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External dichotomous noise: The problem of the mean first-passage time

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A retarded backward equation for a non-Markovian process induced by dichotomous noise (the random telegraphic signal) is deduced. The mean first-passage time of this process is exactly obtained. The Gaussian white-noise and the white-shot-noise limits are studied. Explicit physical results in first approximation are evaluated.

The study of the dynamics of stochastic processes induced by an external nonwhite noise has recently received an increasing interest not only from a theoretical point of view[1–8] but also from the experimental one.[9,10] The dynamical quantities to focus on are, among others, the relaxation time of the correlation function,[1–3,9] the activation rate of a metastable state,[5,7] and the mean first-passage time (MFPT).[7,8] The MFPT is widely used for evaluating different time scales in dynamical problems such as the decay of an initial state in nonequilibrium situations. A particularly important one is the study of the decay of a metastable or unstable state in a bistable system.[5,7]

Many systems of practical interest[6,11] can be described by a single relevant variable. This variable obeys a first-order differential equation of motion which depends on external control parameters. If it is assumed that there exists a fluctuating control parameter (external noise[6]) the deterministic equation of motion transforms into a stochastic one of the form

$$\dot{x}=f(x)+q(x)\xi(t) \quad , \tag{1}$$

where $\xi(t)$ is the external noise, $f(x)$ is the deterministic flow, and $g(x)$ represents a linear coupling of the fluctuating parameter with the variable x. This is the most general case[6] in a variety of real situations.[1–12] If one assumes that $\xi(t)$ is a white noise then the process $x(t)$ is Markovian[11] and the problem of the MFPT has been already solved.[7,11] When $\xi(t)$ is a nonwhite noise (noise with finite correlation time) the process $x(t)$ is non-Markovian[4–8,11] and the problem of the MFPT remains unsolved.

In this Rapid Communication the problem of calculating the MFPT is addressed for the first time when the external noise $\xi(t)$ is a dichotomous noise. A backward equation for the probability density of the process $x(t)$ is derived. From this equation an exact formula for the MFPT is obtained exactly. The main conclusion is that the effect of the nonwhite noise is important as compared with that of the white noise.

The mathematical properties of this stochastic process are well known:[6,7] The process $\xi(t)$ takes two possible values, $a'>0$ and $a<0$, with transition rates μ' and μ, respectively. The stationary mean value of $\xi(t)$ will be assumed equal to zero: i.e., $\mu a'+\mu' a=0$. The correlation time of $\xi(t)$ is then $\tau=\lambda^{-1}=(\mu+\mu')^{-1}$. Modeling $\xi(t)$ in (1) by a dichotomous noise has several practical advantages. The stationary distribution of the process $x(t)$ can be evaluated exactly. This is not the case in other situations, for example, if $\xi(t)$ is taken to be an Ornstein-Uhlenbeck noise. Also, a dichotomous noise can be easily modeled in the laboratory[6,12] and in particular limits it reduces to other interesting noise models.

Up to now the only dynamical result available for nontrivial systems influenced by dichotomous noise is the approximate evaluation of the MFPT through the calculation of the activation rate of a metastable state.[5,7] Nevertheless the solution of this problem is not well defined for all the values of the intensity of the noise.[5] On the contrary, the problem of the MFPT is well formulated. For a Markovian process the standard approach starts with the deduction of a backward equation of motion for the probability density $P(x,t)$ $[P(x,0)=\delta(x-x_0)]$,

$$\dot{P}(x,t)=\Gamma^{+}_{x_0}P(x,t) \quad , \tag{2}$$

where $\Gamma^{+}_{x_c}$ is the adjoint (backward) operator. The mean first-passage time $T_1(x_0)$ obeys then the equation[11]

$$\Gamma^{+}_{x_0}T(x_0)=-1 \quad , \tag{3}$$

where the boundary conditions should be defined in each particular situation. The backward equation for $P(x,t)$ of a non-Markovian process is only known in a few cases.[13] A first original result of this work is the derivation of the backward equation for the process (1) when $\xi(t)$ is dichotomous noise. This equation will be used here to obtain the exact expression for the mean first-passage time. Approximate backward equations for other types of non-Markovian processes can be found in Ref. 14.

The master equation for the probabilities $P(x,a;t)$ and $P(x,a';t)$ is well known.[6,7] One can also write the equations for the reduced probability $P(x,t)$ and $P(x,a';t)$, where

$$P(x,t)\equiv\langle P(x,\xi;t)\rangle_{\xi}=P(x,a;t)+P(x,a';t) \quad . \tag{4}$$

They explicitly read

$$\dot{P}(x,t)=-\partial_x[f(x)+ag(x)]P(x,t) -(a'-a)\partial_x g(x)P(x,a',t) \quad , \tag{5a}$$

$$\dot{P}(x,a';t)=\mu P(x,t) -\{\lambda+\partial_x[f(x)+a'g(x)]\}P(x,a',t) \quad . \tag{5b}$$

This is a forward master equation but our interest is to find the associated backward equation. The first step in this direction is to take the Laplace transform of (5) with the initial conditions

$$P(x,0)=\delta(x-x_0);\ \ P(x,a';0)=0 \quad . \tag{6}$$

Then the formal solution for the Laplace transform of $P(x,t)$, $\hat{P}(x,s)$ is

$$\hat{P}(x,s)=(s+\partial_x[f(x)+ag(x)]+(a'-a)\mu\partial_x g(x)\{s+\lambda+\partial_x[f(x)+a'g(x)]\}^{-1})^{-1}\delta(x-x_0) \ . \tag{7}$$

Using the properties of the delta function, Eq. (7) can be rewritten also in a backward form

$$\hat{P}(x,s)=(s-[f(x_0)+ag(x_0)]\partial_{x_0}-(a'-a)\mu\{s+\lambda-[f(x_0)+a'g(x_0)]\partial_{x_0}\}^{-1}g(x_0)\partial_{x_0})^{-1}\delta(x-x_0) \ . \tag{8}$$

We define now the auxiliary quantity

$$\hat{H}(x,s)=(a'-a)\{s+\lambda-[f(x_0)+a'g(x_0)]\partial_{x_0}\}^{-1}g(x_0)\partial_{x_0}\hat{P}(x,s) \ . \tag{9}$$

Inverting the Laplace transform in Eqs. (8) and (9) we obtain the backward equation corresponding to Eq. (5):

$$\dot{P}(x,t)=[f(x_0)+ag(x_0)]\partial_{x_0}P(x,t)+\mu H(x,t) \ , \tag{10a}$$

$$\dot{H}(x,t)=(a'-a)g(x_0)\partial_{x_0}P(x,t)+\{-\lambda+[f(x_0)+a'g(x_0)]\partial_{x_0}\}H(x,t) \ , \tag{10b}$$

where $H(x,t=0)=0$. $H(x,t)$ is formally obtained from (10b)

$$H(x,t)=\int_0^t dt'\exp(\{-\lambda+[f(x_0)+a'g(x_0)]\partial_{x_0}\}(t-t'))(a'-a)g(x_0)\partial_{x_0}P(x,t') \ , \tag{11}$$

which is substituted in the equation for $P(x,t)$ to obtain a retarded backward equation which has the useful form

$$\dot{P}(x,t)=\int_0^t F(x_0,\partial_{x_0};t-s)P(x,s)\,ds \ , \tag{12}$$

where

$$F(x_0,\partial_{x_0};t-s)=[f(x_0)+ag(x_0)]\partial_{x_0}2\delta(t-s)+\mu(a'-a)\exp(\{-\lambda+[f(x_0)+a'g(x_0)]\}(t-s))g(x_0)\partial_{x_0} \ . \tag{13}$$

Equations of the type of (12) have been studied by Hänggi and Talkner[13] and they have proved that the mean first-passage time T_1 of the non-Markovian process (12) obeys an equation similar to (3):

$$\Omega^+_{x_0}T(x_0)=-1 \ , \tag{14}$$

where $\Omega^+_{x_0}$ is given now in terms of $F(x_0,\partial_{x_0};t-s)$ by

$$\Omega^+_{x_0}=\int_0^\infty F(x_0,\partial_{x_0};s)\,ds \ . \tag{15}$$

The explicit equation satisfied by T_1 is

$$([f(x_0)+ag(x_0)]\partial_{x_0}+\mu(a'-a)$$
$$\times\{\lambda-[f(x_0)+a'g(x_0)]\partial_{x_0}\}^{-1}g(x_0)\partial_{x_0})T_1=-1 \ , \tag{16}$$

which can be reduced to a second-order differential equation for T_1 (but of first-order for $\partial T_1/\partial x_0$)

$$D_{\text{eff}}(x_0)\partial^2_{x_0}T_1=\{-\lambda f(x_0)+[f(x_0)+a'g(x_0)]$$
$$\times[f'(x_0)+ag'(x_0)]\}\partial_{x_0}T_1-\lambda \ , \tag{17}$$

where

$$D_{\text{eff}}(x_0)=-[f(x_0)+a'g(x_0)][f(x_0)+ag(x_0)] \ . \tag{18}$$

This quantity is positive if $x_0\in(x_a,x_b)$, where x_a is the left natural boundary of this process[6] $[f(x_a)+ag(x_a)=0]$ and x_b is the right natural boundary[6] $[f(x_b)+a'g(x_b)=0]$. Equation (17) for T_1 can be solved with the boundary conditions[7]

$$\partial_{x_0}T_1(x_0)|_{x_c}=0, \quad T_1(x_2)=0 \ . \tag{19}$$

The first condition means that we have a reflecting boundary at $x_c\geq x_a$ and the second one means that there is an absorbing boundary at $x_2\leq x_b$.[7] These conditions physically mean that T_1 is the mean first-passage time of a particle starting in $x_c\in(x_0,x_2)$, which escapes from this domain through the point x_2.[7]

Using (19) the solution of (17) is

$$T_1(x_0)=\lambda\int_{x_0}^{x_2}\frac{dx}{\psi(x)D_{\text{eff}}(x)}\int_{x_0}^{x}\psi(y)\,dy \ , \tag{20}$$

where

$$\psi(x)=\frac{|a|g(x)-f(x)}{g(x)}P_{St}(x)$$
$$=[a'g(x)+f(x)]^{-1}\exp\left[\int^x\frac{\lambda f(x')}{D_{\text{eff}}(x')}dx'\right] \ . \tag{21}$$

This is the second main result of this work.

The solution (20) is also useful to study two Markovian limits.[7]

(i) Gaussian white-noise limit. This is obtained with the following limiting procedure:

$$a'=|a|=\infty, \quad \mu+\mu'=\infty, \quad D=\frac{|a|^2}{\lambda}=\text{const} \ . \tag{22}$$

(ii) White-shot-noise limit. The values of the parameters are now

$$a'=\infty, \quad \mu'=\infty, \quad \frac{a'}{\mu'}=-\frac{a}{\mu}=\frac{D}{|a|}=\text{const} \ , \tag{23}$$

where the parameter D is usually called the intensity of the noise. The difference between these two noises is the Gaussian property.

Simple analytical results for $T_1(x_0)$ are in general difficult to obtain from (20). However, reliable information on scape times can be obtained by standard approximate methods.[5] The way to proceed is the following. One can consider that the parameter $D'=a'|a|/\lambda$ is a small quantity. Physically this means that the stochastic motion of x is dominated by the deterministic flow $f(x)$ in (1) and the relative fluctuations around the deterministic trajectory are small. The deterministic steady states are the real roots of the equation

$$f(x)=0 \quad . \tag{24}$$

For the sake of simplicity one can consider the case of three real roots $x_1 < \tilde{x} < x_2$, where $x_1 > x_a$ and $x_2 < x_b$ are stable steady states and $\tilde{x}$ is unstable (bistable process). Let us now evaluate the MFPT of a particle initially in the stable state x_1, which scapes to the other stable state x_2 when the reflecting bounding at x_c coincides with the left natural boundary of this process x_a. At this point it is convenient to introduce an effective "potential" $\hat{u}(x)$ by

$$\frac{\hat{u}(x)}{D'} \equiv -\lambda \int^x \frac{f(x')}{D_{\text{eff}}(x')} dx' \quad , \tag{25}$$

whose extrema are also the solutions of (24). Using (21) and (25) the result (20) takes the explicit form

$$T_1(x_1) = \frac{1}{D'} \int_{x_1}^{x_2} \frac{dx\, e^{\hat{u}(x)/D'}}{g(x)-f(x)/|a|} \int_{x_a}^{x} dy \frac{e^{-\hat{u}(y)/D'}}{g(y)+f(y)/a'} \quad . \tag{26}$$

The exponential term in the first integral becomes dominant near $\tilde{x}$ and then the integral can be evaluated approximately by the method of steepest descent.[15] For the second integral (the upper limit is now $\tilde{x}$) the exponential term is dominant near x_1 and the way to proceed is the same one. The final result for $T_1(x_1)$ is

$$T_1(x_1) \simeq \frac{2\pi}{|f'(x_1)f'(\tilde{x})|^{1/2}} \exp(\Delta\hat{\mu}/D') \quad , \tag{27}$$

where the exponential term is the Arrhenius factor and

$$\frac{\Delta\hat{u}}{D'} = -\lambda \int_{x_1}^{\tilde{x}} \frac{f(x')}{D_{\text{eff}}(x')} dx' \quad . \tag{28}$$

In (27) one can distinguish two factors. The most important one is the second since the exponential is dominant when D' is small. It has the same formal expression for a Gaussian white noise and shot noise but with different $D_{\text{eff}}(x)$ in each case.[7] It means that for these three noises and small enough D', the process (1) can be approximated by a Markovian Fokker-Planck equation with the corresponding effective diffusion $D_{\text{eff}}(x)$. Taking the limits (22) and (23) in (27) we recover well known results.[7,15]

From (27) one can also conclude that the effect of the correlation time of the noise is dynamically very important since it appears in the exponential factor through (28). A lengthy but straightforward calculation shows that $T_1(x_1)$ in the approximation (27) is larger than the corresponding value for the white-shot-noise limit (23) if $g(x) > 0$. This is not the case if the negative impulses of $\xi(t)$ are larger than the positive ones ($|a| > a'$). This corresponds to a different white-shot-noise limit ($a \to \infty$, $\mu \to \infty$) not considered in (23). The same conclusion is obtained if we compare (27) with the Gaussian white-noise limit (22) but only in the case of symmetric dichotomous noise ($a' = |a|$, $\mu' = \mu$). The fact that the time scale of a non-Markovian process is larger than for a Markovian one with the same D' has been also observed in other situations with symmetric noises.[1-3] This is physically understood since a nonwhite symmetric noise is not an "instantaneous" process like a white noise and hence it slows down the physical variable x in (1). For a nonsymmetric dichotomous noise no general statement can be drawn from (27). Hence one can conclude that the finite relaxation time τ and the lack of symmetry of the random impulses of the noise play a very different role in the dynamics of the noise-induced process (1). This is in agreement with Ref. 7 where an approximate study of (1) is presented. Nevertheless, numerical analysis of (20) will give us exact information about this problem for all the values of the noise parameters.

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