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Proportionality between allocations in asset management

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Abstract

Asset allocation refers to deciding the optimal participation of each asset within a portfolio. Therefore, these participations are a composition, and compositional methods should be used to treat the data and perform analysis over it. When trying to find relationships between parts of a composition, proportions have shown to be more suitable than correlations. In this paper, using a previous proportionality index as starting point, two new indexes are proposed and all of them are used to analyze the asset allocation in a portfolio composed of five stocks from the IBEX 35 (the Spanish stock market index). Results shed light on the connection between volatility, allocations and their proportionality.

JEL Classification: C01, E22, G11.

Keywords: Aitchison geometry, capital allocation, portfolio theory, proportionality.

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1 Introduction

Asset and capital allocation are important parts of portfolio and risk management for any business. From one side, asset allocation is related to the optimal portfolio selection, introduced by Markowitz (1952), and refers to the construction of portfolios that maximize the expected returns while minimizing risk. On the other hand, within Enterprise Risk Management (ERM), risk capital allocation is the partition of a specific capital (usually the capital requirements set by regulatory entities) among the different sources of risk (Dhaene et al., 2012). The set of guidelines to solve this kind of problems are commonly called capital allocation principles.

An important difference between asset and capital allocation is that, while capital allocation is derived from analyzing the contributions of individual risks to potential aggregate losses, asset allocation is an investment decision aiming at maximizing profit with minimum risk. Notwithstanding the differences, both consider interconnectedness of the components and their risk to define participations. This means that, for either analysis, the relative information of the components is highly relevant, which is the core of compositional data analysis. By acknowledging this compositional nature, linear dependency measures, such as the widely used Pearson correlation, become unsuitable to analyze the relationships between the components, an issue that dates back to the spurious correlation revealed by Pearson (1897) and further analyzed by Chayes (1960).

This paper focuses on asset allocation. Following its construction, the Markowitz (1952, 1991) minimum variance portfolio principle has a compositional nature and, hence, any analysis needs to acknowledge this characteristic. However, up until now there has been no compositional analysis on asset allocation. Conversely, although still incipient, the compositional nature of capital allocation has been analyzed by Belles-Sampera et al. (2016), Boonen et al. (2019), Fiori and Porro (2023) and Fiori and Rosazza Gianin (2025), for example, showing how the compositional approach can be a relevant and desirable tool for this matter.

When working with compositional data, it is necessary to use what is known as the Aitchison geometry (Aitchison, 1986), a framework for treating this kind of data according to its particularities. Among other aspects, Aitchison (1986) stated proportionality as a more appropriate way of measuring the relationship between parts of a composition, unraveling the issue of spurious correlation mentioned before. The logratio variance was defined as an indicator of this proportionality, using natural logarithms. Following this approach, Lovell et al. (2015) showed that proportions are more suitable than correlations to find relationships between parts of compositions, whilst Egozcue and Pawlowsky-Glahn (2023) assessed different functional forms to improve the interpretability of the logratio variance and formulated the proportionality index of parts (PIP). Although theoretically bounded between zero and one, the PIP would require the logratio variance to approach infinity to reach zero, which is unlikely in practice. Therefore, the distance of the index value from the lower bound provides only limited information.

In this paper, the logratio variance is revisited from the point of view of subcompositional coherence to propose two new proportionality indexes. This approach focuses on pairs of parts of the composition, which are the smallest subcomposition possible, to derive the maximum and minimum limits for the logratio variance and obtain an interpretable index: the pairwise proportionality of logratios (PPL). Additionally, the two-parts composition can be analyzed directly, without recurring to the logratio variance. Thus, since the maximum and minimum participations are one and zero, respectively, they can be seen as following a Bernoulli distribution. This allows to define limits for the variance of the participations and, therefore, a new proportionality index: the pairwise proportionality index (PPI).

Despite the desirable properties of the logratio variance to measure proportionality, there is still a lifelong issue: the presence of zeros in compositions. This issue has been covered largely in compositional analysis with different approaches such as Martín-Fernández et al. (2014), **?** and Porro et al. (2025), for example. Proportionality measures relying on log-ratio variance are subject to this limitation. In this context, the PPI shows the advantage of being defined even in the presence of zeros in the composition.

The optimal weights for a portfolio composed by the top 5 stocks by market capitalization traded in the Spanish stock market are calculated daily using the minimum variance portfolio principle, from January 1st, 2021, to December 31st, 2024. Then, the three proportionality measures are applied to the compositions obtained and the relationships between pairs of assets are analyzed, as well as the connection of these proportionalities with the asset allocation method used, extending the assessment to an exercise excluding one of the assets.

Results vary for the proportionality measures used. When analyzing the PIP and PPL, the allocations showed low proportionalities, while for the PPI the participations showed moderate and high proportionalities. However, in all cases one of the assets showed consistently higher proportionalities to the others (Iberdrola - IBE). A deeper look into this, revealed that this stock exhibits the lowest volatility for most of the period under analysis, which would be consistent with the expectation that the allocation method would assign the less volatile asset a higher participation and assign the others using this one as benchmark.

The remaining of the paper is structured as follows: Section 2 explains the minimum variance allocation method and section 3 the compositional data framework, as well as the three proportionality measures used for the analysis. Section 4 elaborates on the data and the empirical approach, while section 5 shows the results and section 6 concludes.

2 Mean-Variance portfolio theory

Portfolio and risk management include a large set of components, from which asset and capital allocation are particularly relevant. The first to discuss asset allocation and introduce modern portfolio theory was Markowitz (1952), who analyzed the trade-off between the two objectives of investors: maximize the expected returns and minimize risk. This is the main idea of the Mean-Variance portfolio theory, which tries to find all portfolios that satisfy the two objectives. Therefore, it can be formulated as the maximization of the returns given a maximum accepted risk or as the minimization of risk given a minimum expected return. For the sake of this analysis the latter formulation will be used, which is the minimum variance portfolio approach. Hence, the problem can be expressed as in Markowitz (1952, 1991):

$$\min_{\boldsymbol{w}} \quad \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$

Subject to:

$$\boldsymbol{w}^\top \mathbf{1} = 1 \quad \text{(Fully invested)}$$

where $\boldsymbol{w}$ is the vector of asset weights and $\boldsymbol{\Sigma}$ is the covariance matrix of asset returns[1]. The solution of this problem is a set of weights $\boldsymbol{w}^*$ that contains the participation of each asset within the total of the allocation. Therefore, $\boldsymbol{w}^*$ has a compositional nature, as explained in more detail

[1] It is possible to add the constraint $\boldsymbol{w}^\top \boldsymbol{\mu} = \mu$ for target return μ.

in the next section. As such, when assessing the existence of relationships between the components of the allocation, correlation is not the most suitable tool and proportions are a better approach, as concluded by Lovell et al. (2015).

3 Compositional data and proportionality

A composition is a vector $X = [x_1, \dots, x_n]$ of size $n > 1$, whose elements carry only relative information, such that $\sum_{j=1}^{n} x_j = \kappa$ (Pawlowsky-Glahn et al., 2015). Therefore, it is defined in the simplex:

$$\mathcal{S}^n = \{X \in \mathbb{R}^n | x_i \geq 0, i = 1, ..., n, \sum_{i=1}^{n} x_i = \kappa\} \tag{1}$$

For simplicity, the value of κ is set to 1, without loss of generality, by using the closure operation:

$$\begin{aligned} \mathcal{C}(X) &= \left[\frac{\kappa x_1}{\sum_{i=1}^{n} x_i}, \frac{\kappa x_2}{\sum_{i=1}^{n} x_i}, \dots, \frac{\kappa x_n}{\sum_{i=1}^{n} x_i}\right] \in \mathcal{S}^n \\ \mathcal{C}(X) &= \left[\frac{x_1}{\sum_{i=1}^{n} x_i}, \frac{x_2}{\sum_{i=1}^{n} x_i}, \dots, \frac{x_n}{\sum_{i=1}^{n} x_i}\right] \in \mathcal{S}^n \end{aligned} \tag{2}$$

In the same line, a subcomposition is defined as a subset of components from a composition Aitchison (1982). Thus, for a subset of components $m \subseteq [1, \dots, n]$ the subcomposition is[2]:

$$X_m = [x_j]_{j \in m}, \quad \text{for } m \subseteq [1, \dots, n] \tag{3}$$

This implies that the size of X_m must be larger than one (by the definition of the composition X) and smaller or equal to n. Then, by applying the closure operation:

$$\mathcal{C}(X_m) = \left[\frac{x_i}{\sum_{j \in m} x_j}\right]_{i \in m} \tag{4}$$

This means that the subcomposition can be seen as a composition itself and can be analyzed independently.

Considering the nature of compositional data, Aitchison (1983, 1984, 1986) defined desirable features of compositional methods. One of these characteristics is subcompositional coherence, which implies that the conclusions obtained for elements in a subcomposition should be equal to those obtained in the full composition.

More formally (Egozcue and Pawlowsky-Glahn, 2023), a function $f_n : \mathcal{S}^n \to \mathbb{R}$ is subcompositionally coherent if it is:

1. Scale invariant: For a positive real constant α, if it holds that $f_n(\alpha \cdot X) = f_n(X)$, then it is scale invariant. This, translated into compositional terms means that for $f_m : \mathcal{S}^m \to \mathbb{R}$ containing only the parameters of the elements in the subset m, it should hold that $f_n(X) = f_m(X_m)$ for any subcomposition X_m. This is referred to as an invariant function under subcomposition.

[2]Note that they do not necessarily correspond to the first m parts of the composition.

2. Subcompositionally dominant: If it holds that either $f_n(X) \geq f_m(X_m)$ (non-increasing dominance) or $f_n(X) \leq f_m(X_m)$ (non-decreasing dominance), then f_n is subcompositionally dominant with respect to f_m under the subcomposition X_m.[3]

3.1 Association between components

When assessing relationships between the parts of a composition, Pearson's linear correlation is not an appropriate measure, as concluded by Lovell et al. (2015). The authors refer to the Aitchison (1986) observation that correlations can lead to misleading conclusions because they depend on the specific parts of the composition considered. This issue relates to the problem of spurious correlation highlighted by Pearson (1897) and later addressed by Chayes (1960), who further explored the closure problem for certain variables. Lovell et al. (2015) also provide a geometrical explanation on how correlations are a linear measure, not suitable for the characteristics of the simplex $\mathcal{S}^n$.

To overcome this limitation of linear correlation as a measure of dependence when dealing with compositional data, Aitchison (1986) proposed the logratio variance as a measure of association between components. For two components x_i and x_j, the logratio variance is defined as $\tau_{ij} = Var\,(log(x_i/x_j))$, using the natural logarithm. The logratio variance equals zero when the two variables are exactly proportional. Following Lovell et al. (2015), it is possible to compare pairwise proportionalities among several variables and determine which pairs are more proportional than others. This measure has desirable properties; in particular, it satisfies subcompositional coherence, since the ratios x_i/x_j are independent of the other components.

Despite these advantages, the functional form of the logratio variance has two drawbacks. First, this indicator has no upper bound, which means its magnitude alone provides limited information about the degree of proportionality of two components. Second, it is undefined in the presence of zeros within the composition. This issue has been widely discussed in the compositional data literature; see, for example, Martín-Fernández et al. (2014), **?** and Porro et al. (2025).

3.2 Proportionality Index of Parts

To overcome the lack of scale, Egozcue and Pawlowsky-Glahn (2023) proposed a set of measures of association based on the logratio variance, aiming at surpassing the interpretability issue, and evaluate which measure performs best. The result is the Proportionality Index of Parts (PIP), which is defined as:

$$PIP(i,j) = \frac{1}{1+\sqrt{\tau_{ij}}} \tag{5}$$

The PIP is defined in the range (0, 1), where 0 indicates no proportionality and 1 indicates perfect proportionality. Indeed, these bounds indicate that the PIP takes the value of 1 if $\tau_{ij} = 0$, i.e., when logratios $log(x_i/x_j)$ are constant (full proportionality). However, for the PIP to take the value of zero, it would be necessary that τ_{ij} goes to infinity, which is unlikely to occur in practice.

[3] A full, detailed definition of subcompositional coherence can be found in Section 3 of Egozcue and Pawlowsky-Glahn (2023).

3.3 Pairwise Proportionality of Logratios

As the lower value of PIP is only reached when τ_{ij} goes to infinity, it is worth investigating alternative measures of proportionality. Following the properties of logarithms and variances, it is possible to derive finite upper and lower limits for τ_{ij} by decomposing it as follows:

$$\begin{aligned}
\tau_{ij} &= Var\left(log(x_i/x_j)\right) \\
\tau_{ij} &= Var\left(log(x_i) - log(x_j)\right) \\
\tau_{ij} &= Var\left(log(x_i)\right) + Var\left(log(x_j)\right) - 2Cov\left(log(x_i), log(x_j)\right)
\end{aligned} \tag{6}$$

Now, since the interest is in the relationship between pairs of parts of the composition, it is possible to consider the subcomposition formed only by x_i and x_j, which is the minimal subcomposition possible with two elements, and apply the closure operation, following the definitions in Equations 3 and 4 to have:

$$\begin{aligned}
\tilde{x}_{ij} &= \left(\frac{x_i}{x_i + x_j}\right) \\
\tilde{x}_{ji} &= \left(\frac{x_j}{x_j + x_i}\right)
\end{aligned} \tag{7}$$

Now, replacing x_i and x_j for $\tilde{x}_{ij}$ and $\tilde{x}_{ji}$, respectively, in Equation 6[4]

$$\tau_{ij} = Var\left(log(\tilde{x}_{ij})\right) + Var\left(log(\tilde{x}_{ji})\right) - 2Cov\left(log(\tilde{x}_{ij}), log(\tilde{x}_{ji})\right) \tag{8}$$

Since the covariance in Equation 8 is always negative[5], this new relationship has a lower bound which is attained when the $Cov\left(log(\tilde{x}_{ij}), log(\tilde{x}_{ji})\right) = 0$, implying the highest possible relationship between the variables. Consequently, it is possible to define:

$$\tau_{ij}^{MIN} = Var\left(log(\tilde{x}_{ij})\right) + Var\left(log(\tilde{x}_{ji})\right) \tag{9}$$

On the other hand, regarding the maximum value τ_{ij} can take, following the Cauchy–Schwarz inequality [6], the limit for the covariance is given by the covariance inequality (Keener, 2010) $|Cov(A, B)| \leq \sqrt{Var(A)} \cdot \sqrt{Var(B)} = \sigma_A \cdot \sigma_B$, where $\sigma_A = \sqrt{Var(A)}$ and $\sigma_B = \sqrt{Var(B)}$ are the standard deviations of A and B, respectively. This implies that:

$$\tau_{ij}^{MAX} = Var\left(log(\tilde{x}_{ij})\right) + Var\left(log(\tilde{x}_{ji})\right) + 2\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})} \tag{10}$$

Setting these limits for τ_{ij} allows for a new definition of the proportionality as the difference between its maximum and the observed values, normalized by its full range, the Pairwise Proportionality of Logratios (PPL):

[4]This will not change the relationship τ_{ij}, since it is invariant under the subcomposition X_m (Egozcue and Pawlowsky-Glahn, 2023). It is straightforward to see that since $log(\tilde{x}_{ij}/\tilde{x}_{ji}) = log\left(\frac{x_i/(x_i+x_j)}{x_j/(x_j+x_i)}\right) = log(x_i/x_j)$, the value of τ_{ij} is the same, regardless of the use of x_i and x_j or $\tilde{x}_{ij}$ and $\tilde{x}_{ji}$.

[5]Because of the closure operation, $\tilde{x}_{ji} = 1 - \tilde{x}_{ij}$, which implies that whenever one of the components increases its participation, the other one will decrease and, therefore, the covariance will be negative.

[6]Expressed in its general form as: $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$, where $\mathbf{u}$ and $\mathbf{v}$ are two vectors, $\langle \cdot, \cdot \rangle$ is the inner product and $\| \cdot \|$ is the norm (Ford, 2015).

$$PPL_{ij} = \frac{\tau_{ij}^{MAX} - \tau_{ij}}{\tau_{ij}^{MAX} - \tau_{ij}^{MIN}}$$

$$PPL_{ij} = \frac{Var\Big(log(\tilde{x}_{ij})\Big) + Var\Big(log(\tilde{x}_{ji})\Big) + 2\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})} - \Big[Var\Big(log(\tilde{x}_{ij})\Big) + Var\Big(log(\tilde{x}_{ji})\Big) - 2Cov\Big(log(\tilde{x}_{ij}), log(\tilde{x}_{ji})\Big)\Big]}{Var\Big(log(\tilde{x}_{ij})\Big) + Var\Big(log(\tilde{x}_{ji})\Big) + 2\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})} - \Big[Var\Big(log(\tilde{x}_{ij})\Big) + Var\Big(log(\tilde{x}_{ji})\Big)\Big]}$$

$$PPL_{ij} = \frac{Var\Big(log(\tilde{x}_{ij})\Big) + Var\Big(log(\tilde{x}_{ji})\Big) + 2\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})} - Var\Big(log(\tilde{x}_{ij})\Big) - Var\Big(log(\tilde{x}_{ji})\Big) + 2Cov\Big(log(\tilde{x}_{ij}), log(\tilde{x}_{ji})\Big)}{Var\Big(log(\tilde{x}_{ij})\Big) + Var\Big(log(\tilde{x}_{ji})\Big) + 2\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})} - Var\Big(log(\tilde{x}_{ij})\Big) - Var\Big(log(\tilde{x}_{ji})\Big)}$$

$$PPL_{ij} = \frac{2\Big(\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})} + Cov\Big(log(\tilde{x}_{ij}), log(\tilde{x}_{ji})\Big)\Big)}{2\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})}} = \frac{\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})}}{\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})}} + \frac{Cov\Big(log(\tilde{x}_{ij}), log(\tilde{x}_{ji})\Big)}{\sigma_{log(\tilde{x}_{ij})}\sigma_{log(\tilde{x}_{ji})}}$$

$$PPL_{ij} = 1 + \rho_{log(\tilde{x}_{ij}), log(\tilde{x}_{ji})} = 1 + \rho_{log(\tilde{x}_{ij}), log(1-\tilde{x}_{ij})}$$

where $\rho_{log(\tilde{x}_{ij}), log(\tilde{x}_{ji})}$ is the Pearson's correlation coefficient, defined as: $\rho_{A,B} = \frac{Cov(A,B)}{\sigma_A \cdot \sigma_B}$. Since this correlation is always negative, it will be in the interval $[-1, 0]$, with 0 being the highest proportionality. Thus, PPL_{ij} will be in the interval $[0, 1]$ with 0 meaning no proportionality and 1 meaning complete proportionality. As can be seen, this index depends only on the relative importance of the two elements in question within the subcomposition formed by them, meaning that it is not affected by the other components.

Nevertheless, it is worth mentioning the case when $\tilde{x}_{ij}$ is constant. Consequently, $Var\left(log(\tilde{x}_{ij})\right) = 0$ and $\tau_{ij} = \tau_{ij}^{MAX} = \tau_{ij}^{MIN} = 0$, so PPL_{ij} is not defined. Since this case is the maximum proportionality possible, it is defined as $PPL_{ii} = 1$. Therefore, the definition of the proportionality index would be:

$$PPL_{ij} = \begin{cases} 1 + \rho_{log(\tilde{x}_{ij}), log(1-\tilde{x}_{ij})} & \text{for } \tilde{x}_{ij} \text{ not constant} \\ 1 & \text{for } \tilde{x}_{ij} \text{ constant} \end{cases} \tag{11}$$

3.4 Pairwise Proportionality Index

Still, focusing on $\tilde{x}_{ij}$ and $\tilde{x}_{ji}$ could give a different perspective. Thus, considering the extreme values of $\tilde{x}_{ij}$ and $\tilde{x}_{ji}$ are 0 and 1, one can think of them as following a Bernoulli distribution. Also, since $\tilde{x}_{ji} = 1 - \tilde{x}_{ij}$, then the $Var\left(\tilde{x}_{ij}\right) = Var\left(\tilde{x}_{ji}\right)$, meaning one can focus on analyzing the variance of $\tilde{x}_{ij}$. If the two were fully proportional, $\tilde{x}_{ij}$ would be constant and, therefore, its variance would be zero. On the other hand, the minimum proportionality would occur when $\tilde{x}_{ij}$ takes only the extreme values, with the upper and lower limits having equal likelihood. That is, the maximum variance is the variance of a Bernoulli distribution, which is attained when the parameter p or probability of success is 0.5, and the variance, defined as $p(1-p)$, is equal to 0.25. With this upper bound in mind, it is possible to define the Pairwise Proportionality Index (PPI) as:

$$PPI_{ij} = 1 – (4 \times var(\tilde{x}_{ij})) \tag{12}$$

This index takes the value of 1 when the variance of $\tilde{x}_{ij}$ is 0, or there is full proportionality between the variables and approaches 0 as the variance of $\tilde{x}_{ij}$ goes to 0.25.

An advantage of this approach comes from a recurrent issue in compositional data analysis: the treatment of zeros in compositions. Indeed, since the seminal Aitchison (1986) paper, the use of logarithms to express the compositions in $\mathbb{R}^n$ instead of the simplex $\mathcal{S}^n$ has the drawback of being undefined when the participation of one component is zero. However, this is not an issue in the definition of the PPI, which becomes a very relevant improvement compared to the indexes using the logaratio variance. Nevertheless, it remains necessary that at least one of the two components of the subcomposition is larger than zero, since both being zero would leave the closure operation undefined. Still, this represents an improvement compared to the measures based on the logratio variance.

Another downside of the use of logarithms that can be overcome with the functional form of the PPI is the sensibility to extremely low values. Indeed, the closer the ratios x_i/x_j to zero, the higher the variation of $log(x_i/x_j)$ to small changes of the ratio, which means that the variance will be large and the PIP show lower values because of these higher variations. This also applies to the PPL, since the variability remains the same, while the functional form of the PPI is less susceptible to these changes.

4 Data and empirical approach

When measuring relationships between assets in a portfolio, subcompositional coherence is relevant because of the nature of portfolios. Depending on the policies defined by each investor, the composition of portfolios is updated frequently, and having a measure of dependence that is not affected by the entrance and exit of assets in the portfolio can be useful and desirable.

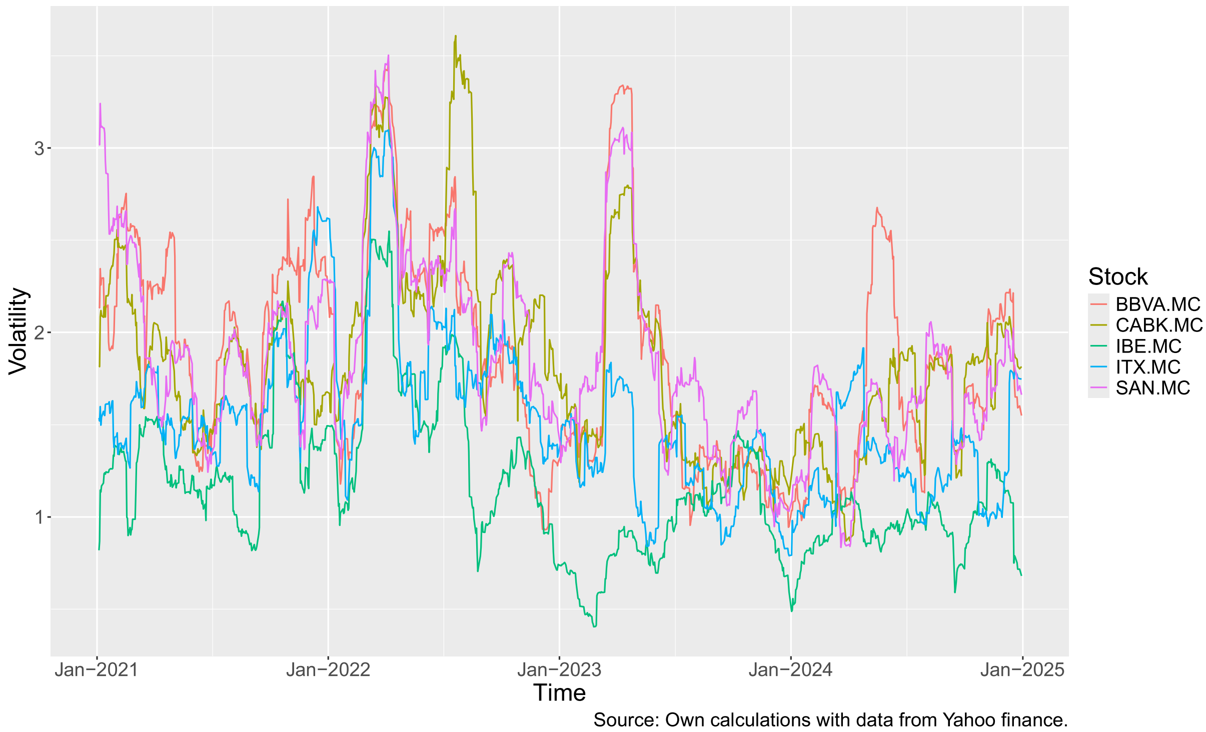


Figure 1: 30-day volatility of the selected stocks.

As an example, the methodologies proposed are used to assess the proportionalities between stocks in a portfolio composed by the five stocks with the highest market capitalization of the IBEX35, the main reference index of the Spanish stock market: Industria de Diseño Textil - INDITEX (ITX), Banco Santander (SAN), Iberdrola (IBE), Caixa Bank (CABK), and Banco Bilbao Vizcaya Argentaria (BBVA). The participations in the portfolios are calculated daily from January 1st, 2021, to December 31st, 2024, following the Markowitz (1952) capital allocation method and using the previous month of observations to estimate the volatility. For the selected stocks, the price used is the daily closing price adjusted for splits, dividends, and capital gain distributions.

Figure 1 shows the 30-day volatility of the stocks in the portfolio. As can be seen, stocks from companies in the banking sector (SAN, CABK and BBVA) tend to show similar behaviors, specially in moments of high volatility, which is a well-known stylized fact in financial markets. Meanwhile, IBE, which is in the energy sector, shows lower volatilities through most of the analyzed period, while ITX (a retail company) stays somewhere in-between.

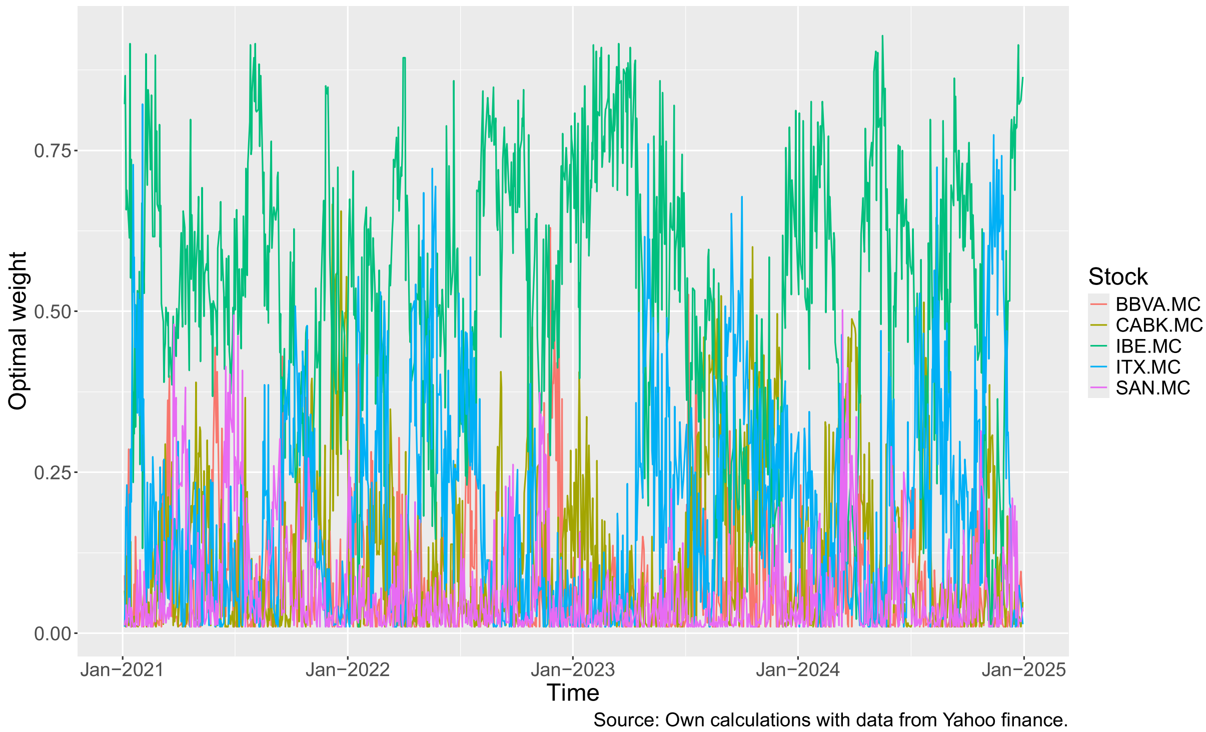


Figure 2: Optimal weights for portfolio with the selected stocks.

Figure 2 shows the optimal weights $\boldsymbol{w}^*$ for the proposed portfolio, obtained using the minimum variance portfolio approach. Apparently, the allocation method puts a higher weight on IBE because of the lower volatility of this asset, and it is possible that the participations of other stocks are connected to this one.

Following the definition in 7, the optimal weights $\boldsymbol{w}^*$ for any pair of assets i and j can be expressed as:

$$\tilde{w}^*_{ij} = \left(\frac{w^*_i}{w^*_i + w^*_j} \right)$$
$$\tilde{w}^*_{ji} = \left(\frac{w^*_j}{w^*_i + w^*_j} \right)$$

Thus, the proportionality index of parts from equation 5 is expressed as:

$$PIP(i,j) = \frac{1}{1+\sqrt{\tau_{ij}}} \text{ with } \tau_{ij} = Var\left(log(\tilde{w}^*_{ij}/\tilde{w}^*_{ji}) \right)$$

Then, following equation 11 the pairwise proporcionality for the logratios of the assets' allocations in risk management will have the form:

$$PPL_{ij} = \begin{cases} 1 + \rho_{log(\tilde{w}^*_{ij}),log(1-\tilde{w}^*_{ij})} & \text{for } \tilde{w}^*_{ij} \text{ not constant} \\ 1 & \text{for } \tilde{w}^*_{ij} \text{ constant} \end{cases}$$

Similarly, from equation 12 the pairquise proportionality for the weights in asset allocation would be:

$$PPI_{ij} = 1–(4 \times var(\tilde{w}^*_{ij}))$$

5 Results

Table 1 shows the results of the PIP calculations for the proposed portfolio. Proportionalities are relatively low for all stocks, although slightly higher for those with IBE.

	ITX	SAN	IBE	BBVA	CABK
ITX	1	0.37	0.39	0.37	0.35
SAN	0.37	1	0.45	0.39	0.37
IBE	0.39	0.45	1	0.43	0.40
BBVA	0.37	0.39	0.43	1	0.36
CABK	0.35	0.37	0.40	0.36	1

Table 1: Proportionality Index of Parts (PIP) matrix for the proposed portfolio.

Table 2 shows the PPL matrix for the stocks in the proposed portfolio. Although the proportionalities are rather low in general, it is worth noticing that those of IBE with the other stocks are the highest ones.

	ITX	SAN	IBE	BBVA	CABK
ITX	1	0.23	0.32	0.25	0.25
SAN	0.23	1	0.26	0.23	0.25
IBE	0.33	0.26	1	0.30	0.26
BBVA	0.25	0.23	0.30	1	0.24
CABK	0.25	0.25	0.26	0.24	1

Table 2: Pairwise proportionality of logratios (PPL) matrix for the proposed portfolio.

Correspondingly, Table 3 shows the PPI matrix for the allocations of the selected portfolio. Here, the proportionalities seem higher than in the PPL case, particularly with respect to IBE, which goes in line with the previous results.

	ITX	SAN	IBE	BBVA	CABK
ITX	1	0.68	0.80	0.69	0.62
SAN	0.68	1	0.92	0.68	0.66
IBE	0.80	0.92	1	0.91	0.84
BBVA	0.69	0.68	0.91	1	0.64
CABK	0.62	0.66	0.84	0.64	1

Table 3: Pairwise proportionality index (PPI) matrix for the proposed portfolio.

Focusing on the proportionalities obtained for IBE, one can notice that even if all of them are high, those with SAN and BBVA are above 0.90, which means that the variability of these allocations is small. This seems to relate to two aspects observed previously. Firstly, the optimal allocations showed in figure 2 show, in general, higher participation of IBE and very low participations for SAN and BBVA. Secondly, figure 1 shows that, for most of the period, IBE exhibits the lowest volatility, while SAN and BBVA the highest, very close between them. Recalling the allocation method used to obtain the optimal weights $\boldsymbol{w}^*$ from section 2, it is the minimization of the variance what defines the optimum. Thus, one can think that the low volatility of IBE makes the allocation method assign a higher weight on this stock and that of SAN and BBVA keep a closer relation to it, assigning lower participations.

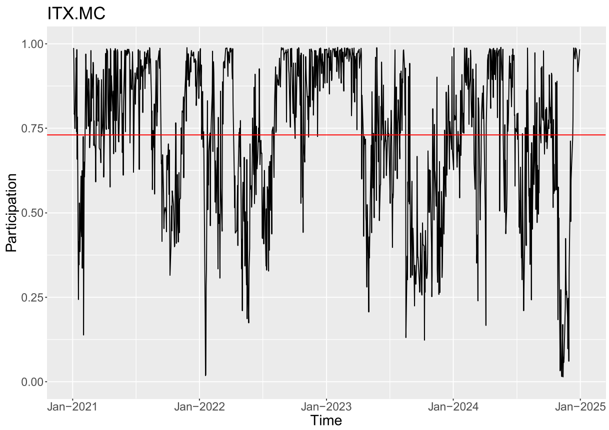
This is also in line with the stock with the third highest proportionality: CABK. It exhibits a high volatility, like those of SAN and BBVA, but less connected than the latter two between them. This causes the allocation method to assign a higher participation to it in some periods and, ultimately, slightly reduce the proportionality with IBE. On the other hand, though still high, the proportionality of IBE with ITX is the lowest, which goes in line with its volatility and participation, compared to the others.

Figure 3 shows the participation IBE in the two-parts subcompositions with each one of the other stocks ($\tilde{w}^*_{IBE,j}$) and the red horizontal line represents the mean for the overall period. In general, IBE has high participations in all subcompositions, in line with the statement that the allocation method assigned this stock high participations, from figure 2. Additionally, the average participation is higher in the subcompositions with SAN and BBVA, which are the ones with the highest proportionality. More important, the graph shows that the variability of these participations in the subcompositions with SAN and BBVA are relatively stable compared to those of ITX and CABK, which explains the higher value for the PPI for the first to, compared to the latter.

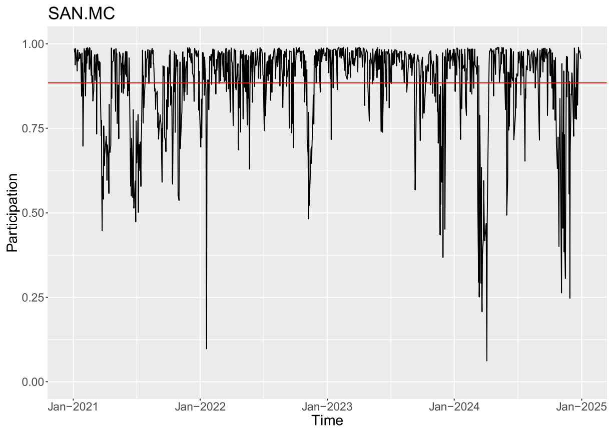
Additionally, figures A.1, A.2, A.3, and A.4 in appendix A show the same graph for the remaining stocks. The figures show that the pairwise proportionality between the stocks is closely related to the average allocations, which is in line with the results mentioned previously. For instance, figure A.1 shows how the variability of BBVA with IBE is much lower than with the other stocks. As a result, the proportionality between these two stocks is much higher than with the others, which goes in line with the results mentioned previously.

5.1 Minimum variance allocation and proportionality of optimal weights

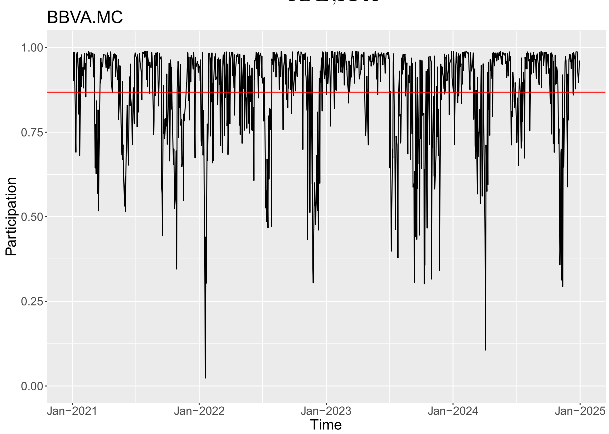
To assess these results, one can verify the consistency of this relationship between the asset allocation method and the proportionalities of the optimal weights. For instance, the exercise can be done excluding IBE from the portfolio, recalculating the optimal weights and analyzing the effect on the three proportionality indexes used previously. Figure 4 shows the optimal weights for the new portfolio. As expected, the allocation method now assigns higher participations to ITX for most of the period (recall that in figure 1 this stock showed lower volatilities than those in the banking sector: SAN, CABK and BBVA).



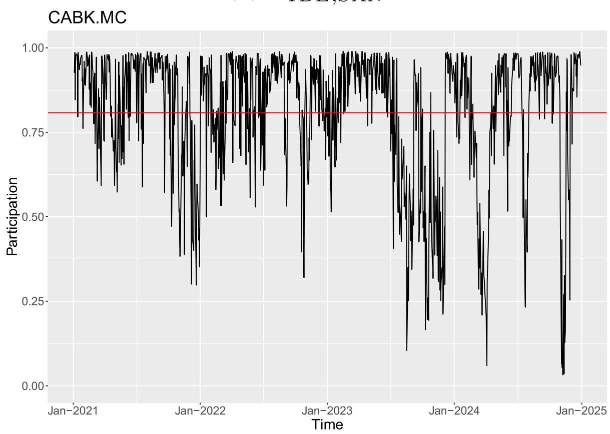
(a) $\tilde{w}^*_{IBE,ITX}$



(b) $\tilde{w}^*_{IBE,SAN}$



(c) $\tilde{w}^*_{IBE,BBVA}$



(d) $\tilde{w}^*_{IBE,CABK}$

Figure 3: Optimal weights for IBE in the subcompositions with each stock.

Tables 4 and 5 show that, though still low, the proportionalities calculated from the PIP and the PPL are slightly higher for ITX than between the other stocks.

	ITX	SAN	BBVA	CABK
ITX	1	0.40	0.38	0.38
SAN	0.40	1	0.33	0.34
BBVA	0.38	0.33	1	0.34
CABK	0.38	0.34	0.34	1

Table 4: Proportionality Index of Parts (PIP) matrix for the new portfolio.

	ITX	SAN	BBVA	CABK
ITX	1	0.28	0.30	0.35
SAN	0.28	1	0.25	0.26
BBVA	0.30	0.25	1	0.27
CABK	0.35	0.26	0.27	1

Table 5: Pairwise proportionality of logratios (PPL) matrix for the new portfolio.

Also, the results for the PPI in table 6 are consistent with the expectation of higher proportionalities of the allocations of ITX with the other stocks than the latter between them.

	ITX	SAN	BBVA	CABK
ITX	1	0.84	0.80	0.77
SAN	0.84	1	0.57	0.64
BBVA	0.80	0.57	1	0.62
CABK	0.77	0.64	0.62	1

Table 6: Pairwise proportionality index (PPI) matrix for the new portfolio.

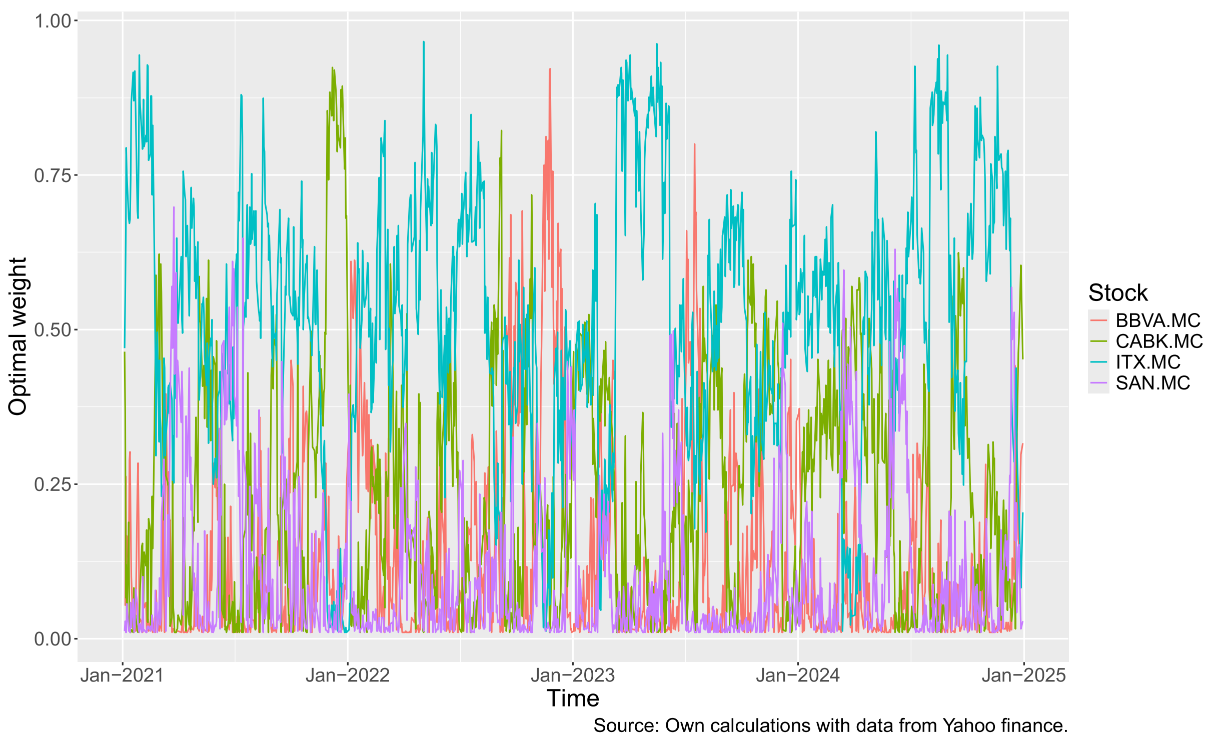
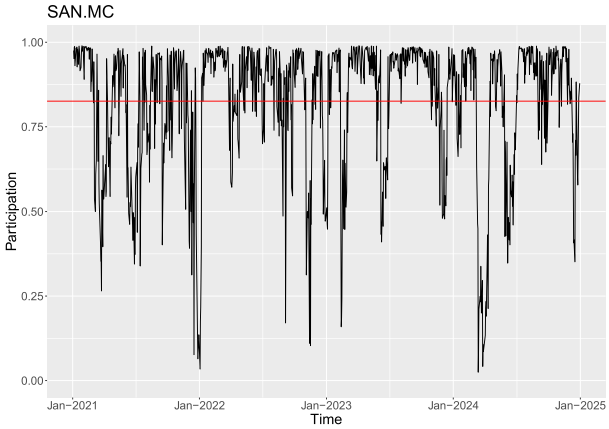
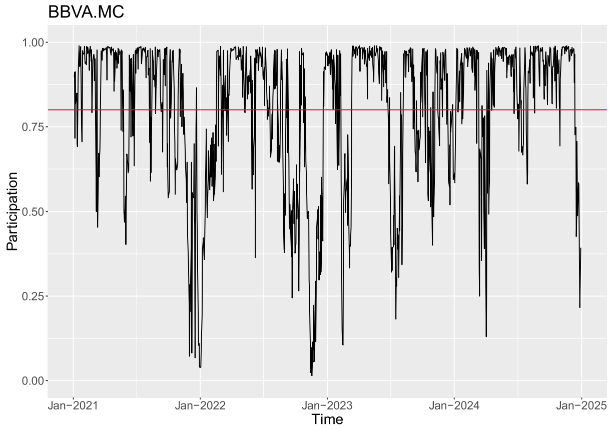


Figure 4: Optimal weights for portfolio with the selected stocks.

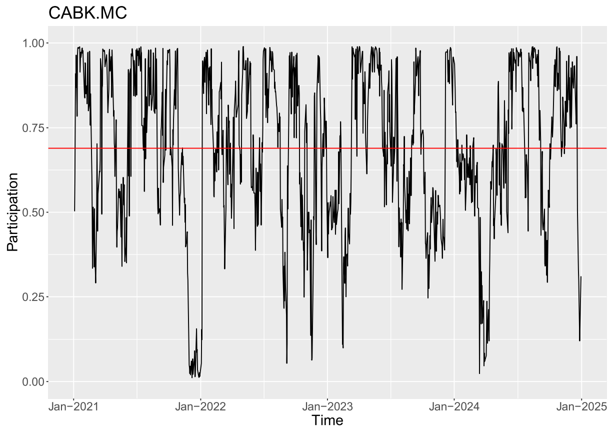
Figure 5 shows the participations of ITX in the subcompositions with each of the other stocks for the new portfolio. As expected, within all the subcompositions ITX holds a high participation and that with SAN seems to be more stable, which goes in line with the higher proportionality showed by the PPI.



(a) $\tilde{w}^*_{ITX,SAN}$



(b) $\tilde{w}^*_{ITX,BBVA}$



(c) $\tilde{w}^*_{ITX,CABK}$

Figure 5: Optimal weights for ITX in the subcompositions with each stock (4 stocks case).

All in all, the results seem consistent, which may allow for drawing conclusions about the relationship between the volatility of returns, the allocations obtained through the minimum variance method and their proportionalities.

6 Conclusions

The logratio variance has shown interesting properties when analyzing compositional data, particularly the proportionality between its parts. Recent work aiming at analyzing proportionality has proposed indexes that overcome the lack of scale of the logratio variance, as done through the proportionality index of parts (PIP). This paper used a different approach, by focusing on the subcomposition formed by only two parts and using properties of the variance to assess proportionality. This allowed to define the maximum and minimum values of the logratio variance and express the index in relative terms to these limits using a different index: the pairwise proportionality of logratios (PPL). Nevertheless, analyzing the properties of the two-parts subcomposition, it was possible to define the pairwise proportionality index (PPI), which is also considers the maximum and minimum values of the variance of the subcomposition. The PPI has the same desirable characteristics of the other two indexes, plus, it does not have one of the main issues of the logratio variance: the presence of zeros in compositions and the sensitivity to small values.

The three indexes were applied to a scenario in portfolio analysis where subcompositional coherence is relevant and proportionality a suitable measure. As a result, Iberdrola (IBE) showed consistently higher proportionalities to the other stocks in the portfolio. This is consistent with the expectation that the allocation method assigns the less volatile asset a higher participation and allocates the others using this one as benchmark. In particular, the PPI showed high proportionality values (closer to 1), which means that applying the closure operation and focusing on the subcomposition led to lower variations of the optimal participations.

To assess the consistency of the results, an additional exercise was performed. The IBE stock was excluded from the portfolio and the optimal weights recalculated, as well as the three proportionality indexes. The results were consistent with those from the previous exercise: the stock with the lowest volatility (now INDITEX - ITX) was assigned the highest participation in the portfolio and it showed the highest proportionalities with all other stocks than the remaining three between them, for each index.

To conclude, by comparing the proportionality values, it is possible to understand how the minimum variance method determines allocations, providing evidence on the advantages of using the compositional framework for this analysis. Among the three proportional measures examined, the PPI shows methodological advantages over the other two. In the illustration, the PPI results were superior in terms of interpretability. Nevertheless, further research is needed to explore the behavior of the proposed indexes in other scenarios.

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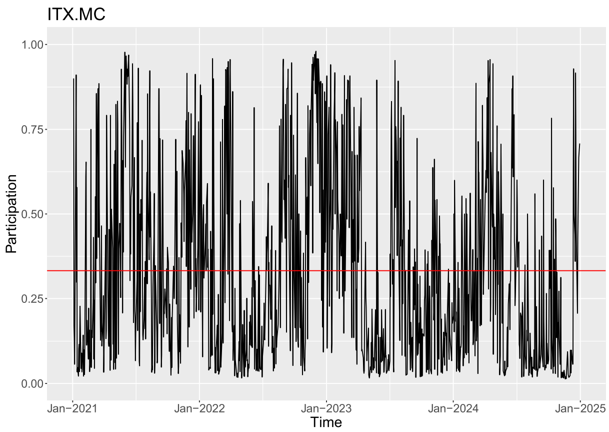
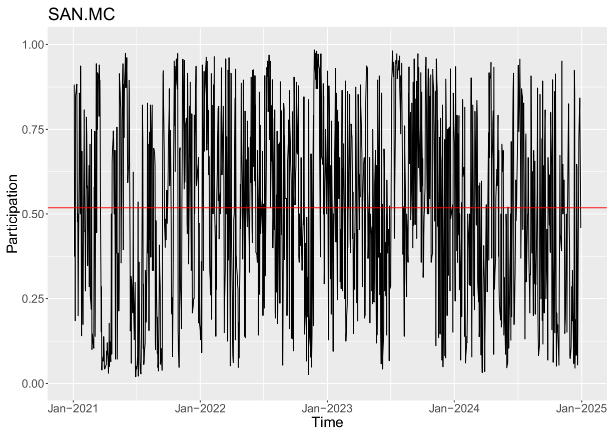
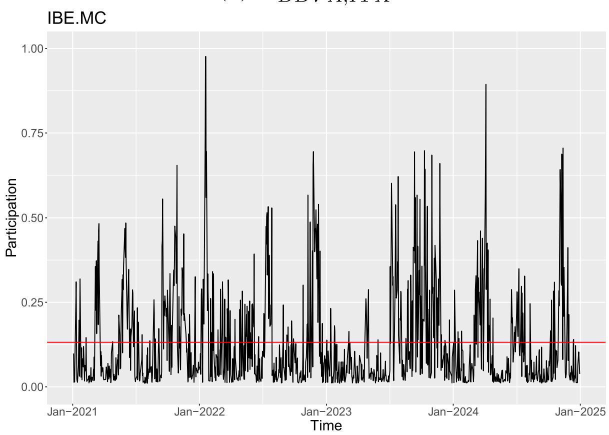
A Appendix: Optimal weights for subcompositions



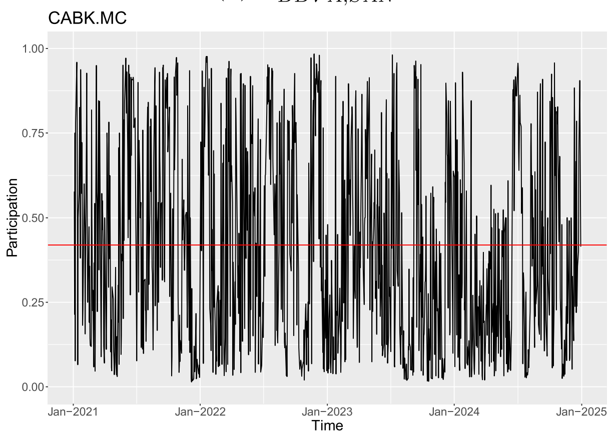
(a) $\tilde{w}^*_{BBVA,ITX}$



(b) $\tilde{w}^*_{BBVA,SAN}$

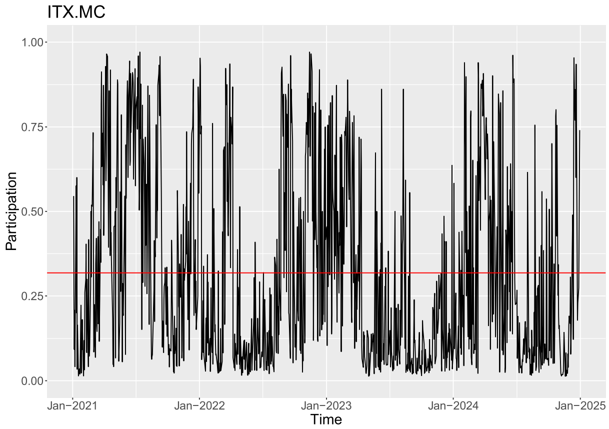


(c) $\tilde{w}^*_{BBVA,IBE}$

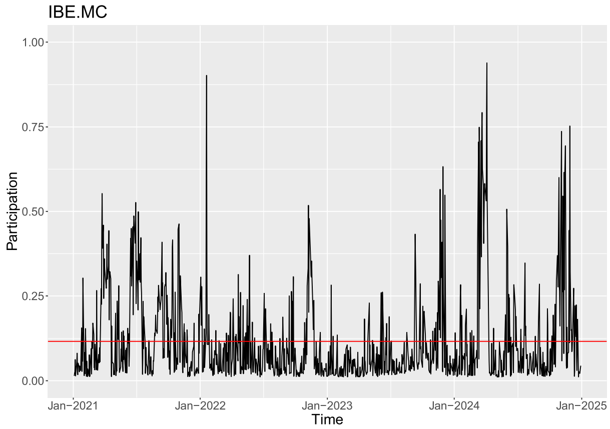


(d) $\tilde{w}^*_{BBVA,CABK}$

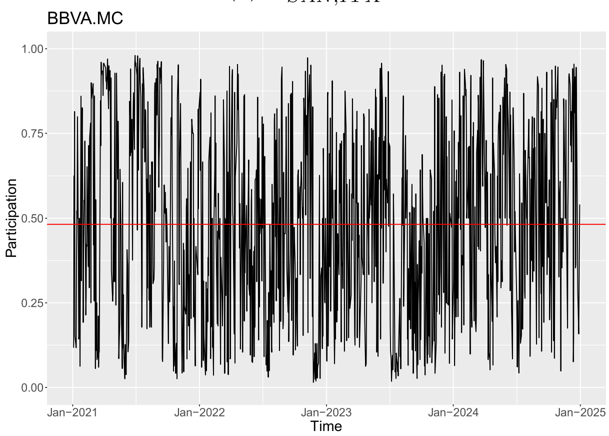
Figure A.1: Optimal weights for BBVA in the subcompositions with each stock.



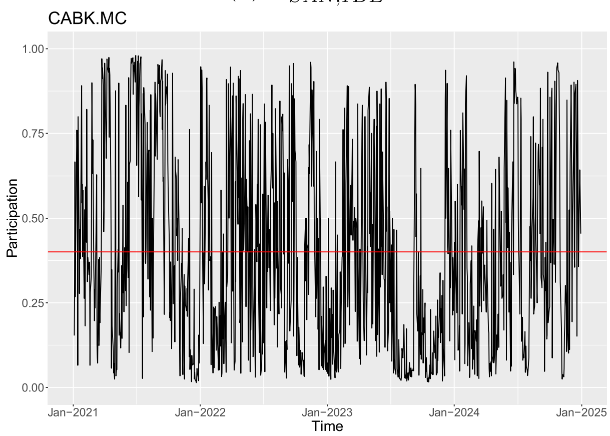
(a) $\tilde{w}^*_{SAN,ITX}$



(b) $\tilde{w}^*_{SAN,IBE}$

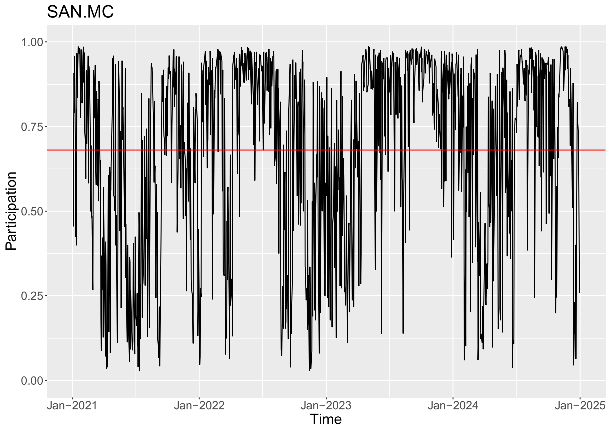


(c) $\tilde{w}^*_{SAN,BBVA}$

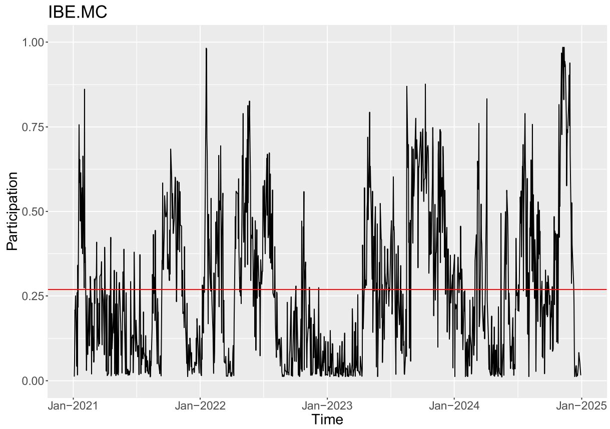


(d) $\tilde{w}^*_{SAN,CABK}$

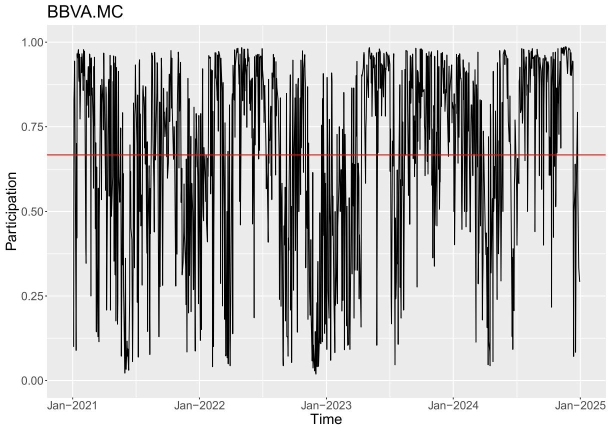
Figure A.2: Optimal weights for SAN in the subcompositions with each stock.



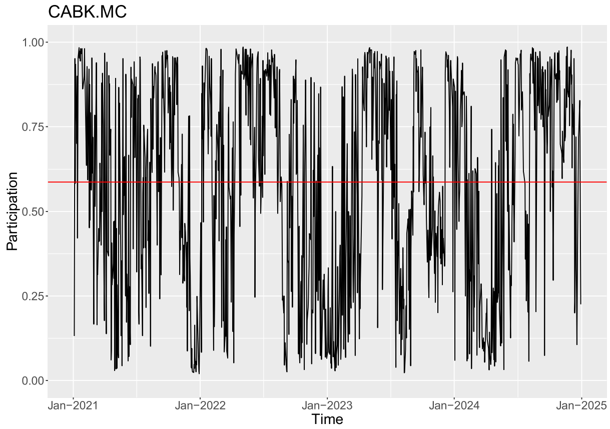
(a) $\tilde{w}^*_{ITX,SAN}$



(b) $\tilde{w}^*_{ITX,IBE}$

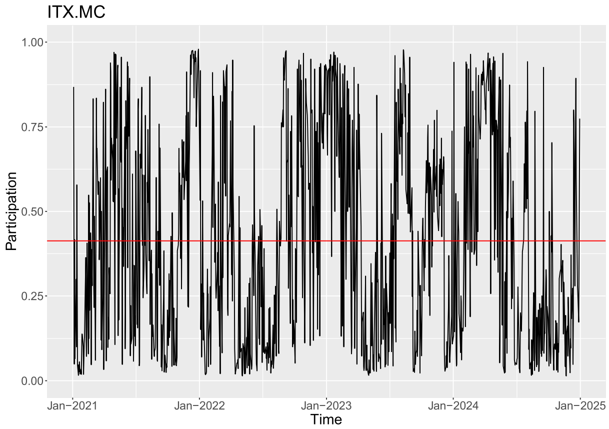


(c) $\tilde{w}^*_{ITX,BBVA}$

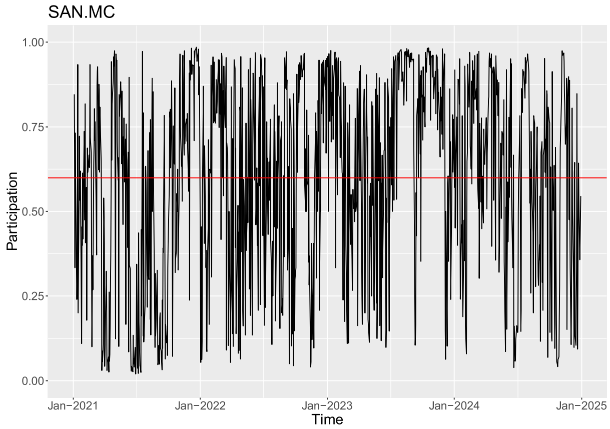


(d) $\tilde{w}^*_{ITX,CABK}$

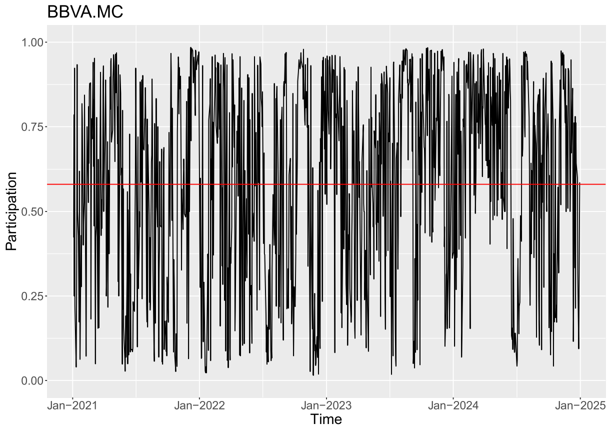
Figure A.3: Optimal weights for ITX in the subcompositions with each stock.



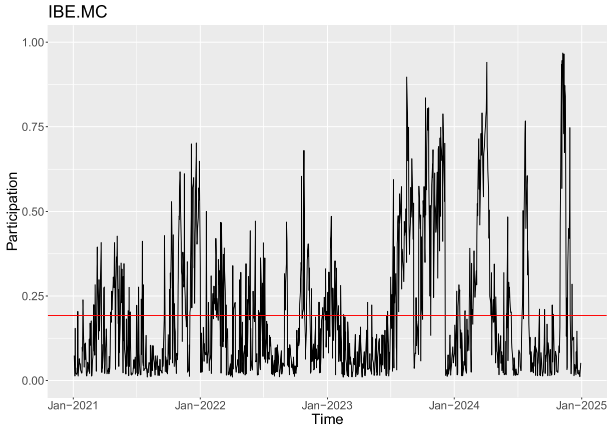
(a) $\tilde{w}^*_{CABK,ITX}$



(b) $\tilde{w}^*_{CABK,SAN}$



(c) $\tilde{w}^*_{CABK,BBVA}$



(d) $\tilde{w}^*_{CABK,IBE}$

Figure A.4: Optimal weights for CABK in the subcompositions with each stock.

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