2.27}$$

- ***Consistency****: An estimator is consistent if, as N tends to infinity, the estimator tends to the population parameter. In this case, since $\hat{\beta}_{OLS}$, consistency requires that the variance tends to 0:*

$$\lim_{N\to\infty} Var(\hat{\beta}) = \lim_{N\to\infty} \sigma_u^2(X'X)^{-1} = \lim_{N\to\infty} \frac{\sigma_u^2}{N}\left(\frac{X'X}{N}\right)^{-1} = 0 \tag{2.28}$$

Proof. Written above. □

2.3.4 Interpretation of results

This section proposes certain statistical indicators that aim to asses how well do the explanatory variables explain the dependent variable.

Definition 2.20. *Consider an MLRM model with the usual assumptions. Then, the Total Sum of Squares (SST), the Explained Sum of Squares (SSE) are defined:*

$$SST = \sum_{i=1}^{N} (y_i - \bar{y})^2 = Y'Y - N\bar{Y}^2 \tag{2.29}$$

$$SSE = \sum_{i=1}^{N} (\hat{y}_i - \bar{y})^2 = \hat{Y}'\hat{Y} - N\bar{Y}^2 \tag{2.30}$$

SST is a measure of the total sample variation in the y_i. On the other hand, SSE is a measure of the sample variation in the fitted values; part of variation in y explained by the model.

Definition 2.21. *Consider a MLRM model with the usual assumptions. The R-squared/Coefficient of determination is defined as:*

$$R^2 := \frac{SSE}{SST} = 1 - \frac{SSR}{SST} \tag{2.31}$$

Since in the model with an intercept, $SST = SSE + SSR$. In this framework, $0 \leq R^2 \leq 1$ is the ratio of the explained variation compared to the total variation. Thus, the coefficient of determination is a good measure of the goodness of fit of the model.

Another important aspect of analysis is the individual and joint significance of parameters, which help us to objectively test if certain explanatory variables are relevant or not, as well as the joint model, in explaining variable y.

1. Individual Significance Test

We use the t-test to determine whether an individual regressor x_j is statistically significant.

Hypotheses:

$$H_0 : \beta_j = 0 \qquad \text{vs.} \qquad H_A : \beta_j \neq 0$$

Test Statistic:

$$t_0 = \frac{\hat{\beta}_j}{\sqrt{\widehat{\mathrm{Var}}(\hat{\beta}_j)}} = \frac{\hat{\beta}_j}{\sqrt{\hat{\sigma}_u^2 a_{jj}}}, \quad \hat{\sigma}_u^2 = \frac{e'e}{N-k}$$

where a_{jj} is the j-th diagonal element of $(X'X)^{-1}$ and $\hat{\sigma}_u^2$ is the estimated variance of the error term. **Decision Rule:**

- If $|t_0| \geq t_{N-k,\alpha/2}$, we reject H_0 at significance level α: the variable x_j is statistically significant.
- If $|t_0| < t_{N-k,\alpha/2}$, we fail to reject H_0 at significance level α: the variable x_j is not statistically significant.

2. Joint Significance Test

We use the F-test to evaluate whether a group of regressors (or all) are jointly significant.

Hypotheses:

$$H_0 : \beta_2 = \beta_3 = \cdots = \beta_k = 0 \qquad \text{vs.} \qquad H_A : H_0 \text{ is not true}$$

Test Statistic:

$$F_0 = \frac{SSE/(k-1)}{SSR/(N-k)} = \frac{(\hat{Y}'\hat{Y} - N\bar{Y}^2)/(k-1)}{(Y'Y - \hat{Y}'Y)/(N-k)} \sim F_{k-1,N-k}$$

Decision Rule:

- If $F_0 \geq F_{k-1,N-k;\alpha}$, we reject H_0 at significance level α: explanatory variables are jointly significant.
- If $F_0 < F_{k-1,N-k;\alpha}$, we fail to reject H_0 at significance level α: explanatory variables are not jointly significant.

Chapter 3

A macroeconomic model of uncertainty

The following model aims to overcome all these limitations present in classical models of the macroeconomy. In order to do so, it is important to equip the model with proper micro-foundations that consider the stochasticity of agents. Markov process theory will help in introducing such features to the model. This section mainly uses reasonings from [3].

3.1 Microeconomic foundations

3.1.1 General assumptions of the model

In this section, the economy to model is described. This model tries to be as complete as possible, taking as a starting point a microeconomic analysis of the agents, to then aggregate all the behaviors, resulting in a new approach to modeling the macroeconomy. We shall start by the microeconomic modeling of economic agents.

Conjecture 3.1. *Let N be the number of agents in the economy, which we consider fixed.*

Notice that this assumption may seem too strong, as the population is changing over time because of changes in net natality rates and immigration. However, since the purpose of the model is to study the effects of uncertainty onto policies, we are obviously in the short term (Where, precisely, there is uncertainty). As our focus of study is the sort term, it seems reasonable to assume that population is constant, since it simplifies the model.
We now turn to describe the microeconomic behavior of each agent in the economy.

Conjecture 3.2. *Agents in the economy can choose between two levels of production:* $0 < y < y^*$.

Now, it is convenient for future calculations to define x:

Definition 3.3. ***(Share of bulls)*** *Let* $x := n/N$ *denote the share of economic agents which produce at* y^*, *namely.*

To set the microeconomic foundations of the model, we recognize that we are not able to explain why an economic agent changes his decisions. This is, although as we saw in sections 2.1 and 2.2, it is impossible to infer any knowledge on all agent's preferences and constraints. Namely, some agents might change their decisions even if the macroeconomic environment remains unchanged. However, details of the microeconomic behavior of an individual agent are irrelevant in macroeconomics. We can assume specific decisions of agents in the economy as given by a Markov process.
The microeconomic basis of the model is described in the following conjecture.

Conjecture 3.4. *Changes in* x *are assumed to follow a jump Markov process.*

As we've learnt, to specify the Markov process means specifying the Markov density function. To find it, some more assumptions need to be made.

Conjecture 3.5. *In an infinitesimal time interval* dt*, there are only three possibilities: no economic agent changes its production level, or one either raises or lowers its production level.*

Note that this conjecture directly implies that, even though the stochastic process $x(t)$ takes multiple values between 0 and 1, namely, $0 \leq 1/N \leq 2/N \leq \cdots \leq N/N = 1$, the Markov state density function takes values 0 everywhere except for transitions to the state after the current one, and the one before. This is, it is just needed to define transition rates $r(y \to y^*)$ and $l(y^* \to y)$. Therefore, the microeconomic foundations of the model reduce to defining microeconomic foundations for the transition rates.

3.1.2 Microeconomic foundations for transition rates

To find expressions for the transition rates of the model, we will recur to a common economic strategy of modeling: cost and benefit analysis. In reality, rational agents, which act in the maximization of their own utility, take decisions comparing the benefits and costs of each option. This is also the logic here.

Definition 3.6. *Let* $V_1(x)$ *demote the expected return from choice 1 (*$y^* > 0$*), given that fraction x have selected choice 1.* $V_2(x)$ *is defined analogously.*

As announced, agents choose the option with lays the most expected return, so the focus is on modeling $\Delta V := V_1 - V_2$. However, we maintain our assumption that we cannot directly infer choices for all agents, therefore we assume that ΔV is a random variable following a normal distribution. The only concreteness we can point to is that it's mean and variance depend on fraction x (the share of bulls).

Conjecture 3.7. ΔV *is a real and continuous random variable following a normal distribution with mean* $g(x)$ *and variance* $\sigma^2(x)$*:*

$$\mathbb{P}(\Delta V = v) = \frac{1}{\sqrt{2\pi}\sigma(x)} exp\left[-\frac{(v - g(x))^2}{2\sigma^2(x)}\right] \tag{3.1}$$

This distribution represents microeconomic fluctuations, unpredictable but irrelevant for macroeconomic outcomes. In the standard approach, the agent has deterministic values for V_1 and V_2, and thus, there is no need to speak of the probability of V_1 exceeding V_2.

We can introduce now some notation that will be important

Definition 3.8.

$$\beta := \sqrt{\frac{2}{\pi}}\left(\frac{1}{\sigma}\right) \tag{3.2}$$

where σ *is the variance of random variable* ΔV

Note that this offers a very intuitive interpretation of β as a measure of uncertainty. For example, large variances mean large uncertainty in the expected difference of the alternative choices, and this is associated with a low β. Conversely, small variances mean more precise knowledge about the difference in the values of the two choices. This is associated with large values of β.

We now introduce a parameter that will be crucial in the characterization of transition rates.

Definition 3.9.

$$\eta_1(x) := \mathbb{P}(V_1(x) \geq V_2(x)) \tag{3.3}$$

$$\eta_2(x) = \mathbb{P}(V_2(x) > V_1(x)) \tag{3.4}$$

Conjectures 3.1 to 3.7, together with definitions 3.8 and 3.9 characterize the model. From now on, we will refer to an economy described by such assumptions with a Markov Economy. Note that clearly $\eta_1(x) + \eta_2(x) = 1$. It seems reasonable that transition rates are proportional to these magnitudes, since if the probability that producing at a high level will lay grater returns is higher, transition rate

$r(y \rightarrow y^*)$ should be high as well. The following proposition specifies a particular dependency for $\eta_1(x)$ and, by extension, $\eta_2(x)$.

Proposition 3.10. *Consider a Markov economy. Then:*

$$\eta_1(x) \approx X^{-1} exp[\beta g(x)] \tag{3.5}$$

where $X = e^{\beta g(x)} + e^{-\beta g(x)}$.

Proof. Note that:

$$\eta_1(x) = \mathbb{P}(\Delta V \geq 0) = \frac{1}{2}[1 + \text{erf}(u)]$$

with $u = g(x)/\sqrt{2}\sigma(x)$. Now, using Ingber's approximation (see [9]), and defining $\kappa := \frac{2}{\sqrt{\pi}}$:

$$\text{erf}(u) \approx \tanh(\kappa u) = \frac{e^{\kappa u} - e^{-\kappa u}}{e^{\kappa u} + e^{-\kappa u}}$$

Operating, the desired result is obtained:

$$\eta_1(x) \approx \frac{1}{2}\left[1 + \frac{e^{\kappa u} - e^{-\kappa u}}{e^{\kappa u} + e^{-\kappa u}}\right] = \frac{1}{2}\frac{2e^{\kappa u}}{e^{\kappa u} + e^{-\kappa u}} = \frac{e^{\kappa u}}{e^{\kappa u} + e^{-\kappa u}} = \frac{e^{\beta g(x)}}{e^{\beta g(x)} + e^{-\beta g(x)}}$$

$$= X^{-1}\exp[\beta g(x)]$$

□

This model also offers a clear interpretation of $g(x)$. It is the mean of the "prospect" that choice 1 is better than choice 2, conditional on the fraction x that have chosen to produce high. Note that function $g(x)$ indicates how advantageous a switch of strategy from bear to bull is. The greater $g(x)$ is, the more advantageous a switch from bear to bull is, and vice versa. Notice first that $g(x)$ describes economic behavior. If $g(x)$ is positive, then for a given share of bulls, it is more advantageous for agents to turn bull, as for proportion 2.9, $\eta_1(x) > 1/2$, and x rises over time. Analogously, when $g(x)$ is negative, it is more advantageous for agents to turn bear, and x will lower. Finally, when $g(x) = 0$, $\eta_1(x) = \eta_2(x) = 1/2$, a switch form $y \rightarrow y^*$ or $y^* \rightarrow y$ is equally probable, x will remain stable. In this context, many assumptions can be made for $g(x)$, specially regarding to the number of critical points (see figures 3.1 and 3.2).

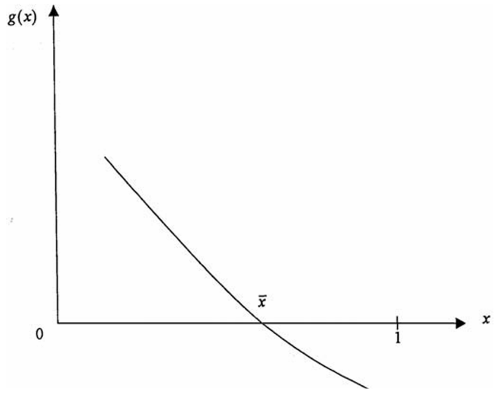


Figure 3.1: function g(x) with 1 critical point, taken from [3].

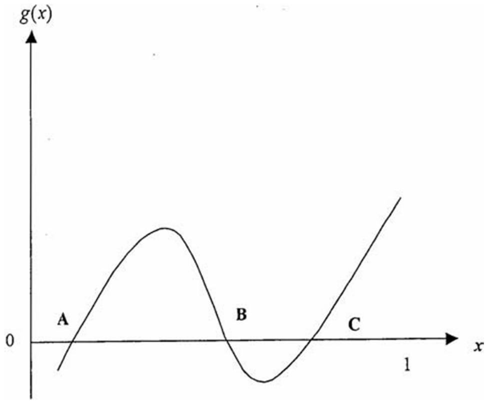


Figure 3.2: Function g(x) with 3 critical points, taken from [3].

In this thesis, a function with 1 critical point is assumed, for simplicity. Finally, output in the economy is:

Proposition 3.11. *The total output in a Markov economy is:*

$$Y = N[xy^* + (1-x)y] \tag{3.6}$$

Proof. Let n be the number of economic agents which produce at y^*. Then, it is clear that:

$$Y = ny^* + (N-n)y = Nxy^* + (1-x)Ny = Y = N[xy^* + (1-x)y]$$

where $x = n/N$ is the share of "bulls" ☐

The model is now fully characterized. However, the results that have been provided some intuitive about how uncertainty and policy can be understood in this model.
With this result, the transition rates can be defined:

Definition 3.12. *Consider a Markov economy. The output decision for each agent in the economy is governed by a stochastic process, with the following transition rates:*

$$r(y \to y^*) = \lambda N(1-x)\eta_1(x) \tag{3.7}$$

$$l(y^* \to y) = \mu Nx\eta_2(x) \tag{3.8}$$

with λ and μ positive parameters.

The interpretation of this transition rates is immediate. On the one hand, the probability that one agent currently producing at a low level (y) raises its production to high level (y^*), $r(y \to y^*)$ is proportional to the number of agents producing at y, this is $N(1-x)$. Analogously, the transition rate from y^* to y depends on Nx. On the other hand, transition rates depend on $\eta_1(x)$ and $\eta_2(x)$, since the higher the probability that producing at a high level has a greater return, the greater the probability of an agent changing from y to y^* will be, and vice versa.
We are now in position to characterize the Markov process:

Proposition 3.13. *Consider a Markov economy. The Markov state density function for process $x(t)$ is:*

$$P(x_j, t_j | x_{j-1}, t_{j-1}) = r\delta(x_j, x_{j-1} - 1/N) + l\delta(x_j, x_{j-1} + 1/N) \tag{3.9}$$

Proof. Trivial when combining Definition 1.24, Conjecture 2.5 and Definition 2.10. □

3.2 Policy in the model

This framework also allows to incorporate policy via changes in the $g(x)$ function. Ultimately, policies aim to achieve changes in the GDP, and they do so by changing the expected returns of agents, $V_1(x)$ and $V_2(x)$.

Definition 3.14. *A "policy" in a Markov economy corresponds to a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that the new $g'(x)$ is:*

$$g'(x) = g(x) + h(x)$$

This definition of a policy allows incorporating the basic IS-LM framework in the Markov economy. For the sake of simplicity, assume that $g(x)$ only has one critical point $\bar{x}$, such as in Figure 3.1. Assume that a deteriorated expected profitability makes the IS curve shift down. As a consequence, output declines. This situation is equivalent in our model where given x, more economic agents are likely to now find advantageous to switch from bull to bear, this is, $g(x)$ shifts down (see Figure 3.3). The critical point moves to the left accordingly. Next, suppose that the authority lowered the interest rate to fight against the recession. The LM curve moves downward to the right, leading output to rise. In this model, thanks to lower interest rates, given x, economic agents find it more advantageous to switch from bear to bull. The $g(x)$ function shifts to the right as shown in Figure 3.4. The economy returns from $\bar{X}_2$ to $\bar{x}_1$, it recovers from the recession as output goes back tho the previous level.

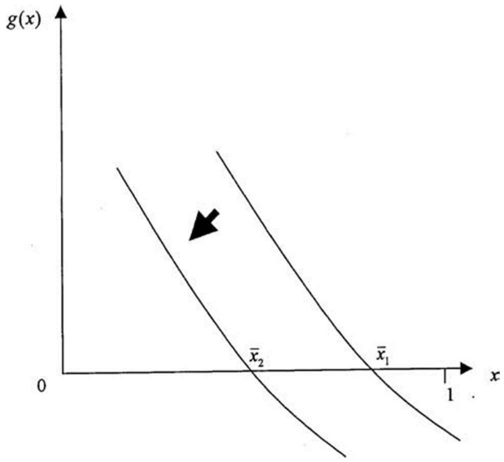


Figure 3.3: Downward shift in $g(x)$, extracted from [3]

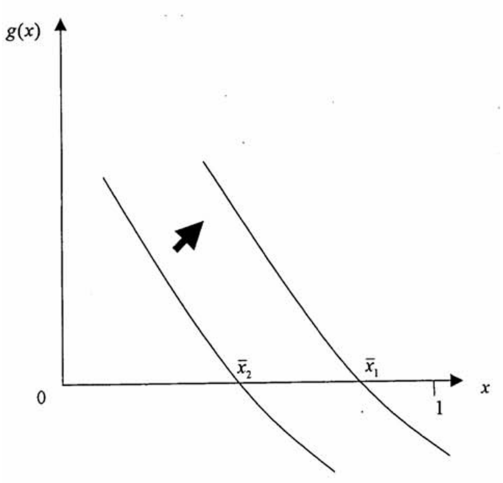


Figure 3.4: upward shift in $g(x)$, extracted from [3]

Therefore, the following definition can be made:

Definition 3.15. *Consider a Markov economy. For simplicity, consider only policies such that* $\dot{h}(x) = 0$*. This is, there is no particular bias in the policy. Then, A policy* $h(x)$ *in this model is called expansionary if* $h(x) > 0$*. On the contrary, a policy is called contractive if* $h(x) < 0$*.*

This definition seems logical as expansionary policies respond to the common framework of stimulative policies, since a policy $h(x) > 0$ would shift $g(x)$ to the right, thus increasing the equilibrium share of bulls $\bar{x}$, which will increase output. Analogously, contractive policies reduce expected output.

3.3 Dynamics of the model

In conjecture 3.4 it is stated that the share of bulls, x, changes stochastically. Specifically, if is modeled by a jump Markov process described by Proposition 3.13. This directly implies that GDP does as well follow a jump Markov process, also described by Proposition 3.13 (since it is only a linear transformation).
To study the dynamics of the Markov process, as it was clear in chapter 1, it is crucial to analyze the forward master equation for the Markov state density function. In this case, the master equation is, simplifying notation:

Proposition 3.16. *Consider a Markov economy with g(x) only having one critical point. Then:*

$$\frac{\partial P(n,t)}{\partial t} = N\big(x+\tfrac{1}{N}\big)\,\eta_2\big(x+\tfrac{1}{N}\big)\,P(n+1,t) + N\big(1-x+\tfrac{1}{N}\big)\,\eta_1\big(x-\tfrac{1}{N}\big)\,P(n-1,t)$$
$$- \Big[Nx\,\eta_2(x) + N(1-x)\,\eta_1(x)\Big]\,P(n,t) \tag{3.10}$$

Proof. Start with Theorem 1.39. Note that, because of conjecture 2.5,

$$W(\nu|n-\nu,t) = l(y^* \to y)\delta(-1,\nu) + r(y \to y^*)\delta(1,\nu)$$
$$W(-\nu|n,t) = l(y^* \to y)\delta(1,\nu) + r(y \to y^*)\delta(-1,\nu)$$

Substituting these expressions and Definition 2.11 into Theorem 1.39 provides the desired result. □

The next step is solving the master equation to describe the dynamics of the model. The reader will have noticed that there is not a general solution procedure detailed to solve it. This is because, more often than not, the discrete state master equations cannot be directly solved for $P(n,t|n_0,t_0)$. There are mathematical techniques to find simpler differential equations for the evolutions of the moments of the stochastic process, but they are of no interest practically. However, the hope is not lost. The master equation obtained in Proposition 3.16 can be solved for the first moment in $x(t)$ using Taylor expansion. This is shown in the next proposition.
For convenience, let's introduce some notation

Definition 3.17. *Consider a Markov economy, define:*

$$\phi_t := E[x(t)] \tag{3.11}$$

Subscript t will often be used now to simplify notation, but it just indicates time dependence. Now, the next proposition reduces the dynamics of the model to an ordinary differential equation.

Proposition 3.18. *Depart from a Markov economy. The dynamics for the first moment of the process $x(t)$ are governed by the following ordinary differential equation:*

$$\dot{\phi}_t := \frac{d\phi_t}{dt} = \phi_t\eta_2(\phi_t) - (1-\phi_t)\eta_1(\phi_t) \tag{3.12}$$

Proof. First, we decompose the stochastic variable x_t into two components:

$$x_t = \phi_t + \frac{\xi_t}{\sqrt{N}}$$

This can be done as ξ_t is the stochastic deviation of x_t from its mean ϕ_t. $\sqrt{N}$ just normalizes its standard deviation. Now, let's rewrite the left-hand side of the master equation, Proposition 3.16.

$$P(n,t) = P(Nx,t) = P(N\phi + \sqrt{N}\xi, t) =: \pi(\xi, t)$$

$\pi(\xi,t)$ Characterizes the density function of ξ at time t. Then, applying the chain rule:

$$\frac{\partial P(n,t)}{\partial t} = \frac{\partial \pi}{\partial t} + \frac{\partial \pi}{\partial \xi}\frac{d\xi_t}{dt} \tag{3.13}$$

Now, at each moment of time, a realized value of x is given, and therefor ($dx/dt = 0$). This allows us to state

$$\frac{d\phi_t}{dt} = -\frac{1}{\sqrt{N}}\frac{d\xi_t}{dt} \tag{3.14}$$

Substituting this expression in equation (3.13), we obtain:

$$\frac{\partial P(n,t)}{\partial t} = \frac{\partial \pi}{\partial t} - \sqrt{N}\frac{d\phi_t}{dt}\left(\frac{\partial \pi}{\partial \xi}\right) \tag{3.15}$$

Next, we rewrite the right-hand side of the master equation in Proposition 3.16. To do so, we expand ϕ ad ξ around the expected value of x, ϕ by the Taylor series with respect $1/\sqrt{N}$ up to the quadratic term (this is, $1/N$):

$$\begin{aligned}\frac{\partial P(n,t)}{\partial t} = {} & N\Big(\phi + \tfrac{\xi}{\sqrt{N}} + \tfrac{1}{N}\Big)\Big[\eta_2(\phi) + \eta_2'(\phi)\big(\tfrac{\xi}{\sqrt{N}} + \tfrac{1}{N}\big)\Big] \\ & \times \Big[\pi(\xi,t) + \tfrac{\partial \pi}{\partial \xi}\big(\tfrac{1}{\sqrt{N}}\big) + \tfrac{1}{2}\tfrac{\partial^2 \pi}{\partial \xi^2}\big(\tfrac{1}{N}\big)\Big] \\ & + N\Big(1 - \phi - \tfrac{\xi}{\sqrt{N}} + \tfrac{1}{N}\Big)\Big[\eta_1(\phi) + \eta_1'(\phi)\big(\tfrac{\xi}{\sqrt{N}} - \tfrac{1}{N}\big)\Big] \\ & \times \Big[\pi(\xi,t) - \tfrac{\partial \pi}{\partial \xi}\big(\tfrac{1}{\sqrt{N}}\big) + \tfrac{1}{2}\tfrac{\partial^2 \pi}{\partial \xi^2}\big(\tfrac{1}{N}\big)\Big] \\ & - \Big[N\Big(\phi + \tfrac{\xi}{\sqrt{N}}\Big)\big(\eta_2(\phi) + \eta_2'(\phi)\tfrac{\xi}{\sqrt{N}}\big) \\ & \quad + N\Big(1 - \phi - \tfrac{\xi}{\sqrt{N}}\Big)\big(\eta_1(\phi) + \eta_1'(\phi)\tfrac{\xi}{\sqrt{N}}\big)\Big]\,\pi(\xi,t)\end{aligned} \tag{3.16}$$

For the sake of simplicity, several steps were omitted. It was also used that when the number of bulls, $n = Nx = N\phi + \sqrt{N}\xi$, increases from n to $n+1$, the change if the stochastic variable ξ must be $1/\sqrt{N}$, because in that case, n increases by $\sqrt{N}\xi = \sqrt{N} \times \frac{1}{\sqrt{N}} = 1$. Analogously, when the number of bulls is $n-1$, the deviation of ξ_t is $-1/\sqrt{N}$.

Now, look at the right-hand side of equation (3.16) as a power series of N. The maximum term if $N^{1/2}$, while the other terms are of lower order. The term of the highest order, namely the one with $\sqrt{N}$, is:

$$\sqrt{N}\left[\phi\eta_2(\phi) - (1-\phi)\eta_1(\phi)\right]\left(\frac{\partial \pi}{\partial \xi}\right) \tag{3.17}$$

Comparing this term to the right-hand side of equation (3.15), we can deduce the enounced relationship. □

This ordinary differential equation determines the dynamics of the expected value of x, namely ϕ. Since ϕ is not stochastic, its dynamics obey an ordinary differential equation.
Proposition 3.18 will be the central point of our analysis. In the following section, we'll use this result to study the steady state of our model economy.

3.4 Equilibrium in the model

This section is dedicated to study the equilibrium of the economy. Because of the heterogeneity of agents introduced in this model, a novelty in macroeconomic theory, we'll find new conclusions not foreseen with the classical models. In particular, both the economic behavior (described by $g(x)$) will be important in determining GDP, as it has been in all traditional models. However, this model will allow us to state that uncertainty is also crucial in the determination of the equilibrium output.
We have announced that the study of the equilibrium of the economy implies important consequences, but first, it is important to prove that it exists, and that it is stable.

Proposition 3.19. *Depart from a Markov economy. There exists a stable equilibrium in the economy, described by:*

$$2\beta g(\phi) = \log\left(\frac{\phi}{1-\phi}\right) \tag{3.18}$$

Proof. To prove the existence, start by Proposition 3.18 and impose $\dot{\phi}_t = 0$. Then:

$$\phi_t \eta_2(\phi_t) - (1-\phi_t)\eta_1(\phi_t) = 0 \Rightarrow \frac{\eta_1(\phi)}{\eta_2(\phi)} = \frac{\phi}{1-\phi}$$

Now, using Proposition 3.10 and Definition 3.9, it is easy to prove that this equation is equivalent to the enounced. Since

$$\lim_{\phi\to 0} \log\left(\frac{\phi}{1-\phi}\right) = 0, \qquad \lim_{\phi\to 1^-} \log\left(\frac{\phi}{1-\phi}\right) = \infty.$$

and it is a continuous function, there exists a ϕ^* verifying the equation.
Now, let's prove that it is stable. If $\phi > \phi^*$, then $\eta_1(\phi)/\eta_2(\phi) > \eta_1(\phi^*)/\eta_2(\phi^*)$

since $\phi/(1-\phi)$ is an increasing function. Thus, because of Proposition 3.18,

$$\frac{\dot{\phi}_t}{\eta_2(\phi)} = \phi_t - (1-\phi_t)\frac{\eta_1(\phi)}{\eta_2(\phi)} < 0$$

Therefore ϕ decreases until going back to the steady state. Analogously, stability is proved for $\phi < \phi^*$. Thus, there exists a steady state, and it is stable. □

Remember that our focus is the study of ϕ, since this characterizes the expected value of the share of bulls, x, and therefore the GDP of the economy. Since there exists a stable steady state, the economy will tend to be there, assuming no frictions. Therefore, we shall reduce the scope of our analysis to the steady state. In this situation, output in the economy will be:

$$Y = N[\phi^* y^* + (1-\phi^*)y] \tag{3.19}$$

Before getting more deep into the steady state dynamics, we shall define some concepts.

Definition 3.20. *Consider a Markov economy. We define the Shannon entropy of the economy as:*

$$H(x) = -x\ln x - (1-x)\ln(1-x) \tag{3.20}$$

The following proposition justifies why $H(x)$ is called the Shannon entropy.

Proposition 3.21. *Consider a Markov economy. Then:*

$$\log\left(\frac{N!}{(N-n)!n!}\right) = NH(x) \tag{3.21}$$

Proof.

$$\log\left(\frac{N!}{(N-n)!n!}\right) = N\left[-\left(\frac{n}{N}\right)\log\left(\frac{n}{N}\right) - \left(1-\frac{n}{N}\right)\log\left(1-\frac{n}{N}\right)\right] = NH(x)$$

where we used the Stirling formula, explained in section 6.2, in the Appendix. □

Thus, $H(x)$ is a measure of the number of cases where n out of N agents are bulls. It expresses the combinatorial aspect of the economy, Precisely, is this combinatorial aspect that standard macroeconomics entirely ignores (since it assumes that all agents are homogeneous), yet it plays a critical role in any physical or social system.

Definition 3.22. *Consider a Markov economy. The potential function of the economy is defined as.*

$$U(x) = -2\int^x g(y)dy - \frac{1}{\beta}H(x) \tag{3.22}$$

Keeping this in mind, let's go back to the analysis of the expected value of the share of bulls in the steady state, ϕ^*.
First, note that we can associate the steady states of the economy to local minima of the potential function.

Proposition 3.23. *Consider a Markov economy. Then:*

$$\dot{U}(x) = 0 \Longleftrightarrow 2\beta g(\phi) = \log\left(\frac{\phi}{1-\phi}\right) \tag{3.23}$$

Proof.

$$\begin{aligned} \dot{U}(x) &= -2g(\phi) - \frac{1}{\beta}\dot{H}(\phi) = -2g(\phi) - \frac{1}{\beta}\log\Big(\tfrac{\phi}{1-\phi}\Big) \\ &= 0 \iff 2\beta g(\phi) = \log\Big(\tfrac{\phi}{1-\phi}\Big). \end{aligned}$$

□

This result provides a very rich interpretation of the equilibrium of the economy. In particular, we can study the equilibrium in the macroeconomy in two different cases.
When there is little uncertainty, β is large. In this limit, the expected value of the share of bulls in the steady state is provided by

$$g(\phi^*) = 0 \tag{3.24}$$

Therefore, if $g(x)$ is like in Figure 3.1, then the steady state corresponds to the unique stable equilibrium $\bar{x}$. In this case, x changes stochastically, but spends most of the time in the neighborhood of $x = \phi^*$. Accordingly, GDP will fluctuate around the value:

$$Y = N[\phi^* y^* + (1-\phi^*)y] \tag{3.25}$$

However, when the degree of uncertainty is non-negligible ($\beta < \infty$) and small, then multiple equilibria may arise. In this case, the steady state values of ϕ^* are determined by both $g(x)$ and $\dot{H}(x)/\beta$. Now, even if function $g(x)$ only has one stable root, $\dot{U}(\phi) = 0$ may have multiple stable roots (Figure 3.5).

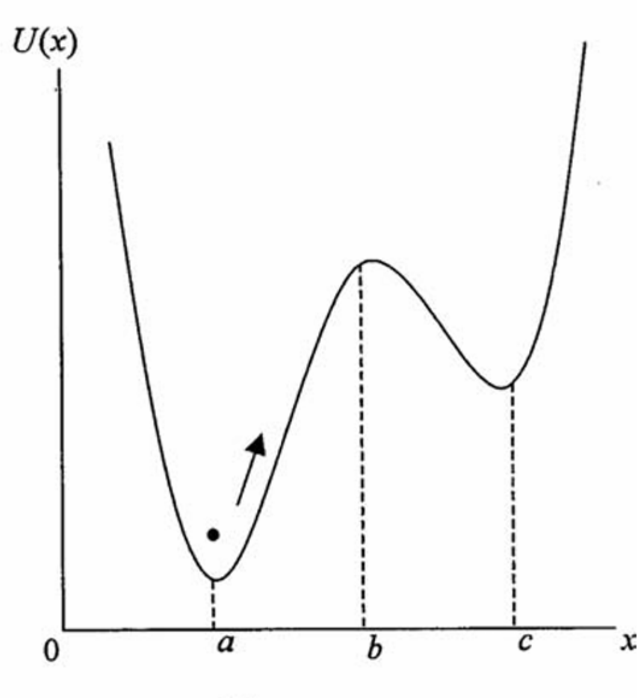


Figure 3.5: Potential function and fluctuations in a Markov economy, taken from [3].

In the second case, the economy stochastically fluctuates, spending most of the time in the neighborhood of two minima of the potential function. Although is not the point of this thesis, the mean passage time t^* between the two minima is:

$$t^* \approx e^{\beta N(U(b)-U(a))} \tag{3.26}$$

Therefore, when uncertainty becomes significant, we cannot ignore the combinatorial aspect of the problem, considered by the Shannon entropy. In this sense, one might say that uncertainty can be a source of fluctuations of the macroeconomy. Moreover, this would essentially provide a justification for Keyne's claim that the economy can be in equilibrium with unemployment, since when there is uncertainty, the economy may be trapped in a equilibria with lower GDP than it's potential.

In particular, in the next section we'll use this result to study the effectiveness of policy depending on the degree of uncertainty in the economy, namely β in our model.

3.5 Uncertainty and Policy ineffectiveness

In the previous section, it was found that by considering heterogeneity between agents in an economy, uncertainty arose as a parameter, β. This parameter had a huge impact in the equilibrium of the macroeconomy, being a source of fluctuations as well as multiple equilibriums.
In this section, a new implication regarding uncertainty will be analyzed, it's interaction with economic policy. In particular, the main result of this section is that β highly conditions the effectiveness of macroeconomic policies proposed.
That being the objective, it seems reasonable to define a parameter measuring the effectiveness of policies.

Definition 3.24. *Consider a Markov economy with function $g(x)$ only having a unique stable equilibrium, with an expected share of bulls in equilibrium ϕ^* and an economic policy $h(x)$. Then, the multiplier of the policy is described by:*

$$E = \frac{\delta\phi}{h(\phi^*)} \tag{3.27}$$

where $\delta\phi$ is the increase in the expected average share of bulls x caused by the policy.

The multiplier provides an intuitive measure of the effectiveness of monetary policy, since it gives the increase in ϕ relative to a shift of $g(x)$, to a policy. It indirectly measures the amount of output change per "unit" of policy.
With our model, we are able to characterize the multiplier as a function of the uncertainty level β. This is what the following theorem proves.

Theorem 3.25. *Depart from a Markov economy, assuming that $g(x)$ has a unique stable equilibrium and an equilibrium expected value of the share of bulls ϕ^*. A policy is applied to this economy such that $h(x) > 0$ and $\dot{h}(x) = 0$ (expansionary policy and no bias). Then, the multiplier is:*

$$E(\beta) = \frac{2}{\frac{1}{\beta}\left(\frac{1}{\phi^*(1-\phi^*)}\right) - 2g'(\phi^*)} > 0 \tag{3.28}$$

Proof. When applying the policy, function $g(x)$ shifts to $g'(x) = g(x) + h(x)$. Therefore, the steady state expected share of bulls shifts from ϕ^* to $\phi^* + \delta\phi$. Because $\phi^* + \delta\phi$ also satisfies Proposition 3.23, then:

$$-2[g(\phi^* + \delta\phi) + h(\phi^* + \delta\phi)] + \frac{1}{\beta}\log\left(\frac{\phi^* + \delta\phi}{1 - \phi^* - \delta\phi}\right) = 0$$

Taking the Taylor series expansion of this last equation to the first order:

$$-2[g(\phi^*) + \dot{g}(\phi^*)\delta\phi + h(\phi^*)] + \frac{1}{\beta}\left[\log\left(\frac{\phi^*}{1-\phi^*}\right) + \left(\frac{1}{\phi^*(1-\phi^*)}\right)\delta\phi\right] = 0$$

As ϕ^* also satisfies Proposition 3.23, the equation is reduced to:

$$-2\dot{g}(\phi^*)\delta\phi - 2h(\phi^* + \frac{1}{\beta}\left(\frac{1}{\phi^*(1-\phi^*)}\right)\delta\phi = 0$$

We can solve this for $\delta\phi$ and obtain:

$$\delta\phi = \frac{2h(x)}{\frac{1}{\beta}\left(\frac{1}{\phi^*(1-\phi^*)}\right) - 2g'(\phi^*)}$$

Dividing by $h(x)$, the theorem is proven. □

Notice that, in this framework, expansionary policies increase the equilibrium expected share of bulls, and because of equation 3.19, they boost expected GDP. However, this increase depends crucially on the uncertainty level, via β.
On the one hand, when uncertainty is negligible, $\beta \to \infty$ and the multiplier takes its maximum value

$$\lim_{\beta\to\infty} E = \frac{-1}{\dot{g}(x)} > 0 \tag{3.29}$$

On the other hand, as the degree of uncertainty rises ($\beta \to 0$), E declines and tends to 0

$$\lim_{\beta\to 0} E = 0 \tag{3.30}$$

Thus, when uncertainty rises, the effectiveness of policies which affect agent's economic incentives necessarily weakens. In the limit, when the economy faces infinite uncertainty (remember that, because of Definition 3.8 $\beta \to 0 \Rightarrow \sigma \to \infty$), it is trapped in a state in which no economic policy works. In fact, no economic decision makes sense in that is no different from tossing a coin. This makes policies that act as incentives for economic agents ineffective. We have proven the following result:

Corollary 3.26. *When the degree of uncertainty rises, the effectiveness of macroeconomic policy necessarily weakens. In the limit ($\beta \to 0$), macroeconomic policy becomes completely ineffective.*

In this sense, we can say that when the degree of uncertainty is extremely high, the economy is caught in an *uncertainty trap.*

Chapter 4

Evidence on Policy Effectiveness under Uncertainty: The Case of Spain

4.1 Introduction

The Markov economy model uncovers some bold conclusions. The fact that policy becomes ineffective in the presence of uncertainty might seem a devastating statement to policy-design institutions that have contributed to economic growth and short-run economic management for decades. Moreover, during crisis, where policy takes a leading role, is precisely when policies become ineffective. Nonetheless, there is no doubt that institutions play a fundamental role in shaping economic development by defining the incentives that guide individual and collective behavior. According to [2], inclusive institutions, those that protect property rights, encourage broad participation in economic activity, and are supported by a pluralistic political system, foster an environment conducive to investment, innovation, and sustained growth. In contrast, extractive institutions, which concentrate power and resources in the hands of a few, tend to discourage productive activity and generate long-term instability. From this perspective, institutional stability is not only a prerequisite for sustained economic growth but also acts as a buffer against the adverse effects of uncertainty, as it reduces the perceived risk associated with future economic policymaking.

The results of this model have to be taken with care. The point of this study, far from discrediting institutions, is precisely to help them with their policy design. Policy-makers need to be aware that, when designing their interventions, uncertainty might play a non-negligible role.

In this chapter, we want to assess if the conclusions of the model apply to the real-world economy. We will restrict our analysis to fiscal policy in Spain during the XXI century and before the COVID-19 crisis. Previous literature already suggested that uncertainty, specially in a crisis context, might play an important role in policy effectiveness. Serrano in [13] examines the fiscal measures adopted by the Spanish government to counteract the sharp decline in aggregate demand during the global financial crisis. One of the most notable actions was the reduction of the Personal Income Tax (IRPF), intended to increase households disposable income and thereby stimulate private consumption. However, as Serrano highlights, this policy did not achieve the desired effect. In a context marked by heightened economic uncertainty and a restrictive monetary stance by the European Central Bank, households chose to increase savings rather than consumption. This precautionary behavior significantly limited the effectiveness of the fiscal stimulus, illustrating how, in times of high uncertainty, expansionary fiscal policies aimed at boosting demand can fail to deliver their expected outcomes due to a shift in private agents' expectations and preferences.

It seems that everything points to the same conclusion, though we shall verify this with data in a more general framework. Ideally, it would be desirable to calibrate the model in Chapter 3, and check if it can generate real data. However, the model presented in Chapter 3 is too abstract to calibrate. For instance, function $g(x)$ is too abstract and there are no methods to estimate it empirically. Moreover, parameter β arises as a variance with is also hard to estimate. The following section will feature an econometric approach, and will aim to estimate the causal effect of uncertainty on policy effectiveness. This analysis will be focused on the effect of uncertainty on fiscal policy. In particular, we restrict our analysis to changes in Government expenditure.

4.2 Data

To estimate the proposed regression model and analyze how economic uncertainty affects the effectiveness of fiscal policy in Spain, we rely on both macroeconomic aggregates and a quantitative measure of policy uncertainty.

The macroeconomic data-covering GDP, private consumption (C), investment (I), government expenditure (G), and net exports (XN)-are obtained from the Spanish National Accounts (*Contabilidad Nacional*), published by the *Instituto Nacional de Estadística* (INE)[10]. All variables are expressed as chained volume indices, which represent real economic activity by adjusting for inflation through a method that links yearly price structures over time. This approach ensures greater accuracy in tracking real output and expenditure dynamics across periods.

To quantify economic uncertainty, we use the Economic Policy Uncertainty (EPU) index, originally developed by Baker, Bloom, and Davis, and later adapted for Spain by Álvarez, Blanco, and Cabrales [8]. The index is constructed based on the frequency of newspaper articles that contain keywords related to the economy, policy, and uncertainty. It has been widely adopted in macroeconomic literature as a reliable proxy for the level of perceived uncertainty in economic policymaking.

4.3 An Econometric model

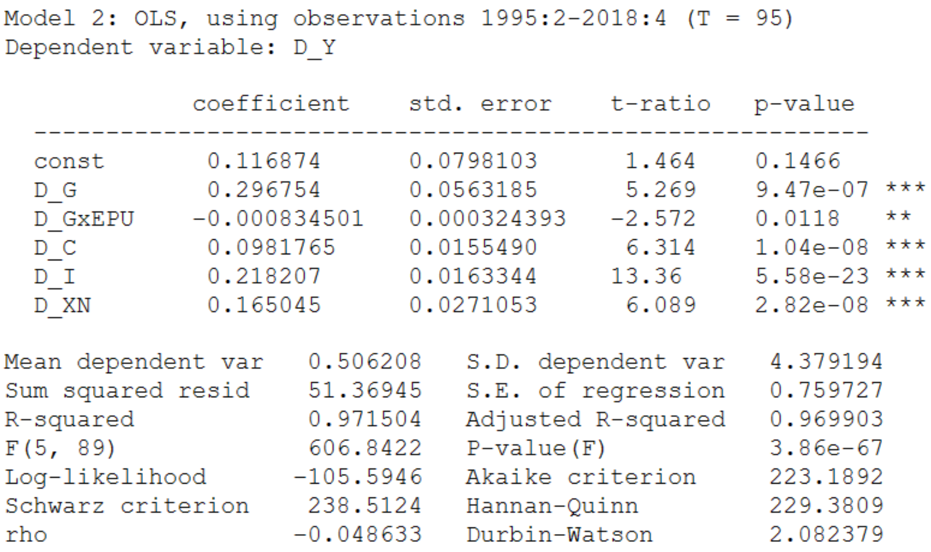
To empirically evaluate the hypothesis that uncertainty undermines the effectiveness of macroeconomic policy, we develop a regression grounded in the fundamental accounting identity of national income:

$$Y = C + I + G + XN \tag{4.1}$$

This identity states that aggregate output (GDP) is the result of private consumption (C), investment (I), public spending (G), and net exports (XN). Although this relationship is purely accounting-based and holds by definition, it provides a natural foundation for building a regression model that explains variations in GDP growth. In particular, we aim to assess whether public spending (G) has a smaller impact on GDP in periods of high economic policy uncertainty, as predicted by the theoretical model developed in chapter 3. This accounting identity suggests running the following regression:

$$\Delta Y_t = \alpha + \beta_1 \Delta C_t + \beta_2 \Delta I_t + \beta_3 \Delta G_t + \beta_4 (\Delta G \times \mathrm{EPU}_t) + \beta_5 \Delta XN_t + u_t \tag{4.2}$$

The regression model applied in this analysis aims to examine how the effectiveness of fiscal policy varies with levels of economic uncertainty. The dependent variable, ΔY_t, represents the variation in real Gross Domestic Product (GDP). The explanatory variables reflect the main components of aggregate demand as classified by the Instituto Nacional de EstadÃstica (INE) using chained volume indices. Specifically, ΔC_t captures changes in final consumption expenditure by households and non-profit institutions serving households (ISFLSH). ΔI_t refers to the variation in gross capital formation, which includes both gross fixed capital formation (FBCF) and changes in inventories and acquisitions less disposals of valuables. The variable ΔG_t denotes changes in final consumption expenditure by general government (AAPP) and serves as a proxy for public spending. To assess whether the effectiveness of this policy depends on uncertainty, the model includes an interaction term $\Delta G_t \times \mathrm{EPU}_t$ where EPU is the Economic Policy Uncertainty index. A negative and statistically significant coefficient on this interaction would suggest that public spending has a lower impact on output in times of high uncertainty. Finally, ΔXN_t represents the change in net exports, measured as the difference between exports and imports of goods and services. The error term u_t captures unobserved shocks and residual variation not explained by the model.



```
Model 2: OLS, using observations 1995:2-2018:4 (T = 95)
Dependent variable: D_Y

             coefficient    std. error    t-ratio   p-value
  -----------------------------------------------------------
  const        0.116874     0.0798103      1.464    0.1466
  D_G          0.296754     0.0563185      5.269    9.47e-07 ***
  D_GxEPU     -0.000834501  0.000324393   -2.572    0.0118   **
  D_C          0.0981765    0.0155490      6.314    1.04e-08 ***
  D_I          0.218207     0.0163344     13.36     5.58e-23 ***
  D_XN         0.165045     0.0271053      6.089    2.82e-08 ***

Mean dependent var   0.506208   S.D. dependent var   4.379194
Sum squared resid    51.36945   S.E. of regression   0.759727
R-squared            0.971504   Adjusted R-squared   0.969903
F(5, 89)             606.8422   P-value(F)           3.86e-67
Log-likelihood      -105.5946   Akaike criterion     223.1892
Schwarz criterion    238.5124   Hannan-Quinn         229.3809
rho                 -0.048633   Durbin-Watson        2.082379
```

Figure 4.1: Outcome of the regression. The estimation was done by econometric software Gretl.

We estimate the model using Ordinary Least Squares (OLS) over quarterly data covering the period from 1995:Q2 to 2018:Q4 (T = 95). The dependent variable is ΔY_t, which represents the quarterly change in real GDP. The explanatory variables correspond to the main components of aggregate demand. Specifically, D_G represents differences in government spending (ΔG_t), D_C corresponds to differences in private consumption (ΔC_t), D_I to investment (ΔI_t), and D_XN to net exports (XN). In addition, the interaction term $D\_G \times EPU$ captures how the effect of government spending is moderated by economic policy uncertainty, proxied by the EPU index.

Regarding the statistical significance of the estimated coefficients, all explanatory variables are individually significant at the 1% or 5% level, as expected since the regression derives from an accounting rule.

The coefficient on the interaction term $\Delta G_t \times \mathrm{EPU}_t$, which captures the effect of economic policy uncertainty on the fiscal multiplier, is negative and statistically significant at the 5% level ($\beta_4 = -0.00083, p = 0.0118$). This result implies that the effectiveness of government spending in stimulating output decreases as economic uncertainty increases. In other words, during periods of high uncertainty-when the EPU index is high-the marginal impact of a fiscal expansion on GDP is reduced. Formally, the marginal effect of government spending on output in this model is given by:

$$\frac{\partial \Delta Y_t}{\partial \Delta G_t} = 0.30 - 0.00083 \cdot \mathrm{EPU}_t \tag{4.3}$$

This expression shows that the impact of fiscal policy is not constant, but depends on the prevailing level of uncertainty. When uncertainty is low (i.e., EPU_t is small), the marginal effect is close to $\beta_3 = 0.30$, the direct coefficient of government spending. However, as uncertainty increases, the term $\beta_4 \cdot \mathrm{EPU}_t = -0.00083$ becomes more negative, reducing the overall impact.

On first sight, β_4 might not seem to constrain the efficiency of government spending since it is 3 orders of magnitudes lower than β_3. However, in situations of high uncertainty, the effectiveness of fiscal policy can be constrained. Figure 4.1 shows a time evolution of uncertainty in Spain during the XXI century:

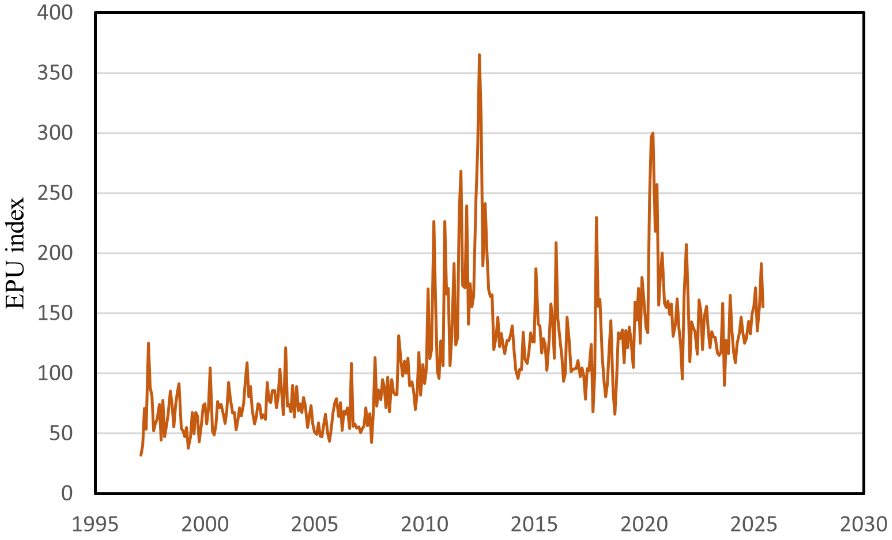


Figure 4.2: Economic Policy Uncertainty index for Spain between 1997 and 2025, monthly data.

During the sovereign debt crisis, the Economic Policy Uncertainty (EPU) index reached record levels, peaking at 360. According to our estimations, then:

$$\frac{\partial \Delta Y_t}{\partial \Delta G_t} = 0.30 - 0.00083 \cdot 360 = 0.30 - 0.2988 \approx 0 \tag{4.4}$$

Therefore, uncertainty can block public policy effectiveness. Empirical evidence also supports our theoretical conclusions extracted from our Markov economy.

Chapter 5

Conclusions

The standard analysis in macroeconomics departs from the assumption of the representative agent. Suppose, for example, that the authority cut the interest rate. In that case, the real interest rate is reduced. For the representative household or firm, a lower real interest rate would raise the optimal level of investment and some other expenditures. Translating this result to macroeconomic analysis, the standard analysis concludes that, ceteris paribus, aggregate demand increases. This kind of analysis is taken by most economists and policymakers as sound guidance for macroeconomic policy, so long as the degree of uncertainty facing the economy is low. However, when uncertainty becomes significant, we must depart from the representative agent assumption and seriously consider that the macroeconomy consists of a large number of heterogeneous agents. In that case, a stochastic approach is necessary. The combinatorial aspect of the system plays a crucial role in the analysis of any complex system, either physical or social, comprising many entities, yet standard economic analysis entirely ignores it.

This thesis has successfully provided a theoretical framework with sufficient mathematical modeling techniques that allow to surpass these limitations presented by the classical analysis. In doing so, a new relationship was found between uncertainty and real economic variables. Two main conclusions can be extracted:

- At a **theoretical level**, we have shown that a properly micro-founded model can provide a relationship between uncertainty and policy effectiveness. This relationship is shown by the multiplier, which is non-linear and bounded 0 and an asymptotic value, $-1/g'(\phi^*)$.
- At an **empirical level**, an econometric model supports this relationship in a Spanish context during the XXIst century. Data also supports the possibility that uncertainty rises to the extent that makes fiscal policy completely ineffective, since estimates show that during the sovereign debt crisis, high

uncertainty constrained policy effectiveness to the point where GDP did not respond to fiscal policy measures.

These findings should not be interpreted as a wholesale repudiation of decades of insights into fiscal and monetary policy. On the contrary, they offer an additional lens through which to design interventions that better align with intended objectives, now paying special attention to the uncertainty level of economic agents. This is, the objective of this thesis is to provide policymakers with a new diagnostic tool, not a replacement for existing models.

Finally, this thesis is aims to invite researchers in mathematics and economic theory to collaborate together in this subject, since we have shown that innovative mathematical techniques are needed to find new economic relationships. Moreover, this work stresses the need for more research effort to be devoted into further investigating this relationship between uncertainty and real GDP.

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Appendix

5.1 The error function

The error function in mathematics is a specific type of function, commonly used in probabilities and statistic because it allows to easily summarize the normal distribution CDF function [1].

Definition 5.1. *The error function, is a function erf* : $\mathbb{R} \to \mathbb{R}$ *defined as:*

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Some relevant properties that will be used in this project are:

Proposition 5.2. *The error function can be well approximated by hte hyperbolic function*

$$erf(x) \approx \tanh \frac{2}{\sqrt{\pi}} x$$

Proof. A numerical proof can be found in [9]. If we calculate the two Taylor series, we find that:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right)$$

$$\tanh(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{2.36} + \frac{x^5}{4.63} + \cdots \right)$$

It is clear that, especially for $|u| < 1$, the approximation is reasonable. □

5.2 Stirling formula

A first expression of the Stirling formula is provided in [6]. To obtain a more functional approximation:

$$\log n! \approx \log \sqrt{2\pi n} (\frac{n}{e})^n$$

Operating and using properties of the logarithm

$$\log n! \approx \frac{1}{2} \log 2\pi n + n \log n - n$$

Since the first term is negligible, we obtain

$$\log n! \approx n \log n - n$$