

Planar radial weakly dissipative diffeomorphisms

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We study the effect of a small dissipative radial perturbation acting on a one parameter family of area preserving diffeomorphisms. This is a specific type of dissipative perturbation. The interest is on the global effect of the dissipation on a fixed domain around an elliptic fixed/periodic point of the family, rather than on the effects around a single resonance. We describe the local/global bifurcations observed in the transition from the conservative to a weakly dissipative case: the location of the resonant islands, the changes in the domains of attraction of the foci inside these islands, how the resonances disappear, etc. The possible ω -limits are determined in each case. This topological description gives rise to three different dynamical regimes according to the size of dissipative perturbation. Moreover, we determine the conservative limit of the probability of capture in a generic resonance from the interpolating flow approximation, hence assuming no homoclinics in the resonance. As a paradigm of weakly dissipative radial maps, we use a dissipative version of the Hénon map. © 2010 American Institute of Physics. [doi:10.1063/1.3515168]

Many physical problems are usually described as a Hamiltonian system plus a (relatively) small perturbation. This work concerns with slightly dissipative perturbations of Hamiltonian systems. In general, the phase space of a weakly dissipative system has different coexistent attractors. A major goal in applications is to determine the corresponding basins of attraction which, in particular, will allow to rigorously define a probability of capture by the attractors. In this direction, we deal with generic area preserving maps and we study the effect of a particular (but general enough) type of dissipative perturbation. We focus on a region around an elliptic fixed point of the conservative map and we describe the topological changes that the dissipation causes on the conservative resonant islands inside this region. For a fixed dissipative perturbation size, we distinguish three main regions in the phase space according to the topological properties of the “resonant” structures which survive the dissipation. We also include several considerations and results on the probability of capture by the attractors.

I. INTRODUCTION

This paper deals with the problem of describing the behavior of weakly dissipative systems. Physically, this type of dynamical systems arises when considering effects such as friction or medium resistance, providing in those cases a more accurate approach to the real dynamics. In celestial mechanics, for instance, weakly dissipative dynamics arise when adding on the gravitational N -body problem the effect of dissipative forces as the interplanetary media drag, tidal torques, the Yarkovsky effect, etc.⁵ Also the effect of dissipative forces is relevant when determining transfer orbits us-

ing a low thrust transfer, where the weak dissipation comes from the loss of mass during the maneuver, or using a solar sailing transfer, where the nonperfect mirror effect of the sail adds a small dissipation on the system, beyond the fact that sailing with arbitrary angle of the sail can move the system away from Hamiltonian (see Ref. 9).

More specifically we study a dissipative perturbation of a one parameter family of area preserving maps (APMs) $F_\nu: \mathcal{U} \rightarrow \mathbb{R}^2$, $\nu \in \mathcal{I} \subset \mathbb{R}$, $\mathcal{U} \subset \mathbb{R}^2$, analytic with respect to the Cartesian variables $x, y \in \mathcal{U}$ for all $\nu \in \mathcal{I}$. We assume that F_ν has one fixed elliptic point in $\mathcal{U}$ for all $\nu \in \mathcal{I}$ (it can be maybe parabolic for concrete ν values) and, without losing generality, the fixed point is considered to be the origin $E_0 = (0, 0) \in \mathcal{U}$.

We denote $\text{Spec}(DF_\nu)(E_0) = \{\lambda, \lambda^{-1}\}$, and we consider $\lambda = \exp(2\pi i \alpha)$ with $\alpha = \alpha(\nu)$ and, generically, $\alpha'(\nu) \neq 0$. We shall assume that we are interested in the dynamics close to the $(q:m)$ -resonance for $q, m \in \mathbb{N}$, with $1 \leq q < m$, $\gcd(q, m) = 1$, and that there is some range of parameters $\mathcal{I}_0 \subset \mathcal{I}$, such that the map F_ν exhibits the $(q:m)$ -resonant islands for all $\nu \in \mathcal{I}_0$. Then, one can write $\alpha = q/m + \delta$ with $\delta \in \mathbb{R}$. In this way, for a fixed $q/m \in \mathbb{Q}$, we can consider a one parameter family F_δ which is completely equivalent to the one parameter family F_ν . From now on, we shall focus on the family F_δ for arbitrary q and m given as above. In Ref. 25 the authors justified, in the conservative case, the possibility of study the relevant $(q:m)$ resonances isolated from the others in a “first” dynamical approximation.

General dissipative perturbations of a map F_δ could give rise to a large variety of different topological changes in the phase space which complicate the theoretical study (see Ref. 4). Therefore, we simply consider a radial dissipative perturbation, that is, we restrict ourselves to maps of the form

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$$F_{\delta,\epsilon}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = (1-\epsilon)F_{\delta}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right), \quad (1)$$

where $0 \leq \epsilon \ll 1$ denotes the dissipation parameter. We note that if the dissipation depends on (x, y) in a smooth way, most of the arguments used along this work can be adapted. For a general overview on weakly dissipative dynamics, the reader is referred to Ref. 2.

Despite the restriction on the dissipation in Eq. (1), the studied maps present a rich phenomenology. Among many other, a physical motivating example to study the dynamics of weakly dissipative maps with constant Jacobian could be the dissipative version of the standard map which arises when modeling the motion in a bouncing ball system if the collisions are assumed to be close to elastic (see Ref. 11).

When studying the evolution from a conservative case to a dissipative one, the phase space suffers some changes that can be understood in terms of the destruction of topological structures such as homoclinic points and periodic orbits. Obviously, this transition depends on the conservative family $F_{\delta,0}$. For instance, if we consider an integrable twist, $(\theta, r) \mapsto (\theta + \alpha(r), r)$ in polar coordinates, then for $\epsilon > 0$ the dynamics of Eq. (1) collapses, in a trivial way, to the origin. Nevertheless, this is not true for a general APM because it is expected to have some periodic orbits outlasting the dissipation effect and organizing the phase space into resonant strips. In particular, as no rotational invariant curves can survive to the dissipation effect, a set of points should be captured by the focus periodic orbit of each surviving resonance. The barriers of the domain of attraction depend on the position of the branches of the invariant manifolds of the corresponding hyperbolic periodic points.

We are especially interested in the behavior of the two-parameter family $F_{\delta,\epsilon}$ when approaching to the conservative case. Observe that, in the conservative case, the set of points with ω -limit E_0 has zero Lebesgue measure (recall that the ω -limit of $x \in \mathbb{R}^2$ is the set $\omega(x) = \{y \in \mathbb{R}^2 \text{ s.t. } \exists \{n_k\}_{k \in \mathbb{N}}, n_k \nearrow +\infty, \lim_{k \rightarrow +\infty} F_{\delta,0}^{n_k}(x) = y\}$). Obviously, this set has finite positive measure for $\epsilon > 0$ if we restrict the domain of initial conditions to a finite domain around the fixed perturbed elliptic point (note that it becomes a focus). An important question to understand near conservative dynamics is which is the limit of this measure when $\epsilon \searrow 0$. In a more specific way, let $\mathcal{C}$ be a rotational invariant curve around E_0 of the conservative map $F_{\delta,0}$ and denote by $\mathcal{A}$ the domain bounded by $\mathcal{C}$. Consider, for δ and ϵ given, the set

$$\Gamma(F_{\delta,\epsilon}) = \{(x, y) \in \mathcal{A} \mid \omega(x, y) = E_0\},$$

where $\omega(x, y)$ denotes the ω -limit of $(x, y) \in \mathbb{R}^2$. Then, the mentioned limit can be expressed as

$$\lim_{\epsilon \rightarrow 0} \mu(\epsilon) = \lim_{\epsilon \rightarrow 0} \frac{\text{meas}_L(\Gamma(F_{\delta,\epsilon}))}{\text{meas}_L(\mathcal{A})}, \quad (2)$$

where meas_L denotes the usual Lebesgue measure of $\mathbb{R}^2$. This value depends on the diffeomorphism $F_{\delta,0}$ considered and on the invariant curve $\mathcal{C}$. Some values of this limit for the Hénon map are numerically illustrated in Sec. II, while in Sec. IV E (see also Ref. 28) we study analytically

the behavior of limit (2) for a generic map. We conjecture that, for any family of maps with the assumptions considered above, the limit when $\epsilon \searrow 0$ of the measure of the set of points which do not go to the origin E_0 , that is, $\lim_{\epsilon \rightarrow 0} \text{meas}_L(\Gamma(F_{\delta,\epsilon}))^c$, is the measure of the resonances in $\mathcal{A}$ of the conservative case plus the measure of the entrance strips bounded by the branches of the invariant manifolds of the resonances (see Fig. 2 and Sec. IV E for definition of the entrance strip). Moreover, for a given resonance and for δ small enough, we conjecture that the measure of the entrance strip is $K \times \text{meas}_L(\mathcal{A}_{\text{ext}})$, with $K = \mathcal{O}(\delta^{m/4-1})$ and where $\mathcal{A}_{\text{ext}}$ is the set of points in $\mathcal{A}$ located further from the origin than the islands of the m -order conservative resonance. In this paper, we provide some preliminary results supporting these conjectures.

Remark: Note that $\lim_{\epsilon \rightarrow 0} \mu(\epsilon) \neq \mu(0)$. For instance, if the conservative map is an integrable twist, then $\lim_{\epsilon \rightarrow 0} \mu(\epsilon) = 1$ but $\mu(0) = 0$.

Depending on the size of the dissipation acting on the system, a resonance can change its topological shape. According to this fact the following cases have to be considered.

- (1) *Strong dissipative perturbation.* If the dissipation is sufficiently strong, the resonance is destroyed in a saddle-focus bifurcation. By a saddle-focus bifurcation, we refer to the dissipative analog to the well-known saddle-center bifurcation or, in other words, to a Bogdanov–Takens (BT) bifurcation for diffeomorphisms when the unfolding parameters are on a suitable domain. A simple model retaining the essential features is given by $(x, y) \mapsto (x_1, y_1)$, where $x_1 = x + y_1$, $y_1 = y + \mu + y(\nu - Ax)$ with A finite and μ, ν small. A bifurcation appears for $\mu = 0$. For $\mu < 0$, two fixed points are created, one of them being a saddle and the other a stable focus if $\nu + A\sqrt{-\mu} < 0$. See Ref. 29 for the BT bifurcation for flows and Ref. 3 for an analysis of the discrete case and alternative formulations. Through the paper we refer simply to saddle-focus bifurcation to keep in mind that one of the points is a saddle and the other a stable focus.
- (2) *Medium size dissipative perturbation.* The resonance survives but does not have any associated homoclinic point because the dissipation has destroyed them. The resonance becomes a flow type one [see Fig. 6(d)].
- (3) *Weak dissipative perturbation.* If we consider a very small dissipation the resonance keeps the topological properties of the conservative case [see Fig. 6(a)]. Increasing the dissipation parameter, the resonance loses its homoclinic structure giving rise to different homoclinic tangles until becoming a flow type resonance with no homoclinic points [see Figs. 6(b) and 6(c)].

To describe the dynamics, it becomes necessary to quantify the dissipation effect in order to determine when a resonance changes its topology or it is destroyed. Note, however, that all the scenarios above coexist when considering ϵ fixed and small enough and we study the dynamics associated with the outlasting resonances (a finite number of them) in a ball of constant radius around the elliptic fixed point E_0 . That is,

for our purpose one has to consider the global effects of the dissipation on a fixed domain containing E_0 and not only the effects on a given resonance.

The paper is organized as follows. In Sec. II we introduce the radial perturbation of the Hénon map that will be considered as a paradigmatic example and for numerical computations. Using this map, we provide an overview of the main characteristics we expect to find in weakly dissipative dynamics. Section III is devoted to clarify the topological transitions taking place in the resonant islands when adding dissipation. After describing the situation in a generic resonance, we explain the evolution of the islands and meandering curves in a conservative reconnection scenario when adding a radial dissipation. In Sec. IV we carry out the study of flow type resonances by means of the normal form approach and a suitable interpolating Hamiltonian. In particular, we discuss on the dissipation parameter value for the destruction of a resonant island, the probability of capture into a resonance, and the limits of this approach when the map is close to conservative and homoclinic points appear. Finally, Sec. V adds some final conclusions and remarks to the theory developed.

II. A NUMERICAL EXPLORATION: WEAKLY DISSIPATIVE HÉNON MAP

Before the theoretical approach of the following sections, we want to illustrate the phenomenology that occurs in weakly dissipative maps around an elliptic fixed point. As an example consider the radial dissipative version of the well-known conservative Hénon map H_α (see Ref. 12) given by

$$H_{\alpha,\epsilon} \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto (1 - \epsilon) H_\alpha \left(\begin{pmatrix} x \\ y \end{pmatrix} \right), \quad H_\alpha \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = R_{2\pi\alpha} \begin{pmatrix} x \\ y - x^2 \end{pmatrix}, \quad (3)$$

where $R_{2\pi\alpha}$ denotes the rigid rotation of angle $2\pi\alpha$. The map $H_{\alpha,\epsilon}$ is equivalent to the well-known dissipative Hénon map,

$$H_{a,b} \left(\begin{pmatrix} u \\ v \end{pmatrix} \right) \mapsto \begin{pmatrix} 1 - au^2 + v \\ -bu \end{pmatrix}, \quad (4)$$

by taking $a = (1 - \epsilon)^2 (\cos^2(2\pi\alpha) - 2 \cos(2\pi\alpha))$ and $b = (1 - \epsilon)^2$.

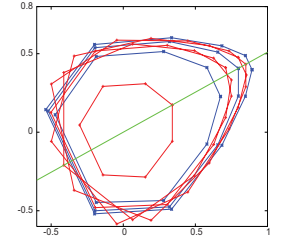
The dissipative version of Hénon map (4) was introduced by Hénon in Ref. 13, where it was observed the existence of a strange attractor for the values $a=1.4$ and $b=-0.3$. However, we are interested in values of b close to 1. A numerical exploration of this map when it is close to the conservative case can be found in Ref. 24.

We try now to describe the effect of the dissipation on a conservative resonance. As we have briefly explained in Sec. I, there are different scenarios to consider, depending on the effect of the dissipation on the topology of the resonance.

First of all, we examine the destruction of the resonances. Table I shows different values of ϵ and the resonances that are destroyed as the parameter increases. The robustness of a resonance to the dissipation effect depends on the “width” of the resonance, which in turn depends on

TABLE I. (Color online) For $\alpha=0.15$, the table contains the values of the dissipation parameter ϵ that correspond to the destruction of a resonance. The picture shows the location of the resonances (1:7), (1:8), (2:17), (1:9), (2:19) and (1:10) in the phase space for the conservative case. As usual, $B_0(0.27)$ denotes the ball of radius 0.27 centered at the elliptic point $E_0 = (0,0)$.

$\log_{10}(\epsilon)$	Resonances destroyed
-6	All inside $B_0(0.27)$
-4.569	(2:19)
-4.625	(1:7)
-3.456	(1:8)
-3.297	(1:9)



the order, on the distance from the resonance to the origin, and on the local twist properties (Sec. IV A). We observe that there exists a neighborhood of the elliptic point $E_0 = (0,0)$, such that all resonances disappear. This can be well observed for small values of α . In this case, the conservative map $H_{\alpha,0}$ can be accurately interpolated by a flow in a large region surrounding E_0 , meaning that the resonant islands are tiny ones, i.e., the map is a close-to-integrable map in a suitable region around E_0 (see also comments below concerning Fig. 4).

For the surviving resonances, one has to distinguish between two topologically different structures: it could be that the dissipation destroys the homoclinic points of the conservative case or it could happen that some of them still exist. This was already noticed in Ref. 15.

In Fig. 1 it is shown how the shape of a resonance changes when increasing the parameter ϵ . To analyze how homoclinic points are destroyed by the effect of the dissipation we focus on a concrete seven-periodic hyperbolic point of Eq. (3). As a general conclusion, different types of trelises can be found in weakly dissipative maps. See Sec. III A for a more detailed description of the process of destruction of homoclinic points on a resonance by means of tangencies between the invariant manifolds. In fact, the reader can verify that the situations illustrated in Fig. 1 from left to right are topologically equivalent to the cases b, c, and d in Fig. 6, respectively.

If there are not homoclinic points in a given resonant island and the separatrices bounding this island are far enough from homoclinic tangency, then the dynamics is like the one generated by a flow. More concretely, the phase space looks like that of a pendulum with damping and an additional constant torque, as described by the equation $\ddot{x} = -\sin(x) - \epsilon_1 - \epsilon_2 \dot{x}$ for suitable $\epsilon_1, \epsilon_2 > 0$ (see Fig. 10 and Sec. IV for proofs). We shall refer to this type of model as “pendulum with torque” for shortness. In that case, the invariant manifolds of the hyperbolic points of the island delimit a strip such that points inside it are captured by the island. The probability of capture is related to the width of this strip. In the three first plots of Fig. 2, these strips are shown for two different resonances. We observe in the

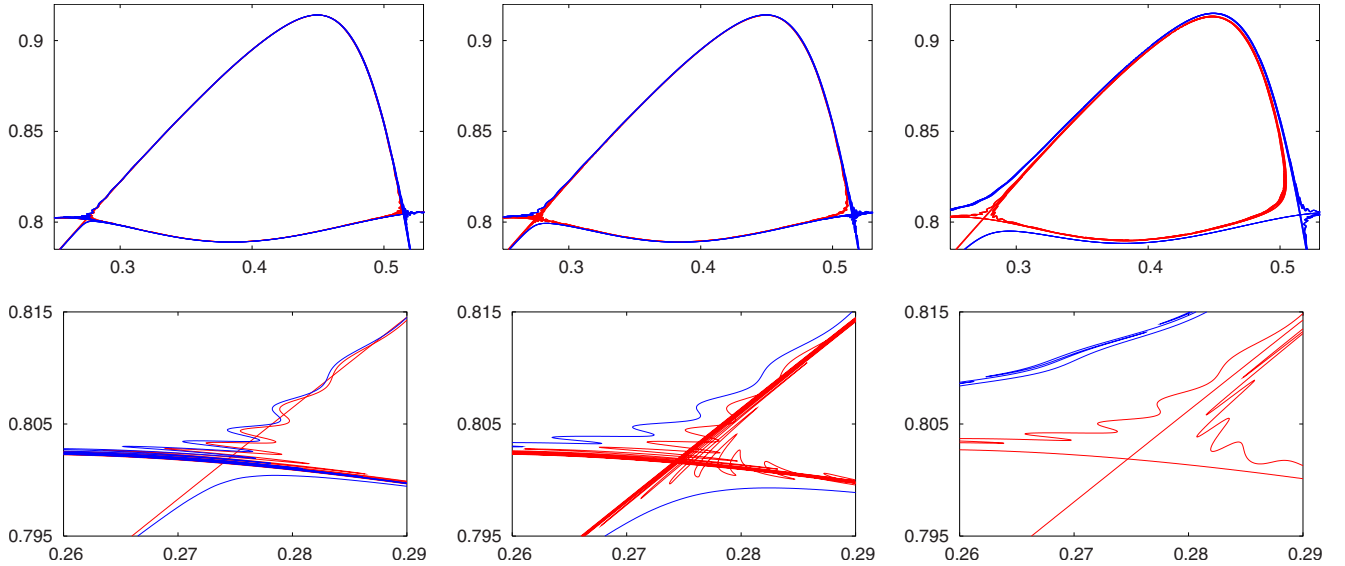


FIG. 1. (Color online) For $\alpha=0.17$ the resonance (1,7) is depicted. From left to right, the dissipative parameters are $\log_{10}(\epsilon)=-6, -5.4, -4$, respectively. Dark gray (blue) and light gray (red) lines show the manifolds of two consecutive points of the same periodic orbit. Top: Topology of the islands. Bottom: Magnification of the above figures around the hyperbolic periodic point. (We have applied a rotation of angle $\pi/6$ on the phase space to display the figures.)

figures how these strips fold when crossing another resonance due to the intersection of the invariant manifolds of the hyperbolic periodic points of different chains of resonant islands.

In order to understand these foldings, the last three illustrations of Fig. 2 show the invariant manifolds of the periodic points. Concretely, the two branches of the stable manifold of the order eight resonance associated with the hyperbolic point H_8 [$W_{8,l}^s$ and $W_{8,r}^s$ (blue and red)] and one of the branches of the stable manifold of the order eight resonance associated with the hyperbolic point H'_8 [$W_{8',l}^s$ (magenta)] are shown. Also one of the branches of the unstable manifold of the order nine resonance which goes away from the hyperbolic point H_9 [W_9^u (green)] is displayed. We observe the folds of the unstable manifold (bottom left figure) that are produced in order to cross the inner resonance. The intersection of the unstable manifold of the outer resonance, which corresponds to the (1:9) resonance, with the stable manifolds of the inner one can also be observed (bottom right figure). The creation of a horseshoe in this situation explains the difficulty of determining whether a concrete point is going to be captured.

For a fixed α , the probability of being captured by a resonance depends on ϵ . For $\epsilon=0$ there is an infinite number of resonant islands. For any $\epsilon>0$ only a finite number of resonant chains from the conservative case survive. The larger the dissipation parameter ϵ , the smaller the number of resonances surviving the perturbation effect. An island of the conservative case evolves according to the pattern shown in Fig. 1 (see also Fig. 6 below, for details) as ϵ increases. Moreover, for a suitable parameter ϵ , the periodic orbits related to the resonance (evolving from the Birkhoff hyperbolic-elliptic orbits of the conservative case) suffer a saddle-focus bifurcation and the resonance itself is destroyed. This explains why, as ϵ goes to zero, more and more points are captured by the successively created resonances, a

fact that can be observed in Fig. 3, where we represent the fraction Ψ of points captured by some resonance with respect to the total number of initial points considered in the iteration (and which do not escape) for different values of ϵ . The parameter α of the map has been taken $\alpha=0.15$. We have considered an initial mesh of pixel size 5×10^{-4} on the domain $[-1, 1] \times [-1, 1]$, but as points outside the ball of radius $r_{\text{ext}}=0.97$ escape quickly, we restrict our attention to the points in that ball. Then we consider that an initial point does not go to the origin if after a suitable number of iterates of the map, ranging from 2^{12} to 2^{21} depending on ϵ (see remark below), no one of them is inside the ball of radius 0.27. This value has been selected according to Table I since for $\alpha=0.15$ and all the values of ϵ used in the computations, all the resonances inside this ball are destroyed. We observe that as ϵ increases the number of points captured by some resonance decreases, as expected. For $\epsilon \sim 10^{-6}$ a large number of resonances survive to the dissipative perturbation and they capture a little bit more than one tenth of the initial points that do not escape. Moreover, the shape of the figure can be explained in terms of the destruction of the resonances given in Table I. The complementary part $1-\Psi$ tends to the origin.

Remark: Assume $\epsilon=10^{-k}$ and $r \approx 1$. After n iterations of the map, the displacement in the radial direction is, on the average, $\mathcal{O}(n\epsilon)$. To reach the origin, one should have $n\epsilon=\mathcal{O}(1)$ and, hence, we need to perform $n \approx 10^k$ iterates. Taking into account that neither the radius nor the radial displacement are constant, we perform $n=2^s \approx 2 \times 10^k$, that is, $s=[(k+1)\log_{10}(2)]+1$ iterates to be confident on the results. For $k=3.5$, one has $s=12$, while for $k=6$, one has $s=21$. This justifies the values used in computations.

Finally, it will be useful to clarify how the probability of capture changes as a function of α . Figure 4 represents the ratio of points captured at some resonance with respect to the number of total points that do not go to infinity under itera-

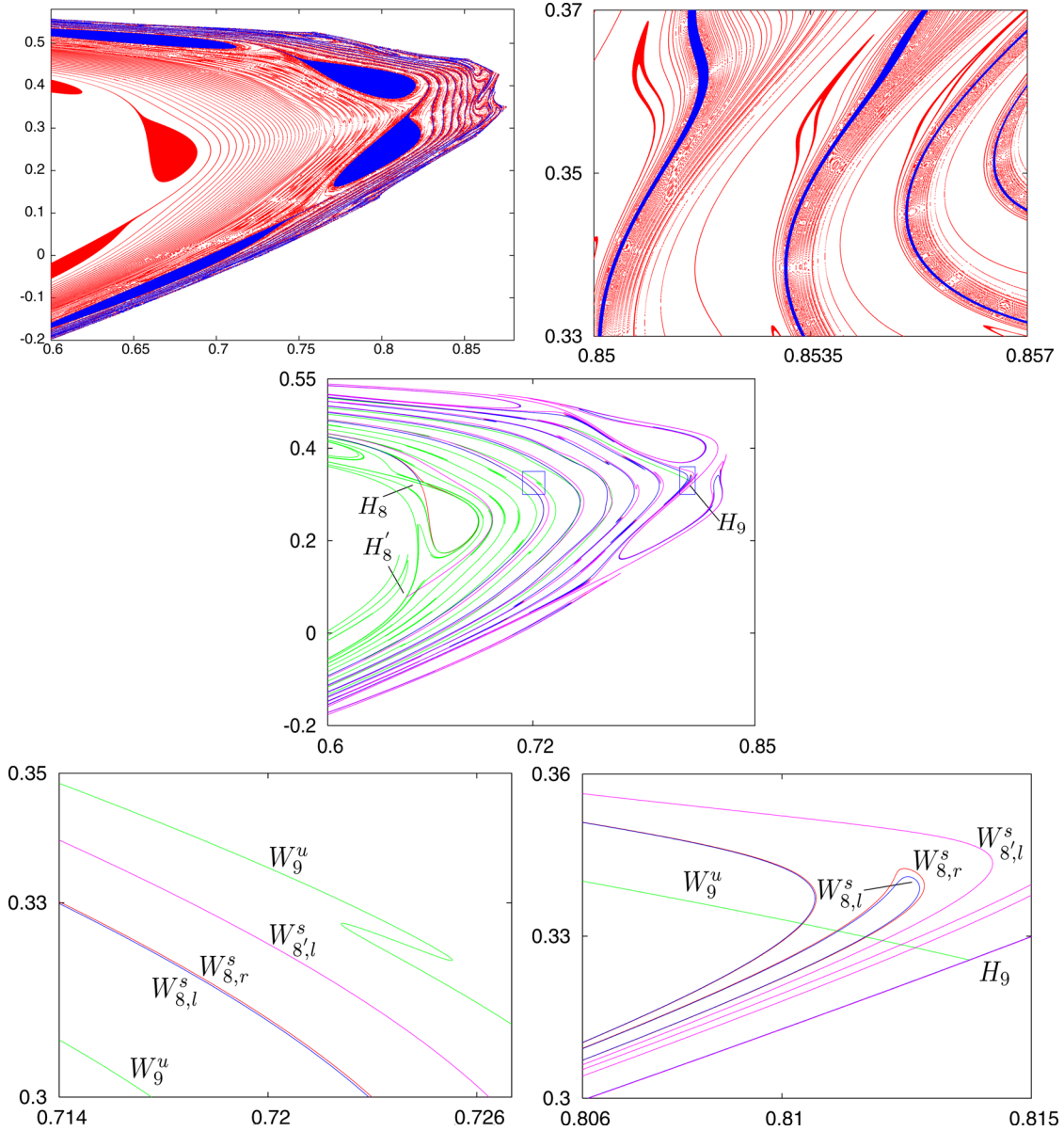


FIG. 2. (Color online) From left to right and from top to bottom: In the first two illustrations the strips of capture into the resonances of orders 8 [light gray (red)] and 9 [dark gray (blue)] are shown. The right figure of the top row and the left picture of the bottom row are magnifications where the folds of the strip of the inner resonance when crossing the outer one can be seen. Last three illustrations show the folds of the invariant manifolds when crossing another resonance. Branches of the stable manifold of the eight order resonance [$W_{8,l}^s$ (blue), $W_{8,r}^s$ (red), and $W_{8,l'}^s$ (magenta)] and a branch of the unstable manifold of the nine order one [W_9^u , (green)] are shown. The two figures of the bottom are magnifications of the central one. The parameters used to obtain these figures are $\epsilon=10^{-4}$ and $\alpha=0.15$. For these values of α , ϵ all the points outside the strips (and which do not escape) have the origin as ω -limit.

tion. The parameter α is moving on the horizontal axis. This representation is done for different values of the dissipation parameter. In order to get one of the curves (ϵ fixed) for each value of α , we have chosen a set of initial points on the symmetry axis $y=\tan(\pi\alpha)x$, equidistributed on an interval centered at the origin, and with amplitude two times the distance to the hyperbolic fixed point. We consider that an initial condition is captured by the origin if any iteration is inside the ball of radius $10^{-3}d_h$, where d_h denotes the distance to the hyperbolic fixed point from the origin. It is important to remark that the probability of capture is well-defined for $\alpha \neq 0.5$ since it corresponds to the case $a=3(1-\epsilon)^2$ in Eq. (4). In particular, the shape of Fig. 4 gives a good evidence that low order resonances, which are the

biggest ones in the phase space, are the ones which capture more points.

Some comments are necessary to explain what is observed in Fig. 4 and to make clear the relation it has with the evolution (as α changes) of the domain of stability of the conservative map (see Ref. 27 for a phenomenological description and Ref. 26 for a formal definition of stability domain).

- We recall that we look for the ratio of points that are captured by the different islands among those points that do not escape under iteration of the map. To understand which points escape requires to look at the global geometry of the phase space and, hence, at the invariant manifolds of

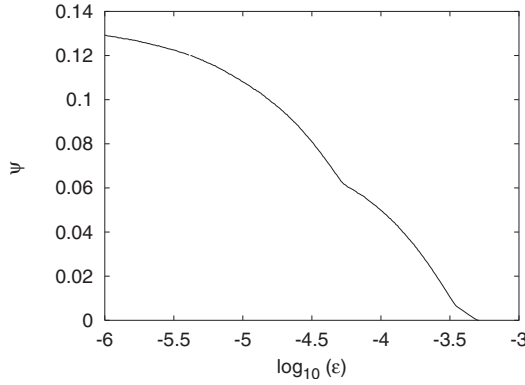


FIG. 3. In the figure one represents on the vertical axis the ratio between the number of points captured by the stable foci of some resonances and the number of points that remain under iteration inside the ball of radius $r_{\text{ext}}=0.97$.

the hyperbolic fixed point $p_{h,\epsilon}$ of the dissipative Hénon map. One has $p_{h,\epsilon} = ((1-2c\mu+\mu^2)/\mu s, (\mu-c)(1-2c\mu+\mu^2)/\mu s^2)$, where $c=\cos(2\pi\alpha)$, $s=\sin(2\pi\alpha)$, and $\mu=1-\epsilon$. The bifurcation diagram of the dissipative Hénon map was described in Ref. 14. The change of variables $x_{\text{new}}=ax$, $y_{\text{new}}=a(1-ax^2+y)$ reduces map (4) to the map $(x,y) \rightarrow (y, -\epsilon x + \mu - y^2)$ used in the cited reference. For completeness, we represent in Fig. 5, left, the first tangency line of the bifurcation diagram in the coordinates we use. Notice that for any value of $\alpha \in (0, 1/2)$, for ϵ large enough, the line of first homoclinic tangency is necessarily crossed. This means that, for ϵ large enough, the homoclinic points are destroyed and the global phase space geometry is topologically equivalent to the one shown in Fig. 5, right. The computations of the bifurcation diagram in Ref. 14 have been recently improved by Rabassa and his advisor J. C. Tatjer, who provided us the data corresponding to Fig. 5 left, see Ref. 19.

- For values of $\alpha < 0.1$ the conservative Hénon map is close to integrable in a domain $\mathcal{D}$ bounded by rotational invariant curves close to the stable/unstable manifolds $W^{u,s}(p_h)$ of the hyperbolic fixed point $p_h = (2 \tan(\pi\alpha), 2 \tan^2(\pi\alpha))$. For $\alpha=0.1$ the angle of the splitting of these stable/unstable separatrices measured at the primary homoclinic point on the symmetry axis is $\approx 1.195 \times 10^{-5}$ and this angle of the splitting decreases exponentially with respect to α for values of $\alpha < 0.1$. Indeed, in Refs. 25 and 26, it was observed that for α small, and after a suitable change of variables, the map can be accurately interpolated in $\mathcal{D}$ by a fish-like Hamiltonian $\mathcal{H}$, hence being close to the integrable dynamics that generates the time-1 map of $\mathcal{H}$. This explains, as observed in Fig. 4, why a small amount of dissipation destroys the tiny islands of the conservative map and all the points end up at the origin.
- The (1:4) resonance of the conservative Hénon map is degenerated: to describe the dynamics around the origin for values of α close to $1/4$, one has to consider not only the third order resonant term in the normal form but also the fifth order resonant ones. A detailed analysis of this bifurcation shows that the invariant manifolds related to the four-islands go to a distance $\mathcal{O}(\delta^{1/4})$ (in the original Car-

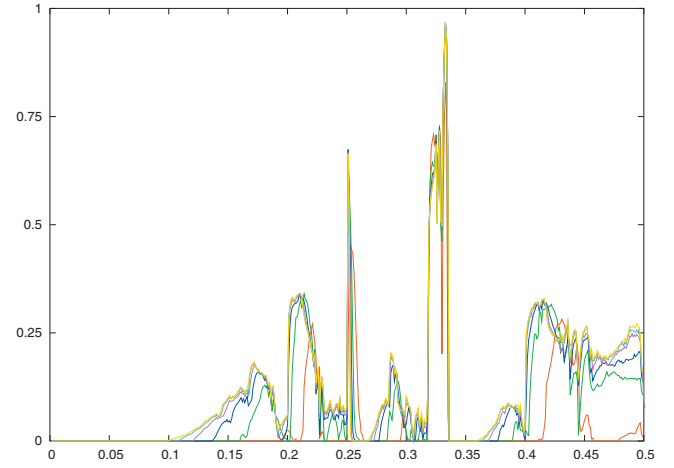


FIG. 4. (Color) For each α (horizontal axis), the figure shows the fraction of points (in the vertical axis), among the ones that do not escape, that are captured by the foci of the islands of the resonances. The different curves are obtained for different values of ϵ : 10^{-2} (red line), 10^{-3} (green line), 10^{-4} (blue line), 10^{-5} (violet line), 10^{-6} (sky blue line), and 10^{-7} (yellow line).

tesian coordinates of the map) instead of $\mathcal{O}(\delta^{1/2})$ as in the generic case.²⁵ Consequently the (1:4) islands are larger than in the generic case and we expect them to capture more points. In Fig. 4 we observe a large peak for $\alpha \approx 1/4$.

- Generically, for $\alpha=1/3$ the elliptic points of the three-periodic orbit remain at a finite distance while the hyperbolic ones collide at the origin which is suffering the local bifurcation. However, there is a nongeneric property of the (1:3) islands of the conservative Hénon map: at the same value $\alpha=1/3$, where the bifurcation at the origin takes place, the elliptic points of the three-islands suffer a period doubling bifurcation.²⁵ Then, for $\alpha > 1/3$ the three-islands are quickly destroyed. This explains why for α between, approximately, $1/3$ and 0.36 there is a gap in Fig. 4, meaning that all nonescaping points reach the origin.
- In Fig. 4 we can distinguish clearly the main resonances by their capture properties. The most relevant ones are the (1:3), the (1:4), the (1:5), and the (2:5), but also the relative effect of the (1:6), the (1:7), the (2:7), and the (3:7) can be well observed.
- The irregularities in the peaks can be understood by the continuous evolution of the islands thrown away from the stability domain. See Ref. 27 for an explanation of this phenomenon.

To sum up what is observed from the numerical experiments performed, we sketch in Table II the different cases to be considered in a general map for a fixed radial dissipation strength determined by ϵ (see the different scenarios described in Sec. I and compare with the sketch in Fig. 12). In the next sections we rigorously derive such a splitting of the phase space into different regions according to the type of islands contained inside. We remark that in a region far enough from the elliptic point, we can find resonances with homoclinic tangle, but, generically, they always coexist with flow type ones.

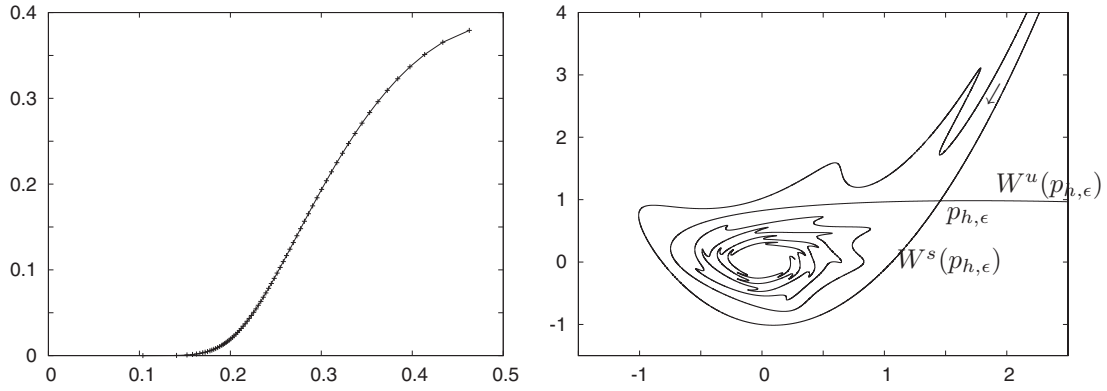


FIG. 5. Left plot: x-axis: value of α , y-axis: ϵ . The line represents the values of ϵ corresponding to a homoclinic tangency between $W^{u,s}(p_{h,\epsilon})$. In particular, for ϵ greater than 0.5 there are no homoclinic points and, for all $\epsilon > 0.5$, the manifolds behave as the ones represented in the right plot. Right plot: We represent $W^{u,s}(p_{h,\epsilon})$ for $\alpha=0.2$ and $\epsilon=0.05$. Note that many points escape under iteration but there is a strip of points that are captured either by the focus E_0 at the origin or by a resonant chain of islands surrounding E_0 (the arrow indicates the strip of capture). Observe also the folds of the manifolds.

III. RADIAL WEAKLY DISSIPATIVE DYNAMICS: A TOPOLOGICAL GLOBAL DESCRIPTION

In Sec. II we have numerically illustrated different geometrical scenarios that appear as a consequence of the weakly dissipative perturbation. In the present one we provide a topological description in terms of the invariant manifolds of some situations observed when adding dissipation. First, we focus on the evolution of a resonant island when adding dissipation. Next, we will discuss the effect of the dissipation on a reconnection scenario. However, our aim is not just to provide qualitative information as the one given in this section but also quantitative information as far as possible. To this end, in the following sections and also in Ref. 28, we develop different strategies adapted to each of the different topological scenarios that produces the radial dissipative perturbation in a resonant island. The different topological cases are detailed below.

A. Transition from a conservative resonance to a dissipative one

We try to describe the transition from the conservative case to a weakly dissipative one. In particular, we focus our attention on understanding how different splittings are successively destroyed for a concrete island in a resonant chain of a given APM. Below we will refer to inner and outer splitting of an island, see Ref. 25 for a rigorous definition. Informally, the inner/outer splitting is the one measured on a primary homoclinic point in the lower/upper part (i.e., below/above the elliptic point E) given in Fig. 6(a). This informal idea is enough for our purposes (see also Sec. IV F 1).

Figure 6 shows one of the possible scenarios of the evolution of an island with respect to the dissipation parameter ϵ . Here we assume that the outer splitting is larger than the inner one, as it is generically the case for resonant islands of APMs in a neighborhood of the elliptic fixed point (see Ref. 25 and also Sec. IV F 1 for a brief explanation). Notice that if the inner splitting is larger than the outer one, the evolution would be slightly different.

The elliptic point related to the resonant conservative island becomes a focus under the effect of dissipation. In any case it is denoted by E in Fig. 6. The fixed point E_0 located at the origin (an elliptic point in the conservative case, a focus in the dissipative one) is assumed to be below, at some distance, of the resonant island shown. For ϵ small, the island is like a periodically perturbed pendulum (a), showing an homoclinic tangle similar to that of the conservative case $\epsilon=0$. Then, for larger ϵ the manifolds creating the inner splitting separate from each other but one of the splittings between an “inner manifold” and an “outer one” still remains (b). This splitting is still present when the outer splitting is destroyed (c). Finally, the dissipation destroys the splittings and the resonant island is a flow type one, like the one that generates a pendulum with torque (d).

Let H_+ and H_- be two consecutive hyperbolic periodic points of the same resonance strip and let W_+^u , W_+^s , W_-^u , and W_-^s be the branches of the unstable and the stable manifolds associated with the points H_+ and H_- , respectively (see Fig. 6). In order to clarify the scenario shown above, Table III contains the possible situations in the evolution of the resonance. It contains also the tangencies $\epsilon=\epsilon_1$, ϵ_2 , and ϵ_3 , that produce different shapes on the resonance. We note the existence of an impossible tangency transition as it is empha-

TABLE II. Different regions to be studied for a generic radial weakly dissipative map for a fixed dissipation parameter ϵ .

Position	Type of resonances
Close to the elliptic point E_0	No resonances survive
At some distance from the elliptic point E_0	Resonances of flow-like type
Far from the elliptic point E_0	Resonances with homoclinic tangle

TABLE III. Possible relative positions of the invariant manifolds related to two consecutive hyperbolic points H_+ and H_- of the resonance. In the table “i” means that there is transversal intersection and “t” that the two corresponding manifolds have homoclinic tangency. The symbol “-” means that no intersection between the manifolds exists. Observe that the case (*) is impossible to achieve. Last row shows the correspondence with Fig. 6.

ϵ	0	>0	(*)	ϵ_1	ϵ_2	ϵ_3	Large
W_+^u, W_-^s	i	i	i	i	i	t	-
W_+^u, W_+^s	i	i	i	i	i	i	t
W_-^u, W_-^s	i	i	t or -	-	-	-	-
W_+^s, W_-^u	i	i	i	t	-	-	-
Figure 6		(a)		(b)	(c)		(d)

sized in the table. This evolution scenario is the one observed for most of the resonances for the Hénon map (see Fig. 1). Moreover, for a generic radially perturbed APM, the same evolution scenario is expected to be the one which holds, at least, for resonant islands located close enough to E_0 because, for these islands, the outer splitting is generically larger than the inner one (see Ref. 25 and Sec. IV F 1).

B. The dissipative effect on a reconnection scenario

When studying globally the phase space of an APM, maybe at some distance of the elliptic fixed point E_0 , it can happen to find a radius such that the twist vanishes if we consider a twist approximation of the map. If this situation occurs inside the domain foliated by invariant curves (in the integrable case), a sequence of global bifurcations (that will be referred to as reconnection scenario) gives rise to the so-called meandering curves.^{8,22} Reconnection scenarios are generic for families of APMs. The twist related to the conservative Hénon map, given by Eq. (3) with $\epsilon=0$, vanishes for a value of α between $1/3$ and $1/4$. Since the conservative Hénon map provides a dynamical model for islands in general systems, it is expected to find them in many systems. In celestial mechanics, such a scenario is observed, for example, in the restricted three body problem for small values of the total mass parameter μ and also in the Hill problem (see Ref. 23).

It is remarkable the role that this global scenario plays in plasma physics to enhance the confinement in a tokamak (see Ref. 6 and references therein). Note that, due to the high velocity of the confined plasma, any small dissipative effect plays a relevant role in the dynamics. Similarly, in a more general setting, one has to look at the long time dynamics on

a reconnection scenario to study transport properties of Rossby waves⁷ (think, for instance, in the kinetic models of the fluid exchange process across the Gulf Stream²⁰). The presence of any small dissipative perturbation in such a processes can change a lot the geometry. Next, we analyze this effect in the particular case of a radial dissipation. Note the topological character of the discussion below, in contrast with the quantitative approach of the following sections.

Figure 7 shows a dissipative reconnection scenario for the (3:10) resonance of Hénon map (3). The corresponding values of α are 0.299 543, 0.299 544, 0.299 545, and 0.299 546, and the dissipation parameter is fixed to be $\epsilon=10^{-6}$.

We observe different scenarios in Fig. 7 and to clarify them we consider the dissipative model,

$$\begin{aligned}\dot{x} &= -b + y^2, \\ \dot{y} &= \sin(x) - \epsilon y_0 - \epsilon y.\end{aligned}\quad (5)$$

For $\epsilon=0$ the above equations simplify to the Hamiltonian model introduced in Ref. 22 for the analysis of the bifurcations in a (conservative) reconnection scenario. The corresponding Hamiltonian is given by $\mathcal{H}(x, y) = -by + y^3/3 + \cos(x)$. For $b = b_{\text{crit}} = (3/2)^{2/3}$, the hyperbolic points have heteroclinic connections. For $b > b_{\text{crit}}$, there is no interaction between the two chains of islands. For $0 < b < b_{\text{crit}}$, there are meandering curves and for $b \leq 0$ the islands disappear.

The parameter y_0 in Eq. (5) corresponds to the average distance (average radius) where the reconnection scenario is located. It is relevant just for the dissipative effect, since the radial dissipation depends on the distance, and in the study below will be fixed (in all the illustrations $y_0=5$).

For a fixed $b > b_{\text{crit}}$ there are defined (at least for $0 \leq \epsilon < \epsilon_c$ for small enough ϵ_c) two resonant, say, top and bottom, chains of islands. Let us denote by p_1^h (p_{-1}^h) the hyperbolic points of the top (bottom) chain of islands. The corresponding elliptic/focus points will be denoted as p_1^e (p_{-1}^e). We will denote by $W_+^u(p_1^h)$ the branch of the unstable manifold of p_1^h that in the conservative case $\epsilon=0$ propagates to the right of p_1^h in the x -coordinate. Analogously, by $W_-^u(p_1^h)$, $W_+^s(p_{-1}^h)$, $W_-^s(p_{-1}^h)$, $W_+^u(p_{-1}^h)$, $W_-^u(p_{-1}^h)$, $W_+^s(p_{-1}^h)$, and $W_-^s(p_{-1}^h)$, we will refer to the corresponding stable/unstable manifolds (s/u) of the point p_1^h/p_{-1}^h that evolve to right/left (+/-) with respect to the point p_1^h/p_{-1}^h itself (see Fig. 8 left for notation).

Clearly, the dissipation will destroy the invariant curves between the islands and hence a heteroclinic connection is expected to occur. In the weakly dissipative regime, the islands have been replaced by basins of attraction of stable

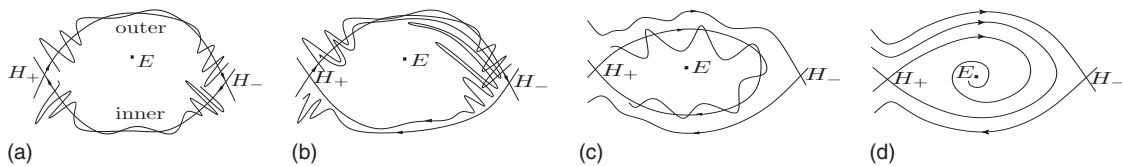


FIG. 6. Transition from a conservative resonance [case (a) is “almost” a conservative island] to a flow type resonance [case (d)]. The domain of attraction is clearly determined by the position of the invariant manifolds of the hyperbolic points of the resonance.

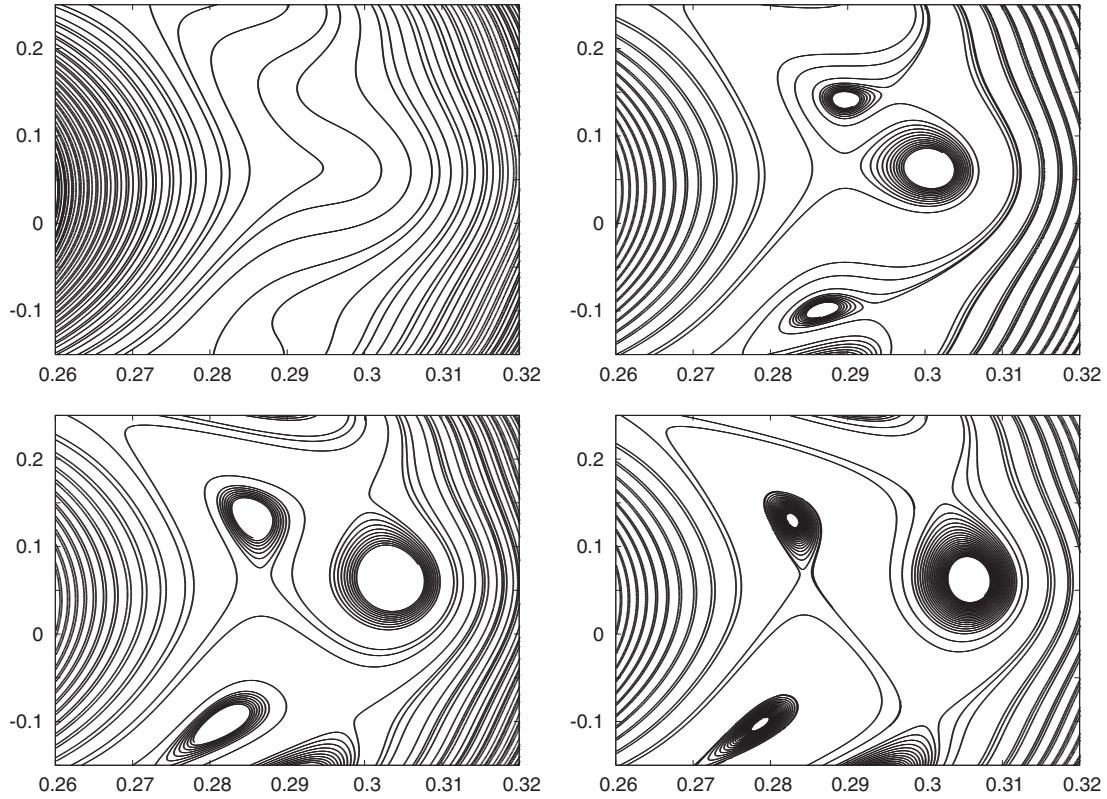


FIG. 7. Hénon map (3) with $\epsilon=10^{-6}$. The values of α are $\alpha=0.299\,543$ (top left), $\alpha=0.299\,544$ (top right), $\alpha=0.299\,545$ (bottom left), and $\alpha=0.299\,546$ (bottom right).

foci. For shortness, from now on in this section and in the following ones, we will still use the name “island” to refer to the dissipative structures evolving from the conservative islands, with the hope that no confusion will be produced. Figure 8 left shows the corresponding situation for $\epsilon=0.01$. When adding dissipation the positions of the manifolds change. Figure 8 center shows the situation for $\epsilon=0.04$. The manifolds $W_-^u(p_1^h)$ and $W_+^s(p_{-1}^h)$ exchange the position. The change in the relative position of the invariant manifolds allows to travel from top to bottom directly. For greater values of the dissipation, the islands evolve until they are destroyed by the dissipation effect (saddle-focus bifurcation). This can be deduced from Fig. 8, right, where the manifolds are shown for the value $\epsilon=0.15$ and where we see how the islands have been reduced by the effect of the dissipation.

Let us analyze the effect of approaching each other the two chains of islands. To this end we fix the dissipation parameter ϵ (in the illustrations $\epsilon=0.04$) and we move the parameter b .

- For b relatively large, e.g., $b=2$, the situation is like the case of Fig. 8, left, as it is shown in Fig. 9, top left. It corresponds to the case illustrated in Fig. 7, bottom right.
- For b smaller, e.g., $b=1.5$, the resonant chains approach each other and the invariant manifolds $W_-^u(p_1^h)$ and $W_+^s(p_{-1}^h)$ exchange position as it was shown before in Fig. 8, right, which is reproduced in Fig. 9, top center. This case corresponds to the illustration 7, bottom left. Hence the effect of moving the resonances closer is almost equivalent to the effect of keeping them far away but adding dissipation.

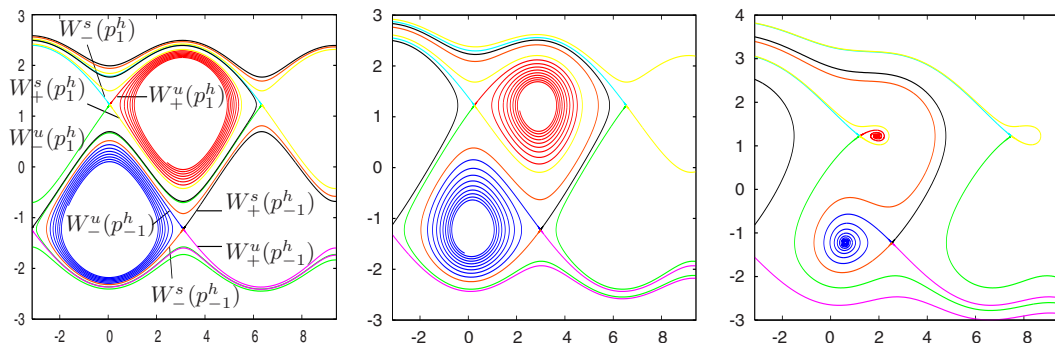


FIG. 8. (Color online) Representation of the invariant manifolds of the hyperbolic points p_1^h and p_{-1}^h of model (5) for $b=1.5$ and $y_0=5$. From left to right $\epsilon=0.01$, 0.04 , and 0.15 . See the change in the position of the invariant manifolds $W_-^u(p_1^h)$ and $W_+^s(p_{-1}^h)$.

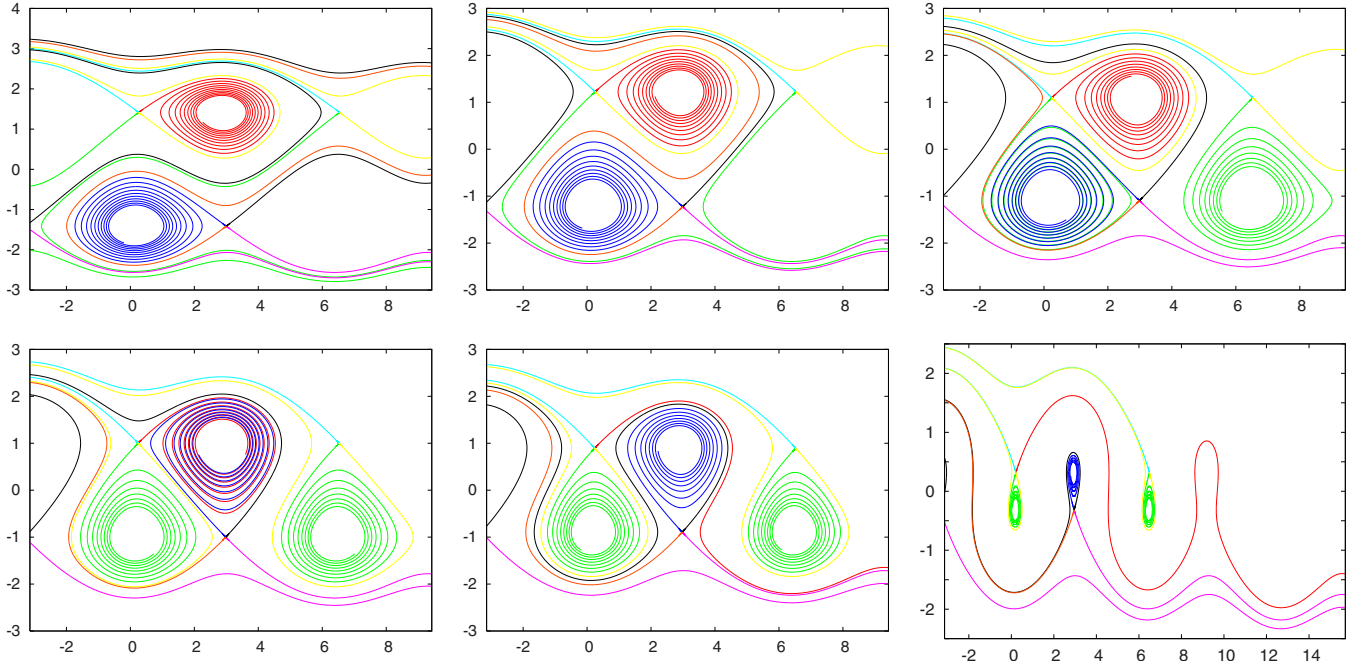


FIG. 9. (Color online) Model (5) for $y_0=5$ and $\epsilon=0.04$. From left to right and from top to bottom the values of b are $b=2, 1.5, 1.2, 1, 0.8$, and 0.01 , respectively. Observe the changes on the invariant manifolds of the hyperbolic points as b evolves (see text for details).

- However, for smaller values of b , e.g., $b=1.2$, the situation changes (see Fig. 9, top right). The first observed effect is that the invariant manifold $W_-^u(p_1^h)$, according to the notation introduced above and shown in Fig. 8 left, spirals around the point p_{-1}^e and has exchanged position with $W_-^s(p_{-1}^h)$.
- Taking b smaller, e.g., $b=1$ (Fig. 9, bottom left), we observe that the invariant manifold $W_-^u(p_{-1}^h)$ exchanges position with the invariant manifold $W_+^s(p_1^h)$. In particular, for $b=1.2$ (top right), it is observed that $W_-^u(p_{-1}^h)$ spirals around p_{-1}^e but for $b=1$ spirals around the point p_1^e .
- For values of b even smaller, e.g., $b=0.8$, the last global bifurcation concerning the manifolds is observed (see Fig. 9, bottom center). The invariant manifold $W_+^u(p_1^h)$ exchanges position with the invariant manifold $W_+^s(p_{-1}^h)$. We observe that for $b=1$, the manifold $W_+^u(p_1^h)$ spirals around the elliptic point p_1^e , while for $b=0.8$ it does not. On the other hand, the invariant manifold $W_+^s(p_{-1}^h)$ has to make a loop around the island around p_{-1}^e , in order to skip the entrance strip associated with this island and before going up far from the resonance chain for reverse time iterations. This explains the situation observed in Fig. 7, top right.
- Finally, for smaller values of $b \searrow 0$, we observe no changes in the position of the manifolds: the islands become smaller as b decreases up to critical values of b where the islands are destroyed through a saddle-focus bifurcation (see Fig. 9, bottom right).

IV. FLOW APPROACH TO WEAKLY DISSIPATIVE DYNAMICS

In this section we perform a theoretical study of the dynamics of a planar weakly dissipative map in order to get quantitative information on the effect of the dissipation on

the dynamics. Our starting point is the Birkhoff normal form and the interpolating Hamiltonian of the conservative map around an elliptic fixed/periodic point. In this way we construct a dissipative vector field modeling the dynamics in the resonance from which information is obtained.

Note that this approach ignores the effect of the splitting of separatrices of the resonance we focus on. Hence, it can be only applied to flow type resonances (situation d of Fig. 6). In Sec. IV F we discuss on the limits of validity of this approach, while in Ref. 28 we derive a suitable model for the return map to deal with the homoclinic type resonances. It turns out, however, that around an elliptic fixed/periodic point, the radial dissipative perturbation generically creates a domain, below referred to as the “flow domain,” where all the surviving resonances are of flow type. This domain is confined between the “first” and “second” critical radii introduced below. The schematic global situation is represented in Fig. 12 and summarizes the formal considerations of this section.

A. Hamiltonian description of the conservative map close to E_0

Consider a one parameter family F_δ of APMs, such that, expressed in Cartesian coordinates $x, y \in \mathbb{R}^2$, it has an elliptic fixed point at the origin $E_0=(0,0)$ for all the values of the parameter δ . Let $\alpha=q/m+\delta$, $q, m \geq 5$, be the rotation number at E_0 and $\lambda=e^{2\pi i \alpha}$ the associated multiplier. From now on, $|\delta|$ will be considered small enough so that the resonance of order m cannot be avoided [note that, generically, the coefficient of the corresponding resonant term in the normal form is expected to be, under generic assumptions, of order $\mathcal{O}(1)$ and the m -resonance plays a role in the dynamics]. We will focus on the analysis of the m order resonance. It can be

shown that the effect of other resonances of similar order can be neglected (i.e., it is well averaged) in a first order analysis of the dynamics in a resonant chain.²⁵

It is well-known that by a suitable change of variables, F_δ can be reduced to a resonant Birkhoff normal form,^{1,21} which can be expressed in $z=x+iy$, $\bar{z}=x-iy$ coordinates as (see Ref. 25)

$$\text{BNF}_m(F): z \mapsto R_{2\pi(q/m)} \circ K(z, \bar{z}, \delta),$$

being $K(z, \bar{z}, \delta) = e^{2\pi i \gamma(r)} z + c \bar{z}^{m-1} + \hat{R}_{m+1}(z, \bar{z})$, $\gamma(r) = \delta + b_1 r^2 + b_2 r^4 + \dots + b_s r^{2s}$, $r = |z|$, $b_i \in \mathbb{R}$ the so-called Birkhoff coefficients, $c \in \mathbb{C}$, $\hat{R}_{m+1}(z, \bar{z})$ the terms of higher order than $m+1$, and where R_β denotes the rigid rotation of angle β . Due to the symmetries of the Birkhoff normal form reduction, the m -jet of K commutes with the rotation $R_{2\pi q/m}$, hence being able to reduce the analysis to the study of the near-the-identity one parameter δ -family of maps K . By a suitable scaling (we are assuming that the generic condition $c \neq 0$ holds) of the z variable the map K can be rewritten as

$$K: z \mapsto e^{2\pi i \gamma(r)} z + i \bar{z}^{m-1} + R_{m+1}(z, \bar{z}). \quad (6)$$

A near-the-identity map, like map (6), can be approximated by a planar autonomous flow. If the map is symplectic, the flow can be chosen to be Hamiltonian. Let us introduce the Poincaré action-angle variables (I, φ) defined by $z = \sqrt{2I} \exp(i\varphi)$. Furthermore, we denote δ as b_0 , define

$$\mathcal{H}_{nr}(I) = \pi \sum_{n=0}^s \frac{b_n}{n+1} (2I)^{n+1} \quad \text{and}$$

$$\mathcal{H}_r(I, \varphi) = \frac{1}{m} (2I)^{m/2} \cos(m\varphi),$$

the nonresonant and resonant parts of the interpolating Hamiltonian, and let r_* be such that $\gamma(r_*) = 0$, that is, $r_* \approx (-\delta/b_1)^{1/2}$, where we assume $b_1 \neq 0$. The following result, which can be found in Ref. 25, defines the interpolating Hamiltonian which allows to study easily the dynamics in a disk surrounding E_0 and containing the m order resonance.

Theorem 4.1: *The flow ϕ generated by the Hamiltonian*

$$\mathcal{H}(I, \varphi) = \mathcal{H}_{nr}(I) + \mathcal{H}_r(I, \varphi) \quad (7)$$

interpolates diffeomorphism (6) with an error of order $m+1$ with respect to the $(z, \bar{z})$ -coordinates, that is,

$$\hat{K}(I, \varphi) = \phi_{t=1}(I, \varphi) + \mathcal{O}(I^{(m+1)/2}),$$

where $\hat{K}$ denotes the original diffeomorphism K expressed in Poincaré variables. If we assume $b_1 \neq 0$ this approximation holds in an annulus centered in the resonance radius r_ of width $r_*^{1+\nu}$ for $\nu > 0$. $\square$*

The following assertions can be easily checked from the Hamiltonian above. For $m \geq 5$, that is, in the nonstrong resonant case, the m -resonant islands exist provided $b_1 \delta < 0$. This m -resonant chain has two periodic orbits of period m located near two concentric circumferences (in the normal form coordinates). The closest orbit to the external one is elliptic, while the nearest orbit to the inner circumference is hyperbolic. The resonant chain consists on pendulum-like islands

and the width of them behaves as $I_*^{m/4}$, where $I_* = -\delta/2b_1 + \mathcal{O}(\delta^2)$ corresponds to the radius r_* expressed in Poincaré variables. The reader is referred to Ref. 25 for details on the normal forms and for rigorous proofs of the statements above and of Theorem 4.1.

B. A dissipative vector field describing the dynamics close to E_0

The effect of the dissipation on the system is now considered. We add the dissipation to flow approximation (7) of $F_{\delta,0}$ to obtain a dissipative flow which interpolates the map $F_{\delta,\epsilon}$.

As was stated before, the dissipation considered is of radial type. It can be interpolated by the flow $\dot{x} = \log(1-\epsilon)x = -\epsilon x + \mathcal{O}(\epsilon^2)$, $\dot{y} = \log(1-\epsilon)y = -\epsilon y + \mathcal{O}(\epsilon^2)$ which is expressed in terms of Poincaré variables as

$$\dot{\varphi} = 0, \quad \dot{I} = 2 \log(1-\epsilon)I = -2\epsilon I + \mathcal{O}(\epsilon^2). \quad (8)$$

We want to construct a vector field which interpolates the composition $T_2 \circ T_1$ of the maps $T_1: (I, \varphi) \mapsto \hat{K}(I, \varphi)$ and $T_2: (I, \varphi) \mapsto ((1-\epsilon)^2 I, \varphi)$, where $\hat{K}$ denotes the original diffeomorphism K in Eq. (6) expressed in Poincaré variables. Above vector field (8) interpolates T_2 and the vector field associated with Eq. (7) interpolates the map T_1 . Observe that the sum of the vector fields in a neighborhood of width $\mathcal{O}(I_*^{m/4})$ of the resonance of order m is a vector field of the type $(\dot{\varphi}, \dot{I}) = (\mathcal{O}(\delta^{m/4}), \mathcal{O}(\delta^{m/2}, \epsilon\delta))$. On the other hand, the Lie bracket of both vector fields is a vector field of the type $(\dot{\varphi}, \dot{I}) = (\mathcal{O}(\epsilon\delta), \mathcal{O}(\epsilon\delta^{m/2}))$. It follows that for $\epsilon \ll \delta^{m/4-1}$ the Lie bracket is of order less than the sum of the vector fields (recall that we assume $m \geq 5$). In particular, to get a first order approximation, it can be assumed that both vector fields commute (that is, the lack of commutativity is small compared to the vector field) and, hence, the sum of the vector fields yields a vector field which interpolates the composition of the maps with a suitable error.

Hence, we consider the “complete” vector field which according to Eq. (7) is given by

$$\dot{I} = (2I)^{m/2} \sin(m\varphi) - 2\epsilon I, \quad (9)$$

$$\dot{\varphi} = 2\pi \sum_{n=0}^s b_n (2I)^n + (2I)^{m/2-1} \cos(m\varphi).$$

Note that the value of ϵ required for the destruction of the m -order resonance is, at least, $\mathcal{O}(\delta^{m/2-1})$ [see first equation of Eq. (9)] and, then, the approximation above is useful to understand the process of destruction of the resonance from a quantitative point of view.

C. Destruction of periodic orbits by the dissipation: first critical radius

Let $F_{\delta,\epsilon}$ be a one parameter family of weakly dissipative maps parametrized by δ and depending on the dissipative parameter $\epsilon > 0$. Assume that for $\epsilon=0$ the map $F_{\delta,0}$ is an APM for all $\delta \in \mathbb{R}$ with an elliptic fixed point located at the origin E_0 , such that $\text{Spec}(F_{\delta,0}(E_0)) = \{\lambda, \lambda^{-1}\}$, being λ

$=\exp(2\pi i\alpha)$ with $\alpha=q/m+\delta$, δ small enough, $q, m \in \mathbb{N} \setminus \{0\}$, $m \geq 5$. Let $b_1(\delta)$ be the first Birkhoff coefficient of the normal form of $F_{\delta,0}$ and assume that $b_1(\delta)=b_1+\mathcal{O}(\delta)$, $b_1=\mathcal{O}(1)$ being the first Birkhoff coefficient of $F_{0,0}$. Under these conditions the following result holds.

Proposition 4.1: Assume that α verifies a Diophantine condition of type $(\tilde{c}, \tau)$, i.e., for $a/b \in \mathbb{Q}$,

$$\left| \alpha - \frac{a}{b} \right| > \frac{\tilde{c}}{|b|^\tau}, \quad \tilde{c}, \tau > 0.$$

A sufficient condition for the nonexistence of the m -order resonance for $F_{\delta,\epsilon}$ is given by

$$\epsilon > \left(\frac{2C}{m^\tau} \right)^{m/2-1}, \quad (10)$$

where $C=\tilde{c}/2|b_1|$.

Proof: A standard application of the implicit function theorem shows that the fixed points also exist if the perturbative dissipative term is sufficiently small. Observe that the existence of fixed points requires $\epsilon < (2I)^{m/2-1}$. Taking into account that $I \sim I_*$ at the fixed points of the periodic orbit, we obtain

$$I_* > \frac{1}{2} \epsilon^{2/(m-2)}.$$

In particular, since $\alpha=q/m+\delta$, it should be $|\delta| > \tilde{c}m^{-\tau}$, and, consequently,

$$I_* \simeq -\frac{\delta}{2b_1} > \frac{\tilde{c}}{2m^\tau|b_1|} = Cm^{-\tau}, \quad \text{with } C = \frac{\tilde{c}}{2|b_1|}.$$

Then, if it is satisfied

$$\frac{C}{m^\tau} < I_* < \frac{\epsilon^{2/(m-2)}}{2}, \quad (11)$$

no fixed points exist and the m -order resonance has been destroyed by the dissipation. $\square$

Remark: Let $\alpha=p/q \in \mathbb{Q}$ be the rotation number at the origin. We observe that, if b_1 is assumed to be negative, the rotation number of an invariant rotational curve inside the region delimited by $\mathcal{C}$ is less than α . Consider an invariant curve such that $\rho=p'/q' < \alpha$. Then,

$$\delta = \alpha - \frac{p'}{q'} = \frac{p}{q} - \frac{p'}{q'} = \frac{pq' - qp'}{qq'} \geq \frac{1}{qq'},$$

and the equality holds only if $q'=kq+m_0$ and $p'=kp+n_0$ for m_0 and n_0 , such that $pm_0 - qn_0 = 1$. This relation could be used to obtain estimates of the radius I_* similar to the ones given in this section.

Expression (10) allows us, for a fixed m , to deduce an approximation of the critical dissipation parameter $\epsilon=\epsilon_c$, such that for this value the resonance of order m disappears. Taking logarithms, condition (10) can be reformulated as

$$\log \epsilon > \frac{m-2}{2} (\tilde{C} - \tau \log m),$$

where $\tilde{C}=\log(2C)$.

Assume ϵ fixed. From the equation

$$\log \epsilon = \frac{m-2}{2} (\tilde{C} - \tau \log m),$$

the value of m can be obtained and, then, an approximation of I_c follows,

$$I_c^\epsilon \sim \frac{C}{m^\tau}. \quad (12)$$

This allows us to conclude that any point in $B(E_0, \sqrt{2I_c^\epsilon})$ has as ω -limit the elliptic point E_0 because all the lower order resonances have disappeared by the effect of the weak dissipation.

Definition 4.1: Given any weakly dissipative planar map of form (1) (i.e., a radial dissipative perturbation acting on an APM) the first critical radius is defined, for a fixed dissipative parameter ϵ , to be the radius r_c^ϵ , such that inside the $B(E_0, r_c^\epsilon)$, no resonances survive the effect of the dissipation.

The value $\sqrt{2I_c^\epsilon}$ provides, according to the reasoning above Eq. (12), an estimate of the first critical radius r_c^ϵ .

D. A simplified model to describe the dynamics around a resonant island

Hamiltonian (7) describes the dynamics in a narrow strip, of width at least $\mathcal{O}(I_*^{m/4})$, containing the resonance of order m . It is of interest to study analytically how the position of the separatrices of the hyperbolic points of an island changes with respect to the dissipation parameter. In order to do this, we localize the Hamiltonian around an island of the m -order resonance. Recall that the resonance is located at a distance $I_* \approx -b_0/2b_1$, value that corresponds to the approximated average between the hyperbolic and elliptic fixed point distances from the origin. Let λ_ϵ be the dominant eigenvalue of the hyperbolic m -periodic point of the resonant island we focus on. In particular, $\hat{\lambda}=\log(\lambda_0)$ is the eigenvalue of the corresponding hyperbolic m -periodic point of the vector field generated by Hamiltonian (7). A straightforward computation shows that $\hat{\lambda}=4\pi|b_1|(2I_*)^{m/4}+\mathcal{O}(\delta)$ and, hence, $\hat{\lambda}=\mathcal{O}(\delta^{m/4})$.²⁵

Proposition 4.2: For a radial dissipative planar diffeomorphism, the dynamics of the m -order resonance, $m \geq 5$, can be approximated (up to order of 2 in ϵ) by the time $\gamma=\hat{\lambda}(1+\mathcal{O}(\delta))$ map related to the flow generated by the vector field,

$$X_{\hat{\epsilon}}: \begin{cases} \dot{J} = -(1+dJ)\sin \psi - \hat{\epsilon} - k\delta^{m/4-1}\hat{\epsilon}J \\ \dot{\psi} = J + cJ^2 - d \cos \psi, \end{cases} \quad (13)$$

where

$$c = \frac{\text{sign}(b_1b_2)|b_2|}{\sqrt{m\pi}|b_1|^{(6+m)/4}} |\delta|^{m/4}(1+\mathcal{O}(\delta)),$$

$$d = \frac{\sqrt{m}}{2\sqrt{\pi}|b_1|^{(m-2)/4}} |\delta|^{m/4-1}(1+\mathcal{O}(\delta)),$$

$$k = \frac{|b_1|^{(2-m)/4}}{m\sqrt{\pi}} (1+\mathcal{O}(\delta)),$$

and

$$\hat{\epsilon} = \text{sign}(b_1)(2I_*)^{1-m/2}\epsilon.$$

The Hamiltonian part of the above vector field, with Hamiltonian function,

$$\mathcal{H}_{\hat{\epsilon}}(J, \psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos \psi + \hat{\epsilon}\psi, \quad (14)$$

provides a family of “averaged” Hamiltonians roughly describing the effect of the dissipation on the resonance.

Proof: The starting point is vector field (9). The time one map of the flow generated by Eq. (9) for a suitable $\nu > 0$ depending on m (e.g., $\nu \leq m/2 - 2$, see Theorem 4.1) provides an approximation of the dynamics in a suitable annulus containing the m -order resonance, $m \geq 5$. The error of this approximation is $\mathcal{O}(\delta^{m+1/2})$ in the (I, φ) variables and $\mathcal{O}(\epsilon^2)$ with respect the dissipative perturbation (assuming $\epsilon \ll \delta^{m/4-1}$).

The symplectic change $J = (I - I_*)/m$, $\psi = m\varphi$ provides the truncated equations

$$\dot{J} = (d_1 + d_2J)\sin(\psi) - 2\epsilon J - 2\epsilon I_*/m,$$

$$\dot{\psi} = CJ + DJ^2 + d_2 \cos(\psi),$$

with $C = 4\pi b_1 m^2 + \mathcal{O}(\delta)$, $D = 8\pi b_2 m^3 + \mathcal{O}(\delta)$, $d_1 = (1/m) \times (2I_*)^{m/2}$, and $d_2 = m(2I_*)^{m/2-1}$.

We shall maintain the notation J , ψ , and t for the action-angle variables and time variable associated to the m -resonant island after introducing new coordinates J_{new} , ψ_{new} , t_{new} , in each of the steps below.

Introducing $J_{\text{new}} = CJ$, the equations can be rewritten as

$$\dot{J} = (\tilde{d}_1 + d_2J)\sin \psi - d_3\epsilon - 2\epsilon J,$$

$$\dot{\psi} = J + \tilde{D}J^2 + d_2 \cos \psi,$$

with $\tilde{d}_1 = Cd_1$, $\tilde{D} = D/C^2$, and $d_3 = 2I_*C/m$.

By means of the symplectic transformation $J_{\text{new}} = \gamma^{-1}J$, $t_{\text{new}} = \gamma t$, with $\gamma = |\tilde{d}_1|^{1/2} = \mathcal{O}(\delta^{m/4})$, the equations become

$$\dot{J} = (1 + dJ)\sin \psi - \tilde{d}_3\epsilon - \bar{k}\epsilon J,$$

$$\dot{\psi} = J + cJ^2 + d \cos \psi,$$

where $d = d_2/\gamma$, $c = \tilde{D}\gamma$, $\tilde{d}_3 = d_3/\gamma^2$, and $\bar{k} = 2/\gamma$. Moreover, $d > 0$, $d = \mathcal{O}(\delta^{m/4-1})$, $\text{sign}(c) = \text{sign}(b_1)$, $c = \mathcal{O}(\delta^{m/4})$, $\tilde{d}_3 = \text{sign}(b_1)(2I_*)^{1-m/2} = \mathcal{O}(\delta^{1-m/2})$, $\bar{k} > 0$, and $\bar{k} = \mathcal{O}(\delta^{-(m/4)})$.

Introducing $\hat{\epsilon} = \tilde{d}_3\epsilon = \text{sign}(b_1)(2I_*)^{1-m/2}\epsilon$, the vector field is transformed to

$$X_{\hat{\epsilon}}: \begin{cases} \dot{J} = (\text{sign}(b_1) + dJ)\sin \psi - \hat{\epsilon} - k\delta^{m/4-1}\hat{\epsilon}J \\ \dot{\psi} = J + cJ^2 + d \cos \psi, \end{cases}$$

where $k = 2 \text{sign}(b_1)\sqrt{m/C} = \mathcal{O}(1)$.

If $b_1 > 0$ (then $\hat{\epsilon} > 0$, $k > 0$), we perform the change $\psi_{\text{new}} = \psi - \pi$ to reduce the vector field to the desired form. Otherwise, if $b_1 < 0$ (then $\hat{\epsilon} < 0$, $k < 0$), we perform the change $J_{\text{new}} = -J$, $t_{\text{new}} = -t$ which reduces the vector field to

the desired form changing the sign of c and with $k > 0$. In any case, $d > 0$, $\text{sign}(c) = \text{sign}(b_1b_2)$, $k > 0$, and $\text{sign}(\hat{\epsilon}) = \text{sign}(b_1)$. $\square$

Remark: The error in the discrete approximation obtained by flow (13) is $\mathcal{O}(\delta^{m+2/4}, \epsilon^2)$ when expressed in the (J, ψ) variables. From Hamiltonian (14), since we have to consider the time γ -map, $\gamma = \mathcal{O}(\delta^{m/4})$, the error in the discrete approximation becomes $\mathcal{O}(\delta^{m+2/4}, \epsilon)$. Note that the dissipative effect included in the Hamiltonian gives a term in the discrete approximation of order $\mathcal{O}(\delta^{1-m/4}\epsilon)$ which is much larger than $\mathcal{O}(\epsilon)$ for $m \geq 5$.

E. Capture probability in resonances of flow type

The effect of the dissipative term on the resonances changes the topology of the elliptic periodic points to be foci. As a consequence, some points of the phase space must be captured by the resonances. We want to analyze the probability of capture in a resonance from the flow approach described in the last sections.

We have seen how some resonances are destroyed by the dissipation (Sec. IV C). Outside the region where resonances are destroyed, we find resonances of two different types. Maybe, the dissipation is sufficiently large so that homoclinic points of the conservative case are destroyed. However, it may happen that homoclinic points from the conservative case survive and the homoclinic tangle also exists but with a slight change on the position of the points and on the intersections of manifolds. The quantitative analysis of this last situation is not considered in this paper but postponed to Ref. 28.

Different approaches to the idea of probability of capture in a resonance can be found in the literature. To simulate the different ω -limit sets that coexist in the case of small dissipation and due to the sensitivity to initial conditions because the intersections of the manifolds (see Sec. III), it is usual to add some random perturbation to the dynamical system (see, for instance, Ref. 30 where the dissipation effect is understood as a white noise perturbation). However, chaoticity is different from noise because geometry plays an important role when describing the motion.

On the other hand, averaging theory provides some perturbative results. For instance, it provides a satisfactory interpretation of the probability of capture in the case that we add a small not necessarily Hamiltonian perturbation to an integrable Hamiltonian planar field (see Ref. 16). Also, it provides an accurate analysis of the probability of capture when the dissipation is considered to be much smaller than the nonintegrable Hamiltonian part of the map (see Ref. 17). Note that in these studies, the Hamiltonian perturbation to get a nonintegrable map is assumed to be constant, that is, it does not depend on the point we choose in the phase space. Nevertheless, this is not the case we are dealing with because the conservative map is closer to be integrable as we approach to the origin. A direct consequence of this fact is that the radial weakly dissipative map for a fixed dissipation parameter is integrable in a small domain close to the origin where the dissipation has destroyed the resonances. Note that since the eigenvalues of the focus fixed point at the origin lie

in the Poincaré domain, there is a small domain surrounding the origin where the dynamics of the map is analytically conjugated to the dynamics of a linear focus, consequently there is a linear vector field interpolating exactly the map in this domain.

As said we are interested in determining the probability of capture in a resonance. Observe that, for a fixed δ , we are dealing with a family of maps depending on the dissipation parameter ϵ . By probability of capture, however, it should be understood the probability of an initial point to have a concrete ω -limit. Obviously, it depends on the parameter ϵ of the map $F_{\delta,\epsilon}$ and on the map $F_{\delta,0}$ itself.

Assume in this section that no homoclinic points appear in the m -order resonance as $\epsilon \searrow 0$ or, in other words, that the flow approximation can be considered to model the dynamics around the resonance for any $\epsilon \geq 0$. The phase space in a neighborhood of an island is like the one represented in Fig. 10 (a pendulum with torque).

The invariant manifolds of the hyperbolic points of the resonance determine a collection of strips. If we fix our attention in a resonant island, we observe two different strips. Denote by H_- and H_+ two consecutive hyperbolic points of the resonance and denote by $W_{\pm}^{s,u}$ the invariant manifolds associated to them (see Fig. 10). We will refer to $W_{\pm}^{s,u}$ as the right branch of $W_{\pm}^{s,u}$, that is, the part of $W_{\pm}^{s,u}$ which is attached by the right to $H_{\pm}$ according to the representation of Fig. 10. Similarly, we will use $W_{\pm,l}^{s,u}$ to denote the “left” branches.

Definition 4.2: The exit strip is the strip obtained by iteration of the subdomain bounded by $W_{-,l}^s$ and $W_{+,r}^s$ of a fundamental domain $\mathcal{D}$ (see below for the construction of this domain) located in the outer part of the (dissipative) island (see Fig. 10). In particular, this strip is also bounded, after passing close to the hyperbolic point H_- , by $W_{-,l}^u$ and $W_{+,r}^s$ in the inner part of the island.

The entrance strip is the strip obtained by iteration of the subdomain bounded by $W_{+,r}^s$ and $W_{+,r}^u$ of a fundamental domain $\mathcal{D}$ in the outer part of the (dissipative) island. Considering negative iterates we observe that the strip passes close to H_+ and then it is bounded by $W_{+,r}^s$ and $W_{+,l}^s$.

Points inside the entrance strip will be captured by the resonance, while points inside the exit strip will be expelled from the resonance (see Fig. 10).

Denote by $\mathcal{P}$ the set of points of the phase space that are located further from the origin than the resonance and such that are not captured by other resonances outside the one we are considering. Observe that a point of $\mathcal{P}$ must be inside one of the strips, either in the entrance strip or in the exit one. In order to define the probability of capture in a concrete way, we construct a fundamental domain (see Fig. 10). Take a line L joining a point p of W_+^u with the one q from W_-^s which is the closest one to p . We require also that L has p as the only point on W_+^u , that is, $L \cap W_+^u = \{p\}$. Let $\hat{L}$ be the image of L under the map. Then, any point of $\mathcal{P}$ has an iterate which is inside the domain delimited by L , $\hat{L}$, W_+^u , and W_-^s . We denote this fundamental domain as $\mathcal{D}$.

The probability of capture in the island is defined to be the ratio between the measure of the set of points of $\mathcal{D}$ inside

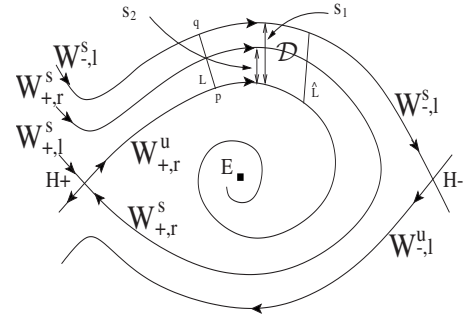


FIG. 10. Sketch of the phase space of a pendulum with torque corresponding to flow type resonances. The probability of capture depends on the width of the strips, that is, on s_1 and s_2 . Exit strip bounded in $\mathcal{D}$ by W_-^s and W_+^s ; points which “pass” through the resonance. Entrance strip bounded in $\mathcal{D}$ by W_+^s and W_+^u ; points which are captured in the resonance.

the entrance strip and the measure of $\mathcal{D}$. We assume also that the resonance is such that the manifolds are relatively “far” from homoclinic tangency, meaning that the lengths of the segments of W_-^s , W_+^u , and W_+^s inside the fundamental domain are of the same order. Then, the probability of being captured by the resonance depends, mainly, on the ratio of the amplitudes of the strips, s_1 , and the strip of capture, s_2 (see Fig. 10).

Remark: Our study holds in a fixed domain $\mathcal{B}$ around the origin E_0 (where the truncated normal form approach of the map F holds). Under iteration of F^{-1} , the domain $\mathcal{D}$ leaves the domain $\mathcal{B}$. Let X be the set of points of $\mathcal{D}$ in the exit strip and let N be the set of points of $\mathcal{D}$ in the entrance strip. It is expected to have some points of N , such that they leave the domain $\mathcal{B}$ in less iterates of F^{-1} than any point of X . On the other hand, there are points $p \in P \subset X$, such that $F^{-k}(p) \in \mathcal{B}$ for all $k \geq 0$, because X contains points on the unstable manifold of one of the hyperbolic points of the island. Note that the approach developed above does not take into account these two effects which, on the other hand, are expected to be irrelevant when computing the probability of capture.

We are interested in the behavior of the map when it is close to the conservative case. In particular, we are interested in the probability of capture when we approach to the conservative case from the flow approximation, that is, we try to understand limit (2). Observe that if the resonance of the conservative case is of flow type the probability of capture when $\epsilon=0$ should be zero. Nevertheless, the following result implies that the measure of the entrance strip plays a role in this limit.

Proposition 4.3: Let $F_{\delta,\epsilon}$ be a radially perturbed family of APMs and consider $\delta \in \mathbb{R}$ (with $|\delta|$ small enough), such that $F_{\delta,0}$ presents an m -resonant chain of islands. Assume that for $\epsilon = \epsilon_h$ the last homoclinic tangency between the invariant manifolds that form the m -order resonant islands of $F_{\delta,\epsilon}$ takes place (in other words, we assume that for $\epsilon < \epsilon_h$ the m -order resonant islands have homoclinic points, while for $\epsilon > \epsilon_h$ they do not). Let $\epsilon_0 > \epsilon_h$ be such that for $\epsilon > \epsilon_0$ the entrance and exit strips of the m -resonant chain of islands of $F_{\delta,\epsilon}$ are well enough approximated, in an annulus of width $\mathcal{O}(\delta^{m/4})$ containing the islands and after introducing the

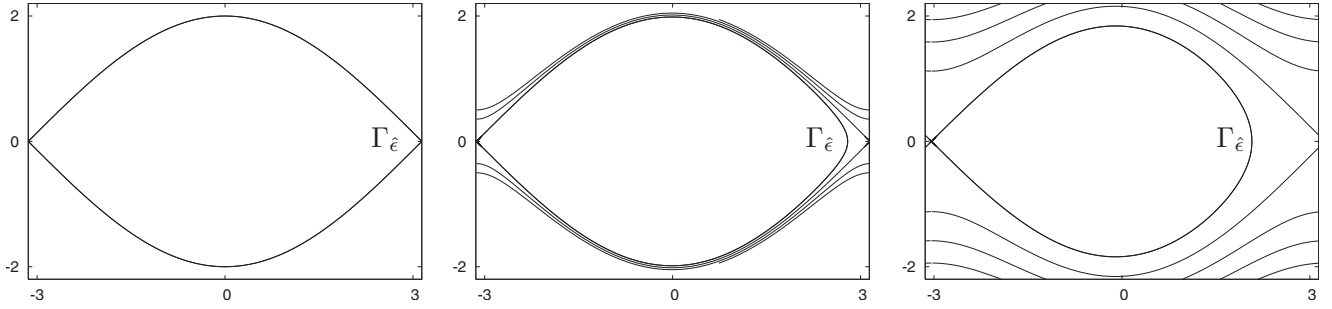


FIG. 11. Shape of the separatrix $\Gamma_{\hat{\epsilon},0,0}$. The parameters chosen in Hamiltonian model (14) to get the illustrations are $c=0$, $d=0$, and $\hat{\epsilon}=0, 0.01, 0.1$ from left to right, respectively.

(J, ψ) -coordinates as in Proposition 4.2, by the entrance and exit strips, respectively, of vector field (13). Then, for values of ϵ such that $\epsilon_0 < \epsilon \ll 1$, the probability of capture by the perturbed elliptic points (stable foci) of the islands of the m -order resonance of $F_{\delta, \epsilon}$ is given by

$$P_{\text{capture}} = \frac{8|b_1|^{(2-m)/4}}{m\pi\sqrt{\pi}} |\delta|^{m/4-1} + \mathcal{O}(\delta^{1-m/2}\epsilon, \delta^{m/2-2}). \quad (15)$$

Remarks:

- (1) We recall that for $|\delta|$ in the present context, the splitting in the conservative case is exponentially small in $|\delta|$. On the other hand, the effect of the dissipation is to separate the “averaged” stable and unstable manifolds by a quantity $\mathcal{O}(\epsilon)$. Hence ϵ_h is also exponentially small in $|\delta|$.
- (2) The ratio ϵ_0/ϵ_h needs not to be very big. In practice, values ranging from 10 to 20 are enough.

Proof of the proposition: One has to measure the widths s_1 and s_2 of the entrance and exit strips and the exit strip (see Fig. 10). To this end we use the Hamiltonian approximation $\mathcal{H}_{\hat{\epsilon}}(J, \psi)$ given by Eq. (14) and estimate the widths as a difference in energy values. We will denote it by $\mathcal{H}_{\hat{\epsilon}, c, d}(J, \psi)$, making explicit the dependence on the c and d parameters. Let $\Gamma_{\hat{\epsilon}, c, d}$ denote the separatrices of $\mathcal{H}_{\hat{\epsilon}, c, d}(J, \psi)$. Figure 11 shows the separatrix $\Gamma_{\hat{\epsilon}, 0, 0}$ for values of $\hat{\epsilon}=0, 0.01, 0.1$, respectively. The geometrical picture is similar for c and d small, with $c \ll d$ (see Proposition 4.2), for the separatrices $\Gamma_{\hat{\epsilon}, c, d}$.

Dissipative interpolating vector field (13) does not preserve the energy $\mathcal{H}_{\hat{\epsilon}, c, d}(J, \psi)$. Hence, we should estimate also the difference between the Hamiltonian approximation and the dissipative vector field.

To measure s_1 we observe that the energy $\mathcal{H}_{0, c, d}(J, \psi)$ changes when integrating along the separatrix $\Gamma_{0, c, d}$ by a quantity $2\pi\hat{\epsilon} + \mathcal{O}(\hat{\epsilon}\delta^{m/4-1})$. This can be easily obtained for $c=d=0$, while in the general case it follows from the fact that $|J_{0,0,0}(\psi) - J_{0,c,d}(\psi)| = \mathcal{O}(\delta^{m/4-1})$, where $J_{0,0,0}(\psi)$ and $J_{0,c,d}(\psi)$ denote the actions as functions of the angle $\psi \in (-\pi, \pi]$ along the separatrices $\Gamma_{0,0,0}$ and $\Gamma_{0,c,d}$.

To measure s_2 we consider the variation of the energy $\mathcal{H}_{\hat{\epsilon}, c, d}(J, \psi)$ along the separatrix $\Gamma_{\hat{\epsilon}, c, d}$. Let $\mathcal{A}(\Gamma_{\hat{\epsilon}, c, d})$ denote the area inside the curve $\Gamma_{\hat{\epsilon}, c, d}$. Using Green’s theorem it can

be reduced to

$$s_2 = \int_{\Gamma_{\hat{\epsilon}, c, d}} \dot{H} = -\Theta \mathcal{A}(\Gamma_{\hat{\epsilon}, c, d}),$$

where $\Theta = k\delta^{m/4-1}\hat{\epsilon}$ [see the reduced dissipative vector field (13)]. One has $|J_{\hat{\epsilon}, c, d}(\psi) - J_{\hat{\epsilon}, 0, 0}(\psi)| = \mathcal{O}(\delta^{m/4-1})$ and then $\mathcal{A}(\Gamma_{\hat{\epsilon}, c, d}) = \mathcal{A}(\Gamma_{\hat{\epsilon}, 0, 0}) + \mathcal{O}(\delta^{m/4-1})$. In order to see that $\mathcal{A}(\Gamma_{\hat{\epsilon}, 0, 0}) = \mathcal{O}(1)$, we observe that the right (left) intersection with the x -axis of the inner loop of the curve $\Gamma_{\hat{\epsilon}, 0, 0}$ for $\hat{\epsilon} > 0$ ($\hat{\epsilon} < 0$), which we assume has coordinates $(\psi_1, 0)$, must verify $\cos \psi_1 - \hat{\epsilon}\psi_1 + h = 0$. Then $\psi_1 = \pi - \sqrt{2\pi}\sqrt{\hat{\epsilon}} + \mathcal{O}(\hat{\epsilon})$ if $\hat{\epsilon} > 0$ [$\psi_1 = -\pi + \sqrt{2\pi}\sqrt{\hat{\epsilon}} + \mathcal{O}(\hat{\epsilon})$ otherwise]. Finally, as the height of the unperturbed separatrix at ψ_1 is of the order $\mathcal{O}(\sqrt{\hat{\epsilon}})$, one has $\mathcal{A}(\Gamma_{\hat{\epsilon}, 0, 0}) = \mathcal{A}(\Gamma_{0, 0, 0}) + \mathcal{O}(\hat{\epsilon})$, and it is easily found $\mathcal{A}(\Gamma_{0, 0, 0}) = 16$. As a consequence, $s_2 = 16\Theta + \mathcal{O}(\hat{\epsilon}^2, \delta^{2(m/4-1)})$.

From these considerations we deduce that the probability of capture for $\epsilon_0 < \epsilon \ll 1$ (i.e., when approaching to the conservative case but without having homoclinics) is given by the ratio

$$P_{\text{capture}} = \frac{s_2}{s_1} \simeq \frac{k\delta^{m/4-1}}{\pi} = \mathcal{O}(\delta^{m/4-1}).$$

□

Remarks:

- (1) The above result gives a good approximation of the probability of capture provided $\epsilon_0 < \epsilon \ll \delta^{(3m-8)/4}$ (see Ref. 15). For instance, assume that for $\alpha=0.15$ (see Fig. 2) we consider the eighth and ninth order resonances. Then, for $m=8$ one has $\delta=1/40$ and it is required $\epsilon \ll 10^{-6}$, and for $m=9$ one has $\delta=7/180$ and it should be $\epsilon \ll 10^{-7}$. This makes numerical simulations, to check the theoretical results, somewhat cumbersome. Moreover, the outer splittings (the largest one) of the (1:8) and of the (1:9) resonances for $\alpha=0.15$ are of orders $\mathcal{O}(10^{-10})$ and $\mathcal{O}(10^{-3})$, respectively. As a consequence the flow approximation is not justified in this case (i.e., $\alpha=0.15$) for the $m=9$ resonance (see considerations in Sec. IV F), while for the $m=8$ resonance it holds. In Ref. 28 we claim, and we prove under suitable assumptions,

that the homoclinic tangle plays no relevant role in the computation of the probability of capture being of order $\delta^{m/4-1}$ in any case.

- (2) Note that, for values of ϵ for which the expression of the theorem holds and the flow approach is justified, it is expected to have $P_8 > P_9$ for $\alpha=0.15$, being P_m the probability of capture in the m -order resonance. For greater values of ϵ , this does not necessarily hold. As an example, we have numerically estimated P_8 and P_9 for $\epsilon=10^{-4}$. We have taken 12 345 points on the line $y=\tan(\pi\alpha)x$ with $x \in [0.8234, 0.8516]$ and we have performed enough iterates to be at a distance $\nu=10^{-4}$ of one of the elliptic periodic points of the $m=8$ or $m=9$ resonance. Then we have checked if the point is captured or not by doing 10^6 more iterates and checking the distance to the elliptic periodic points requiring to be smaller than ν . If the radius of some iterate of the point is less than $r=0.17$, it is considered to be trapped by the origin. In this way it is found that 365 points are captured by the eight-order resonance and 373 by the nine-order one. Hence, it seems to be P_8 slightly less than P_9 for this value of α .
- (3) We can proceed in the opposite way in the above remark. Instead of fixing δ and look for the value of ϵ needed so that Eq. (15) be a good approximation, we can fix ϵ and look for the minimum value of δ so that Eq. (15) holds. One has $\delta > \delta_{\min} = \epsilon^{4/(3m-8)}$. This means that for $\alpha=0.15$ and $\epsilon=10^{-4}$ since $\delta_{\min} > 10^{3m-8}$, Eq. (15) holds for resonances located further than $I_{\min} = \delta_{\min}/2b_1$ from the origin. For $\alpha=0.15$ we have $b_1 \approx 8.42 \times 10^{-2}$. In particular, for $m=8$ one has $\delta_{\min} \approx 0.87$, hence $I_{\min} \sim 5.15$. This turns out to be relatively far from the origin and the resonance is destroyed. We conclude that there is no flow type eight-order resonance for $\epsilon=10^{-4}$ in the stability domain where approximation (15) holds.

F. Region of validity of the flow approximation

The theory developed in the previous sections has two main restrictions. First, because it is only valid in a small neighborhood of the elliptic point E_0 since it is obtained after an approximation of the map by a Birkhoff normal form. Second, because to apply it requires to have a flow type resonance. For a fixed dissipation, it has been stated the existence of a neighborhood, say U_{r_c} , surrounding the elliptic fixed point E_0 where no resonances survive. This gave rise to the idea of the first critical radius introduced before. On the other hand, we can extend this domain to a domain $U_{r_{cc}}$ where only flow type resonances exist. This will give rise to the idea of second critical radius in the present section. Outside this domain $U_{r_{cc}}$, we find resonances with homoclinics, like the ones described in the cases (a), (b), and (c) of Fig. 6, together with flow type resonances, either close to have homoclinics or close to be destroyed. In this domain far from the elliptic point E_0 also an infinite number of small resonances have been destroyed by the dissipation effect but there are more and more surviving resonances as we move away from the origin E_0 .

As said we want to determine the existence of the second critical radius, say r_{cc} , that delimits the region where just flow type resonances exist. This will provide an annular region, between the first r_c and second critical r_{cc} radius, where the theory developed using the interpolating flow of the Birkhoff normal form holds.

In Sec. IV F 1 we review some recent results in Ref. 25 concerning the splitting of separatrices for generic m -order resonances. Then, as an application of these results, we will address the problem of the existence of the second critical radius.

1. The splitting of separatrices of the conservative map: an overview

We briefly review, for reader's convenience, some recent results concerning the splitting of separatrices that form a resonant island for an APM. For details the reader is referred to Ref. 25. Consider a one parameter δ -family of area preserving maps having an elliptic fixed point at $E_0=(0,0)$ with rotation number $\alpha=q/m+\delta$, $q \in \mathbb{N} \setminus \{0\}$, $m \geq 5$.

In a resonant island there are two splittings that play a role: the inner and the outer splitting. The inner/outer splitting is related to the closest/furthest primary intersection of the corresponding separatrices to the fixed elliptic point E_0 . Generically, they are of different magnitudes being, for $|\delta|$ small enough, the outer splitting the largest.

The behavior of the splittings is determined by the position of the singularities of the separatrices (considered as complex functions of the complex time, see Ref. 10) of an interpolating flow. The relevant singularities are close to the pendulum ones, that is, at a distance $\pi/2$ from the real time axis, since Hamiltonian (14) for $\hat{\epsilon}=0$ can be considered as an interpolating Hamiltonian. The correction due to the d parameter provides a difference $\mathcal{O}(\delta^{m/4-1})$ between the position of the singularity of the inner separatrices and the one of the outer separatrices. The c parameter plays a minor role in the splitting behavior. Hence, at first order both splittings behave in a similar way [as $\sim \exp(-C/\delta^{m/4})$, with $C=\mathcal{O}(1)$] and the higher order terms in the Hamiltonian produce a ratio of order $\mathcal{O}(\exp(1/\delta))$, for δ small enough, between the splittings.

Remark: According to the theory “generically” it is expected to have the outer splitting larger than the inner one. This explains why the transition from conservative to dissipative resonance described in Sec. III A is observed for most of the resonances in the Hénon map. This is also the situation for any APM under suitable generic assumptions.

2. The second critical radius

We observe that for any resonance there exists a critical dissipation parameter ϵ_{cc} , such that for $\epsilon > \epsilon_{cc}$ there is no intersection between the separatrices. To determine it we fix our attention on the family of “averaged” Hamiltonians (14). Note that the value of ϵ_{cc} should be, according to the splitting theory described above, exponentially small with respect $\delta^{m/4}$.

For $\hat{\epsilon}=0$, the points,

$$(J_0^0, \psi_0^0) = ((-1 + \sqrt{1 + 4cd})/2c, 0),$$

$$(J_0^\pi, \psi_0^\pi) = ((-1 + \sqrt{1 - 4cd})/2c, \pi),$$

are elliptic and hyperbolic fixed points, respectively, of vector field (14).

For $\hat{\epsilon} \neq 0$, a point $(J_*, \psi_*) = (J_0 + \Delta J, \psi_0 + \Delta \psi)$ is a fixed point if it verifies

$$-(1 + d(J_0 + \Delta J))\sin(\psi_0 + \Delta \psi) = \hat{\epsilon},$$

$$J_0 + \Delta J + c(J_0 + \Delta J)^2 - d \cos(\psi_0 + \Delta \psi) = 0.$$

Second equation implies $\Delta J = \mathcal{O}(\Delta \psi^2)$. Then, if $\psi_0 = 0$ and $J_0 = J_0^0$, one has $\Delta \psi^0 = -\hat{\epsilon}/(1 + dJ_0^0) + \mathcal{O}(\hat{\epsilon}^2)$. Otherwise, if $\psi_0 = \pi$ and $J_0 = J_0^\pi$ one has $\Delta \psi^\pi = \hat{\epsilon}/(1 + dJ_0^\pi) + \mathcal{O}(\hat{\epsilon}^2)$. In particular, for $\hat{\epsilon} > 0$ the dissipation moves the elliptic points to the left and the hyperbolic ones to the right (if $\hat{\epsilon} < 0$ they behave in the opposite way). When they collide, which turns out to be for $\hat{\epsilon} = \mathcal{O}(1)$, the resonance is destroyed.

Let Γ_ϵ denote the separatrix of vector field (14) corresponding to the energy level of the hyperbolic point. One easily checks that $d(\Gamma_\epsilon, \Gamma_0) \approx \Delta \psi^\pi$.

Let $\sigma_{\delta, \epsilon}$ denote the angle of the splitting of the outer separatrices (which according to the comments above is the largest one when $\epsilon \approx 0$ and δ is small enough). The position of the unstable invariant manifold with respect to the stable invariant manifold, in a suitable parametrization, is locally given by an expression of the form $\sum_{n \geq 1} a_n \sin(nz + \phi_n)$, where $a_n = \mathcal{O}(\exp(-nk\nu^{-\delta}))$, being k and $\delta = \delta(\nu)$ positive constants and ν the parameter of perturbation from the integrable case (see Ref. 26 for definition). Moreover, generically, $\delta(\nu) \rightarrow 0$ as $\nu \rightarrow 0$ (see Ref. 4). We will assume that it is a graph of the x coordinate (on W^s) of the first order function given by the first harmonic of the above representation, that is,

$$\Gamma(x) = a_\delta \sin(2\pi x/\Omega),$$

where a_δ is the amplitude of the main phase of the conservative splitting and Ω is the distance between two consecutive iterates of the homoclinic orbit. Consequently, $a_\delta \approx 0.25 \Omega \tan \sigma_{\delta, 0}$.

To compute Ω we need a homoclinic point. Hamiltonian (14) for $\hat{\epsilon} = 0$ admits the symmetry $\psi \rightarrow -\psi$ which means that a point of the form $(J_h, 0)$ must be homoclinic. We compute it by imposing the corresponding energy level of the separatrix. Then the distance between two consecutive iterates is given in a first order approximation by $\Omega = \gamma(J_h + cJ_h^2 - d)$, since the Hamiltonian flow is the time $\gamma = \sqrt{d_1}$ of the map skipping high order terms (see Sec. IV C).

A heuristic condition for the nonexistence of homoclinic points could be $d(\Gamma_\epsilon, \Gamma_0) > a_\delta$. By imposing $\Delta \psi^\pi > a_\delta$ in order to destroy homoclinic intersections, last condition gives

$$\hat{\epsilon} > \frac{4ca_\delta}{-2c + d(-1 + \sqrt{1 - 4dc})}. \quad (16)$$

The above inequality is exponentially small with respect $\delta^{m/4}$ as stated for a_δ in Sec. IV F 1.

Condition (16) defines a second critical radius r_{cc} . For a fixed ϵ , the value δ_{cc} is obtained from the equality $\hat{\epsilon} = 4ca_\delta/(-2c + d(-1 + \sqrt{1 - 4dc}))$. Note that all the variables

involved are explicitly known as a function of δ . Taking logarithms in this equality, one obtains $\log(\epsilon) = -K/\delta^{m/4} + \mathcal{O}(\log(\delta))$, with $K > 0$ a suitable constant. Then, $\delta_{cc} \approx (-K/\log(\epsilon))^{4/m}$, which defines the radius I_{cc} by taking $I_{cc} = |\delta_{cc}/2b_1|$. For values $I < I_{cc}$ there are no homoclinic points in the resonances.

Definition 4.3: Given a weakly dissipative planar map the second critical radius is defined, for a fixed dissipative parameter ϵ , to be the radius r_{cc}^ϵ , such that inside the ball $B(E_0, r_{cc}^\epsilon)$ all resonances that survive the effect of the dissipation are of flow type.

Condition (16) gives, as explained, an approximation of the second critical radius, I_{cc}^ϵ . The flow approximation holds in the annulus defined by I_c^ϵ (see remark in Sec. IV C) and I_{cc}^ϵ . We will refer to this domain as the flow domain.

Remarks:

- (1) From Eq. (11) it is obtained that the first critical radius is defined by a $\delta_c \sim \epsilon^{2/(m-2)}$. On the other hand, $\delta_{cc} \sim (-1/\log(\epsilon))^{4/m}$ as noticed above. Then, $r_c^\epsilon \ll r_{cc}^\epsilon$ for ϵ small enough (indeed it is expected to hold for a relatively large range of values of $\epsilon > 0$).
- (2) Outside the second critical radius, a finite number of resonances survive. They could be of flow type but some of them present homoclinic points (depending on the width of the resonance and on the size of the splitting of separatrices). A different approach to analyze the effect of dissipation is needed in this case. In Ref. 28 we use return models to this end. Using this approach, one can determine the effect of the dissipation on the invariant manifolds. For ϵ small enough, it produces a $\mathcal{O}(\epsilon)$ vertical translation between the position of the manifolds. Consequently, the above exponentially δ -small condition of ϵ to destroy the homoclinic points can be reformulated and the same conclusions concerning the second critical radius can be obtained.

V. CONCLUSIONS AND FINAL REMARKS

We have described the different dynamical scenarios taking place around a focus fixed/periodic point of a radial weakly dissipative planar map. We have distinguished different regimes according to the existence/nonexistence of attractors coming from the islands of stability of the conservative case and to the existence/nonexistence of transversal intersections of the invariant manifolds delimiting the basis of attraction. The general picture is sketched in Fig. 12.

In particular, we have distinguished different regions of interest according to the existence either of resonances or homoclinic points (regions II and III in Fig. 12). Also a trivial region where there are no resonances surviving the dissipation has been described (region I in Fig. 12).

Moreover, from Birkhoff normal form, we have obtained a suitable model describing the dynamics in a flow type resonance. Using this model we have addressed the capture probability question and we have established how this probability behaves, in a conservative limit sense, in terms of the distance $\sqrt{-\delta/2b_1}$ where is located the conservative island.

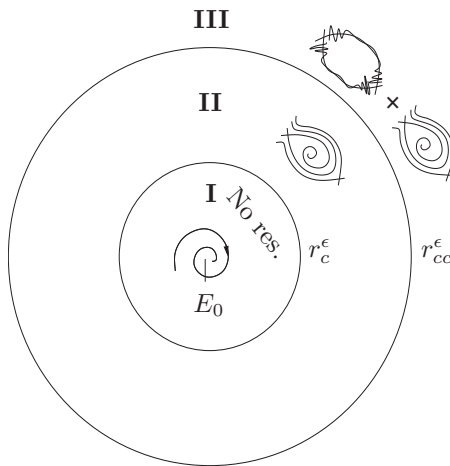


FIG. 12. Schematic representation of the different scenarios around a radially dissipative perturbed elliptic fixed point. We distinguish three regions. In region I there are no resonances of the conservative case surviving the dissipative effect. This region, in normal form coordinates, is bounded by the first critical radius r_c . In region II, we expect only flow type resonances without homoclinic points. This region is bounded by the critical radius r_c and r_{cc} . Finally, region III contains resonances which have homoclinic points despite the dissipation.

In a forthcoming paper,²⁸ the authors will address the study of the resonances with homoclinics in region III. We note that, in this case, the model to consider is a return map model, similar to the separatrix map (see Refs. 18 and 26 for recent developments), but containing the dissipative effects. Preliminary results show that the limit probability of capture does not depend, in a first order approximation, on the existence of homoclinic points in the resonant structure (provided that in the conservative limit case the island is located close enough to the origin).

On the other hand, we have shown some difficulties of dealing with weakly dissipative dynamics when performing numerics. Clearly, weakly dissipative dynamics deserve further investigations even in the planar case. We think that the methodology shown in this paper is the proper one to deal with these systems, and also with generalizations to weakly dissipative perturbations of four dimensional symplectic maps. Nevertheless, we note that in higher dimensions, there is a lack of knowledge even in the conservative setting since, to address the type of questions addressed in this paper, it is required not only a topological approach but also a quantitative one.

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- ¹Arnold, V. I., *Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires* (Mir, Moscow, 1978).
- ²Arrowsmith, D. K. and Place, C. M., *An Introduction to Dynamical Systems* (Cambridge University Press, Cambridge, 1990).
- ³Broer, H., Roussarie, R., and Simó, C., "Invariant circles in the Bogdanov-Takens bifurcation for diffeomorphisms," *Ergod. Theory Dyn. Syst.* **16**, 1147–1142 (1996).
- ⁴Broer, H., Simó, C., and Tatjer, J. C., "Towards global models near homoclinic tangencies of dissipative diffeomorphisms," *Nonlinearity* **11**, 667–770 (1998).
- ⁵Celletti, A., "Weakly dissipative systems in celestial mechanics," *Lecture Notes in Physics* **729**, 67 (2007).
- ⁶del Castillo-Negrete, D., Greene, J. M., and Morrison, P. J., "Area preserving nontwist maps: Periodic orbits and transition to chaos," *Physica D* **91**, 1–23 (1996).
- ⁷del Castillo-Negrete, D. and Morrison, P. J., "Chaotic transport by Rossby waves in shear flow," *Phys. Fluids A* **5**, 948–965 (1993).
- ⁸Dullin, H. R., Meiss, J. D., and Sterling, D., "Generic twistless bifurcations," *Nonlinearity* **13**, 203–224 (1999).
- ⁹Farrés, A. and Jorba, Á., "Periodic and quasi-periodic motions of a solar sail close to SL_1 in the Earth-Sun system," *Celest. Mech. Dyn. Astron.* **107**, 233–253 (2010).
- ¹⁰Fontich, E. and Simó, C., "The splitting of separatrices for analytic diffeomorphism," *Ergod. Theory Dyn. Syst.* **10**, 295–318 (1990).
- ¹¹Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Natural Sciences, Vol. 42 (Springer, New York, 1997).
- ¹²Hénon, M., "Numerical study of quadratic area-preserving mappings," *Q. Appl. Math.* **27**, 291–312 (1969).
- ¹³Hénon, M., "A two dimensional mapping with a strange attractor," *Commun. Math. Phys.* **50**, 69–77 (1976).
- ¹⁴Holmes, P. and Whitley, D., "Bifurcations of one- and two-dimensional maps," *Philos. Trans. R. Soc. London, Ser. A* **311**, 43–102 (1984).
- ¹⁵McLaughlin, J., "Stochastic behavior in slightly dissipative systems," *Phys. Rev. A* **20**, 2114–2119 (1979).
- ¹⁶Neishtadt, A. I., "On probabilistic phenomena in perturbed systems," *Sel. Math. Sov.* **12**, 195–210 (1993).
- ¹⁷Neishtadt, A. I., "On averaging in two-frequency systems with small Hamiltonian and much smaller non-Hamiltonian perturbations," *Mosc. Math. J.* **3**, 1039–1052 (2003).
- ¹⁸Piftankin, G. N. and Treschev, D. V., "Separatrix maps in Hamiltonian systems," *Russ. Math. Surveys* **2**, 219–322 (2007).
- ¹⁹Rabassa, P., "Contribution to the study of perturbations of low dimensional maps," Ph.D. thesis, Universitat de Barcelona, 2010.
- ²⁰Samelson, R. M., "Fluid exchange across a meandering jet," *J. Phys. Oceanogr.* **22**, 431–444 (1992).
- ²¹Siegel, C. L. and Moser, J. K., *Lectures on Celestial Mechanics* (Springer-Verlag, Berlin, 1971).
- ²²Simó, C., "Invariant curves of analytic perturbed nontwist area preserving maps," *Regular Chaotic Dyn.* **3**, 180–195 (1998).
- ²³Simó, C. and Stuchi, T., "Central stable/unstable manifolds and the destruction of KAM tori in the planar Hill problem," *Physica D* **140**, 1–32 (2000).
- ²⁴Simó, C. and Viero, A., "A numerical exploration of weakly dissipative two-dimensional maps," *Proceedings of ENOC'05*, 2005.
- ²⁵Simó, C. and Viero, A., "Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps," *Nonlinearity* **22**, 1191–1245 (2009).
- ²⁶Simó, C. and Viero, A., "Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones," *Physica D* (submitted).
- ²⁷Simó, C. and Viero, A., "Dynamical description of the domain of stability of area preserving maps," (unpublished).
- ²⁸Simó, C. and Viero, A., "Planar radial weakly-dissipative diffeomorphisms: Homoclinic-type resonances"(unpublished).
- ²⁹Takens, F., "Forced oscillations and bifurcations," *Applications of Global Analysis, I* (Math. Inst. Rijksuniv., Utrecht, 1974), pp. 1–59.
- ³⁰Wolansky, G., "Limit theorem for a dynamical system in the presence of resonances and homoclinic orbits," *J. Differ. Equations* **83**, 300–335 (1990).