Thermal cloaking via heat flux transformation

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The methods used in Transformation optics could be extrapolated to thermal cloaking studies thanks to invariance of Heat Conduction Equation under mappings. On this basis, we study different transformations looking for the most suitable mapping for our interests. Then, we apply an asymptotic homogenization and diverse experimental constraints with the aim to implement a thermal cloak using exclusively homogeneous isotropic materials.

I. INTRODUCTION

Since science fiction irruption, humanity always have dreamed with handle the light at will. Although the results are so far from the invisibility concept provided by movies, in the last decades have been a very important progress to achieve invisibility. In 1967 L. S. Dolin [1], exploiting the invariance of Maxwell's equations under coordinate transformations, presented the idea of study three dimensional electromagnetic systems comparing them with other more simple systems through mappings and, although the technology of 1960's was unable to create the materials designed by the mappings, this established the first mention to Transformation Optics. All of these materials become possible since Sir J. B. Pendry, asked from *Marconi Materials Technology* about the features of carbon for stealth technology, realized that the behavior of a material can be altered by changing its internal structure on a very fine scale. This new materials will be known as metamaterials. This discovery [2] presents an inverse problem: up to now we study materials and then we figure out what to use them, but since Pendry's ascertainment we can design the materials in function of our requirements.

Cloaking appears for the first time in 2006 with the simultaneously published articles (both of them appears at the same *Science Express* number) of U. Leonhardt [3] and J. B. Pendry [4]. In both articles cloaking was described as the phenomenon *"which guide light around an object as if nothing were there"*. Raising the inverse problem, we should find a mathematical description of the behavior that should describe the electromagnetic wave so to avoid the cloaking zone in order to reflect it at the metamaterial properties. This could be achieved using the strategy illustrated in Fig.(1): knowing the invariance under mappings of Maxwell's equation, we can design a suitable mapping that describes a new space where a constant κ, *pulled back* to our original space, will reflect the behavior we are looking for.

In practice appears some complications like dispersion problems derived from broadband expansion or the polarization dependence which appears when using certain not-conformal mappings. During last years the Transformation Optics has been used to study many other optical devices such as rotators, concentrators or even artificial black holes. Also exist studies that exports Transformation Optics to other physical branches like acoustic or matter waves transmission as well as to DC fields or thermal conduction.

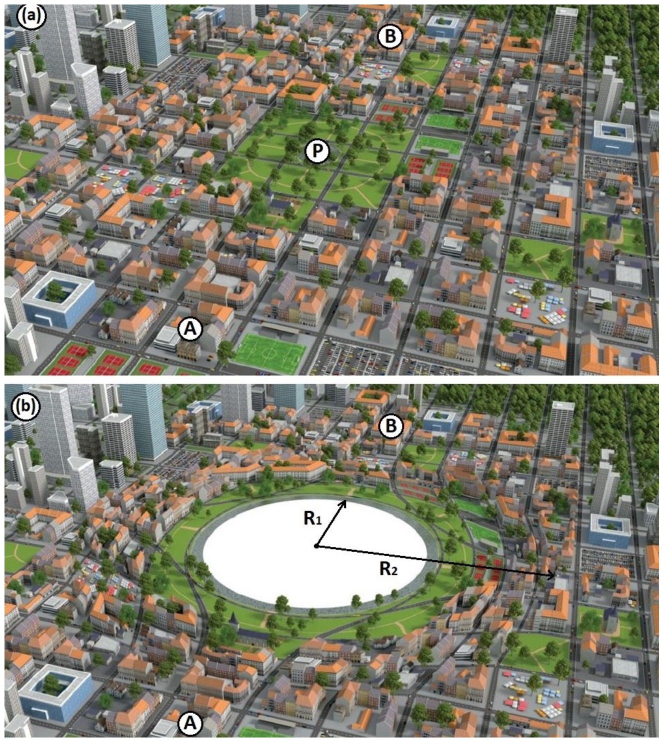

FIG. 1: We can intuitively assimilate the concept underlying coordinate-transformations-based cloaking to city streets grid. Here we use the transformation that we are going to end up using, which maps the point P onto a circle of radius R_1 and leaves the area outside R_2 unchanged. Two metric spaces are involved in this mapping. First of them, represented in *(a)* and also known as *Physical space*, is described by ordinary cartesian coordinates and represents our real material. The second metric space, also known as *Virtual space*, illustrated in *(b)*, is a immaterial space we define through our mapping with the intention of being able to practice a *pull back* to a constant conductivity (κ in our case) up to *Physical space*. The inverse problem consists in reflect the velocities of the cars in *(b)* onto the conductivity of our new material.

The latter is the one that occupies us in this work. Invariance of the heat conduction equation under coordinate transformation (demonstrated later) allow us to apply the same mapping-based recipe of Transformation Optics with the aim of achieve Thermal Cloaking. This time, we want to keep a region also known as *Cloaking zone* colder than its environment without modifying the exterior distribution of T. This latter condition is essential, because it represents the main difference with a typical isolation. Although the same strategy is used in both optical and thermal cloaking, there are many differences between them. The most significant lies in the fact that the equation that governs the propagation of electromagnetic waves, *i. e.* the wave equation,

$$\left.\begin{aligned} \nabla^2\vec{E} &= \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}, \\ \nabla^2\vec{B} &= \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}, \end{aligned}\right\} \quad \nabla^2\vec{f} = \alpha\frac{\partial^2\vec{f}}{\partial t^2}, \qquad (1)$$

is a hyperbolic partial differential equation (PDE) from which waves are solution. On the other hand, heat conduction equation in absence of sources,

$$\nabla\cdot(\kappa\nabla T) = \rho c\frac{\partial^2 T}{\partial t}, \quad \rightarrow \quad \nabla^2\vec{f} = \alpha\frac{\partial\vec{f}}{\partial t}, \qquad (2)$$

is a parabolic PDE that doesn't accept waves like one of their solutions. This implies that many phenomenons typical from waves studies like scattering, interference or reflection which cause many problems in Optical Cloaking do not occur in Thermal Cloaking.

First research on Thermal Cloaking dates from 2008. Chen *et al* [5] studied bidimensional and three-dimensional thermal cloaking. Guenneau *et al* [6] extended the previous work onto thermodynamics and accomplish a theoretical method to solve the inhomogeneity in thermal conductivity using a multilayered cloak. Later, Schittny *et al* [7] and Ma *et al* [8] implemented a transient thermal cloak through tailoring the anisotropy of conductivity. Recently the scattering cancellation method has araised, by which it's possible to obtain a bilayered thermal cloaking using a layer of ferromagnetic material and a layer of superconductor material. Other works in this area like rotators, concentrators or illusion thermal devices have been published through last years.

In this study, we first deduce the heat conduction equation and demonstrate its invariance under a arbitrary coordinate transformation. Then we apply a general annular transformation that will be restricted to a circular transformation due to certain constraints we detail. In the next chapters, we solve the inhomogeneity and anisotropy that mapping requires using different layered techniques. Finally, we analyze the experimental data and numerical simulations obtained by Schittny *et al* [7] using the same model we detail.

II. HEAT CONDUCTION EQUATION

In this section we will deduce the heat conduction equation and we will demonstrate its invariance under coordinate mappings.

A. Deduction

Let us consider a conductive 2D surface, say Ω. Let ν be the thermal velocity and u be the thermal energy volume density at any point P. Let us now regard a closed circular subregion Ω_r centered at P with radius R_1. We can define the thermal energy concentrated in Ω_r like,

$$E_r = \iint_{\Omega_r} u\,dxdy. \qquad (3)$$

Applying now the First Principle of Thermodynamics, which implies that the rate of decrease of E_r is the flux of the vector field $\vec{J} = u\vec{\nu}$ across the boundary of Ω_r,

$$\frac{d}{dt}E_r = \frac{d}{dt}\iint_{\Omega_r} u\,dxdy = -\int_{\Gamma_r}\vec{J}\cdot\vec{n}\,dl, \qquad (4)$$

where Γ_r is the boundary of subregion Ω_r that thermal flux is crossing, $\vec{n}$ is the unit outward normal and dl is the arc lenght. Using now the First Fundamental Theorem of Calculus on the left-hand side and the Green's theorem on the right-hand side, we obtain,

$$\iint_{\Omega_r}\frac{\partial u}{\partial t}\,dxdy = -\iint_{\Omega_r}\nabla\cdot\vec{J}\,dxdy, \qquad (5)$$

reorganizing our expression we get ,

$$\iint_{\Omega_r}(\frac{\partial u}{\partial t} + \nabla\cdot\vec{J})\,dxdy = 0 \quad \rightarrow \quad \frac{\partial u}{\partial t} + \nabla\cdot\vec{J} = 0, \qquad (6)$$

also known as Continuity Equation. Introducing now,

$$\vec{J} = -\kappa\nabla T \qquad u = c\rho T, \qquad (7)$$

where κ is the heat conductivity, c is the specific heat per mass unit and ρ is the mass density, we obtain the Heat Conductivity Equation in absence of sources:

$$\nabla\cdot(\kappa\nabla T) - c\rho\frac{\partial T}{\partial t} = 0. \qquad (8)$$

For the steady-state case, for which the temporal derivative of T is canceled, this expression reduces to,

$$\nabla\cdot(\kappa\nabla T) = 0. \qquad (9)$$

B. Invariance under coordinate transformations

Upon an arbitrary coordinate transformation $\mathbf{x} = (x, y) \rightarrow \mathbf{x}' = (x', y')$ described by the Jacobian matrix,

$$J = \frac{\partial\mathbf{x}'}{\partial\mathbf{x}}, \qquad (10)$$

considering $c(t)\rho$ as a whole, Eq (8) is expressed as:

$$\frac{\partial \rho c(T)T}{\partial t} = \frac{\partial}{\partial x^i}\kappa^{ij}(T)\frac{\partial}{\partial x^j}T + \Gamma^i_{ik}\kappa^{kj}(T)\frac{\partial}{\partial x^j}T, \quad (11)$$

where Γ^i_{ik} is the Christoffel symbol. Using the identity [9]:

$$\Gamma^i_{ik} = \frac{1}{2}g^{il}\frac{\partial}{\partial x^k}g_{il} = |J|\frac{\partial}{\partial x^k}|J|^{-1}, \quad (12)$$

where g is the metric tensor, we can write:

$$\frac{\partial}{\partial t}\frac{\rho c(T)T}{|J|} = \frac{\partial}{\partial x'^i}\left[\frac{J\kappa(T)J^T}{|J|}\right]^{ij}\frac{\partial}{\partial x'^j}T. \quad (13)$$

Now, identifying,

$$\kappa' = \frac{J\kappa J^T}{|J|}, \qquad c(T)'\rho' = \frac{c(T)\rho}{|J|}, \quad (14)$$

our equation recover the Heat Conduction Equation's form:

$$\frac{\partial c(T)'\rho' T}{\partial t} - \frac{\partial}{\partial x'^i}\kappa'\frac{\partial}{\partial x'^j}T = \frac{\partial c(T)'\rho' T}{\partial t} - \nabla'\cdot(\kappa'\nabla' T) = 0. \quad (15)$$

III. COORDINATE TRANSFORMATIONS

Now we study the suitableness of some appropriate transformations. First we apply a general annular transformation to the Heat Conduction Equation that we will end up restricting to a circular transformation making use of detailed mathematical criteria.

For simplicity, we decide to develop our study in polar coordinates hereinafter, so we apply first the common transformation: $r = \sqrt{x^2+y^2}$, $\theta = \arctan(\frac{y}{x})$. This transformation will impose the need to practice two additional coordinate changes,

$$(x,y) \underset{J_{xr}}{\rightarrow} (r,\theta) \underset{J_{rr'}}{\rightarrow} (r',\theta') \underset{J_{r'x'}}{\rightarrow} (x',y') \quad (16)$$

with the total Jacobian,

$$J_{xx'} = J_{xr}J_{rr'}J_{r'x'} = \\ = \left[M(\theta)\mathrm{diag}\left(1,\frac{1}{r}\right)\right]J_{rr'}\left[\mathrm{diag}(1,r')M(\theta)^T\right], \quad (17)$$

where $M(\theta)$ denotes the rotation matrix through an angle θ. It's important to mention that we are practicing a *pull back* to a tensorial κ, so the Jacobian matrix used here will correspond to the inverse transformation.

A. Radial transformation

We consider a general radial stretch with an azimuthal dependence,

$$\begin{cases} r \rightarrow r' = \alpha(\theta)r + \beta(\theta), \\ \theta \rightarrow \theta' = \theta. \end{cases} \quad (18)$$

Using (14) and (17) we obtain,

$$\kappa' = \kappa\left[M(\theta)\begin{pmatrix} \frac{(r'-\beta)^2+q^2\cdot\alpha^2}{(r'-\beta)r'} & \frac{-q\cdot\alpha}{(r'-\beta)} \\ \frac{-q\cdot\alpha}{(r'-\beta)} & \frac{r'}{(r'-\beta)} \end{pmatrix}M(\theta)^T\right], \quad (19)$$

where, $q = \frac{\partial r}{\partial \theta'}$. One can see that when $r' = \beta$ in the transformed coordinates, *i. e.* $r = 0$ in the original system, the first coefficient vanishes and the other three tend to infinity. We will solve it specifying our mapping.

B. Annular transformation

If we restrict our transformation to the following mapping

$$\begin{cases} r \rightarrow r' = \frac{R_2(\theta)-R_1(\theta)}{R_2(\theta)}r + R_1(\theta), \\ \theta \rightarrow \theta' = \theta. \end{cases} \quad (20)$$

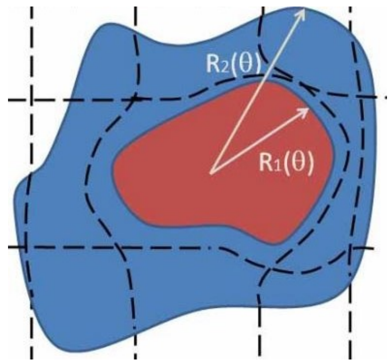

FIG. 2: General annular transformation. This conversion maps the domain $r \leq R_2(\theta)$ onto an annular region contained in $R_2(\theta) \geq r' \geq R_1(\theta)$ leaving the remaining space undisturbed.

In these conditions [10],

$$q = \frac{-1}{(\Delta R(\theta))^2}\left[\frac{dR_1(\theta)}{d\theta}(\Delta R(\theta))^2(R_2(\theta)-r')R_2(\theta) + \frac{dR_2(\theta)}{d\theta}R_1(\theta)(R_1(\theta)-r')\right], \quad (21)$$

where $\Delta R(\theta) = R_2(\theta) - R_1(\theta)$. If we restrict now our transformation to a mapping that draws an annular region without azimuthal dependence in its radius, *i.e.* a circular transformation, we get $\frac{dR_1(\theta)}{d\theta} = \frac{dR_2(\theta)}{d\theta} = 0$, and the coefficient q vanishes. Now,

$$J_{xx'} = M(\theta)diag\left(\frac{R_2-R_1}{R_2},\frac{r'}{r}\right)M(\theta)^T, \quad (22)$$

$$\kappa' = \kappa\begin{pmatrix} \frac{r'-R_1}{r'} & 0 \\ 0 & \frac{r'}{r'-R_1} \end{pmatrix}, \\ \rho' c' = \rho c\left(\frac{r'-R_1}{r'}\right)\left(\frac{R_2}{R_2-R_1}\right)^2. \quad (23)$$

Although our expression for κ' has been really simplified, it still present a singularity for the last term. Guenneau [6] proposes to get reduced κ'' and ρ'' dividing κ' and multiplying the ρ' by the determinant of Eq.(22),

$$\kappa'' = \begin{pmatrix} \left(\frac{r'-R_1}{r'}\right)^2 \left(\frac{R_2}{R_2-R_1}\right)^2 & 0 \\ 0 & \left(\frac{R_2}{R_2-R_1}\right)^2 \end{pmatrix}, \qquad (24)$$
$$\rho'' c'' = \rho c.$$

In this approximation, appears a new term in Heat Conduction Equation, inasmuch as

$$\nabla\cdot(\kappa''\nabla T) = |J|^{-1}\nabla\cdot(\kappa'\nabla T) + \nabla|J|^{-1}\cdot(\kappa''\nabla T). \qquad (25)$$

The second term is conceived as a perturbation, and when it's small enough we are able to say that our approximations is suitable.

IV. HOMOGENIZATION MODEL

Even after having obtained our reduced variables, our thermal conductivity still presents anisotropy and r dependence in κ_r term. Although this parameters could be mapped onto a practical metamaterial, it's a really hard task [11]. With the aim of approach our parameters in a more manageable materials, we apply techniques of homogenization to Heat Conduction Equation.

Let us consider that our temperature field inside our cloak, with no sources, complies,

$$\nabla\cdot(\kappa_\eta \nabla T_\eta) - c_\eta \rho_\eta \frac{\partial T_\eta}{\partial t} = 0, \qquad (26)$$

where we have used a microscopic new variables $(\frac{r}{\eta}, \theta)$ where η is a small positive real parameter which we use to deal with the fast anisotropic variation of κ inside our cloak. In this last expression we have identified,

$$\kappa_\eta = \kappa\left(\frac{r}{\eta}\right), \qquad c_\eta\rho_\eta = c\left(\frac{r}{\eta}\right)\rho\left(\frac{r}{\eta}\right). \qquad (27)$$

Once we have expressed the equation in our new parameter-dependent variables, we apply a formal expansion in η

$$T_\eta\left(\frac{r}{\eta},\theta\right) = \sum_{i=0}^{\infty} \eta^i T^{(i)}\left((r,\theta), \left(\frac{r}{\eta},\theta\right)\right), \qquad (28)$$

and rescaling the differential operator in accordance with [12], we obtain,

$$\langle \rho c\rangle \frac{\partial u_0}{\partial t} = \nabla\cdot(\kappa_{hom}\nabla T_0), \qquad (29)$$

where

$$\kappa_{hom} = \mathrm{Diag}(\langle \kappa^{-1}\rangle^{-1}, \langle\kappa\rangle), \qquad (30)$$

and where $\langle f\rangle$ represents the arithmetic expected value over a unit cell along the radial axis and $T_0 = T^{(i=0)}((r,\theta),(\frac{r}{\eta},\theta))$. Following [6], we note that if the cloak is designed as a periodic structure of alternative layers of two homogeneous isotropic materials A and B, we obtain:

$$\kappa_r = \frac{1+\eta}{\frac{1}{\kappa_A}+\frac{\eta}{\kappa_B}}, \qquad \kappa_\theta = \frac{\kappa_A + \eta\kappa_B}{1+\eta},$$
$$\langle \rho c\rangle = \frac{\rho_A c_A + \eta\rho_B c_B}{1+\eta}. \qquad (31)$$

where $\eta = d_A/d_B$, being d_i the thickness of the layer made up by the material i. This process designs a cloak formed by two layers, one of material A and another made up of material B. A more detailed mathematical treatment of asymptotic homogenization is given at [13].

Applying now the model proposed by Huang [14] to a Thermal Cloaking work, we proposed a cloak formed by N layers, divided in turn each one of them into two layers, one formed with material A and another formed with material B. This structure allow us to reflect the radial dependence of κ_r on our cloak by enforcing κ_A and κ_B to vary with the radius.

V. EXPERIMENTAL IMPLEMENTATION

Robert Schittny *et al* [7] implemented a 2D transient thermal cloak following the model developed by Huang and that we have used in the previous chapter: they fabricated a cloak with $R_1 = 2.5$cm and $R_2 = 5$cm formed by $N = 10$ concentric rings with alternating large and small effective heat conductivities which contrast $|\kappa_A - \kappa_B|$ gradually decreases with the radial coordinate. This conductivities are obtained as a combination of two homogeneous isotropic materials: copper ($\kappa_{Cu} = 394W/Km$) and polydimethylsiloxane or PDMS ($\kappa_{PDMS} = 0.15W/Km$). The composition required to obtain the suitable thermal conductivity is expressed by $\kappa_i = f_i\kappa_{Cu} + (1-f_i)\kappa_{PDMS}$. The product ρc that must be constant according to (24) cannot be adjusted independently. Knowing that $(\rho c)_{PDMS} = 1.4MJ/(Km^3)$ and $(\rho c)_{Cu} = 3.4MJ/(Km^3)$, we can compensate this approximation choosing the effective heat conductivity of the surrounding as $\kappa_0 = 85W/KM$. Schittny *et al* implement this structure into a copper plate (Fig.(3)(a)). On the basis of an experimental optimization they choose an hexagonal lattice of holes for the surrounding that will be filled with PDMS.

Figure (3)(b1-2) shows the results obtained by Schittny *et al* with their homogeneous isotropic multilayered transient thermal cloak after 120 seconds (b1) compared with a common isolated zone (b2). The cloak considerably achieve its objective to keep the cloaking zone colder than its surrounding leaving the temperature distribution outside the cloak undisturbed. We can see though the contrast between the curvature of the isothermal lines in both cases, inflicted by the presence of the cloak. In Fig.(3)(b3-4) we present numerical simulations (COMSOL) provided by [7] that in general agree well with

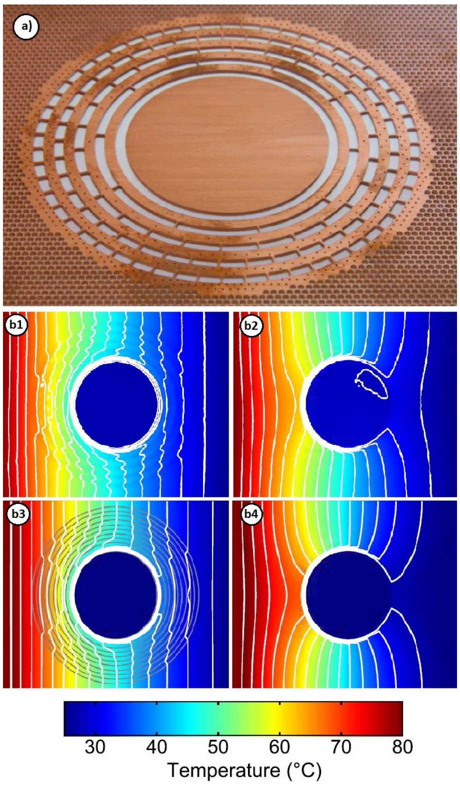

FIG. 3: (a) Blueprint of Schittny's thermal cloak. (b1-2) Measures and (b3-4) numerical simulations obtained by Schittny for a Huang modeled cloak (left side) and for a ordinary isolated surface (right side).

the experimental data collected, although they present a greater cleanliness thanks to be able to completely obviate the heat conduction to the surrounding air.

VI. CONCLUSIONS

In conclusion, we have studied cloaking phenomena in a transient thermal system, exploiting the invariance under coordinate transformations of the Heat Conduction Equation to get the values of thermal conductivity and of the product ρc which lead to this behavior.

First we have analyzed a general mapping, that we have been particularizing into a circular transformation. Then, we have got rid of the principal singularities of our κ and ρc obtaining a reduced pair of variables. Later we have solved the inhomogenity and anisotropy that presented our parameters using the asymptotic homogenization of the Heat Conduction Equation. Finally we have analyzed the experimental and numerical data obtained by Schittny *et al* using the same model that we have studied along this work.

We can conclude that we can implement a well functioning transient thermal cloaking using only homogeneous isotropic materials, using a circular transformation that maps a point in the Virtual space into a annular region in the Physical space to get a set of parameters that we can turn into suitable with an appropriate mathematical and experimental methods.

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