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Topology and dynamics of the escaping set

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Abstract

Topology and dynamics of the escaping set

by Sergio Hernández Antón

The set of points that escape to infinity under iteration is a fundamental object in complex dynamics, often providing valuable insight into the global behavior of iterates and their relationship with the Fatou and Julia sets. The aim of this thesis is to study the topological and dynamical properties of the escaping set for polynomials, as well as transcendental entire and meromorphic functions.

As an original contribution, we show that for the function $f(z) = z - \tan(z)$, the Julia set becomes connected upon adjoining infinity and contains a the escaping set, which is totally disconnected. This is the first known nontrivial example exhibiting such behavior.

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Contents

Introduction

This thesis fits withing the field of complex dynamics, also known as holomorphic dynamics, which deals with the study of discrete dynamical systems generated by the iteration of holomorphic maps. More precisely, given a holomorphic function $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$, one aims to understand the sequence

$$\{f^n(z)\}_{n\geq 0},$$

where $z \in \mathbb{C}$ and f^n denotes the n-th iterate of f.

The roots of holomorphic dynamics can be traced back to the late 19th century. During the 1870s, Schröder and Cayley pioneered the dynamical study of holomorphic function iteration by examining complex versions of Newton's root-finding method, as documented in [1, 2]. Their work marked the beginning of a new mathematical perspective, setting the groundwork for future explorations in this area.

While early investigations by Böttcher [3], Koenigs [4], and Siegel [5] primarily addressed the local dynamics near fixed points, a major advancement came in 1912 with Montel's Theorem on normal families [6]. This powerful result became a cornerstone that enabled Fatou and Julia to independently build the foundations of modern holomorphic dynamics between 1910 and 1920, largely spurred by the prestigious *Grand Prix des Sciences Mathématiques* of 1918.

Employing Montel's Theorem, Fatou [7] and Julia [8] demonstrated that the Riemann sphere, denoted by $\widehat{\mathbb{C}} \coloneqq \mathbb{C} \cup \{\infty\}$, could be split into two disjoint sets characterized by drastically different dynamical properties. These sets are now known as the *Fatou set* and the *Julia set*. Informally, for a holomorphic function f, the Fatou set $\mathcal{F}_f$ consists of the points whose nearby orbits under iteration exhibit similar behavior, reflecting stability. In contrast, the Julia set $\mathcal{J}_f$ comprises the points where such regularity breaks down, giving rise to chaotic dynamics.

In addition to this classical partition of the Riemann sphere, another key concept in the field is the *escaping set*, denoted by $\mathcal{I}_f$ and defined as the set of points in the complex plane whose iterates diverge to infinity

$$\mathcal{I}_f \coloneqq \left\{ z \in \mathbb{C} \mid f^n(z) \xrightarrow[n\to\infty]{} \infty \right\}.$$

Here, we are assuming that f is a polynomial or that ∞ is an essential singularity, hence ∞ is a "special" point in the dynamical plane. While the Fatou and Julia sets emphasize local stability and chaos, respectively, understanding the escaping set often provides a structure to the dynamical plane, crucial for the combinatorial analysis.

The central goal of this thesis is to examine the topological structure and dynamical features of the escaping set, particularly for polynomials and transcendental functions, both entire and meromorphic (maps holomorphic except at poles). Recall that transcendental functions have an essential singularity at infinity, while rational maps are the quotient of two polynomials and extend analytically to $\widehat{\mathbb{C}}$. For clarity and consistency, we will use specific notation throughout the thesis to refer to these classes of holomorphic functions.

- $\text{Rat} := \{f \mid f \text{ is rational}\}$,
- $\text{Pol} := \{f \mid f \text{ is a polynomial of degree } d \geq 2\} \subset \text{Rat}$,
- $\text{Ent} := \{f \mid f \text{ is transcendental entire}\}$,
- $\mathcal{M} := \{f \mid f \text{ is transcendental meromorphic}\}$.

For rational functions, the point at infinity is no different from any other. In contrast, for any polynomial p of degree $d \geq 2$, the point at infinity is always a superattracting fixed point. As a result, there exists an open set of points whose orbit under iteration tends to infinity; this set is known as the basin of attraction of infinity, and it is defined as

$$\mathcal{A}_p(\infty) := \left\{ z \in \widehat{\mathbb{C}} \mid p^n(z) \xrightarrow[n\to\infty]{} \infty \right\}.$$

We will show that, in the case of polynomials, the Fatou and Julia sets admit a particularly simple and elegant characterization in terms of the basin of attraction of infinity. Specifically, if we define the *filled Julia set* as

$$\mathcal{K}_p := \mathbb{C} \setminus \mathcal{A}_p(\infty) = \left\{ z \in \mathbb{C} \mid \{p^n(z)\}_{n\geq 0} \text{ is bounded} \right\},$$

then the Julia set $\mathcal{J}_p$ coincides with the common boundary of these two complementary sets

$$\mathcal{J}_p = \partial\mathcal{K}_p = \partial\mathcal{A}_p(\infty).$$

Consequently, the Fatou set can be expressed as $\mathcal{F}_p = \mathcal{A}_p(\infty) \cup \text{int}(\mathcal{K}_p)$. An illustrative example is shown in Figure 1.

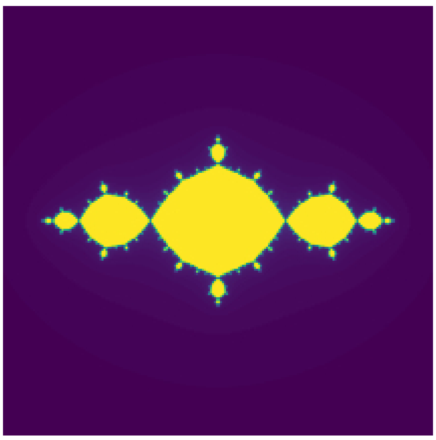


FIGURE 1: Dynamics of $p(z) = z^2 - 1$. The yellow region represents the points whose orbits remain bounded (the interior of the filled Julia set), while the purple region is composed by the points escaping to infinity. The boundary between these regions is the Julia set.

Therefore, if p is a polynomial of degree $d \geq 2$, then the escaping set coincides with the basin of attraction of infinity, i.e. $\mathcal{I}_p = \mathcal{A}_p(\infty)$. In particular, $\mathcal{I}_p$ is an open set containing infinity in its interior.

A natural question then arises regarding the connectivity properties of $\mathcal{A}_p(\infty)$. Interestingly, the connectivity of $\mathcal{A}_p(\infty)$ is determined entirely by the dynamical behavior of the *critical points*, i.e. the set of points $c \in \mathbb{C}$ for which $p'(c) = 0$. Theorems A and B highlight the deep relationship between the topology of the escaping set and the dynamics of the critical points.

Theorem A. *Let* $p \in \mathrm{Pol}$. *Then,* $\mathcal{A}_p(\infty)$ *is connected.*

Theorem B. *Let* $p \in \mathrm{Pol}$.

i) If no critical point of p *belongs to* $\mathcal{A}_p(\infty)$*, then* $\mathcal{K}_p$ *is a connected set.*

ii) If at least one critical point of p *belongs to* $\mathcal{A}_p(\infty)$*, then* $\mathcal{K}_p$ *is disconnected and has infinitely many connected components. Additionally, if* $\mathcal{A}_p(\infty)$ *contains all critical points of* p*, then* $\mathcal{K}_p = \mathcal{J}_p$ *is totally disconnected, i.e. a Cantor set.*

In sharp contrast to polynomials, transcendental entire functions display far more intricate and unpredictable dynamical behavior. This increased complexity stems from the fact that infinity is an essential singularity, as emphasized by the Great Picard Theorem. Building upon this foundational result, Fatou made the first substantial advances in the iteration theory of transcendental entire functions in 1926. The subject was further developed through the influential work of Baker, particularly in [9, 10], and was later formalized by Eremenko in the late 1980s. Notably, Eremenko was the first to rigorously define the escaping set in the context of transcendental entire dynamics. However, this concept had already appeared implicitly in earlier works, such as McMullen's study concerning the area of Julia sets for transcendental entire functions in [11].

The Great Picard Theorem implies that every neighborhood of infinity is mapped by a transcendental entire function f onto the entire complex plane, with the possible exception of a single point. This remarkable behavior emphasizes the extreme complexity introduced by the essential singularity at infinity, distinguishing transcendental entire functions sharply from polynomial maps. At first glance, such chaotic behavior near infinity might suggest that orbits $\{f^n(z)\}_{n\geq 0}$ could never truly escape, as they would be repeatedly "thrown back" into the finite complex plane after approaching infinity. In this light, one might be led to conjecture that escaping to infinity is dynamically obstructed for transcendental entire functions.

Surprisingly, Eremenko demonstrated in his seminal 1989 paper [12] that this intuition is incorrect. He proved that there always exist points whose orbits under iteration tend to infinity, establishing the nonemptiness of the escaping set. This groundbreaking result marked a turning point in the study of transcendental dynamics and established the foundation for much of the subsequent research in the field. This theorem reads as follows.

Theorem C. *Let* $f \in \mathrm{Ent}$. *Then, the escaping set* $\mathcal{I}_f$ *is nonempty.*

Although Eremenko's proof of Theorem C is both the earliest and perhaps the most well-known approach to establishing the existence of escaping points for transcendental entire functions, in this thesis we adopt the method introduced by Domínguez in [13]. The primary motivation for this choice lies in the broader applicability of Domínguez's construction: her method is not only elegant and intuitive, but it also generalizes to the transcendental meromorphic setting, which is also treated in this work. This flexibility makes it a particularly suitable foundation for our study, which encompasses both entire and meromorphic dynamics. Moreover, Eremenko's original proof, while highly influential, relies on Wiman-Valiron theory: an advanced analytical framework that, although powerful, can obscure the underlying ideas for those unfamiliar with its technical machinery. In contrast, Domínguez's constructive and more accessible argument offers clearer insights into the behavior of escaping orbits.

Eremenko's contributions to the field, however, extend far beyond proving that the escaping set is nonempty. Notably, he demonstrated that the Julia set of any transcendental entire function coincides with the boundary of its escaping set, a result that underscores the centrality of the escaping set in understanding global dynamical behavior. Furthermore, he proposed what became known as Eremenko's conjecture, which posits that every connected component of the escaping set of a transcendental entire function is unbounded. This conjecture inspired a vast body of research, leading to numerous partial results and refined formulations. The original conjecture remained open for decades, until it was recently disproven in [14], thereby raising further questions regarding the fine structure and topology of escaping sets in transcendental dynamics.

Once the existence of the escaping set is established, it is natural to investigate its topological and dynamical properties. In contrast to the polynomial case, the escaping set of transcendental entire functions exhibits a much wider variety of behaviors. For example, there exist transcendental entire functions f for which the escaping set contains open subsets, while for others, $\mathcal{I}_f$ consists of curves with empty interior (see Figure 2). Another remarkable distinction is that, in general, the escaping set of a transcendental entire function is not contained in the Fatou set. In fact, more commonly we have $\mathcal{I}_f \subseteq \mathcal{J}_f$, as we will present with an example later in this thesis.

(A) $f(z) = 1 + z + e^{-z}$ (B) $f(z) = \frac{1}{4}e^{z}$

FIGURE 2: Dyamics of two transcendental entire functions. In the left image, the escaping set contains open regions, as the orbits of points in the blue area tend towards infinity. In contrast, in the right image, the escaping set consists of a collection of curves with no interior, while points in the yellow region converge to an attracting fixed point. In both cases, the Julia set is depicted in purple, and its interior is a result of numerical approximation.

As with transcendental entire functions, transcendental meromorphic functions have an essential singularity at infinity. However, the presence of poles introduces additional layers of complexity. Unlike in the polynomial or entire settings, orbits under iteration in the meromorphic case may approach, accumulate at, or even pass through poles, which can significantly influence the global dynamics. This added intricacy makes the study of transcendental meromorphic functions particularly rich.

A natural question arises: which dynamical properties known for transcendental entire functions persist in the meromorphic setting?

In recent years, the study of Julia and Fatou sets for meromorphic functions has been developed considerably. The interplay between essential singularities and poles gives rise to complex dynamical structures, including Baker domains, wandering domains, and disconnected Julia sets. Current research focuses on both topological and measure-theoretical aspects of these sets, investigating their connectivity, dimensional properties, and relationships with singular values.

Unlike in the polynomial or entire cases, the escaping set of a transcendental meromorphic function can be totally disconnected. This behavior was already implicit in [15, 16], where the authors proved that for real parameters $\lambda \in \mathbb{R}$ with $0 < |\lambda| < 1$, the Julia set of the map $T_\lambda(z) := \lambda \tan(z)$ can be totally disconnected and contain the escaping set.

This observation naturally leads to the following question.

Question. Is there any transcendental meromorphic function $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ such that $\mathcal{J}_f$ is connected (in $\mathbb{C}$ or $\widehat{\mathbb{C}}$), $\mathcal{I}_f \subseteq \mathcal{J}_f$ and $\mathcal{I}_f$ is totally disconnected?

Although it is believed that a broad class of transcendental meromorphic functions could exhibit such behavior, a general characterization remains elusive. In Section 5.1, we show that this situation can occur in the trivial case where the Julia set coincides with the real line. However, no explicit examples with a nontrivial Julia set (for instance, a fractal) have been identified until now. In this thesis we present the following.

Theorem D. *Let $f(z) = z - \tan(z)$. Then, $\mathcal{J}_f \cup \{\infty\}$ is connected, while $\mathcal{I}_f \subseteq \mathcal{J}_f$ is totally disconnected.*

To the best of our knowledge, this is the first nontrivial example of a transcendental meromorphic function answering the previous question affirmatively, leading to a better understanding of the topological dynamics of transcendental meromorphic functions.

Under the framework outlined, the purpose of this thesis is to analyse, compare, and understand the escaping set of polynomials, and transcendental entire and meromorphic functions. These three classes of functions each exhibit distinct dynamical behaviors, particularly in relation to the escaping set, which motivates a comparative approach. To facilitate this analysis, we have organized the thesis as follows.

In Chapter 1, we introduce several foundational concepts that support the material presented in the subsequent chapters. This chapter is divided into three sections, each addressing a distinct topic: complex analysis, Riemann surfaces, and hyperbolic geometry. These tools are indispensable for understanding various results throughout the thesis and, in particular, are required to prove Theorem B.

Chapter 2 focuses on both the local and global theories of complex dynamics, which together form the theoretical backbone for analysing the behavior of escaping sets. In the former, we examine periodic points and the linearization problem, highlighting how the local behavior near fixed or periodic points influences global dynamics. In the latter, we define the Julia and Fatou sets and establish several of their key properties.

Of special importance is the classification of connected components of the Fatou set, i.e. Fatou components, which is crucial for understanding the decomposition of the complex plane under iteration.

The remaining chapters are devoted to the escaping set. Given the added complexity of transcendental functions, we begin with the polynomial case, which provides a clearer and more structured starting point. We then transition to transcendental entire functions.

The primary objective of Chapter 3 is to prove Theorems A and B. Along the way, we demonstrate that infinity is a superattracting fixed point for any polynomial of degree at least two, and we analyse the associated basin of attraction. The chapter also includes additional insights that lay the groundwork for the transcendental cases.

Chapter 4 opens with the proof of the non-emptiness of the escaping set for transcendental entire functions. Building on this, we explore several important results derived from Theorem C. Then, we introduce the Cantor bouquet (a set homeomorphic to the Cartesian product of a Cantor set and the real line) and present an example of a transcendental entire function whose Julia set exhibits this structure.

Finally, Chapter 5 turns to transcendental meromorphic functions. We begin with a concise overview of their key properties and illustrative examples. We then extend several results from the entire case to the meromorphic setting. The chapter culminates with Theorem D, which demonstrates the possibility of a nontrivial and connected Julia set containing a totally disconnected escaping set. The final section of the chapter is devoted to the proof of this result.

Chapter 1

Preliminaries

The aim of this chapter is to introduce all relevant concepts and results that will serve as important tools later on. It is divided into three sections: complex analysis, Riemann surfaces, and hyperbolic geometry.

1.1 Complex analysis

This first section, which is divided into three parts, provides the necessary background in complex analysis for our study. The first part introduces the concept of normal families, with the main material drawn from [17]. Next, we discuss singularities of the inverse, i.e. points at which some local branch of the inverse of a holomorphic map is not well-defined; for a more in-depth treatment, see [18]. The final part presents a miscellany of results that do not fall under the previous categories; references for each statement are provided therein.

1.1.1 Normal families

The notion of normality was introduced by Montel in 1912 and played a crucial role in the work of Fatou and Julia in the field of complex iteration.

Definition 1.1 (Normal family). Let $\mathcal{F}$ be a family of meromorphic functions, we say $\mathcal{F}$ is a *normal family* if and only if every sequence $\{f_n\}_n$ in $\mathcal{F}$ contains a subsequence that converges uniformly on compact subsets.

A useful criterion for determining normality is the following.

Theorem 1.2 (Montel Theorem). *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $\Omega \subseteq \widehat{\mathbb{C}}$. If there exist three distinct points in $\widehat{\mathbb{C}}$ which are omitted by every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

1.1.2 Singularities of the inverse

To understand the global dynamics of holomorphic functions, it is essential to study the singularities of their inverses. These singularities govern the behavior of orbits and often determine the structure of the Julia and Fatou sets. Here we introduce the concept of singularity and provide some of their properties.

Definition 1.3 (Conformal, univalent). Let Ω be an open set and let $f\colon \Omega \longrightarrow \mathbb{C}$. We say that f is *conformal* or *univalent* if and only if it is holomorphic and one-to-one.

Definition 1.4 (Regular value, singular value). Let $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be meromorphic. We say v_0 is a *regular value* if and only if there exists a neighborhood V of v_0 such that all branches of f^{-1} are well-defined (and, hence, conformal) in V. Otherwise we say that v_0 is a *singular value*. We denote the set of singular values of f by $\mathcal{S}(f)$.

These singularities can be classified in three different types: critical values, asymptotic values and the accumulation thereof.

Definition 1.5 (Critical value, asymptotic value). Given a value $v_0 \in \widehat{\mathbb{C}}$, we say that v_0 is a *critical value* if and only if there exists $z_0 \in \mathbb{C}$ such that $f'(z_0) = 0$ and $f(z_0) = v_0$. We say that z_0 is a *critical point*. Moreover, we say that v_0 is an *asymptotic value* if and only if there exists an unbounded curve $\gamma(t) \longrightarrow \infty$ such that $f(\gamma(t)) \xrightarrow[t\to\infty]{} v_0$. Then, we say that γ is an *asymptotic path* (or *curve*) of v_0.

The following lemma makes explicit the relation between, singular, critical and asymptotic values.

Lemma 1.6. *Let $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be meromorphic. Then,*

$$\mathcal{S}(f) = \overline{\{\text{critical and asymptotic values}\}}.$$

The next proposition ensures the existence of a proper local branch in a neighborhood of a critical point, which becomes crucial in Chapter 3.

Proposition 1.7. *Let f be a meromorphic function, z_0 a critical point of f and $k \geq 1$ satisfying $f^{(i)}(z_0) = 0$ for all $1 \leq i < k$ and $f^{(k)}(z_0) \neq 0$. Then, for any neighborhood V of $v_0 = f(z_0)$, there exists a neighborhood U of z_0 such that $f : U \longrightarrow V$ is a proper map of degree k.*

We now state two lemmas related to singular values and covering maps, which are introduced in the next section.

Lemma 1.8. *Let $U \subseteq \widehat{\mathbb{C}}$ and $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$.*

i) If f is a holomorphic covering from $U \longrightarrow \mathbb{D}$, then U is simply connected and f is univalent.

ii) If f is a holomorphic covering from $U \longrightarrow \mathbb{D}^*$, then either U is conformal to $\mathbb{D}^*$ and f is equivalent to z^d, or U is simply connected and f is the universal covering, hence equivalent to the exponential map.

Lemma 1.9. *Let $U, V \subseteq \widehat{\mathbb{C}}$ and assume U is a connected component of $f^{-1}(V)$. If $V \cap \mathcal{S}(f) = \emptyset$, then $f : U \longrightarrow V$ is a covering map. Moreover, if V is a simply connected open set, then the covering f is conformal.*

The latter lemma is a consequence of the former and the notions of local branches of f^{-1} and singular values.

Finally, we introduce the notion of proper map.

Definition 1.10 (Proper map). Let U, V be open domains and let $f : U \longrightarrow V$ be meromorphic. We say f is a *proper map* of degree $k \geq 1$ if and only if every point in V has exactly k preimages in U, counting multiplicity.

An easier way to understand this kind of maps is the following. One may find it proven in [19, Section 1.2].

Theorem 1.11. *Let U, V be open domains. A meromorphic map $f : U \longrightarrow V$ is proper if $f(\partial U) = \partial V$.*

1.1.3 Other results

The Open Mapping Theorem ensures that holomorphic functions send open sets to open sets, which may be used to prove topological statements (such as Proposition 4.3). One can find its proof in [20, Theorem 9.6.14].

Theorem 1.12 (Open Mapping Theorem). *Let* (X, Φ) *and* (Ω, Ψ) *be analytic manifolds and let* $f\colon X \longrightarrow \Omega$ *be a nonconstant analytic function. If* U *is an open subset of* X*, then* $f(U)$ *is open in* Ω*.*

A crucial result in the study of both transcendental entire and meromorphic functions is the Great Picard Theorem, which shows the chaotic behavior of a holomorphic map near an essential singularity. Its proof can be found in [20, Theorem 12.4.2].

Theorem 1.13 (Great Picard Theorem). *Let* f *be holomorphic with an essential singularity at* z_0*. Then in any neighborhood* U *of* z_0*,* $f(U)$ *assumes each complex number, with one possible exception, an infinite number of times.*

The next one is a beautiful result due to Wolff and Denjoy (1926), which characterizes the behavior of holomorphic maps from the unit disk to itself. A proof due to Beardon (1990) is included in [17, Theorem 4.3.1].

Theorem 1.14 (Denjoy-Wolff Theorem). *Let* f *be a holomorphic self-map of the unit disk* $\mathbb{D}$*, not conjugate to a rotation. Then, there exists* $p \in \overline{\mathbb{D}}$ *such that for all* $z \in \mathbb{D}$ *we have* $f^n(z) \xrightarrow[n\to\infty]{} p$*.*

1.2 Riemann surfaces

The main goal of this section is to introduce the concept of Riemann surfaces. We begin by focusing on the simply connected case, providing a classification that serves as the foundation for understanding general Riemann surfaces. In addition, we examine the structure of the group of conformal automorphisms associated with these surfaces. Unless otherwise noted, the results and proofs presented here follow [21].

1.2.1 Simply connected surfaces

Here we focus on simply connected Riemann surfaces, which serve as the building blocks for understanding the general theory. Using the uniformization Theorem, we classify all simply connected surfaces up to conformal equivalence. This classification is a key step toward the broader study of arbitrary Riemann surfaces.

Definition 1.15 (Riemann surface). We say S is a *Riemann surface* if and only if it is a connected complex analytic manifold of complex dimension 1.

Definition 1.16 (Conformally isomorphic). Two Riemann surfaces S, S' are *conformally isomorphic* (or *biholomorphically equivalent*) if and only if there exists a conformal map from S to S'. In the special case $S = S'$, it is called a *conformal automorphism* of S.

Consequently, it suffices to study the simplest representative within each conformal equivalence class. This motivates the result stated below.

Theorem 1.17 (Riemann Mapping Theorem). *Let* $\Omega \subset \widehat{\mathbb{C}}$ *be a nonempty, simply connected open set such that* $\widehat{\mathbb{C}} \setminus \Omega$ *consists of more than one point. There exists a conformal map* $\varphi : \Omega \longrightarrow \mathbb{D}$*, which we refer to as the* Riemann map*. Moreover, given* $z_0 \in \Omega$*, there exists a unique Riemann map such that* $\varphi(z_0) = 0$ *and* $\varphi'(z_0) > 0$*.*

Proof. Here, we focus on proving uniqueness, as it captures the essence of this type of arguments. A proof of existence can be found in [20, Theorem 7.4.2].

Let us fix $z_0 \in \Omega$ and let φ_1, φ_2 be two Riemann maps satisfying $\varphi_i(z_0) = 0$ and $\varphi_i'(z_0) > 0$, for $i = 1, 2$. Then, $f\colon \mathbb{D} \longrightarrow \mathbb{D}$ given by $f(z) = \varphi_2\left(\varphi_1^{-1}(z)\right)$ is a conformal automorphism of $\mathbb{D}$ with $f(0) = 0$. By Theorem 1.21, there exists $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$. Thus, $f'(0) = e^{i\theta}$ and by the chain rule

$$f'(z) = \varphi_2'\left(\varphi_1^{-1}(z)\right) \frac{1}{\varphi_1'\left(\varphi_1^{-1}(z)\right)} \Longrightarrow e^{i\theta} = f'(0) = \frac{\varphi_2'(z_0)}{\varphi_1'(z_0)} \in \mathbb{R}_+.$$

Hence, $\theta \in 2\pi\mathbb{Z}$, i.e. $f(z) = z$ for all $z \in \mathbb{D}$, which implies $\varphi_1 = \varphi_2$. □

Although there are uncountably many conformally distinct Riemann surfaces, there are only three different surfaces in the simply connected case. The following result can also be seen as a generalization of Theorem 1.17.

Theorem 1.18 (Uniformization Theorem). *Any simply connected Riemann surface is conformally isomorphic to the unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$ or the Riemann sphere $\widehat{\mathbb{C}}$.*

We refer to these three cases as the *hyperbolic*, *Euclidean*, and *spherical* types, respectively. The proof of Theorem 1.18 is nontrivial and is not included in [21], but it can be found in [22]. Thanks to this result, we can proceed more efficiently to the key concepts in holomorphic dynamics.

1.2.2 Groups of conformal automorphisms

For any Riemann surface S, the notation $\mathcal{G}(S)$ will denote the group of all conformal automorphisms of S. The identity map will be refered to as $I = I_S \in \mathcal{G}(S)$.

We first consider the case of the Riemann sphere $\widehat{\mathbb{C}}$ and show that $\mathcal{G}(\widehat{\mathbb{C}})$ can be identified with a well-known complex Lie group. Thus, not only is it a group, but also a complex manifold; therefore, the product and inverse operations for this group are both holomorphic maps.

Lemma 1.19 (Möbius transformations). *The group $\mathcal{G}(\widehat{\mathbb{C}})$ is equal to the group of all Möbius transformations*

$$f(z) = \frac{az + b}{cz + d},$$

where the coefficients are complex numbers with $ad - bc \neq 0$.

Next, we show that both $\mathcal{G}(\mathbb{C})$, $\mathcal{G}(\mathbb{D})$ can be considered as Lie subgroups of $\mathcal{G}(\widehat{\mathbb{C}})$.

Corollary 1.20 (The affine group). *The group $\mathcal{G}(\mathbb{C})$ consists of all affine transformations*

$$f(z) = \lambda z + c,$$

with complex coefficients $\lambda \neq 0$ and c.

Theorem 1.21 (Automorphisms of $\mathbb{D}$). *The group $\mathcal{G}(\mathbb{D})$ can be identified with the subgroup of $\mathcal{G}(\widehat{\mathbb{C}})$ consisting of all maps*

$$f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a} z},$$

where a ranges over the open disk $\mathbb{D}$ and $e^{i\theta}$ ranges over the unit circle $\partial\mathbb{D}$.

To conclude this section, we provide a deeper study into the structure of these three groups. For any holomorphic map $f : X \longrightarrow X$, it will be convenient to use the notation $\mathrm{Fix}\,(f) \subseteq X$ to denote the set of all fixed points $x = f(x)$. If f and g are commuting maps from X to itself, i.e. $f \circ g = g \circ f$, note that

$$f\,(\mathrm{Fix}(g)) \subseteq \mathrm{Fix}(g).$$

If $x \in \mathrm{Fix}(g)$, then $f(x) = f \circ g(x) = g \circ f(x)$, hence $f(x) \in \mathrm{Fix}(g)$. We first apply these ideas to the group $\mathcal{G}(\mathbb{C})$ of all affine transformations of $\mathbb{C}$.

Lemma 1.22 (Commuting elements of $\mathcal{G}(\mathbb{C})$). *Two nonidentity affine transformations of $\mathbb{C}$ commute if and only if they have the same fixed point set.*

Now we consider the group of $\mathcal{G}(\widehat{\mathbb{C}})$, though first we require the following definition.

Definition 1.23 (Involution). An automorphism g is called an *involution* if and only if $g \circ g = I$ and $g \neq I$.

Theorem 1.24 (Commuting elements of $\mathcal{G}(\widehat{\mathbb{C}})$). *For every $f \neq I$ in $\mathcal{G}(\widehat{\mathbb{C}})$, the set $\mathrm{Fix}(f) \subseteq \widehat{\mathbb{C}}$ contains either one point or two points. In general, nonidentity elements $f, g \in \mathcal{G}(\widehat{\mathbb{C}})$ commute if and only if $\mathrm{Fix}(f) = \mathrm{Fix}(g)$. The only exceptions being pairs of commuting involutions, each of which interchanges the two fixed points of the other.*

We want a corresponding statement for $\mathbb{D}$. However, it is better to work with the closed disk $\overline{\mathbb{D}}$, in order to obtain a richer set of fixed points. By Theorem 1.21, every automorphism of $\mathbb{D}$ extends uniquely to an automorphism of $\overline{\mathbb{D}}$.

Theorem 1.25 (Commuting elements of $\mathcal{G}(\mathbb{D})$). *For every $f \neq I$ in $\mathcal{G}(\mathbb{D}) \cong \mathcal{G}(\overline{\mathbb{D}})$, the set $\mathrm{Fix}(f) \subseteq \overline{\mathbb{D}}$ consists of either a single point of $\mathbb{D}$, a single point of $\partial\mathbb{D}$, or two points of $\partial\mathbb{D}$. Two nonidentity automorphisms $f, g \in \mathcal{G}(\mathbb{D})$ commute if and only if they have the same fixed point set in $\overline{\mathbb{D}}$.*

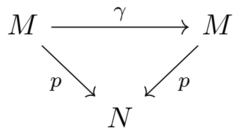
1.2.3 Classifying arbitrary Riemann surfaces

In order to classify any arbitrary Riemann surface, we require the concepts of covering map and deck transformation.

Definition 1.26 (Covering map). A map $p : M \longrightarrow N$ between connected manifolds is called a *covering map* if and only if every point in N has a connected open neighborhood U within N which is *evenly covered*, i.e. each component of $p^{-1}(U)$ must map to U by a homeomorphism. The manifold N is simply connected if and only if it has no nontrivial coverings, i.e. if and only if every such covering map $M \longrightarrow N$ is a homeomorphism.

Definition 1.27 (Universal covering). For any connected manifold N, there exists a covering map $\widetilde{N} \longrightarrow N$ such that $\widetilde{N}$ is simply connected. This is called the *universal covering* of N and is unique up to homeomorphism.

Definition 1.28 (Deck transformation). Given a covering map $p : M \longrightarrow N$, we say $\gamma : M \longrightarrow M$ is a *deck transformation* associated to p if and only if it satisfies the identity $p \circ \gamma = p$, i.e. the following diagram commutes:



Since the *fundamental group* $\pi_1(N)$ and the group Γ consisting of all deck transformations for the universal covering $p : \widetilde{N} \longrightarrow N$ are isomorphic (see [23, Proposition 1.39]), for our purposes we shall use Γ as $\pi_1(N)$. Note that this universal covering p is always a *normal covering* of N. That is, given two points $x, x' \in M = \widetilde{N}$ with $p(x) = p(x')$, there exists one and only one deck transformation mapping x to x'. It follows that N can be identified with the quotient $\widetilde{N}/\Gamma$ of $\widetilde{N}$ by this action of Γ.

A given group Γ of homeomorphisms of a connected manifold M gives rise in this way to a normal covering $M \longrightarrow M/\Gamma$ if and only if the following conditions are satisfied.

i) Γ acts *properly discontinuously*, i.e. any compact set $K \subseteq M$ intersects only finitely many of its translates $\gamma(K)$ under the action of Γ.

ii) Γ acts *freely*, i.e. all nonidentity elements of Γ act without fixed points on M.

Now let S be a Riemann surface. Then the universal covering manifold $\widetilde{S}$ inherits the structure of a Riemann surface, and every deck transformation is a conformal automorphism of $\widetilde{S}$. According to Theorem 1.18, since this universal covering surface $\widetilde{S}$ is simply connected, it must be conformally isomorphic to one of the three model surfaces. Thus, we have the following statement.

Theorem 1.29 (Uniformization for arbitrary Riemann surfaces). *Every Riemann surface S is conformally isomorphic to a quotient of the form $\widetilde{S}/\Gamma$, where $\widetilde{S}$ is a simply connected Riemann surface (which is necessarily isomorphic to either $\mathbb{D}$, $\mathbb{C}$, or $\widehat{\mathbb{C}}$) and where $\Gamma \cong \pi_1(S)$ is a group of conformal automorphisms which acts freely and properly discontinuously on $\widetilde{S}$.*

The group $\mathcal{G}(\widetilde{S})$ has been studied in Section 1.2.2. It is a Lie group, and in particular has a natural topology. Since the action of Γ on $\widetilde{S}$ is properly discontinuous, Γ must be a *discrete* subgroup of $\mathcal{G}(\widetilde{S})$.

We can now give a rough catalogue of all possible Riemann surfaces. The discussion will be divided into two easy cases and one hard case.

By Theorem 1.24, every conformal automorphism of $\widehat{\mathbb{C}}$ has at least one fixed point. Therefore, if $S \cong \widehat{\mathbb{C}}/\Gamma$ is a Riemann surface with universal covering $\widetilde{S} \cong \widehat{\mathbb{C}}$, then the group $\Gamma \subseteq \mathcal{G}(\widehat{\mathbb{C}})$ must be trivial. Hence, S is also conformally isomorphic to $\widehat{\mathbb{C}}$.

By Corollary 1.20, the group $\mathcal{G}(\mathbb{C})$ consists of all affine transformations $z \longmapsto \lambda z + c$ with $\lambda \neq 0$. Every such transformation with $\lambda \neq 1$ has a fixed point. Thus, if $S \cong \mathbb{C}/\Gamma$ is a surface with universal covering $\widetilde{S} \cong \mathbb{C}$, then Γ must be a discrete group of translations $z \longmapsto z + c$ of $\mathbb{C}$. There are three subcases, which are listed below.

- If Γ is trivial, then S itself is isomorphic to $\mathbb{C}$.
- If Γ has just one generator, then S is isomorphic to the infinite *cylinder* $\mathbb{C}/\mathbb{Z}$, where $\mathbb{Z} \subset \mathbb{C}$ is the additive subgroup of integers. Note that this cylinder is isomorphic to the *punctured plane* $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ under the isomorphism

$$z \longmapsto \exp(2\pi i z).$$

- If Γ has two generators, then it can be described as a 2-dimensional *lattice* $\Lambda \subset \mathbb{C}$, i.e. an additive group generated by two complex numbers which are linearly independent over $\mathbb{R}$ (two generators which are dependent over $\mathbb{R}$ would not generate a discrete group). The quotient $\mathbb{T} = \mathbb{C}/\Lambda$ is called a *torus*.

In all three subcases, note that our surface inherits a locally Euclidean geometry from the Euclidean metric $|dz|$ on its universal covering surface. As an example, the punctured plane $\mathbb{C}^*$ has a complete locally Euclidean metric $2\pi|dz| = |dw| \,/\, |w|$. It is convenient to use the term *Euclidean surface* for these Riemann surfaces, though the term *parabolic surface* is also commonly used in the literature.

In all other cases, the universal covering $\widetilde{S}$ must be conformally isomorphic to the unit disk. Such Riemann surfaces are said to be *hyperbolic*. It follows from the discussion above that $\widehat{\mathbb{C}}$, $\mathbb{C}$, $\mathbb{C}/\mathbb{Z}$, and the tori $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ are the only nonhyperbolic Riemann surfaces, up to conformal isomorphism.

The inclusions $\mathbb{D} \longrightarrow \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ provide examples of nonconstant holomorphic maps from the hyperbolic surface $\mathbb{D}$ to the Euclidean surface $\mathbb{C}$ and, then, to the Riemann sphere $\widehat{\mathbb{C}}$. However, no maps in the other direction are possible.

Lemma 1.30. *Every holomorphic map from a Euclidean Riemann surface to a hyperbolic one is necessarily constant. Similarly, every holomorphic map from the Riemann sphere to a Euclidean or hyperbolic surface is necessarily constant.*

A particular interesting property is that any Riemann surface can be made hyperbolic by removing at least three points.

Lemma 1.31. *Let $a, b, c \in \widehat{\mathbb{C}}$ be distinct. Then, $S := \widehat{\mathbb{C}} \setminus \{a, b, c\}$ is hyperbolic, with universal covering $\widetilde{S}$ conformally isomorphic to $\mathbb{D}$.*

Theorem 1.32 (Little Picard Theorem). *Every holomorphic map $f\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ which omits three different values must necessarily be constant.*

Proof. It follows immediately from Lemmas 1.30 and 1.31. If f omits three distinct values a, b, c, then it can be considered as a map from the Riemann sphere $\widehat{\mathbb{C}}$ to the hyperbolic surface $\mathbb{C} \setminus \{a, b, c\}$. □

1.3 Hyperbolic geometry

The primary objective of this section is to extend the definition of the hyperbolic metric from the unit disk $\mathbb{D}$ to an arbitrary hyperbolic surface S. In addition, we present three formulations of the Schwarz–Pick Lemma: one for the unit disk, another for simply connected domains, and a third for the general case. Finally, employing the comparison principle, we demonstrate that the hyperbolic metric on S exhibits behavior inversely proportional to the Euclidean distance from a point to the boundary ∂S. As in the previous section, the material presented here is primarily based on [24], which also includes all proofs unless explicitly stated otherwise.

1.3.1 The unit disk as the hyperbolic plane

The *hyperbolic plane* is the unit disk $\mathbb{D}$ endowed with the *hyperbolic metric*

$$\lambda_{\mathbb{D}}(z)\,|dz| := \frac{2\,|dz|}{1 - |z|^2}.$$

This metric induces a *hyperbolic distance* $d_{\mathbb{D}}(z, w)$ between two points z and w in $\mathbb{D}$ in the following way. We join z to w by a smooth curve γ in $\mathbb{D}$, and define the *hyperbolic length* $l_{\mathbb{D}}(\gamma)$ of γ by

$$l_{\mathbb{D}}(\gamma) := \int_\gamma \lambda_{\mathbb{D}}(z)\,|dz|.$$

Finally, we set

$$d_{\mathbb{D}}(z,w) := \inf_{\gamma} l_{\mathbb{D}}(\gamma),$$

where the infimum is taken over all smooth curves γ joining z to w in $\mathbb{D}$.

We need to identify the isometries of both the hyperbolic metric and the hyperbolic distance. A holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ is an *isometry of the metric* $\lambda_{\mathbb{D}}(z)\,|dz|$ if and only if for all z in $\mathbb{D}$,

$$\lambda_{\mathbb{D}}\left(f(z)\right)\left|f'(z)\right| = \lambda_{\mathbb{D}}(z),$$

and it is an *isometry of the distance* $d_{\mathbb{D}}$ if and only if for all z and w in $\mathbb{D}$,

$$d_{\mathbb{D}}\left(f(z), f(w)\right) = d_{\mathbb{D}}(z,w).$$

In fact, the two classes of isometries coincide, and each isometry is a Möbius transformation from $\mathbb{D}$ to itself.

Theorem 1.33. *For any holomorphic map $f : \mathbb{D} \longrightarrow \mathbb{D}$, the following are equivalent.*

- *i) f is a conformal automorphism of $\mathbb{D}$;*
- *ii) f is an isometry of the metric $\lambda_{\mathbb{D}}$;*
- *iii) f is an isometry of the distance $d_{\mathbb{D}}$.*

In summary, relative to the hyperbolic metric and the hyperbolic distance, the group $\mathcal{G}(\mathbb{D})$ becomes a group of isometries.

Theorem 1.34. *The hyperbolic distance $d_{\mathbb{D}}(z,w)$ in $\mathbb{D}$ is given by*

$$d_{\mathbb{D}}(z,w) = \log\frac{1+p_{\mathbb{D}}(z,w)}{1-p_{\mathbb{D}}(z,w)} = 2\tanh^{-1} p_{\mathbb{D}}(z,w),$$

where $p_{\mathbb{D}}(z,w)$ denotes the pseudo-hyperbolic distance and is defined as

$$p_{\mathbb{D}}(z,w) := \left|\frac{z-w}{1-z\overline{w}}\right|.$$

As hyperbolic isometries map circles into circles, preserve orthogonality, and leave the unit circle invariant, we arrive at the following definition.

Definition 1.35 (Hyperbolic geodesic). Suppose z and w are in $\mathbb{D}$. Then the *hyperbolic geodesic*, i.e. minimal longitude path, through z and w is $C \cap \mathbb{D}$, where C is the unique Euclidean circle which passes through z and w and is orthogonal to the unit circle $\partial\mathbb{D}$. If γ is any smooth curve joining z to w in $\mathbb{D}$, then the hyperbolic length of γ is $d_{\mathbb{D}}(z,w)$ if and only if γ is the simple arc of C in $\mathbb{D}$ that joins z to w.

The unit disk together with the hyperbolic metric is also called the *Poincaré model* of the hyperbolic plane.

We now show that the hyperbolic distance $d_{\mathbb{D}}$ is *additive along geodesics*. By contrast, the pseudo-hyperbolic distance $p_{\mathbb{D}}$ is never additive along geodesics.

Theorem 1.36. *If u, v, w are three distinct points in $\mathbb{D}$ that lie (in this order) along a geodesic, then $d_{\mathbb{D}}(u,w) = d_{\mathbb{D}}(u,v) + d_{\mathbb{D}}(v,w)$. For any three distinct points u, v, w in $\mathbb{D}$, $p_{\mathbb{D}}(u,w) < p_{\mathbb{D}}(u,v) + p_{\mathbb{D}}(v,w)$.*

Theorem 1.37. *The topology induced by $d_{\mathbb{D}}$ on $\mathbb{D}$ coincides with the Euclidean topology. The space $\mathbb{D}$ with the distance $d_{\mathbb{D}}$ is a complete metric space.*

The Euclidean metric on $\mathbb{D}$ arises from its embedding in the ambient space $\mathbb{C}$, but is not complete when restricted to $\mathbb{D}$. In contrast, a notable property of the hyperbolic distance $d_{\mathbb{D}}$ is that it tends to infinity as a point z approaches to the boundary $\partial\mathbb{D}$; informally, every point on $\partial\mathbb{D}$ is "infinitely far away" from any point in $\mathbb{D}$. This behavior reflects the fact that $\mathbb{D}$, when equipped with the hyperbolic metric $\lambda_{\mathbb{D}}$, forms a complete metric space. This completeness provides a compelling reason to favor the hyperbolic metric over the Euclidean one on $\mathbb{D}$.

Lemma 1.38 (Schwarz-Pick Lemma). *Suppose that $f : \mathbb{D} \longrightarrow \mathbb{D}$ is holomorphic. Then either*

i) f is a hyperbolic contraction; that is, for all z and w in $\mathbb{D}$,

$$d_{\mathbb{D}}\left(f(z), f(w)\right) < d_{\mathbb{D}}(z, w), \qquad \lambda_{\mathbb{D}}\left(f(z)\right)\left|f'(z)\right| < \lambda_{\mathbb{D}}(z);$$

ii) f is a hyperbolic isometry; that is, $f \in \mathcal{G}(\mathbb{D})$ and for all z and w in $\mathbb{D}$,

$$d_{\mathbb{D}}\left(f(z), f(w)\right) = d_{\mathbb{D}}(z, w), \qquad \lambda_{\mathbb{D}}\left(f(z)\right)\left|f'(z)\right| = \lambda_{\mathbb{D}}(z).$$

Hyperbolic geometry had been used in complex analysis by Poincaré in his proof of Theorem 1.18. The work of Pick is a milestone in geometric function theory, it shows that the hyperbolic metric is the natural metric for much of the subject. Although the definition of this metric might seem arbitrary at first, up to multiplication by a positive scalar it is the only metric on the unit disk that makes every holomorphic self-map a contraction, or every conformal automorphism an isometry.

Theorem 1.39. *For a metric $\rho(z)\,|dz|$ on the unit disk, the following are equivalent.*

i) For any holomorphic self-map of $\mathbb{D}$ and all $z \in \mathbb{D}$, $\rho\left(f(z)\right)|f'(z)| \leq \rho(z)$.

ii) For any $f \in \mathcal{G}(\mathbb{D})$ and all $z \in \mathbb{D}$, $\rho\left(f(z)\right)|f'(z)| = \rho(z)$.

iii) $\rho(z) = c\,\lambda_{\mathbb{D}}$ for some $c > 0$.

1.3.2 The hyperbolic metric on simply connected regions

Theorem 1.17 enables us to transfer the hyperbolic metric from $\mathbb{D}$ to any simply connected region S of $\widehat{\mathbb{C}}$. Let f be a conformal map of a simply connected plane region S to $\mathbb{D}$. Then, the hyperbolic metric $\lambda_S(z)\,|dz|$ of S is defined as

$$\lambda_S(z) \coloneqq \lambda_{\mathbb{D}}\left(f(z)\right)\left|f'(z)\right|. \tag{1.1}$$

We need to show that λ_S is independent of the choice of the conformal map f used in (1.1), since this implies that λ_S is determined by S alone. Assume f is a conformal map of S to $\mathbb{D}$. The set of all conformal maps of S to $\mathbb{D}$ is given by $h \circ f$, where h ranges over $\mathcal{G}(\mathbb{D})$. Any conformal automorphism h of $\mathbb{D}$ is a hyperbolic isometry, so that for all w in $\mathbb{D}$,

$$\lambda_{\mathbb{D}}(w) = \lambda_{\mathbb{D}}\left(h(w)\right)\left|h'(w)\right|.$$

If we now let $g = h \circ f$, $w = f(z)$ and use the chain rule, we get

$$\lambda_{\mathbb{D}}\left(g(z)\right)\left|g'(z)\right| = \lambda_{\mathbb{D}}\left(h\left(f(z)\right)\right)\left|h'\left(f(z)\right)\right|\left|f'(z)\right| = \lambda_{\mathbb{D}}\left(f(z)\right)\left|f'(z)\right|,$$

so λ_S as defined in (1.1) is independent of the choice of the conformal map f.

Thus, every conformal map of a simply connected region of $\mathbb{C}$ to the unit disk is converted into an isometry of the hyperbolic metric. The hyperbolic distance d_S can be defined in two equivalent ways. First, one can pull-back the hyperbolic distance on $\mathbb{D}$ to S by setting $d_S(z,w) := d_{\mathbb{D}}(f(z), f(w))$ for any conformal map $f : S \longrightarrow \mathbb{D}$ and verify this is independent of the choice of conformal mapping to $\mathbb{D}$. Alternatively, the hyperbolic length of a path γ in S is

$$l_S(\gamma) := \int_\gamma \lambda_S(z)\,|dz|,$$

and one can define

$$d_S(z,w) := \inf l_S(\gamma),$$

where the infimum is taken over all piecewise smooth curves γ in S that join z and w. These two definitions of the hyperbolic distance, which is complete on S, are equivalent. Moreover, a path γ in S connecting z and w is a hyperbolic geodesic in S if and only if $f \circ \gamma$ is a hyperbolic geodesic in $\mathbb{D}$. Also, for any $a \in S$ and $r > 0$, $f(D_S(a,r)) = D_{\mathbb{D}}(f(a), r)$.

In fact, the entire body of geometric facts about the Poincaré model $\mathbb{D}$ transfers, without any essential change, to an arbitrary simply connected region of $\mathbb{C}$ with its own hyperbolic metric. If $f : S \longrightarrow \mathbb{D}$ is a conformal map, then f is an isometry relative to the hyperbolic metrics and distances of S and $\mathbb{D}$. As an immediate consequence of this, we can assert that all conformal maps between simply connected regions are isometries relative to the hyperbolic metrics and distances of said regions.

Theorem 1.40 (Conformal invariance). *Let S_1, S_2 be simply connected regions of $\mathbb{C}$ and f a conformal map from S_1 to S_2. Then f is a hyperbolic isometry, i.e. for all $z, w \in S_1$,*

$$\lambda_{S_2}(f(z))\,|f'(z)| = \lambda_{S_1}(z); \qquad d_{S_2}(f(z), f(w)) = d_{S_1}(z,w).$$

Note that if γ is a smooth curve in S_1, then $l_{S_2}(f \circ \gamma) = l_{S_1}(\gamma)$.

This result implies that each element of $\mathcal{G}(S)$ is a hyperbolic isometry, which we use to state an extension of Lemma 1.38 for simply connected regions.

Lemma 1.41 (Schwarz-Pick Lemma - simply connected regions). *Let S_1, S_2 be simply connected regions of $\mathbb{C}$ and let f be a holomorphic map from S_1 to S_2. Then either*

i) f is a hyperbolic contraction, i.e. for all z and w in S_1,

$$d_{S_2}(f(z), f(w)) < d_{S_1}(z,w), \qquad \lambda_{S_2}(f(z))\,|f'(z)| < \lambda_{S_1}(z);$$

ii) f is a hyperbolic isometry, i.e. f is a conformal map from S_1 to S_2 and for all z and w in S_1,

$$d_{S_2}(f(z), f(w)) = d_{S_1}(z,w), \qquad \lambda_{S_2}(f(z))\,|f'(z)| = \lambda_{S_1}(z).$$

This is the appropriate place to point out that neither the complex plane nor the Riemann sphere have a metric analogous to the hyperbolic in the sense that the metric is invariant under the group of conformal automorphisms.

Theorem 1.42. *Let d be a distance function on the complex plane or the Riemann sphere that is invariant under the action of the full group of conformal automorphisms, then there exists $t > 0$ such that $d(z,w) = 0$ if $z = w$ and $d(z,w) = t$ otherwise.*

1.3.3 The hyperbolic metric on a hyperbolic region

To transfer the hyperbolic metric from the unit disk to multiply connected regions, we require the following result (which uses the concepts presented in Section 1.2.3).

Theorem 1.43. *Let S be a hyperbolic region and let $h\colon \mathbb{D} \longrightarrow S$ be a holomorphic universal covering. Then, there exists a unique metric $\lambda_S(w)\,|dw|$ on S such that $\lambda_S(h(z))\,|h'(z)|\,|dz| = \lambda_{\mathbb{D}}(z)\,|dz|$. Moreover, the metric is independent of the covering.*

The unique metric $\lambda_S(w)\,|dw|$ is called the hyperbolic metric on S. The hyperbolic distance d_S, which is complete on a hyperbolic region, is defined by

$$d_S(z,w) \coloneqq \inf l_S(\gamma),$$

where the infimum is taken over all piecewise smooth paths γ in S joining z and w. Unlike the case of simply connected regions, a holomorphic covering $f : \mathbb{D} \longrightarrow S$ to a multiply connected hyperbolic region is not an isometry, but only a *local isometry*. That is, each point $a \in S$ has a neighborhood U such that $f|_U$ is an isometry. In general, one can only assert that $d_S(f(z), f(w)) \le d_{\mathbb{D}}(z,w)$ for $z, w \in \mathbb{D}$. When S is multiply connected f is not injective, so there exist distinct $z, w \in \mathbb{D}$ with $f(z) = f(w)$. Thus, $d_S(f(z), f(w)) = 0 < d_{\mathbb{D}}(z,w)$. In particular, each holomorphic self-covering h of S is a local isometry of the hyperbolic metric, and every $h \in \mathcal{G}(S)$ is an isometry.

Lemma 1.44 (Schwarz-Pick Lemma - general version). *Let S_1, S_2 be hyperbolic regions and let $f : S_1 \longrightarrow S_2$ be a holomorphic map. Then, for all $z \in S_1$,*

$$\lambda_{S_2}(f(z))\,\left|f'(z)\right| \le \lambda_{S_1}(z). \tag{1.2}$$

Moreover, if $f : S_1 \longrightarrow S_2$ is a covering, then $\lambda_{S_2}(f(z))\,|f'(z)| = \lambda_{S_1}(z)$ for all $z \in S_1$; and if there exists a point in S_1 such that equality holds in (1.2), then f is a covering.

1.3.4 The comparison principle

There is a powerful, and very general, comparison principle for hyperbolic metrics, which we state here only for simply connected plane regions. This principle allows us to estimate the hyperbolic metric of a region in terms of other hyperbolic metrics which are known or, at least, can be easily estimated. In general, it is not possible to explicitly calculate the density of the hyperbolic metric, so estimates are useful.

Theorem 1.45 (Comparison principle). *Let S_1, S_2 be simply connected regions of $\mathbb{C}$. If $S_1 \subseteq S_2$, then $\lambda_{S_2} \le \lambda_{S_1}$ on S_1. Furthermore, if $\lambda_{S_1}(z) = \lambda_{S_2}(z)$ at any point z of S_2, then $S_1 = S_2$ and $\lambda_{S_1} = \lambda_{S_2}$. Moreover, if $\overline{S_1} \subseteq S_2$, then there exists some $\mu = \mu(S_1, S_2) < 1$ such that $\lambda_{S_2} \le \mu\,\lambda_{S_1}$.*

The last part of Theorem 1.45 is based on the Definite Schwarz Lemma, covered in [25, Section 7.5].

In other words, the comparison principle asserts that the hyperbolic metric on a simply connected region decreases as the region increases. Such metric on the disk $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ is $2r|dz|/(r^2 - |z|^2)$, which decreases to zero as r goes to ∞.

The comparison principle is used in the following way. Suppose that we want to estimate the hyperbolic metric λ_S of a region S. We attempt to find regions S_1, S_2 with known $\lambda_{S_1}, \lambda_{S_2}$ such that $S_1 \subseteq S \subseteq S_2$, to then get $\lambda_{S_2} \leq \lambda_S \leq \lambda_{S_1}$. The next result is probably the simplest application of Theorem 1.45, and it gives an upper bound of the hyperbolic metric of a region S in terms of the Euclidean distance to the boundary of S.

Proposition 1.46. *Let S be a simply connected region of $\mathbb{C}$. Then, for all $z \in S$*

$$\lambda_S(z) \leq \frac{2}{d\,(z, \partial S)},$$

and equality holds if and only if S is a disk with centre z.

To obtain a lower bound we first require the following result due to Bieberbach (1916), which was first proved in [26].

Theorem 1.47 (Koebe 1/4 Theorem). *Let f be holomorphic and univalent in $\mathbb{D}$. Then the region $f(\mathbb{D})$ contains the open Euclidean disk with centre $f(0)$ and radius $|f'(0)|/4$.*

Proposition 1.48. *Let S be a simply connected region of $\mathbb{C}$. Then, for all $z \in S$*

$$\frac{1}{2\,d\,(z, \partial S)} \leq \lambda_S(z),$$

and equality holds if and only if S is a slit-plane.

Therefore, Propositions 1.46 and 1.48 show that the hyperbolic metric in S, in fact, behaves as the inverse of the Euclidean distance up to the boundary ∂S

$$\frac{1}{2\,d\,(z, \partial S)} \leq \lambda_S(z) \leq \frac{2}{d\,(z, \partial S)}.$$

Chapter 2

Holomorphic dynamics

We now turn to the study of dynamical systems arising from the iteration of complex analytic maps. We begin by examining the local behavior near periodic points, with particular emphasis on the existence of canonical coordinate systems, also known as *normal forms*, near such points. Subsequently, we partition the phase space into the Fatou and Julia sets to introduce key concepts from the global theory. The material presented in this chapter is part of the standard curriculum of this master's program. Therefore, we provide detailed proofs only for those results that are of particular relevance to the thesis. For further details, the reader is referred to [17, 21, 27].

2.1 Local theory

In this section we define periodic points and the notion of conjugacy, both of which are essential for introducing normal forms. Unless stated otherwise, f denotes a non-constant meromorphic map as explained before.

2.1.1 Periodic points

We first introduce the concepts of forward and backward orbits, as they are fundamental for defining periodic points.

Definition 2.1 (Forward and backward orbit). We say the *forward orbit* of a point z_0 is the set $\mathcal{O}^+(z_0) \coloneqq \{f^n(z_0)\}_{n\geq 0}$. Similarly, if f is invertible, we can define the *backward orbit* of z_0, denoted by $\mathcal{O}^-(z_0) \coloneqq \{f^{-n}(z_0)\}_{n\geq 0}$. Furthermore, we often write $\mathcal{O}(z_0) \coloneqq \mathcal{O}^+(z_0)$.

Definition 2.2 (Fixed, periodic and preperiodic point). We say z_0 is a *periodic point* of f with minimal period $p > 0$ if and only if $f^p(z_0) = z_0$ and $f^k(z_0) \neq z_0$ for all $0 < k < p$. The set $\langle z_0 \rangle \coloneqq \mathcal{O}(z_0) = \{z_0, \ldots, f^{p-1}(z_0)\}$ is called the *periodic orbit* of z_0. In the special case $p = 1$, z_0 is a *fixed point* of f. Additionally, if $f^k(z_0)$ is periodic for some $k \in \mathbb{N}$ but z_0 is not periodic, we say z_0 is *preperiodic*.

Given a periodic point z_0 of period p, we define its *multiplier* as $\lambda \coloneqq (f^p)'(z_0)$. By the chain rule, one can see that

$$\lambda = \prod_{n=0}^{p-1} f'\left(f^n(z_0)\right)$$

and, hence, the multiplier is the same for every point of the periodic orbit. Thus, we call it the multiplier of the orbit, and periodic points can be classified using it.

Definition 2.3 (Classification of periodic points). Let $\langle z_0 \rangle$ be a periodic orbit and λ its multiplier. The cycle $\langle z_0 \rangle$ is called

- *attracting* if and only if $0 < |\lambda| < 1$;
- *superattracting* if and only if $\lambda = 0$;
- *repelling* if and only if $|\lambda| > 1$;
- *rationally indifferent* (or *parabolic*) if and only if $\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{Q}$;
- *irrationally indifferent* if and only if $\lambda = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

The reason behind the name attracting is the following. Let z_0 be an attracting (or superattracting) fixed point. We start from

$$F(z_0) := \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \xrightarrow[z \to z_0]{} |f'(z_0)| = |\lambda| < 1.$$

For any $\varepsilon > 0$, there exists $|\lambda| < \rho < 1$ such that if $|z - z_0| < \varepsilon$, then $F(z_0) < \rho$. Hence, $|f(z) - f(z_0)| \leq \rho\,|z - z_0|$. By repeating this argument, and because $f(z_0) = z_0$ and $\rho < 1$, we obtain

$$|f^n(z) - z_0| = |f^n(z) - f^n(z_0)| \leq \rho^n |z - z_0| \xrightarrow[n \to \infty]{} 0,$$

which implies that the iterates $\{f^n(z)\}_{n \geq 0}$ converge to z_0 in a neighborhood of z_0. This justifies the name (super)attracting and leads to the definition below.

Definition 2.4 (Basin of attraction). Let $\langle z_0 \rangle = \{z_0, z_1, \dots, z_{p-1}\}$ be an attracting periodic orbit. We define the *basin of attraction* of $\langle z_0 \rangle$ as

$$\mathcal{A}(\langle z_0 \rangle) := \{z \mid f^{np}(z) \xrightarrow[n \to \infty]{} z_i \ \text{ for some } \ 0 \leq i \leq p-1\}.$$

The connected component of $\mathcal{A}(\langle z_0 \rangle)$ containing the cycle is called the *immediate basin of attraction* of $\langle z_0 \rangle$ and is denoted by $\mathcal{A}^*(\langle z_0 \rangle)$.

2.1.2 Conjugacies and equivalences

Conjugacies are the main tool used for comparison and classification of dynamical systems. Unless stated otherwise, let X, Y be topological spaces and let $f : X \longrightarrow X$, $g : Y \longrightarrow Y$ be continuous maps.

Definition 2.5 (Topological conjugacy). We say f and g are *topologically conjugate* if and only if there exists a homeomorphism $h : X \longrightarrow Y$ such that $h \circ f = g \circ h$, i.e. the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

We write $f \underset{top}{\sim} g$ if the topological conjugacy h needs to be specified.

If h can be chosen to be $\mathcal{C}^r$, with $0 < r \leq \infty$, we say that f and g are $\mathcal{C}^r$*-conjugate*. If $X, Y \subset \widehat{\mathbb{C}}$ we shall also use the term *conformally conjugate* if the map h is conformal. Notice that the definitions also make sense locally.

Perhaps their most important property is that conjugacies preserve orbits. Indeed, if $f \underset{top}{\sim} g$, then $f^n \underset{top}{\sim} g^n$. Hence, $\mathcal{O}_f(z_0)$ is mapped bijectively onto $\mathcal{O}_g(h(z_0))$ by h. In particular, periodic orbits are mapped to periodic orbits of the same period. We can therefore name the properties preserved under conjugacies.

Definition 2.6 (Conjugacy invariants). A property or a quantity associated to a dynamical system which is preserved under topological (resp. $\mathcal{C}^r$, conformal, etc.) conjugacy is called a *topological* (resp. $\mathcal{C}^r$, conformal, etc.) *invariant*.

An example of a $\mathcal{C}^1$-invariant is the multiplier of a periodic orbit.

Many properties of dynamically defined sets are preserved under topological conjugacy. We are also interested in sets which remain invariant upon iteration.

Definition 2.7 (Invariant set). A set $U \subseteq X$ is called *forward invariant* under f (or f*-invariant*) if and only if $f(U) \subseteq U$. Similarly, U is *backward invariant* under f if and only if $f^{-1}(U) \subseteq U$. Additionally, U is *totally invariant* (or *completely invariant*) under f if and only if $f(U) = U = f^{-1}(U)$.

Lemma 2.8. *Topological conjugacies between f and g map f-invariant sets to g-invariant sets. The same statement holds for backward and totally invariant sets.*

Dynamical systems which are not conjugate may still be *semi-conjugate*. This happens when there exists a continuous function $h : X \longrightarrow Y$ satisfying $h \circ f = g \circ h$ without being a homeomorphism. As a result, many properties are still transferred from f to g, but some are lost. For instance, orbits of f are mapped to orbits of g, but a p-cycle of f might be mapped to a p'-cycle of g, with $p'|p$.

A weaker but still useful concept is that of equivalence between dynamical systems.

Definition 2.9 (Topological equivalence). We say f and g are *topologically equivalent* if and only if there exist homeomorphisms h_1 and h_2 such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h_2} \\ Y & \xrightarrow{g} & Y \end{array}$$

Equivalences do not preserve orbits, so the dynamics of f and g may differ considerably. Nonetheless, if f and g are topologically equivalent, there exists a bijection between the critical points of f and the ones of g, and the same is true for critical values. We shall see that these concepts play a crucial role in holomorphic dynamics. In this context, being equivalent means to belong to the same "family" of maps.

2.1.3 Normal forms

We now focus on the dynamical behavior of a function near a periodic point. Since periodic points of f correspond to fixed points of some iterate f^n, with $n \in \mathbb{N}$, it suffices to consider fixed points. Our objective is to represent f in its simplest form near these fixed points; that is, to determine its normal form. To this end, let z_0 be a fixed point of f with multiplier λ.

We also want to know in which cases we can conjugate f to a linear function. This is known as the *linearization problem*, and consists on finding the conditions under which there exists a conformal map φ such that f is φ-conjugate to the linear map $w \longmapsto \lambda w$, i.e.

$$\varphi(f(z)) = \lambda\,\varphi(z) \Longrightarrow f\left(\varphi^{-1}(w)\right) = \varphi^{-1}(\lambda w);$$

which is also known as the *Schröder equation.*

(a) Attracting and repelling fixed points. In these cases, f is locally conformally conjugate to its linear part $z \longmapsto \lambda z$. The precise statement is due to Koenigs (1884), and one can find its proof in [21, Theorem 8.2].

Theorem 2.10 (Koenigs' linearization). *Assume $|\lambda| \neq 0, 1$, then there exists a neighborhood U of 0 and a local conformal conjugacy $w = \varphi(z)$, where $\varphi : U \longrightarrow \varphi(U)$ satisfies $\varphi(0) = 0$ and $\varphi \circ f \circ \varphi^{-1}(w) = \lambda w$ in $\varphi\left(U \cap f^{-1}(U)\right)$. The conjugacy φ is called a linearizing map of f at the fixed point and is unique up to multiplication by a non-zero constant.*

This is clearly a local result. However, for $0 < |\lambda| < 1$, the linearizing map can actually be extended to the whole basin of attraction of the fixed point, though at the expense of its bijectivity.

(b) Superattracting fixed points. The superattracting case corresponds to a holomorphic map f with a fixed critical point at the origin. Then f is locally conformally conjugate to the normal form $w \longmapsto w^m$, where $m-1$ is the multiplicity of the critical point. The original statement is due to Böttcher (1904) and, because of its relevance in Chapter 3, we include its proof here.

Theorem 2.11 (Böttcher coordinates). *Let $f(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots$, with $a_m \neq 0$ and $m \geq 2$. There exists a neighborhood U of 0 and a local conformal conjugacy $\varphi : U \longrightarrow \mathbb{C}$ conjugating f to $w \longmapsto w^m$. Moreover, the conjugacy φ is unique up to multiplication by an $(m-1)$st root of unity.*

Proof. Let us prove existence and uniqueness separately.

- **Existence:** Given a neighborhood $D(0,r)$ with $r > 0$ small enough, there exists $C > 1$ such that $|f(z)| \leq C|z|^m$, for all $z \in D(0,r)$. By induction, we obtain

$$|f^n(z)| \leq C^{\sum_{j=0}^{n-1} m^j} |z|^{m^n} \leq (C|z|)^{m^n},$$

and we can choose r such that $Cr < 1$, which implies $f^n(z) \longrightarrow 0$ for all $z \in D(0,r)$. Applying the change of variables $\phi(z) = bz$, where $b^{m-1} = 1/a_m$, we obtain

$$\phi^{-1}\left(f(\phi(z))\right) = \frac{1}{b}\left(a_m (bz)^m + \cdots\right) = a_m b^{m-1} z^m + \cdots = z^m + \cdots,$$

so we can assume $a_m = 1$. Now, consider for any $n \in \mathbb{N}$

$$\varphi_n(z) = \left(f^n(z)\right)^{m^{-n}} = \left(z^{m^n} + \cdots\right)^{m^{-n}} = z\left(1 + \cdots\right)^{m^{-n}},$$

which are well-defined in a neighborhood of 0 and satisfy

$$\varphi_n(f(z)) = \left(f^{n+1}(z)\right)^{m^{-n}} = \left(\varphi_{n+1}(z)\right)^m.$$

Therefore, if we prove that $\varphi_n \longrightarrow \varphi$, then we will get $\varphi \circ f = \varphi^m$ and $\varphi'(0) \neq 0$ (since $\varphi_n'(0) = 1$), obtaining a non-constant conjugacy φ. Since

$$\frac{\varphi_{n+1}}{\varphi_n} = \left(\frac{\left(f(f^n)\right)^{m^{-1}}}{f^n}\right)^{m^{-n}} = \left(\frac{\varphi_1 \circ f^n}{f^n}\right)^{m^{-n}}$$

and

$$\frac{\varphi_1 \circ f^n}{f^n} = \frac{\left((f^n(z))^m + a_{m+1}\left(f^n(z)\right)^{m+1} + \cdots\right)^{\frac{1}{m}}}{f^n(z)} = \left(1 + a_{m+1} f^n(z) + \cdots\right)^{\frac{1}{m}},$$

we obtain, for any $z \in D(0,r)$ with $Cr < 1$,

$$\frac{\varphi_{n+1}}{\varphi_n} = \left(\left(1 + a_{m+1} f^n(z) + \cdots\right)^{\frac{1}{m}}\right)^{m^{-n}} = \left(1 + O\left(|f^n(z)|\right)\right)^{m^{-n}} =$$

$$= 1 + O\left(m^{-n}\right) O\left((Cz)^{m^n}\right) = 1 + O\left(m^{-n}\right).$$

Hence, on $D(0,r)$ the product

$$\prod_{n=1}^{\infty} \frac{\varphi_{n+1}}{\varphi_n}$$

converges uniformly. Thus, $\{\varphi_n\}_n$ converges to a local conformal map φ.

- **Uniqueness:** Assume φ, ϕ satisfy

$$\varphi \circ f \circ \varphi^{-1}(w) = w^m = \phi \circ f \circ \phi^{-1}(w).$$

By considering $\Phi := \varphi \circ \phi^{-1}$, we obtain

$$\Phi\left(w^m\right) = \Phi \circ \phi \circ f \circ \phi^{-1}(w) = \varphi \circ f \circ \varphi^{-1} \circ \Phi(w) = \left(\Phi(w)\right)^m.$$

Since $\Phi(0) = 0$, then $\Phi(z) = c_1 z + \cdots$. The condition above tells us that $c_1^{m-1} = 1$ and the other coefficients are 0. Therefore, $\varphi = c_1 \phi$, where c_1 is a $(m-1)$st root of unity. □

The map φ is called a *Böttcher map* or *Böttcher coordinate*. Since $w \longmapsto w^m$ is not an invertible map, we cannot extend φ to the whole basin of attraction as it can be done for $0 < |\lambda| < 1$. However, it is still possible to extend the real map $|\varphi| : U \longrightarrow \mathbb{R}_+ \cup \{0\}$ to the whole basin.

Theorem 2.11 has important applications to the dynamics of polynomials near infinity, which is a superattracting fixed point for any $p \in \text{Pol}$. We will deepen into it in the following chapter.

(c) Parabolic fixed points. In this case we assume that the origin is a fixed point whose multiplier λ is a qth root of unity. By taking the qth iterate of the map we reduce to the case $\lambda = 1$. Hence, we consider maps of the form

$$f(z) = z + az^{m+1} + O\left(z^{m+2}\right), \text{ where } m > 0 \text{ and } a \neq 0.$$

The integer $m+1$ is called the *multiplicity* of the parabolic point, which is the order of the zero of $f - \text{Id}$ at the fixed point.

In order to describe the dynamics around the parabolic fixed point, some additional concepts are required.

Definition 2.12 (Attracting and repelling petals). Assume f is defined and conformal in a neighborhood U of the origin. An open set $\mathcal{P} \subseteq U$ is called an *attracting petal* for f at the fixed point if and only if

i) $f\left(\overline{\mathcal{P}}\right) \subseteq \mathcal{P} \cup \{0\}$;

ii) $\bigcap_n f^n\left(\overline{\mathcal{P}}\right) = \{0\}$.

An open set $\mathcal{P} \subseteq f(U)$ is called a *repelling petal* for f at the fixed point if and only if $\mathcal{P}$ is an attracting petal for $f^{-1} : f(U) \longrightarrow U$, where f^{-1} denotes the branch of the inverse of f fixing the origin.

The following result is due to Leau, Fatou and Julia in successive approximations.

Theorem 2.13 (Parabolic flower Theorem). *Assume f has a parabolic fixed point with multiplier $\lambda = 1$ at the origin of multiplicity $m+1$. Then there exist $2m$ petals $\{\mathcal{P}_j\}_{j=1}^{2m}$, numbered cyclically around the origin and such that $\mathcal{P}_j$ is attracting or repelling according to whether j is odd or even, respectively. Each petal $\mathcal{P}_j$ intersects only its two immediate neighbors $\mathcal{P}_{j-1}$ and $\mathcal{P}_{j+1}$ (indices are taken mod $2m$), and are disjoint from the rest. The petals can be chosen so that the union*

$$\{0\} \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{2m}$$

forms an open neighborhood of the origin, as illustrated in Figure 2.1.

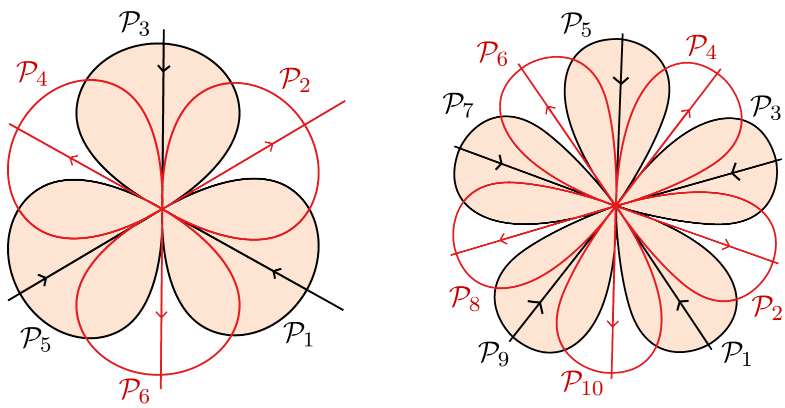


FIGURE 2.1: Distribution of the invariant petals around a parabolic point with multiplier $\lambda = 1$ and multiplicity 4 (left) and 6 (right).

Suppose f is globally defined and has a parabolic fixed point at the origin of multiplier $\lambda = 1$. If the orbit of $z \neq 0$ is infinite and converges to 0, it needs to belong to one of the attracting petals $\mathcal{P}$ in Theorem 2.13 from some iterate and onwards. Then, we say that the orbit converges to 0 through $\mathcal{P}$. This motivates the following definition.

Definition 2.14 (Parabolic basin of attraction). Let z_0 be a parabolic fixed point of f with multiplier $\lambda = 1$ and let $\mathcal{P}$ be an attracting petal. We define the *parabolic basin of attraction* of z_0 associated to $\mathcal{P}$ as

$$\mathcal{A}_{\mathcal{P}}(z_0) := \left\{ z \in \mathbb{C} \setminus \bigcup_{n>0} f^{-n}(z_0) \;\middle|\; f^n(z) \xrightarrow[n\to\infty]{} z_0 \text{ through } \mathcal{P} \right\}.$$

Additionally, we define the *immediate parabolic basin of attraction*, denoted by $\mathcal{A}^*_{\mathcal{P}}(z_0)$, as the forward invariant connected component of the basin.

Notice that if the parabolic fixed point has multiplicity $m+1$, then it has exactly m disjoint parabolic basins. Even though f is not conjugate to its linear part in a neighborhood of the parabolic fixed point, it turns out that some kind of linearization is possible inside each of the petals.

Theorem 2.15 (Parabolic linearization). *For every attracting and repelling petal $\mathcal{P}$, there exists a conformal map $\varphi : \mathcal{P} \longrightarrow \mathbb{C}$, called the Fatou coordinate in $\mathcal{P}$, which conjugates f to the translation $w \longmapsto w+1$ on $\mathcal{P} \cap f^{-1}(\mathcal{P})$. That is, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{P} \cap f^{-1}(\mathcal{P}) & \xrightarrow{\ f\ } & \mathcal{P} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathbb{C} & \xrightarrow{\ w \longmapsto w+1\ } & \mathbb{C} \end{array}$$

If 0 is a parabolic fixed point of f with multiplier $\lambda = e^{2\pi i \frac{p}{q}} \neq 1$ and if f^q has multiplicity $m+1$ at 0, then the number of attracting and repelling petals in Theorem 2.13 is $m = kq$, for some $k \in \mathbb{N}$. The m attracting petals are forward invariant under f^q and can be chosen so that they form k forward invariant cycles of petals of period q under f. The petals in a cycle are mapped among themselves with *rotation number* p/q, see Figure 2.2. Moreover, the k cycles of petals of period q give rise to k disjoint parabolic basins of attraction for the parabolic fixed point.

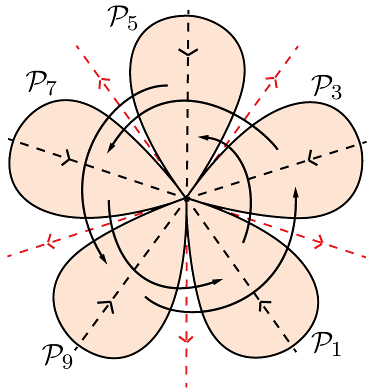


Figure 2.2: A parabolic fixed point of multiplier $\lambda = e^{2\pi i \frac{2}{5}}$ and multiplicity 6 for f^5 (hence $k = 1$). There is one cycle of attracting petals with combinatorial rotation number $\frac{2}{5}$.

(d) Irrationally indifferent fixed points. Here the origin is a fixed point with multiplier $\lambda = e^{2\pi i\theta}$, with $\theta \in \mathbb{R}\setminus\mathbb{Q}$. The fundamental question is which conditions on θ ensure that f is locally conformally conjugate to its linear part $z \longmapsto \lambda z$. If so, we say f is *conformally linearizable* around the fixed point. In contrast to the previous cases, the complete answer is still open in some particular scenarios.

Definition 2.16 (Siegel and Cremer points). We say that an irrationally indifferent fixed point is either a *Siegel point* or a *Cremer point* depending on whether a local linearization is possible or not. The connected component on which f is conformally linearizable is called a *Siegel disk*, with the fixed point z_0 as its *centre*.

The linearization problem depends on the number theoretical properties of θ. Notice that θ is taken modulus 1 and, therefore, we can assume without loss of generality $\theta \in (0,1)$, though it is not required for the definitions below. We recall here that $\theta \in (\mathbb{R}\setminus\mathbb{Q}) \cap (0,1)$ can be expressed as a *continued fraction*

$$\theta = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \ddots}} = [a_0, a_1, \ldots, a_n, \ldots],$$

and one can consider the rational number representing a finite sequence

$$\frac{p_n}{q_n} = [a_1, \dots, a_n] \xrightarrow[n\to\infty]{} \theta.$$

Several classes of numbers have been defined using these concepts. Although they are not required for our purposes, we include the definitions of both *Diophantine* and *Bryuno* numbers for completeness.

Definition 2.17 (Diophantine numbers). Let $\theta \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$, we say θ is *Diophantine of order* k, denoted by $\mathcal{D}(k)$, if and only if there exists $c > 0$ such that for all $n \geq 0$ we have

$$q_{n+1} < c q_n^{k-1}.$$

Definition 2.18 (Bryuno numbers). Let $\theta \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,1)$, we say θ is a *Bryuno number*, denoted by $\mathcal{B}$, if and only if

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty.$$

Using these, in 1942 Siegel showed that if θ is Diophantine, then there is a Siegel disk around the fixed point. This condition was later improved by Bryuno and Rüssman in the 1960s, and it was shown to be the best possible optimization by Yoccoz in 1988.

Theorem 2.19. *Let $f(z) = e^{2\pi i\theta} z + O(z^2)$ be defined in a neighborhood of the origin and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. If $\theta \in \mathcal{B}$, then f is holomorphically linearizable in a neighborhood of* 0. *Conversely, for any $\theta \notin \mathcal{B}$, the quadratic polynomial $f(z) = e^{2\pi i\theta} z + z^2$ is not locally holomorphically linearizable.*

In other words, for the family of quadratic polynomials, the Bryuno condition is optimal for linearization. It is conjectured by Douady this is also the case for the whole class of rational maps.

The local dynamics around Cremer points is complicated and interesting. When linearization is not possible, there is always a compact invariant set with nontrivial topology around the fixed point. For a deeper study on Cremer points, see [28].

2.2 Fatou and Julia sets

The aim of this section is to explore global properties of the dynamics of holomorphic functions by combining Montel's theory of normal families with the local dynamical theory developed in the previous section. Unlike earlier, where the focus was broader, here the holomorphic map f is restricted to belong to either the class Rat or Ent, since some results that do not hold for the $\mathcal{M}$ class are included.

We will revisit these concepts in Chapter 5, where the emphasis shifts to meromorphic functions. There, we discuss the main challenges and differences that arise when extending the theory to this broader setting, as well as identify which results from the current section remain valid in the more general context.

2.2.1 Definitions and properties

We begin by defining the Fatou and Julia sets, which partition the phase space into regions of stable and chaotic behavior, respectively. This section also summarizes their essential properties that underpin much of the theory in complex dynamics.

Definition 2.20 (Fatou and Julia sets). Let f be holomorphic. We define the *Fatou set* of f as the points z such that there exists a neighborhood of z where $\{f^n\}_{n\geq 0}$ is normal, we denote it by $\mathcal{F}_f$. In contrast, the *Julia set* is denoted by $\mathcal{J}_f$ and is defined as the complement of the Fatou set, i.e. $\mathcal{J}_f := \mathbb{C} \setminus \mathcal{F}_f$.

Example 2.21. Consider the quadratic map $f(z) = z^2$, which is holomorphic in the Riemann sphere. Its fixed points are determined by $f(z) = z$, whose solutions are 0, 1 and ∞. In order to classify them, we shall study their multipliers. Since $f'(z) = 2z$, 1 is repelling, while 0 and ∞ are superattracting (notice that for ∞ we must take its respective local chart $f^*(w) = 1/f(z)$).

Let us now express the complex number z in polar coordinates, i.e. $z = re^{i\theta}$, and observe the following:

- If $z \in \mathbb{D}$, then $f^n(z) = z^{2^n} = r^{2^n} e^{2^n i\theta} \xrightarrow[n\to\infty]{} 0$.
- If $z \notin \overline{\mathbb{D}}$, then $f^n(z) = z^{2^n} = r^{2^n} e^{2^n i\theta} \xrightarrow[n\to\infty]{} \infty$.
- Finally, if $z \in \partial\mathbb{D} = \mathbb{S}^1$, then $f(z)$ is just a rotation and $f^n(z) \in \mathbb{S}^1$.

One could now study the normality of $\{f^n\}_{n\geq 0}$ in these sets in order to determine the Fatou and Julia sets. However, by Proposition 2.26, we immediately obtain $\mathcal{F}_f = \widehat{\mathbb{C}} \setminus \mathbb{S}^1$ (since $f \in \text{Rat}$) and $\mathcal{J}_f = \mathbb{S}^1$, which is illustrated in Figure 2.3.

FIGURE 2.3: Representation of the dynamics of $f(z) = z^2$. Points whose orbits converge to 0 are shown in yellow, while the ones escaping to ∞ are in purple. The Julia set is the boundary of both regions, which corresponds to the unit circle $\mathbb{S}^1$.

In order to state properties of the Fatou and Julia sets, we first need to define grand orbits and exceptional points.

Definition 2.22 (Grand orbit). The *grand orbit* of z_0 is defined as the set consisting of all z whose orbits eventually intersect the orbit of z_0. That is,

$$\text{GO}(z, f) = \{z \mid f^n(z_0) = f^m(z) \text{ for some } n, m \geq 0\}.$$

Definition 2.23 (Exceptional points). A point z_0 is called *exceptional* under f if and only if its grand orbit $\mathrm{GO}(z_0, f)$ is a finite set. The set of exceptional points is denoted by $\mathcal{E}(f)$.

Lemma 2.24 (Finite grand orbits). *The set $\mathcal{E}(f)$ of a rational map has at most two points which, if exist, must always be superattracting periodic points of f. If f is transcendental entire, then $\mathcal{E}(f)$ has at most one point.*

We finish this section with the statement of some results for the Fatou and Julia sets in the form of two propositions. The first one contains some of their general properties, while the second one classifies periodic points into one of them.

Proposition 2.25. *Let $f \in \{\mathrm{Rat}, \mathrm{Ent}\}$. Then,*

i) The Fatou set is open, while the Julia set is closed.

ii) The Fatou and Julia sets are completely invariant.

iii) For any $k > 0$, $\mathcal{F}_{f^k} = \mathcal{F}_f$ and $\mathcal{J}_{f^k} = \mathcal{J}_f$.

iv) Let $z_0 \in \mathcal{J}_f$ and U be a neighborhood of z_0 disjoint from $\mathcal{E}(f)$. Then

$$S \setminus \mathcal{E}(f) \subset \bigcup_n f^n(U).$$

v) Either $\mathcal{J}_f$ has no interior point, or $\mathcal{J}_f = S$.

vi) The Julia set is perfect (closed, nonempty and without isolated points).

vii) The Julia set is either connected or has uncountably many components.

viii) If $z_0 \in \mathbb{C} \setminus \mathcal{E}(f)$, then $\mathcal{J}_f \subset \overline{\bigcup_{n>0} f^{-n}(z_0)}$.

ix) If A is a closed, backward invariant subset of $\mathbb{C}$ containing at least three points, then the Julia set is contained in A.

Proof. We only prove *ix*) due to its particular relevance in Chapter 4.

ix) Let S be the complex plane $\mathbb{C}$ or the Riemann sphere $\widehat{\mathbb{C}}$ depending on whether f is entire or rational, respectively. Since A is backward invariant, we have $f^{-1}(A) \subset A$ and $f(S \setminus A) \cap A = \emptyset$. Hence,

$$\bigcup_{n\geq 1} f^n(S \setminus A) \cap A = \emptyset.$$

Since A contains at least three points, by Theorem 1.2 we get that $S \setminus A \subset \mathcal{F}_f$, which implies $\mathcal{J}_f \subset A$. □

Proposition 2.26. *Let $f \in \{\mathrm{Rat}, \mathrm{Ent}\}$.*

i) Every attracting periodic orbit and their attracting basin are contained in the Fatou set. Moreover, the topological boundary of the basin of attraction is equal to the entire Julia set.

ii) Every repelling periodic orbit is contained in the Julia set. Additionally, repelling periodic points are dense in $\mathcal{J}_f$.

iii) Parabolic periodic points are contained in the Julia set. However, their basins of attraction are contained in the Fatou set.

iv) All Siegel points and their respective disks are contained in the Fatou set. On contrast, Cremer points are in the Julia set.

2.2.2 Fatou components

To gain a deeper understanding of the possible limit functions arising from the iteration of f within its set of normality, we introduce the concept of Fatou components. These components are the connected subsets of the Fatou set, where the family of iterates $\{f^n\}_{n\geq 0}$ forms a normal family and exhibits stable, well-behaved dynamics. Analyzing these components allows us to classify the local and global behaviors of the system and provides insight into the different types of dynamical phenomena that can occur in holomorphic iteration.

Definition 2.27 (Fatou component). A *Fatou component* U is a connected component of the Fatou set.

Since the Julia set always surrounds Fatou components, they must be mapped to another components with the exception of at most one point, which must be an asymptotical value (because it is locally omitted). Therefore, rational maps send Fatou components to Fatou components properly, i.e. $f : U \longrightarrow V$ holomorphic such that $f(\partial U) = \partial V$.

Thus, Fatou components can only be:

- **Periodic:** $f^p(U) = U$ for some minimal $p \geq 1$. If $p = 1$, it is a *fixed component*.
- **Preperiodic:** $f^n(U)$ is periodic for some $n \geq 1$.
- **Wandering:** $f^k(U) \cap f^j(U) = \emptyset$ for all $k \neq j$.

The behavior of successive iterates of f on periodic components is well understood. The following result, originally stated by Fatou, summarizes the different possibilities we might have (see Figure 2.4).

Theorem 2.28 (Classification of Fatou components). *If U is a periodic Fatou component of period p, then exactly one of the following holds.*

i) U is an attracting domain*, i.e. there exists an attracting periodic point $z_0 \in U$ of period p such that $f^{np}(z) \xrightarrow[n\to\infty]{} z_0$, for all $z \in U$.*

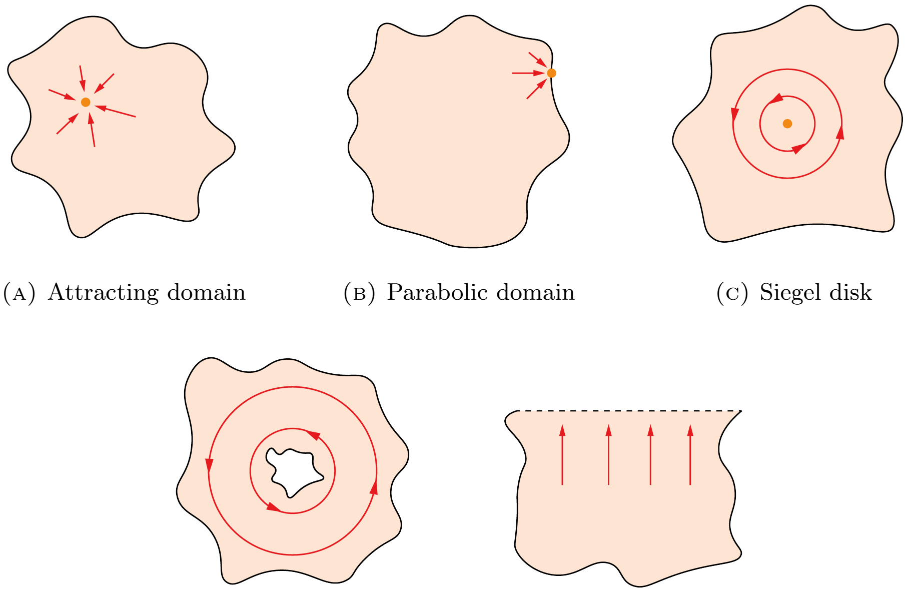
ii) U is a parabolic *or* Leau domain*, i.e. there exists a parabolic periodic point $z_0 \in \partial U$ of period p such that $f^{np}(z) \xrightarrow[n\to\infty]{} z_0$, for all $z \in U$.*

iii) U is a Siegel disk*, i.e. U is simply connected and there exists a conformal map $\phi : U \longrightarrow \mathbb{D}$ such that $\phi\left(f^p\left(\phi^{-1}(z)\right)\right) = e^{2\pi i\theta}z$, for some $\theta \in \mathbb{R}\setminus\mathbb{Q}$.*

iv) U is a Herman ring*, i.e. U is a doubly connected set and there exist an annulus $A_r = \{1 < |z| < r\}$ and a conformal mapping $\phi : U \longrightarrow A_r$ such that $\phi\left(f^p\left(\phi^{-1}(z)\right)\right) = e^{2\pi i\theta}z$, for some $\theta \in \mathbb{R}\setminus\mathbb{Q}$.*

v) U is a Baker domain*, i.e. there exists $z_0 \in \partial U$ such that $f^{np}(z) \xrightarrow[n\to\infty]{} z_0$ for all $z \in U$, but $f^p(z_0)$ is not defined.*

Although the following result does not rely on the notion of Fatou components, we include it here due to its significance in the study of attracting and parabolic domains. Its proof can be found in [21, Lemma 8.5, Theorem 10.15].



(A) Attracting domain (B) Parabolic domain (C) Siegel disk

(D) Herman ring (E) Baker domain

FIGURE 2.4: Schematic representation of the different types of Fatou components.

Theorem 2.29. *Let z_0 be an attracting or a parabolic fixed point. Then, the immediate basin of attraction of z_0 contains at least one singular value.*

In contrast to periodic Fatou components, wandering domains can display far more intricate dynamical behavior. Their study provides insight into the global structure of the Fatou set and the function's long-term dynamics. The existence of wandering domains strongly depends on the class of holomorphic functions under consideration. For rational maps, Sullivan established in [29] that wandering domains do not exist in this setting, resolving a major open question in one-dimensional complex dynamics. The formal statement is as follows.

Theorem 2.30 (No wandering domains). *Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational map with* $\deg f = d \geq 2$*. Then, every Fatou component is either periodic or preperiodic.*

This result highlights a deep rigidity in the dynamics of rational maps. However, in the setting of transcendental entire and meromorphic functions, wandering domains not only exist but can exhibit diverse behaviors. Their presence reflects the richness and complexity of transcendental dynamics, which is not constrained by the compactness of the Riemann sphere.

For a foundational exposition of these ideas, see Sullivan's original paper [29]. However, may you be interested in an overview of meromorphic dynamics, including the role of wandering domains, see [30]. On the other hand, Milnor's [21] provides an accessible introduction to the subject, including a clear discussion of Theorem 2.30. For more advanced results on wandering domains in transcendental settings, the work of Mihaljević-Brandt and Rempe [31] is highly recommended, particularly in connection with escaping sets and the Eremenko–Lyubich class, i.e. transcendental entire functions with bounded singular set.

Chapter 3

The escaping set of polynomials

The primary goal of this chapter is to prove Theorems A and B, the latter through the use of hyperbolic geometry. In the first section, we apply Theorem 2.11 to establish that infinity is a superattracting fixed point, leading to the conclusion that its basin of attraction is an open and connected set. The second section addresses the connectivity of the Julia set for polynomials, providing a complete characterization and illustrating each case with explicit examples. Further discussions on these topics can be found in [17, 21].

3.1 The dynamics of polynomials

Throughout this chapter, we assume that $p \in \text{Pol}$, i.e. a polynomial of degree at least 2. This case is of particular importance, as it is the only one in which infinity lies in both the domain and the image of the functions under consideration. In fact, this provides a natural context in which to introduce our main object of study.

Definition 3.1 (Escaping set). Let f be a non-constant meromorphic map, we define the *escaping set* of f, denoted by $\mathcal{I}_f$, as the set of points whose orbit tends to infinity. That is,

$$\mathcal{I}_f := \{z \in \mathbb{C} \mid f^n(z) \xrightarrow[n\to\infty]{} \infty\}.$$

In the case of polynomials, the phase space can be partitioned into the escaping set and *filled Julia set*, denoted by $\mathcal{K}_p$. This set consists of all points whose orbits under p remain bounded, and it includes both the Julia set and the bounded components of the Fatou set. Formally,

$$\mathcal{K}_p := \{z \in \mathbb{C} \mid \mathcal{O}_p(z) \text{ is bounded}\}.$$

This leads to a fundamental property for our analysis

$$\partial\mathcal{I}_p = \partial\mathcal{K}_p = \mathcal{J}_p,$$

which highlights the Julia set as the common boundary between the escaping and non-escaping dynamics.

The following result guarantees that we may refer to the escaping set and the basin of attraction of infinity interchangeably.

Proposition 3.2. *Let $p \in \text{Pol}$, then $z = \infty$ is a superattracting fixed point of p.*

Proof. We first see that $z = \infty$ is indeed a fixed point of $p(z)$:

$$p(\infty) = \lim_{z\to\infty} p(z) = \infty.$$

We shall now prove it is also superattracting. We consider the local chart $\varphi(z) = 1/z$ in order to obtain the expression of $p(z)$ in a neighborhood of ∞, i.e.

$$\widetilde{p}(z) = \varphi \circ p \circ \varphi^{-1}(z) = \frac{1}{p(1/z)}.$$

Since $p(z) = a_0 + a_1 z + \cdots + a_d z^d$, with $a_d \neq 0$ and $d \geq 2$, we get the following:

$$\widetilde{p}(z) = \frac{1}{p(1/z)} = \frac{1}{a_0 + a_1/z + \cdots + a_d/z^d} = z^d \left(\frac{1}{a_0 z^d + a_1 z^{d-1} + \cdots + a_d} \right).$$

Thus, $\widetilde{p}(0) = 0$ (i.e. $p(\infty) = \infty$) and $(\widetilde{p})'(0) = 0$ (i.e. $p'(\infty) = 0$). □

Remark 3.3. Notice that $\mathcal{A}_p(\infty) = \mathcal{I}_p$, since both share the same definition.

The following are direct implications of the previous remark:

- $\mathcal{K}_p = \widehat{\mathbb{C}} \setminus \mathcal{A}_p(\infty)$,
- $\mathcal{J}_p = \partial \mathcal{K}_p = \partial \mathcal{A}_p(\infty)$,
- $\mathcal{F}_p = \mathcal{A}_p(\infty) \cup \operatorname{int}(\mathcal{K}_p)$.

Moreover, the escaping set is connected when considered within the Riemann sphere.

Theorem A. *Let $p \in \mathrm{Pol}$. Then, $\mathcal{A}_p(\infty)$ is connected.*

Proof. Since ∞ is superattracting, there exists a neighborhood U of ∞ such that $f(U) \subseteq U$. We can assume U is the largest open connected subset of $\mathcal{A}_p(\infty)$ satisfying this property by enlarging it if necessary. If $\mathcal{A}_p(\infty)$ is disconnected, then there exists a connected component $V \subset \mathcal{A}_p(\infty)$ such that $U \cap V = \emptyset$. Since Fatou components are mapped to Fatou components and $\infty \in U$, there exists an integer $k > 0$ such that $p^k(V) = U$. Thus, V contains a (pre)pole, which leads to contradiction. □

Remark 3.4. From $\mathcal{F}_p = \mathcal{A}_p(\infty) \cup \operatorname{int}(\mathcal{K}_p)$, we get that the escaping set is open. In fact, by Theorem A, it is a Fatou component. Therefore, K_p is closed and both $\mathcal{K}_p$ and $\mathcal{J}_p = \partial \mathcal{K}_p$ are compact.

Hence, the basin of attraction of infinity, i.e. the escaping set, is an open and connected set that surrounds the Julia set. What remains is to determine whether $\mathcal{A}_p(\infty)$ is simply connected, which will follow from an analysis of the connectivity of $\mathcal{J}_p$.

3.2 Connectivity of the Julia set

Since $\mathcal{J}_p = \partial \mathcal{I}_p$, its topological structure plays a crucial role in understanding the global dynamics of the polynomial p. In this section, we address this question by first establishing a few preliminary results that will allow us to characterize precisely when $\mathcal{J}_p$ is connected.

Lemma 3.5. *Let $p \in \mathrm{Pol}$. There exist $R \in \mathbb{R}_+$ and $\lambda > 1$ such that if $|z| > R$, then $|p(z)| \geq \lambda |z|$.*

Proof. Since $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$, with $d \geq 2$ and $a_d \neq 0$, we have for all $z \neq 0$

$$|p(z)| = |z| \left| z^{d-1} \right| \left| a_d + \frac{a_{d-1}}{z} + \cdots + \frac{a_1}{z^{d-1}} + \frac{a_0}{z^d} \right|.$$

Therefore,

$$|p(z)| \geq \lambda|z| \iff \left|z^{d-1}\right|\left|a_d + \frac{a_{d-1}}{z} + \cdots + \frac{a_1}{z^{d-1}} + \frac{a_0}{z^d}\right| \geq \lambda,$$

for some $\lambda > 1$. Note that

$$\lim_{|z|\to\infty} \lambda = \lambda < \infty = \lim_{|z|\to\infty} \left|z^{d-1}\right|\left|a_d + \frac{a_{d-1}}{z} + \cdots + \frac{a_1}{z^{d-1}} + \frac{a_0}{z^d}\right|,$$

so there exists $R \in \mathbb{R}_+$ satisfying the statement. □

Prior to the next lemma, we shall introduce the concepts of Jordan arc and curve.

Definition 3.6 (Jordan arc, Jordan curve). Let $a, b \in \mathbb{R}$ such that $a < b$ and let $C : [a, b] \longrightarrow \mathbb{C}$ be continuous. We say C is a *Jordan arc* if and only if C is injective. Also, we say C is a *Jordan curve* if and only if $C(a) = C(b)$ but C is injective in (a, b), i.e. C is a closed and simple curve. Moreover, if C is a Jordan curve, we denote by $\widehat{C}$ the bounded component of $\mathbb{C}$ delimited by C, including its boundary.

Notice that for a complex polynomial p, the set of singular values $\mathcal{S}(p)$ only contains critical values, i.e. images of critical points. Let us also denote the set of critical points as $\operatorname{Crit}(p)$.

Lemma 3.7. *Let $p \in \operatorname{Pol}$ and let C be a Jordan curve.*

i) If $C \cap \mathcal{S}(p) = \emptyset$, then $p^{-1}(C)$ is a set of Jordan curves $C_1, \ldots, C_d$ which are either equal or disjoint.

ii) If $C \cap \mathcal{S}(p) = \{v_0\}$, then $p^{-1}(C)$ is a set of Jordan curves $C_1, \ldots, C_d$ which are either equal or intersecting at most on $p^{-1}(v_0)$, the latter satisfied at least once.

In both cases, $C_1, \ldots, C_d$ satisfy that $p|_{\widehat{C_i}} \colon \widehat{C_i} \longrightarrow \widehat{C}$ is a proper map. Its degree is given by

$$\deg p\big|_{C_i} =: d_i = \#\left(\operatorname{Crit}(p) \cap \widehat{C_i}\right) + 1,$$

where critical points are counted with multiplicity. Moreover, if we only count repeated curves once, the sum of all d_i is d.

Proof. Since polynomials are proper maps and C is compact, $p^{-1}(C)$ is compact. Therefore, every connected component of $p^{-1}(C)$ is closed and bounded.

i) Let us fix $w \in C$. Since $w \notin \mathcal{S}(p)$, $p^{-1}(w) = \{z_1, \ldots, z_d\}$ are distinct points. Let W be a neighborhood of w such that $W \cap \mathcal{S}(p) = \emptyset$, which exists because the number of singular values is finite. By Proposition 1.7, there exist $U_1, \ldots, U_d$ neighborhoods of $z_1, \ldots, z_d$, respectively, such that $p\colon U_i \longrightarrow W$ is a proper map of degree 1, i.e. biholomorphic. By reducing W if necessary, we may assume that $U_1, \ldots, U_d$ are pairwise disjoint. Therefore, $p^{-1}(C) \cap U_i$ is a Jordan arc containing z_i in its interior. This implies that $p^{-1}(C)$ has no self-intersections and no endpoints, hence $p^{-1}(C)$ is a Jordan curve. Since p has degree d, there exist d preimages of C counted with multiplicity, all of them Jordan curves.

ii) Now suppose that C contains one singular value v_0 and consider $C' = C \setminus \{v_0\}$. By the same argument above, $p^{-1}(C')$ cannot have self-intersections. Thus, the connected components of $p^{-1}(C)$ can only self-intersect at critical points in $p^{-1}(v_0)$. Since v_0 must have at least one critical preimage, let $c_0 \in p^{-1}(v_0)$ such that $p'(c_0) = 0$. Let W be a neighborhood of v_0 such that $\mathcal{S}(p) \cap W = \{v_0\}$ and let V be the connected component of $p^{-1}(W)$ containing c_0. Then $p|_V : V \longrightarrow W$ is proper of degree $d' \geq 2$. Hence, every point of W (except v_0) must have at least two distinct preimages. This implies that at least two preimages of C must intersect at c_0.

In both cases, if C_i is a Jordan curve mapping to C, $p\left(\widehat{C_i}\right) = \widehat{C}$ by Theorem 1.12. Moreover, since p is proper, $p|_{\widehat{C_i}} : \widehat{C_i} \longrightarrow \widehat{C}$ has degree $d_i = \#\left(\mathrm{Crit}(p) \cap \widehat{C_i}\right)$. □

Lemma 3.8. *Let $U, V \subset \mathbb{C}$ be open, simply connected sets satisfying $\overline{V} \subseteq U$, let φ be a conformal map from U to V and let d_U denote the hyperbolic distance on U. Then, for all $z, w \in U$*

$$d_U(\varphi(z), \varphi(w)) \leq \mu\, d_U(z, w),$$

where $0 < \mu < 1$ only depends on the domains U, V.

Proof. Let us denote by d_V the hyperbolic distance on V. Then, for all $z, w \in U$

$$d_U\left(\varphi(z), \varphi(w)\right) \leq \mu\, d_V\left(\varphi(z), \varphi(w)\right) = \mu\, d_U(z, w),$$

where we have applied Theorem 1.45 on the inequality (which also provides μ) and part *ii*) of Theorem 1.41 on the equality, since φ is conformal. □

We now present the result that addresses the question of the connectivity of $\mathcal{J}_p$. One may either state a general theorem valid for all $p \in \mathrm{Pol}$, or restrict to the quadratic case in order to obtain a more precise characterization. Both versions are given below; however, only the first part of the statement is proven in the general case.

Theorem B. *Let $p \in \mathrm{Pol}$.*

i) If no critical point of p belongs to $\mathcal{A}_p(\infty)$, then $\mathcal{K}_p$ is a connected set.

ii) If at least one critical point of p belongs to $\mathcal{A}_p(\infty)$, then $\mathcal{K}_p$ is disconnected and has infinitely many connected components. Additionally, if $\mathcal{A}_p(\infty)$ contains all critical points of p, then $\mathcal{K}_p = \mathcal{J}_p$ is totally disconnected, i.e. a Cantor set.

Proof. i) Lemma 3.5 provides us a radius $R > 0$ such that if $|z| > R$, then $|p(z)| \geq \lambda |z|$ for some $\lambda > 1$. Let us define $D_0 \coloneqq \{z \in \mathbb{C} \,|\, |z| < R{+}1\}$, which satisfies $\partial D_0 \subseteq \mathcal{A}_p(\infty)$ is a Jordan curve. Notice that, by Lemma 3.5, every point outside D_0 goes further away; thus, D_0 contains all its preimages. Moreover, every $z_0 \in D_0 \setminus p^{-1}(D_0)$ is also in $\mathcal{A}_p(\infty)$, since $p(z_0) \notin D_0$ implies $|p(z_0)| > R$.

From Lemma 3.7 and the fact that ∂D_0 does not contain any singular value, we get that $p^{-1}\left(\partial D_0\right)$ is composed of d Jordan curves, which must either coincide or be disjoint. We are going to prove the latter is not possible. Thus, let us assume $d-1$ preimages coincide and $p^{-1}(\partial D_0) = C_1 \cup C_2$, where C_1, C_2 are two disjoint Jordan curves. Since ∂D_0 is contained in the escaping set, no critical value belongs to $C_1 \cup C_2$. Hence, $p\colon \widehat{C_i} \longrightarrow \overline{D_0}$ is a proper map for both $i = 1, 2$. Their degree is given by the number of critical points contained in $\widehat{C_i}$ plus 1. Since we have $d-1$ critical points and they must all be contained in either $\widehat{C_1}$ or $\widehat{C_2}$ by hypothesis (otherwise such point would belong to $\mathcal{A}_p(\infty)$), the sum of both degrees gives $d+1$, which leads to contradiction (it should not exceed d). Therefore, $C_1 = C_2$ and $D_1 \coloneqq p^{-1}(D_0)$ is an open, simply connected set such that $\overline{D_1} \subset \overline{D_0}$ and $p\colon D_1 \longrightarrow D_0$ is a proper map of degree d.

Iterating through the same process, one gets a chain

$$D_0 \supset D_1 \supset \cdots \supset D_n \supset \cdots$$

such that $\overline{D_n} \subset \overline{D_{n-1}}$, which means $\{\overline{D_n}\}_{n\geq 0}$ is a decreasing nested sequence of nonempty compact and connected subsets. Therefore,

$$\mathcal{K}_p = \bigcap_{n=0}^{\infty} \overline{D_n}$$

is also a nonempty compact and connected set.

ii) Let us assume $p(z) = z^2 + c$ (with unique critical value c) without loss of generality and consider the same radius $R > 0$ as before. We define $D_0 \coloneqq \{z \in \mathbb{C} \mid |z| < R+1\}$ and, by enlarging R if necessary, we may also assume $p^m(0) \in \partial D_0$, for some $m > 1$. Then, $c \in p^{-m+1}(\partial D_0)$ and $p^{-m+1}\left(\overline{D_0}\right) \subseteq \cdots \subseteq p^{-1}\left(\overline{D_0}\right) \subseteq D_0$ for similar arguments as in *i)*. Therefore, $D_n \coloneqq p^{-1}(D_{n-1})$ is an open, simply connected set such that $\overline{D_n} \subseteq \overline{D_{n-1}}$, the map $p\colon D_n \longrightarrow D_{n-1}$ is proper of degree 2 and $0, c \in \overline{D_n}$, for all $1 \leq n \leq m-1$.

Since $c \in \partial D_{m-1}$, we cannot iterate in the same way any further. Let us denote $D \coloneqq D_{m-1}$ for the sake of simplicity. By part *ii)* of Lemma 3.7, the whole preimage of ∂D consists of two Jordan curves, say C_1, C_2, such that $C_1 \cap C_2 = \{0\}$. Moreover, since $p(z)$ is an even map, the curves C_1, C_2 are symmetric with respect to the origin.

By denoting $I_i \coloneqq \operatorname{int}\left(\widehat{C_i}\right)$, we define the proper biholomorphic maps $\varphi_i : D \longrightarrow I_i$, for $i = 1, 2$, such that $p \circ \varphi_i = \mathrm{Id}$. Notice that since I_1, I_2 are conformally conjugate to D, they also contain an infinity-shaped region mapped to $I_1 \cup I_2$ inside each of them. Let us denote by I_{ij} the subset in I_i mapped to I_j via p, with $i, j \in \{1, 2\}$. By iterating this procedure, we obtain

$$I_{s_0 s_1 \cdots s_n} \coloneqq \{z \in I_1 \cup I_2 \mid p^k(z) \in I_{s_k}, \text{ for all } 0 \leq k \leq n\},$$

where $s_i \in \{1, 2\}$. Another way to understand these sets is by considering images of φ_1, φ_2. For instance, we have $I_1 = \varphi_1(D)$ and $I_{12} = \varphi_1(I_2) = \varphi_1 \circ \varphi_2(D)$, see Figure 3.1 for a showcase of $n = 1$. In general,

$$I_{s_0 s_1 \cdots s_n} = \varphi_{s_0} \circ \varphi_{s_1} \circ \cdots \circ \varphi_{s_n}(D) \Longrightarrow p\left(I_{s_0 s_1 \cdots s_n}\right) = \varphi_{s_1} \circ \cdots \circ \varphi_{s_n}(D) = I_{s_1 \cdots s_n}.$$

Let us now define $I^n \coloneqq \{I_{s_0 s_1 \cdots s_n} \mid s_i \in \{1, 2\}\}$ and $d_n \coloneqq \max_{I \in I^n} \operatorname{diam}(I)$, which exists because the diameter of I and $\overline{I}$ coincide. Notice that a point z_0 is contained in $\mathcal{K}_p$ if and only if $z_0 \in \bigcap_{n=0}^{\infty} I^n$, so it suffices to prove $d_n \xrightarrow[n\to\infty]{} 0$. If we prove that there exists some $0 < \mu < 1$ such that $d_n \leq \mu\, d_{n-1}$ for all $n \geq 1$, we will have $d_n \leq \mu^n\, d_0 \xrightarrow[n\to\infty]{} 0$. Therefore, all elements of $\mathcal{K}_p$ are points and $\mathcal{K}_p = \mathcal{J}_p$ is a totally disconnected set.

By definition of d_n, there exist $I^* \in I^n$ and $z_0, w_0 \in \overline{I^*}$ such that $d_D(z_0, w_0) = d_n$, where d_D denotes the hyperbolic distance on D. By applying p, we get two points $z_1 = p(z_0)$ and $w_1 = p(w_0)$ belonging to the same connected component of I^{n-1}. Let $i \in \{1, 2\}$ such that $z_0 = \varphi_i(z_1)$ and $w_0 = \varphi_i(w_1)$. Since φ_i is a conformal map, $d_D(z_0, w_0) = d_D(\varphi_i(z_1), \varphi_i(w_1))$. By applying Lemma 3.8, we get, with $0 < \mu < 1$,

$$d_n = d_D(\varphi_i(z_1), \varphi_i(w_1)) \leq \mu\, d_D(z_1, w_1) \leq \mu \max_{I \in I^{n-1}} \operatorname{diam}(I) = \mu\, d_{n-1}. \qquad \square$$

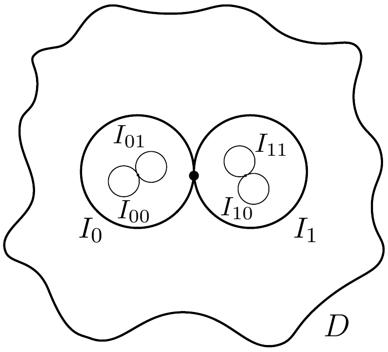


FIGURE 3.1: Showcase of the construction $I_{s_0 s_1 \cdots s_n}$ for $n = 1$.

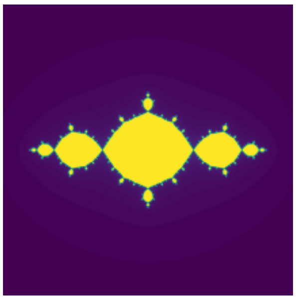
In the case of a quadratic polynomial, there is a unique finite critical point, say z_0. This leads to a natural dichotomy based on whether z_0 lies in the escaping set.

Theorem 3.9. *Let p be a quadratic polynomial and let z_0 be its finite critical point.*

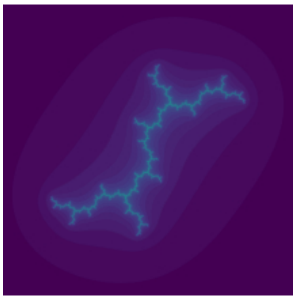
i) $\mathcal{K}_p$ is connected if and only if $z_0 \notin \mathcal{A}_p(\infty)$.

ii) $\mathcal{K}_p = \mathcal{J}_p$ is totally disconnected if and only if $z_0 \in \mathcal{A}_p(\infty)$.

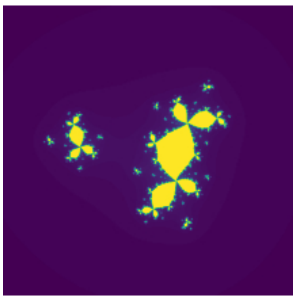
We conclude this chapter with Figure 3.2, which illustrates the various types of Julia sets described in Theorems B and 3.9.



(A) $p(z) = z^2 - 1$



(B) $p(z) = z^2 + i$



(C) $p(z) = z^3 - 0.48z + 0.706 + 0.503i$

(D) $p(z) = z^2 + 1 + i/2$

FIGURE 3.2: Filled Julia sets of polynomials of degree 2 and 3. Bounded orbits are represented in yellow, and the boundary between the yellow and purple regions is the Julia set of p. Notice that (A) and (B) show connected Julia sets, while the ones in (C) and (D) are disconnected and totally disconnected, respectively.

Chapter 4

The escaping set of a transcendental entire function

In this chapter we study the escaping set for $f \in \text{Ent}$, i.e. entire maps with an essential singularity at infinity. A key feature of such functions is that infinity is always an asymptotic value. This motivates the study of the escaping set, which captures the long-term behavior of points whose orbits tend to infinity.

The first section is devoted to the topological structure of the escaping set, including key results that distinguish this setting from the polynomial case. In the second section, we explore how the escaping set can contain rich and intricate structures such as continua. In particular, we construct a Cantor bouquet using the exponential family to illustrate how connected components may be organized in this context. The contents of each section are primarily based on [32, 33], respectively.

4.1 The topology of the escaping set

The central result of this section is Theorem C, which establishes the non-emptiness of the escaping set for transcendental entire functions. Although this theorem was originally proved by Eremenko in [12], we present here the proof given by Domínguez in [13], as it offers a more accessible and instructive approach. Following this, we explore several key topological properties of both the escaping and Julia sets, laying the groundwork for understanding their complex structure and interplay in transcendental dynamics.

4.1.1 Non-emptiness of the escaping set

Before proving Theorem C, we must first establish three foundational lemmas. These intermediate results clarify the asymptotic behavior of transcendental functions and provide the necessary tools for constructing points whose orbits tend to infinity.

Lemma 4.1 (Bohr's Lemma). *For all $\rho \in (0,1)$, there exists a positive constant c such that if $h : \mathbb{D} \longrightarrow \mathbb{C}$ is holomorphic with $h(0) = 0$ and*

$$M(\rho, h) := \max_{|z|=\rho} |h(z)| \geq 1,$$

then $\partial D(0,R) \subseteq h(\mathbb{D})$ for some $R \geq c$.

Proof. Let us assume that such c does not exist for some $\rho \in (0,1)$. Then there exists a sequence $\{h_n\}_n$ of holomorphic maps satisfying the hypothesis such that

$$c_n := \sup\{R > 0 \,|\, \partial D(0,R) \subseteq h_n(\mathbb{D})\} \xrightarrow[n\to\infty]{} 0.$$

Hence there exist $a_n, b_n \in \mathbb{C}$ with $|a_n| = c_n$ and $|b_n| = 2c_n$ such that $h_n(z) \neq a_n$ and $h_n(z) \neq b_n$ for all $z \in \mathbb{D}$. The functions $g_n : \mathbb{D} \longrightarrow \mathbb{C}$ defined by

$$g_n(z) := \frac{h_n(z) - a_n}{b_n - a_n}$$

satisfy $g_n(z) \neq 0$ and $g_n(z) \neq 1$ for all $z \in \mathbb{D}$. By Theorem 1.2, $\{g_n\}_n$ forms a normal family. However,

$$|g_n(0)| = \left| \frac{-a_n}{b_n - a_n} \right| \leq \frac{|a_n|}{|b_n| - |a_n|} = 1,$$

while

$$M(\rho, g_n) \geq \frac{M(\rho, h_n) - |a_n|}{|b_n| + |a_n|} \geq \frac{1 - c_n}{3c_n},$$

and thus $M(\rho, g_n) \xrightarrow[n\to\infty]{} \infty$. This contradicts the normality of the sequence $\{g_n\}_n$. □

The optimal constant c in Lemma 4.1 was established by Hayman in [34] and is given by

$$c = \frac{(1-\rho)^2}{4\rho}.$$

For a bounded domain $G \subset \mathbb{C}$, we denote by $U(G)$ the unbounded connected component of $\mathbb{C} \setminus G$. Moreover, we call $T(G) = \mathbb{C} \setminus U(G)$ the *topological hull* of G. Thus, $T(G)$ is the union of G and the bounded connected components of its complement. One could informally consider $T(G)$ is obtained from G by "filling holes". Notice that a bounded domain G is simply connected if and only if $T(G) = G$.

Let us also define

$$\mu(G) := \min_{z \in U(G)} |z|.$$

If $\mu(G) > 0$, then

$$\mu(G) = \sup\{R > 0 \,|\, D(0, R) \subseteq T(G)\}.$$

Moreover, if h satisfies the hypothesis of Lemma 4.1, then $\mu(h(\mathbb{D})) \geq c$.

The following technical results enable the construction of orbits that tend to infinity and establish an important property of topological hulls, respectively.

Lemma 4.2. *Let $f \in \text{Ent}$, $r > 0$ and $0 < \rho < 1$. Let also $c > 0$ be the one provided in Lemma 4.1. Then,*

$$\mu(f(D(0, r))) \geq cM(\rho r, f) - (1 + c)|f(0)|.$$

Proof. We denote $M := \max_{|z|=\rho r} |f(z) - f(0)|$ and consider

$$h(z) := \frac{f(rz) - f(0)}{M},$$

with $M(\rho, h) \geq 1$ for any $0 < \rho < 1$. Furthermore,

$$h(\mathbb{D}) = \frac{f(r\mathbb{D}) - f(0)}{M} \Longleftrightarrow f(D(0, r)) = h(\mathbb{D})M + f(0).$$

By using $\mu(kG) = |k|\,\mu(G)$ and $\mu(G+k) \geq \mu(G) - |k|$ for any $k \in \mathbb{C}$, we get

$$\mu\left(f\left(D(0,r)\right)\right) = \mu\left(h(\mathbb{D})M + f(0)\right) \geq \mu\left(h(\mathbb{D})M\right) - |f(0)| = \mu\left(h(\mathbb{D})\right)M - |f(0)| \geq$$

$$\geq cM - |f(0)| = c\max_{|z|=\rho r}|f(z) - f(0)| - |f(0)| \geq c\left|\max_{|z|=\rho r}|f(z)| - |f(0)|\right| - |f(0)| \geq$$

$$\geq c\max_{|z|=\rho r}|f(z)| - c\,|f(0)| - |f(0)| = c\,M(\rho r, f) - (1+c)\,|f(0)|\,. \qquad \square$$

Lemma 4.3. *Let $f \in \mathrm{Ent}$ and let G be a bounded domain in $\mathbb{C}$. Then*

$$f(T(G)) \subseteq T(f(G)) \tag{4.1}$$

and

$$\partial T(f(G)) \subseteq f(\partial T(G)). \tag{4.2}$$

Proof. Since both (4.1) and (4.2) share arguments, we only prove the former. We first notice the following:

i) $\partial f(G) \subseteq f\left(\partial G\right)$, by Theorem 1.12.

ii) $\mathrm{int}\,(U(G)) \cap \overline{G} = \emptyset$, since $G \in T(G)$ and $T(G) \cap U(G) = \emptyset$.

iii) $\partial U(G) \subseteq \partial G$, due to $\partial T(G) \subseteq \partial G$.

iv) Any curve from $z_1 \in G$ to $z_2 \notin G$ must meet ∂G.

By the sake of contradiction, let us assume $x \in T(G)$ and $f(x) = y \notin T(f(G))$, so $y \in U\left(f(G)\right)$. Let us denote $W = \mathrm{int}\,(U(f(G)))$ and let $V \subseteq T(G)$ be a neighborhood of x. By Theorem 1.12, $f(V)$ is a neighborhood of y in $f\left(T(G)\right)$. Thus $f(V) \cap W \neq \emptyset$, which means that there exists some $x' \in T(G)$ such that $y' = f(x') \in f(T(G)) \cap W$.

Now let γ be a curve in W connecting y' and ∞. Since $\infty \notin f(T(G))$, there exists some $z \in \gamma \subseteq W$ such that $z \in \partial f(T(G))$. However,

$$\partial f(T(G)) \subseteq f(\partial T(G)) \subseteq f\left(\partial G\right) \subseteq f(\overline{G}) = \overline{f(G)}.$$

Hence, $z \in \mathrm{int}\,(U(f(G))) \cap \overline{f(G)}$, which is a contradiction. $\square$

We are now prepared to present the proof of Theorem C, having established the key preliminary results.

Theorem C. *Let $f \in \mathrm{Ent}$. Then, the escaping set $\mathcal{I}_f$ is nonempty.*

Proof. Let us fix $\rho \in (0,1)$. It follows from Lemma 4.2 that if $r > 0$ is large enough, say $r \geq r_0$, then

$$\mu\left(f\left(D(0,r)\right)\right) \geq c\,M(\rho r, f) - (1+c)\,|f(0)| \geq \frac{c}{2}M(\rho r, f) \geq 2r. \tag{4.3}$$

By (4.2) and the defintion of $\mu(G)$ we have

$$\mu\left(f(G)\right) = \mu\left(T\left(f(G)\right)\right) \geq \mu\left(f\left(T(G)\right)\right) \geq \mu\left(f\left(D(0,\mu(G))\right)\right) \tag{4.4}$$

for every bounded domain G. We deduce from (4.3) and (4.4) that if $\mu(G) \geq r_0$, then

$$\mu\left(f(G)\right) \geq 2\mu(G).$$

We now fix a bounded domain G_0 satisfying $\mu(G_0) \geq r_0$, for instance $G_0 := D(0, r_0)$. By denoting $G_n := f(G_{n-1}) = f^n(G_0)$ for $n \geq 0$, we conclude

$$\mu(G_n) \geq 2\,\mu(G_{n-1}) \geq \cdots \geq 2^n\,\mu(G_0) = 2^n\,r_0. \tag{4.5}$$

Moreover, by Lemma 4.3

$$\partial T(G_n) \subseteq f(\partial T(G_{n-1})) \subseteq \cdots \subseteq f^n(\partial G_0) = f^n(\partial D(0, r_0)).$$

Hence,

$$A_n := f^{-n}(\partial T(G_n)) \cap \partial D(0, r_0)$$

is a compact set such that $A_{n+1} \subseteq A_n$ for all $n \geq 0$. Therefore,

$$A := \bigcap_{n=0}^{\infty} A_n \neq \emptyset.$$

For $z \in A$, we have $f^n(z) \in \partial T(G_n)$. Thus, $|f^n(z)| \geq 2^n\,r_0$ by (4.5) and $z \in \mathcal{I}_f$. □

4.1.2 Further results

A fundamental result in holomorphic dynamics is the non-emptiness of the Julia set. While proving this can be challenging in the case of transcendental entire functions, Theorem C offers a viable approach. Nevertheless, an additional auxiliary result is required to complete the argument.

Lemma 4.4. *Any* $f \in \text{Ent}$ *has at least one periodic point of period* 2.

Proof. Let us assume by the sake of contradiction that there are no points of period 2. Thus there are no fixed points, so $f(z) \neq z$ and $f(f(z)) \neq f(z)$ for all $z \in \mathbb{C}$.

Let us also consider

$$h(z) := \frac{f(f(z)) - z}{f(z) - z},$$

by the remarks above $h(z) \neq 0, 1$ for all $z \in \mathbb{C}$. However, by Theorem 1.13, h can omit at most one point. So it must be constant, i.e. $h \equiv c \in \mathbb{C} \setminus \{0, 1\}$.

Now let $a \in \mathbb{C}$ be any non-exceptional point of f (which exists by Lemma 2.24) and let $z_1, z_2 \in \mathbb{C}$ be two different preimages, i.e. $f(z_1) = f(z_2) = a$. Notice that both z_1, z_2 exist by Theorem 1.13. For $i = 1, 2$, we have

$$h(z_i) = \frac{f(f(z_i)) - z_i}{f(z_i) - z_i} = \frac{f(a) - z_i}{a - z_i} = c,$$

which implies

$$z_i = \frac{f(a) - ac}{1 - c}$$

is independent of i. Hence, a has only one preimage, leading to contradiction. □

Theorem 4.5. *Let* $f \in \text{Ent}$*, then the Julia set* $\mathcal{J}_f$ *is nonempty. In fact,* $\mathcal{J}_f$ *contains infinitely many points.*

Proof. By Theorem C, there exists $z_1 \in \mathcal{I}_f$. If we also have $z_1 \in \mathcal{J}_f$, then $\mathcal{J}_f \neq \emptyset$ and, since the Julia set is completely invariant, $\mathcal{J}_f$ contains infinitely many points.

Let us assume then that $z_1 \in \mathcal{F}_f$. By Lemma 4.4, f^2 has a fixed point z_2. Since z_1 and z_2 cannot be in the same Fatou component of f, any curve connecting them will meet the Julia set, so $\mathcal{J}_f \neq \emptyset$.

To prove that the Julia set is infinite we note that $\mathcal{I}_f$ contains the orbit of z_1 and hence is infinite. Thus, the same argument as above will show that $\mathcal{J}_f$ is infinite once we show that the set of points with bounded orbit is infinite (since z_2 could belong to the Julia set and be the common meeting point).

For the sake of contradiction, assume that this set is finite. Then f^2 has only finitely many fixed points. Assuming without loss of generality that $z_2 = 0$ so that $f^2(0) = 0$, we find that $g(z) := f^2(z)/z$ defines a transcendental entire function g which takes the value 1 only finitely often. By Theorem 1.13, g has infinitely many zeros. However, a zero of g is also a zero of f^2 and, then, has bounded orbit with respect to f. □

Once we know that $\mathcal{I}_f$ and $\mathcal{J}_f$ are infinite sets, the following result is easily proven.

Theorem 4.6. *Let $f \in \text{Ent}$, then $\partial\mathcal{I}_f = \mathcal{J}_f$.*

Proof. If a Fatou component intersects $\mathcal{I}_f$, then, by normality, it is completely contained in $\mathcal{I}_f$. This implies that $\partial\mathcal{I}_f \subseteq \mathcal{J}_f$.

To prove the opposite inclusion, notice that $\mathcal{I}_f$, $\mathbb{C} \setminus \mathcal{I}_f$ and $\partial\mathcal{I}_f$ are all backward invariant. As seen in the proof of Theorem 4.5, the set of points with bounded orbit and hence $\mathbb{C}\setminus\mathcal{I}_f$ are infinite. This implies that $\partial\mathcal{I}_f$ is also infinite. The conclusion now follows since every closed, backward invariant set with at least three points contains the Julia set (see Theorem 2.25, property *ix*)). □

As shown in Chapter 3, the Julia set of a rational map can, in some cases, be a Cantor set. This phenomenon also occurs in the setting of meromorphic functions. However, the following result, originally proved in [10], demonstrates that such behavior is impossible for transcendental entire functions.

Theorem 4.7. *Let $f \in \text{Ent}$, then $\mathcal{F}_f$ has no unbounded multiply connected component.*

Corollary 4.8. *Let $f \in \text{Ent}$, then $\mathcal{J}_f$ contains nondegenerate continua.*

Since the Julia set is the boundary of the escaping set, it is natural to ask whether similar topological properties hold for $\mathcal{I}_f$ itself. It is relatively straightforward to show that $\mathcal{I}_f$ may contain continua; for instance, the entire real line is contained in the escaping set of the exponential map $f(z) = e^z$. Moreover, even open subsets can lie entirely within I_f, as in the case for $g(z) = 1 + z + e^{-z}$ (see Figure 2). In fact, a result from [32, Corollary 7.21] (presented here as a proposition) confirms that the escaping set necessarily contains continua.

Proposition 4.9. *Let $f \in \text{Ent}$, then $\mathcal{I}_f$ has at least one unbounded connected component. In particular, it contains continua.*

4.2 The exponential family

In the previous section we have established that both the Julia and escaping sets of a transcendental entire function always contain continua. Using the exponential family $E_\lambda = \lambda e^z$, we now aim to show that the escaping set of $f \in \text{Ent}$ can, in fact, consist of infinitely many disjoint continua. In this case, $\mathcal{I}_f$ is structured as the collection of tails of a Cantor bouquet.

4.2.1 Cantor bouquets

Cantor bouquets arise naturally in the study of the escaping set of certain transcendental entire functions, particularly within the exponential family. These topological structures consist of uncountably many disjoint curves, known as hairs, each tending to infinity and accumulating only at infinity. Despite their seemingly intricate nature, Cantor bouquets exhibit a highly organized and well-understood geometry, making them a central object in transcendental dynamics.

Let $E \in \text{Ent}$ be critically finite, i.e. which has finitely many singular values, and denote the set of finite singular values by V. Let $D \subset \mathbb{C}$ be an open disk containing V and let also Γ be the complement of D, i.e. $\Gamma := \mathbb{C} \setminus D$.

Theorem 4.10. *Let $f \in \text{Ent}$ be critically finite. Then,*

i) any connected component T of $E^{-1}(\Gamma)$ is a disk whose closure contains ∞;

ii) the map $E : T \longrightarrow \Gamma$ is a universal covering.

We call a component T of $E^{-1}(\Gamma)$ an *exponential tract*. In such a region, we may write $E(z) = \exp(\phi(z))$, for $z \in T$ and some analytic map ϕ.

Our goal is to understand the behavior of E on the Julia set near infinity. Points leaving $E^{-1}(\Gamma)$ upon iteration fall into D and hence leave a neighborhood of ∞, by Theorem 4.10, so we exclude such points from consideration. Given an exponential tract T, let $J_E(T)$ be the points of the Julia set whose orbit is contained in T, which is a closed and invariant subset of $\mathcal{J}_E$. Our aim is to analyze the topological structure of $J_E(T)$ as well as the dynamics of E on this set.

As in other types of dynamical systems, Cantor sets and shift automorphisms often arise as invariant sets for entire maps. Let $N \in \mathbb{N}$ and define

$$\Sigma_N := \{(s) = (s_0 s_1 s_2 \ldots) \mid s_j \in \mathbb{Z},\ |s_j| \leq N\}.$$

Notice that Σ_N consists of all infinite sequences of integers less than or equal to N in absolute value. It is well-known that, with the product topology, Σ_N is homeomorphic to a Cantor set. The map $\sigma\colon \Sigma_N \longrightarrow \Sigma_N$ defined by $\sigma(s_0 s_1 s_2 s_3 \ldots) := (s_1 s_2 s_3 \ldots)$, i.e. σ "forgets" the first entry of a sequence, is called the shift automorphism. The following properties of σ are commonly known, see [35] for more details.

Proposition 4.11. *Let σ denote the shift map. Then,*

i) periodic points of σ are dense in Σ_N;

ii) σ has a dense orbit, i.e. σ is topologically transitive.

Definition 4.12 (N-bouquet). We call an invariant subset C of $\mathcal{J}_E$ an *N-bouquet* for E if and only if all following conditions are satisfied.

- There exists a homeomorphism $h\colon \Sigma_N \times [0, \infty) \longrightarrow C$.
- $\pi \circ h^{-1} \circ E \circ h(s, t) = \sigma(s)$, where $\pi\colon \Sigma_N \times [0, \infty) \longrightarrow \Sigma_N$ is the projection map.
- $\lim_{t\to\infty} h(s, t) = \infty$.
- $\lim_{n\to\infty} E^n \circ h(s, t) = \infty$ if $t \neq 0$.

An N-bouquet is included naturally in an $(N + 1)$-bouquet by considering only sequences with entries less than or equal to N in absolute value.

The invariance of C requires that $E(h(s,0)) = h(\sigma(s),0)$. Hence, the set of points $\Lambda := h(s,0)$ is an invariant set on which E is topologically conjugate to the shift. We call Λ the *crown* of C. The curve $h_s(t)$ for $t > 0$ is called the *tail* associated to s. We view each tail as comprising a piece of the "stable manifold" of infinity. A *hair* is composed of the union of a point in the crown and its corresponding tail.

The following situation often arises for entire maps. Let C_n be an n-bouquet and assume $C_n \subseteq C_{n+1} \subseteq \cdots$ is an increasing sequence of bouquets. Then, the set

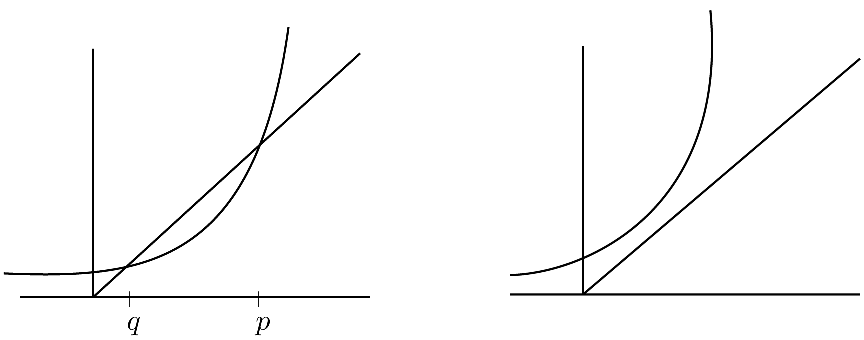
$$C := \overline{\bigcup_{n\geq 0} C_n}$$

is called a *Cantor bouquet.*

In the following section, we construct a Cantor bouquet for the complex exponential family and demonstrate that it coincides with the Julia set. Moreover, we show that the escaping set is precisely composed of the tails of this bouquet.

4.2.2 Constructing a Cantor bouquet

Cantor bouquets arise often in the dynamics of entire maps which are critically finite. We describe the simplest example of a Cantor bouquet through the exponential family. Let $E_\lambda = \lambda e(z)$, where $0 < \lambda < 1/e$. The graph E_λ on $\mathbb{R}$ is shown in Figure 4.1.



(A) $0 < \lambda < 1/e$ (B) $\lambda > 1/e$

FIGURE 4.1: Representation of E_λ in the real line.

Notice that E_λ has two fixed points $q < 1$ and $p > 1$ which are attracting and repelling, respectively. Since E_λ maps the vertical line $x = p$ onto the circle of radius p, it follows that all points with $\operatorname{Re} z < p$ lie in the basin of attraction of q. Hence, $\mathcal{J}_{E_\lambda}$ lies to the right of this line. In fact, more can be said.

Proposition 4.13. *The basin of attraction of q is open and dense in $\mathbb{C}$. Moreover, $\mathcal{J}_{E_\lambda}$ is the complement of this basin.*

Proof. Let us assume that D is an open disk in the complement of the basin of q. Hence, $\operatorname{Re}(E_\lambda^n(z)) \geq p > 1$ for all $n \geq 0$ and all $z \in D$. Therefore, $\left|(E_\lambda^n)'(z)\right| > 1$, so E_λ^n expands D for each n. In particular, there exists $m \geq 0$ such that $E_\lambda^m(D)$ contains a disk of radius larger than 2π. Consequently, there exists $z_0 \in D$ such that $\operatorname{Im}(E_\lambda^m(z_0)) = (2k+1)\pi$, for some $k \in \mathbb{Z}$. Thus, $E_\lambda^{m+1}(z_0) \in \mathbb{R}_-$, contradicting the fact that $\operatorname{Re}\left(E_\lambda^{m+1}(z_0)\right) > 1$. This proves that the complement of the basin of q is nowhere dense. It also shows that the family of iterates of E is not a normal family at any point in the complement, which establishes the second part of the statement. $\square$

We end this section proving that $\mathcal{J}_{E_\lambda}$ is a Cantor bouquet. We first describe the crown of an N-bouquet. Let $N \in \mathbb{N}$, for each $k \in \mathbb{Z}$ with $|k| \leq N$, we construct a rectangle B_k whose boundaries are

- $x = \log(1/\lambda)$ on the left;
- $y = (2k \pm 1)\,\pi$ above and below;
- $x = \nu$ on the right, where ν satisfies $\lambda e^{\nu} - \nu > (2N+1)\,\pi$.

Clearly, there exists ν_0 such that, if $\nu > \nu_0$, (3) holds. The construction described above is illustrated in Figure 4.2. Let us define

$$\Lambda := \left\{ z \in \mathbb{C} \,\middle|\, E_\lambda^i(z) \in \bigcup_{k=-N}^{N} B_k \text{ for all } i \geq 0 \right\}.$$

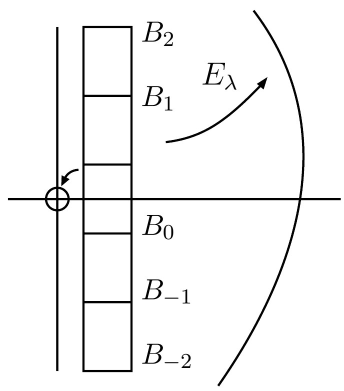


FIGURE 4.2: Representation of the crown construction.

We now show that Λ is homeomorphic to a Cantor set.

Proposition 4.14 (Crown of an N-bouquet).

i) Λ *is homeomorphic to* Σ_N.

ii) $E_\lambda|_\Lambda$ *is conjugate to the shift on* Σ_N.

Proof. Each B_k is mapped by Σ_N to the annulus $1 \leq r \leq \lambda e^{\nu}$ and covers each other B_j. Moreover, $|E_\lambda'(z)| > 1$ for all $z \in \bigcup_{k=-N}^{N} B_k$. Therefore, the preimage of B_j in B_k is a quadrilateral completely contained in the interior of B_k. Standard arguments similar to the Smale horseshow construction, see [35], now complete the proof. A nested sequence of appropriate inverse images, by Theorems 1.14 and 1.17, converges to a unique point in Λ with pre-assigned symbol sequence. $\square$

Therefore, Λ is a Cantor set. We now turn our look to the existence of the tails for points in Λ. Let S_ν denote the strip

$$S_\nu := \{ z \in \mathbb{C} \,|\, \mathrm{Re}(z) \geq \nu,\ |\mathrm{Im}(z)| \leq (2N+1)\,\pi \}.$$

We want to show that if ν is large enough, the set of points whose orbits remain for all time in S_ν is homeomorphic to $\Sigma_N \times [\nu, \infty)$. In order to do so, we require three technical lemmas not proven here. One can see their proofs in [33].

Lemma 4.15. *There exists ν_1 such that if $\nu \geq \nu_1$ and if both $z, E_\lambda(z) \in S_\nu$, then* $\operatorname{Re}(E_\lambda(z)) > 2\operatorname{Re}(z)$.

Lemma 4.16. *Choose $\nu_2 > \nu_1$ such that $\lambda e^{\nu_2} > 1$. Suppose $\nu \geq \nu_2$ and $E_\lambda^j \in S_\nu$ for $j = 0, \ldots, k$, then $\left|\left(E_\lambda^k\right)'(z)\right| \geq \exp \nu\left(2^{k-1} - 1\right)$.*

Let $t \in \mathbb{R}$ and define for any integer k the square $\Delta(t + 2\pi ik)$ of sidelength 2π centred at $t + 2\pi ik$. Notice that the horizontal boundaries of $\Delta(t + 2\pi ik)$ lie along the lines $y = (2k \pm 1)\pi$ and hence are mapped to $\mathbb{R}_-$ by E_λ. We also define the substrip $S_\nu(j) \subseteq S_\nu$ to be

$$S_\nu(j) \coloneqq \{z \in S_\nu \mid (2j - 1)\pi \leq \operatorname{Im}(z) \leq (2j + 1)\pi\}.$$

Lemma 4.17. *Let $\Delta(t + 2\pi ik)$ denote the square of sidelength 2π centred at $t + 2\pi ik$.*

i) Suppose $\nu_3 > \log\left((2N + 2)\pi e^\pi / [\lambda(e^\pi - 1)]\right)$. If $t > \nu_3$, then the image of $\Delta(t + 2\pi ik)$ covers $\Delta(E_\lambda(t) + 2\pi ij)$ for any j with $|j| \leq N$.

ii) Let $\nu > \nu_3 + \pi$. Suppose $E_\lambda^j(t) \in S_\nu$ for $j = 0, 1, \ldots, k$. Let $s_0, s_1, \ldots, s_{k-1}$ be integers with $|s_j| \leq N$. Then, there exists $z \in S_\nu$ with $E_\lambda^j(z) \in S_\nu(s_j)$ and $E_\lambda^k(z) = E_\lambda^k(t)$.

Remark 4.18. The content of the second part of Lemma 4.17 is that E_λ behaves very much like $E_\lambda|_{\mathbb{R}}$ on the set of points whose orbits remain for all time in S_ν.

Let Log be the branch of the natural logarithm taking values in $|\operatorname{Im}(z)| < \pi$. Additionally, let L_k be the branch of the inverse of E_λ which takes values in $S_\nu(k)$, i.e. $L_k(z) \coloneqq \operatorname{Log}(z/\lambda) + 2\pi ik$. Let $s = (s_0 s_1 s_2 \ldots) \in \Sigma_N$. For each $k \geq 0$ we define

$$\Phi_k(s, t) \coloneqq L_{s_0} \circ \cdots \circ L_{s_k} \circ E_\lambda^{k+1}(t),$$

with t sufficiently large. Note that each Φ_k is well-defined by Lemma 4.17.

Proposition 4.19. *Let $\tau > \nu_i + \pi$ for $i = 0, 1, 2, 3$. Then, for each $s \in \Sigma_N$, $\Phi_k(s, t)$ converges uniformly on $[\tau, \infty)$ as k grows to infinity.*

Proof. We have

$$|\Phi_{k+1}(s, t) - \Phi_k(s, t)| = \left|L_{s_0} \circ \cdots \circ L_{s_{k+1}} \circ E_\lambda^{k+2}(t) - L_{s_0} \circ \cdots \circ L_{s_k} \circ E_\lambda^{k+1}(t)\right| =$$

$$= \left|L_{s_0} \circ \cdots \circ L_{s_k}\left(E_\lambda^{k+1}(t) + 2\pi i s_{k+1}\right) - L_{s_0} \circ \cdots \circ L_{s_k}\left(E_\lambda^{k+1}(t)\right)\right|.$$

By Lemma 4.16,

$$|\Phi_{k+1}(s, t) - \Phi_k(s, t)| \leq e^{-\tau\left(2^k - 1\right)}(2N + 1)\pi. \tag{4.6}$$

This gives uniform convergence in t. □

In particular, this proposition defines, for $t \geq \tau$, a continuous curve $\Phi(s, t)$ in S_τ for each $s \in \Sigma_N$. If $r = (s_0 s_1 \cdots s_k r_{k+1} r_{k+2} \cdots) \in \Sigma_N$, then (4.6) implies that $\Phi(s, t)$ is close to $\Phi(r, t)$, thus Φ is continuous in s. Moreover, Φ is easily seen as a one-to-one map on $\Sigma_N \times [\tau, \infty)$. Let $\Lambda_\tau := \{z \in S_\tau \mid E_\lambda^j(z) \in S_\tau \text{ for all } j\}$. We now prove that $\Omega\colon \Sigma_N \times [\tau, \infty) \longrightarrow \Lambda_\tau$ is also surjective. If $z \in \Lambda_\tau$, then there exists a well-defined sequence $s \in \Sigma_N$ associated to z, since the horizontal boundaries of $S_v(s_i)$ are mapped out of S_ν into $\mathbb{R}_-$ by E_λ. Furthermore,

$$\left|\left(E_\lambda^j\right)'(z)\right| \xrightarrow[j\to\infty]{} \infty.$$

Then, any small disk U about z is eventually expanded so that the diameter of its image exceeds 2π. Hence, there is a sequence of points $\{z_k\}_k$ in U which converges to z and which satisfies $E_\lambda^j(z_k) \in S_\nu(s_j)$ for $j < k$ but $E_\lambda^k(z_k) \in \mathbb{R}$. Let us denote by t^* the supremum of the real parts of points in U. It follows immediately that

$$E_\lambda^k(t^*) \geq E_\lambda^k(z_k).$$

Consequently, $z_k = \Phi_k(r_k, t_k)$ for some $r_k \in \Sigma_N$ and $t_k \leq t^*$, and the sequence $\Phi_k(r_k, t_k)$ converges to z. We have proved the following.

Proposition 4.20 (Tails of an N-bouquet). Λ_τ *is homeomorphic to* $\Sigma_N \times [\tau, \infty)$.

Let z be any point whose orbit lies for all time in $S_{\log 1/\lambda}$ and which does not lie in Λ, i.e. z does not lie in the crown of the N-bouquet. It follows that there exists an m for which $E_\lambda^m(z) \in \Lambda_\tau$. Hence, we can pull back the curve in Λ_τ through any such point. It follows that

$$C_n := \left\{z \in \mathbb{C} \,\middle|\, E_\lambda^j(z) \in S_{\log 1/\lambda} \text{ for all } j\right\} = \bigcup_{\tau \geq 0} \Lambda_\tau \cup \Lambda$$

is an N-bouquet, see Figure 4.3. Thus, the Cantor bouquet

$$C := \overline{\bigcup_{n\geq 0} C_n}$$

is the Julia set $\mathcal{J}_{E_\lambda}$, while $\mathcal{I}_f$ is composed by all tails.

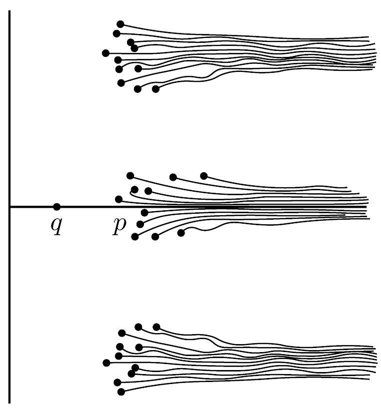


FIGURE 4.3: Schematic representation of the Cantor bouquet C.

Chapter 5

The escaping set of a transcendental meromorphic function

Transcendental meromorphic functions exhibit significantly richer behavior than their entire counterparts. In particular, the presence of poles introduces new dynamical phenomena, including prepoles and logarithmic tracts, which strongly influence the geometry of both the escaping and Julia sets. We begin with foundational results on the escaping set, emphasizing its non-emptiness, structure, and relation to singular values. These topics build upon and extend the theory developed for transcendental entire functions. The first two sections are based on [30, 13], respectively.

5.1 Definitions and properties

In this section, we introduce the basic terminology and fundamental properties relevant to the study of transcendental meromorphic functions. In contrast to entire functions, meromorphic maps may exhibit isolated singularities in the form of poles. The presence and distribution of these poles, as well as of their preimages, i.e. prepoles, play a central role in shaping the global dynamics of these functions. In particular, they significantly influence the structure and topology of the escaping set. We begin by formally defining these notions and exploring their implications for the iteration of meromorphic maps.

Definition 5.1 (Poles, prepoles). Let $f \in \mathcal{M}$, we say a point $z \in \mathbb{C}$ is a *pole* if and only if $f(z) = \infty$. Similarly, we call z a *prepole* if and only if there exists $n \geq 2$ such that $f^n(z) = \infty$. We denote the set of poles and prepoles by $P(f)$.

With the inclusion of poles, some definitions must be updated. For instance, since now orbits $\{f^n(z)\}_{n\geq 0}$ may be truncated at some $m \in \mathbb{N}$, we need to add a condition in the definition of the Fatou set.

Definition 5.2 (Fatou set - meromorphic case). We define the *Fatou set* of f as the points z such that there exists a neighborhood of z where $\{f^n\}_{n\geq 0}$ is defined and normal.

The definition of the Julia set remains unchanged: it is the complement of the Fatou set. However, in the context of transcendental meromorphic functions, this definition implies that the Julia set also contains all poles and prepoles. Notably, not only is $P(f)$ contained in the Julia set, but is in fact dense within it. This fundamental property is formalized in the following result, originally established in [36, 37].

Proposition 5.3. *Let $f \in \mathcal{M}$. Then, $\mathcal{J}_f = \overline{P(f)}$.*

The right inclusion follows from $P(f) \subset \mathcal{J}_f$, since the Julia set is closed. We do not discuss the left inclusion, since it is a direct consequence of [30, Theorem 4]. However, we remark this only holds because f is transcendental, otherwise another assumptions should be made for it to be true.

In general, the theory presented in Chapter 2 remains the same with the aforementioned restrictions. We want to highlight, though, that proofs for transcendental meromorphic functions must be divided into two cases, since functions with finitely many poles behave very differently to the ones with infinitely many. Some results might even hold for only one case, like the following result proven in [13].

Theorem 5.4. *Let $f \in \mathcal{M}$ with finitely many poles. Then, the Julia set cannot be totally disconnected.*

We end this section by introducing the tangent family $f_\lambda(z) := \lambda \tan(z)$, which is an important and elementary example of a transcendental meromorphic function. Though it is already studied in detail in [15, 16], we require some remarks for the proof of Theorem D. In particular, that both the escaping and Julia sets can be totally disconnected in this setting (by Theorem 5.4, f must have an infinity of poles), differing from the case of transcendental entire functions.

In the aforementioned works, the authors proved that if $\lambda \in \mathbb{R}$ and $0 < |\lambda| < 1$, then the Julia set of $f_\lambda(z)$ is a Cantor set. Since the Julia set is the boundary of the escaping set (Theorem 5.9) and the latter is nonempty (Theorem 5.6), the escaping set is also totally disconnected.

To show that the Julia set can be connected (in $\widehat{\mathbb{C}}$) and contain a totally disconnected escaping set, we consider $f_\lambda(z)$ when $\lambda \in \mathbb{R}$ and $\lambda > 1$. In this case, J_{f_λ} is the real line and all other points tend asymptotically to one of two fixed sinks located on the imaginary axis. Hence, all escaping points must belong to the Julia set. However, repelling periodic points are dense in the Julia set (see [30, Theorem 4]), so the escaping set must be totally disconnected.

Therefore, transcendental meromorphic functions are more complicated to study, since their behavior changes drastically when altering their parameters. This also makes the study of escaping points essential to understand global dynamics.

5.2 The escaping set

The aim of this section is to prove the basic results from Section 4.1 in the transcendental meromorphic case. Hence, we shall prove the non-emptiness of both the Julia and escaping sets, in addition to showing that $\mathcal{J}_f$ is the topological boundary of $\mathcal{I}_f$. Additionally, we present Theorem D, whose proof is given in the following section.

We have already mentioned in the previous section that the behavior of transcendental meromorphic functions varies heavily depending on whether the set of poles is finite or not. Therefore, to prove that the escaping set is nonempty we require both Theorem 5.5 and Theorem 5.6.

Theorem 5.5. *Let $f \in \mathcal{M}$ with only finitely many poles. Then, $\mathcal{I}_f \neq \emptyset$. In fact, for any curve γ with $n(\gamma, 0) \neq 0$ and such that $d(0, \gamma)$ is sufficiently large, we have $\mathcal{I}_f \cap \gamma \neq 0$.*

The proof of Theorem 5.5 follows the same approach as Theorem C. Nonetheless, instead of a disk $D(0, r)$ we now consider the annulus $A_\rho := \{R \leq |z| \leq \rho\}$, where $R > 0$ is such that $D(0, R)$ contains the finite set of poles of $f(z)$. One can check [13, Theorem F, Theorem G] for the particular details.

Theorem 5.6. *Let $f \in \mathcal{M}$ with infinitely many poles. Then, $\mathcal{I}_f \neq \emptyset$. In fact, in any neighborhood of a pole there are points of $\mathcal{I}_f$.*

Proof. Let p_1 be a pole of $f(z)$ of order $k \in \mathbb{N}$ and define $D_1 := D(p_1, r_1)$, with $r_1 > 0$. Hence, $f(D_1)$ is a neighborhood of ∞. Since p_1 is a k-fold pole of $f(z)$, for r_1 sufficiently small there exists a simply connected neighborhood $V_1 \subseteq D_1$ of p_1 which is mapped locally univalently except at p_1 onto $\{|f| > R_1\}$, for some $R_1 > 1$. Therefore, we can find a pole p_2 in $\{|z| > R_1\}$ and a disk $D_2 := D(p_2, r_2) \subseteq \{|z| > R_1\}$, with r_2 very small, such that $|p_2| > |p_1|$. By the aforementioned argument, $p_2 \in V_2$, where V_2 is a simply connected neighborhood of D_2 which is mapped locally univalently by f (except at p_2) onto $\{|f| > R_2\}$, for some $R_2 > 2R_1$. We may take a branch of f^{-1} analytic in $\overline{V_2}$ so that $U_1 := f^{-1}\left(\overline{V_2}\right) \subseteq \overline{V_1}$.

By repeating the same argument, we can take a pole p_3 in $\{|z| > R_3\}$ and the disk $D_3 := D(p_3, r_3)$ (with r_3 very small) such that $p_3 \in V_3$, where $V_3 \subseteq D_3$ is a simply connected neighborhood mapped locally univalently by f (except at p_3) onto $\{|f| > R_3\}$, where $R_3 > 2R_2$. We might take a branch of f^{-1} analytic in $\overline{V_3}$ so that $f^{-1}\left(\overline{V_3}\right) \subseteq \overline{V_2}$.

By induction, we get $f^{-n}\left(\overline{V_{n+1}}\right) = U_n \subseteq \overline{V_1} \subseteq D_1$ and $D_n := D(p_n, r_n) \xrightarrow[n\to\infty]{} \infty$, with $n \in \mathbb{N}$. Then,

$$U := \bigcap_{n=1}^{\infty} U_n \neq \emptyset$$

and $\beta \in U$ satisfies $f^n(\beta) \in D_{n+1}$. Therefore, D_1 contains $\beta \in \mathcal{I}_f$. □

Remark 5.7. In Theorem 5.6, we have $p_3 \in V_3$ so that $U_2 = f^{-2}\left(\overline{V_3}\right)$ contains a point $f^{-2}(p_3) = f^{-3}(\infty)$. In general, U_n contains a preimage $f^{-n}(\infty)$ and we could choose r_n so small that U is a single point β. Additionally, since preimages of poles are in the Julia set, we then have $\beta \in \mathcal{J}_f$.

In order to prove Theorem 5.9, we require Lemma 5.8, which is included in [37, Lemma 1]. Notice that there is only one exceptional point for transcendental meromorphic functions, which is the omitted Picard value of Theorem 1.13.

Lemma 5.8. *Let $f \in \mathcal{M}$. For any $q \in \mathcal{J}_f$ and any $p \notin \mathcal{E}(f)$, q is an accumulation point of $\mathcal{O}^-(p)$.*

Theorem 5.9. *Let $f \in \mathcal{M}$. Then, $\mathcal{J}_f = \partial\mathcal{I}_f$.*

Proof. It follows from Theorems 5.5 and 5.6 that $\mathcal{I}_f \neq \emptyset$. Moreover, since for any $z \in \mathcal{I}_f$ all $f^n(z)$, $n \in \mathbb{N}$, are different and in the escaping set, $\mathcal{I}_f$ is infinite. Thus, we can choose three different points $\beta, \gamma, \delta \in \mathcal{I}_f$ such that $f(\beta) = \gamma$ and $f(\gamma) = \delta$. Now take a point $\alpha \in \mathcal{J}_f$ and let V be a neighborhood of α. Since there are at most two exceptional points, by Lemma 5.8 there exists a preimage $\alpha^* \in V$ of one of the points β, γ, δ. The point α^* belongs to the escaping set, so $\mathcal{J}_f \subseteq \overline{\mathcal{I}_f}$. However, periodic points are dense in the Julia set and do not belong to the escaping set. Hence, the interior of $\mathcal{I}_f$ is contained in $\mathcal{F}_f$ and $\mathcal{J}_f \subseteq \partial\mathcal{I}_f$.

To prove the opposite inclusion, let $\alpha \in \partial\mathcal{I}_f$ and assume that $\alpha \in \mathcal{F}_f$ for the sake of contradiction. Let $V \subseteq \mathcal{F}_f$ be a neighborhood of α. Then, V contains points of $\mathcal{I}_f$, all f^n are analytic in V and $f^n \xrightarrow[n\to\infty]{} \infty$ in V. This implies that $V \subseteq \mathcal{I}_f$, which contradicts the assumption that $\alpha \in \partial\mathcal{I}_f$. □

The following result establishes that the escaping set is not an open subset of the complex plane. In the case where the meromorphic function has infinitely many poles, this can be shown easily using Remark 5.7. For functions with only finitely many poles, the argument relies on techniques similar to those employed in Theorem 5.5. For this reason, we omit its proof here.

Theorem 5.10. *Let $f \in \mathcal{M}$, then $\mathcal{I}_f \cap \mathcal{J}_f \neq \emptyset$.*

During this thesis we have seen that the escaping set is always connected for polynomials, and that it contains a continua for transcendental entire functions. We have also seen in the previous section that in the transcendental meromorphic case both the Julia and escaping sets can be totally disconnected. However, there are no instances of a transcendental meromorphic function with a nontrivial Julia set connected (in $\mathbb{C}$ or $\widehat{\mathbb{C}}$), and containing a totally disconnected escaping set. Theorem D provides the first such example.

Theorem D. *Let $f(z) = z - \tan(z)$. Then, $\mathcal{J}_f \cup \{\infty\}$ is connected, while $\mathcal{I}_f \subseteq \mathcal{J}_f$ is totally disconnected.*

Notice that the same argument provided in Section 5.1 cannot work here, since the Julia set is now a fractal.

5.3 Proof of Theorem D

Let us consider

$$f(z) = z - \tan(z)$$

and $k \in \mathbb{Z}$ for the remainder of this section.

The proof of Theorem D is structured in three parts. First, we analyze the Fatou and Julia sets to understand the global dynamics of f. Next, we characterize the escaping set by showing that it consists of points whose real part tend to infinity upon iteration. Finally, we confine $\mathcal{I}_f$ to specific regions where f acts expansively, allowing us to apply the contraction properties of its inverse branches to conclude the argument.

Since $f(z)$ is the Newton's method of $\sin(z)$, we have the result below (which was firstly proven in [38]).

Proposition 5.11. *The Julia set of $f(z)$ is a connected subset of $\widehat{\mathbb{C}}$, i.e. $\mathcal{J}_f \cup \{\infty\}$ is connected. Equivalently, every Fatou component is simply connected.*

It is clear that Proposition 5.11 plays an important role in our proof. Notice that we can restrict our study to a strip of width π because $\tan(z)$ has period π. Indeed,

$$f(z + k\pi) = z + k\pi - \tan(z + k\pi) = z - \tan(z) + k\pi = f(z) + k\pi, \tag{5.1}$$

$$f'(z + k\pi) = 1 - (1 + \tan^2(z + k\pi)) = -\tan^2(z + k\pi) = -\tan^2(z) = f'(z). \tag{5.2}$$

Our next step is to compute the fixed points, poles and asymptotic values of $f(z)$. In fact, since

$$f(z) = z \iff z - \tan(z) = z \iff \tan(z) = 0 \iff z = k\pi,$$

the points $c_k = k\pi$ are the only fixed points of $f(z)$. Moreover, since

$$f'(k\pi) = 0 \ \text{ and } \ f''(k\pi) = 0,$$

they are all superattracting fixed points of order 3 (it is easy to check that $f'''(k\pi) \neq 0$). We also have an infinity of poles at $p_k = \frac{\pi}{2} + k\pi$, i.e. the zeros of $\cos(z)$. In fact, all c_k and p_k are contained in invariant lines, as shown in the following result (see [39]).

Proposition 5.12. *Let $t \in \mathbb{R}$, $r_k := \{c_k + it\}$ and $l_k := \{p_k + it\}$. Then,*

$$f(r_k) = f(c_k + it) = c_k + ig(t) = r_k \subset \mathcal{F}_f,$$

$$f(l_k) = f(p_k + it) = p_k + ih(t) = l_k \cup \{\infty\} \subset \mathcal{J}_f \cup \{\infty\},$$

where $g(t) := t - \tanh(t)$ and $h(t) := t - \operatorname{cotanh}(t)$.

Proof. We begin with the computation for the lines r_k.

$$f(c_k + it) = c_k + it - \tan(c_k + it) = c_k + it - \tan(it) = c_k + it - \frac{1}{i}\frac{e^{-t} - e^{t}}{e^{-t} + e^{t}} =$$

$$= c_k + it - i\frac{e^{t} - e^{-t}}{e^{t} + e^{-t}} = c_k + i\left(t - \tanh(t)\right).$$

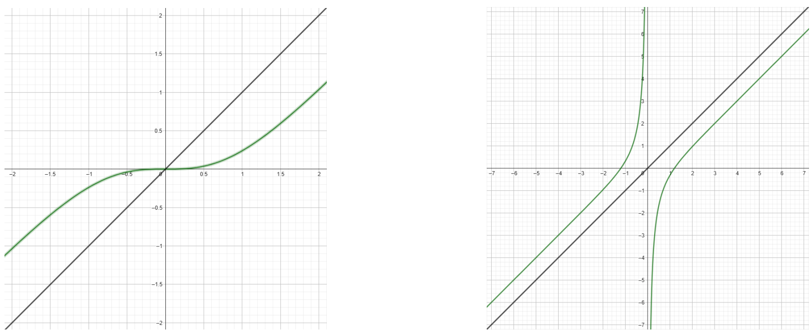
Similarly, we now prove the invariance of l_k.

$$f\left(p_k + it\right) = p_k + it - \tan\left(p_k + it\right) = p_k + it - \tan\left(\frac{\pi}{2} + it\right) =$$

$$= p_k + it - \frac{1}{i}\frac{\exp\left(\frac{\pi}{2}i - t\right) - \exp\left(-\frac{\pi}{2}i + t\right)}{\exp\left(\frac{\pi}{2}i - t\right) + \exp\left(-\frac{\pi}{2}i + t\right)} = p_k + it + i\frac{ie^{-t} + ie^{t}}{ie^{-t} - ie^{t}} =$$

$$= p_k + it - i\frac{1}{\tanh(t)} = p_k + i\left(t - \operatorname{cotanh}(t)\right). \qquad \square$$

We can see the behavior of r_k and l_k by plotting both $g(t)$ and $h(t)$ (see Figure 5.1). Notice that the only special points are c_k and p_k, both at $t = 0$.



(A) $g(t) = t - \tanh(t)$ (B) $h(t) = t - \operatorname{cotanh}(t)$

FIGURE 5.1: Plots of $g(t)$ and $h(t)$ from Proposition 5.12

On the other hand, in [16] the authors showed that the only asymptotic values for $\lambda \tan(z)$ are $\pm\lambda i$, with asymptotic paths $\operatorname{Im}(z) \longrightarrow \pm\infty$. Nonetheless,

$$\lim_{\operatorname{Im}(z)\to\pm\infty} f(z) = \lim_{\operatorname{Im}(z)\to\pm\infty} z - \tan(z) = z \mp i. \tag{5.3}$$

This means that any asymptotic path $\gamma(t)$ must satisfy $\operatorname{Re}(\gamma(t)) \longrightarrow \pm\infty$, but by Proposition 5.12 the path $\gamma(t)$ intersects some invariant line l_k. Thus, $f(z)$ has no asymptotic values aside from ∞.

The following proposition can be found in [39, Example 7.2] and [40, Proposition 4.1].

Proposition 5.13. *Let U_k be the immediate basin of attraction of c_k and denote by J_k the connected component of $\mathcal{J}_f$ in $\mathbb{C}$ containing the line l_k. Then,*

i) the basins U_k are the only periodic components of $\mathcal{F}_f$,

ii) the map $f(z)$ has no wandering domains,

iii) the Julia set $\mathcal{J}_f$ is the union of all J_k.

Therefore, $\mathcal{I}_f \subset \mathcal{J}_f$ and the Fatou and Julia sets (illustrated in Figures 5.2 and 5.3) are of the form

$$\mathcal{F}_f = \bigcup_{k \in \mathbb{Z}} \bigcup_{n \geq 0} f^{-n}(U_k) \quad \text{and} \quad \mathcal{J}_f = \bigcup_{k \in \mathbb{Z}} J_k.$$

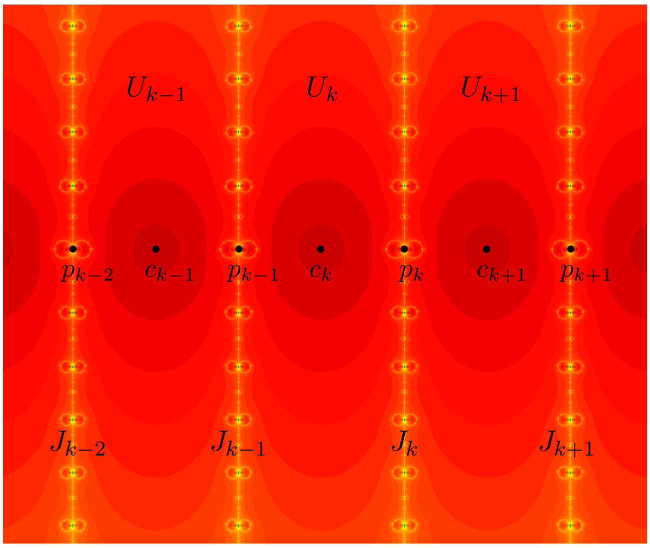


FIGURE 5.2: Dynamical plane of $f(z) = z - \tan(z)$. The Fatou set is composed by the red shaded regions, while the Julia set is the union of the yellow lines and their decorations dividing these regions.

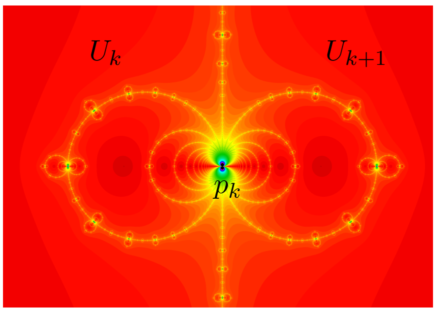


FIGURE 5.3: Dynamics of $f(z) = z - \tan(z)$ near a pole p_k. The red region outside the biggest yellow ∞-shaped line is contained in some basin U_k, while the regions inside of it are preimages of other basins.

Inspired by (5.3), we now show that the absolute value of the real part of z upon iteration grows to infinity if and only if z is in the escaping set. To proceed with the proof, we first require the following lemma, which is a quantitative version of (5.3).

Lemma 5.14. *For all* $\epsilon > 0$*, there exists* $M = M(\epsilon) = \frac{1}{2}\log\left(1+\frac{\sqrt{\epsilon^2+4}}{\epsilon}\right) > \frac{1}{2}\log(2)$ *such that*

i) if $\operatorname{Im}(z) > M$*, then* $|\tan(z) - i| < \epsilon$*;*

ii) if $\operatorname{Im}(z) < -M$*, then* $|\tan(z) + i| < \epsilon$*.*

Proof. i) Let $\epsilon > 0$, we have $|\tan(z) - i| = |\tan(x+iy) - i| =$

$$= \left|\frac{\sin(x+iy)}{\cos(x+iy)} - i\right| = \left|\frac{1}{i}\frac{e^{-y}(\cos(x)+i\sin(x)) - e^{y}(\cos(x)-i\sin(x))}{e^{-y}(\cos(x)+i\sin(x)) + e^{y}(\cos(x)-i\sin(x))} - i\right| =$$

$$= \left|\frac{2\,e^{-y}(\cos(x)+i\sin(x))}{\cos(x)(e^{y}+e^{-y}) + i\sin(x)(e^{-y}-e^{y})}\right| \leq \frac{2\,e^{-y}}{|\cos(x)(e^{y}+e^{-y}) + i\sin(x)(e^{-y}-e^{y})|}.$$

Since

$$\left|\cos(x)\left(e^{y}+e^{-y}\right) + i\sin(x)\left(e^{-y}-e^{y}\right)\right| =$$

$$= \sqrt{\cos^2(x)\,(e^{2y}+e^{-2y}+2) + \sin^2(x)\,(e^{2y}+e^{-2y}-2)} =$$

$$= \sqrt{e^{2y}+e^{-2y}+2\left(\cos^2(x)-\sin^2(x)\right)} = \sqrt{e^{2y}+e^{-2y}+2\cos(2x)} \geq \sqrt{e^{2y}-2}$$

if $y > \frac{1}{2}\log(2)$ (which we show below), we have

$$|\tan(z) - i| \leq \frac{2\,e^{-y}}{\sqrt{e^{2y}-2}} < \epsilon \iff \left(\frac{2\,e^{-y}}{\sqrt{e^{2y}-2}}\right)^2 < \epsilon^2 \iff 4\,e^{-2y} < e^{-2y}\,\epsilon^2 - 2\epsilon.$$

By denoting $e^{2y} = u$ and $e^{-2y} = 1/u$, we get

$$|\tan(z) - i| < \epsilon \iff u^2\epsilon^2 - 2u\epsilon - 4 > 0 \iff u > 1 + \frac{\sqrt{\epsilon^2+4}}{\epsilon},$$

due to $u > 0$. Therefore,

$$e^{2y} > 1 + \frac{\sqrt{\epsilon^2+4}}{\epsilon} \iff 2y > \log\left(1+\frac{\sqrt{\epsilon^2+4}}{\epsilon}\right) \iff y > \frac{1}{2}\log\left(1+\frac{\sqrt{\epsilon^2+4}}{\epsilon}\right).$$

It remains to see

$$M = \frac{1}{2}\log\left(1+\frac{\sqrt{\epsilon^2+4}}{\epsilon}\right) > \frac{1}{2}\log(2), \quad \forall\epsilon > 0.$$

Indeed,

$$M > \frac{1}{2}\log(2) \iff 1 + \sqrt{\frac{\epsilon^2+4}{\epsilon^2}} > 2 \iff \sqrt{1+\frac{4}{\epsilon^2}} > 1,$$

which is satisfied for all $\epsilon > 0$.

If $\operatorname{Im}(z) < -M$, then $y < 0$ and we take the hypothesis $-y > \frac{1}{2}\log(2)$. Thus, the proof of *ii)* is analogous. ☐

Proposition 5.15. *A point* z *belongs to* $\mathcal{I}_f$ *if and only if* $|\operatorname{Re}(f^n(z))| \xrightarrow[n\to\infty]{} \infty$*.*

Proof. The right implication follows from $|\operatorname{Re}(f^n(z))| \leq |f^n(z)|$.

Let us now assume that $z_0 \in \mathcal{I}_f$ and consider $z_n := f^n(z_0) = x_n + iy_n$. Now fix $\epsilon > 0$ very small and $M > 0$ be as in Lemma 5.14. Then, for all z such that $\text{Im}(z) > M$, there exists $N \in \mathbb{N}$ such that $\text{Im}\left(f^n(z)\right) < M$ for all $n \geq N$. Indeed,

$$|z - i - f(z)| = |\tan(z) - i| < \epsilon \Longrightarrow \text{Im}(z) - 1 - \epsilon \leq \text{Im}\left(f(z)\right) \leq \text{Im}(z) - 1 + \epsilon.$$

On the other hand, for all z such that $\text{Im}(z) < -M$, there exists $N' \in \mathbb{N}$ such that $\text{Im}\left(f^n(z)\right) > -M$ for all $n \geq N'$ due to

$$|z + i - f(z)| = |\tan(z) + i| < \epsilon \Longrightarrow \text{Im}(z) + 1 - \epsilon \leq \text{Im}\left(f(z)\right) \leq \text{Im}(z) + 1 + \epsilon.$$

Therefore, there exists $m \in \mathbb{N}$ such that $\{y_n\}_{n\geq m}$ is bounded, so it must be $|x_n|$ who grows to ∞. □

We now want to determine the regions on which $|f'(z)| > 1$. To tell such points apart, let us define the following strips (represented in Figure 5.4)

$$A_k := \left\{-\frac{\pi}{4} + k\pi \leq \text{Re}(z) \leq \frac{\pi}{4} + k\pi\right\},$$
$$T_k := \left\{\frac{\pi}{4} + k\pi < \text{Re}(z) < \frac{3\pi}{4} + k\pi\right\}.$$

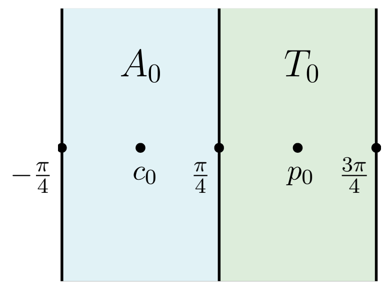


FIGURE 5.4: Representation of the strips A_0 and T_0.

Proposition 5.16. *Let $t \in \mathbb{R}$ and $l_\pm = \{\pm\frac{\pi}{4} + it\}$. Then,*

i) $|f'(z)| \leq 1$ *if and only if* $z \in A_k$, *and hence* $|f'(z)| > 1$ *if and only if* $z \in T_k$;

ii) $f(l_\pm) = f\left(\pm\frac{\pi}{4} + it\right) = \pm\left(\frac{\pi}{4} - \frac{1}{\cosh(2t)}\right) + i\left(t - \tanh(2t)\right)$;

iii) $A_k \subset U_k \subset \mathcal{F}_f$.

Proof. *i*) By taking the expression in (5.2), we get

$$|f'(z)| = |-\tan^2(z)| > 1 \Longleftrightarrow |\tan(z)|^2 > 1 \Longleftrightarrow |\sin(z)|^2 > |\cos(z)|^2.$$

Since

$$|\sin(z)|^2 = |\sin(x + iy)|^2 = \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y)$$
$$|\cos(z)|^2 = |\cos(x + iy)|^2 = \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y),$$

we have

$$|f'(z)| > 1 \iff \left(\sin^2(x) - \cos^2(x)\right)\left(\cosh^2(y) - \sinh^2(y)\right) > 0.$$

But $\cosh^2(y) - \sinh^2(y) = 1$, so

$$|f'(z)| > 1 \iff \sin^2(x) > \cos^2(x) \iff |\sin(x)| > |\cos(x)| \iff |\sin(x)| > \frac{\sqrt{2}}{2},$$

and the result follows from

$$|\sin(x)| > \frac{\sqrt{2}}{2} \iff z \in T_k.$$

ii) Let $z = \frac{\pi}{4} + it$, then

$$f(z) = \frac{\pi}{4} + it - \frac{\sin\left(\frac{\pi}{4} + it\right)}{\cos\left(\frac{\pi}{4} + it\right)} = \frac{\pi}{4} + it - \frac{\sin\left(\frac{\pi}{4}\right)\cosh(t) + i\cos\left(\frac{\pi}{4}\right)\sinh(t)}{\cos\left(\frac{\pi}{4}\right)\cosh(t) - i\sin\left(\frac{\pi}{4}\right)\sinh(t)} =$$

$$= \frac{\pi}{4} + it - \frac{\cosh(t) + i\sinh(t)}{\cosh(t) - i\sinh(t)}\,\frac{\cosh(t) + i\sinh(t)}{\cosh(t) + i\sinh(t)} =$$

$$= \frac{\pi}{4} + it - \frac{\cosh^2(t) - \sinh^2(t) + 2i\sinh(t)\cosh(t)}{\cosh^2(t) + \sinh^2(t)} =$$

$$= \frac{\pi}{4} + it - \frac{1 + i\sinh(2t)}{\cosh(2t)} = \frac{\pi}{4} - \frac{1}{\cosh(2t)} + i\left(t - \tanh(2t)\right).$$

The case when $z = -\frac{\pi}{4} + it$ is analogous.

iii) Since $\cosh(2t) \geq 1$ for all t, the points in the boundary of A_k are sent to the interior of A_k. Thus, the strips A_k are contained in $U_k \subset \mathcal{F}_f$. □

Let us define

$$d := \min_{z \in J_0}\left|\mathrm{Re}(z) - \frac{\pi}{4}\right| = \min_{z \in J_0}\left|\frac{3\pi}{4} - \mathrm{Re}(z)\right|,$$

where both terms coincide by the symmetry of the function $f(z)$.

Now denote by q_- and q_+ the points where d is achieved in $\{\frac{\pi}{4} < \mathrm{Re}(z) \leq \frac{\pi}{2}\}$ and $\{\frac{\pi}{2} \leq \mathrm{Re}(z) < \frac{3\pi}{4}\}$, respectively, and consider $\epsilon = d$ and $M(d)$ from Lemma 5.14. We define $B_k := B_0 + k\pi$, where B_0 is the rectangle delimited by the horizontal lines $\{\mathrm{Im}(z) = \pm M(d)\}$ and the vertical lines $\{\mathrm{Re}(z) = q_\pm\}$ (see Figure 5.5). The following result ensures that only points in $J_k \cap B_k$ can leave J_k after one iterate.

Proposition 5.17. *Let $z \in J_k \setminus B_k$, then $f(z) \in J_k$.*

Proof. By Lemma 5.14, for all $z \in J_k$ with $\mathrm{Im}(z) > M(d)$, we have $|\tan(z) - i| < d$. Then, $w := z - i - f(z) = \tan(z) - i$ also satisfies $|w| < d$, while $z - f(z) = w + i$. Thus,

$$|\mathrm{Re}\left(z - f(z)\right)| = |\mathrm{Re}(w)| \leq |w| < d. \tag{5.4}$$

The same argument works for all $z \in J_k$ with $\mathrm{Im}(z) < -M(d)$. In particular, (5.4) holds for all $z \in J_k \setminus B_k$. The result follows from the fact that J_k is the only connected component of $\mathcal{J}_f$ in the strip T_k. □

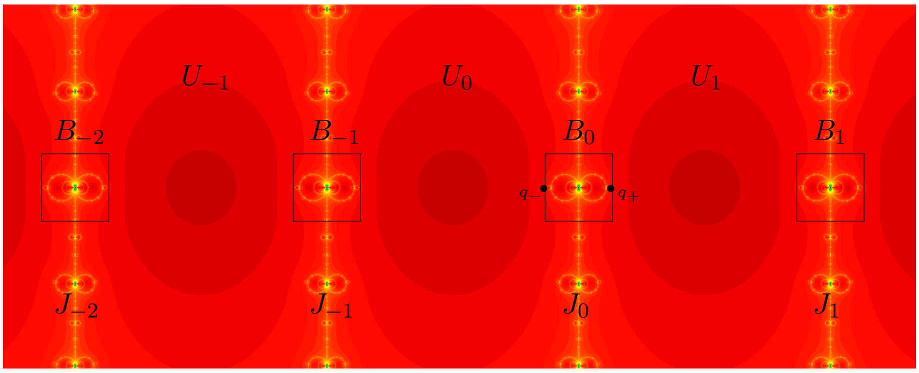


FIGURE 5.5: Dynamical plane of $f(z) = z - \tan(z)$ highlighting the construction of the regions B_k (the black boxes).

Since $\mathcal{I}_f \cap P(f) = \emptyset$, we may define an *itinerary* for $z \in \mathcal{J}_f \setminus P(f)$

$$s(z) := \{(s_0 s_1 s_2 \cdots) \,|\, s_i \in \mathbb{Z}\},$$

where $s_i(z) = k$ if and only if $f^i(z) \in J_k$. Let us also denote by $\rho_n(z)$, with $n \in \mathbb{N}$, the number of changes in the entries of the itinerary of z after n iterates. Formally,

$$\rho_n(z) := \#\{0 \leq i < n \,|\, s_i(z) \neq s_{i+1}(z)\}.$$

The following proposition, which is a direct consequence of Propositions 5.15 and 5.17, ensures that any $z \in \mathcal{I}_f$ satisfies that $\rho_n(z) \xrightarrow[n\to\infty]{} \infty$.

Proposition 5.18. *Let $z \in \mathcal{J}_f \setminus P(f)$. Then, the following are equivalent.*

i) $z \in \mathcal{I}_f$,

ii) $|s_n(z)| \xrightarrow[n\to\infty]{} \infty$,

iii) $\#\{i \in \mathbb{Z} \,|\, f^n(z) \in J_i,$ *for some* $n \geq 0\} = \infty$.

In particular, for any $z \in \mathcal{I}_f$ we have

$$\lim_{n\to\infty} \rho_n(z) = \infty.$$

Our next step is to define some branches of the inverse map. We start with the branches of the inverse f^{-1} mapping T_k to itself, there are two of them. If we denote by T_k^+ the connected component of $f^{-1}(T_k)$ containing $l_k \cap \{\mathrm{Im}(z) > 0\}$ and by T_k^- the one containing $l_k \cap \{\mathrm{Im}(z) < 0\}$, there exist φ_k^+, φ_k^- such that

- $f\colon T_k^\pm \longrightarrow T_k$ is conformal,
- $\varphi_k^\pm\colon T_k \longrightarrow T_k^\pm$ is conformal,
- $f \circ \varphi_k^\pm = \mathrm{Id}$.
- $\left|\left(\varphi_k^\pm\right)'(z)\right| < 1.$

Let us now define

$$\Omega_k' := \left\{\left|\mathrm{Re}(z) - \left(\frac{\pi}{2} + k\pi\right)\right| > \frac{3\pi}{4}\right\}.$$

Since a neighborhood of a pole is mapped to the whole complex plane one-to-one, there exist $\Omega_k \subset T_k$ and ϕ_k, such that

- $f\colon \Omega_k \longrightarrow \Omega'_k$ is conformal,
- $\phi_k\colon \Omega'_k \longrightarrow \Omega_k$ is conformal,
- $f \circ \phi_k = \mathrm{Id}$.

This construction is illustrated in Figure 5.6. We now show that the branches of the inverse $\varphi_k^+, \varphi_k^-, \phi_k$ are contractive in T_k.

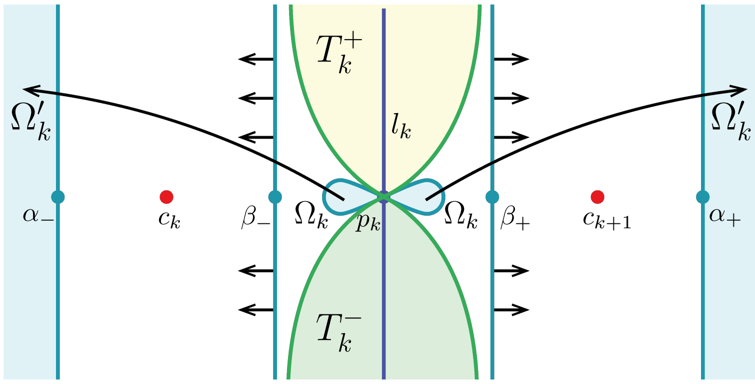


FIGURE 5.6: Representation of the dynamics of $f(z) = z - \tan(z)$ near a pole p_k. The points $\alpha_\pm, \beta_\pm$ denote $p_k \pm \frac{3\pi}{4}$ and $p_k \pm \frac{\pi}{4}$, respectively.

Lemma 5.19. *Let $* \in \{+,-\}$. Then, there exists $\lambda < 1$ such that*

i) $\left|(\varphi_k^*)'(z)\right| < 1$ *for all* $z \in T_k$,

ii) $\left|(\phi_k)'(z)\right| \leq \lambda$ *for all* $z \in \Omega'_k \cap f(B_k)$.

Proof. *i*) Follows from Proposition 5.16.

ii) Since $\frac{\pi}{2}$ is a pole of $|f'(z)|$, there exist $\delta > 0$ and $\mu_1 > 1$ such that $|f'(z)| \geq \mu_1$ for all $z \in D\left(\frac{\pi}{2}, \delta\right)$ and $B_0 \cap D\left(\frac{\pi}{2}, \delta\right) \neq \emptyset$. By (5.2), we may take the same δ, μ_1 for all k. Hence, we can apply Weierstrass Theorem in $D_k := B_k \setminus D(p_k, \delta)$ in order to obtain $|f'(z)| \geq \mu_2 > 1$ for all $z \in D_k$.

Now define $\mu := \min\{\mu_1, \mu_2\}$, which satisfies $|f'(z)| \geq \mu > 1$ for all $z \in B_k$. Therefore,

$$\left|(\phi_k)'(z)\right| \leq \frac{1}{\mu} =: \lambda < 1,$$

for all $z \in \Omega'_k \cap f(B_k)$. □

We can now prove the main result.

Proof of Theorem D. We have already proven that the Julia set is connected in $\widehat{\mathbb{C}}$ and that $\mathcal{I}_f \subset \mathcal{J}_f$. It remains to see that $\mathcal{I}_f$ is a totally disconnected set.

Let us denote by V_0 a connected component of the escaping set and $V_n := f^n(V_0)$. Since the images of connected components are contained in connected components, we have $s(z) = s(w)$ for all $z, w \in V_0$. Thus, we shall consider $s(V_0) := s_0 s_1 s_2 \ldots$ and $\rho(V_0) := \rho_1 \rho_2 \rho_3 \ldots$

Since $s(V_0)$ undergoes infinitely many changes by Proposition 5.18, we can assume without loss of generality that for a given $n \geq 0$ we have $s_n \neq s_{n+1}$. Then, $V_n \subset B_{s_n}$ and

$$V_0 \subset \psi_0 \circ \cdots \circ \psi_{\rho_n} \left(B_{s_n} \right),$$

where ψ_0 is the composition of the φ_k^*'s prior to the first change in $s(V_0)$. Moreover, for all $i \in \{1, \ldots, \rho_n\}$, ψ_i denotes the composition of one ϕ_k and the φ_k^*'s before the $(i+1)$-th change in $s(V_0)$. Hence, we have exactly one change in $s(V_0)$ for each ψ_k applied (except ψ_0).

Now denote by $m_i \geq 0$, with $i \in \mathbb{N}$, the number of iterations between the $(i-1)$-th and i-th changes in $s(V_0)$. For $n = m_1$ and $n = m_1 + m_2 + 1$, respectively, we have

$$V_0 \subset \psi_0 \left(B_{s_{m_1}} \right) = \varphi_{s_0}^* \circ \overset{m_1)}{\cdots} \circ \varphi_{s_0}^* \left(B_{s_0} \right),$$

$$V_0 \subset \psi_0 \circ \psi_1 \left(B_{s_{m_1+m_2+1}} \right) = \psi_0 \circ \phi_{s_0} \circ \varphi_{s_{m_1+1}}^* \circ \overset{m_2)}{\cdots} \circ \varphi_{s_{m_1+1}}^* \left(B_{s_{m_1+m_2+1}} \right),$$

where $* \in \{+, -\}$ depending on each particular case.

The result now follows from Lemma 5.19, since

$$\operatorname{diam}(V_0) \leq \operatorname{diam}\left(\psi_0 \circ \cdots \circ \psi_{\rho_n}\left(B_{s_n}\right)\right) \leq \lambda^{-\rho_n} \operatorname{diam}\left(B_{s_n}\right) \xrightarrow[n \to \infty]{} 0. \qquad \square$$

Notice that we have also proved that points are the only connected components of the Julia set which undergo an infinite number of changes in their itinerary.

We end this thesis by showing, as a curiosity, that every point of $\mathcal{I}_f$ is a buried point. For a deeper study on such points, see [41].

Definition 5.20 (Buried points). We say $z \in \mathcal{J}_f$ is a *buried point* if and only if it does not belong to the boundary of a Fatou component. The set of all buried points of $\mathcal{J}_f$ is called the *residual Julia set*, and is denoted by $\mathcal{J}_f'$.

Proposition 5.21. *The escaping set $\mathcal{I}_f$ is a subset of the residual Julia set $\mathcal{J}_f'$.*

Proof. We shall see that if $z \in \mathcal{J}_f$ is not a buried point, then $z \notin \mathcal{I}_f$. By Proposition 5.13, the Fatou set is the union of U_k and their preimages, i.e.

$$\mathcal{F}_f = \bigcup_{k \in \mathbb{Z}} \bigcup_{n \geq 0} f^{-n}(U_k).$$

Now assume $z \in \partial f^{-n^*}(U_{k^*})$, for some $k^* \in \mathbb{Z}$, $n^* \geq 0$. Since the Julia set is invariant and all U_k are periodic, we have

$$z \in \partial f^{-n^*}(U_{k^*}) \Longrightarrow f^{n^*}(z) \in \partial U_{k^*} \Longrightarrow f^m(z) \in \partial U_{k^*}, \text{ for all } m \geq n^*.$$

However, $\partial U_{k^*} \subset J_{k^*-1} \cup J_{k^*}$ and the result follows from Proposition 5.18. $\square$

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