Universitat de Barcelona
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GRAU DE MATEMÀTIQUES

Treball final de grau

# A Mathematical Perspective on 0-Dimensional Quantum Field Theory

Autor: Elena Gijón Ruiz

Director: Dr. Bartomeu Fiol
Dra. Joana Cirici
Realitzat a: Departament de Física Quàntica i Astrofísica

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# Contents

## Abstract

This thesis explores 0-dimensional quantum field theory from a combinatorial and diagrammatic perspective. In this setting, fields reduce to ordinary variables, and path integrals become standard integrals. This serves as a toy model to develop mathematical tools that are then applicable to higher-dimensional theories.

The central objects of study are partition functions and correlation functions. We focus on their perturbative expansion, which is interpreted combinatorially in terms of Feynman diagrams. This is developed in a rigorous mathematical framework with the use of generating functions, group theory, and graph enumeration techniques. Specifically, we analyse the role of labelled, unlabelled, connected, and bridgeless graphs in encoding the structure of these expansions. The first part of this work establishes this foundation. Then, this theory is applied to certain models in 0-dimensional quantum field theory, showing how the formal power series expansions lead to diagrammatic representations.

In the second part, we acknowledge the divergence of these formal series and address the issue by introducing Padé approximants. These are rational approximants that often yield better convergence properties. We prove their convergence in a specific interacting model and show their efficiency in recovering accurate numerical results.

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## Resum

Títol del TFG: Teoria Quàntica de Camps en Dimensió 0 des d'una Perspectiva Matemàtica.

Aquesta tesi explora la teoria quàntica de camps en dimensió zero des d'una perspectiva combinatòria i diagramàtica. En aquest context, els camps es redueixen a variables ordinàries i les integrals de camí esdevenen integrals estàndard. Això serveix com a model simplificat per desenvolupar eines matemàtiques aplicables posteriorment a teories de dimensions superiors.

Els objectes centrals d'estudi són les funcions de partició i les funcions de correlació. Ens centrem en l'anàlisi en la seva expansió pertorbativa, que s'interpreta de manera combinatòria en termes de diagrames de Feynman. Aquesta anàlisi es desenvolupa dins d'un marc matemàtic rigorós mitjançant funcions generadores, teoria de grups i tècniques d'enumeració de grafs. En particular, s'analitza el paper dels grafs etiquetats, no etiquetats, connexos i sense ponts en la codificació de l'estructura d'aquestes expansions. La primera part del treball estableix aquest fonament. Posteriorment, aquesta teoria s'aplica a certs models de teoria quàntica de camps en dimensió zero, mostrant com les expansions en sèries de potències formals condueixen a representacions diagramàtiques.

En la segona part, es considera la divergència d'aquestes sèries formals i s'aborda aquesta qüestió mitjançant la introducció dels aproximants de Padé. Aquests són aproximants racionals que sovint presenten millors propietats de convergència. Es demostra la seva convergència en un model d'interacció concret i es posa de manifest la seva eficàcia a l'hora d'obtenir resultats numèrics precisos.

## Acknowledgements

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# Introduction

## Why Quantum Field Theory?

Electrons and photons exhibit similar quantum behaviour, such as wave-particle duality, yet their origins appear fundamentally different. In classical physics, electrons are treated as elementary particles, while photons emerge as excitations of the electromagnetic field. This distinction leads to a deeper question: What is truly fundamental, particles or fields?

Quantum Field Theory (QFT) provides a powerful framework to address this question. Rather than viewing fields as emerging from the collective behaviour of many fundamental particles, QFT reverses the perspective: fields are the fundamental entities, and their quantised excitations give rise to particles. In this picture, there should be (at least theoretically) an electron field, a neutrino field, and so on.

This shift in viewpoint allows QFT to address some fundamental questions of the quantum world:

1. *Locality*: As in classical physics, nothing happens in a non-local way. The primary reason for introducing the concept of a field is to construct local laws of physics. Particles influence and are influenced through local changes in their respective fields.

2. *Indistinguishability*: It is a well-established fact that all particles of the same type are inherently indistinguishable—a natural outcome when particles are treated as identical excitations of a common underlying field.

3. *Variable particle number*: The combination of special relativity and quantum mechanics leads to scenarios in which particle number is not conserved. QFT provides a framework for describing such processes, allowing us to handle quantum states with an arbitrary number of particles.

## Quantum Field Theory Setup

There are several formulations of Quantum Field Theory, each based on a distinct mathematical framework. In this work, we will consider the Functional Quantisation formulation of QFT. In this formalism, the central objects of study are integrals over infinite-dimensional spaces of field configurations. In particular, we are interested in computing integrals such as

$$\langle \phi_{j_1}(x_1,t_1)\cdots\phi_{j_n}(x_n,t_n)\rangle = \frac{1}{Z}\int \phi_{j_1}(x_1,t_1)\cdots\phi_{j_n}(x_n,t_n)e^{\frac{iS(\phi)}{\hbar}}D\phi, \tag{1}$$

where

$$Z = \int e^{\frac{iS(\phi)}{\hbar}}D\phi, \tag{2}$$

and the term $S(\phi)$ is the *action* of the system. Integrals like (1) are called *correlation functions* or *n-point functions*, and $Z$ is the *partition function*. Although these integrals completely describe the theory of interest, they are mathematically ill-defined in most cases. The reason is that the symbol $D\phi$ in (1) and (2) generally lacks a rigorous mathematical definition: it is intended to represent an extension of the familiar Lebesgue measure on finite-dimensional spaces to infinite-dimensional ones. However, it can be argued that a generalisation preserving all the standard properties of the Lebesgue measure does not exist in infinite-dimensional spaces, and a suitable definition of $D\phi$ remains unknown. The only exception is QFTs in one dimension, for which the Wiener measure [22] provides a rigorous definition of $D\phi$.

A possible approach to address these issues is to study simplified models where the core ideas of QFT can be explored without adding the complexities that arise in higher-dimensional settings. The goal of this thesis is to explore QFT in zero dimensions from a mathematical perspective. In this framework, fields reduce to ordinary variables, and integrals (1) and (2) become ordinary integrals:

$$\langle \phi^n\rangle = \frac{1}{Z}\int \phi^n e^{\frac{iS(\phi)}{\hbar}}d\phi, \tag{3}$$

where

$$Z = \int e^{\frac{iS(\phi)}{\hbar}}d\phi. \tag{4}$$

This might seem like a drastic simplification, but 0-dimensional QFT preserves some of the essential features, serving as a toy model to develop the main techniques. As such, it provides a fertile ground for the development of mathematical tools that are applicable to higher-dimensional theories.

From a physical point of view, the action $S(\phi)$ is typically of the form

$$S(\phi) = \frac{1}{2}m\phi^2 - j\phi - V(\phi), \tag{5}$$

where $j\phi$ is called the *source term* and $V(\phi) \in \phi^3\mathbb{R}[[\phi]]$ is the *potential energy*. The key idea is that these terms can be thought of as a perturbation. In this work, the main focus is on exploring the combinatorial and algebraic structure underlying these perturbative expansions in 0-dimensional QFT. Treating integrals (3) and (4) as formal power series and interpreting the coefficients in terms of graphs gives rise to Feynman diagrams in 0 dimensions.

## Historical background

The diagrammatic expansion for the evaluation of correlation functions in QFT was introduced by Feynman in his ground-breaking and Nobel-prize deserving work [10]. He introduced a set of graphs—known as *Feynman diagrams*—along with a corresponding set of rules—*Feynman rules*—that associate each diagram with an analytical expression. The simplest of these rules assigns to each graph a weight equal to the inverse of the order of its automorphism group, a quantity referred to as *symmetry factor* in physics literature. The remaining rules involve more intricate integrals over space and time.

Interest in the large-order behaviour of perturbative series has led to the study of QFT in lower dimensions, where the enumeration and symmetry factors of Feynman diagrams remain independent of spacetime dimension. In particular, in 0-dimensional QFT—where there is no spacetime to integrate over—Feynman rules reduce to assigning each diagram a weight equal to its symmetry factor. This simplified setting isolates the combinatorial and group-theoretic structure inherent in more realistic quantum field theories, putting aside the complexities introduced by spacetime dependence. One of the earliest works to highlight this perspective is [6]. More recent developments in 0-dimensional QFT are presented in the works of Yeats [23], Borinsky [5], and Etingof [8], among others.

## Structural Outline of the Thesis

The central theme of the thesis is the use of graph theory and combinatorics to understand the expansion of partition functions and correlation functions. In Chapter 1, following the work of Borinsky [5] and Yeats [23], we introduce a less conventional but useful definition of a graph, which will serve as the foundation for the following chapters, and we outline general aspects of generating functions. Then, we introduce the combinatorial notions of labelled and unlabelled graphs and use specific generating functions to count these graphs. In particular, we see how to transition from counting labelled graphs to unlabelled ones, making use of elementary group theory. Another significant concept introduced in this chapter is the *symmetry factor* of a Feynman diagram, which is closely related to the automorphism group of the graph and plays a crucial role in the generating functions of graphs. The treatment of symmetry factors and their computation is based on the detailed analysis given in [7].

Following this combinatorial and algebraic groundwork, we apply the developed theory to concrete models of 0-dimensional QFT in Chapter 2. We start by studying *free field theories*, where the source term and the potential energy in the action (5) are absent, and the integrals reduce to Gaussians. In this context, and following Etingof's lecture notes [8], Wick's theorem gives an explicit formula to compute the correlation functions, which can be interpreted diagrammatically. This serves as a preface to Feynman diagrams.

We then turn to *interacting theories*, in which the action $S(\phi)$ takes its full form. This leads to the main integrals (3) and (4), which are no longer analytically solvable in general but admit a formal expansion in terms of graphs. The main reference for this approach is Borinsky's thesis [5]. We show that the partition function $Z$ serves as the generating function for all Feynman diagrams. The remainder of the chapter explores related partition functions. In particular, we show how $Z$ can be reduced to the generating function of *connected* diagrams, denoted $W$, thereby simplifying the combinatorial enumeration. This refinement can be taken a step further by isolating the contribution of *bridgeless* graphs, leading to a more compact generating function, $G$.

This first part considers the perturbative expansions as formal power series, so convergence is not a concern. However, it turns out that these series are not convergent, but rather asymptotic. Therefore, in the second part of the thesis (Chapter 3), we address this issue by introducing *Padé approximants*, which are extensively studied in the work of Baker and Graves-Morris [3]. These approximants provide rational function approximations to a formal power series, often yielding better convergence properties and analytic insights. We then apply them to a specific case in 0-dimensional QFT, proving that these approximants converge and yield extremely accurate values compared to the exact solution. This final example is, to the best of our knowledge, an original contribution. It demonstrates how tools from classical analysis can be combined with the combinatorial framework of 0-dimensional QFT to provide accurate, convergent approximations to divergent perturbative series.

Overall, this work contributes to the theoretical understanding of combinatorial and algebraic structures that arise in 0-dimensional quantum field theory. The aim is to examine the mechanisms behind perturbative expansions from a rigorous mathematical perspective and then to explore an alternative approximation, which yields better numerical results.

# Chapter 1

# Graphs and Generating Functions

Before we get to 0-dimensional quantum field theory, a brief introduction to graphs and generating functions is necessary. The main references for this chapter are [5], [23], and [19].

In this section, $K$ is the base field, and we assume that the characteristic is 0.

## 1.1 Combinatorial Classes and Generating Functions

**Definition 1.1.** *A* ***combinatorial class*** $\mathscr{C}$ *is a set and a size function* $||\cdot|| : \mathscr{C} \to \mathbb{Z}_{\geq 0}$ *such that* $\mathscr{C}_n = \{c \in \mathscr{C} : |c| = n\}$ *are all finite.*

$\{C_n\}_{n\geq 0}$ *is the* ***counting sequence*** *of a combinatorial class* $\mathscr{C}$*, where* $C_n = |\mathscr{C}_n|$.

Let's look at some examples.

**Example 1.2.** *Binary words*. Let $\mathscr{W}$ be the set of binary words, which are sequences built from 0 and 1,

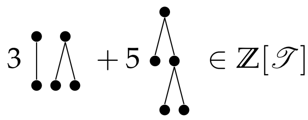
$$\mathscr{W} = \{e, 0, 1, 00, 10, 01, 11, 000, 001, 010, \dots\},$$

where $e$ is the empty word. This set, along with a size function which returns the number of letters of a word, gives the combinatorial class of binary words. Notice that the counting sequence $\{W_n\}_{n\geq 0}$ is given by $W_n = 2^n$ (for each letter there are two possibilities).

The following is a crucial example and will be essential for future chapters.

**Example 1.3.** *Rooted trees*. Let us define a rooted tree as a finite simple graph (i.e., with no loops and no multi-edges), with no cycles, and with a certain vertex called the root. For each vertex $v$, different from the root, there is a unique vertex adjacent to $v$ and closer to the root. This vertex is called $v$'s *parent*. If there are any vertices with $v$ as a parent, they are called $v$'s *children*. Vertices with no children are called *leaves*. Given a tree $t$, the subtree consisting of $v$ and all its children, and all their children, and so on, is called the *subtree* rooted at $v$, and we denote it as $t_v$. Given the set of rooted trees and a size function that *counts* the number of vertices of a tree, $t \mapsto$ number of vertices of $t$, we get the combinatorial class of rooted trees. We will denote this class as $\mathscr{T}$.

Note that any combinatorial class $\mathscr{C}$ can be made into an algebra by simply taking the polynomial algebra $K[\mathscr{C}]$ with generators the elements of $\mathscr{C}$. Here, addition is completely formal—a sum of trees is just a sum of trees, it is not identified with any other object. If $\mathscr{T}$ is the class of rooted trees, then




is a formal sum of trees.

The main problem in enumerative combinatorics is counting. In general, we deal with a collection of finite sets $S_i$, where $i$ ranges over some index set $I$, and we wish to count the number $a_i$ of elements in each $S_i$ "simultaneously". In some rare cases, these $a_i$ can be given as an explicit formula. For instance, say we want to count the number of subsets of the set $[n] = \{1, 2, 3, \ldots, n\}$, then $a_n = 2^n$. However, in general, these formulas do not exist or are extremely difficult to obtain.

The problem of obtaining these $a_i$ can be tackled in different ways. One can define a recurrence so that each $a_i$ can be calculated in terms of the previous ones. Algorithms may also be given for computing $a_i$. But the most useful method for evaluating them is to provide their ***generating function***. Informally, a generating function is an "object" that represents a counting function $a_i$, and usually, this object is a *formal power series*. The most common generating functions are the ***ordinary*** generating functions and the ***exponential*** generating functions. Let's take $I = \mathbb{N}$ for simplicity.

**Definition 1.4.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. The **ordinary generating function** (OGF) of this sequence is the formal power series*

$$O(x) = \sum_{n=0}^{\infty} a_n x^n.$$

**Definition 1.5.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. The **exponential generating function** (EGF) of this sequence is the formal power series*

$$E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

The power series is called *formal* because we do not let $x$ take on particular values, so we are not concerned with convergence or divergence.

We denote the ***coefficient extraction*** for an ordinary generating function as

$$a_n = [x^n]O(x).$$

Analogously, for an exponential generating function

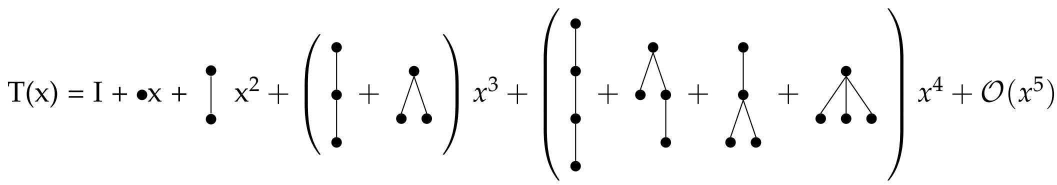
$$a_n = n![x^n]E(x).$$

Going back to combinatorial classes, we can define generating functions that keep the objects of our set in the sums.

**Definition 1.6.** *Let $\mathscr{C}$ be a combinatorial class. The* ***augmented generating function*** *of $\mathscr{C}$ is the formal power series*

$$C(x) = \sum_{c \in \mathscr{C}} c x^{|c|} \in (K[\mathscr{C}])[[x]]$$

If we continue with our previous example $\mathscr{T}$, the augmented generating function of the class of rooted trees begins as follows:



The next thing we would like to do is define a map which assigns a "weight" to each element of $\mathscr{C}$.

**Definition 1.7.** *A map $\varphi : K[\mathscr{C}] \to A$, where $A$ is some algebraic structure over $K$, is called an* ***evaluation map****.*

The most simple example is the evaluation map $\varphi_{\text{or}}(c) = 1, \forall c \in \mathscr{C}$. With this map, one gets the ordinary generating function

$$\sum_{c \in \mathscr{C}} x^{|c|} = \varphi_{\text{or}}(C(x)).$$

For instance, for the combinatorial class of rooted trees, $\varphi_{\text{or}}(T(x)) = 1 + x + x^2 + 2x^3 + 4x^4 + \mathcal{O}(x^5)$.

Another evaluation map, which will be useful for future sections, is $\varphi_{\text{ex}}(c) = \frac{1}{|c|!}, \forall c \in \mathscr{C}$. This map yields the exponential generating function

$$\sum_{c \in \mathscr{C}} \frac{x^{|c|}}{|c|!} = \varphi_{\text{ex}}(C(x)).$$

Again, for $T(x)$ we get $\varphi_{\text{ex}}(T(x)) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \mathcal{O}(x^5)$.

Until now, we only considered generating functions of one variable. However, sometimes one wants to keep track of various parameters. So, we can generalize the previous definitions.

**Definition 1.8.** *Let $\{a_{n_1,\dots,n_k}\}_{n_i \in \mathbb{N}}$ be a sequence. The* ***multivariate ordinary generating function*** *of this sequence is the formal power series*

$$O(x_1, \dots, x_k) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} a_{n_1,\dots,n_k} x_1^{n_1} \cdots x_k^{n_k}.$$

**Definition 1.9.** *Let $\{a_{n_1,\dots,n_k}\}_{n_i \in \mathbb{N}}$ be a sequence. The* ***multivariate exponential generating function*** *of this sequence is the formal power series*

$$E(x_1, \dots, x_k) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} a_{n_1,\dots,n_k} \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_k^{n_k}}{n_k!}.$$

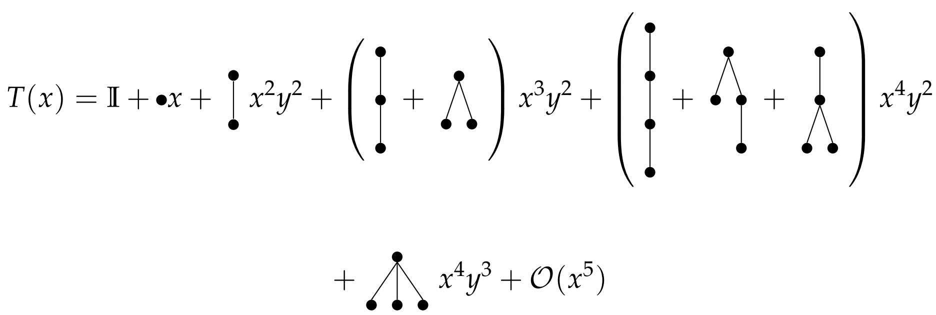
The ***coefficient extraction*** is denoted as before, but now we deal with more variables:

$$a_{n_1,\dots,n_k} = [x_1^{n_1} \cdots x_k^{n_k}] O(x_1, \dots, x_k)$$

and,

$$a_{n_1,\dots,n_k} = n_1! \cdots n_k! [x_1^{n_1} \cdots x_k^{n_k}] E(x_1, \dots, x_k).$$

For instance, say we want to count, a part from the number of vertices, the number of leaves of a tree, then we can introduce an additional variable $y$ and an additional size function $|\cdot|_{\text{leaves}} : \mathscr{T} \to \mathbb{Z}_{\geq 0}$, which assigns to each graph $t$ the number of leaves. Then, the **multivariate generating function** is given by




Again, we could define evaluation maps for each parameter.

## 1.2 Graphs

As I have mentioned, trees and graphs in general play a huge role in the most common computational approach in quantum field theory. The most common definition of a graph involves *vertices* and *edges*. However, here it will be more optimal to use an alternative definition with *half-edges*, instead of *edges* and *vertices*.

**Definition 1.10.** *A **graph** is a tuple* $(H, V, \nu, E)$*, where*

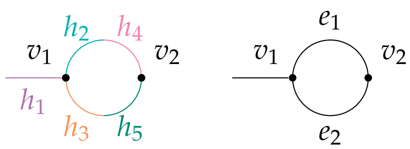
- $H$ *is a set of half-edges.*
- $V$ *is a set of vertices.*
- $\nu : H \to V$ *is a map that assigns half-edges to vertices.*
- $E \subset 2^H$*, such that* $\forall e_1, e_2 \in E$, $e_1 \cap e_2 = \emptyset$ *and* $|e| = 2\ \forall e \in E$.

In this definition, we explicitly give the edges as a set of disjoint subsets of half-edges of cardinality 2. In the following definition, we get rid of the set $E$ and instead include an involution.

**Definition 1.11.** *A **graph** is a tuple* $(H, V, \nu, \iota)$*, where*

- $H$ *is a set of half-edges.*
- $V$ *is a set of vertices.*
- $\nu : H \to V$ *is a map that assigns half-edges to vertices.*
- $\iota : H \to H$ *is an involution on* $H$ *(such that* $\iota \circ \iota = id$*).*

The involution $\iota$ pairs half-edges to obtain "full" edges. Note that we do not impose that the involution is *fixed-point free* (i.e., a half-edge can be paired with itself). Notice, also, that there are no restrictions on the number of edges connecting two vertices (i.e., there can be multiple-edge connections and multiple loops).




In the example above we have a graph where $V = \{v_1, v_2\}$, $H = \{h_1, \ldots, h_5\}$, the first three half-edges are mapped to $v_1$ by $\nu$, and the last two are mapped to $v_2$. The involution $\iota$ is defined as $\iota(h_1) = h_1$, $\iota(h_2) = h_4$, $\iota(h_3) = h_5$, $\iota(h_4) = h_2$, and $\iota(h_5) = h_3$. Edges are defined such that $e_1 = \{h_2, h_4\}$ and $e_2 = \{h_3, h_5\}$.

**Proposition 1.12.** *Definitions 1.10 and 1.11 are equivalent.*

*Proof.* We have to show that giving an involution $\iota$ or a set of edges $E$ is equivalent. The orbits of the involution $\iota : H \to H$ give a partition of $H$ into sets of cardinality 1 and 2. The sets of cardinality 2 are the edges $E$. Given a set of edges $E$, we can build an involution $\iota$ by mapping each half-edge to its partner, if it has one, and to itself, if it has none. □

The first definition is more helpful for the diagrammatic representation of a graph, whilst the second is more useful for proofs.

Let's continue with several definitions.

**Definition 1.13.** *The half-edges that are not contained in any edge are called* ***legs****. If we denote the set of half-edges as $H_G$, then $H_G^{legs} := H_G \setminus \bigcup_{e \in E_G} e$.*

**Definition 1.14.** *Let $v \in V_G$ be a vertex of a graph $G$. Then, the preimage $\nu_G^{-1}(v)$ is a subset of half-edges. We say that this subset is the* ***corolla*** *of the vertex $v$. The cardinality of this set $d_G^{(v)} = |\nu_G^{-1}(v)|$ is the degree of the vertex $v$. Let $\{v \in V_G \,|\, d_G^{(v)} = d\}$ be the set of vertices with degree $d$. The cardinality of this set is the number of vertices with degree $d$ in $G$, and we will denote it as $k_G^{(d)}$.*

**Definition 1.15.** *A* ***path*** *between two vertices $u, v \in V_G$ is a sequence of half-edges $h_1, \ldots, h_{2n}$ and vertices $v_1, \ldots, v_{n-1}$ such that $v_1 = u$, $v_{n-1} = v$, $\iota(h_{2k+1}) = h_{2k+2}$, and $h_{2k}, h_{2k+1} \in \nu_G^{-1}(v_k)$. A path is closed if $v_1 = v_{n-1}$.*

**Definition 1.16.** *We can define an equivalence relation on the vertex set $V_G$ of a graph $G$. We say that two vertices $v, u \in V_G$ are in the same* ***connected component*** *if we can find a path between them. The set of equivalence classes based on this relation $C_G := V_G / \sim$ is the set of connected components of $G$. A graph is connected if it has exactly one component.*

**Definition 1.17.** *An edge $e \in E_G$ of a connected graph $G$ is called a* ***bridge*** *if the graph $G \setminus e$ is disconnected.*

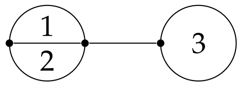
**Definition 1.18.** *An* ***isomorphism*** *$\varphi : G_1 \to G_2$ between to graphs $G_1$ and $G_2$ is a pair of bijections $\varphi = (\varphi_H, \varphi_V)$, with $\varphi_H : H_{G_1} \to H_{G_2}$ and $\varphi_V : V_{G_1} \to V_{G_2}$, such that $\nu_{G_2} = \varphi_V \circ \nu_{G_1} \circ \varphi_H^{-1}$ and $\iota_{G_2} = \varphi_H \circ \iota_{G_1} \circ \varphi_H^{-1}$.*

**Definition 1.19.** *An isomorphism from a graph to itself is called an* ***automorphism****.*

### 1.2.1 Euler's Formula

**Definition 1.20.** *We define $L_G$ as the number of* ***faces*** *of a graph $G$. A face is an independent cycle of the graph.*

For example, the following graph has three faces.




**Proposition 1.21.** *Let $G$ be a connected graph, with $|E_G|$ and $|V_G|$ the number of edges and vertices of the graph, respectively. The following identity holds:*

$$|V_G| - |E_G| + L_G = 1 \tag{1.1}$$

*This is called* ***Euler's formula****.*

## 1.3 Labelled Graphs

In this section, we will consider labelled graphs, where the labelled entities of the graph will be the half-edges and the vertices. We will consider the sets of half-edges and vertices to be intervals of integers.

**Definition 1.22.** *A graph $G = (H, V, \nu, \iota)$ is* ***labelled*** *if the sets $H$ and $V$ are intervals starting from 1: $H = \{1, \dots, |H|\}$ and $V = \{1, \dots, |V|\}$.*

**Definition 1.23.** *We write $\mathcal{G}^{lab}_{m,k}$ the set of labelled graphs with $m$ half-edges and $k$ vertices. $\mathcal{G}^{lab}_{m,k}$ is the set of all tuples $([m], [k], \nu, \iota)$, where here $[n]$ denotes the set of positive integers from 1 to $n$, with some map $\nu : [m] \to [k]$ and some involution $\iota : [m] \to [m]$.*

*We denote the set of all labelled graphs as $\mathcal{G}^{lab} = \cup_{m,k\geq 0}\mathcal{G}^{lab}_{m,k}$.*

### 1.3.1 Generating Functions

The elements of $\mathcal{G}^{\text{lab}}$ are completely determined by the number of half-edges, the number of vertices, and the maps $\nu$ and $\iota$, so it is relatively easy to find generating functions. If we know how to count the number of maps $\nu$ and $\iota$, we can count the respective labelled graphs. In this section, we will give two generating functions that take into account different parameters of the elements $G \in \mathcal{G}^{\text{lab}}$.

**Proposition 1.24.** *The following enumeration identity holds:*

$$\sum_{G\in\mathcal{G}^{lab}} \frac{x^{|H_G|}\lambda^{|V_G|}}{|H_G|!|V_G|!} = \sum_{k\geq 0} e^{kx + \frac{k^2x^2}{2}} \frac{\lambda^k}{k!}.$$

*Proof.* Instead of summing over all graphs $G \in \mathcal{G}^{\text{lab}}$, we want to sum over the number of half-edges $m$ and the number of vertices $k$. For that, we have to count the number of graphs for a fixed $k$ and $m$. The number of labelled graphs on a set of half-edges $H = [m]$ and a set of vertices $V = [k]$ is equal to the number of maps $\nu : H \to V$ times the number of involutions $\iota : H \to H$. It is easy to see that the number of maps $\nu : [m] \to [k]$ is $k^m$. Each of the $m$ half-edges can be independently assigned to any of the $k$ vertices, so for each half-edge, there are $k$ options, which yields the result.

For the number of involutions $\iota : H \to H$, first let's restrict to fixed-point free involutions. This means that every half-edge is mapped to a different half-edge. Say we have a set of $2n$ elements, and we want to count the number of fixed-point free involutions. Take one element. This element has $2n-1$ options left to be paired with. Say you choose one, and now you are left with $2n-2$ elements. Take another element. Now, this element has $2n-3$ elements left to be paired with. By iteration, we get that the number of fixed-point free involutions on a set of $2n$ elements is given by $\prod_{k=0}^{n}(2k+1) = (2n-1)!!$, the double factorial. Therefore, the total number of involutions $\iota : [m] \to [m]$ is $\sum_{n\geq 0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2n}(2n-1)!!$, where each summand is the number of involutions with $m-2n$ fixed points, and $\binom{m}{2n}$ represents the ways in which we can choose the $2n$ non-fixed points in the labelled set $k$.

Now we can obtain the generating function of the elements in $\mathcal{G}^{\text{lab}}$, where the number of half-edges and vertices are marked:

$$\sum_{G \in \mathcal{G}^{\text{lab}}} \frac{x^{|H_G|}\lambda^{|V_G|}}{|H_G|!|V_G|!} = \sum_{m,k\geq 0} \frac{x^m\lambda^k}{m!k!} k^m \sum_{n\geq 0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2n}(2n-1)!! =$$

$$= \sum_{n,k\geq 0}\sum_{m\geq 2n} x^m\lambda^k \frac{k^m(2n-1)!!}{k!(2n)!(m-2n)!} = \sum_{n,k\geq 0}\sum_{m\geq 0} x^{m+2n}\lambda^k \frac{k^{m+2n}(2n-1)!!}{k!(2n)!m!} =$$

$$= \sum_{n,k\geq 0} e^{kx}x^{2n}\lambda^k \frac{k^{2n}(2n-1)!!}{k!(2n)!} = \sum_{n,k\geq 0} e^{kx}x^{2n}\lambda^k \frac{k^{2n}}{k!2^n n!} = \sum_{k\geq 0} e^{kx+\frac{k^2x^2}{2}} \frac{\lambda^k}{k!}$$

where we used that $(2n-1)!! = \frac{(2n)!}{2^n n!}$ and $\sum_{n\geq 0} \frac{x^n}{n!} = e^x$. $\square$

This generating function marks the number of half-edges $|H_G|$ and the number of vertices $|V_G|$ of a graph $G$. However, it will be useful for future results to include information about the degree of the vertices in the graph. So, we want to generalize this identity. Instead of half-edges and vertices, the following result marks the number of vertices with degree $d$ (with $\lambda_d$), the number of legs (with $j$), and the number of edges (with $a$).

**Proposition 1.25.** *The following enumeration identity holds:*

$$\sum_{G\in\mathcal{G}^{lab}} \frac{\varphi_c^{|H_G^{legs}|} a^{|E_G|} \prod_{v\in V_G} \lambda_{d^{(v)}}}{|H_G|!|V_G|!} = \sum_{m\geq 0} m![x^m y^m] e^{a\frac{y^2}{2}+\varphi_c y} e^{\sum_{d\geq 0}\lambda_d \frac{x^d}{d!}} .$$

*Proof.* The idea for this proof is similar to the previous one. Instead of summing over $G \in \mathcal{G}^{\text{lab}}$, we want to sum over the number of vertices $k$, the number of half-edges $m$, and the number of edges $n$.

From the proof of Proposition 1.24, we know that the number of involutions on $m$ elements with $m-2n$ fixed points is $I_{n,m} := \binom{m}{2n}(2n-1)!!$. Let $y$ mark the number of elements $m$ (i.e., the number of half-edges), $j$, the number of fixed points $m-2n$ (i.e., the number of legs), and $a$, the number of pairs $n$ (i.e., the number of edges). We define the generating function of these involutions as follows:

$$\sum_{n,m\geq 0} I_{n,m}\frac{y^m}{m!}\varphi_c^{m-2n}a^n.$$

This generating function may be considered as exponential in $y$ and ordinary in $j$ and $a$.

$$\sum_{n,m\geq 0} I_{n,m}\frac{y^m}{m!}\varphi_c^{m-2n}a^n = \sum_{n,m\geq 0}\binom{m}{2n}(2n-1)!!\frac{y^m}{m!}\varphi_c^{m-2n}a^n = \sum_{n\geq 0}\sum_{m\geq 2n}\frac{y^m\varphi_c^{m-2n}a^n}{(m-2n)!2^n n!}$$
$$= \sum_{n,m\geq 0}\frac{y^{m+2n}\varphi_c^m a^n}{m!2^n n!} = e^{a\frac{y^2}{2}+\varphi_c y}$$

The difference from the previous proof comes when counting the number of maps $\nu : [m] \to [k]$. There, we had no restrictions, but here we are taking into account the degrees of the vertices. So, we have to count the number of maps $\nu$ with prescribed sizes of the preimages $|\nu^{-1}(v)| = d^{(v)}$, and such that $\sum_{v\in V_G} d^{(v)} = m$. This is given by the multinomial coefficient $\binom{m}{d^{(1)},\dots,d^{(k)}} = \frac{m!}{d^{(1)}!\cdots d^{(k)}!}$, let's work on this. From the $m$ half-edges, we have to choose $d^{(1)}$ which will be "attached" to vertex 1, and there are $\binom{m}{d^{(1)}}$ ways to do this. From the remaining $m-d^{(1)}$ half-edges, we have to choose $d^{(2)}$ and "attach" them to vertex 2. There are $\binom{m-d^{(1)}}{d^{(2)}}$ ways to do this. Iterating this process, we get that the number of maps $\nu$ with prescribed sizes of the preimages is given by

$$\binom{m}{d^{(1)}}\binom{m-d^{(1)}}{d^{(2)}}\binom{m-d^{(1)}-d^{(2)}}{d^{(3)}}\cdots\binom{m-d^{(1)}-d^{(2)}-\cdots-d^{(k-1)}}{d^{(k)}},$$

which, after cancellation of terms, becomes

$$\frac{m!}{d^{(1)}!\cdots d^{(k)}!},$$

which is the multinomial coefficient, as expected. With this, the following expression generates the number of maps $\nu$ with marked sizes of the preimages.

$$M_{m,k}(\lambda_0, \lambda_1, \dots) := \sum_{\substack{d^{(1)},\dots,d^{(k)} \geq 0 \\ d^{(1)}+\dots+d^{(k)}=m}} \binom{m}{d^{(1)}, \dots, d^{(k)}} \prod_{i=1}^{k} \lambda_{d^{(i)}}$$

Note that for $\lambda_d = 1 \; \forall d \in \mathbb{Z}_{\geq 0}$, this reduces to the number of maps $\nu$ from the proof of Proposition 1.24 by the multinomial theorem:

$$M_{m,k}(1, 1, \dots) := \sum_{\substack{d^{(1)},\dots,d^{(k)} \geq 0 \\ d^{(1)}+\dots+d^{(k)}=m}} \binom{m}{d^{(1)}, \dots, d^{(k)}} = k^m.$$

We define the exponential generating function as follows:

$$\sum_{m,k \geq 0} \frac{x^m}{m!k!} M_{m,k}(\lambda_0, \lambda_1, \dots),$$

where we added an additional variable $x$ which marks the number of half-edges $m$, and we sum over all possible numbers of vertices $k$ and half-edges $m$. Then,

$$\begin{aligned} \sum_{m,k \geq 0} \frac{x^m}{m!k!} M_{m,k}(\lambda_0, \lambda_1, \dots) &= \sum_{m,k \geq 0} \frac{x^m}{m!k!} \sum_{\substack{d^{(1)},\dots,d^{(k)} \geq 0 \\ d^{(1)}+\dots+d^{(k)}=m}} \binom{m}{d^{(1)}, \dots, d^{(k)}} \prod_{i=1}^{k} \lambda_{d^{(i)}} \\ &= \sum_{k \geq 0} \frac{1}{k!} \sum_{d^{(1)},\dots,d^{(k)} \geq 0} x^{\sum_{i=1}^{k} d^{(i)}} \prod_{i=1}^{k} \frac{\lambda_{d^{(i)}}}{d^{(i)}!} \\ &= \sum_{k \geq 0} \frac{1}{k!} \left( \sum_{d \geq 0} \frac{\lambda_d x^d}{d!} \right)^k = e^{\sum_{d \geq 0} \lambda_d \frac{x^d}{d!}}. \end{aligned}$$

Therefore, we can write the generating function of graphs in $\mathcal{G}^{\text{lab}}$ as a sum over the number of edges $n$, the number of vertices $k$, and the number of half-edges $m$:

$$\sum_{G \in \mathcal{G}^{\text{lab}}} \frac{\varphi_c^{|H_G^{\text{legs}}|} a^{|E_G|} \prod_{v \in V_G} \lambda_{d(v)}}{|H_G|!|V_G|!} = \sum_{n,m,k \geq 0} \frac{\varphi_c^{m-2n} a^n}{m!k!} I_{n,m} M_{m,k}(\lambda_0, \lambda_1, \dots),$$

where $I_{n,m} = m![y^m]e^{a\frac{y^2}{2}+jy}$ and $M_{m,k}(\lambda_0, \lambda_1, \dots) = m!k![x^m]e^{\sum_{d \geq 0} \lambda_d \frac{x^d}{d!}}$. By substitution, we get the result.

□

## 1.4 Graph Isomorphisms and the Orbit-Stabilizer Theorem

In the last section, we considered labelled graphs. Therefore, two isomorphic graphs with different labellings were considered distinct graphs. For example, in the following graphs, we permuted edges $e_1$ and $e_2$, but seen as labelled graphs, they are different.


When it comes to drawing graphs diagrammatically, it is more practical to consider isomorphism classes of graphs. Otherwise, we would have to include all the possible labellings of vertices and half-edges!

A necessary condition, but not sufficient, for two graphs to be isomorphic is that they have the same number of half-edges and vertices. Recall Definition 1.18: an isomorphism between two graphs $G_1, G_2 \in \mathcal{G}^{\text{lab}}_{m,k}$ with the half-edge set $[m]$ and the vertex set $[k]$ is a pair of bijections $\varphi_H : [m] \to [m]$, $\varphi_V : [k] \to [k]$ such that $\nu_{G_2} = \varphi_V \circ \nu_{G_1} \circ \varphi_H^{-1}$ and $\iota_{G_2} = \varphi_H \circ \iota_{G_1} \circ \varphi_H^{-1}$. The goal is to isolate the isomorphism classes in $\mathcal{G}^{\text{lab}}_{m,k}$.

The bijections $\varphi_H$ and $\varphi_V$ are a pair of permutations of the half-edge and vertices labels, respectively. All the possible bijections from the sets $[m]$ and $[k]$ to themselves are represented by the symmetric group $S_m$ and $S_k$, which are of order $m!$ and $k!$. Let $P_{m,k} := S_m \times S_k$ be the product of all pairs of permutations which can be used to relabel the graphs in $\mathcal{G}^{\text{lab}}_{m,k}$. The order of $P_{m,k}$ is $m!k!$. We define the group action $*$ of $P_{m,k}$ on the set $\mathcal{G}^{\text{lab}}_{m,k}$ as follows.

$$\begin{aligned} * \quad : \qquad P_{m,k} \times \mathcal{G}^{\text{lab}}_{m,k} &\to \mathcal{G}^{\text{lab}}_{m,k} \\ (\varphi_V, \varphi_H), ([m], [k], \nu, \iota)) &\mapsto ([m], [k], \varphi_V \circ \nu \circ \varphi_H^{-1}, \varphi_H \circ \iota \circ \varphi_H^{-1}) \end{aligned}$$

**Definition 1.26.** *The **orbit** of an element $G \in \mathcal{G}^{lab}_{m,k}$ by the group $P_{m,k}$ is the set of all graphs in $\mathcal{G}^{lab}_{m,k}$ that can be obtained from $G$ by a permutation of the half-edges and vertex labels. We denote this set as $Orb_{P_{m,k}}(G) := \{p * G : p \in P_{m,k}\}$.*

Notice that, if we ignore the labelling of a graph, this set can be interpreted as an *unlabelled* graph.

**Definition 1.27.** *We define the set of unlabelled graphs $\mathcal{G}_{m,k}$ with half-edge set $[m]$ and vertex set $[k]$ as the set of all orbits of $G \in \mathcal{G}^{lab}_{m,k}$.*

$$\mathcal{G}_{m,k} := \{Orb_{P_{m,k}}(G) : G \in \mathcal{G}^{lab}_{m,k}\}$$

*The set of **unlabelled graphs**, $\mathcal{G}$, is then*

$$\mathcal{G} = \bigcup_{m,k\geq 0} \mathcal{G}_{m,k} \, .$$

Notice that the set of unlabelled graphs $\mathcal{G}_{m,k}$ is a partition of $\mathcal{G}^{\text{lab}}_{m,k}$ into subsets of isomorphic graphs. We could alternatively define this set as the quotient of $\mathcal{G}^{\text{lab}}_{m,k}$ under the group action $*$: $\mathcal{G}_{m,k} = \mathcal{G}^{\text{lab}}_{m,k}/P_{m,k}$.

In Proposition 1.25, we sum over all labelled graphs $G \in \mathcal{G}^{\text{lab}}$, but the goal is to sum over all unlabelled graphs. Let us denote by $\Gamma \in \mathcal{G}_{m,k}$ a representative graph from an orbit. Then, $\sum_{G\in\mathcal{G}^{\text{lab}}_{m,k}}$ is equivalent to $\sum_{\Gamma\in\mathcal{G}_{m,k}} \sum_{G\in\text{Orb}_{P_{m,k}}(\Gamma)}$. Since the action of the group $P_{m,k}$ on a graph $G \in \mathcal{G}^{\text{lab}}_{m,k}$ simply changes the labelling of half-edges and vertices, the term

$$\frac{\varphi_c^{|H_G^{\text{legs}}|} a^{|E_G|} \prod_{v\in V_G} \lambda_{d(v)}}{|H_G|!|V_G|!}$$

will be identical for all graphs in the same orbit. Hence, for Proposition 1.25, the sum $\sum_{G\in\mathcal{G}^{\text{lab}}_{m,k}}$ is equivalent to $\sum_{\Gamma\in\mathcal{G}_{m,k}} |\text{Orb}_{P_{m,k}}(\Gamma)|$. This means that summing over all labelled graphs is the same as summing over all unlabelled graphs and multiplying by the order of their orbits. Nevertheless, this does not solve the problem of counting how many possible labellings there are for a certain graph. Computing the term $|\text{Orb}_{P_{m,k}}(\Gamma)|$ can be quite tedious. Luckily, we can use elementary group theory to get rid of this term. By the Orbit-Stabilizer Theorem (*cf.* [2, Proposition 6.8.4]), we get:

**Proposition 1.28.** *For every $G \in \mathcal{G}^{lab}_{m,k}$ we have*

$$\frac{|Orb_{P_{m,k}}(G)|}{m!k!} = \frac{1}{|Aut(G)|} \, ,$$

*where $Aut(G)$ is the set of all automorphisms of the graph $G$.*

*Proof.* We have the group $P_{m,k} = S_m \times S_k$ acting on the set or labelled graphs $\mathcal{G}^{\text{lab}}_{m,k}$. We define the stabilizer of $G \in \mathcal{G}^{\text{lab}}_{m,k}$ under the action by $P_{m,k}$ as $\text{Stab}_{P_{m,k}}(G) := \{p \in P_{m,k} : p * G = G\}$. It is the set of permutations $p \in P_{m,k}$ that map the graph to itself. Thus, the elements of the stabilizer of $G$ are the automorphisms of $G$, $\text{Stab}_{P_{m,k}}(G) = \text{Aut}(G)$. Then, by the Orbit-Stabilizer Theorem,

$$m!k! = |P_{m,k}| = |\text{Orb}_{P_{m,k}}(G)||\text{Stab}_{P_{m,k}}(G)| = |\text{Orb}_{P_{m,k}}(G)||\text{Aut}(G)|.$$

□

Now we are ready to rewrite Proposition 1.25 as a sum over unlabelled graphs.

**Corollary 1.29.** *The following enumeration identity holds:*

$$\sum_{\Gamma\in\mathcal{G}} \frac{\varphi_c^{|H_\Gamma^{legs}|} a^{|E_\Gamma|} \prod_{v\in V_\Gamma} \lambda_{d(v)}}{|Aut(\Gamma)|} = \sum_{m\geq 0} m![x^m y^m] e^{a\frac{y^2}{2}+\varphi_c y} e^{\sum_{d\geq 0}\lambda_d \frac{x^d}{d!}}.$$

*Proof.*

$$\begin{aligned}
\sum_{G\in\mathcal{G}^{\text{lab}}} \frac{\varphi_c^{|H_G^{\text{legs}}|} a^{|E_G|} \prod_{v\in V_G} \lambda_{d(v)}}{|H_G|!|V_G|!} &= \sum_{m,k\geq 0} \sum_{\Gamma\in\mathcal{G}_{m,k}} \sum_{G\in \text{Orb}_{P_{m,k}}(\Gamma)} \frac{\varphi_c^{|H_G^{\text{legs}}|} a^{|E_G|} \prod_{v\in V_G} \lambda_{d(v)}}{m!k!} \\
&= \sum_{m,k\geq 0} \sum_{\Gamma\in\mathcal{G}_{m,k}} \frac{\varphi_c^{|H_G^{\text{legs}}|} a^{|E_G|} \prod_{v\in V_G} \lambda_{d(v)}}{m!k!} |\text{Orb}_{P_{m,k}}(\Gamma)| \\
&= \sum_{m,k\geq 0} \sum_{\Gamma\in\mathcal{G}_{m,k}} \frac{\varphi_c^{|H_G^{\text{legs}}|} a^{|E_G|} \prod_{v\in V_G} \lambda_{d(v)}}{|\text{Aut}(\Gamma)|} \\
&= \sum_{\Gamma\in\mathcal{G}} \frac{\varphi_c^{|H_G^{\text{legs}}|} a^{|E_G|} \prod_{v\in V_G} \lambda_{d(v)}}{|\text{Aut}(\Gamma)|}.
\end{aligned}$$

□

This last result tells us that the exponential generating function of half-edge labelled graphs is identical to the ordinary generating function of half-edge unlabelled graphs weighted with $1/|\text{Aut}(\Gamma)|$. This factor is called the ***symmetry factor*** of the graph, and we will see how to obtain it in Section 1.6.

## 1.5 Graph Algebra

In Section 1.1, we claimed that any combinatorial class $\mathscr{C}$ can be made into an algebra by taking the polynomial algebra $K[\mathscr{C}]$, where addition is completely formal. Here we want to further formalise this idea for the combinatorial class of all graphs $\mathscr{G}$.

**Definition 1.30.** *Let $G_1$ and $G_2$ be two graphs. We define the **disjoint union** $G_1 \sqcup G_2$ as the graph $(H_{G_1} \sqcup H_{G_2}, V_{G_1} \sqcup V_{G_2}, \nu_{G_1} \sqcup \nu_{G_2}, \iota_{G_1} \sqcup \iota_{G_2})$.*

The disjoint union of two sets $A \sqcup B$ can be explicitly constructed by multiplying the respective sets with a unique symbol: $A \sqcup B := (\star \times A) \cup (\diamond \times B)$. For maps, $f : A \to C$ and $g : B \to D$, the disjoint union $f \sqcup g : A \sqcup B \to C \sqcup D$ is the unique map whose restriction on $\star \times A$ is equal to $f$ and on $\diamond \times B$ is equal to $g$.

**Definition 1.31.** *We define* $\mathbb{Q}[\mathscr{G}]$ *as the* $\mathbb{Q}$*-algebra generated by all elements of* $\mathscr{G}$ *with the following multiplication defined on its generators:*

$$\begin{aligned} m \quad : \quad \mathbb{Q}[\mathscr{G}] \otimes \mathbb{Q}[\mathscr{G}] \quad &\to \mathbb{Q}[\mathscr{G}] \\ \Gamma_1 \otimes \Gamma_2 \quad &\mapsto \Gamma_1 \sqcup \Gamma_2 \end{aligned}$$

*where* $\Gamma_1 \sqcup \Gamma_2$ *denotes the unlabelled graph associated to the disjoint union of the representatives* $\Gamma_1$ *and* $\Gamma_2$*. This multiplication is commutative and associative, and the empty graph* $\mathbb{1}$*, with* $|H_{\mathbb{1}}| = |V_{\mathbb{1}}| = \emptyset$*, is the neutral element of* $\mathbb{Q}[\mathscr{G}]$*.*

### 1.5.1 The Exponential Formula

One of the key objects of the algebra $\mathbb{Q}[\mathscr{G}]$ in QFT is the sum over all graphs weighted by the cardinality of their automorphism group. We will denote this element by Y.

$$\mathrm{Y} = \sum_{\Gamma \in \mathcal{G}} \frac{\Gamma}{|\mathrm{Aut}\ \Gamma|} \tag{1.2}$$

If we restrict the sum to graphs with only one connected component, then we denote it as $\mathrm{Y}^c$.

$$\mathrm{Y}^c = \sum_{\substack{\Gamma \in \mathcal{G} \\ |C_\Gamma| = 1}} \frac{\Gamma}{|\mathrm{Aut}\ \Gamma|} \tag{1.3}$$

**Theorem 1.32.** *(The exponential formula) The following algebraic identity holds in* $\mathbb{Q}[\mathscr{C}]$*:*

$$\mathrm{Y} = e^{\mathrm{Y}^c}, \quad \mathrm{Y}^c = \log(\mathrm{Y}).$$

*Proof.* Every graph with $n$ connected components can be thought of as the disjoint union of $n$ graphs with one connected component. Recall that we defined the product between graphs as their disjoint union. So, every graph with $n$ connected components can be written as the product of $n$ graphs with one connected component:

$$\sum_{\substack{\Gamma \in \mathcal{G} \\ |C_\Gamma| = n}} \frac{\Gamma}{|\mathrm{Aut}\ \Gamma|} = \frac{1}{n!} \sum_{\substack{\gamma_1, \ldots, \gamma_n \in \mathcal{G} \\ |C_{\gamma_i}| = 1}} \prod_{i=1}^{n} \frac{\gamma_i}{|\mathrm{Aut}\ \gamma_i|},$$

where the factorial $\frac{1}{n!}$ accounts for over-counting symmetries between these components. Summing over $n$ and using $e^x = \sum_{n \geq 0} \frac{x^n}{n!}$ results in the statement. □

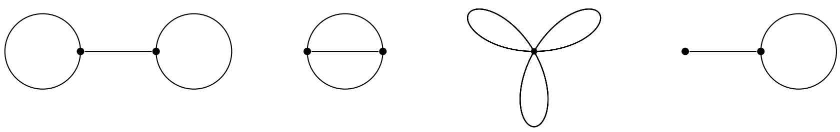
### 1.5.2 Algebra Homomorphisms

As a brief note, the evaluation maps from Definition 1.7 can be regarded as algebra homomorphisms if we include an additional condition that preserves the algebra structure.

**Definition 1.33.** *A linear map $\varphi : K[\mathscr{C}] \to A$, where $A$ is another commutative algebra, is an* ***algebra homomorphism*** *if $\varphi$ is compatible with the multiplication of $K[\mathscr{C}]$ and $A$. This means that $\forall a, b \in K[\mathscr{C}]$, $\varphi(a)\varphi(b) = \varphi(ab)$. In particular, the neutral element of $K[\mathscr{C}]$ is mapped to the neutral element of $A$.*

## 1.6 Symmetry Factor of a Graph

In 0-dimensional quantum field theory, we will deal with graphs such as the following:

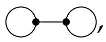


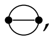
One of the most important characteristics of a graph is its *symmetry factor*, given by $\frac{1}{|\text{Aut }\Gamma|}$. To compute this quantity, we need to understand how to count graph automorphisms. Throughout, we consider graphs as consisting of a set of vertices and a set of half-edges. In this context, flipping a loop is considered an automorphism, as it corresponds to interchanging its half-edges.

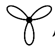
Graph automorphisms include permutations of both vertices and half-edges that leave the graph unchanged. Our task is to count all such permutations. To proceed systematically, we introduce some notation: let $g$ denote the number of vertex permutations; let $\alpha_n$ be the number of pairs of vertices connected by $n$ identical self-conjugate lines; let $\beta$ be the number of lines connecting a vertex to itself (i.e., loops); and let $d_m$ denote the number of vertices with $m$ loops. Then,

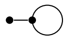
$$|\text{Aut }\Gamma| = g 2^{\beta} \prod_m (m!)^{d_m} \prod_n (n!)^{\alpha_n}.$$

The term $g$ accounts for the number of vertex permutations that leave the graph invariant. The factor $2^{\beta}$ arises from flipping loops, as each loop can be flipped independently. If a vertex has $m$ loops, these loops can be permuted among themselves, contributing a factor of $(m!)^{d_m}$. Finally, the term $(n!)^{\alpha_n}$ accounts for the permutations of $n$ indistinguishable edges connecting the same pair of vertices.

Let's work out $|\text{Aut }\Gamma|$ for the examples above. For the graph , $g = 2! = 2$ because we could either leave the vertices as they are or exchange them, and the graph would stay the same, so there are two possible permutations. Since it has two loops, $\beta = 2$, and this contributes a factor of $2^2 = 4$ corresponding to flipping none, flipping one or the other, and flipping both at the same time. In this case, $d_m = 0\ \forall m \geq 2$, because there are no vertices with more than one loop, and $\alpha_n = 0\ \forall n \geq 2$, because there are no multi-edges between vertices. Altogether, $|\mathrm{Aut}\ \Gamma| = 2!2^2 = 8$.

For , $g = 2$ as in the previous example, $\beta = 0$ because there are no loops and, consequently, $d_m = 0\ \forall m \geq 2$. In this case, $\alpha_3 = 1$, meaning that there is one pair of vertices joined by three equivalent edges, and this contributes with a factor of 3!, which accounts for the number of ways in which we can exchange these edges. So, $|\mathrm{Aut}\ \Gamma| = 2(3!)^1 = 12$.

For the graph , we only have one vertex, so the only permutation possible is the identity, $g = 1$. Then, $\beta = 3$ and $\alpha_n = 0\ \forall n \geq 2$. Here we have one vertex with three loops, so $d_3 = 1$, and we have $|\mathrm{Aut}\ \Gamma| = 2^3(3!)^1 = 48$.

Lastly, the graph  only has 2 automorphisms, which correspond to flipping the loop.

# Chapter 2

# 0-Dimensional Quantum Field Theory

Besides a few particular cases, it is not possible to solve the infinite-dimensional integrals (1) and (2), and they can only be computed approximately. A standard approach in quantum field theory to tackle this problem is to start in lower dimensions and then generalize the tools and results to higher dimensions. In fact, when fields do not depend on time nor space (i.e., 0-dimensional quantum field theory), they become ordinary variables $\phi(x,t) \to \phi$, and the ill-defined Lebesgue measure $D\phi$ reduces to the well-known $d\phi$. It may seem a bit extreme to reduce our fields to simple variables, but this theory serves as a toy model to introduce perturbation theories and see how Feynman diagrams arise from them.

From a physics perspective, the action introduced before $S(\phi)$ will be of the form

$$S(\phi) = \frac{1}{2}m\phi^2 - V(\phi),$$

where $V(\phi)$ is the *potential energy*. Typically, $V(\phi) \in \phi^3\mathbb{R}[[\phi]]$, so it is some polynomial in $\mathbb{R}$ without the first three coefficients. When the action also involves a linear term in $\phi$, it is called the *source term*. In general,

$$S(\phi) = \frac{1}{2}m\phi^2 - j\phi - V(\phi)\,.$$

The partition function will be

$$Z(m,\lambda_i,j,\hbar) = \int \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{1}{2}m\phi^2 + j\phi + V(\phi)\right)}\, d\phi\,. \tag{2.1}$$

Notice that the prefactor $\frac{1}{\sqrt{2\pi\hbar}}$ was not included in expression (2). Some texts add it, others don't. Here we will keep it for convenience. In addition, we removed the $i$ factor from the exponential. This means that we work in what is called the *Euclidean signature*, instead of the *Lorentzian signature*. The oscillatory term is replaced by a damping term, which tends to improve the behaviour of these integrals. Nonetheless, integral (2.1) is ill-defined for general $V(\phi)$. For example, for $V(\phi) = \phi^4/4!$, it is not integrable over $\mathbb{R}$ since it is divergent. However, here we will focus on computing the coefficients of the expansion of the integral, and we will treat it as a formal power series, without concerning ourselves with convergence.

To obtain a power series expansion from this integral, the term $V(\phi) + j\phi$ is treated as a perturbation, and the exponential $e^{j\phi+V(\phi)}$ is expanded as a Taylor series. The next step is an unjustified exchange of integration and summation, so the resulting perturbative power series generally fails to agree with the non-perturbative result. The "integration" will be performed by evaluating integrals of the form

$$\int \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{1}{2\hbar}m\phi^2} \phi^{2n} \, d\phi \,. \tag{2.2}$$

In the following section, we will see how to solve these. Notice that for any even power $\phi^{2n+1}$, it is equal to 0 since the integrand would be an even function.

## 2.1 Free Field Theory. Wick's Theorem

To be consistent with the notation of the previous chapter, let us denote the integration variable by $x$, and change the constant $m$ to a general parameter $1/a$. Before jumping to the integral (2.1), let us set the potential to $V(x) = 0$ and remove any sources $j = 0$. These theories are called *free field theories*, and the partition function becomes

$$Z = \int \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{S(x)}{\hbar}} \, dx = \int \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{x^2}{2a\hbar}} \, dx,$$

which is a Gaussian integral. It is straightforward that $Z = \sqrt{a}$.

Here, we will take a more general approach. Let $V$ be a real vector space of dimension $d$ and $A : V \times V \to \mathbb{R}$ a bilinear form on $V$. Take $A$ to be non-degenerate and symmetric with a non-negative definite real part. The partition function is given by

$$Z = \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{A(x,x)}{2\hbar}} \, dx$$

and the *n-point function* (1) will be

$$\langle x^n \rangle = \frac{1}{Z} \int \frac{1}{\sqrt{(2\pi\hbar)^d}} x^n e^{-\frac{A(x,x)}{2\hbar}} \, dx = \frac{\int x^n e^{-\frac{A(x,x)}{2\hbar}} \, dx}{\int e^{-\frac{A(x,x)}{2\hbar}} \, dx}. \tag{2.3}$$

Let us add now a source term, $j \neq 0$, so that the partition function becomes

$$Z(j) = \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{A(x,x)}{2\hbar} + \frac{jx}{\hbar}}\, dx.$$

Notice that we can obtain the n-point functions from $Z(j)$ by taking the derivative over $j$:

$$\langle x^n \rangle = \hbar^n \frac{\partial^n}{\partial j^n} \frac{Z(j)}{Z(0)}\bigg|_{j=0} = \hbar^n \frac{\partial^n}{\partial j^n} \frac{\int e^{-\frac{A(x,x)}{2\hbar} + \frac{jx}{\hbar}}\, dx}{\int e^{-\frac{A(x,x)}{2\hbar}}\, dx}\Bigg|_{j=0}. \tag{2.4}$$

Computing (2.3) can be a difficult task, but if we know how to obtain $Z(j)$, we can easily get $\langle x^n \rangle$. That is why $Z(j)$ is called the *generating function of n-point functions*.

Let's compute $Z(j)$. Since $A$ is real and symmetric, it is diagonalizable over $\mathbb{R}$, so we can assume that $A$ is diagonal (otherwise, we change basis). Then, for $Z(0)$ we get $d$ independent one-dimensional Gaussian integrals:

$$\begin{aligned} Z(0) &= \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{A(x,x)}{2\hbar}}\, dx = \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{1}{2\hbar}\sum_{i=1}^d A_{ii} x_i^2}\, dx_1 \cdots dx_d \\ &= \frac{1}{\sqrt{A_{11}\cdots A_{dd}}} = \frac{1}{\sqrt{\det A}} \end{aligned}$$

For the case $j \neq 0$, we use the trick of "completing squares" for each term:

$$\frac{A(x,x)}{2} + jx = \sum_{i=1}^d \left( \frac{A_{ii}}{2} x_i^2 + j_i x_i \right) = \sum_{i=1}^d \left( \frac{A_{ii}}{2} \left( x_i - \frac{j_i}{A_{ii}} \right)^2 + \frac{j_i^2}{2A_{ii}} \right),$$

and making the change of variable $x_i' = x_i - \frac{j_i}{A_{ii}}$, we get the result:

$$\begin{aligned} Z(j) &= \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{A(x,x)}{2\hbar} + \frac{jx}{\hbar}}\, dx = \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{1}{\hbar}\sum_{i=1}^d \left( \frac{A_{ii}}{2} x_i^2 + j_i x_i \right)}\, dx_1 \cdots dx_d \\ &= \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{1}{\hbar}\sum_{i=1}^d \left( \frac{A_{ii}}{2} x_i^2 + j_i x_i \right)}\, dx_1 \cdots dx_d \\ &= e^{\frac{A^{-1}(j,j)}{2\hbar}} \int \frac{1}{\sqrt{(2\pi\hbar)^d}} e^{-\frac{1}{\hbar}\sum_{i=1}^d \frac{A_{ii}}{2} x_i'^2}\, dx_1' \cdots dx_d' = \frac{1}{\sqrt{\det A}} e^{\frac{A^{-1}(j,j)}{2\hbar}}, \end{aligned}$$

where $A^{-1}$ denotes the inverse form to $A$. Then,

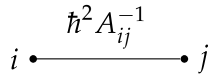
$$\langle x^n \rangle = \hbar^n \frac{\partial^n}{\partial j^n} \frac{Z(j)}{Z(0)}\bigg|_{j=0} = \hbar^n \frac{\partial^n}{\partial j^n} e^{\frac{A^{-1}(j,j)}{2\hbar}}\bigg|_{j=0}.$$

For example,

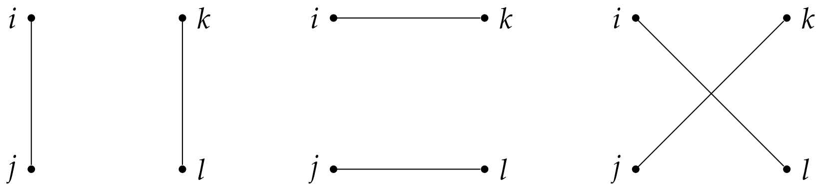
$$\langle x_i x_j \rangle = \hbar^2 A_{ij}^{-1} \tag{2.5}$$

$$\langle x_i x_j x_k x_l \rangle = \hbar^4 (A_{ij}^{-1} A_{kl}^{-1} + A_{ik}^{-1} A_{jl}^{-1} + A_{il}^{-1} A_{jk}^{-1}) \tag{2.6}$$

There is a way of representing this diagrammatically. For each $x_i$, add a vertex and denote it by the subindex $i$. Then, each $A_{ij}^{-1}$ represents an edge connecting vertices $i$ and $j$, and it is weighted by $\hbar^2 A_{ij}^{-1}$. Example (2.5) is represented by




and expression (2.6), as follows:




We are representing all possible pairings for the set of $n$ elements, where $n$ represents the number of variables $x_i$. Let's denote the set of matchings on a set $\{1, \ldots, 2k\}$ by $\Pi_k$. We can think of the elements $\sigma \in \Pi_k$ as permutations of the set $\{1, \ldots, 2k\}$ such that $\sigma^2 = 1$ and $\sigma$ has no fixed points. So, every $\sigma \in \Pi_k$ is an involution on the set $\{1, \ldots, 2k\}$, and we know from the proof of Proposition 1.24 that the number of fixed point free involutions on a set of $2k$ elements is $(2k-1)!!$. The following theorem generalizes results (2.5) and (2.6).

**Theorem 2.1.** *(Wick's theorem) Let $A^{-1}$ denote the inverse form to $A$ on $V^*$, and $\ell_1, \cdots, \ell_N \in V^*$. Then, if N is even, we have*

$$\int_V \ell_1(x) \cdots \ell_N(x) e^{-\frac{A(x,x)}{2\hbar}} dx = \frac{(2\pi\hbar)^{\frac{d}{2}} \hbar^{\frac{N}{2}}}{\sqrt{\det A}} \sum_{\sigma \in \Pi_{N/2}} \prod_{i \in \{1, \cdots, N\}/\sigma} A^{-1}(\ell_i, \ell_{\sigma(i)})$$

*If N is odd, the integral is zero.*

*Proof.* If $N$ is odd, then the integrand is an odd function. Hence, the integral is zero. So, let's assume $N = 2k$ is even. Since both sides of the equation are symmetric polylinear forms in $\ell_1, \cdots, \ell_N$, it suffices to prove it for $\ell_1 = \cdots = \ell_N = \ell$. Moreover, the expression is stable under linear changes of variable. So, we can choose a coordinate system in which $A(x, x) = x_1^2 + \cdots + x_d^2$ and $l(x) = x_1$. Consequently, we can assume that $d = 1$ and $l(x) = x$. The above expression reduces to

$$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2\hbar}} dx = (2\pi\hbar)^{\frac{1}{2}} \hbar^k \sum_{\sigma \in \Pi_k} 1 = (2\pi\hbar)^{\frac{1}{2}} \hbar^k (2k-1)!!$$

Let us make the change of variable $y = x^2/2\hbar$,

$$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2}}\, dx = (2\hbar)^{k+\frac{1}{2}} \int_0^{\infty} y^{k-\frac{1}{2}} e^{-y}\, dy = (2\hbar)^{k+\frac{1}{2}} \Gamma\left(k+\frac{1}{2}\right) = (2\pi\hbar)^{\frac{1}{2}} \hbar^k (2k-1)!!$$

where we used that $\Gamma(k+1) = k\Gamma(k)$ and $\Gamma(1/2) = \sqrt{\pi}$. □

## 2.2 Interacting Theories

Now we are ready to study the integral (2.1). When the potential is not null, we talk about *interacting theories*. First, let us consider a theory without source terms, $j = 0$. The partition function is

$$Z = \int \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{1}{\hbar}\left(-\frac{x^2}{2a}+V(x)\right)}\, dx\,.$$

For free field theories, we were able to compute the integral analytically, without approximations. However, now the additional term $V(x)$ makes this a difficult task. To solve this, we can expand the exponent of $V(x)$ and exchange integration and summation.[1] We will have to compute integrals like (2.2), which can be solved using Wick's theorem 2.1:

$$\int \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{x^2}{2a\hbar}} x^{2n}\, dx = \sqrt{a}(a\hbar)^n (2n-1)!!$$

We can "define" the zero-dimensional path integral as follows.

**Definition 2.2.** *Let $S(x) \in x\mathbb{R}[[x]]$ be a power series with vanishing constant and with strictly positive quadratic term, $S(x) = \frac{x^2}{2a} - V(x) - jx$. We define $\mathcal{F} : x\mathbb{R}[[x]] \to \mathbb{R}[[\lambda_i, j]]$ as the operator that maps $S(x) \in x\mathbb{R}[[x]]$ to a power series in $\lambda_i$ and $j$, $\mathcal{F}[S(x)] \in \mathbb{R}[[\lambda_i, j]]$, where $\lambda_i$ are possible parameters that appear in $V(x)$, as*

$$\mathcal{F}[S(x)](\lambda_i, j) = \sqrt{a} \sum_{n=0}^{\infty} (a\hbar)^n (2n-1)!! [x^{2n}] e^{\frac{1}{\hbar}(V(x)+jx)}. \tag{2.7}$$

The key property of $\mathcal{F}$, which connects to the previous chapter, is that $\mathcal{F}[S(x)](\hbar)$ enumerates multigraphs.

[1] The **Dominated Convergence Theorem** provides sufficient conditions under which the exchange of the integral and the limit is valid, but in the examples considered here, these conditions are not satisfied. In fact, in such cases, the full integral is non-analytic in the parameters appearing in $V(x)$, and therefore cannot be expressed as a power series in those parameters.

### 2.2.1 Feynman Diagrams

In this section, we will assume that the potential is of the form

$$V(x) = \sum_{d=0}^{\infty} \frac{\lambda_d}{d!} x^d.$$

Later, we will set $\lambda_0 = \lambda_2 = 0$ and $\lambda_1 = j$. But, for now, we will work in this more general framework.

**Definition 2.3.** *Let $\mathbb{Q}[\mathcal{G}]$ be a graph algebra. Let $\mathbb{R}[[\lambda_i]]$ be the ring of formal power series in $\lambda_i$, $i \in \mathbb{Z}_{\geq 0}$, with coefficients in $\mathbb{R}$. We define the evaluation map $\varphi_S$ as*

$$\begin{aligned} \varphi_S \quad : \quad \mathbb{Q}[\mathcal{G}] \quad &\to \mathbb{R}[[\lambda_i]] \\ \Gamma \quad &\to \hbar^{|E_\Gamma| - |V_\Gamma|} a^{|E_\Gamma|} \textstyle\prod_{v \in V_\Gamma} \lambda_{d(v)} \end{aligned}$$

**Proposition 2.4.** *If $S(x) = \frac{x^2}{2a} - \sum_{d=0}^{\infty} \frac{\lambda_d}{d!} x^d$ with $a > 0$, then*

$$\mathcal{F}[S(x)](\hbar) = \sqrt{a}\varphi_S(\Upsilon),$$

*where $\Upsilon$ is given by expression (1.2).*

*Proof.* Recall the identity from Corollary 1.29. If we restrict to graphs without legs, we get the following identity:

$$\sum_{\substack{\Gamma \in \mathcal{G} \\ |H_\Gamma^{\text{legs}}|=0}} \frac{a^{|E_\Gamma|} \prod_{v \in V_\Gamma} \lambda_{d(v)}}{|\text{Aut}(\Gamma)|} = \sum_{m \geq 0} m! [x^m y^m] e^{a\frac{y^2}{2}} e^{\sum_{d \geq 0} \lambda_d \frac{x^d}{d!}}.$$

If we evaluate the coefficient extraction in $y$, we get

$$\sum_{\substack{\Gamma \in \mathcal{G} \\ |H_\Gamma^{\text{legs}}|=0}} \frac{a^{|E_\Gamma|} \prod_{v \in V_\Gamma} \lambda_{d(v)}}{|\text{Aut}(\Gamma)|} = \sum_{m \geq 0} a^m (2m-1)!! [x^{2m}] e^{\sum_{d \geq 0} \lambda_d \frac{x^d}{d!}},$$

where we used that $e^{a\frac{y^2}{2}} = \sum_{n=0}^{\infty} \frac{a^n y^{2n}}{2^n n!}$, and we applied the identity $(2m-1)!! = \frac{(2m)!}{2^m m!}$. Notice that it is starting to look like expression (2.7), we are just missing the $\hbar$ factors. Let us rescale $a \to \hbar a$ and $\lambda_d \to \frac{\lambda_d}{\hbar}$. This gives,

$$\sum_{\substack{\Gamma \in \mathcal{G} \\ |H_\Gamma^{\text{legs}}|=0}} \hbar^{|E_\Gamma| - |V_\Gamma|} \frac{a^{|E_\Gamma|} \prod_{v \in V_\Gamma} \lambda_{d(v)}}{|\text{Aut}(\Gamma)|} = \sum_{m \geq 0} (a\hbar)^m (2m-1)!! [x^{2m}] e^{\frac{1}{\hbar} \sum_{d \geq 0} \lambda_d \frac{x^d}{d!}}.$$

On the left-hand side, we identify the evaluation map $\varphi_S$, and on the right-hand side, we recover expression (2.7) except for $\sqrt{a}$. By adding this factor, the result follows. □

If $S(x) = \frac{x^2}{2a} - \sum_{d=3}^{\infty} \frac{\lambda_d}{d!} x^d - jx$ then we simply denote $\lambda_1$ as $j$ and $\lambda_0 = \lambda_2 = 0$.

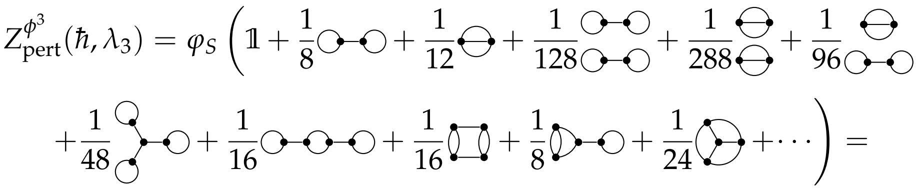
This result tells us that the terms in the expansion of the integral (2.1) can be interpreted as a sum over *Feynman diagrams* weighted by their symmetry factor. We will refer to evaluation maps such as $\varphi_S$ as *Feynman rules*.

Let's work out some examples.

**Example 2.5.** ($\phi^3$-theory) In $\phi^3$-theory, if we let $a = 1$, the action takes the form $S(x) = \frac{x^2}{2} - \frac{\lambda_3 x^3}{3!}$, and the partition function is given by

$$Z_{\text{pert}}^{\phi^3}(\hbar, \lambda_3) = \mathcal{F}\left[\frac{x^2}{2} - \frac{\lambda_3 x^3}{3!}\right](\hbar, \lambda_3).$$

The diagrammatic expansion starts with




$$= 1 + \left(\frac{1}{8} + \frac{1}{12}\right)\lambda_3^2\hbar + \frac{385}{1152}\lambda_3^4\hbar^2 + \cdots = 1 + \frac{5}{24}\lambda_3^2\hbar + \frac{385}{1152}\lambda_3^4\hbar^2 + \cdots$$

Notice that all the graphs have no external legs (because there is no source term, $j = 0$). These are called *vacuum bubbles*. In this case, we can compute the partition function directly from Definition 2.2:

$$Z_{\text{pert}}^{\phi^3}(\hbar, \lambda_3) = \mathcal{F}\left[\frac{x^2}{2} - \frac{\lambda_3 x^3}{3!}\right](\hbar, \lambda_3) = \sum_{m \geq 0} \hbar^m (2m-1)!! [x^{2m}] e^{\frac{1}{\hbar}\frac{\lambda_3 x^3}{3!}}.$$

Since $e^{\frac{1}{\hbar}\frac{\lambda_3 x^3}{3!}} = \sum_{n \geq 0} \frac{\lambda_3^n x^{3n}}{\hbar^n (3!)^n n!}$, when we evaluate the coefficient extraction, only the coefficients $[x^{6m}]$ survive, and we obtain

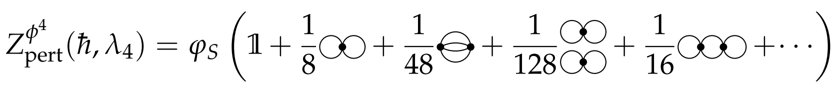
$$Z_{\text{pert}}^{\phi^3}(\hbar, \lambda_3) = \sum_{m \geq 0} \frac{(6m-1)!!}{(3!)^{2m}(2m)!} \lambda_3^{2m} \hbar^m.$$

The first terms of the series are

$$Z^{\phi^3}(\hbar, \lambda_3) = 1 + \frac{5}{24}\lambda_3^2\hbar + \frac{385}{1152}\lambda_3^4\hbar^2 + \cdots,$$

which is the same as we obtained with the diagrammatic expansion, as expected.

**Example 2.6.** ($\phi^4$-theory) In $\phi^4$-theory, the action takes the form $S(x) = \frac{x^2}{2} - \frac{\lambda_4 x^4}{4!}$, and the partition function is given by $Z_{\text{pert}}^{\phi^4}(\hbar, \lambda_4) = \mathcal{F}\left[\frac{x^2}{2} - \frac{\lambda_4 x^4}{4!}\right](\hbar, \lambda_4)$. The diagrammatic expansion starts with



$$= 1 + \frac{1}{8}\lambda_4\hbar + \frac{35}{384}\lambda_4^2\hbar^2 + \cdots$$

As in the previous example, we can compute the partition function directly from Definition 2.2:

$$Z^{\phi^4}_{\text{pert}}(\hbar,\lambda_4) = \mathcal{F}\left[\frac{x^2}{2} - \frac{\lambda_4 x^4}{4!}\right](\hbar,\lambda_4) = \sum_{m\geq 0}\hbar^m(2m-1)!![x^{2m}]e^{\frac{1}{\hbar}\frac{\lambda_4 x^4}{4!}}.$$

Since $e^{\frac{1}{\hbar}\frac{\lambda_4 x^4}{4!}} = \sum_{n\geq 0}\frac{\lambda_4^n x^{4n}}{\hbar^n(4!)^n n!}$, when we evaluate the coefficient extraction, only the coefficients $[x^{4m}]$ survive, and we get

$$Z^{\phi^4}_{\text{pert}}(\hbar,\lambda_4) = \sum_{m\geq 0}\frac{(4m-1)!!}{(4!)^m m!}\lambda_4^m\hbar^m.$$

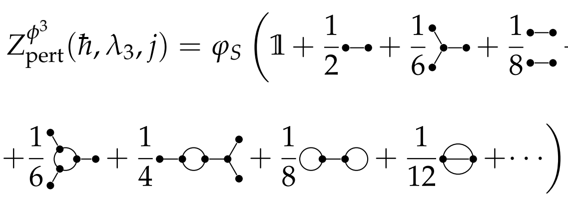
Then, the first terms of the series are

$$Z^{\phi^4}_{\text{pert}}(\hbar,\lambda_4) = 1 + \frac{1}{8}\lambda_4\hbar + \frac{35}{384}\lambda_4^2\hbar^2 + \cdots,$$

which is the same as we obtained with the diagrammatic expansion, as expected.

If we add a source term, then we get graphs with external legs. Let's look at an example.

**Example 2.7.** Consider a $\phi^3$-theory with a source term, so that $S(x) = \frac{x^2}{2} - \frac{\lambda_3 x^3}{3!} - jx$. Then the diagrammatic expansion starts with



$$= 1 + \frac{1}{2}\frac{j^2}{\hbar} + \frac{1}{6}\frac{j^3\lambda_3}{\hbar} + \frac{1}{8}\frac{j^4}{\hbar^2} + \frac{1}{2}j\lambda_3 + \frac{1}{8}j^2\lambda_3^2 + \frac{1}{4}j^2\lambda_3^2 + \frac{1}{6}j^3\lambda_3^3 + \frac{1}{4}j^3\lambda_3^3 + \frac{1}{8}\lambda_3^2\hbar + \frac{1}{12}\lambda_3^2\hbar + \cdots$$

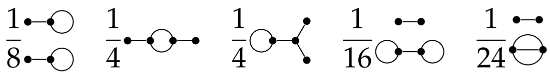
Similarly to (2.4), $Z(j,\lambda_3,\dots) := Z(j,\lambda)$ can be seen as the generating function of $n$-point functions $\langle x^n\rangle$:

$$\langle x^n\rangle = \hbar^n\frac{\partial^n}{\partial j^n}\frac{Z(j,\lambda)}{Z(0,\lambda)}\bigg|_{j=0}.$$

For $\phi^3$-theory (see Example 2.7),

$$\langle x^2 \rangle_{\text{pert}} = \hbar^2 \frac{\frac{1}{\hbar} + \frac{1}{4}\lambda_3^2 + \frac{1}{2}\lambda_3^2 + \frac{1}{2}\lambda_3^2 + \frac{1}{8}\lambda_3^2 + \frac{1}{12}\lambda_3^2 + \cdots}{1 + \frac{5}{24}\lambda_3^2 \hbar + \cdots} = \hbar + \frac{5}{4}\lambda_3^2 \hbar^2 + \cdots$$

where the terms from the numerator correspond to all graphs with two vertices with degree 3 (related to $\lambda_3^2$ term) and two vertices with degree 1 (related to $j^2$ term, which vanishes after derivation):




### 2.2.2 Partition Function of Connected Diagrams

In the previous examples, there was no restriction on the number of connected components of the graphs. For the first terms of the series, where we only have a few vertices and edges, it is easy to obtain all graphs for a certain coefficient. But, for higher-order terms, the number of graphs starts to escalate quickly. One solution is restricting the sum to connected diagrams, since the number of connected graphs with a given number of vertices and edges is significantly smaller than the number of all graphs. We already know how to do this: Theorem 1.32 instructs us to take the logarithm of the partition function for all diagrams.

**Proposition 2.8.** *Let $S(x) = \frac{x^2}{2a} - \sum_{d=0}^{\infty} \frac{\lambda_d}{d!} x^d$, with $a \geq 0$. The partition function over connected diagrams is given by*

$$W(\hbar, \lambda_i, a) := \sqrt{a}\varphi'_S(\Upsilon^c), \tag{2.8}$$

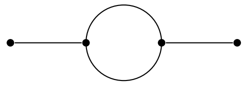
*where $\Upsilon^c$ is given by (1.3), and we assigned the slightly modified Feynman rules $\varphi'_S : \Gamma \mapsto \hbar^{L_\Gamma} a^{|E_\Gamma|} \prod_{v \in V_\Gamma} \lambda_{d(v)}$, where $L_\Gamma$ is the number of independent cycles or faces of the graph $\Gamma$.*

*Proof.* The result follows from Euler's formula (1.1) and by applying Theorem 1.32 to $\hbar \log Z(\hbar, \lambda_i, a)$. □

Therefore, we have that

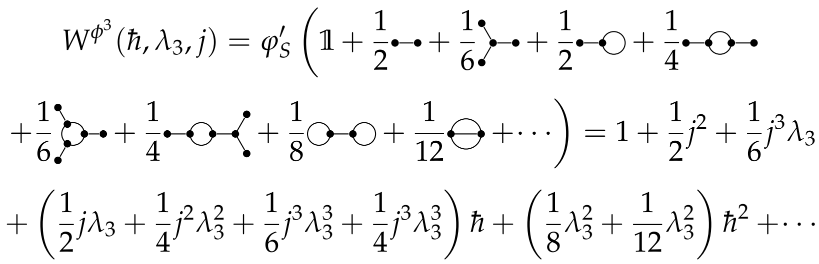
$$W(\hbar, \lambda_i, a) = \hbar \log Z(\hbar, \lambda_i, a). \tag{2.9}$$

Some texts define $L_\Gamma$ as the number of *loops* of the graph $\Gamma$. That is why physicists refer to (2.8) as the *loop expansion*. In particular, the lowest coefficient in $\hbar$ (that is, $\hbar^0$) is the sum over all trees, the coefficient on $\hbar$ is the sum over all graphs with exactly one "loop", and so on. Nonetheless, following the conventional definition of a loop—an edge connecting a vertex to itself—this is not entirely true. In Proposition 1.21, not only do we consider loops to count faces of a graph, but also independent cycles. So the graph



would be part of the $\hbar$ coefficient, since it has one independent cycle (or face).

**Example 2.9.** Consider a $\phi^3$-theory with a source term, so that $S(x) = \frac{x^2}{2} - \frac{\lambda_3 x^3}{3!} - jx$. Then the diagrammatic expansion of the connected partition function starts with



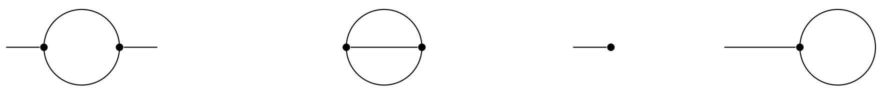
$$W^{\phi^3}(\hbar, \lambda_3, j) = \varphi'_S\left(\mathbb{1} + \frac{1}{2}\,\cdots + \frac{1}{6}\,\cdots + \frac{1}{2}\,\cdots + \frac{1}{4}\,\cdots + \frac{1}{6}\,\cdots + \frac{1}{4}\,\cdots + \frac{1}{8}\,\cdots + \frac{1}{12}\,\cdots + \cdots\right) = 1 + \frac{1}{2}j^2 + \frac{1}{6}j^3\lambda_3$$
$$+ \left(\frac{1}{2}j\lambda_3 + \frac{1}{4}j^2\lambda_3^2 + \frac{1}{6}j^3\lambda_3^3 + \frac{1}{4}j^3\lambda_3^3\right)\hbar + \left(\frac{1}{8}\lambda_3^2 + \frac{1}{12}\lambda_3^2\right)\hbar^2 + \cdots$$

Similarly to (2.4), $W(j)$ can be seen as the generating function of connected $n$-point functions[2] $\langle x^n \rangle_c$:

$$\langle x^n \rangle_c = \left.\frac{\partial^n W(a, j, \lambda)}{\partial j^n}\right|_{j=0}.$$

### 2.2.3 The Effective Action and 1-Particle Irreducible Diagrams

We say that a graph $\Gamma$ without bridges is a *1-particle irreducible* (1PI) diagram. In particular, all external edges are absent in a 1PI diagram. As an example, the following graphs are 1PI:



In this section, we want to show the connection between 1PI diagrams and the generating function of connected diagrams $W(j)$. Let's start with a straightforward result from graph theory.

**Lemma 2.10.** *Any connected graph $\Gamma$ can be uniquely represented as a tree whose vertices are 1-particle irreducible subgraphs (with external edges), and the edges are the bridges of $\Gamma$. The tree corresponding to the graph $\Gamma$ is called the **skeleton** of $\Gamma$.*

*Proof.* Remove all bridges from $\Gamma$, so that it turns into a disjoint union of bridgeless graphs. Take these as the vertices of the tree, and the result follows. □

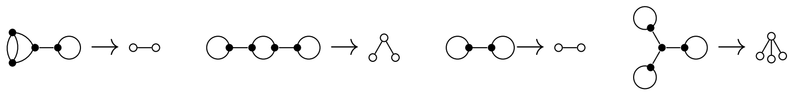


Figure 2.1: Connected diagrams without external legs and their corresponding trees, where each vertex (white circle) represents a 1PI diagram.

Figure 2.1 shows examples of connected diagrams and their corresponding *skeleton*.

**Lemma 2.11.** *The generating function of trees, marked by the degrees of their vertices, fulfils the identity*

$$\sum_{\substack{\Gamma \in \mathcal{G} \\ \text{st } \Gamma \text{ is a tree}}} \frac{\prod_{v \in V_\Gamma} \lambda_{d(v)}}{|Aut\ \Gamma|} = -\frac{\varphi_c^2}{2} + V(\varphi_c),$$

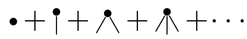
*where* $V(x) = \sum_{d=0}^{\infty} \frac{\lambda_d}{d!} x^d$ *and* $\varphi_c = \mathbb{Q}[[\lambda_1, \lambda_2, \dots]]$ *is the unique power series solution of* $\varphi_c = V'(\varphi_c)$.

*Proof.* The proof is based on a combinatorial argument.

Notice that $V(\varphi_c)$ is the generating function of rooted trees with one vertex fixed, where $\varphi_c$ enumerates legs and $d!$ accounts for symmetry factors:

$$V(\varphi_c) = \lambda_0 + \lambda_1 \varphi_c + \lambda_2 \frac{\varphi_c^2}{2!} + \lambda_3 \frac{\varphi_c^3}{3!} + \cdots,$$

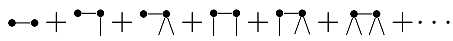
which corresponds to the graphical representation



Notice that these diagrams are bridgeless. Since $\varphi_c = V'(\varphi_c) = \lambda_1 + \lambda_2 \varphi_c + \lambda_3 \frac{\varphi_c^2}{2!} + \lambda_4 \frac{\varphi_c^3}{3!} + \cdots$, the expression $\frac{\varphi_c^2}{2}$ counts the number of trees with one edge fixed, which is the number of pair of rooted trees where the two roots are joined to an edge:

$$\frac{\varphi_c^2}{2} = \frac{\lambda_1^2}{2} + \lambda_1 \lambda_2 \varphi_c + \lambda_1 \lambda_3 \frac{\varphi_c^2}{2!} + \frac{\lambda_2^2 \varphi_c^2}{2} + \lambda_2 \lambda_3 \frac{\varphi_c^3}{2!} + \lambda_3^2 \frac{\varphi_c^4}{2(2!)^2} + \cdots,$$

which corresponds to the diagrammatic representation



Applying Euler's formula (1.1) for trees $\Gamma$, we get that $|V_\Gamma| - |E_\Gamma| = 1$. Therefore,

$$\frac{\varphi_c^2}{2} + V(\varphi_c) = -\sum_{\substack{\Gamma \in \mathcal{G} \\ \text{st } \Gamma \text{ is a tree}}} |E_\Gamma| \frac{\lambda_{d(v)}}{|\text{Aut } \Gamma|} + \sum_{\substack{\Gamma \in \mathcal{G} \\ \text{st } \Gamma \text{ is a tree}}} |V_\Gamma| \frac{\lambda_{d(v)}}{|\text{Aut } \Gamma|}$$

which proves the statement. □

[2]Connected $n$-point functions correspond to cumulants in probability theory.

Using Lemmas 2.10 and 2.11, we can deduce that

$$W(\lambda_i) = -\frac{\varphi_c^2}{2} + \widetilde{V}(\varphi_c), \tag{2.10}$$

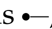
where $W(\lambda_i)$ is the generating function of connected graphs without legs, $\widetilde{V}(\varphi_c)$ is the generating function of connected bridgeless graphs with the number of legs marked by $\varphi_c$

$$\widetilde{V}(\varphi_c) = \sum_{\substack{\Gamma\in\mathcal{G}_{\text{bl}} \\ |C_\Gamma|=1}} \frac{\varphi_c^{|H_\Gamma^{\text{legs}}|}\prod_{v\in V_\Gamma}\lambda_{d(v)}}{|\text{Aut}\,\Gamma|},$$

and $\varphi_c = \widetilde{V}'(\varphi_c)$. We denoted by $\mathcal{G}_{\text{bl}}$ the set of bridgeless graphs.

Now, let's see how to obtain an explicit expression for $\varphi_c$. Let's take the derivative of (2.10) with respect to the formal $\lambda_1$ variable. Following the same notation as in previous sections, let us write $j := \lambda_1$. Then,

$$\frac{\partial}{\partial j}W(\lambda_0, j, \lambda_2, \cdots) = \frac{\partial\varphi_c}{\partial j}\frac{\partial}{\partial\varphi_c}\left(-\frac{\varphi_c^2}{2} + \widetilde{V}(\varphi_c)\right) + \sum_{\substack{\Gamma\in\mathcal{G}_{\text{bl}} \\ |C_\Gamma|=1}}\left(\frac{\partial}{\partial j}\frac{\prod_{v\in V_\Gamma}\lambda_{d(v)}}{|\text{Aut}\,\Gamma|}\right)\varphi_c^{|H_\Gamma^{\text{legs}}|}$$

The first term of the right-hand side vanishes because $\varphi_c = \widetilde{V}'(\varphi_c)$. Notice that the only connected bridgeless graph with a one-valent vertex is , which corresponds to the term $j\varphi_c$. Hence,

$$\frac{\partial}{\partial j}W(\lambda_0, j, \lambda_2, \cdots) = \frac{\partial}{\partial j}(j\varphi_c) = \varphi_c$$

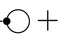
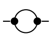
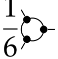
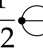
Finally, we may write equation (2.10) as

$$G(\varphi_c) = W(j) - \varphi_c j, \tag{2.11}$$

where $G(\varphi_c) := -\frac{\varphi_c}{2} + \widetilde{V}(\varphi_c) - \varphi_c j$ and $\varphi_c = \frac{\partial}{\partial j}W(j)$. The functional $G(\varphi_c)$ is called the *effective action*, and we just showed that it is related to $W(j)$ by a Legendre transformation. In particular $G(\varphi_c)$ is a generating function in $\varphi_c$ and the coefficients gives us the 1PI $n$-point functions:

$$\langle x^n\rangle_{\text{1PI}} = \left.\frac{\partial^n G(\varphi_c)}{\partial\varphi_c^n}\right|_{\varphi_c=0}.$$

**Example 2.12.** Consider the connected partition function of Example 2.9. The effective action can be depicted diagrammatically as

$G^{\phi^3}(\hbar, \varphi_c) = -\frac{\varphi_c^2}{2} + \rho_S\Big(\frac{1}{6}$  $+ \frac{1}{2}$  $+ \frac{1}{4}$  $+ \frac{1}{6}$  $+ \frac{1}{12}$  $+ \cdots\Big),$

where $\rho_S : \Gamma \mapsto \hbar^{L_\Gamma}\varphi_c^{|H_\Gamma^{\text{legs}}|}$, which assigns a $\varphi_c$ to every leg of the 1PI diagrams.

To summarize, we showed how to obtain three partition functions: $Z(j)$, $W(j)$, and $G(\varphi_c)$, which are, at the same time, the generating functions of $n$-point functions, connected $n$-point functions, and 1PI $n$-point functions, respectively. $W(j)$ is related to $Z(j)$ by $W(j) = \hbar \log Z(j)$, and $G(\varphi_c)$ is the Legendre transformation of $W(j)$.

# Chapter 3

# Padé Approximants

Up to this point, we have not addressed the issue of convergence. The expansion of the partition function has been treated as a formal power series, without assigning numerical values to the couplings $\lambda_i$. However, once we do assign such values, the series discussed in Chapter 2 turn out to be asymptotic rather than convergent. One approach to extracting meaningful information from these divergent series is to compute Padé approximants. In this chapter, we introduce PadÃ© approximants and apply them to a concrete example from zero-dimensional quantum field theory, comparing the results with those obtained from the perturbative expansion. The main reference for this section is [21].

## 3.1 Introduction to Padé Approximants

The starting point to define a Padé approximant of a function $f(z)$ is computing its Maclaurin expansion–that is, its Taylor series at $z = 0$:

$$f(z) = \sum_{i=0}^{\infty} c_i z^i. \tag{3.1}$$

In an informal approach, the Padé approximant is a rational fraction

$$[L/M] = \frac{a_0 + a_1 z + \cdots + a_L z^L}{b_0 + b_1 z + \cdots + b_M z^M}$$

such that its Maclaurin expansion agrees with (3.1) as far as possible. For definiteness, we take $b_0 = 1$, so that we end up with $L + M + 1$ unknown independent coefficients, $L + 1$ from the numerator and $M$ from the denominator. Normally, the $[L/M]$ should fit the power series (3.1) up to order $z^{L+M}$.

$$\sum_{i=0}^{\infty} c_i z^i = \frac{a_0 + a_1 z + \cdots + a_L z^L}{b_0 + b_1 z + \cdots + b_M z^M} + \mathcal{O}(z^{L+M+1})$$

Cross-multiplying, we get that

$$(b_0 + b_1 z + \cdots + b_M z^M)(c_0 + c_1 z + \cdots) = a_0 + a_1 z + \cdots + a_L z^L + \mathcal{O}(z^{L+M+1}). \quad (3.2)$$

We can equate the coefficients of $z^{L+1}, \ldots, z^{L+M}$ to obtain

$$\begin{aligned} b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + \cdots + b_0 c_{L+1} &= 0 \\ b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + \cdots + b_0 c_{L+2} &= 0 \\ &\vdots \\ b_M c_L + b_{M-1} c_{L+1} + \cdots + b_0 c_{L+M} &= 0 \end{aligned}$$

Since we did not impose any restrictions on the powers $L$ and $M$, some $c_j$ above have $j < 0$. In that case, for consistency, we define $c_j = 0$ for $j < 0$. Considering $b_0 = 1$, we have a set of $M$ linear equations for $M$ unknown denominator coefficients.

$$\begin{pmatrix} c_{L-M+1} & c_{L-M+2} & c_{L-M+3} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & c_{L-M+4} & \cdots & c_{L+1} \\ c_{L-M+3} & c_{L-M+4} & c_{L-M+5} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_L & c_{L+1} & c_{L+2} & \cdots & c_{L+M-1} \end{pmatrix} \begin{pmatrix} b_M \\ b_{M-1} \\ b_{M-2} \\ \vdots \\ b_1 \end{pmatrix} = - \begin{pmatrix} c_{L+1} \\ c_{L+2} \\ c_{L+3} \\ \vdots \\ c_{L+M} \end{pmatrix} \quad (3.3)$$

Hence, the $b_i$ can be found. The coefficients from the numerator follow from (3.2) by equating the coefficients of $1, z, \ldots, z^L$:

$$\begin{aligned} a_0 &= c_0 \\ a_1 &= c_1 + b_1 c_0 \\ a_2 &= c_2 + b_1 c_1 + b_2 c_0 \\ &\vdots \\ a_L &= c_L + \sum_{i=0}^{\min(L,M)} b_i c_{L-i} \end{aligned} \quad (3.4)$$

Equations (3.3) and (3.4) are normally referred to as *Padé equations*. Eventually, we expect that a sequence of Padé approximants will approximate a certain function $f(z)$, but it is important to distinguish between problems of convergence of Padé approximants and problems of construction of Padé approximants. The only thing we needed to define these equations was a power series, so we do not need to know if a function $f(z)$ with $\sum_{i=0}^{\infty} c_i z^i$ as its Maclaurin series exists to construct $[L/M]$.

Let us consider an alternative definition of Padé approximants. Using Cramer's rule [15, p. 157], we may compute the denominator of $[L/M]$. Aside from a common factor, we get

$$Q^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-2} & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M-1} & c_{L+M} \\ z^M & z^{M-1} & \cdots & z & 1 \end{vmatrix}. \tag{3.5}$$

Now, let's consider

$$Q^{[L/M]}(z) \sum_{i=0}^{\infty} c_i z^i = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ \sum_{i=0}^{\infty} c_i z^{M+i} & \sum_{i=0}^{\infty} c_i z^{M+i-1} & \cdots & \sum_{i=0}^{\infty} c_i z^i \end{vmatrix}.$$

If we subtract $z^{L+1}$ times the first row from the last, we eliminate the term $z^{L+1}$ from the series of the last row. Analogously, by subtracting $z^{L+2}$ times the second row from the last, the term $z^{L+2}$ vanishes. We can iterate this process up to $z^{L+M}$ times the penultimate row. By doing this, we reduce the series in the last row, creating a gap of $M$ missing terms. Taking the initial terms of these series, we define the following determinant:

$$P^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ \sum_{i=0}^{L-M} c_i z^{M+i} & \sum_{i=0}^{L-M-1} c_i z^{M+i-1} & \cdots & \sum_{i=0}^{L} c_i z^i \end{vmatrix}. \tag{3.6}$$

**Theorem 3.1.** *Let $Q^{[L/M]}(z)$ and $P^{[L/M]}(z)$ be defined by (3.5) and (3.6), respectively. Then,*

$$Q^{[L/M]}(z) \sum_{i=0}^{\infty} c_i z^i - P^{[L/M]}(z) = \mathcal{O}(z^{L+M+1}). \tag{3.7}$$

*Proof.* First, notice that $\deg\{P^{[L/M]}\} \leq L$ and $\deg\{Q^{[L/M]}\} \leq M$. Now, using that the determinant is multilinear, we get that

$$Q^{[L/M]}(z)\sum_{i=0}^{\infty} c_i z^i - P^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ \sum\limits_{i=L-M+1}^{\infty} c_i z^{M+i} & \sum\limits_{i=L-M}^{\infty} c_i z^{M+i-1} & \cdots & \sum\limits_{i=L+1}^{\infty} c_i z^i \end{vmatrix}$$

By performing the subtraction of $z^{L+i}$ times the $i$-th row from the last, we obtain

$$Q^{[L/M]}(z)\sum_{i=0}^{\infty} c_i z^i - P^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ \sum\limits_{i=L+1}^{\infty} c_i z^{M+i} & \sum\limits_{i=L+2}^{\infty} c_i z^{M+i-1} & \cdots & \sum\limits_{i=L+M+1}^{\infty} c_i z^i \end{vmatrix}$$

$$= \sum_{i=1}^{\infty} z^{L+M+i} \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-1} \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ c_{L+i} & c_{L+i+1} & \cdots & c_{L+M+i} \end{vmatrix}.$$

□

From this result follows an important theorem.

**Theorem 3.2.** *(Jacobi's theorem) Let $Q^{[L/M]}(z)$ and $P^{[L/M]}(z)$ be defined by (3.5) and (3.6), respectively. If $Q^{[L/M]}(0) \neq 0$, the $[L/M]$ Padé approximant of $\sum_{i=0}^{\infty} c_i z^i$ is given by*

$$[L/M] = \frac{P^{[L/M]}(z)}{Q^{[L/M]}(z)}. \tag{3.8}$$

*Proof.* It is clear that if $Q^{[L/M]}(0) \neq 0$, we may divide (3.7) by $Q^{[L/M]}(z)$, yielding the result. □

Table 3.1: The Padé table.

| M \ L | 0 | 1 | 2 | $\cdots$ |
|---|---|---|---|---|
| 0 | [0/0] | [1/0] | [2/0] | $\cdots$ |
| 1 | [0/1] | [1/1] | [2/1] | $\cdots$ |
| 2 | [0/2] | [1/2] | [2/2] | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

It is common to display the approximants in what is called a Padé table, shown in Table 3.1.

**Definition 3.3.** *We define*

$$C(L/M) := Q^{[L/M]}(0) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L-1} & c_L & \cdots & c_{L+M-2} \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \end{vmatrix}. \tag{3.9}$$

$C(L/M)$ is called a *Hankel determinant*, which is defined as follows:

**Definition 3.4.** *Let $A$ be a matrix. Assume $i \leq j$, then $A$ is a **Hankel matrix** if $A_{ij} = A_{i+k,j-k}$ for all $k = 0, \ldots, j-i$. That is, $A$ is a rectangular matrix in which each ascending skew-diagonal from left to right is constant. The determinants of such a matrix is called a **Hankel determinant**.*

In the following section, we will see how to relate these Hankel determinants to the numerators and denominators of Padé approximants, giving a set of recursion formulas.

## 3.2 Recursion Formulas

In this section, we will give several identities that apply to either the Padé approximant itself, or to the numerators and denominators alone. First, let us prove the following theorem.

**Theorem 3.5.** *(Sylvester's theorem) Let $A$ be a matrix. We denote by $A_{rp}$ the matrix that remains when deleting row $r$ and column $p$ from $A$, and by $A_{rs;pq}$ the matrix that remains when deleting rows $r$ and $s$ and columns $p$ and $q$. Provided $r < s$ and $p < q$,*

$$\det A \det A_{rs;pq} = \det A_{rp} \det A_{sq} - \det A_{rq} \det A_{sp}. \tag{3.10}$$

*Proof.* Let $A$ be an $(n+2)\times(n+2)$ matrix and consider the deletion of its last two rows and columns—that is, take $r=p=n+1$ and $s=q=n+2$. Let us denote $M=A_{n+1,n+2;n+1,n+2}$. We can write the matrices in a block form:

$$A=\left(\begin{array}{c|c|c} M & h & g \\ \hline f & e & d \\ \hline c & b & a \end{array}\right) \qquad A_{n+2,n+2}=\left(\begin{array}{c|c} M & h \\ \hline f & e \end{array}\right) \qquad A_{n+2,n+1}=\left(\begin{array}{c|c} M & g \\ \hline f & d \end{array}\right)$$

Consider a $(2n+2)\times(2n+2)$ block matrix, with determinant given by

$$\left|\begin{array}{c|c|c|c} M & h & g & 0 \\ \hline f & e & d & 0 \\ \hline c & b & a & c \\ \hline 0 & 0 & 0 & M \end{array}\right| = \det A \det A_{n+1,n+2;n+1,n+2}.$$

Now,

$$\begin{vmatrix} M & h & g & 0 \\ f & e & d & 0 \\ c & b & a & c \\ 0 & 0 & 0 & M \end{vmatrix} = \begin{vmatrix} M & h & g & 0 \\ f & e & d & 0 \\ c & b & a & c \\ M & h & g & M \end{vmatrix} = \begin{vmatrix} M & h & g & 0 \\ f & e & d & 0 \\ 0 & b & a & c \\ 0 & h & g & M \end{vmatrix} =$$

$$= \begin{vmatrix} M & h & g & 0 \\ f & e & d & 0 \\ 0 & 0 & a & c \\ 0 & 0 & g & M \end{vmatrix} + \begin{vmatrix} M & 0 & g & 0 \\ f & 0 & d & 0 \\ 0 & b & a & c \\ 0 & h & g & M \end{vmatrix} =$$

$$= \begin{vmatrix} M & h \\ f & e \end{vmatrix} \cdot \begin{vmatrix} a & c \\ g & M \end{vmatrix} - \begin{vmatrix} M & g \\ f & d \end{vmatrix} \cdot \begin{vmatrix} b & c \\ h & M \end{vmatrix}$$

Hence,

$$\det A \det A_{n+1,n+2;n+1,n+2} = \det A_{n+2,n+2} \det A_{n+1,n+1} - \det A_{n+1,n+2} \det A_{n+2,n+1}.$$

By interchanging rows $r, n+1$ and $s, n+2$ and columns $p, n+1$ and $q, n+2$, the result follows. □

**Corollary 3.6.**

$$C(L/M+1)C(L/M-1) = C(L+1/M)C(L-1/M) - C(L/M)^2 \tag{3.11}$$

*Proof.* Let $\det A = C(L/M+1)$. With $r=p=1$ and $s=q=M+1$, the result follows from the previous theorem. □

Recall definition (3.5) for $Q^{[L/M]}(z)$. This is a $(M+1)\times(M+1)$ determinant whose general structure is preserved after deleting the first or last column and the first row. With either of these deletions, we end up with another $Q^{[l/m]}(z)$. If we apply Theorem 3.5 with deletion of the first and last rows and columns ($r=p=1$ and $s=q=M+1$), we obtain the following identity:

$$Q^{[L/M]}(z)C(L+1/M-1) = Q^{[L+1/M-1]}(z)C(L/M) - zQ^{[L/M-1]}(z)C(L+1/M). \tag{3.12}$$

This equation is referred to as a

$$\begin{pmatrix} * & * \\ * & \end{pmatrix}$$

identity, since it connects denominators of Padé approximants with the configuration shown in Figure 3.1.

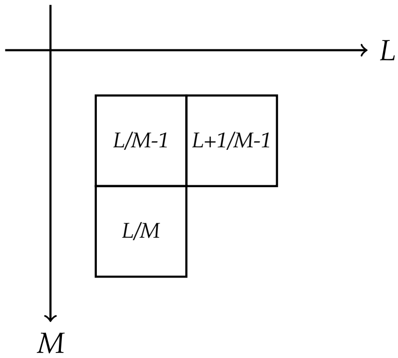



Figure 3.1: Locations in the Padé table of the denominators in (3.12).

We can obtain a similar result with the deletion of rows $M$ and $M+1$ and the first and last columns.

$$Q^{[L/M]}(z)C(L/M-1) = Q^{[L/M-1]}(z)C(L/M) - zQ^{[L-1/M-1]}(z)C(L+1/M) \tag{3.13}$$

This is a

$$\begin{pmatrix} * & * \\ & * \end{pmatrix}$$

identity.

In (3.12), let $L \mapsto L-1$, and rewrite the identity as

$$C(L/M)zQ^{[L-1/M-1]}(z) - C(L-1/M)Q^{[L/M-1]}(z) + C(L/M-1)Q^{[L-1/M]}(z) = 0.$$

Similarly, rewrite (3.13) as

$$C(L+1/M)zQ^{[L-1/M-1]}(z) - C(L/M)Q^{[L/M-1]}(z) + C(L/M-1)Q^{[L/M]}(z) = 0.$$

We can eliminate terms to obtain two new identities. If we eliminate the first terms, then

$$\begin{aligned}&[C(L/M)^2 - C(L+1/M)C(L-1/M)]Q^{[L/M-1]}(z)+\\&+C(L+1/M)C(L/M-1)Q^{[L-1/M]}(z)-\\&-C(L/M-1)C(L/M)Q^{[L/M]}(z) = 0.\end{aligned}$$

By deleting the second terms, we obtain

$$\begin{aligned}&[C(L/M)^2 - C(L+1/M)C(L-1/M)]zQ^{[L-1/M-1]}(z)+\\&+C(L/M)C(L/M-1)Q^{[L-1/M]}(z)-\\&-C(L-1/M)C(L/M-1)Q^{[L/M]}(z) = 0.\end{aligned}$$

We can now use the relation (3.11) to find

$$-C(L/M+1)Q^{[L/M-1]}(z) + C(L+1/M)Q^{[L-1/M]}(z) - C(L/M)Q^{[L/M]}(z) = 0 \tag{3.14}$$

and

$$-C(L/M+1)zQ^{[L-1/M-1]}(z) + C(L/M)Q^{[L-1/M]}(z) - C(L-1/M)Q^{[L/M]}(z) = 0, \tag{3.15}$$

which are

$$\begin{pmatrix} & * \\ * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & \\ * & * \end{pmatrix}$$

identities, respectively. Equations (3.12)-(3.15) are the *Frobenius identities* for the Padé denominators. For the numerators, we can proceed analogously with deletion of the first and last rows and columns to get a similar result to (3.12), but for $P^{[l/m]}(z)$. Hence, we may write the Frobenius identities for a general $S^{[L/M]}(z)$, where $S$ can be either $Q$ or $P$.

**Frobenius identities**

$$\begin{aligned}C(L/M)zS^{[L-1/M-1]}(z) - C(L-1/M)S^{[L/M-1]}(z) + C(L/M-1)S^{[L-1/M]}(z) &= 0\\ C(L+1/M)zS^{[L-1/M-1]}(z) - C(L/M)S^{[L/M-1]}(z) + C(L/M-1)S^{[L/M]}(z) &= 0\\ C(L/M+1)S^{[L/M-1]}(z) - C(L+1/M)S^{[L-1/M]}(z) + C(L/M)S^{[L/M]}(z) &= 0\\ C(L/M+1)zS^{[L-1/M-1]}(z) - C(L/M)S^{[L-1/M]}(z) + C(L-1/M)S^{[L/M]}(z) &= 0\end{aligned}$$

$$\begin{pmatrix} S^{[L-1/M-1]} & S^{[L/M-1]} \\ S^{[L-1/M]} & S^{[L/M]} \end{pmatrix} \leftrightarrow \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

Figure 3.2: Scheme for the Frobenius identities.

## 3.3 Hankel Determinants

In the previous section, we developed several recursion formulas known as Frobenius identities from which we can obtain numerators and denominators of Padé approximants. These identities involve the determinants $C(L/M)$ from Definition 3.3, known as Hankel determinants. In this section, we will see how to compute them.

First, we need to briefly discuss *continued fractions*. The theory of continued fractions is wide and extensive (refer to [21]), but here we are going to focus on the Jacobi (or $J$-type) and the Stieltjes (or $S$-type) continued fractions. In particular, we want to see how these relate to power series and Hankel determinants.

A *Stieltjes continued fraction* is represented as

$$\cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cdots}}}. \tag{3.16}$$

If the series is truncated at order $n$, the result is called the $n$-th approximant. A *Jacobi continued fraction* is represented as

$$\cfrac{1}{1+a_0 x-\cfrac{b_1 x^2}{1+a_1 x-\cfrac{b_2 x^2}{1+a_2 x-\cdots}}}. \tag{3.17}$$

The following result gives a direct correspondence between both continued fractions.

**Proposition 3.7.** *The following identity holds:*

$$\cfrac{1}{1-\cfrac{\alpha_1 x}{1-\cfrac{\alpha_2 x}{1-\cfrac{\alpha_3 x}{1-\cdots}}}} = \cfrac{1}{1-\alpha_1 x-\cfrac{\alpha_1\alpha_2 x^2}{1-(\alpha_2+\alpha_3)x-\cfrac{\alpha_3\alpha_4 x^2}{1-(\alpha_4+\alpha_5)x-\cfrac{\alpha_5\alpha_6 x^2}{1-\cdots}}}}$$

*Hence, the coefficients from the Jacobi continued fraction* (3.17) *are*

$$\begin{aligned} a_0 &= -\alpha_1 \\ a_n &= -(\alpha_{2n} + \alpha_{2n+1}) \\ b_n &= \alpha_{2n-1}\alpha_{2n} \end{aligned}$$

*Proof.* Define

$$t_0(w) = w \text{ and } t_p(w) = \frac{1}{1 - \alpha_p x w}, \; p = 1, 2, 3, \ldots$$

so that $t_0 t_1 \cdots t_n(1)$ is the $n$-th approximant of the $S$-type continued fraction. Define

$$s_p(w) = t_{2p} t_{2p+1}(w), \; p = 0, 1, 2, \cdots$$

so that

$$s_0(w) = \frac{1}{1 - \alpha_1 x w} \text{ and } s_p(w) = 1 + \frac{\alpha_{2p} x}{1 - \alpha_{2p} x - \alpha_{2p+1} x w}, \; p = 1, 2, 3, \cdots$$

Notice that $s_0 s_1 \cdots s_p(1) = t_0 t_1 \cdots t_{2p+1}(1)$, which is the $(2p+1)$-th approximant. Since

$$s_0 s_1 \cdots s_p(1) = \cfrac{1}{1 - \alpha_1 x - \cfrac{\alpha_1 \alpha_2 x^2}{1 - (\alpha_2 + \alpha_3) x - \cfrac{\alpha_3 \alpha_4 x^2}{1 - (\alpha_4 + \alpha_5) x - \cdots}}},$$

the result follows. □

If $\{c_k\}_k$ is a sequence of real numbers, we will denote the Hankel determinants associated with the sequence as follows:

$$\det_{0 \le i,j \le n-1} (c_{i+j}) = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-2} \end{vmatrix}$$

The following theorem (*cf.* [13, Theorem 11] or [14, Theorem 29]) gives a direct relationship between Hankel determinants and Jacobi continued fractions. For a detailed proof, you can refer to [21, Theorem 51.1].

**Theorem 3.8.** *Let $\{c_k\}_k$ be a sequence of real numbers and $f(x) = \sum_{k=0}^{\infty} c_k x^k$ its ordinary generating function, written in the form*

$$\sum_{k=0}^{\infty} c_k x^k = \cfrac{c_0}{1 + a_0 x - \cfrac{b_1 x^2}{1 + a_1 x - \cfrac{b_2 x^2}{1 + a_2 x - \cdots}}}.$$

*The Hankel determinant* $\det_{0 \le i,j \le n-1}(c_{i+j})$ *equals* $c_0^n b_1^{n-1} b_2^{n-2} \cdots b_{n-2}^2 b_{n-1}$.

## 3.4 Application to 0-Dimensional Quantum Field Theory

In this section we want to compute the Padé approximants for the 2-point function $\langle \phi^2 \rangle$ of $\phi^4$-theory, which is given by

$$\langle \phi^2 \rangle = \frac{1}{Z} \int \frac{1}{\sqrt{2\pi}} \phi^2 e^{-\frac{\phi^2}{2} - \lambda \frac{\phi^4}{4}} d\phi = \frac{I}{Z},$$

with

$$Z = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2} - \lambda \frac{\phi^4}{4}} d\phi$$

and where $I$ denotes the integral from the numerator. Notice that we set $m = \hbar = 1$, so $\langle \phi^2 \rangle$ is a function that depends on $\lambda > 0$. Moreover, for this example, we erased the factorial 4! of the exponential from Example 2.6, and left a 4, simply because the computations become easier to manage.

Using the perturbative expansions of $Z$ and $I$, we can obtain the Taylor series for $\langle \phi^2 \rangle$ and compute the Padé approximants. With a direct computation analogous to Example 2.6, we get that

$$Z(\lambda)_{\text{pert}} = \sum_{n=0}^{\infty} \frac{(4n-1)!!}{n! 4^n} (-\lambda)^n \quad \text{and} \quad I(\lambda)_{\text{pert}} = \sum_{m=0}^{\infty} \frac{(2(2m+1)-1)!!}{m! 4^m} (-\lambda)^m.$$

So, the first terms of $\langle \phi^2 \rangle_{\text{pert}}$ are

$$\langle \phi^2 \rangle_{\text{pert}} = 1 - 3\lambda + 24\lambda^2 - 297\lambda^3 + 4896\lambda^4 + \mathcal{O}(\lambda^5) \ . \tag{3.18}$$

We are now going to make use of The On-Line Encyclopedia of Integer Sequences (OEIS) (see [16]). The sequence number is A292186, and here we find that the Stieltjes continued fraction of this generating function is the following:

$$\langle \phi^2 \rangle_{\text{pert}} = \cfrac{1}{1 + \cfrac{3\phi}{1 + \cfrac{5\phi}{1 + \cfrac{7\phi}{\cdots}}}}$$

From expression (3.16) we see that $\alpha_n = -(2n+1)$. Now we can use Proposition 3.7 to obtain the coefficients of the Jacobi continued fraction:

$$\begin{aligned} a_0 &= 3 \\ a_n &= [2(2n)+1+2(2n+1)+1] = (8n+4) \\ b_n &= [2(2n-1)+1][2(2n)+1] = (4n-1)(4n+1) \end{aligned} \tag{3.19}$$

It turns out that the $[N/N]$ and the $[N/N+1]$ Padé approximants work nicely for the 2-point function. Applying Theorem 3.8, we can obtain the Hankel determinants $C(N/N)$ and $C(N/N+1)$, which we will use in the Frobenius identities to obtain the numerators and denominators of the Padé approximants. We want to build the Padé table 3.2.

Table 3.2: Padé table structure for $\langle\phi^2\rangle$.

| M \ L | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|
| 0 | [0/0] | | | | |
| 1 | [0/1] | [1/1] | | | |
| 2 | | [1/2] | [2/2] | | |
| 3 | | | [2/3] | [3/3] | |
| ⋮ | | | | | ⋱ |

First, notice that the determinant

$$C(N/N+1) = \begin{vmatrix} c_0 & c_1 & \cdots & c_N \\ c_1 & c_2 & \cdots & c_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_N & c_{N+1} & \cdots & c_{2N} \end{vmatrix}$$

corresponds to the series from $\langle\phi^2\rangle_{\text{pert}}$, that has the coefficients (3.19) for its Jacobi continued fraction. Since $c_0 = 1$,

$$C(N/N+1) = \prod_{j=1}^{N} [(4j-1)(4j+1)]^{N+1-j}$$

whereas

$$C(N/N) = \begin{vmatrix} c_1 & c_2 & \cdots & c_N \\ c_2 & c_3 & \cdots & c_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_N & c_{N+1} & \cdots & c_{2N-1} \end{vmatrix}$$

Table 3.3: Padé table for the 2-point function $\langle \phi^2 \rangle$ of $\phi^4$ theory.

| M \ L | 0 | 1 | 2 | 3 | $\cdots$ |
|---|---|---|---|---|---|
| 0 | $1$ | | | | |
| 1 | $\frac{1}{1+3\lambda}$ | $\frac{1+5\lambda}{1+8\lambda}$ | | | |
| 2 | | $\frac{1+12\lambda}{1+15\lambda+21\lambda^2}$ | $\frac{1+21\lambda+45\lambda^2}{1+24\lambda+93\lambda^2}$ | | |
| 3 | | | $\frac{1+32\lambda+177\lambda^2}{1+35\lambda+258\lambda^2+231\lambda^3}$ | $\frac{1+45\lambda+450\lambda^2+585\lambda^3}{1+48\lambda+570\lambda^2+1440\lambda^3}$ | |
| $\vdots$ | | | | | $\ddots$ |

corresponds to the series from $(\langle \phi^2 \rangle_{\text{pert}} - 1)/x$. In this case, the first term of the series is $c_1 = -3$, $a_n = -(\alpha_{2n+1} + \alpha_{2n+1}) = 8(n+1)$, and $b_n = \alpha_{2n}\alpha_{2n+1} = (4n+1)(4n+3)$. Hence,

$$C(N/N) = (-3)^N \prod_{j=1}^{N-1} [(4j+1)(4j+3)]^{N-j} \,.$$

Now, using the Frobenius identities (3.13) and (3.15) and simplifying common factors, we get that

$$\frac{P^{[N/N]}(\lambda)}{Q^{[N/N]}(\lambda)} = \frac{P^{[N-1/N]}(\lambda) + (4N+1)\lambda P^{[N-1/N-1]}(\lambda)}{Q^{[N-1/N]}(\lambda) + (4N+1)\lambda Q^{[N-1/N-1]}(\lambda)} \quad \begin{pmatrix} * & \\ * & * \end{pmatrix} \tag{3.20}$$

$$\frac{P^{[N-1/N]}(\lambda)}{Q^{[N-1/N]}(\lambda)} = \frac{P^{[N-1/N-1]}(\lambda) + (4N-1)\lambda P^{[N-2/N-1]}(\lambda)}{Q^{[N-1/N-1]}(\lambda) + (4N-1)\lambda Q^{[N-2/N-1]}(\lambda)} \quad \begin{pmatrix} * & * \\ & * \end{pmatrix} \tag{3.21}$$

with $P^{[0/0]}(\lambda) = P^{[0/1]}(\lambda) = Q^{[0/0]}(\lambda) = 1$ and $Q^{[0/1]}(\lambda) = 1 + 3\lambda$. From this, we can built a very simple recursion formula for the Padé approximants. First, let's define $n = L + M$ and denote $P^{[L/M]}(\lambda) = p_{L+M}(\lambda) = p_n(\lambda)$ and $Q^{[L/M]}(\lambda) = q_{L+M+1}(\lambda) = q_{n+1}(\lambda)$. Then, either from (3.20) or (3.21), we obtain that $p_n(\lambda) = p_{n-1}(\lambda) + (2n+1)\lambda p_{n-2}(\lambda)$, with $p_0(\lambda) = p_1(\lambda) = 1$. For the denominators, we get that $q_n(\lambda) = q_{n-1}(\lambda) + (2n-1)\lambda q_{n-2}(\lambda)$, with $q_0(\lambda) = q_1(\lambda) = 1$. Notice that $q_0(\lambda)$ does not correspond to any denominator. However, we still obtain that $Q^{[0/1]}(\lambda) = q_2(\lambda) = 1 + 3\lambda$, as expected. All in all, we defined the recursion formula

$$\begin{array}{ll} p_0(\lambda) = p_1(\lambda) = 1 & p_n(\lambda) = p_{n-1}(\lambda) + (2n+1)\lambda p_{n-2}(\lambda) \\ q_0(\lambda) = q_1(\lambda) = 1 & q_n(\lambda) = q_{n-1}(\lambda) + (2n-1)\lambda q_{n-2}(\lambda) \end{array} \tag{3.22}$$

and the Padé approximants are

$$\frac{p_n(\lambda)}{q_{n+1}(\lambda)},$$

which are $[N/N]$ approximants if $n$ is even and $[N/N+1]$ if $n$ is odd.

### 3.4.1 A Note on Convergence

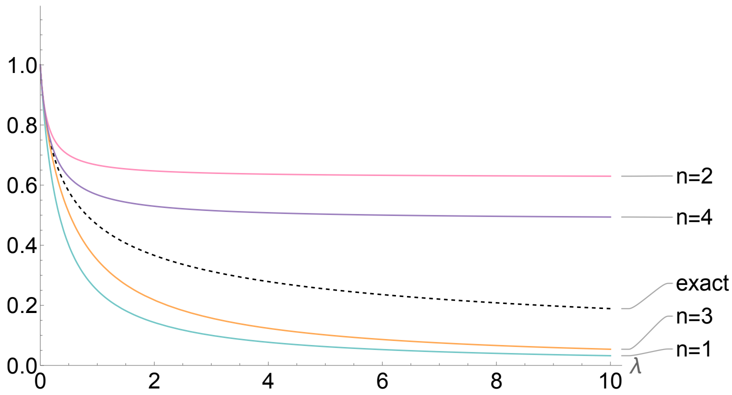



Figure 3.3: First Padé approximants with the exact solution (dashed curve).

Notice that each approximant is between the other two. This can be seen graphically in Figure 3.3.

If we denote

$$r_n = \frac{p_n}{q_{n+1}},$$

this means that the subsequences $\{r_{2n}\}_n$ and $\{r_{2n+1}\}_n$ of even and odd terms, respectively, are monotone. In particular $\{r_{2n}\}_n$ is strictly decreasing (*i.e.* $r_{2n+2} < r_{2n}\ \forall n \in \mathbb{N}$), and $\{r_{2n+1}\}_n$ is strictly increasing (*i.e.* $r_{2n+3} > r_{2n+1}\ \forall n \in \mathbb{N}$).

Now, notice that both subsequences are bounded. Since all coefficients are positive and so is $\lambda$, the decreasing subsequence $\{r_{2n}\}_n$ is bounded below by 0. Moreover, since $q_{n+1} > p_n$, the increasing subsequence $\{r_{2n+1}\}_n$ is bounded above by 1. By the Monotone Convergence Theorem (*cf.* [1, Theorem 2.4.2]), we have that both $\{r_{2n}\}_n$ and $\{r_{2n+1}\}_n$ converge.

This does not immediately imply that the sequence $\{r_n\}_n$ converges, since the subsequences could have different limits, so we need an additional condition. We want to see that the difference between two consecutive terms tends to 0. That is,

$$|r_{n-1} - r_n| \to 0.$$

This is equivalent to saying that $|r_{2n} - r_{2n+1}| \to 0$, and hence saying that $\{r_n\}_n$ converges to a function.

The difference between two consecutive terms is

$$\left| \frac{p_n}{q_{n+1}} - \frac{p_{n-1}}{q_n} \right| = (2n+1)!!\ \lambda^n \frac{1}{q_{n+1}q_n}. \tag{3.23}$$

First, we want to prove that the right-hand side of this equation is a monotone increasing function, thus the maximum is reached as $\lambda \to \infty$. Then, we will see that in this limit, the difference tends to 0 when $n \to \infty$. Since the maximum of this difference is reached for $\lambda \to \infty$, this ensures that it must converge to 0 when $n \to \infty$ for any value of $\lambda$.

**Step 1:** The right-hand side of (3.23) is a monotone increasing function.

Let us denote $Q_n := q_{n+1}q_n$. Notice that $Q_n$ is a polynomial of degree $n$ with all its coefficients positive. We can write

$$Q_n(\lambda) = \sum_{j=0}^{n} c_j \lambda^j$$

with $c_j > 0 \; \forall j \in \{0, \dots, n\}$. Then, the first derivative of

$$\frac{\lambda^n}{Q_n(\lambda)}$$

with respect to $\lambda$ is given by

$$\frac{\lambda^{n-1}(nQ_n - \lambda Q_n')}{Q_n^2} > 0,$$

which is strictly positive since $nQ_n - \lambda Q_n' = \sum_{j=0}^{\infty}(n-j)c_j\lambda^j > 0$. Hence, this first step is proven. Figure 3.4 shows the curves for the difference between two consecutive Padé approximants for $n = 2, \dots, 29$ in (3.23).

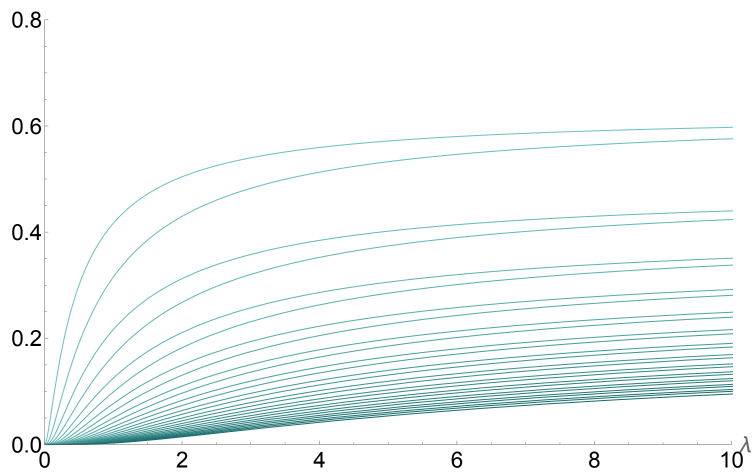



Figure 3.4: Differences between consecutive Padé approximants for the first 30. As $\mathfrak{n} \to \infty$, the curves tend to 0, and as $\lambda \to \infty$, the differences increase monotonously.

**Step 2**: In the limit $\lambda \to \infty$, the difference tends to 0 when $n \to \infty$.

If $n$ is even, say $n = 2k$, then both $q_{n+1}$ and $q_n$ are of order $\lambda^k$, which means that $q_{2k+1}q_{2k}$ is of order $\lambda^{2k}$. This implies that the numerator and denominator of

(3.23) are polynomials of the same degree, so the limit as $\lambda \to \infty$ is a finite number. The same happens if $n$ is odd. In this case, if $n = 2k+1$, $q_{n+1}$ is of order $k+1$ and $q_n$ of order $k$. So, this difference always tends to a finite number, which depends on $n$, as $\lambda \to \infty$.

Now we want to see that as $n \to \infty$, this number goes to 0. For that, we need the leading terms of the polynomials $q_{2k}$ and $q_{2k+1}$. From the recursive definition (3.22), we learn that

$$[\lambda^k]q_{2k} = \prod_{j=1}^{k}(4j-1) \quad \text{and} \quad [\lambda^k]q_{2k+1} = \frac{1}{2}\left(\prod_{j=0}^{k}(4j+3) - \prod_{j=0}^{k}(4j+1)\right),$$

where $[\lambda^k]q_{2k}$ and $[\lambda^k]q_{2k+1}$ denote the leading term of each polynomial. From the properties of the Gamma function $\Gamma$, we can deduce the following:

$$[\lambda^k]q_{2k} = \frac{4^k\Gamma(k+\frac{3}{4})}{\Gamma(\frac{3}{4})} \quad \text{and} \quad [\lambda^k]q_{2k+1} = \frac{4^k}{2}\left(3\frac{\Gamma(k+\frac{7}{4})}{\Gamma(\frac{7}{4})} - \frac{\Gamma(k+\frac{5}{4})}{\Gamma(\frac{5}{4})}\right).$$

Let $f(k) = \frac{4^k\Gamma(k+\frac{3}{4})}{\Gamma(\frac{3}{4})}\frac{4^k}{2}\left(3\frac{\Gamma(k+\frac{7}{4})}{\Gamma(\frac{7}{4})} - \frac{\Gamma(k+\frac{5}{4})}{\Gamma(\frac{5}{4})}\right)$. Now we can compute the limit

$$\lim_{k\to\infty} \frac{(2(2k)+1)!!}{f(k)},$$

which, with the use of *Mathematica*, we get that it is 0, as expected.

Altogether, we have seen that the subsequences $\{r_{2n}\}_n$ and $\{r_{2n+1}\}_n$ converge. Since the difference of two consecutive terms tends to 0 for all values of $\lambda$, this limit has to be the same, and this means that the sequence of Padé approximants converges to a function. □

Figure 3.5 includes the exact solution (dashed curve) along with the first 30 approximants. As $\lambda$ increases, these approximants tend to deviate from the exact solution, but for small values of $\lambda$, they are extremely accurate.

If we recall, the main goal of this chapter was to improve the results from the perturbative series of Chapter 2. So, the question now is: have we? Graphically, we can already see the improvement in Figure 3.6, where the perturbative series rapidly deviates from the exact solution. Table 3.4 shows the values for Padé approximants and perturbative expansions with their corresponding relative errors (RE). For small values of $\lambda$, both approximations are accurate. As $\lambda$ increases, the Taylor expansion completely disagrees with the exact solution, whilst the Padé approximant for $n = 29$ still gives reliable results.

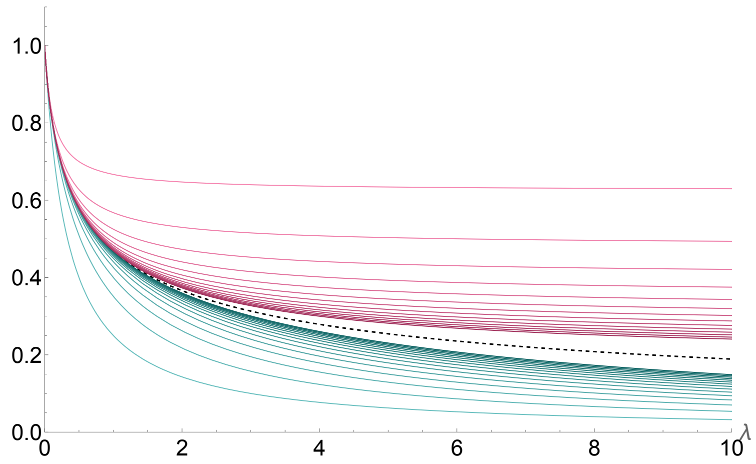



Figure 3.5: Exact solution (dashed) and the first 30 Padé approximants. The subsequences $\{r_{2n}\}_n$ and $\{r_{2n+1}\}_n$ correspond to the pink and blue curves, respectively, and as $n$ increases, the curves get closer to the exact solution.

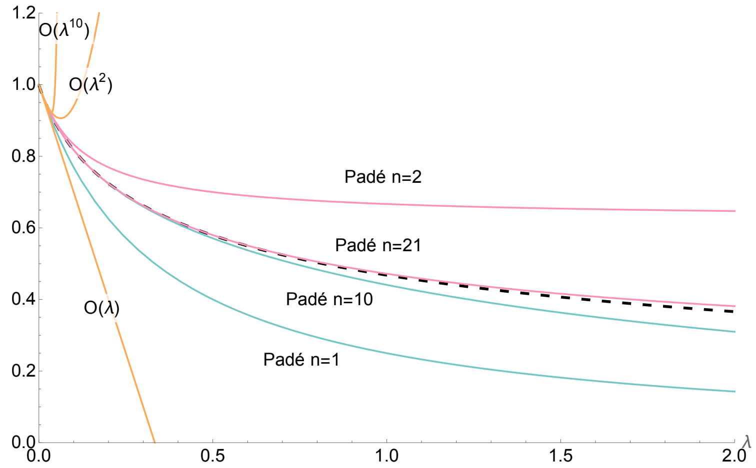



Figure 3.6: Exact solution (dashed), perturbative expansion of order $\lambda$, $\lambda^2$ and $\lambda^{10}$ (in orange), and Padé approximants for $n = 1, 2, 10, 21$.

Table 3.4: $\langle \phi^2 \rangle$ for different $\lambda$ values. Showing the exact values along with different approximations and their corresponding relative error (RE).

| $\lambda$ | Exact | Padé $n = 3$ | RE | Padé $n = 29$ | RE | Taylor $\mathcal{O}(\lambda^3)$ | RE |
|---|---|---|---|---|---|---|---|
| 0.001 | 0.996037 | 0.997024 | 0.099 | 0.997024 | 0.099 | 0.997024 | 0.099 |
| 0.01 | 0.971664 | 0.972222 | 0.057 | 0.972144 | 0.049 | 0.972103 | 0.045 |
| 0.05 | 0.888682 | 0.892857 | 0.470 | 0.888705 | 0.003 | 0.872875 | 1.779 |
| 0.1 | 0.817561 | 0.833333 | 1.929 | 0.817561 | $8 \times 10^{-5}$ | 0.643000 | 21.35 |
| 0.5 | 0.579205 | 0.700000 | 20.85 | 0.579110 | 0.016 | -31.6250 | 5560 |
| 1.0 | 0.467920 | 0.666667 | 42.47 | 0.469231 | 0.280 | -275.000 | 58870 |
| 2.0 | 0.365957 | 0.647059 | 76.81 | 0.372870 | 1.889 | -2285.00 | 624490 |

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