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MASTER THESIS

An Algebraic Study of Admissible Rules

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Chapter 1

Introduction

In this thesis we shall study admissible rules within the general framework of Abstract Algebraic Logic (AAL). Following Lorenzen [22], we say that a rule is *admissible* for a logic $\mathcal{S}$ whenever it does not add new theorems to $\mathcal{S}$. Despite the seemingly natural definition, the determination of admissible rules in particular logics is usually a difficult problem and requires a deep understanding of the structural properties of the logic. Our purpose is not to study particular cases but instead, we intent to present algebraic conditions of the admissibility of a rule for a logic both in the general case and also depending on its classification in the Leibniz hierarchy. Particular cases will be presented as examples or counter-examples, whenever it is necessary.

The reason that justifies a study on admissible rules is the even more interesting question concerning the derivability of such rules in a logic. Following Pogorzelski, we say that a logic $\mathcal{S}$ is *structurally complete* when all of its admissible *finite* rules are derivable in $\mathcal{S}$ (however, its infinitary analogue will also be considered in this study). The essence of the notion of structural completeness of a logic is of great logical and metalogical importance; it reveals some internal self-sufficiency of the logic. However, it is a property very sensitive and strongly dependent on various changes of the language. For example, it is well-known that classical logic is structurally complete while intuitionistic logic is not, although certain of its fragments are structurally complete (in fact, hereditarily so). The structural completeness of several logics (such as intermediate, modal and fuzzy logics) has been studied thoroughly in Rybakov's monograph [36], while Cintula and Metcalfe have investigated extensively structural completeness in fuzzy logics in [11]. Results concerning structural completeness in substructural and relevance logics can be found in [34] and [35].

Although proofs are not included for well-known results from AAL and Universal Algebra, all necessary preliminary knowledge is accessible from the reader in Chapter 2. In Chapter 3 we shall introduce formally the notion of an admissible rule and of a structurally complete logic and we will present some syntactic conditions concerning these notions. Furthermore, we will see how these notions are inherited (or not) in fragments of a logic and finally, their connections with the finite model property and the deduction theorem will be established.

In Chapter 4, admissible rules will be studied in purely algebraic terms. A general condition of admissibility will be given, namely Theorem 4.1.2, and some sufficient conditions for hereditary structural completeness and its infinitary analogue. By the end of this thesis we shall realize that structural completeness (and its infinitary analogue) can be fully characterized algebraically in logics for which there is an isomorphism theorem between the (finitary) extensions of the logic and certain kinds of subclasses, either of their non-trivial matrix semantics or their algebraic semantics (if the logic has such semantics), depending on each case. This is established in the cases of equivalential and finitely equivalential logics and also in algebraizable and BP-algebraizable logics. Finally, in the last section of Chapter 4 we shall study overflow rules (and passive structural completeness), which is a weaker notion than structural completeness.

This thesis essentially surveys the current research on admissible rules and structural completeness in abstract algebraic terms. However, certain generalizations have been established mostly in the section 4.5 on truth-equational logics (and are the ones without citation), which yielded some corollaries in the case of weakly algebraizable logics, in section 4.6.

Chapter 2

Preliminaries

2.1 Sentential logics

Let L be an arbitrary algebraic similarity type and Var an infinite set of variables (possibly uncountable), both assumed to be well ordered. The *formulas* of L over Var are the elements of the absolutely free algebra of type L, denoted by $\mathbf{Fm}_L$ generated by Var. The elements of Var are usually denoted by $x, y, z, ...$ or $p, q, r, ...$ A **substitution** is any endomorphism on $\mathbf{Fm}_L$. A set $\Gamma \subseteq Fm_L$ is *closed under substitutions* if for every substitution σ, $\sigma[\Gamma] \subseteq \Gamma$.
Most of the following definitions and fundamental results can be found in [15] or the classic book [13]. We include them for convenience.

Definition 2.1.1. A (*sentential*) ***logic*** (of type L) is a pair $\mathcal{S} = \langle L, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(Fm_L) \times Fm_L$ is a *structural* (or invariant under substitutions) consequence relation. That is, for any sets of formulas Γ, Δ and any formulas φ, ψ, the following conditions hold:

1. If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$ (Identity)

2. If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for all $\psi \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \psi$, then $\Delta \vdash_{\mathcal{S}} \varphi$ (Cut)

3. If $\Gamma \vdash_{\mathcal{S}} \varphi$ and σ is a substitution, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$ (invariance under substitutions or structurality)

From the first two conditions follows:

4. If $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathcal{S}} \varphi$ (Monotonicity)

A sentential logic $\mathcal{S} = \langle L, \vdash_{\mathcal{S}} \rangle$ is **finitary** if $\vdash_{\mathcal{S}}$ additionally satisfies:

5. If $\Gamma \vdash_{\mathcal{S}} \varphi$, then there is a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{S}} \varphi$.

A **theory** of $\mathcal{S}$ is any set of formulas closed under consequence. That is, any set $T \subseteq Fm_L$ such that $T \vdash_{\mathcal{S}} \varphi$ implies $\varphi \in T$. A theory T is called **inconsistent** if $T = Fm_L$. The set of theories of $\mathcal{S}$ is denoted by $Th(\mathcal{S})$.

A **rule** is a pair $\langle \Gamma, \varphi \rangle$, where $\Gamma \cup \{\varphi\} \subseteq Fm_L$. It is a *finite rule* if Γ is a finite set. Otherwise, we say it is an *infinite rule*. A rule $\langle \Gamma, \varphi \rangle$ is *derivable* in $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$. The **theorems** of $\mathcal{S}$ are the formulas φ for which $\varnothing \vdash_{\mathcal{S}} \varphi$, and we abbreviate it as $\vdash_{\mathcal{S}} \varphi$.

NOTATIONAL CONVENTIONS: We use the notation $\Gamma, \varphi \vdash_{\mathcal{S}} \psi$ instead of $\Gamma \cup \{\varphi\} \vdash_{\mathcal{S}} \psi$. Similarly, we abbreviate $\varphi \vdash_{\mathcal{S}} \psi$ for $\{\varphi\} \vdash_{\mathcal{S}} \psi$. Finally, for sets of formulas $\Gamma, \Delta \subseteq Fm_L$, we abbreviate $\Gamma \vdash_{\mathcal{S}} \Delta$ and we mean that for all $\varphi \in \Delta$, $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \dashv\vdash_{\mathcal{S}} \Delta$, for $\Gamma \vdash_{\mathcal{S}} \Delta$ and $\Delta \vdash_{\mathcal{S}} \Gamma$.

Definition 2.1.2. Let A be a set. A **closure operator** on A is a map $C : \mathcal{P}(A) \to \mathcal{P}(A)$ that satisfies the following conditions:

1. $X \subseteq C(X)$ for every $X \subseteq A$ (extensive)
2. If $X \subseteq Y$, then $C(X) \subseteq C(Y)$, for every $X, Y \subseteq A$ (monotone)
3. $C(C(X)) = C(X)$ for every $X \subseteq A$ (idempotent)

and it is called **finitary** if moreover it satisfies:

4. For every $X \subseteq A$, $C(X) = \bigcup\{C(Y) : Y \subseteq X,\ Y \text{ finite }\}$.

A closure operator C on Fm_L is called *structural* if for any substitution σ and any $\Gamma \cup \{\varphi\} \subseteq Fm_L$, if $\varphi \in C(\Gamma)$, then $\sigma(\varphi) \in C(\sigma[\Gamma])$, i.e. when $\sigma[C(X)] \subseteq C(\sigma[X])$.
Let $\mathcal{S}$ be a logic (of type L). We define a mapping $Cn_{\mathcal{S}} : \mathcal{P}(Fm_L) \to \mathcal{P}(Fm_L)$ by:

$$Cn_{\mathcal{S}}(\Gamma) := \{\varphi \in Fm_L : \Gamma \vdash_{\mathcal{S}} \varphi\}, \text{ for any } \Gamma \subseteq Fm_L.$$

It is almost immediate to see that $Cn_{\mathcal{S}}$ is a structural closure operator on Fm_L and it is called the associated closure operator of the consequence relation $\vdash_{\mathcal{S}}$. It is a finitary closure operator if $\mathcal{S}$ is finitary. Under this definition, the set of theorems of $\mathcal{S}$ is the set $Cn_{\mathcal{S}}(\varnothing)$. Finally, given any $\Gamma \subseteq Fm_L$, the smallest $\mathcal{S}$-theory containing Γ is $Cn_{\mathcal{S}}(\Gamma)$ and it is called the **theory generated by** Γ.
Given any *structural* closure operator C on Fm_L, we can define the relation $\vdash \subseteq \mathcal{P}(Fm_L) \times Fm_L$ by

$$\Gamma \vdash \varphi \ \text{ iff } \ \varphi \in C(\Gamma)$$

which is easily seen to be a consequence relation and thus defines a logic. Clearly, if C is finitary, then $\vdash$ is finitary as well. Thus, there is a bijection between all logics of type L and the *structural* closure operators on Fm_L.

Definition 2.1.3. Let $\mathcal{S}$ be a logic of type L.

1. Let $\mathcal{S}'$ be a logic of the same type. We say that $\mathcal{S}'$ is an **extension** of $\mathcal{S}$, and we write $\mathcal{S} \leq \mathcal{S}'$, if for any $\Gamma \cup \{\varphi\} \subseteq Fm_L$, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\Gamma \vdash_{\mathcal{S}'} \varphi$. The extension $\mathcal{S}'$ is **axiomatic** if there is a set of formulas Δ closed under substitutions such that for every $\Gamma \cup \{\varphi\} \subseteq Fm_L$, $\Gamma \vdash_{\mathcal{S}'} \varphi$ iff $\Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi$. In this case $\mathcal{S}'$ is the least logic that extends $\mathcal{S}$ and has as new theorems the formulas in Δ.

2. Let $L' \supseteq L$ be an algebraic similarity type and $\mathcal{S}'$ a logic of type L'. The logic $\mathcal{S}'$ is called an **expansion** of $\mathcal{S}$ if for any set of formulas Γ of type L and any formula φ of type L, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\Gamma \vdash_{\mathcal{S}'} \varphi$. It is called a **conservative expansion** if the converse also holds.

3. Let $L' \subseteq L$ be an algebraic similarity type. The L'-**fragment** of $\mathcal{S}$ is the logic $\mathcal{S}'$ of type L', whose consequence relation $\vdash_{\mathcal{S}'}$ is defined by $\Gamma \vdash_{\mathcal{S}'} \varphi$ iff $\Gamma \vdash_{\mathcal{S}} \varphi$, for any $\Gamma \cup \{\varphi\}$ of type L'.

Notation: Given a logic $\mathcal{S}$ and a rule $\langle\Gamma, \varphi\rangle$, we denote by $\mathcal{S} + \langle\Gamma, \varphi\rangle$ the smallest extension of $\mathcal{S}$ containing the rule $\langle\Gamma, \varphi\rangle$. That is, $\mathcal{S} + \langle\Gamma, \varphi\rangle$ is the logic whose consequence relation is: $\vdash_{\mathcal{S}+\langle\Gamma,\varphi\rangle} = \bigcap\{\vdash_{\mathcal{S}'} : \mathcal{S} \leq \mathcal{S}' \text{ and } \Gamma \vdash_{\mathcal{S}'} \varphi\}$.

Definition 2.1.4. A logic $\mathcal{S}$ (of type L) is **trivial** when it satisfies $x \vdash_{\mathcal{S}} y$ for two different variables x, y. An **inconsistent** logic is a trivial logic with theorems and an **almost inconsistent** logic is a trivial logic without theorems. Equivalently, $\mathcal{S}$ is inconsistent when $Th(\mathcal{S}) = \{Fm_L\}$ and is almost inconsistent when $Th(\mathcal{S}) = \{\varnothing, Fm_L\}$.
A logic $\mathcal{S}$ is called **consistent** if $Cn_{\mathcal{S}}(\varnothing) \neq Fm$ and it is called **strongly consistent** if $\varphi \nvdash_{\mathcal{S}} \psi$, for some $\varphi, \psi \in Fm$.

2.2 Matrix semantics and Leibniz congruences

We fix an arbitrary algebraic similarity type L and an (arbitrarily) infinite set of variables Var, as in 2.1. Unless stated otherwise, all algebras and logics that shall be considered they will all be of type L. Thus, from now on we write $\mathbf{Fm}$ instead of $\mathbf{Fm_L}$. Following standard practice, we denote algebras by $\mathbf{A}, \mathbf{B}, \mathbf{C}$ etc. and their universes by A, B, C etc. Finally, given two algebras $\mathbf{A}, \mathbf{B}$ we denote the set of all (algebraic) homomorphisms from $\mathbf{A}$ into $\mathbf{B}$ by $Hom(\mathbf{A}, \mathbf{B})$.

Definition 2.2.1. Let $\mathcal{S}$ be a logic, $\mathbf{A}$ an algebra and $F \subseteq A$. The set F is a filter of $\mathcal{S}$ or for short an $\mathcal{S}$**-filter**, if F is closed under the rules of $\mathcal{S}$. That is, for any $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$, and any homomorphism $h : \mathbf{Fm} \to \mathbf{A}$, if $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. The set of $\mathcal{S}$-filters of the algebra $\mathbf{A}$ is denoted by $Fi_{\mathcal{S}}\mathbf{A}$ and it is closed under arbitrary intersections. Hence, it becomes a complete lattice when ordered by inclusion.

A trivial checking reveals that the $\mathcal{S}$-theories are exactly the $\mathcal{S}$-filters on $\mathbf{Fm}$, i.e. $Th(\mathcal{S}) = Fi_{\mathcal{S}}\mathbf{Fm}$. The following basic facts are well-known (see eg. [15] or [13]):

Proposition 2.2.1. *Let $\mathcal{S}$ be a logic.*

i Let $\mathbf{A}, \mathbf{B}$ be algebras and $h \in Hom(\mathbf{A}, \mathbf{B})$. If $F \in Fi_{\mathcal{S}}\mathbf{B}$, then $h^{-1}[F] \in Fi_{\mathcal{S}}\mathbf{A}$.

ii φ is a theorem of $\mathcal{S}$ iff for any $\mathbf{A}$, any $F \in Fi_{\mathcal{S}}\mathbf{A}$ and any $h \in Hom(\mathbf{Fm}, \mathbf{A})$, $h(\varphi) \in F$.

iii $\mathcal{S}$ does not have theorems iff $\varnothing \in Fi_{\mathcal{S}}\mathbf{A}$, for every algebra $\mathbf{A}$.

A **matrix** is any pair $\langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is an algebra and $F \subseteq A$. The elements of F are called the *designated elements* of the matrix. A **submatrix** of $\langle \mathbf{A}, F\rangle$ is any matrix $\langle \mathbf{B}, G\rangle$ where $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ (denoted by $\mathbf{B} \subseteq \mathbf{A}$) and $G = B \cap F$. Notice that an L-matrix can be seen as a first-order structure of type $L \cup \{P\}$, where P is a unary relation symbol. We say that $\langle \mathbf{A}, F\rangle$ is a *model* of the rule (or that *validates* the rule) $\langle \Gamma, \varphi\rangle$, and we write $\langle \mathbf{A}, F\rangle \vDash \langle \Gamma, \varphi\rangle$, if for any homomorphism $h : \mathbf{Fm} \to \mathbf{A}$, if $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. Given a logic $\mathcal{S}$, a **matrix model** of $\mathcal{S}$ is any matrix $\langle \mathbf{A}, F\rangle$, with $F \in Fi_{\mathcal{S}}\mathbf{A}$. In such case, we write $\langle \mathbf{A}, F\rangle \vDash \mathcal{S}$, meaning that for every $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$, $\langle \mathbf{A}, F\rangle \vDash \langle \Gamma, \varphi\rangle$. We denote the class of all matrix models of $\mathcal{S}$ by $Matr(\mathcal{S})$. Moreover, we denote by $LMatr(\mathcal{S}) := \{\langle \mathbf{Fm}, T\rangle : T \in Th(\mathcal{S})\}$ the set of *Lindenbaum matrices* of $\mathcal{S}$. Clearly, $LMatr(\mathcal{S}) \subseteq Matr(\mathcal{S})$.
A *trivial algebra* is an algebra whose universe is a singleton. All trivial algebras are isomorphic. A *trivial matrix* is a matrix $\mathcal{M} = \langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is a trivial algebra and $F = A$. We usually denote a trivial algebra by $\mathbf{A}_{tr}$ and a trivial matrix by $\mathcal{M}_{tr}$. It is obvious that a trivial matrix is a matrix model of any logic $\mathcal{S}$, so it belongs to $Matr(\mathcal{S})$.
Given a class of matrices M, the relation $\vDash_M \subseteq \mathcal{P}(Fm) \times Fm$, defined by

$$\Gamma \vDash_M \varphi \text{ iff for all } \langle \mathbf{A}, F\rangle \in M, \langle \mathbf{A}, F\rangle \vDash \langle \Gamma, \varphi\rangle$$

is a consequence relation invariant under substitutions and thus induces a logic which is called *the logic induced by* M. This logic is usually denoted by $\mathcal{S}_M$.
We say that a logic $\mathcal{S}$ is *complete* with respect to a class of matrices M, or that M is a *matrix semantics* of $\mathcal{S}$, if $\mathcal{S} = \mathcal{S}_M$. In other words, if $\vdash_{\mathcal{S}} = \vDash_M$.

Theorem 2.2.1. *Let $\mathcal{S}$ be a logic. Any class of matrices M such that $LMatr(\mathcal{S}) \subseteq M \subseteq Matr(\mathcal{S})$ is a matrix semantics for $\mathcal{S}$.*

Definition 2.2.2. A *matrix homomorphism* from $\langle \mathbf{A}, F\rangle$ into $\langle \mathbf{B}, G\rangle$ is an (algebraic) homomorphism $h : \mathbf{A} \to \mathbf{B}$, such that $h[F] \subseteq G$, i.e. $F \subseteq h^{-1}[G]$. It is a *surjective* (or *onto*) matrix homomorphism if $h[A] = B$. It is a **strict** matrix homomorphism if $h^{-1}[G] = F$. Finally, it is an **embedding**, if it is an injective algebraic homomorphism that **additionally is strict** and it is an **isomorphism** if it is a surjective embedding.
The **kernel** of the homomorphism h is the set $ker(h) := \{\langle a, b\rangle \in A \times A : h(a) = h(b)\}$.
We say that $\langle \mathbf{B}, G\rangle$ is a *homomorphic image* of $\langle \mathbf{A}, F\rangle$, if there is a surjective matrix homomorphism h from $\langle \mathbf{A}, F\rangle$ onto $\langle \mathbf{B}, G\rangle$.

Lemma 2.2.1. *If $f : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism, then for each $h : \mathbf{Fm} \to \mathbf{B}$, there is a homomorphism $g : \mathbf{Fm} \to \mathbf{A}$ such that $f \circ g = h$. Graphically:*

$$\begin{tikzcd}
\mathbf{Fm} \arrow[r, dashed, "g"] \arrow[dr, "h"'] & \mathbf{A} \arrow[d, two heads, "f"] \\
& \mathbf{B}
\end{tikzcd}$$

By applying the lemma above we can easily prove:

Lemma 2.2.2. *Let $h : \langle \mathbf{A}, F\rangle \to \langle \mathbf{B}, G\rangle$ be a matrix homomorphism.*

i If h is strict, then every rule validated by $\langle \mathbf{B}, G\rangle$ is validated by $\langle \mathbf{A}, F\rangle$.

ii If moreover h is surjective, then the converse also holds, i.e. $\langle \mathbf{A}, F\rangle$ and $\langle \mathbf{B}, G\rangle$ validate the same rules.

Given any n-ary function symbol $f \in L$ with $f = f(x_0, \dots, x_{n-1})$, any algebra $\mathbf{A}$ and any $a_0, \dots, a_{n-1} \in A$, we follow [16] and denote by $f^{\mathbf{A}}(a_0, \dots, a_{n-1})$ the image of $f(x_0, \dots, x_{n-1})$ by any homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ with $h(x_i) = a_i$, for $i < n$. Often we use the notation $\varphi(\bar{x})$ instead of $\varphi(x_0, \dots, x_{n-1})$. We extend the previous notation naturally for arbitrary formulas. For example $\varphi^{\mathbf{A}}(a, b, \bar{c})$ stands for the image of $\varphi(x, y, \bar{z})$ by any $h \in Hom(\mathbf{Fm}, \mathbf{A})$, with $h(x) = a$, $h(y) = b$ and $h(\bar{z}) = \bar{c}$ (assuming $\bar{z}$ and $\bar{c}$ have the same length and h maps each variable in $\bar{z}$ coordinatewise). This notation will also be used for sets of formulas $\Delta(\bar{x})$, i.e. for $\bar{a} \in A$ (of the same length as $\bar{x}$), $\Delta^{\mathbf{A}}(\bar{a})$ stands for $\{\varphi^{\mathbf{A}}(\bar{a}) : \varphi \in \Delta\}$.
A **congruence** on an L-algebra $\mathbf{A}$ is an equivalence relation θ on A which for every n-ary ($n > 0$) function symbol $f \in L$ satisfies the following compatibility condition: for every $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in A$,

$$\text{if for every } i < n,\ \langle a_i, b_i\rangle \in \theta, \text{ then } \langle f^{\mathbf{A}}(a_0, \dots, a_{n-1}), f^{\mathbf{A}}(b_0, \dots, b_{n-1})\rangle \in \theta.$$

The set of congruences of an algebra $\mathbf{A}$ will be denoted by $Con\mathbf{A}$.
Given an algebra $\mathbf{A}$, a subset $F \subseteq A$ and a congruence $\theta \in Con\mathbf{A}$, we say that θ is *compatible* with F if $a\theta b$ and $a \in F$, implies $b \in F$. The largest congruence of $\mathbf{A}$ compatible with F always exists and it is called the **Leibniz congruence** of F in $\mathbf{A}$. It is denoted by $\Omega^{\mathbf{A}}(F)$. The *Leibniz operator* is the map $\Omega^{\mathbf{A}} : Fi_{\mathcal{S}}\mathbf{A} \to Con\mathbf{A}$, that sends every $\mathcal{S}$-filter of $\mathbf{A}$ to its Leibniz congruence. If $\mathbf{A} = \mathbf{Fm}$, we omit the superscript. The following characterization can be found in [12] or [15]:

Proposition 2.2.2. *Let* $\mathbf{A}$ *be an algebra and* $F \subseteq A$. *For every* $a, b \in A$,

$$\langle a, b\rangle \in \Omega^{\mathbf{A}}(F) \text{ iff for every formula } \phi(x, z_1, \dots, z_n) \text{ and every } \bar{c} = c_1, \dots, c_n \in A,$$
$$\varphi^{\mathbf{A}}(a, \bar{c}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b, \bar{c}) \in F.$$

Notation: Let A, B be sets and let $h : A \to B$ be a mapping. We extend the natural notation of images and preimages of subsets of A and B, to subsets of $A \times A$ and $B \times B$ respectively. That is, for $X \subseteq A \times A$ and $Y \subseteq B \times B$, we define $h[X] := \{\langle h(a), h(b)\rangle : \langle a, b\rangle \in X\}$ and $h^{-1}[Y] := \{\langle a, b\rangle : \langle h(a), h(b)\rangle \in Y\}$.
The following facts are well-known. Only item ii relies on Proposition 2.2.2.

Proposition 2.2.3. *Let* $h : \mathbf{A} \to \mathbf{B}$ *be an algebraic homomorphism and* $G \subseteq B$. *Then,*

i $h^{-1}[\Omega^{\mathbf{B}}(G)] \subseteq \Omega^{\mathbf{A}}(h^{-1}[G])$

ii If moreover h *is surjective, then* $\Omega^{\mathbf{A}}(h^{-1}[G]) = h^{-1}[\Omega^{\mathbf{B}}(G)]$

Definition 2.2.3. A matrix $\langle \mathbf{A}, F\rangle$ is *reduced* if $\Omega^{\mathbf{A}}(F) = Id_A = \{\langle a, a\rangle : a \in A\}$. We denote by $Matr^*(\mathcal{S})$ the class of all reduced matrix models of $\mathcal{S}$.

Let $\mathcal{M} = \langle \mathbf{A}, F\rangle$ be a matrix. A *congruence of the matrix* $\mathcal{M}$ is any congruence $\theta \in Con\mathbf{A}$ compatible with F, i.e. $\theta \subseteq \Omega^{\mathbf{A}}(F)$. We sometimes refer to the Leibniz congruence of F in $\mathbf{A}$ by $\Omega(\mathcal{M})$. The set of congruences of the matrix $\mathcal{M}$ shall be denoted by $Con\mathcal{M}$. It is the interval $[id_A, \Omega^{\mathbf{A}}(F)]$ of the lattice $Con\mathbf{A}$.
Given a matrix $\mathcal{M} = \langle \mathbf{A}, F\rangle$ and $\theta \in Con\mathcal{M}$, the *quotient matrix* $\mathcal{M}/\theta$ is the matrix whose underlying algebra is the quotient algebra $\mathbf{A}/\theta$ and whose set of designated elements is $F/\theta = \{a/\theta : a \in F\}$.
The following fact is elementary:

Proposition 2.2.4. *Let* $\mathcal{M} = \langle \mathbf{A}, F\rangle$ *be a matrix and* $\theta \in Con\mathcal{M}$. *The natural map* $\pi_\theta : \mathbf{A} \to \mathbf{A}/\theta$ *is a strict, surjective homomorphism from* $\langle \mathbf{A}, F\rangle$ *onto* $\langle \mathbf{A}/\theta, F/\theta\rangle$.

Given a matrix $\mathcal{M} = \langle \mathbf{A}, F\rangle$, the quotient matrix $\mathcal{M}/\Omega(\mathcal{M}) = \langle \mathbf{A}/\Omega^{\mathbf{A}}(F), F/\Omega^{\mathbf{A}}(F)\rangle$ is called the *reduction* of $\mathcal{M}$. We denote it by $\mathcal{M}^*$ or $\langle \mathbf{A}, F\rangle^*$ and it is easy to prove that it is reduced.
A reduced matrix $\langle \mathbf{A}, F\rangle$ is non-trivial if $F \neq A$ (since $\Omega^{\mathbf{A}}(A) = A \times A = \Omega^{\mathbf{A}}(\varnothing)$). In particular, if $\mathcal{S}$ is consistent, then $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ is always non-trivial, since in this case either $Cn_{\mathcal{S}}(\varnothing) = \varnothing$ or $\Omega(Cn_{\mathcal{S}}(\varnothing)) \neq Fm \times Fm$.

Theorem 2.2.2 ([37]). *A rule is derivable in* $\mathcal{S}$ *iff it is validated by all reduced matrix models of* $\mathcal{S}$. *In particular, the theorems of* $\mathcal{S}$ *are exactly the formulas taking only designated values in all reduced matrix models of* $\mathcal{S}$.

Thus, it immediately follows that every logic $\mathcal{S}$ is complete with respect to the class of its reduced matrix models $Matr^*(\mathcal{S})$. As opposed to the class $Matr(\mathcal{S})$, the class $Matr^*(\mathcal{S})$ is a non-trivial (matrix) semantics for $\mathcal{S}$.
Given a logic $\mathcal{S}$, the class $Alg^*(\mathcal{S})$ is the class of algebra reducts of reduced matrix models of $\mathcal{S}$, i.e. $Alg^*(\mathcal{S}) = \{\mathbf{A} : (\exists F \in Fi_{\mathcal{S}}\mathbf{A})\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})\}$. The *algebraic counterpart* of $\mathcal{S}$ is the class $Alg(\mathcal{S})$ which is the closure of $Alg^*(\mathcal{S})$ under isomorphisms and subdirect products, i.e. $Alg(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}Alg^*(\mathcal{S})$.

2.3 Varieties and Quasivarieties

For a detailed exposition of the fundamentals of universal algebra we refer the reader to [10] or [3]. We consider as known the algebraic notions of ultraproducts, direct and subdirect products. These notions can be extended naturally to matrices. For example, given a family of matrices $\{\langle \mathbf{A_i}, F_i\rangle : i \in I\}$, its *direct product* is the matrix $\prod\limits_{i\in I}\langle \mathbf{A_i}, F_i\rangle = \langle \prod\limits_{i\in I} \mathbf{A_i}, \prod\limits_{i\in I} F_i\rangle$ and a *subdirect product* of the family is a submatrix $\langle \mathbf{A}, F\rangle$ of $\prod\limits_{i\in I}\langle \mathbf{A_i}, F_i\rangle$ such that for each $i \in I$, the restriction of the projection map π_i to $\mathbf{A}$ is onto $\mathbf{A_i}$.

Following standard notation, we use the class operators $\mathbb{H}, \mathbb{S}, \mathbb{I}, \mathbb{P}, \mathbb{P}_{\mathbb{SD}}, \mathbb{P}_{\mathbb{U}}$ to indicate closure under homomorphic images, subalgebras, isomorphisms, direct and subdirect products, and ultraproducts respectively, when applied to classes of algebras. Given a first-order language $\mathfrak{L}$ we shall use the same notation for class operators applied on a class K of $\mathfrak{L}$-structures. That is, $\mathbb{H}, \mathbb{S}, \mathbb{I}, \mathbb{P}, \mathbb{P}_{\mathbb{SD}}, \mathbb{P}_{\mathbb{U}}$ indicate respectively the closure under homomorphic images, substructures, direct and subdirect products and ultraproducts. We interpret the direct product (and any ultraproduct) of the empty family of $\mathfrak{L}$-structures as the trivial $\mathfrak{L}$-structure with universe $\{\varnothing\}$. Therefore, if K is closed under $\mathbb{P}$ (or $\mathbb{P}_{\mathbb{SD}}$ or $\mathbb{P}_{\mathbb{U}}$), then K contains a trivial structure. Whenever we use a first-order language $\mathfrak{L}$ we shall make it explicit. Otherwise, it is assumed that we work within our fixed but arbitrary algebraic similarity type L.

A *valuation* on an algebra $\mathbf{A}$ is any homomorphism $h \in Hom(\mathbf{Fm}, \mathbf{A})$. An **equation** is just a pair $\langle\alpha, \beta\rangle \in Fm \times Fm$, which we write as $\alpha \approx \beta$. An equation $\alpha \approx \beta$ is satisfied by a valuation h on $\mathbf{A}$, in symbols $\mathbf{A} \vDash \alpha \approx \beta\ [[h]]$ if $h(\alpha) = h(\beta)$. An equation $\alpha \approx \beta$ is *valid* in the algebra $\mathbf{A}$, in symbols $\mathbf{A} \vDash \alpha \approx \beta$, if for all $h \in Hom(\mathbf{Fm}, \mathbf{A})$, $h(\alpha) = h(\beta)$. We say that the equation is valid in a class of algebras K, in symbols $K \vDash \alpha \approx \beta$, if it is valid in every algebra $\mathbf{A} \in K$. Thus, summarizing our notation, given an algebra $\mathbf{A}$, a class of algebras K and $\alpha, \beta \in Fm$,

$$\begin{aligned} \mathbf{A} \vDash \alpha \approx \beta\ [[h]] &\iff h(\alpha) = h(\beta) \quad h \in Hom(\mathbf{Fm}, \mathbf{A}) \\ \mathbf{A} \vDash \alpha \approx \beta &\iff \mathbf{A} \vDash \alpha \approx \beta\ [[h]] \text{ for all } h \in Hom(\mathbf{Fm}, \mathbf{A}) \\ K \vDash \alpha \approx \beta &\iff \mathbf{A} \vDash \alpha \approx \beta \text{ for all } \mathbf{A} \in K \end{aligned}$$

A **quasiequation** is a first-order formula of the form $\big(\alpha_0 \approx \beta_0 \wedge \cdots \wedge \alpha_{n-1} \approx \beta_{n-1}\big) \to \alpha \approx \beta$, for some $n \in \omega$. It is understood that for $n = 0$, this expression denotes the equation $\alpha \approx \beta$. A **generalized quasiequation** is an infinitary formula of the form $\big(\bigwedge\limits_{i\in I} \alpha_i \approx \beta_i\big) \to \alpha \approx \beta$, for some set I. The validity of a (generalized) quasiequation is defined in the obvious way: a quasiequation $\big(\alpha_0 \approx \beta_0 \wedge \cdots \wedge \alpha_n \approx \beta_n\big) \to \alpha \approx \beta$ is valid in an algebra $\mathbf{A}$, in symbols $\mathbf{A} \vDash \big(\alpha_0 \approx \beta_0 \wedge \cdots \wedge \alpha_n \approx \beta_n\big) \to \alpha \approx \beta$, if for any $h \in Hom(\mathbf{Fm}, \mathbf{A})$, if $h(\alpha_i) = h(\beta_i)$, for all $i < n$, then $h(\alpha) = h(\beta)$. Similarly for a generalized quasiequation. As above, a (generalized) quasiequation is valid in a class of algebras K, if it is valid in every algebra in K, in symbols $K \vDash \big(\alpha_0 \approx \beta_0 \wedge \cdots \wedge \alpha_n \approx \beta_n\big) \to \alpha \approx \beta$.

Lemma 2.3.1. *Let h be an embedding from the algebra $\mathbf{A}$ into the algebra $\mathbf{B}$. Then every equation and every (generalized) quasiequation valid in $\mathbf{B}$ is valid in $\mathbf{A}$.*

The following theorem is a version of Łoś's theorem for ultraproducts and it will be used in subsequent arguments.

Theorem 2.3.1. *Let $\{\langle \mathbf{A}_i, F_i\rangle : i \in I\}$ be a non-empty family of matrices and $\mathcal{U}$ an ultrafilter over I. Then, for any finite rule $\langle \Gamma, \varphi\rangle$,*

$$\{i \in I : \langle \mathbf{A}_i, F_i\rangle \models \langle \Gamma, \varphi\rangle\} \in \mathcal{U} \quad \textit{iff} \quad \prod_{i \in I} \langle \mathbf{A}_i, F_i\rangle/\mathcal{U} \models \langle \Gamma, \varphi\rangle.$$

The following are classic notions of universal algebra (see eg. [10],[3], [13]):

Definition 2.3.1. A **variety** V is a class of algebras axiomatized by a set of equations, i.e. if there is a set of equations E such that V contains all algebras (of the fixed similarity type) in which all equations in E are valid.
A **quasivariety** Q is a class of algebras axiomatized by a set of quasiequations, i.e. if there is a set of quasiequations QE such that Q contains all algebras (of the fixed type) in which all quasiequations in QE are valid.

Given a class of algebras K, the smallest variety containing K is denoted by $\mathbb{V}(K)$ and it is called the *variety generated by* K. Analogously, the smallest quasivariety containing K is denoted by $\mathbb{Q}(K)$ and it is called the *quasivariety generated by* K.
The following well-known results will be used later on:

Theorem 2.3.2. *(Tarski) For any class of algebras K, $\mathbb{V}(K) = \mathbb{HSP}(K)$.*

Theorem 2.3.3. *(Maltsev) For any class of algebras K, $\mathbb{Q}(K) = \mathbb{ISPP}_{\mathbb{U}}(K \cup \{\mathbf{A}_{tr}\})$.*

2.4 Atomic and universal Horn classes

For the purposes of this section, fix a first-order language with equality $\mathfrak{L}$. The *atomic $\mathfrak{L}$-formulas* are expressions of the form $\alpha = \beta$ between $\mathfrak{L}$-terms, or expressions $R(t_1, \dots, t_n)$ where $R \in \mathfrak{L}$ is a relation symbol of rank $n > 0$ and $t_1, \dots, t_n$ are $\mathfrak{L}$-terms. *Atomic sentences* are the universal closures $\forall \bar{x}\Phi$ of atomic formulas Φ.
A first-order structure will be called *κ-generated*, for some cardinal κ, if its pure algebra reduct has a generating set with **at most** κ-elements. It will be called *finitely generated* if it is κ-generated for some $\kappa < \omega$. Finally, it will be called *finite* if its universe is a finite set.

Definition 2.4.1. An **atomic class** is a class of structures axiomatized by a set of atomic sentences. The **atomic closure** of a class K of $\mathfrak{L}$-structures is the smallest atomic class containing K.

Theorem 2.4.1 ([25])**.** *Given a class K of $\mathfrak{L}$-structures, the atomic closure of K is $\mathbb{HSP}(K)$.*

The case of algebras in the following theorem was proved in [21], while the case of first-order $\mathfrak{L}$-structures was proved in [18]. It is accessible in [17] (Cor. 2.3.9).

Theorem 2.4.2 ([18])**.** *For any class K, $\mathbb{HSP}(K) = \mathbb{HP}_{\mathbb{SD}}(K)$.*

Thus, K is an atomic class iff it is closed under $\mathbb{H}, \mathbb{S}$ and $\mathbb{P}$, or equivalently under $\mathbb{H}$ and $\mathbb{P}_{\mathbb{SD}}$. Finally, if $\mathfrak{L}$ does not contain relation symbols, then atomic classes are just varieties of algebras.
Given two classes K_1, K_2 of $\mathfrak{L}$-structures, both closed under $\mathbb{I}, \mathbb{S}$ and $\mathbb{P}$, we say that K_1 is a **relative atomic subclass** of K_2, if $K_1 = K_2 \cap K$ for some atomic class K. The smallest relative atomic subclass of K_2 containing K_1 is $K_2 \cap \mathbb{H}(K_1)$.

The *strict universal Horn sentences* of $\mathfrak{L}$ are the first-order $\mathfrak{L}$-sentences of the form

$$\forall \bar{x}\big((\bigwedge_{i<n} \Phi_i) \to \Psi \big)$$

where $n \in \omega$ and $\Phi_0, \dots, \Phi_{n-1}, \Psi$ are atomic $\mathfrak{L}$-formulas. (If these atomic formulas are variable-free, then the quantifier is not required, i.e. $\bar{x}$ could be empty.)

Definition 2.4.2. A class K of $\mathfrak{L}$-structures is a (strict) **universal Horn class** if it can be axiomatized by a set of universal Horn $\mathfrak{L}$-sentences.

Theorem 2.4.3 ([19])**.** *Given a class K of $\mathfrak{L}$-structures, the smallest (strict) universal Horn class containing K is $\mathbb{ISPP}_{\mathbb{U}}(K)$.*

Thus, a class K of $\mathfrak{L}$-structures is a universal Horn class iff it is closed under $\mathbb{I}, \mathbb{S}, \mathbb{P}$ and $\mathbb{P}_{\mathbb{U}}$. Sometimes, in the literature, the term quasivarieties is used for strict universal Horn classes even though they do not consist of pure algebras. We shall keep using the term universal Horn class for them and use the other term only for classes of algebras.
A class K of $\mathfrak{L}$-structures has the **joint embedding property** if for any non-trivial structures $\mathcal{A}, \mathcal{B} \in K$, there is some $\mathcal{C} \in K$ such that both $\mathcal{A}, \mathcal{B}$ can be embedded into $\mathcal{C}$.

Theorem 2.4.4 ([24])**.** *A universal Horn class has the joint embedding property iff it is generated by a single structure.*

An $\mathfrak{L}$-structure $\mathcal{A}$ is said to be **locally embeddable** into a class of $\mathfrak{L}$-structures K if for every *finite* $X \subseteq A$, there is some structure $\mathcal{B} \in K$ and a mapping $h : X \to B$ such that:

- For every constant $c \in \mathfrak{L}$, if $c^{\mathcal{A}} \in X$, then $f(c^{\mathcal{A}}) = c^{\mathcal{B}}$.
- For any n-ary function symbol $f \in \mathfrak{L}$ and any $a_1, \dots, a_n \in X$ with $f^{\mathcal{A}}(a_1, \dots, a_n) \in X$, it holds that

$$h(f^{\mathcal{A}}(a_1, \dots, a_n)) = f^{\mathcal{B}}(h(a_1), \dots, h(a_n)).$$

- For any n-ary relation symbol $R \in \mathfrak{L}$ and any $a_1, \dots, a_n \in X$, if $R^{\mathcal{A}}(a_1, \dots, a_n)$, then $R^{\mathcal{B}}(h(a_1), \dots, h(a_n))$.

For a single structure $\mathcal{B}$, we say that $\mathcal{A}$ is locally embeddable into $\mathcal{B}$, meaning that $\mathcal{A}$ is locally embeddable into $\{\mathcal{B}\}$.

Theorem 2.4.5 ([17])**.** *Let $\mathcal{A}$ be an $\mathfrak{L}$-structure and K a class of $\mathfrak{L}$-structures. If $\mathcal{A}$ is locally embeddable into K, then $\mathcal{A}$ can be embedded into an ultraproduct of a non-empty subfamily of K. The converse holds if $\mathfrak{L}$ is finite.*

A universal Horn class K is called **primitive** if every universal Horn subclass of K is a relative atomic subclass of K, i.e. if for every universal Horn subclass K' of K, $K \cap \mathbb{H}(K') = K'$.

Proposition 2.4.1 ([17])**.** *If K is a primitive universal Horn class, then the lattice of universal Horn subclasses of K is distributive.*

Given an $\mathfrak{L}$-structure $\mathcal{A}$ and a class of $\mathfrak{L}$-structures K, we say that $\mathcal{A}$ is **relatively subdirectly irreducible** in K if whenever $\mathcal{A}$ is a subdirect product of a subfamily $\{\mathcal{B}_i : i \in I\}$ of K, then at least one of the projections $\pi_i : \prod_{i \in I} \mathcal{B}_i \to \mathcal{B}_i$, restricts to an isomorphism from $\mathcal{A}$ onto $\mathcal{B}_i$.

We say that $\mathcal{A}$ is **weakly projective** in a class of $\mathfrak{L}$-structures K, if for every $\mathcal{B} \in K$ such that $\mathcal{A}$ is a homomorphic image of $\mathcal{B}$, there is an embedding from $\mathcal{A}$ into $\mathcal{B}$ (recall that embeddings between $\mathfrak{L}$-structures are injective homomorphisms that reflect relations).
A universal Horn class K is called **locally finite** if every finitely generated member of K is finite.

Theorem 2.4.6 ([17]). *A locally finite universal Horn class K is primitive iff every finite relatively subdirectly irreducible member of K is weakly projective in K.*

2.5 Deduction theorem and the finite model property

For a thorough discussion on the deduction theorem we refer the reader to [13] and [4]. Recall that for a set of formulas $\Delta = \Delta(p,q)$ in at most two variables p, q and two formulas φ, ψ, the set $\Delta(\varphi, \psi)$ stands for $\{\chi(\varphi, \psi) : \chi \in \Delta\}$.

Definition 2.5.1. A logic $\mathcal{S}$ has the **local deduction-detachment theorem**, $\mathcal{LDDT}$ for short, with respect to a set of sets of formulas $\mathcal{E}$ in at most two variables p, q, if for every set of formulas $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$,

$$\Gamma, \varphi \vdash_{\mathcal{S}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \Delta(\varphi, \psi), \text{ for some } \Delta \in \mathcal{E}.$$

We say that $\mathcal{S}$ has the **global deduction-detachment theorem**, $\mathcal{GDDT}$ for short, with respect to $\mathcal{E}$ as above, if $\mathcal{E}$ contains only one set of formulas.
The set $\mathcal{E}$ is called an $\mathcal{LDDT}$-set (resp. $\mathcal{GDDT}$-set) for $\mathcal{S}$.
We say that $\mathcal{S}$ has the $\mathcal{LDDT}$ (resp. $\mathcal{GDDT}$) if there exists an $\mathcal{LDDT}$-set (resp. $\mathcal{GDDT}$-set) for $\mathcal{S}$.

The "hereditary version" of the deduction theorem can be naturally defined as follows:

Definition 2.5.2 ([11]). A *finitary* logic $\mathcal{S}$ has the **hereditary local (global) deduction-detachment theorem**, $\mathcal{HLDDT}$ (resp. $\mathcal{HGDDT}$) for short, with respect to a set of sets of formulas $\mathcal{E}$ if each of its **finitary** extensions has the $\mathcal{LDDT}$ (resp. $\mathcal{GDDT}$) with respect to $\mathcal{E}$.

For the purposes of our study we shall use the following notions:

Definition 2.5.3. We say that a logic $\mathcal{S}$ has the **finite (matrix) model property**, f.m.p. for short, with respect to a matrix semantics M of $\mathcal{S}$, if every non-theorem of $\mathcal{S}$ is invalidated by some finite matrix in M. We say that $\mathcal{S}$ has the **strong finite (matrix) model property**, s.f.m.p. for short, with respect to M, if every finite and underivable rule of $\mathcal{S}$ is invalidated by some finite matrix in M.
A logic $\mathcal{S}$ which has the f.m.p. with respect to $Matr(\mathcal{S})$ is also called **tabular**. (see eg. [32])

Although the notions of f.m.p. and s.f.m.p. in principle do not coincide, we will see later that for a finitary and structurally complete logic $\mathcal{S}$, $\mathcal{S}$ has the f.m.p. (with respect to some class of matrices M) iff it has the s.f.m.p. (see Corollary 3.3.1).

Definition 2.5.4. A logic $\mathcal{S}$ is **strongly finite** if $\vdash_{\mathcal{S}} = \vDash_M$, where M is a finite set of finite matrices.

Theorem 2.5.1 ([37]). *Every strongly finite logic is finitary.*

It is quite obvious that any strongly finite logic is tabular. As a partial converse, if a finitary and tabular logic $\mathcal{S}$ has the $\mathcal{GDDT}$, then it is strongly finite. This can be found in the second chapter of [13].

2.6 The Leibniz hierarchy

In this section will shall review a *portion* of the Leibniz hierarchy. We will not provide a detailed exposition of the hierarchy neither we will exhaust all its levels. The presentation that will be given is rather one that fits our framework. For a systematic and thorough analysis of the Leibniz hierarchy we refer the reader to [15] and [13].
Recall that the notation $\varphi(x,y)$ for formulas, or in general $\Delta(x,y)$ for sets of formulas, means that the variables of φ (resp. Δ) are among x, y.

Theorem 2.6.1 ([8]). *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i There is a set of formulas $\Delta(x,y) \subseteq Fm$ such that $\vdash_{\mathcal{S}} \Delta(x,x)$ and $x, \Delta(x,y) \vdash_{\mathcal{S}} y$.

ii For every algebra $\mathbf{A}$, $\Omega^{\mathbf{A}}$ is order-preserving on $Fi_{\mathcal{S}}\mathbf{A}$. That is, for every algebra $\mathbf{A}$ and every $F, G \in Fi_{\mathcal{S}}\mathbf{A}$, if $F \subseteq G$, then $\Omega^{\mathbf{A}}(F) \subseteq \Omega^{\mathbf{A}}(G)$.

iii $Matr^*(\mathcal{S})$ is closed under subdirect products.

A set that satisfies the conditions in i is often called a ***protoimplication set****. If $\mathcal{S}$ is finitary, then such a protoimplication set can be chosen finite.*

Definition 2.6.1. A logic $\mathcal{S}$ is **protoalgebraic** if it satisfies the equivalent conditions in Theorem 2.6.1.

Since $Alg(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}Alg^*(\mathcal{S})$, as an immediate corollary of condition iii of Theorem 2.6.1 we obtain:

Corollary 2.6.1. *If $\mathcal{S}$ is a protoalgebraic logic, then $Alg^*(\mathcal{S}) = Alg(\mathcal{S})$.*

Remark 2.6.1. Observe that any extension $\mathcal{S}'$ of a protoalgebraic logic $\mathcal{S}$ will also be protoalgebraic since the $\mathcal{S}'$-filters are also $\mathcal{S}$-filters and hence they satisfy the second condition of Theorem 2.6.1.

Definition 2.6.2. A set of formulas $\Delta(x,y) \subseteq Fm$ is a **set of equivalence formulas** for a logic $\mathcal{S}$ if for every $\langle \mathbf{A}, F\rangle \in Matr(\mathcal{S})$ and every $a, b \in A$,

$$\langle a,b\rangle \in \Omega^{\mathbf{A}}(F) \quad \text{iff} \quad \Delta^{\mathbf{A}}(a,b) \subseteq F.$$

We say that $\mathcal{S}$ is **equivalential** if it has a set of equivalence formulas.

Remark 2.6.2. As in the remark 2.6.1, any extension $\mathcal{S}'$ of an equivalential logic $\mathcal{S}$ will also be equivalential since $\mathcal{S}'$-filters are $\mathcal{S}$-filters and therefore they would satisfy the definition 2.6.2.

Thus, equivalence formulas play the role of generalized biconditional $\leftrightarrow$ and by Theorem 2.2.2 it follows that if $\Delta(x,y)$ and $\Delta'(x,y)$ are two sets of equivalence formulas for $\mathcal{S}$, then they are $\mathcal{S}$-interderivable, i.e. $\Delta(x,y) \dashv\vdash_{\mathcal{S}} \Delta'(x,y)$.
Equivalential logics firstly appear in [31] were the following lemma was taken as a definition:

Lemma 2.6.1 ([38]). *A set of formulas $\Delta(x,y)$ is a set of equivalence formulas for $\mathcal{S}$ iff*

- $\vdash_{\mathcal{S}} \Delta(x,x)$
- $x, \Delta(x,y) \vdash_{\mathcal{S}} y$
- $\Delta(x_1,y_1),\ldots,\Delta(x_n,y_n) \vdash_{\mathcal{S}} \Delta(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n))$*, for every connective $f \in L$, where n is the rank of f.*

The following summarizes a collection of theorems which can be found in [13], [5], [20].

Theorem 2.6.2. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *$\mathcal{S}$ is equivalential.*

ii *$\mathcal{S}$ is protoalgebraic and for every algebra $\mathbf{A}$, the Leibniz operator $\Omega^{\mathbf{A}}$ commutes with homomorphic inverse images of $\mathcal{S}$-filters of $\mathbf{A}$, i.e. for every $F \in Fi_{\mathcal{S}}\mathbf{A}$ and every $h \in Hom(\mathbf{B}, \mathbf{A})$*

$$h^{-1}[\Omega^{\mathbf{A}}(F)] = \Omega^{\mathbf{B}}(h^{-1}[F]).$$

(even if h is not surjective, in contrast with Proposition 2.2.3)

iii *$\mathcal{S}$ is protoalgebraic and whenever $\langle \mathbf{B}, G\rangle$ is a submatrix of some $\langle \mathbf{A}, F\rangle \in Matr(\mathcal{S})$, then $\Omega^{\mathbf{B}}(G) = (B \times B) \cap \Omega^{\mathbf{A}}(F)$.*

iv *$Matr^*(\mathcal{S})$ is closed under submatrices and direct products.*

Definition 2.6.3. A logic $\mathcal{S}$ is called **finitely equivalential** if it has a finite set of equivalence formulas.

Given a set A and a family $\mathcal{F}$ of subsets of A, we say that $\mathcal{F}$ is $\subseteq$-directed if for any $X, Y \in \mathcal{F}$, there is some $Z \in \mathcal{F}$ such that $X, Y \subseteq Z$.

Theorem 2.6.3 ([5],[20])**.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *$\mathcal{S}$ is finitely equivalential.*

ii *For every algebra $\mathbf{A}$, the Leibniz operator $\Omega^{\mathbf{A}}$ is continuous on $Fi_{\mathcal{S}}\mathbf{A}$, that is $\Omega^{\mathbf{A}}\big(\bigcup_{i\in I} F_i\big) = \bigcup_{i\in I}\Omega^{\mathbf{A}}(F_i)$, for any $\subseteq$-directed set $\{F_i : i \in I\}$ of $\mathcal{S}$-filters of $\mathbf{A}$, such that $\bigcup_{i\in I} F_i$ is still an $\mathcal{S}$-filter (which of course will be if $\mathcal{S}$ is finitary).*

Theorem 2.6.4 ([5],[12])**.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *$\mathcal{S}$ is finitary and finitely equivalential.*

ii *$Matr^*(\mathcal{S})$ is a universal Horn class (see 2.4.3).*

Clearly, any extension of a finitely equivalential logic will still be finitely equivalential.

Theorem 2.6.5 ([13])**.** *Assume $\mathcal{S}$ is finitary and finitely equivalential. All subclasses of $Matr^*(\mathcal{S})$ which are a matrix semantics for $\mathcal{S}$ generate the same universal Horn class. In other words, for any $K \subseteq Matr^*(\mathcal{S})$ such that $\mathcal{S}$ is complete with respect to K, it holds that $Matr^*(\mathcal{S}) = \mathbb{ISPP}_{\mathbb{U}}(K)$.*

We will now introduce a branch of the Leibniz hierarchy (in principle) incompatible with protoalgebraic logics, namely the truth-equational logics.

Theorem 2.6.6 ([33])**.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *There is a set of pairs of formulas (equations) $\tau(x)$ that defines the filters in $Matr^*(\mathcal{S})$, i.e. such that for every $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})$ and every $a \in A$*

$$a \in F \quad \textit{iff} \quad \delta_l^{\mathbf{A}}(a) = \delta_r^{\mathbf{A}}(a)\,, \textit{for all } \langle \delta_l, \delta_r\rangle \in \tau.$$

ii *For every algebra $\mathbf{A}$, $\Omega^{\mathbf{A}}$ is completely order-reflecting on $Fi_{\mathcal{S}}\mathbf{A}$. That is, for every algebra $\mathbf{A}$ and every family $\{F_i : i \in I\} \cup \{G\} \subseteq Fi_{\mathcal{S}}\mathbf{A}$, if $\bigcap_{i\in I}\Omega^{\mathbf{A}}(F_i) \subseteq \Omega^{\mathbf{A}}(G)$, then $\bigcap_{i\in I} F_i \subseteq G$.*

Definition 2.6.4. A logic $\mathcal{S}$ is **truth-equational** if it satisfies the equivalent conditions of Theorem 2.6.6.

Remark 2.6.3. As it happened in all levels of the Leibniz hierarchy that we've seen so far, being truth-equational persists in extensions since for any extension $\mathcal{S}'$ of a truth-equational logic $\mathcal{S}$, the set of equations that defines the filters in $Matr^*(\mathcal{S})$ defines also the filters in $Matr^*(\mathcal{S}')$.

Notation: Let $\tau(x)$ be a set of equations in at most the variable x, $\mathbf{A}$ an algebra and $F \subseteq A$. We will use the following notation:

$$\tau\mathbf{A} := \{a \in A : \delta_l^{\mathbf{A}}(a) = \delta_r^{\mathbf{A}}(a), \text{ for all } \langle\delta_l, \delta_r\rangle \in \tau\}$$

and

$$\tau^{\mathbf{A}}[F] := \{\langle\delta_l^{\mathbf{A}}(a), \delta_r^{\mathbf{A}}(a)\rangle : \langle\delta_l, \delta_r\rangle \in \tau, a \in F\}.$$

When $F = \{a\}$, we just write $\tau^{\mathbf{A}}(a)$ instead of $\tau^{\mathbf{A}}[\{a\}]$. Under this notation, $\tau^{\mathbf{A}}[F]$ becomes $\bigcup_{a \in F} \tau^{\mathbf{A}}(a)$. Following standard practice, when $\mathbf{A} = \mathbf{Fm}$, we omit the superscript. Moreover, as an immediate consequence of the first item of Theorem 2.6.6 it follows that if $\langle\mathbf{A}, F\rangle$ and $\langle\mathbf{A}, G\rangle$ are reduced matrix models of a truth-equational logic $\mathcal{S}$, then $F = G = \tau\mathbf{A}$. Finally, a set of equations $\tau(x)$ that defines the filters in $Matr^*(\mathcal{S})$ of a logic $\mathcal{S}$, is also called *a set of defining equations for* $\mathcal{S}$. If such a set of equations exists for $\mathcal{S}$, then we say that *truth is equationally definable in* $Matr^*(\mathcal{S})$.
Given a set of equations $\tau(x)$ and a class of algebras K, the logic induced by the class of matrices $\{\langle\mathbf{A}, \tau\mathbf{A}\rangle : \mathbf{A} \in K\}$ is called the **τ-assertional logic of K** and is usually denoted by $\mathcal{S}_K^\tau$. If a logic $\mathcal{S}$ is the τ-assertional logic of a class of algebras K, i.e. $\mathcal{S} = \mathcal{S}_K^\tau$, then the class K is called a **τ-algebraic semantics** for $\mathcal{S}$. It is proven in [33] that if $\mathcal{S}$ is a truth-equational logic with $\tau(x)$ a set of defining equations, then both classes $Alg(\mathcal{S})$ and $Alg^*(\mathcal{S})$ are τ-algebraic semantics for $\mathcal{S}$.
The following can be found in section 6.4 of [15]:

Proposition 2.6.1. *For any truth-equational logic $\mathcal{S}$ and any algebra $\mathbf{A}$, the Leibniz operator $\Omega^{\mathbf{A}} : Fi_{\mathcal{S}}\mathbf{A} \to Co\mathbf{A}$ is injective.*

Remark 2.6.4. In particular, it follows by the above proposition that every truth - equational logic has theorems, since otherwise $\varnothing \in Fi_{\mathcal{S}}\mathbf{A}$, for every algebra $\mathbf{A}$, by the third item of Proposition 2.2.1. But then, if $\mathbf{A} = \mathbf{Fm}$, it follows that $\Omega(\varnothing) = Fm \times Fm = \Omega(Fm)$ and thus Ω is not injective on $Th(\mathcal{S})$, contradicting Proposition 2.6.1.

Theorem 2.6.7 ([14])**.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *$\mathcal{S}$ is both protoalgebraic and truth-equational.*

ii *For every algebra $\mathbf{A}$ the map $F \mapsto \Omega^{\mathbf{A}}(F)$ is injective and order-preserving on $Fi_{\mathcal{S}}\mathbf{A}$.*

iii *For every algebra $\mathbf{A}$, the map $F \mapsto \Omega^{\mathbf{A}}(F)$ defines a lattice isomorphism from $Fi_{\mathcal{S}}\mathbf{A}$ onto the $Alg(\mathcal{S})$-congruences of $\mathbf{A}$, that is, the congruences $\theta \in Con\mathbf{A}$ such that $\mathbf{A}/\theta \in Alg(\mathcal{S})$.*

Definition 2.6.5 ([14])**.** A logic $\mathcal{S}$ is **weakly algebraizable** if it satisfies the equivalent conditions of Theorem 2.6.7.

By combining the remarks 2.6.1 and 2.6.3, it follows that any extension of a weakly algebraizable logic will be weakly algebraizable itself.

We shall close this section with the final levels of the Leibniz hierarchy that concern our study, namely the algebraizable and the finitely algebraizable logics, which were extensively studied in [7]. Firstly, we will review some basic facts.
Recall that an equation is just a pair of formulas $\langle\varphi,\psi\rangle$. Following standard practice, we write an equation as $\varphi \approx \psi$ but in general we will use both notations without distinction. Moreover recall that we fixed an arbitrary algebraic similarity type L. Thus, formulas are L-formulas and L-equations are pairs of L-formulas. We denote by Eq the set of L-equations of our fixed L.

Definition 2.6.6. The **relative equational consequence** associated with a class of algebras K is the relation $\vDash_K$ between sets of equations and equations, defined by:

$$\begin{aligned}\Pi \vDash_K \varepsilon \approx \delta \iff & \text{ For every } \mathbf{A} \in K \text{ and every } h \in Hom(\mathbf{Fm}, \mathbf{A}),\\ & \text{if } \mathbf{A} \vDash \alpha \approx \beta\ [[h]] \text{ for all } \alpha \approx \beta \in \Pi, \text{ then } \mathbf{A} \vDash \varepsilon \approx \delta\ [[h]].\end{aligned}$$

Given a class of algebras K and two sets of equations Π and Π', we write $\Pi \vDash_K \Pi'$ meaning that $\Pi \vDash_K \alpha \approx \beta$, for any equation $\alpha \approx \beta \in \Pi'$. Moreover, a substitution $\sigma \in End\,\mathbf{Fm}$ can be applied to an equation $\alpha \approx \beta$ by defining $\sigma(\alpha \approx \beta) := \sigma(\alpha) \approx \sigma(\beta)$. For sets of equations Π we set $\sigma[\Pi] := \{\sigma(\alpha \approx \beta) : \alpha \approx \beta \in \Pi\}$. It is easy to check that given a class of algebras K, the equational consequence relative to K has the following property: for any substitution σ, if $\Pi \vDash_K \alpha \approx \beta$, then $\sigma[\Pi] \vDash_K \sigma(\alpha \approx \beta)$. Finally, recall that for a set of equations $\tau(x)$ and $\Gamma \cup \{\gamma\} \subseteq Fm$, $\tau[\Gamma]$ and $\tau(\gamma)$ are just $\tau^{\mathbf{Fm}}[\Gamma]$ and $\tau^{\mathbf{Fm}}(\gamma)$, respectively. We shall also use the following notation: given a set of formulas in two variables $\rho(x,y)$ and given a set of equations Π, we set $\rho[\Pi] := \bigcup\{\rho(\alpha,\beta) : \alpha \approx \beta \in \Pi\}$.

Lemma 2.6.2. *Let $\mathcal{S}$ be a logic and K a class of algebras. Let $\tau(x)$ and $\rho(x,y)$ be a set of equations and a set of formulas respectively. For any $\Gamma \cup \{\varphi\} \subseteq Fm$ and any $\Pi \cup \{\alpha \approx \beta)\} \subseteq Eq$, the following are pairwise equivalent:*

1. $\Gamma \vdash_{\mathcal{S}} \varphi \iff \tau[\Gamma] \vDash_K \tau(\varphi)$
2. $\alpha \approx \beta \mathrel{=\joinrel\mathrel|\joinrel\vDash}_K \tau[\rho(\alpha,\beta)]$

and

3. $\Pi \vDash_K \alpha \approx \beta \iff \rho[\Pi] \vdash_{\mathcal{S}} \rho(\alpha,\beta)$
4. $\varphi \dashv\vdash_{\mathcal{S}} \rho[\tau(\varphi)]$

Definition 2.6.7. A logic $\mathcal{S}$ is **algebraizable** if there is class of algebras K, a set of equations $\tau(x)$ and a set of formulas $\rho(x,y)$ (usually called a set of equivalence formulas for $\mathcal{S}$), that satisfy the pairwise equivalent conditions of Lemma 2.6.2. It is called **finitely algebraizable** if ρ is a finite set.

Definition 2.6.8. Let $\mathcal{S}$ be an algebraizable logic. Its **equivalent algebraic semantics** is the largest class of algebras K for which there is a set of equations $\tau(x)$ and a set of formulas $\rho(x,y)$ that satisfy the pairwise equivalent conditions of Lemma 2.6.2 with respect to K.

A **generalized quasivariety** is a class of algebras defined by generalized quasiequations (see section 2.3). For a class of algebras K, the class $\mathbb{GQ}(K)$ is the class of *all* algebras where all the generalized quasiequations that hold in K are valid. Clearly, $\mathbb{GQ}(K)$ is itself a generalized quasivariety and it is the least generalized quasivariety containing K.

Lemma 2.6.3. *Assume that $\mathcal{S}$ is an algebraizable logic with respect to K. Then $\mathbb{GQ}(K)$ is the largest class of algebras, and the only generalized quasivariety, with respect to which $\mathcal{S}$ is algebraizable.*

In particular, if $\vDash_K$ is finitary, then we know that $\mathbb{GQ}(K) = \mathbb{Q}(K)$. In such case, we say that $\mathbb{Q}(K)$ is the **equivalent quasivariety of** $\mathcal{S}$.

Theorem 2.6.8. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *$\mathcal{S}$ is algebraizable,*

ii *$\mathcal{S}$ is equivalential and weakly algebraizable.*

iii *$\mathcal{S}$ is equivalential and truth-equational.*

Corollary 2.6.2. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *$\mathcal{S}$ is finitely algebraizable.*

ii *$\mathcal{S}$ is finitely equivalential and weakly algebraizable.*

iii *$\mathcal{S}$ is finitely equivalential and truth-equational.*

The behaviour of the Leibniz operator in algebraizable and finitely algebraizable logics can be easily determined by combining Theorem 2.6.8 and Corollary 2.6.2 with the Theorems 2.6.2, 2.6.7, 2.6.6 and 2.6.3.

Definition 2.6.9. A logic $\mathcal{S}$ is **algebraizable in the sense of Blok and Pigozzi** (for short, **BP-algebraizable**) if it is finitary and finitely algebraizable.

The following diagram summarizes the *portion* of the Leibniz hierarchy that concern our study. Arrows correspond to class inclusion.

$$
\begin{tikzcd}
 & \textbf{finitely algebraizable} \arrow[dl] \arrow[dr] & & & \\
\textbf{finitely equivalential} \arrow[dr] & & \textbf{algebraizable} \arrow[dl] \arrow[dr] & & \\
 & \textbf{equivalential} \arrow[dr] & & \textbf{weakly algebraizable} \arrow[dl] \arrow[dr] & \\
 & & \textbf{protoalgebraic} & & \textbf{truth-equational}
\end{tikzcd}
$$

Figure 2.1: A portion of the Leibniz hierarchy.

Chapter 3

Structural Completeness

3.1 Syntactic conditions

Definition 3.1.1 ([22])**.** A rule $\langle \Gamma, \varphi \rangle$ is **admissible** for a logic $\mathcal{S}$ if $\mathcal{S}$ and $\mathcal{S} + \langle \Gamma, \varphi \rangle$ have the same theorems.

Clearly, if $\mathcal{S}$ is finitary and $\langle \Gamma, \varphi \rangle$ is a finite rule, then $\mathcal{S} + \langle \Gamma, \varphi \rangle$ is still finitary. However, as explained in Chapter 2 when we speak of a logic $\mathcal{S}$ we do not assume its finitarity. Throughout the years, admissible rules and structurally complete systems have been studied extensively by several researchers. Although finitary logics and finite rules were mostly considered in these investigations, several obtained results could be generalized without assuming finitarity. The following is such an example of a well-known result:

Proposition 3.1.1. *A rule $\langle \Gamma, \varphi \rangle$ is admissible for a logic $\mathcal{S}$ iff for any substitution $\sigma \in End\,\mathbf{Fm}$, if $\vdash_{\mathcal{S}} \sigma[\Gamma]$, then $\vdash_{\mathcal{S}} \sigma(\varphi)$.*

Definition 3.1.2 ([27])**.** A logic $\mathcal{S}$ is called **structurally complete** if every proper *finitary* extension of $\mathcal{S}$ has some new theorems.

Definition 3.1.3 ([26])**.** A *finitary* logic $\mathcal{S}$ is called **hereditarily structurally complete** if all of its *finitary* extensions are structurally complete.

The first question that naturally arises from the definitions is whether if the notions of structural completeness and hereditary structural completeness are essentially different (for finitary logics), i.e. is there any finitary logic which is structurally complete but not hereditarily so? Although it is not a common phenomenon, the answer is affirmative. *Medvedev's logic of finite problems* is an example of a finitary logic that is structurally complete but not hereditarily so (see [29]).
The following characterization is an immediate consequence of the definitions:

Theorem 3.1.1. *A logic $\mathcal{S}$ is structurally complete iff every finite rule admissible for $\mathcal{S}$ is derivable in $\mathcal{S}$.*

The next theorem was essentially proved in [23]. Observe that by omitting or by adding all parenthesis we can obtain two theorems by it.

Theorem 3.1.2. *Let $\mathcal{S}$ be a (finitary) logic. The following conditions are equivalent:*

i Every admissible (finite) rule of $\mathcal{S}$ is derivable in $\mathcal{S}$.

ii For every (finitary) logic $\mathcal{S}'$ (of the same type), if $\mathcal{S}$ and $\mathcal{S}'$ have the same theorems, then $\mathcal{S}' \leq \mathcal{S}$.

Proof. We will prove it for the general case. The proof can be adjusted accordingly for the finitary case.
$i \Rightarrow ii$: Pick any logic $\mathcal{S}'$ (of the same type) with the same theorems as $\mathcal{S}$. Pick any $\Gamma \cup \{\varphi\} \subseteq Fm$ with $\Gamma \vdash_{\mathcal{S}'} \varphi$. Since $\mathcal{S}$ and $\mathcal{S}'$ have the same theorems, by Proposition 3.1.1 it follows that $\langle \Gamma, \varphi \rangle$ is an admissible rule for $\mathcal{S}$ and therefore by i, it is derivable in $\mathcal{S}$, i.e. $\Gamma \vdash_{\mathcal{S}} \varphi$. Hence $\mathcal{S}' \leq \mathcal{S}$.
$ii \Rightarrow i$: Pick any admissible rule $\langle \Gamma, \varphi \rangle$ for $\mathcal{S}$. Then $\mathcal{S} + \langle \Gamma, \varphi \rangle$ is an extension of $\mathcal{S}$ with the same theorems. By ii, $\mathcal{S} + \langle \Gamma, \varphi \rangle \leq \mathcal{S}$. Hence $\mathcal{S} = \mathcal{S} + \langle \Gamma, \varphi \rangle$ and thus the rule $\langle \Gamma, \varphi \rangle$ is derivable in $\mathcal{S}$. □

A *monoid* is a pair $\langle A, * \rangle$, where A is a set and $*$ is a binary operation $* : A \times A \to A$ which is associative, i.e. $(a * b) * c = a * (b * c)$, for all $a, b, c \in A$, and there is an element $e \in A$ (called *neutral*) such that $e * a = a * e = a$, for all $a \in A$. A *submonoid* of a monoid $\langle A, * \rangle$ is any pair $\langle B, * \rangle$ such that $B \subseteq A$ and B is closed under $*$ and contains the neutral element e of A. That is, if $e \in B$ and for all $x, y \in B$, $x * y \in B$. Finally, if $\langle A, * \rangle$ is a monoid and $X \subseteq A$, then the *monoid generated by* X is the least submonoid of $\langle A, * \rangle$ containing X.
For example, the set of self-maps $Sm\big(\mathcal{P}(Fm)\big)$ on $\mathcal{P}(Fm)$ becomes a monoid when we endow it with the operation $\circ$ of function composition. Its neutral element is the identity function $id_{\mathcal{P}(Fm)}$ on $\mathcal{P}(Fm)$.

Lemma 3.1.1. *The composition $C_1 \circ C_2$ of two structural and finitary closure operators on Fm is an extensive, monotone, finitary and structural self-map on $\mathcal{P}(Fm)$.*

Proof. Suppose C_1, C_2 are two structural and finitary closure operators on Fm. Denote $C = C_1 \circ C_2 : \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ and let $X, Y \subseteq Fm$.

- $X \subseteq C(X)$: Clearly, $X \subseteq C_2(X)$ and thus by monotonicity of C_1, $C_1(X) \subseteq C_1(C_2(X))$. But since $X \subseteq C_1(X)$, it follows that $X \subseteq C(X)$.
- $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$: Immediate. Apply first monotonicity of C_2 and then monotonicity of C_1.
- Finitarity of C: Clearly, for any finite $Z \subseteq X$, $C(Z) \subseteq C(X)$ by monotonicity of C. Thus, $\bigcup\{C(Z) : Z \subseteq X \text{ and } Z \text{ is finite}\} \subseteq C(X)$. Conversely, let $\varphi \in C(X)$. Then, $\varphi \in C_1(C_2(X))$. By finitarity of C_1, there is a finite $A \subseteq C_2(X)$ such that $\varphi \in C_1(A)$. Since for each $\psi \in A$, $\psi \in C_2(X)$, then by finitarity of C_2, there are B_ψ finite subsets of X such that $\psi \in C_2(B_\psi)$. Thus, $A \subseteq \bigcup\{C_2(B_\psi) : \psi \in A\}$. Let $B = \bigcup\{B_\psi : \psi \in A\}$. Since A is finite and each B_ψ is finite, then B is a finite union of finite sets. Thus, it is itself a finite subset of X. Moreover, $B_\psi \subseteq B$ for each $\psi \in A$ and therefore $C_2(B_\psi) \subseteq C_2(B)$. Hence $\bigcup\{C_2(B_\psi) : \psi \in A\} \subseteq C_2(B)$ and consequently, $A \subseteq C_2(B)$. Now, apply monotonicity of C_1. It follows that $C_1(A) \subseteq C_1(C_2(B)) = C(B)$. Hence, $\varphi \in C(B)$ and B is a finite subset of X.
- Structurality of C: Let σ be any substitution and suppose $\varphi \in C(X)$. Then $\varphi \in C_1(C_2(X))$ and therefore by structurality of C_1, $\sigma(\varphi) \in C_1(\sigma[C_2(X)]$. By structurality of C_2, $\sigma[C_2(X)] \subseteq C_2(\sigma[X])$ and by monotonicity of C_1, $C_1(\sigma[C_2(X)]) \subseteq C_1(C_2[\sigma[X]))$. Thus, $\sigma(\varphi) \in C(\sigma[X])$.

Obviously, the above result can be generalized for the composition of finitely-many such operators $C_1, \ldots, C_n$, $n \in \omega$. □

Before stating the following Proposition we shall make two observations. Firstly, the identity function on $\mathcal{P}(Fm)$ is a structural and finitary (every set can be written as a union of its finite subsets) closure operator on Fm. Secondly, if for some $X \subseteq Fm$, $C_1(X) = C_2(X)$, where C_1, C_2 are two structural and finitary closure operators on Fm, then $\big(C_1 \circ C_2\big)(X) = \big(C_2 \circ C_1\big)(X) = C_1(X) = C_2(X)$, due to idempotence of C_1, C_2. Clearly, this can be generalized for $C_1 \circ \cdots \circ C_n$ with $C_1(X) = \cdots = C_n(X)$, $n \in \omega$.

For convenience, we use the notation $\mathcal{S} \leq^{fin} \mathcal{S}'$ for *finitary* extensions $\mathcal{S}'$ of $\mathcal{S}$.

Proposition 3.1.2 ([2]). *Every finitary logic $\mathcal{S}$ has a unique structurally complete finitary extension with the same theorems.*

Proof. Let $\mathcal{S}$ be a finitary logic. Let $X = \{Cn_{\mathcal{S}'} : \mathcal{S} \leq^{fin} \mathcal{S}' \text{ and } Cn_{\mathcal{S}}(\varnothing) = Cn_{\mathcal{S}'}(\varnothing)\}$. Observe that X is a subset of (the universe of) the monoid $\langle Sm\big(\mathcal{P}(Fm)\big), \circ\rangle$. However, it is not necessarily a submonoid since for example it may not be closed under $\circ$ or maybe $id_{\mathcal{P}(Fm)} \notin X$ if $\mathcal{S}$ contains more than one theorem (or more generally if $\varphi \vdash_{\mathcal{S}} \psi$ for some $\varphi \neq \psi$). So, let X^* be the submonoid of $\langle Sm\big(\mathcal{P}(Fm)\big), \circ\rangle$ generated by X. We define a mapping $C : \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ by:

$$C(\Gamma) := \bigcup\{G(\Gamma) : G \in X^*\}.$$

Claim: $C \in X$

Observe that extensiveness, monotonicity, finitarity and structurality of C follow immediately by Lemma 3.1.1 since the elements of X^* are of the form $Cn_{\mathcal{S}_1} \circ \cdots \circ Cn_{\mathcal{S}_n}$, where $Cn_{\mathcal{S}_1}, \ldots, Cn_{\mathcal{S}_n} \in X$, $n \in \omega$, together with $id_{\mathcal{P}(Fm)}$. Thus, we only check that C is idempotent. Clearly, since C is extensive, $C(\Gamma) \subseteq C(C(\Gamma))$. Firstly, we check that the family $\{G(\Gamma) : G \in X^*\}$ is $\subseteq$-directed. Let $G_1, G_2 \in X^*$. The map $D = G_1 \circ G_2$ is in X^* since X^* is closed under $\circ$. Moreover, for every $\Delta \subseteq Fm$, $\Delta \subseteq G_2(\Delta)$ by extensiveness of G_2 and hence $G_1(\Delta) \subseteq G_1(G_2(\Delta))$. Clearly also $G_2(\Delta) \subseteq G_1(G_2(\Delta))$ by extensiveness of G_1. Thus, $G_1(\Gamma), G_2(\Gamma) \subseteq D(\Gamma)$ and $D \in X^*$. Clearly this property can be generalized inductively, for any finitely-many $G_1(\Gamma), \ldots, G_n(\Gamma)$, $n \in \omega$.
Now, for the other inclusion, let $\varphi \in C(C(\Gamma))$. By finitarity of C, there is some finite $Y \subseteq C(\Gamma)$ such that $\varphi \in C(Y)$; hence for some $G \in X^*$, $\varphi \in G(Y)$. Since for each $\psi \in Y$, $\psi \in C(\Gamma)$, there are $G_\psi \in X^*$ such that $\psi \in G_\psi(\Gamma)$ and therefore $Y \subseteq \bigcup\{G_\psi(\Gamma) : \psi \in Y\}$. Since Y is finite, then by $\subseteq$-directedness there is some $D \in X^*$ such that $G_\psi(\Gamma) \subseteq D(\Gamma)$ for all $\psi \in Y$ and thus $Y \subseteq \bigcup\{G_\psi(\Gamma) : \psi \in Y\} \subseteq D(\Gamma)$. Apply monotonicity of G. It follows that $G(Y) \subseteq G(D(\Gamma))$ and since X^* is closed under $\circ$, then $G \circ D \in X^*$. Thus, $\varphi \in C(\Gamma)$.
So far we have proven that C is a finitary and structural closure operator on Fm. Let $\mathcal{S}_C$ be the logic induced by C, i.e. $\mathcal{S}_C$ is the logic whose consequence relation is defined by:

$$\Gamma \vdash_{\mathcal{S}_C} \varphi \ \text{ iff } \ \varphi \in C(\Gamma).$$

It is obvious that $\mathcal{S} \leq \mathcal{S}_C$ since $Cn_{\mathcal{S}} \in X^*$. In order to see that $\mathcal{S}_C$ and $\mathcal{S}$ have the same theorems, we consider two cases: If $\mathcal{S}$ has no theorems, then for all $G \in X$, $G(\varnothing) = \varnothing$ and therefore $\big(G_1 \circ \cdots \circ G_n\big)(\varnothing) = \varnothing$, for any $G_1, \ldots, G_n \in X$, as we remarked earlier. Moreover, it is obvious that the identity mapping on $\mathcal{P}(Fm)$ induces a logic with no theorems. Thus, under the observation we did for X^*, it follows that $C(\varnothing) = \varnothing$ and therefore $\mathcal{S}_C$ has no theorems either (if $\mathcal{S}$ has no theorems, then $\mathcal{S}_C$ is the almost inconsistent logic, see Proposition 3.1.4).
Now assume that $\mathcal{S}$ has theorems and let $\varphi \in C(\varnothing)$. Then $\varphi \in G(\varnothing)$ for some $G \in X^*$.

Clearly, $G(\varnothing) \neq \varnothing$ and therefore G cannot be the identity mapping on $\mathcal{P}(Fm)$. Thus, $G = G_1 \circ \cdots \circ G_n$, for some $G_1, \ldots, G_n \in X$ and therefore $\varphi \in G(\varnothing) = \big(G_1 \circ \cdots \circ G_n\big)(\varnothing) = Cn_{\mathcal{S}}(\varnothing)$, since $G_1(\varnothing) = \cdots = G_n(\varnothing) = Cn_{\mathcal{S}}(\varnothing)$. Consequently, $\mathcal{S}$ and $\mathcal{S}_C$ have the same theorems in any case and therefore $C \in X$.
In order to see that $\mathcal{S}_C$ is structurally complete pick any proper finitary extension $\mathcal{S}'$ of $\mathcal{S}_C$. If $\mathcal{S}'$ has the same theorems as $\mathcal{S}_C$ (which are the same as $\mathcal{S}$) then $Cn_{\mathcal{S}'} \in X$ and therefore $\mathcal{S}' \leq \mathcal{S}_C$, a contradiction. Thus, $\mathcal{S}'$ cannot have the same theorems as $\mathcal{S}_C$. Finally, for the uniqueness of $\mathcal{S}_C$, let $\mathcal{S}'$ be a structurally complete finitary extension of $\mathcal{S}$ with the same theorems. Since $\mathcal{S}$, $\mathcal{S}_C$ and $\mathcal{S}'$ all have the same theorems and $\mathcal{S}'$ and $\mathcal{S}_C$ both are structurally complete, then by Theorem 3.1.2 it follows that $\mathcal{S}' \leq \mathcal{S}_C$ and $\mathcal{S}_C \leq \mathcal{S}'$ and hence they are equal. □

Proposition 3.1.1 established that a rule is admissible for a logic $\mathcal{S}$ iff it is validated by the matrix $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle$ which is equivalent to being validated by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ by Lemma 2.2.2 and Proposition 2.2.4. Under this observation, the infinitary analogue of Proposition 3.1.2, which could be stated as, "Every logic $\mathcal{S}$ has a unique extension with the same theorems in which every admissible rule is derivable", becomes a triviality since such an extension would be the logic induced by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle$ (or $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$). However, such a logic is normally *not* finitary and the next example witnesses this fact:

Example 3.1.1 ([32])**.** The *intermediate implicational logics* are the finitary extensions of the $\{\to\}$-fragment of intuitionistic logic. All of these logics are structurally complete (hence hereditarily so) but only the tabular logics among them can derive *all* of their own admissible rules, even the infinite ones (see [28] and [30]). Thus, every non-tabular logic in this class is a finitary logic whose unique extension that derives all of its admissible rules and has the same theorems, is not finitary. There are $2^{\aleph_0}$ non-tabular logics of this kind (see [39]).

Proposition 3.1.3 ([2])**.** *Let $\mathcal{S}$ be a finitary logic. The following conditions are equivalent:*

i $\mathcal{S}$ is hereditarily structurally complete.

ii For any $\mathcal{S}'$ with $\mathcal{S} \leq^{fin} \mathcal{S}'$ and any $\Gamma \subseteq Fm$, $Cn_{\mathcal{S}'}(\Gamma) = Cn_{\mathcal{S}}(\Gamma \cup Cn_{\mathcal{S}'}(\varnothing))$.

Proof. $i \Rightarrow ii$: Pick any $\mathcal{S}'$ with $\mathcal{S} \leq^{fin} \mathcal{S}'$ and denote $\Lambda = Cn_{\mathcal{S}'}(\varnothing)$. Observe that $\Lambda \subseteq Cn_{\mathcal{S}}(\Lambda) \subseteq Cn_{\mathcal{S}'}(\Lambda) = \Lambda$. Define a closure operator C on Fm by setting $C(\Gamma) := Cn_{\mathcal{S}}(\Lambda \cup \Gamma)$. A trivial checking shows that C is indeed a finitary and structural closure operator. For any $\Gamma \subseteq Fm$,

$$C(\Gamma) = Cn_{\mathcal{S}}(\Lambda \cup \Gamma) \subseteq Cn_{\mathcal{S}'}(\Lambda \cup \Gamma) \subseteq Cn_{\mathcal{S}'}(\Lambda \cup Cn_{\mathcal{S}'}(\Gamma)) = Cn_{\mathcal{S}'}\big(Cn_{\mathcal{S}'}(\Gamma)\big) = Cn_{\mathcal{S}'}(\Gamma).$$

Thus, if $\mathcal{S}_C$ is the logic induced by C, then $\mathcal{S} \leq \mathcal{S}_C \leq \mathcal{S}'$. Moreover, by setting $\Gamma = \varnothing$ we obtain that $\Lambda = C(\varnothing) \subseteq Cn_{\mathcal{S}'}(\varnothing) = \Lambda$. Hence $\mathcal{S}'$ and $\mathcal{S}_C$ have the same theorems. Finally, since $\mathcal{S}$ is hereditarily structurally complete, it follows that $C = Cn_{\mathcal{S}'}$, i.e. $\mathcal{S}' = \mathcal{S}_C$.
$ii \Rightarrow i$: Pick any finitary extension $\mathcal{S}'$ of $\mathcal{S}$ and any finitary extension $\mathcal{S}''$ of $\mathcal{S}'$ with the same theorems as $\mathcal{S}'$. We show that $\mathcal{S}' = \mathcal{S}''$. Since they have the same theorems, $\Lambda = Cn_{\mathcal{S}'}(\varnothing) = Cn_{\mathcal{S}''}(\varnothing)$. Thus, $Cn_{\mathcal{S}''}(\Gamma) = Cn_{\mathcal{S}}(\Gamma \cup \Lambda) = Cn_{\mathcal{S}'}(\Gamma)$ and we are done. □

The next Theorem has a double meaning, as it happened with Theorem 3.1.2. Its finitary case was proven in [34]. We will prove it for the general case. Notice that under our definition of axiomatic extensions (see definition 2.1.3 if needed), any logic $\mathcal{S}$ is considered as an axiomatic extension of itself.

Theorem 3.1.3 ([34]). *Let $\mathcal{S}$ be a (finitary) logic. The following are equivalent:*

i For every (finitary) extension $\mathcal{S}'$ of $\mathcal{S}$, all admissible (finite) rules of $\mathcal{S}'$ are derivable in $\mathcal{S}'$.

ii For every axiomatic extension $\mathcal{S}'$ of $\mathcal{S}$, all admissible (finite) rules of $\mathcal{S}'$ are derivable in $\mathcal{S}'$.

iii Every (finitary) extension of $\mathcal{S}$ is an axiomatic extension.

Proof. $i \Rightarrow ii$: This is clear. For the finitary case, just observe that an axiomatic extension of $\mathcal{S}$ is also a finitary extension of $\mathcal{S}$.
$ii \Rightarrow iii$: Pick an extension $\mathcal{S}'$ of $\mathcal{S}$. If $\mathcal{S}$ and $\mathcal{S}'$ have the same theorems, then by Theorem 3.1.2 and since condition ii holds for $\mathcal{S}$ itself, it follows that $\mathcal{S} = \mathcal{S}'$ and we have nothing to prove. Thus, we may assume that $\mathcal{S}'$ has more theorems. Let $\Delta := Cn_{\mathcal{S}'}(\varnothing)$. Clearly, Δ is closed under substitution. By our assumption, $\Delta \smallsetminus Cn_{\mathcal{S}}(\varnothing) \neq \varnothing$. Let $\mathcal{S}_\Delta$ be the logic whose consequence relation is such that

$$\Gamma \vdash_{\mathcal{S}_\Delta} \varphi \ \text{ iff } \ \Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi.$$

It is quite obvious that $\mathcal{S}_\Delta \leq \mathcal{S}'$ and that $\mathcal{S}_\Delta$ is the axiomatic extension of $\mathcal{S}$ by the set of axioms Δ (which essentially adds the axioms $\Delta \smallsetminus Cn_{\mathcal{S}}(\varnothing)$ to $\mathcal{S}$). Almost immediately it follows that $\mathcal{S}_\Delta$ and $\mathcal{S}'$ have the same theorems since for $\Gamma = \varnothing$:

$$\varphi \in Cn_{\mathcal{S}_\Delta}(\varnothing) \ \text{ iff } \ \Delta \vdash_{\mathcal{S}} \varphi \ \text{ iff } \ \varphi \in Cn_{\mathcal{S}'}(\varnothing).$$

By ii, all admissible rules of $\mathcal{S}_\Delta$ are derivable in $\mathcal{S}_\Delta$. Finally, since $\mathcal{S}_\Delta$ and $\mathcal{S}'$ have the same theorems, by Theorem 3.1.2, it follows that $\mathcal{S}' \leq \mathcal{S}_\Delta$ and therefore $\mathcal{S}' = \mathcal{S}_\Delta$, concluding that the arbitrary extension $\mathcal{S}'$ we picked is indeed axiomatic.
$iii \Rightarrow i$: Pick any extension $\mathcal{S}'$ of $\mathcal{S}$. Pick any admissible rule $\langle \Gamma, \varphi \rangle$ for $\mathcal{S}'$. Then, $\mathcal{S}'$ and $\mathcal{S}' + \langle \Gamma, \varphi \rangle$ have the same theorems. By iii, $\mathcal{S}' + \langle \Gamma, \varphi \rangle$ is an axiomatic extension of $\mathcal{S}'$. So there is some $\Delta \subseteq Fm$ closed under substitution such that

$$\Sigma \vdash_{\mathcal{S}' + \langle \Gamma, \varphi \rangle} \psi \ \text{ iff } \ \Sigma \cup \Delta \vdash_{\mathcal{S}'} \psi \ \text{ for any } \Sigma \cup \{\psi\} \subseteq Fm.$$

Since Δ is the set of axioms of $\mathcal{S}' + \langle \Gamma, \varphi \rangle$, and $\mathcal{S}'$, $\mathcal{S}' + \langle \Gamma, \varphi \rangle$ have the same theorems due to the admissibility of $\langle \Gamma, \varphi \rangle$ for $\mathcal{S}'$, it follows that $\Delta = Cn_{\mathcal{S}'}(\varnothing)$. Thus, $\mathcal{S}' = \mathcal{S}' + \langle \Gamma, \varphi \rangle$ and the rule $\langle \Gamma, \varphi \rangle$ is derivable in $\mathcal{S}'$. $\square$

The next useful but quite obvious observation shows that the notion of structural completeness becomes interesting only in logics with theorems:

Proposition 3.1.4. *Let $\mathcal{S}$ be a logic with no theorems. The following are equivalent:*

i $\mathcal{S}$ is structurally complete.

ii $\mathcal{S}$ is almost inconsistent.

iii For every extension $\mathcal{S}'$ of $\mathcal{S}$, every rule admissible for $\mathcal{S}'$ is derivable in $\mathcal{S}'$.

Proof. The direction $iii \Rightarrow i$ is obvious.
$i \Rightarrow ii$: It is enough to show that $x \vdash_{\mathcal{S}} y$ for two different variables x, y. Since $\mathcal{S}$ has no theorems, then the matrix $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing) \rangle$ vacuously validates the rule $\langle \{x\}, y \rangle$ and therefore it is an admissible finite rule for $\mathcal{S}$. By i, it is derivable in $\mathcal{S}$.
$ii \Rightarrow iii$: Since $\mathcal{S}$ is almost inconsistent, then $\varphi \vdash_{\mathcal{S}} \psi$ for any $\varphi, \psi \in Fm$. Moreover, for any admissible rule $\langle \Gamma, \varphi \rangle$, the set Γ is non-empty since otherwise the rule adds theorems.

But clearly, in an almost inconsistent logic, any rule $\langle\Gamma,\varphi\rangle$ with $\Gamma \neq \varnothing$ is derivable. Finally, the only extension of the almost inconsistent logic is the inconsistent logic in which trivially every rule is derivable. □

3.2 Structural (in)completeness in fragments of a logic

Recall that we work within a fixed but arbitrary algebraic similarity type L. In Chapter 2 we defined the L'-fragment of a logic $\mathcal{S}$ of type L, where $L' \subseteq L$ is an algebraic similarity type. Regarding structural completeness in fragments of a logic, a natural question arises: "If $\mathcal{S}$ is a structurally complete logic and $\mathcal{S}'$ is its L'-fragment, then is $\mathcal{S}'$ also structurally complete?"
A possible answer to this question would be: "Not necessarily". The notion of structural completeness is very sensitive even to small changes of the language. There are plenty of examples that witness this fact. One of them is the following:

Example 3.2.1 ([34]). First some notation. Given a logic $\mathcal{S}$ in language L and L' a sublanguage of L (i.e. L' is an algebraic similarity type and $L' \subseteq L$), denote by $\mathcal{S}_{L'}$ the L'-fragment of $\mathcal{S}$.
Let $\mathbf{R}$ be the principal *relevance logic* formulated in the language $\{\rightarrow,\wedge,\vee,\cdot,\neg,t\}$. Under our notation, $\mathbf{R}_{\{\rightarrow,\wedge\}}$ is the $\{\rightarrow,\wedge\}$-fragment of $\mathbf{R}$ and similarly for other subsets of connectives. Let $\mathbf{RM}$ (read $\mathbf{R}$-mingle) be the logic obtained from $\mathbf{R}$ by discarding the two axioms of $\mathbf{R}$ involving t (which are t and $t \rightarrow (p \rightarrow p)$) and by adding the *mingle axiom* $p \rightarrow (p \rightarrow p)$. The logic $\mathbf{RM}$ is *not* structurally complete (as it happens with $\mathbf{R}$) since the *disjunctive syllogism* $\langle\{p,\neg p \vee q\},q\rangle$ is admissible but not derivable in $\mathbf{RM}$ (and also in $\mathbf{R}$). However, the fragments $\mathbf{RM}_{\{\rightarrow,\wedge,\vee\}}$ and $\mathbf{RM}_{\{\rightarrow,\wedge\}}$ of $\mathbf{RM}$ are hereditarily structurally complete (the proof can be found in [34]). Finally, the $\{\rightarrow\}$- and the $\{\rightarrow,\neg\}$-fragments or $\mathbf{RM}$, that is $\mathbf{RM}_{\{\rightarrow\}}$ and $\mathbf{RM}_{\{\rightarrow,\neg\}}$, are not structurally complete since in these fragments the rule $\langle\{p,(p \rightarrow (q \rightarrow q)) \rightarrow (p \rightarrow q)\},q\rangle$ is admissible but not derivable (see [1] and [9]).

The example above establishes that no general theorems can be proven about the structural completeness of fragments or expansions of a structurally complete logic. However, a partial preservation result for fragments is the following:

Proposition 3.2.1 ([34]). *Let $\mathcal{S}$ be a (finitary) logic of type L and let L' be a sublanguage of L. Let $\langle\Gamma,\varphi\rangle$ be a (finite) admissible rule of $\mathcal{S}$, such that $\Gamma\cup\{\varphi\}$ consists of L'-formulas. Then:*

i *$\langle\Gamma,\varphi\rangle$ is also admissible in the L'-fragment of $\mathcal{S}$, and*

ii *If $\langle\Gamma,\varphi\rangle$ is underivable in $\mathcal{S}$, then the L'-fragment is not structurally complete.*

Proof. Denote by $\mathcal{S}'$ the L'-fragment of $\mathcal{S}$.

i Since $\Gamma\cup\{\varphi\}$ consists of L'-formulas, the L'-fragment of $\mathcal{S}+\langle\Gamma,\varphi\rangle$ is just $\mathcal{S}'+\langle\Gamma,\varphi\rangle$. In particular, any theorem of $\mathcal{S}+\langle\Gamma,\varphi\rangle$ of type L' is also an $\mathcal{S}'+\langle\Gamma,\varphi\rangle$ - theorem and vice versa. Thus, $\mathcal{S}+\langle\Gamma,\varphi\rangle$ and $\mathcal{S}'+\langle\Gamma,\varphi\rangle$ have the same theorems of type L' and the same happens with $\mathcal{S}$ and $\mathcal{S}'$. By hypothesis, $\langle\Gamma,\varphi\rangle$ is admissible for $\mathcal{S}$ and hence $\mathcal{S}$ and $\mathcal{S}+\langle\Gamma,\varphi\rangle$ have the same theorems. Thus $\mathcal{S}'$ and $\mathcal{S}'+\langle\Gamma,\varphi\rangle$ have the same theorems and therefore $\langle\Gamma,\varphi\rangle$ is admissible for $\mathcal{S}'$.

ii This is an immediate consequence of i.

□

3.3 Connections with the f.m.p. and the deduction theorem

In what follows we shall see that under the scope of structural completeness the finite (matrix) model property (f.m.p.) and the strong finite (matrix) model property (s.f.m.p.) coincide (see definition 2.5.3). But first a lemma:

Lemma 3.3.1 ([34]). *Let $\mathcal{S}$ be a logic with the f.m.p. with respect to a matrix semantics M of $\mathcal{S}$ and let $\Gamma \cup \{\varphi\} \subseteq Fm$ be a finite set of formulas. If all finite matrices in M satisfy $\langle \Gamma, \varphi \rangle$, then $\langle \Gamma, \varphi \rangle$ is admissible for $\mathcal{S}$.*

Proof. We apply Proposition 3.1.1. Pick any substitution σ such that $\vdash_{\mathcal{S}} \sigma[\Gamma]$. If $\sigma(\varphi)$ is a non-theorem of $\mathcal{S}$, then by the f.m.p. there is a finite matrix $\langle \mathbf{A}, F \rangle$ in M that invalidates $\sigma(\varphi)$. That is, there is some $h \in Hom(\mathbf{Fm}, \mathbf{A})$ such that $h(\sigma(\varphi)) \notin F$. By hypothesis, $\langle \mathbf{A}, F \rangle$ validates $\langle \Gamma, \varphi \rangle$. Moreover, since $\sigma[\Gamma] \subseteq Cn_{\mathcal{S}}(\varnothing)$, then also $h[\sigma[\Gamma]] \subseteq h[Cn_{\mathcal{S}}(\varnothing)] \subseteq F$. Thus, $h \circ \sigma$ is a homomorphism from $\mathbf{Fm}$ into $\mathbf{A}$ such that $h \circ \sigma[\Gamma] \subseteq F$. Hence, $h(\sigma(\varphi)) \in F$, a contradiction. Consequently, $\sigma(\varphi)$ is a theorem of $\mathcal{S}$ and the rule $\langle \Gamma, \varphi \rangle$ is admissible for $\mathcal{S}$. □

Corollary 3.3.1 ([34]). *Let $\mathcal{S}$ be a structurally complete logic and let M be a matrix semantics for $\mathcal{S}$. Then $\mathcal{S}$ has the f.m.p. with respect to M iff it has the s.f.m.p. with respect to M.*

Proof. The direction $\Leftarrow$ is clear and it holds always. For the other direction, assume $\mathcal{S}$ has the f.m.p. with respect to M. Assume $\langle \Gamma, \varphi \rangle$ is a finite and underivable rule in $\mathcal{S}$. Since $\mathcal{S}$ is structurally complete, by Theorem 3.1.1, it follows that $\langle \Gamma, \varphi \rangle$ is inadmissible for $\mathcal{S}$. Finally, by Lemma 3.3.1, there is a finite matrix in M that invalidates $\langle \Gamma, \varphi \rangle$. Hence, $\mathcal{S}$ has the s.f.m.p. □

We shall close this chapter by presenting the connections between the notion of hereditary structural completeness and the hereditary local (global) deduction-detachment theorem denoted by $\mathcal{HLDDT}$ (resp. $\mathcal{HGDDT}$) for finitary logics (if necessary see definition 2.5.2).

Lemma 3.3.2. *Let $\mathcal{S}$ be a logic with the $\mathcal{LDDT}$ (resp. $\mathcal{GDDT}$) with respect to a set of sets of formulas $\mathcal{E}(p,q)$. Then, any axiomatic extension of $\mathcal{S}$ has the $\mathcal{LDDT}$ (resp. $\mathcal{GDDT}$) with respect to $\mathcal{E}$.*

Proof. Pick any axiomatic extension $\mathcal{S}'$ of $\mathcal{S}$. Then, there is a set $\Delta \subseteq Fm$ closed under substitution such that for any $\Gamma \cup \{\varphi\} \subseteq Fm$:

$$\Gamma \vdash_{\mathcal{S}'} \varphi \text{ iff } \Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi$$

In order to check that $\mathcal{S}'$ has the $\mathcal{LDDT}$, let $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$. Then:

$$\begin{aligned}
\Gamma, \varphi \vdash_{\mathcal{S}'} \psi &\text{ iff } \Gamma \cup \Delta, \varphi \vdash_{\mathcal{S}} \psi \\
&\text{ iff } \Gamma \cup \Delta \vdash_{\mathcal{S}} E(\varphi, \psi)\ , \text{ for some } E \in \mathcal{E} \text{ (since } \mathcal{S} \text{ has the } \mathcal{LDDT}) \\
&\text{ iff } \Gamma \vdash_{\mathcal{S}'} E(\varphi, \psi)\ , \text{ for some } E \in \mathcal{E}
\end{aligned}$$

Thus, $\mathcal{S}'$ has the $\mathcal{LDDT}$ with respect to $\mathcal{E}$. The proof for the $\mathcal{GDDT}$ is similar. □

Theorem 3.3.1 ([11]). *Let $\mathcal{S}$ be a finitary logic having the $\mathcal{LDDT}$ with respect to some set of sets of formulas $\mathcal{E}(p,q)$. The following conditions are equivalent:*

i *$\mathcal{S}$ is hereditarily structurally complete.*

ii *$\mathcal{S}$ has the $\mathcal{HLDDT}$ with respect to $\mathcal{E}(p,q)$.*

Proof. $i \Rightarrow ii$: Pick any finitary extension $\mathcal{S}'$ of $\mathcal{S}$. Since $\mathcal{S}$ is hereditarily structurally complete, by Theorem 3.1.3, $\mathcal{S}'$ is an axiomatic extension of $\mathcal{S}$ and by Lemma 3.3.2 it follows that $\mathcal{S}'$ has the $\mathcal{LDDT}$ with respect to $\mathcal{E}$.
$ii \Rightarrow i$: By Theorem 3.1.3, is is enough to show that any finitary extension of $\mathcal{S}$ is an axiomatic extension of $\mathcal{S}$. So, let $\mathcal{S}'$ be a finitary extension of $\mathcal{S}$. Since $\mathcal{S}$ has the $\mathcal{HLDDT}$ with respect to $\mathcal{E}$, then $\mathcal{S}'$ has the $\mathcal{LDDT}$ with respect to $\mathcal{E}$. Thus, for any $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$, such that $\Gamma, \varphi \vdash_{\mathcal{S}'} \psi$, there is some set $E \in \mathcal{E}$ such that $\Gamma \vdash_{\mathcal{S}'} E(\varphi, \psi)$. We denote such a set by $E^{\psi}_{\Gamma,\varphi}$. For convenience on notation, we denote a rule of $\mathcal{S}'$, i.e. an element of $\vdash_{\mathcal{S}'}$, by $R = \langle \Gamma_R, \varphi_R \rangle$.
For each finite rule $R = \langle \Gamma_R, \varphi_R \rangle$ of $\mathcal{S}'$, where $\Gamma_R = \{\gamma^R_1, \dots, \gamma^R_n\}$, $n \in \omega$, we define sets of formulas $A^R_{n+1}, \dots, A^R_1$ inductively as follows:

- $A^R_{n+1} := \{\varphi_R\}$
- $A^R_i := \{\chi(\gamma^R_i, \varepsilon) : \chi \in E^{\varepsilon}_{\{\gamma^R_1,\dots,\gamma^R_{i-1}\},\gamma^R_i} \text{ and } \varepsilon \in A^R_{i+1}\} = \bigcup_{\varepsilon \in A^R_{i+1}} E^{\varepsilon}_{\{\gamma^R_1,\dots,\gamma^R_{i-1}\},\gamma^R_i}(\gamma^R_i, \varepsilon)$

Despite the seemingly complicated definition, the idea behind it is simple: We apply exhaustively the $\mathcal{LDDT}$ of $\mathcal{S}'$, to the rule R. Observe that from the step $i+1$ to the step i, we apply the $\mathcal{LDDT}$ to each ε of the previous step (i.e. from A^R_{i+1}). The result of this application is the set A^R_1 all whose formulas are theorems of $\mathcal{S}'$. If $\mathcal{S}'$ has the $\mathcal{GDDT}$ and $\mathcal{E} = \{\{\chi\}\}$, then $A^R_1 = \{\chi(\gamma^R_1, \chi(\dots(\chi(\gamma^R_{n-1}, \chi(\gamma^R_n, \varphi_R))\dots)\}$, i.e. the n-time application of the $\mathcal{GDDT}$ results to a single formula which is again a theorem of $\mathcal{S}'$.
Let $\Delta := \bigcup\{A^R_1 : R \in \vdash_{\mathcal{S}'} \text{ and } R \text{ is a finite rule }\}$. Observe that for any rule $R = \langle \Gamma_R, \varphi_R \rangle$ of $\mathcal{S}'$ and any substitution σ, the rule $\langle \sigma[\Gamma_R], \sigma(\varphi_R) \rangle$ is still a rule of $\mathcal{S}'$, by invariance under substitutions of $\mathcal{S}'$. Thus, if $\varphi \in \Delta$, then $\varphi \in A^R_1$ for some rule $R = \langle \Gamma, \psi \rangle$ of $\mathcal{S}$, and therefore for any substitution σ, $\sigma(\varphi) \in A^{\sigma[R]}_1$, where $\sigma[R] = \langle \sigma[\Gamma], \sigma(\psi) \rangle$. Thus Δ is closed under substitutions. So let $\mathcal{S}_\Delta$ be the logic defined by:

$$\Gamma \vdash_{\mathcal{S}_\Delta} \varphi \text{ iff } \Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi.$$

Clearly, $\mathcal{S}_\Delta$ is the axiomatic extension of $\mathcal{S}$ by the set of axioms Δ. We will show that $\mathcal{S}_\Delta = \mathcal{S}'$ and conclude that $\mathcal{S}'$ is indeed an axiomatic extension of $\mathcal{S}$.

- $\mathcal{S}_\Delta \leq \mathcal{S}'$: Assume $\Gamma \vdash_{\mathcal{S}_\Delta} \varphi$. Then $\Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi$ and therefore $\Gamma \cup \Delta \vdash_{\mathcal{S}'} \varphi$ since $\mathcal{S}'$ is an extension of $\mathcal{S}$. Hence, $\Gamma \vdash_{\mathcal{S}'} \varphi$ since all formulas in Δ are theorems of $\mathcal{S}'$.

- $\mathcal{S}' \leq \mathcal{S}_\Delta$: Assume $\Gamma \vdash_{\mathcal{S}'} \varphi$. Since $\mathcal{S}'$ is finitary, there is some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{S}'} \varphi$. Assume $\Gamma' = \{\gamma_1, \dots, \gamma_n\}$ and denote by R the rule $\langle \Gamma', \varphi \rangle$ of $\mathcal{S}'$. We show by induction on i that $\gamma_1, \dots, \gamma_{i-1} \vdash_{\mathcal{S}_\Delta} A^R_i$. The base case is immediate since every formula in A^R_1 is a theorem of $\mathcal{S}_\Delta$. So, assume it is true for i. Observe that by the fact that $E(\varphi, \psi) \vdash_{\mathcal{S}_\Delta} E(\varphi, \psi)$, for any $E \in \mathcal{E}$ and any $\varphi, \psi \in Fm$, we obtain that $E(\varphi, \psi), \varphi \vdash_{\mathcal{S}_\Delta} \psi$, because $\mathcal{S}_\Delta$ is an axiomatic extension of $\mathcal{S}$ and therefore it inherits the $\mathcal{LDDT}$ with respect to $\mathcal{E}$, in view of Lemma 3.3.2 . Thus, we obtain that $E^{\varepsilon}_{\{\gamma_1,\dots,\gamma_{i-1}\},\gamma_i}(\gamma_i, \varepsilon)$, $\gamma_i \vdash_{\mathcal{S}_\Delta} \varepsilon$, for any $\varepsilon \in A^R_{i+1}$. Hence, $A^R_i, \gamma_i \vdash_{\mathcal{S}_\Delta} A^R_{i+1}$. By induction hypothesis, $\gamma_1, \dots, \gamma_{i-1} \vdash_{\mathcal{S}_\Delta} A^R_i$ and therefore $\gamma_1, \dots, \gamma_i \vdash_{\mathcal{S}_\Delta} A^R_{i+1}$.

□

By applying the above theorem for a DDT-set $\mathcal{E}(p, q)$ containing only one set of formulas, we immediately obtain:

Corollary 3.3.2 ([11]). *Let $\mathcal{S}$ be a finitary logic with the $\mathcal{GDDT}$ with respect to a set $\mathcal{E}(p,q)$. The following conditions are equivalent:*

i $\mathcal{S}$ is hereditarily structurally complete.

ii $\mathcal{S}$ has the $\mathcal{HGDDT}$ with respect to $\mathcal{E}$.

Chapter 4

Admissible Rules

4.1 General algebraic conditions of admissibility

In this section we will present some general characterization of the admissibility of a rule for a logic $\mathcal{S}$, despite the classification of $\mathcal{S}$ in the Leibniz hierarchy. Throughout the chapter, matrices will be considered as first-order $L \cup \{P\}$-structures, where L is our fixed but arbitrary algebraic similarity type and P is a unary relation symbol.

Definition 4.1.1 ([13]). Let K be a class of algebras. An algebra $\mathbf{A} \in K$ is called *relatively subdirectly irreducible* (or briefly **RSI**) in K, if whenever $\mathbf{A}$ is a subdirect product of algebras $\{\mathbf{A}_i : i \in I\}$ in K, then at least one of the projections $\pi_i : \prod_{i \in I} A_i \to A_i$ restricted to A, is an isomorphism from A onto A_i. It is common that when the class K is clear from the context we just say that $\mathbf{A}$ is RSI.

The next definition naturally extends the one just given for algebras. Observe that essentially we could define it for an arbitrary class of matrices M and a matrix $\langle \mathbf{A}, F \rangle \in M$ but we only do it for $M = Matr^*(\mathcal{S})$. This is reasonable since the only notion of RSI matrices that we will use in this study is the following:

Definition 4.1.2 ([13]). A reduced matrix model $\langle \mathbf{A}, F \rangle$ of a logic $\mathcal{S}$ is called *relatively subdirectly irreducible* (with respect to $\mathcal{S}$), or briefly **RSI**, if whenever $\langle \mathbf{A}, F \rangle$ is a subdirect product of *reduced* matrices $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ of $\mathcal{S}$, then at least one of the projections $\pi_i : \prod_{i \in I} A_i \to A_i$, when restricted to $\mathbf{A}$, it is an isomorphism from $\langle \mathbf{A}, F \rangle$ onto $\langle \mathbf{A}_i, F_i \rangle$.

Recall that a partially ordered set (poset) $\mathbb{L}$ is a *complete lattice* if for any $X \subseteq L$, $\bigwedge X$ and $\bigvee X$ exists. An element q of a complete lattice $\mathbb{L}$ is called *completely meet-irreducible* if for every $X \subseteq L$, if $q = \bigwedge X$, then $q \in X$. An element q of a complete lattice $\mathbb{L}$ is called *compact* if whenever $q \leq \bigvee X$ for some $X \subseteq L$, then there is a *finite* $Y \subseteq X$ such that $q \leq \bigvee Y$. Finally, a complete lattice $\mathbb{L}$ is *algebraic* if every element of L is a join of a set of compact elements.

Lemma 4.1.1 ([38]). *Let $\mathcal{S}$ be a logic. A reduced matrix model $\langle \mathbf{A}, F \rangle$ of $\mathcal{S}$ is RSI iff F is completely meet-irreducible in $Fi_{\mathcal{S}}\mathbf{A}$.*

Proof. $\Rightarrow$: By contraposition. Assume F is not completely meet-irreducible in $Fi_{\mathcal{S}}\mathbf{A}$ and let $X = \{F_i : i \in I\}$ be a family of $\mathcal{S}$-filters of $\mathbf{A}$ such that $F = \bigcap_{i \in I} F_i$ but $F \neq F_i$ for any $i \in I$. Thus, F is properly contained in all F_i's. It is easy to see that the map $h : \langle \mathbf{A}, F \rangle \to \prod_{i \in I} \langle \mathbf{A}, F_i \rangle$ defined by $h(a) := \langle a : i \in I \rangle$ is a matrix homomorphism. Clearly it is an injective mapping and moreover it is strict since for any $a \in A$,

$a \in h^{-1}[\prod_{i\in I} F_i]$ iff $h(a) \in \prod_{i\in I} F_i$ iff $h(a)(i) \in F_i$, $\forall i \in I$ iff $a \in F_i$, $\forall i \in I$ iff $a \in \bigcap_{i\in I} F_i = F$

Thus, h is an embedding. We define a map $g : \prod_{i\in I}\langle \mathbf{A}, F_i\rangle \to \prod_{i\in I}\langle \mathbf{A}/\Omega^{\mathbf{A}}(F_i), F_i/\Omega^{\mathbf{A}}(F_i)\rangle$ by $g(b) := \langle b(i)/\Omega^{\mathbf{A}}(F_i) : i \in I\rangle$, which is easily seen to be a matrix homomorphism.

Claim: $g \circ h$ is an embedding

Pick any $a, b \in A$ with $g(h(a)) = g(h(b))$. Then, $g(h(a))(i) = g(h(b))(i)$ for all $i \in I$ and therefore $h(a)(i)/\Omega^{\mathbf{A}}(F_i) = h(b)(i)/\Omega^{\mathbf{A}}(F_i)$ for all $i \in I$. That is $a/\Omega^{\mathbf{A}}(F_i) = b/\Omega^{\mathbf{A}}(F_i)$ for all $i \in I$ and hence $\langle a, b\rangle \in \bigcap_{i\in I}\Omega^{\mathbf{A}}(F_i) \subseteq \Omega^{\mathbf{A}}\big(\bigcap_{i\in I} F_i\big) = \Omega^{\mathbf{A}}(F)$. Finally, since $\langle \mathbf{A}, F\rangle$ was reduced, it immediately follows that $a = b$.
For convenience denote $F_i/\Omega^{\mathbf{A}}(F_i)$ by F_i^*. Observe finally that $g \circ h$ is strict since:

$$\begin{aligned} a \in (g\circ h)^{-1}\Big[\prod_{i\in I} F_i^*\Big] &\text{ iff } g(h(a)) \in \prod_{i\in I} F_i^* \\ &\text{ iff } h(a)(i)/\Omega^{\mathbf{A}}(F_i) \in F_i^* \ , \forall i \in I \\ &\text{ iff } a/\Omega^{\mathbf{A}}(F_i) \in F_i^* \ , \forall i \in I \\ &\text{ iff } a \in \bigcap_{i\in I} F_i = F \end{aligned}$$

Thus, $g \circ h$ is indeed an embedding and therefore the matrix $\langle g\circ h[\mathbf{A}], g\circ h[F]\rangle$ is a subdirect product of $\prod_{i\in I}\langle \mathbf{A}, F_i\rangle^*$. However, observe that for any $a \in A$ and any $i \in I$

$$g(h(a))(i) = \langle h(a)(i) : i \in I\rangle = \langle a/\Omega^{\mathbf{A}}(F_i) : i \in I\rangle.$$

Hence $g \circ h[F] = \{\langle a/\Omega^{\mathbf{A}}(F_i) : i \in I\rangle : a \in F\}$. Finally, since F is properly contained in all F_i's, it follows that there is no $i \in I$ for which the projection π_i restricts to an isomorphism from $\langle \mathbf{A}, F\rangle$ onto $\langle \mathbf{A}, F_i\rangle^*$. Thus, $\langle \mathbf{A}, F\rangle$ is not RSI.
$\Leftarrow$: By contraposition. Assume $\langle \mathbf{A}, F\rangle$ is not RSI. So there are reduced matrix models $\{\langle \mathbf{A}_i, F_i\rangle : i \in I\}$ of $\mathcal{S}$ such that $\langle \mathbf{A}, F\rangle$ is a subdirect product of $\prod_{i\in I}\langle \mathbf{A}_i, F_i\rangle$ but for no $i \in I$, the projection map π_i restricts to isomorphism from $\langle \mathbf{A}, F\rangle$ onto $\langle \mathbf{A}_i, F_i\rangle$. Since each $\pi_i : \mathbf{A} \to \mathbf{A}_i$ is onto, and both $\langle \mathbf{A}, F\rangle$ and $\langle \mathbf{A}_i, F_i\rangle$ are reduced it follows that for each $i \in I$, $\pi_i^{-1}[F_i]$ is an $\mathcal{S}$-filter of $\mathbf{A}$ (by Proposition 2.2.1) which properly contains F, since otherwise we know that strict surjective mappings between reduced matrices are in fact isomorphisms. Denote for each $i \in I$, $G_i = \pi_i^{-1}[F_i]$. As we said, $F \subsetneq G_i$ for all $i \in I$ and therefore $F \subseteq \bigcap_{i\in I} G_i$. Conversely, if $a \in \bigcap_{i\in I} G_i$, then $\pi_i(a) \in F_i$ for all $i \in I$ and therefore $a \in \prod_{i\in I} F_i$. Since $a \in A$ and $\langle \mathbf{A}, F\rangle$ was a submatrix of $\langle \prod_{i\in I}\mathbf{A}_i, \prod_{i\in I} F_i\rangle$, it follows that $a \in A \cap \prod_{i\in I} F_i = F$. Thus, $F = \bigcap_{i\in I} G_i$ but $F \neq G_i$ for all $i \in I$. Hence F is not completely meet-irreducible in $Fi_{\mathcal{S}}\mathbf{A}$. □

Theorem 4.1.1 ([38]). *(**Subdirect Representation Theorem**) Let $\mathcal{S}$ be a finitary logic. Every reduced matrix model of $\mathcal{S}$ is isomorphic to a subdirect product of RSI reduced matrix models of $\mathcal{S}$.*

Proof. Let $\langle \mathbf{A}, F\rangle$ be a reduced matrix model of $\mathcal{S}$. Since $\mathcal{S}$ is finitary we know that F is the intersection of all completely meet-irreducible $\mathcal{S}$-filters of $\mathbf{A}$ including F (see for example [15], Prop. 1.63). Thus, there are $\{F_i : i \in I\} \subseteq Fi_{\mathcal{S}}\mathbf{A}$ such that each F_i is completely meet-irreducible in $Fi_{\mathcal{S}}\mathbf{A}$ and $F = \bigcap_{i\in I} F_i$. We define $h : \langle \mathbf{A}, F\rangle \to \prod_{i\in I}\langle \mathbf{A}, F_i\rangle$ by $h(a) := \langle a : i \in I\rangle$ for each $a \in A$, which is easily seen to be a matrix homomorphism. Clearly it is injective and moreover it is strict since:

$$a \in h^{-1}[\prod_{i\in I} F_i] \text{ iff } h(a) \in \prod_{i\in I} F_i \text{ iff } h(a)(i) \in F_i\ , \forall i \in I \text{ iff } a \in F_i\ , \forall i \in I \text{ iff } a \in \bigcap_{i\in I} F_i = F$$

Thus, h is an embedding. Moreover, it is obvious that for each $i \in I$, $\pi_i[h[A]] = A$. We define a mapping $g : \prod_{i\in I}\langle \mathbf{A}, F_i\rangle \to \prod_{i\in I}\langle \mathbf{A}, F_i\rangle^*$ by $g(b) := \langle b(i)/\Omega^{\mathbf{A}}(F_i) : i \in I\rangle$. Easily we can see that it is a matrix homomorphism. Additionally, the map $g \circ h : \langle \mathbf{A}, F\rangle \to \prod_{i\in I}\langle \mathbf{A}, F_i\rangle^*$ is an embedding and the proof is obtained as in Lemma 4.1.1. Thus, $\langle \mathbf{A}, F\rangle$ is isomorphic to a subdirect product of $\{\langle \mathbf{A}, F_i\rangle^* : i \in I\}$. Finally, it is easy to see that since each F_i is completely meet-irreducible in $Fi_{\mathcal{S}}\mathbf{A}$, then each $F_i/\Omega^{\mathbf{A}}(F_i)$ is completely meet-irreducible in $Fi_{\mathcal{S}}\mathbf{A}/\Omega^{\mathbf{A}}(F_i)$. Then, by Lemma 4.1.1, each $\langle \mathbf{A}, F_i\rangle^*$ is RSI and we have obtained the required. □

Remark 4.1.1. If $\langle \mathbf{A}, F\rangle$ is a *finite* matrix model of a logic $\mathcal{S}$, regardless of whether the logic is finitary or not, the set $Fi_{\mathcal{S}}\mathbf{A}$ is inductive, i.e. for any $\subseteq$-directed family $\mathcal{F} \subseteq Fi_{\mathcal{S}}\mathbf{A}$, $\bigcup \mathcal{F} \in Fi_{\mathcal{S}}\mathbf{A}$. In such case, be repeating the arguments of the finitary case (as in [15]) we can see that F is the intersection of all completely meet-irreducible $\mathcal{S}$-filters of $\mathbf{A}$ that include F and therefore, as in the proof of the Subdirect Representation Theorem, we obtain that $\langle \mathbf{A}, F\rangle$ is indeed isomorphic to a subdirect product of reduced matrix models of $\mathcal{S}$.

Lemma 4.1.2 ([32]). *Let $\mathcal{S}$ be a logic. If a rule $\langle \Gamma, \varphi\rangle$ is underivable in $\mathcal{S}$, then it is invalidated by some κ-generated reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$, where κ is the number of variables occurring in formulas in $\Gamma \cup \{\varphi\}$. If in addition, $\mathcal{S}$ is finitary or $\langle \mathbf{A}, F\rangle$ is finite, then $\langle \mathbf{A}, F\rangle$ can be chosen RSI as well.*

Proof. Assume $\langle \Gamma, \varphi\rangle$ is underivable in $\mathcal{S}$ and let $T = Cn_{\mathcal{S}}(\Gamma)$. Let $X \subseteq Var$ be the set of all variables occurring in formulas from $\Gamma \cup \{\varphi\}$ and assume $|X| = \kappa$. Since $\varphi \notin T$, then clearly the matrix $\langle \mathbf{A}, F\rangle = \langle \mathbf{Fm}(X), T \cap Fm(X)\rangle$ invalidates $\langle \Gamma, \varphi\rangle$, and it is a κ-generated submatrix of $\langle \mathbf{Fm}, T\rangle$. Consequently, $\langle \mathbf{A}, F\rangle$ is still a matrix model of $\mathcal{S}$. Moreover, we know that a matrix and its reduction validate the same rules and therefore $\langle \mathbf{A}, F\rangle^*$ is a reduced matrix model of $\mathcal{S}$ which invalidates $\langle \Gamma, \varphi\rangle$ and obviously it is still κ-generated.
Finally, if $\mathcal{S}$ is finitary or $\langle \mathbf{A}, F\rangle$ is finite, then by the Subdirect Representation Theorem or remark 4.1.1, it follows that $\langle \mathbf{A}, F\rangle^*$ is isomorphic to a subdirect product of a family $\{\langle \mathbf{A}_i, F_i\rangle : i \in I\}$ of RSI and reduced matrix models of $\mathcal{S}$. Observe that each $\langle \mathbf{A}_i, F_i\rangle$ is still κ-generated since it is a homomorphic image of $\langle \mathbf{A}, F\rangle$. However, since $\langle \mathbf{A}, F\rangle^*$ validates any rule that is validated by all $\langle \mathbf{A}_i, F_i\rangle$, but invalidates $\langle \Gamma, \varphi\rangle$, it follows that there is some $i \in I$ such that $\langle \mathbf{A}_i, F_i\rangle$ invalidates $\langle \Gamma, \varphi\rangle$. □

We shall prove now the main theorem of this section:

Theorem 4.1.2 ([32]). *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

i *$\langle \Gamma, \varphi\rangle$ is an admissible rule for $\mathcal{S}$.*

ii *Every matrix model of $\mathcal{S}$ is a homomorphic image of a matrix model of $\mathcal{S} + \langle \Gamma, \varphi\rangle$.*

iii *Every reduced matrix model of $\mathcal{S}$ is a homomorphic image of a matrix model of $\mathcal{S} + \langle \Gamma, \varphi\rangle$ that is itself a subdirect product of reduced matrix models of $\mathcal{S}$.*

In ii and iii "Every" can be replaced by "Every finitely generated" without loss of strength (even when Γ is infinite). If $\mathcal{S}$ is finitary, then in iii we can replace "Every" by "Every RSI" (with or without "finitely generated").

Proof. $i \Rightarrow ii$: Let $\langle \mathbf{A}, F\rangle \in Matr(\mathcal{S})$. Let Y be a set of variables of cardinality $|Y| = \max\{|Var|, |A|\}$ and consider the absolutely free algebra $\mathbf{Fm}(Y)$. So that there is a surjective homomorphism $h : \mathbf{Fm}(Y) \to \mathbf{A}$. Let $G = \min Fi_{\mathcal{S}}\mathbf{Fm}(Y)$. By Proposition 2.2.1, $h^{-1}[F] \in Fi_{\mathcal{S}}\mathbf{Fm}(Y)$ and hence $G \subseteq h^{-1}[F]$. Thus, $\langle \mathbf{A}, F\rangle$ is a homomorphic image of $\langle \mathbf{Fm}(Y), G\rangle$. Clearly, $\langle \mathbf{Fm}(Y), G\rangle$ is a matrix model of $\mathcal{S}$ and we show that it validates $\langle \Gamma, \varphi\rangle$. So assume that $f : \mathbf{Fm} \to \mathbf{Fm}(Y)$ is a homomorphism such that $f[\Gamma] \subseteq G$. Since $|Y| \geq |Var|$, let $g : Y \to Var$ be a surjective mapping and extend it naturally to a surjective homomorphism $\widetilde{g} : \mathbf{Fm}(Y) \to \mathbf{Fm}$. By Proposition 2.2.1, $\widetilde{g}^{-1}[Cn_{\mathcal{S}}(\varnothing)] \in Fi_{\mathcal{S}}\mathbf{Fm}(Y)$ and therefore $G \subseteq \widetilde{g}^{-1}[Cn_{\mathcal{S}}(\varnothing)]$. Thus, $\widetilde{g} \circ f$ is a substitution with $(\widetilde{g} \circ f)[\Gamma] \subseteq Cn_{\mathcal{S}}(\varnothing)$ and since $\langle \Gamma, \varphi\rangle$ is admissible for $\mathcal{S}$, i.e. it is validated by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle$, it follows that $\widetilde{g}(f(\varphi)) \in Cn_{\mathcal{S}}(\varnothing)$. However, it is not necessary that $G = \widetilde{g}^{-1}[Cn_{\mathcal{S}}(\varnothing)]$, and therefore we cannot infer immediately that $f(\varphi) \in G$. We proceed to show that this is the case.
Observe that $f[\mathbf{Fm}]$ is the universe of a subalgebra of $\mathbf{Fm}(Y)$ and since $\mathbf{Fm}$ is $|Var|$-generated, then so is $f[\mathbf{Fm}]$. So pick a set $X \subseteq Y$ of cardinality $|X| \leq |Var|$ such that $f[\mathbf{Fm}] = Sg^{\mathbf{Fm}(Y)}(X)$ (the usual operator for subuniverse generation). From Universal Algebra (see for example [10]) we know that:

$$Sg^{\mathbf{Fm}(Y)}(X) = \{t^{\mathbf{Fm}(Y)}(y_1, \dots, y_n) : t \text{ is an } n\text{-ary } L\text{-term }, \ n < \omega \text{ and } y_1, \dots, y_n \in X\}.$$

Thus, there are $y_1, \dots, y_n \in X$ such that $f(\varphi) = t^{\mathbf{Fm}(Y)}(y_1, \dots, y_n)$. By composing with $\widetilde{g}$, we obtain that $\widetilde{g}(f(\varphi)) = \widetilde{g}(t^{\mathbf{Fm}(Y)}(y_1, \dots, y_n)) = t(g(y_1), \dots, g(y_n))$ since g and $\widetilde{g}$ agree on X (generally on Y) and $\widetilde{g}$ is a homomorphism. Recall that $\widetilde{g}(f(\varphi))$ was an $\mathcal{S}$-theorem and for each $i \in \{1, \dots, n\}$, $g(y_1), \dots, g(y_n)$ are all variables. Thus, t itself must be an $\mathcal{S}$-theorem since $\vdash_{\mathcal{S}}$ is substitution invariant and the variables can be assigned arbitrarily by homomorphisms. In such case, $f(\varphi) \in G$ since G is an $\mathcal{S}$-filter and the $\mathcal{S}$-theorems are the formulas taking only designated values in all matrix models of $\mathcal{S}$.
$ii \Rightarrow iii$: Let $\langle \mathbf{A}, F\rangle$ be a reduced matrix model of $\mathcal{S}$. By ii, there is a matrix model $\langle \mathbf{B}, G\rangle$ of $\mathcal{S} + \langle \Gamma, \varphi\rangle$ and a surjective matrix homomorphism $h : \langle \mathbf{B}, G\rangle \to \langle \mathbf{A}, F\rangle$. Consider

$$\theta := \bigcap\{\Omega^{\mathbf{B}}(G') : G' \in Fi_{\mathcal{S}}\mathbf{B} \text{ and } G \subseteq G'\}$$

which is called the *Suszko congruence* of $\langle \mathbf{B}, G\rangle$ with respect to $\mathcal{S}$. By Proposition 2.2.1 $h^{-1}[F] \in Fi_{\mathcal{S}}\mathbf{A}$ and since h is a matrix homomorphism, $G \subseteq h^{-1}[F]$. Thus, by Proposition 2.2.3,

$$\theta \subseteq \Omega^{\mathbf{B}}(h^{-1}[F]) = h^{-1}[\Omega^{\mathbf{A}}(F)] = ker(h)$$

since $\langle \mathbf{A}, F\rangle$ is reduced and therefore $\Omega^{\mathbf{A}}(F) = Id_A$, while clearly $h^{-1}[Id_A] = ker(h)$. Thus, the map $g : \mathbf{B}/\theta \to \mathbf{A}$ defined by $g(a/\theta) = h(a)$ is a well-defined surjective homomorphism. In fact, it is a matrix homomorphism from $\langle \mathbf{B}/\theta, G/\theta\rangle$ onto $\langle \mathbf{A}, F\rangle$ since $g[G/\theta] = h[G] \subseteq F$. Moreover, $\theta \subseteq \Omega^{\mathbf{B}}(G)$, and therefore it follows by Proposition 2.2.4 that the natural map $\pi_\theta : \langle \mathbf{B}, G\rangle \to \langle \mathbf{B}/\theta, G/\theta\rangle$ is a strict and surjective matrix homomorphism and therefore these two matrices validate the same rules. Thus, $\langle \mathbf{B}/\theta, G/\theta\rangle$ is a matrix model of $\mathcal{S} + \langle \Gamma, \varphi\rangle$ since $\langle \mathbf{B}, G\rangle$ is such. Finally, $\langle \mathbf{B}/\theta, G/\theta\rangle$ can be embedded into $\prod\limits_{G' \in \left(Fi_{\mathcal{S}}\mathbf{B}\right)^G} \langle \mathbf{B}, G'\rangle^*$, where $\left(Fi_{\mathcal{S}}\mathbf{B}\right)^G := \{G' \in Fi_{\mathcal{S}}\mathbf{B} : G \subseteq G'\}$, and the embedding is given by: $b/\theta \mapsto \langle b/\Omega^{\mathbf{B}}(G') : G' \in \left(Fi_{\mathcal{S}}\mathbf{B}\right)^G\rangle$. Consequently, $\langle \mathbf{B}/\theta, G/\theta\rangle$ is isomorphic

to a subdirect product of reduced matrix models of $\mathcal{S}$.
$iii \Rightarrow i$: Let ψ be a non-theorem of $\mathcal{S}$. By Theorem 2.2.2, there is $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})$ and a homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h(\psi) \notin F$. By iii, $\langle \mathbf{A}, F\rangle$ is a homomorphic image of some $\langle \mathbf{B}, G\rangle \in Matr(\mathcal{S} + \langle \Gamma, \varphi\rangle)$ which is itself a subdirect product of reduced matrix models of $\mathcal{S}$. So pick a surjective matrix homomorphism $g : \langle \mathbf{B}, G\rangle \to \langle \mathbf{A}, F\rangle$. By Lemma 2.2.1, there is $f \in Hom(\mathbf{Fm}, \mathbf{B})$ such that $g \circ f = h$. If ψ is an $\mathcal{S} + \langle \Gamma, \varphi\rangle$-theorem, then again by Theorem 2.2.2 we should have $f(\psi) \in G$ and therefore $g(f(\psi)) \in g[G] \subseteq F$, i.e. $h(\psi) \in F$, a contradiction. Thus, ψ is not an $\mathcal{S} + \langle \Gamma, \varphi\rangle$-theorem and hence $\mathcal{S}$ and $\mathcal{S} + \langle \Gamma, \varphi\rangle$ have the same theorems, i.e. $\langle \Gamma, \varphi\rangle$ is admissible for $\mathcal{S}$.
Finally, in view of Lemma 4.1.2 and the fact that only finitely-many variables occur in each formula $\psi \in Fm$, the comments for items ii and iii are easily justified. For example, it is obvious that ii implies ii': every finitely generated matrix model of $\mathcal{S}$ is a homomorphic image of a matrix model of $\mathcal{S} + \langle \Gamma, \varphi\rangle$. In order to see that ii' implies i: pick a non-theorem ψ of $\mathcal{S}$. As only finitely-many variables occur in ψ, by Lemma 4.1.2, there is a finitely generated reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$ that invalidates $\langle \varnothing, \psi\rangle$, which can be chosen RSI if $\mathcal{S}$ is finitary. By ii', there is a matrix model $\langle \mathbf{B}, G\rangle$ of $\mathcal{S} + \langle \Gamma, \varphi\rangle$ and a surjective matrix homomorphism $h : \langle \mathbf{B}, G\rangle \to \langle \mathbf{A}, F\rangle$. By repeating the same argument as in the direction $iii \Rightarrow i$, we can see that ψ cannot be an $\mathcal{S} + \langle \Gamma, \varphi\rangle$ - theorem. □

It is natural to think if it is possible to characterize the admissibility of a rule $\langle \Gamma, \varphi\rangle$ for a logic $\mathcal{S}$, just in terms of reduced matrix models of $\mathcal{S}$ and $\mathcal{S} + \langle \Gamma, \varphi\rangle$. The next example will show that it is not possible in the general case, not even for finitary logics. However, such a characterization can be obtained for protoalgebraic logics (see Theorem 4.2.2).

Example 4.1.1 ([32]). There is a finitary logic $\mathcal{S}$, a finite rule $\langle \Gamma, \varphi\rangle$ admissible for $\mathcal{S}$ and a finite reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$ such that $\langle \mathbf{A}, F\rangle$ is not a homomorphic image of any reduced matrix model of $\mathcal{S} + \langle \Gamma, \varphi\rangle$.

Proof. Consider the algebraic similarity type $L = \{\Box, \Diamond, \top\}$ and the logic $\mathcal{S}$, given by the Hilbert-style calculus with $\top$ as the only axiom and inference rule $\Diamond x \ / \ \Box \Diamond x$. The logic $\mathcal{S}$ is finitary and its set of theorems is $\{\top\}$. It is quite obvious that $\{\top\}/\Omega(\{\top\}) = \{\top\}$ and therefore for any $\psi \in Fm$, $\psi/\Omega(\{\top\}) = Fm \smallsetminus \{\top\}$, by using Proposition 2.2.2. Thus, $Fm/\Omega(\{\top\}) = \{\{\top\}, Fm \smallsetminus \{\top\}\}$. Consider the algebra $\mathbf{A} = \langle \{\bot, a, \top\}, \Box, \Diamond, \top\rangle$, where $\bot, a, \top$ are all distinct and $\Box x = x$, for any $x \in \{\bot, a, \top\}$, while $\Diamond \bot = \bot$ and $\Diamond a = \Diamond \top = \top$. It is easy to see that for any $x, y \in \{\bot, a, \top\}$, $\langle x, y\rangle \in \Omega^{\mathbf{A}}(\{\top\})$ iff $x = y$ and therefore $\Omega^{\mathbf{A}}(\{\top\}) = Id_A$. Hence $\langle \mathbf{A}, \{\top\}\rangle \in Matr^*(\mathcal{S})$. The rule $\langle \{\Box x\}, y\rangle$ is validated vacuously by $\langle \mathbf{Fm}, \{\top\}\rangle$ and thus by $\langle \mathbf{Fm}, \{\top\}\rangle^*$. Hence it is a finite rule admissible for $\mathcal{S}$. However, it is clearly not derivable in $\mathcal{S}$ since $\langle \mathbf{A}, \{\top\}\rangle$ invalidates $\langle \{\Box x\}, y\rangle$ and this fact is witnessed by any homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ with $h(x) = \top$ and $h(y) \neq \top$.
Now, pick any reduced matrix model $\langle \mathbf{B}, G\rangle$ of $\mathcal{S}$ that validates $\langle \{\Box x\}, y\rangle$. We show that there is no surjective homomorphism from $\langle \mathbf{B}, G\rangle$ onto $\langle \mathbf{A}, \{\top\}\rangle$. Suppose instead that h is such a homomorphism. Then $h[G] \subseteq \{\top\}$ since h is a matrix homomorphism and moreover $|B| \geq |A| = 3$. Thus, $G \subsetneq B$ since not all elements of B can be mapped in $\top$. Since $\langle \mathbf{B}, G\rangle$ validates both $\Diamond x \ / \ \Box \Diamond x$ and $\Box x \ / \ y$, then it validates $\Diamond x \ / \ y$. Hence, for any $b \in B$, it must hold that $\Box b, \Diamond b \notin G$, because otherwise we would have $G = B$. Let $b, b' \in B \smallsetminus h^{-1}[\{\top\}]$. By a simple induction on the L-formulas we can see that for any $\varphi(x, \bar{y}) \in Fm$ and any $\bar{c} \in B$ we have: $\varphi^{\mathbf{B}}(b, \bar{c}) \in G$ iff $\varphi^{\mathbf{B}}(b', \bar{c}) \in G$ and conclude by Proposition 2.2.2 that $\langle b, b'\rangle \in \Omega^{\mathbf{B}}(G)$. Thus, $b = b'$ since $\langle \mathbf{B}, G\rangle$ is reduced. This fact witnesses that at most one element of B is not mapped to $\top$ by h, contradicting the surjectivity of h. □

We shall close this section by providing some sufficient conditions for structural completeness and its infinitary analogue, for finitary logics. Some partial converses will be obtained later, in the sections of equivalential and finitely equivalential logics.

Theorem 4.1.3 ([32]). *Let $\mathcal{S}$ be a finitary logic.*

i If for every finitely generated RSI and reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$, there is a strict matrix homomorphism from $\langle \mathbf{A}, F\rangle$ into an ultrapower of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$, then $\mathcal{S}$ is structurally complete.

ii If for every $|Var|$-generated RSI and reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$, there is a strict matrix homomorphism from $\langle \mathbf{A}, F\rangle$ into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$, then every rule (finite or infinite) admissible for $\mathcal{S}$ is derivable in $\mathcal{S}$.

Proof. i: Let $\langle \Gamma, \varphi\rangle$ be a finite and underivable rule for $\mathcal{S}$. We show that $\langle \Gamma, \varphi\rangle$ is inadmissible for $\mathcal{S}$. Since $\mathcal{S}$ is finitary, by Lemma 4.1.2, there is a finitely generated RSI and reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$ that invalidates $\langle \Gamma, \varphi\rangle$. By hypothesis there is a strict matrix homomorphism from $\langle \mathbf{A}, F\rangle$ into an ultrapower of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. Since the mapping is strict, $\langle \Gamma, \varphi\rangle$ is not validated by the ultrapower. By Theorem 2.3.1, $\langle \Gamma, \varphi\rangle$ is not validated by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. Thus, $\langle \Gamma, \varphi\rangle$ is inadmissible for $\mathcal{S}$.
ii: Let $\langle \Gamma, \varphi\rangle$ be any underivable rule for $\mathcal{S}$. Clearly, the variables appearing in formulas of $\Gamma \cup \{\varphi\}$ have cardinality at most $|Var|$. Thus, due to finitarity of $\mathcal{S}$ and Lemma 4.1.2, there is a $|Var|$-generated RSI and reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$ that invalidates $\langle \Gamma, \varphi\rangle$. By hypothesis, there is a strict matrix homomorphism from $\langle \mathbf{A}, F\rangle$ into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. Since the mapping is strict, $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ also invalidates $\langle \Gamma, \varphi\rangle$ and therefore the rule is inadmissible for $\mathcal{S}$. □

4.2 Admissibility in protoalgebraic logics

Theorem 4.2.1 ([32]). *If $\mathcal{S}$ is a protoalgebraic logic, then $\mathbb{H}(Matr^*(\mathcal{S}))$ is the atomic closure of $Matr^*(\mathcal{S})$.*

Proof. In view of Theorems 2.4.1 and 2.4.2, the atomic closure of $Matr^*(\mathcal{S})$ is just $\mathbb{HSP}(Matr^*(\mathcal{S})) = \mathbb{HP}_{\mathbb{SD}}(Matr^*(\mathcal{S}))$. By Theorem 2.6.1, $Matr^*(\mathcal{S})$ is closed under subdirect products and therefore the atomic closure of $Matr^*(\mathcal{S})$ is indeed $\mathbb{H}(Matr^*(\mathcal{S}))$. □

Theorem 4.2.2 ([32]). *Let $\mathcal{S}$ be a protoalgebraic logic. The following conditions are equivalent:*

i $\langle \Gamma, \varphi\rangle$ is admissible for $\mathcal{S}$.

ii $Matr^*(\mathcal{S}) \subseteq \mathbb{H}(Matr^*(\mathcal{S} + \langle \Gamma, \varphi\rangle))$.

iii $Matr^*(\mathcal{S})$ and $Matr^*(\mathcal{S} + \langle \Gamma, \varphi\rangle)$ have the same atomic closure.

As it happened with Theorem 4.1.2, it would be enough to check only the finitely generated reduced matrix models in ii and iii or the (finitely generated) RSI, in the case $\mathcal{S}$ is finitary.

Proof. $i \Rightarrow ii$: Assume $\langle \Gamma, \varphi\rangle$ is admissible for $\mathcal{S}$ and let $\langle \mathbf{A}, F\rangle$ be a reduced matrix model of $\mathcal{S}$. By Theorem 4.1.2, $\langle \mathbf{A}, F\rangle$ is a homomorphic image of a matrix model $\langle \mathbf{B}, G\rangle$ of $\mathcal{S} + \langle \Gamma, \varphi\rangle$ that is itself a subdirect product of reduced matrix models of $\mathcal{S}$. Since $\mathcal{S}$ is protoalgebraic, $Matr^*(\mathcal{S})$ is closed under subdirect products and therefore $\langle \mathbf{B}, G\rangle \in$

$Matr^*(\mathcal{S})$. Finally, since it validates $\langle\Gamma,\varphi\rangle$, it follows that $\langle\mathbf{B},G\rangle \in Matr^*(\mathcal{S}+\langle\Gamma,\varphi\rangle)$. Thus, $\langle\mathbf{A},F\rangle \in \mathbb{H}(Matr^*(\mathcal{S}+\langle\Gamma,\varphi\rangle))$.
$ii \Rightarrow iii$: Clearly, since $Matr^*(\mathcal{S}+\langle\Gamma,\varphi\rangle) \subseteq Matr^*(\mathcal{S})$ we have that $\mathbb{H}(Matr^*(\mathcal{S}+\langle\Gamma,\varphi\rangle)) \subseteq \mathbb{H}(Matr^*(\mathcal{S}))$. Conversely, by ii we know that $Matr^*(\mathcal{S}) \subseteq \mathbb{H}(Matr^*(\mathcal{S}+\langle\Gamma,\varphi\rangle))$. Thus, $\mathbb{H}(Matr^*(\mathcal{S})) \subseteq \mathbb{H}(Matr^*(\mathcal{S}+\langle\Gamma,\varphi\rangle))$.
$iii \Rightarrow i$: This direction is immediate since iii implies the weaker third condition of Theorem 4.1.2. □

Given a logic $\mathcal{S}$, an algebra $\mathbf{A}$ and $F \in Fi_{\mathcal{S}}\mathbf{A}$, recall our notation:

$$\left(Fi_{\mathcal{S}}\mathbf{A}\right)^F := \{G \in Fi_{\mathcal{S}}\mathbf{A} : F \subseteq G\}.$$

Proposition 4.2.1 ([13]). *Let $\mathcal{S}$ and $\mathcal{S}'$ be two logics of the same type. The following conditions are equivalent:*

i $\mathcal{S}'$ is an axiomatic extension of $\mathcal{S}$.

ii For every algebra $\mathbf{A}$ and every $F \in Fi_{\mathcal{S}'}\mathbf{A}$, $\left(Fi_{\mathcal{S}}\mathbf{A}\right)^F = \left(Fi_{\mathcal{S}'}\mathbf{A}\right)^F$, i.e. for every $G \supseteq F$, $G \in Fi_{\mathcal{S}}\mathbf{A}$ iff $G \in Fi_{\mathcal{S}'}\mathbf{A}$.

iii For every theory $T \in Th(\mathcal{S}')$ and any $T' \supseteq T$, $T' \in Th(\mathcal{S})$ iff $T' \in Th(\mathcal{S}')$.

Proof. $i \Rightarrow ii$: Let $\mathbf{A}$ and $F \in Fi_{\mathcal{S}'}\mathbf{A}$. Since in general $Fi_{\mathcal{S}'}\mathbf{A} \subseteq Fi_{\mathcal{S}}\mathbf{A}$, the inclusion $\left(Fi_{\mathcal{S}'}\mathbf{A}\right)^F \subseteq \left(Fi_{\mathcal{S}}\mathbf{A}\right)^F$ is immediate. For the other inclusion, let $G \in \left(Fi_{\mathcal{S}}\mathbf{A}\right)^F$. Since $\mathcal{S}'$ is an axiomatic extension of $\mathcal{S}$, there is a set of formulas Δ closed under substitutions such that for any $\Gamma \cup \{\varphi\} \subseteq Fm$:

$$\Gamma \vdash_{\mathcal{S}'} \varphi \text{ iff } \Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi.$$

We show that G is an $\mathcal{S}'$-filter of $\mathbf{A}$. So assume $\Gamma \vdash_{\mathcal{S}'} \varphi$ and let $h \in Hom(\mathbf{Fm},\mathbf{A})$ such that $h[\Gamma] \subseteq G$. Then, $\Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi$. Since Δ is the set of axioms of $\mathcal{S}'$ and F is an $\mathcal{S}'$-filter of $\mathbf{A}$, then by Proposition 2.2.1 it follows that $h[\Delta] \subseteq F \subseteq G$. Thus, $h[\Gamma\cup\Delta] \subseteq G$. Hence, $h(\varphi) \in G$ since G is an $\mathcal{S}$-filter of $\mathbf{A}$ and $\Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi$.
$ii \Rightarrow iii$: Apply ii to the formula algebra $\mathbf{Fm}$.
$iii \Rightarrow i$: Firstly, condition iii implies that $Th(\mathcal{S}') \subseteq Th(\mathcal{S})$ and therefore $\mathcal{S} \leq \mathcal{S}'$. Consider $\Delta = Cn_{\mathcal{S}'}(\varnothing)$, which is clearly closed under substitutions.

Claim: $\Gamma \vdash_{\mathcal{S}'} \varphi$ iff $\Gamma \cup \Delta \vdash_{\mathcal{S}} \varphi$.

Observe that the claim is equivalent to: $Cn_{\mathcal{S}'}(\Gamma) = Cn_{\mathcal{S}}(\Gamma\cup\Delta)$, for any $\Gamma \subseteq Fm$. So pick any $\Gamma \subseteq Fm$. Obviously, $Cn_{\mathcal{S}'}(\Gamma) = Cn_{\mathcal{S}'}(\Gamma\cup\Delta)$. Moreover, $Cn_{\mathcal{S}'}(\varnothing) \subseteq Cn_{\mathcal{S}}(Cn_{\mathcal{S}'}(\varnothing))$ since $Cn_{\mathcal{S}}$ is extensive. Conversely, observe that $Cn_{\mathcal{S}}(Cn_{\mathcal{S}'}(\varnothing)) \subseteq Cn_{\mathcal{S}'}(Cn_{\mathcal{S}'}(\varnothing))$ since $Cn_{\mathcal{S}} \leq Cn_{\mathcal{S}'}$. However, $Cn_{\mathcal{S}'}(\varnothing) = Cn_{\mathcal{S}'}(Cn_{\mathcal{S}'}(\varnothing))$ and therefore we have that

$$\Delta = Cn_{\mathcal{S}}(Cn_{\mathcal{S}'}(\varnothing)) \subseteq Cn_{\mathcal{S}}(\Gamma\cup\Delta)$$

By iii, we have that $Cn_{\mathcal{S}}(\Gamma\cup\Delta) \in Th(\mathcal{S}')$. Hence, $Cn_{\mathcal{S}'}(\Gamma) \subseteq Cn_{\mathcal{S}'}\big(Cn_{\mathcal{S}}(\Gamma\cup\Delta)\big) = Cn_{\mathcal{S}}(\Gamma\cup\Delta)$. The other inclusion is immediate since $Cn_{\mathcal{S}} \leq Cn_{\mathcal{S}'}$ and all formulas in Δ are $\mathcal{S}'$-theorems.
Thus, $\mathcal{S}'$ is indeed an axiomatic extension of $\mathcal{S}$. □

Proposition 4.2.2 ([32]). *Assume $\mathcal{S}$ is a protoalgebraic logic. Every $|Var|$-generated reduced matrix model of $\mathcal{S}$ is a homomorphic image of $\langle\mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.*

Proof. Pick a $|Var|$-generated $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})$. Then, there is a surjective function from Var onto a generating set for $\mathbf{A}$ which extends naturally to a surjective homomorphism $h : \mathbf{Fm} \to \mathbf{A}$. By the *First Homomorphism Theorem* of Universal Algebra, $\mathbf{A} \cong \mathbf{Fm}/ker(h)$. By Proposition 2.2.3, $h^{-1}[\Omega^{\mathbf{A}}(F)] = \Omega(h^{-1}(F))$. Since $\langle \mathbf{A}, F\rangle$ is reduced, $\Omega^{\mathbf{A}}(F) = Id_A$ and therefore $h^{-1}[\Omega^{\mathbf{A}}(F)] = h^{-1}[Id_A] = ker(h)$. Thus, $\mathbf{A} \cong \mathbf{Fm}/\Omega(h^{-1}(F))$. Moreover, by Proposition 2.2.1, $h^{-1}[F] \in Th(\mathcal{S})$ and clearly $Cn_{\mathcal{S}}(\varnothing) \subseteq h^{-1}[F]$. Since $\mathcal{S}$ is protoalgebraic Ω is order preserving and therefore $\Omega(Cn_{\mathcal{S}}(\varnothing)) \subseteq \Omega(h^{-1}[F])$. Thus, the map $\pi : \mathbf{Fm}/\Omega(Cn_{\mathcal{S}}(\varnothing)) \to \mathbf{Fm}/\Omega(h^{-1}[F])$, given by $\varphi/\Omega(Cn_{\mathcal{S}}(\varnothing)) \mapsto \varphi/\Omega(h^{-1}[F])$, is a well-defined surjective homomorphism. Clearly, $\pi[Cn_{\mathcal{S}}(\varnothing)/\Omega(Cn_{\mathcal{S}}(\varnothing))] \subseteq h^{-1}[F]/\Omega(h^{-1}[F])$ and hence π is a surjective matrix homomorphism from $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ onto $\langle \mathbf{Fm}, h^{-1}[F]\rangle^*$. But since $\langle \mathbf{A}, F\rangle \cong \langle \mathbf{Fm}, h^{-1}[F]\rangle^*$, it follows that $\langle \mathbf{A}, F\rangle$ is indeed a homomorphic image of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. □

Definition 4.2.1 ([32]). A *reduced* matrix model $\langle \mathbf{A}, F\rangle$ of a logic $\mathcal{S}$ is called **weakly projective** (with respect to $\mathcal{S}$) if whenever $\langle \mathbf{A}, F\rangle$ is a homomorphic image of a *reduced* matrix model $\langle \mathbf{B}, G\rangle$ of $\mathcal{S}$, then there is an embedding from $\langle \mathbf{A}, F\rangle$ into $\langle \mathbf{B}, G\rangle$.

Theorem 4.2.3 ([32]). *Assume $\mathcal{S}$ is a protoalgebraic and finitary logic. If every finitely generated RSI and reduced matrix model of $\mathcal{S}$ is weakly projective, then $\mathcal{S}$ is hereditarily structurally complete.*

Proof. In view of Theorem 3.1.3 it is enough to show that every axiomatic extension of $\mathcal{S}$ is structurally complete. So let $\mathcal{S}'$ be an axiomatic extension of $\mathcal{S}$ and $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S}')$. Proposition 4.2.1 establishes that $Fi_{\mathcal{S}'}\mathbf{A}$ is an interval of the lattice $Fi_{\mathcal{S}}\mathbf{A}$ and therefore the $\mathcal{S}'$-filter F is completely meet-irreducible in $Fi_{\mathcal{S}'}\mathbf{A}$ iff it is completely meet-irreducible in $Fi_{\mathcal{S}}\mathbf{A}$. Thus, F is RSI with respect to $\mathcal{S}'$ iff it is RSI with respect to $\mathcal{S}$, by Lemma 4.1.1. Clearly, if $\langle \mathbf{A}, F\rangle$ is weakly projective with respect to $\mathcal{S}$, then it is weakly projective with respect to $\mathcal{S}'$ since $Fi_{\mathcal{S}'}\mathbf{A} \subseteq Fi_{\mathcal{S}}\mathbf{A}$. Hence, all assumptions persist in $\mathcal{S}'$ and therefore it is enough to show that $\mathcal{S}$ is structurally complete.
Let $\langle \Gamma, \varphi\rangle$ be a *finite* underivable rule for $\mathcal{S}$. We show that the rule is inadmissible for $\mathcal{S}$. Since $\mathcal{S}$ is finitary and the rule is underivable and finite, then by Lemma 4.1.2 it follows that there is a finitely generated RSI reduced matrix model $\langle \mathbf{B}, G\rangle$ of $\mathcal{S}$ that invalidates $\langle \Gamma, \varphi\rangle$. By Proposition 4.2.2, $\langle \mathbf{B}, G\rangle$ is a homomorphic image of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and finally by weak projectivity assumption, there is an embedding from $\langle \mathbf{B}, G\rangle$ into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. Consequently, $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ invalidates $\langle \Gamma, \varphi\rangle$ since embeddings between matrices are strict, injective homomorphisms and therefore they preserve invalidity of rules. Thus, $\langle \Gamma, \varphi\rangle$ is inadmissible for $\mathcal{S}$. □

4.3 Admissibility in equivalential logics

Lemma 4.3.1 ([32]). *Assume $\mathcal{S}$ is equivalential and $h : \langle \mathbf{B}, G\rangle \to \langle \mathbf{A}, F\rangle$ is a matrix homomorphism between matrix models of $\mathcal{S}$, where $\langle \mathbf{B}, G\rangle$ is reduced. If h is strict, then h is injective and therefore an embedding.*

Proof. Since $\mathcal{S}$ is equivalential, let $\Delta(x, y)$ be a set of equivalence formulas for $\mathcal{S}$. Pick any $b, b' \in$ with $h(b) = h(b')$. Then, $h[\Delta^{\mathbf{B}}(b, b')] = \Delta^{\mathbf{A}}(h(b), h(b')) \subseteq F$ since $h(b) = h(b')$ and $\vdash_{\mathcal{S}} \Delta(x, x)$ by Lemma 2.6.1. Thus, $\Delta^{\mathbf{B}}(b, b') \subseteq h^{-1}[F] = G$, since h is strict. Finally, Δ is a set of equivalence formulas for $\mathcal{S}$ and therefore $\langle b, b'\rangle \in \Omega^{\mathbf{B}}(G)$. Consequently, $b = b'$ as $\langle \mathbf{B}, G\rangle$ is reduced. □

Definition 4.3.1. Given a first-order language $\mathfrak{L}$, a class K of $\mathfrak{L}$-structures and a cardinal λ, we define:

$$\mathbb{U}_\lambda(K) := \{\mathcal{A} : \text{every } \lambda-\text{generated substructure of } \mathcal{A} \text{ belongs to } K\}.$$

In our context, this operator shall be applied to classes of matrices which are structures of type $L \cup \{P\}$, where P is a unary relation symbol and L is our fixed algebraic similarity type with a fixed infinite set of variables Var. Moreover, it will be applied only for $\lambda = |Var|$ and therefore for convenience we denote $\mathbb{U}_{|Var|}$ by $\mathbb{U}$.
Recall the following well-known fact:

Lemma 4.3.2. *If* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A}$*, then for any* n*-ary* L*-term* $t = t(x_1, \dots, x_n)$*,* $n \in \omega$*, and any* $b_1, \dots, b_n \in B$*, it holds that* $t^{\mathbf{A}}(b_1, \dots, b_n) = t^{\mathbf{B}}(b_1, \dots, b_n)$*.*

Lemma 4.3.3 ([32])**.** *For any logic* $\mathcal{S}$*,* $\mathbb{U}(Matr^*(\mathcal{S})) \subseteq Matr^*(\mathcal{S})$*.*

Proof. Let $\langle \mathbf{A}, F\rangle \in \mathbb{U}(Matr^*(\mathcal{S}))$. Then, every $|Var|$-generated substructure of $\langle \mathbf{A}, F\rangle$ is a reduced matrix model of $\mathcal{S}$. To see that $\langle \mathbf{A}, F\rangle$ is a matrix model of $\mathcal{S}$, assume $\Gamma \vdash_{\mathcal{S}} \varphi$ and suppose $h \in Hom(\mathbf{Fm}, \mathbf{A})$ is such that $h[\Gamma] \subseteq F$. Clearly, $h[\mathbf{Fm}]$ is the universe of a subalgebra of $\mathbf{A}$, which is $|Var|$-generated since $\mathbf{Fm}$ is such. Thus, $\langle h[\mathbf{Fm}], h[Fm] \cap F\rangle$ is a $|Var|$-generated submatrix of $\langle \mathbf{A}, F\rangle$ and therefore by hypothesis it belongs to $Matr^*(\mathcal{S})$. Finally, h can be seen as a homomorphism from $\mathbf{Fm}$ onto $h[\mathbf{Fm}]$, and since $h[\Gamma] \subseteq h[Fm] \cap F$ and $h[Fm] \cap F \in Fi_{\mathcal{S}} h[\mathbf{Fm}]$, it follows that $h(\varphi) \in h[Fm] \cap F \subseteq F$. Thus, $F \in Fi_{\mathcal{S}}\mathbf{A}$.
In order to see that $\langle \mathbf{A}, F\rangle$ is reduced, let $\langle a, b\rangle \in \Omega^{\mathbf{A}}(F)$ and $\mathbf{B} = Sg^{\mathbf{A}}(\{a, b\})$. By Proposition 2.2.2 and Lemma 4.3.2, it follows that $\langle a, b\rangle \in \Omega^{\mathbf{B}}(F \cap B)$ by an easy induction. Since $\langle \mathbf{B}, F \cap B\rangle$ is a 2-generated submatrix of $\langle \mathbf{A}, F\rangle$, then by hypothesis it is reduced and therefore $a = b$. □

A class K of first-order $\mathfrak{L}$-structures is called a $\mathbb{UISP}$*-class* if it is closed under the operators $\mathbb{U}, \mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. The smallest such class containing K is $\mathbb{UISP}(K)$.

Theorem 4.3.1 ([32])**.** *Let* $\mathcal{S}$ *be an equivalential logic. The map* $\mathcal{S}' \mapsto Matr^*(\mathcal{S}')$ *is a bijection from the extensions of* $\mathcal{S}$ *to the* $\mathbb{UISP}$*-subclasses of* $Matr^*(\mathcal{S})$*. Its inverse sends a* $\mathbb{UISP}$*-class* $K \subseteq Matr^*(\mathcal{S})$ *to the logic induced by* K*.*

Proof. Since $\mathcal{S}$ is equivalential, let $\Delta(x, y)$ be a set of equivalence formulas of $\mathcal{S}$. As explained in section 2.6, any extension of $\mathcal{S}$ will also be equivalential. Regardless of equivalentiality, by Theorem 2.2.2 it follows that for any extensions $\mathcal{S}', \mathcal{S}''$ of $\mathcal{S}$

$$\mathcal{S}' \leq \mathcal{S}'' \ \text{ iff } \ Matr^*(\mathcal{S}'') \subseteq Matr^*(\mathcal{S}').$$

In particular, if $\mathcal{S}''$ is a proper extension of $\mathcal{S}'$, then there is $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{S}''} \varphi$ but $\Gamma \nvdash_{\mathcal{S}'} \varphi$. Again by Theorem 2.2.2, it follows that there is some $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S}')$ that invalidates $\langle \Gamma, \varphi\rangle$ and therefore $\langle \mathbf{A}, F\rangle \notin Matr^*(\mathcal{S}'')$. Hence, the map $\mathcal{S}' \mapsto Matr^*(\mathcal{S}')$ is injective. Moreover, as each $\mathcal{S}' \geq \mathcal{S}$ is also equivalential, by Theorem 2.6.2, $Matr^*(\mathcal{S}')$ is closed under submatrices and direct products, while it is always (regardless of equivalentiality) closed under isomorphisms. Finally, by Lemma 4.3.3 it is closed under $\mathbb{U}$ and therefore it is indeed a $\mathbb{UISP}$-subclass of $Matr^*(\mathcal{S})$.
In order to see that it is a surjective mapping, let K be a $\mathbb{UISP}$-subclass of $Matr^*(\mathcal{S})$. Let $\mathcal{S}_K$ be the logic induced by K. That is:

$$\Gamma \vdash_{\mathcal{S}_K} \varphi \ \text{ iff } \ \langle \mathbf{A}, F\rangle \vDash \langle \Gamma, \varphi\rangle \text{ , for all } \langle \mathbf{A}, F\rangle \in K.$$

Observe that each $\langle \mathbf{A}, F\rangle \in K$ is reduced, since it belongs to $Matr^*(\mathcal{S})$, and clearly a matrix model of $\mathcal{S}_K$. Thus, $K \subseteq Matr^*(\mathcal{S}_K)$. For the other inclusion, let $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S}_K)$. The logic $\mathcal{S}_K$ is an extension of $\mathcal{S}$ and therefore it is equivalential. Hence $Matr^*(\mathcal{S}_K)$ is closed under submatrices. Moreover, K is closed under $\mathbb{U}$ and therefore we may assume that $\langle \mathbf{A}, F\rangle$ is $|Var|$-generated, in which case, there is a surjective homomorphism $h : \mathbf{Fm} \to \mathbf{A}$. By Proposition 2.2.1, $h^{-1}[F] \in Th(\mathcal{S}_K)$. For each $\psi \notin h^{-1}[F]$, the rule $\langle h^{-1}[F], \psi\rangle$ is underivable in $\mathcal{S}_K$ and since $\mathcal{S}_K$ is complete with respect to K, there are $\langle \mathbf{B}_\psi, G_\psi\rangle \in K$ that invalidate $\langle h^{-1}[F], \psi\rangle$. That is, there are homomorphisms $f_\psi : \mathbf{Fm} \to \mathbf{B}_\psi$ such that $f_\psi[h^{-1}[F]] \subseteq G_\psi$ but $f_\psi(\psi) \notin G_\psi$. Let $f : \mathbf{Fm} \to \prod\limits_{\psi\notin h^{-1}[F]} \mathbf{B}_\psi$ be the homomorphism induced by all f_ψ's, i.e. $f(\varphi) = \langle f_\psi(\varphi) : \psi \notin h^{-1}[F]\rangle$. Observe that:

$$\varphi \in h^{-1}[F] \;\Rightarrow\; f_\psi(\varphi) \in G_\psi\ , \text{ for all } \psi \notin h^{-1}[F] \;\Rightarrow\; f(\varphi) \in \prod_{\psi\notin h^{-1}[F]} G_\psi$$

and conversely

$$\varphi \notin h^{-1}[F] \;\Rightarrow\; f_\psi(\varphi) \notin G_\psi\ , \text{ for } \psi = \varphi \;\Rightarrow\; f(\varphi) \notin \prod_{\psi\notin h^{-1}[F]} G_\psi$$

Consequently, $f : \langle \mathbf{Fm}, h^{-1}[F]\rangle \to \langle \prod\limits_{\psi\notin h^{-1}[F]} \mathbf{B}_\psi\ ,\ \prod\limits_{\psi\notin h^{-1}[F]} G_\psi\rangle$ is a strict matrix homomorphism. Observe that the matrix $\langle \prod\limits_{\psi\notin h^{-1}[F]} \mathbf{B}_\psi\ ,\ \prod\limits_{\psi\notin h^{-1}[F]} G_\psi\rangle$ belongs to K, since K is closed under $\mathbb{P}$, and therefore it is a reduced matrix model of $\mathcal{S}_K$ since K is contained in $Matr^*(\mathcal{S}_K)$. Moreover, since $\mathcal{S}_K$ is equivalential, the Leibniz operator commutes with homomorphic inverse images (not necessarily surjective) of $\mathcal{S}_K$-filters, by Theorem 2.6.2. Hence,

$$f^{-1}\big[\Omega^{\prod_\psi \mathbf{B}_\psi}\big(\textstyle\prod_\psi G_\psi\big)\big] = \Omega\big(f^{-1}\big[\textstyle\prod_\psi G_\psi\big]\big)$$

Finally, since f is strict and $\langle \prod\limits_{\psi\notin h^{-1}[F]} \mathbf{B}_\psi\ ,\ \prod\limits_{\psi\notin h^{-1}[F]} G_\psi\rangle$ is reduced, it follows that

$$ker(f) = f^{-1}[Id_{\prod_\psi \mathbf{B}_\psi}] = f^{-1}\big[\Omega^{\prod_\psi \mathbf{B}_\psi}\big(\textstyle\prod_\psi G_\psi\big)\big] = \Omega\big(f^{-1}\big[\textstyle\prod_\psi G_\psi\big]\big) = \Omega(h^{-1}[F]).$$

But clearly, since $\langle \mathbf{A}, F\rangle$ is reduced, then again by Theorem 2.6.2, we obtain:

$$ker(h) = h^{-1}[Id_A] = h^{-1}[\Omega^{\mathbf{A}}(F)] = \Omega(h^{-1}[F]) = ker(f)$$

Thus, the mapping $h(a) \mapsto f(a)$ is a well-defined isomorphism from $\langle \mathbf{A}, F\rangle$ onto a submatrix of $\langle \prod\limits_{\psi\notin h^{-1}[F]} \mathbf{B}_\psi\ ,\ \prod\limits_{\psi\notin h^{-1}[F]} G_\psi\rangle$. Consequently, $\langle \mathbf{A}, F\rangle \in K$ since K is closed under submatrices and isomorphisms. □

Theorem 4.3.2 ([31],[32]). *Let $\mathcal{S}$ be an equivalential logic. The following conditions are equivalent:*

i Every rule (finite or infinite) admissible for $\mathcal{S}$ is derivable in $\mathcal{S}$.

ii For any proper $\mathbb{UISP}$-subclass K of $Matr^*(\mathcal{S})$, $\mathbb{H}(K) \subsetneq \mathbb{H}(Matr^*(\mathcal{S}))$.

iii $Matr^*(\mathcal{S}) = \mathbb{UISP}(\{\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*\})$.

Moreover, these conditions imply the next one:

iv Every $|Var|$-generated RSI and reduced matrix model of $\mathcal{S}$ can be embedded into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.

If $\mathcal{S}$ is finitary, then all the above conditions are equivalent.

Proof. $i \Leftrightarrow ii$:
($\Rightarrow$) By contraposition, assume K is a proper $\mathbb{UISP}$-subclass of $Matr^*(\mathcal{S})$ such that $\mathbb{H}(K) = \mathbb{H}(Matr^*(\mathcal{S}))$. Let $\mathcal{S}_K$ be the logic induced by K. By Theorem 4.3.1 $K = Matr^*(\mathcal{S}_K)$. Since K is proper, then $\mathcal{S}_K$ is a proper extension of $\mathcal{S}$. So there is $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \vdash_{\mathcal{S}_K} \varphi$ but $\Gamma \nvdash_{\mathcal{S}} \varphi$. Since $\mathcal{S}$ is equivalential, then it is protoalgebraic by Theorem 2.6.2. Moreover, $\mathcal{S} \leq \mathcal{S} + \langle\Gamma,\varphi\rangle \leq \mathcal{S}_K$ and $K = Matr^*(\mathcal{S}_K) \subseteq Matr^*(\mathcal{S} + \langle\Gamma,\varphi\rangle)$. Finally, since $\mathbb{H}(K) = \mathbb{H}(Matr^*(\mathcal{S}))$, it follows that every reduced matrix model of $\mathcal{S}$ is a homomorphic image of a reduced matrix model of $\mathcal{S} + \langle\Gamma,\varphi\rangle$ and therefore by Theorem 4.2.2 we obtain that $\langle\Gamma,\varphi\rangle$ is an admissible rule for $\mathcal{S}$. However, it is clearly an underivable rule in $\mathcal{S}$.
($\Leftarrow$) By contraposition. Assume that $\langle\Gamma,\varphi\rangle$ is an admissible but underivable rule for $\mathcal{S}$. Then, $Matr^*(\mathcal{S} + \langle\Gamma,\varphi\rangle)$ is a $\mathbb{UISP}$-subclass of $Matr^*(\mathcal{S})$, which is in fact proper, since $\mathcal{S} + \langle\Gamma,\varphi\rangle$ is a proper extension of $\mathcal{S}$. However, $\mathcal{S}$ is also protoalgebraic, and the rule is admissible for $\mathcal{S}$. Hence, by Theorem 4.2.2, it follows that $\mathbb{H}(Matr^*(\mathcal{S} + \langle\Gamma,\varphi\rangle)) = \mathbb{H}(Matr^*(\mathcal{S}))$.
$i \Leftrightarrow iii$:
($\Rightarrow$) By contraposition, denote $K = \mathbb{UISP}(\{\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*\})$ and suppose K is properly contained in $Matr^*(\mathcal{S})$. Let $\mathcal{S}_K$ be the logic induced by K. By Theorem 4.3.1, $K = Matr^*(\mathcal{S}_K)$ and since it is properly contained in $Matr^*(\mathcal{S})$, then $\mathcal{S}_K$ is a proper extension of $\mathcal{S}$. So let $\Gamma \cup \{\varphi\} \subseteq Fm$ be such that $\Gamma \vdash_{\mathcal{S}_K} \varphi$ but $\Gamma \nvdash_{\mathcal{S}} \varphi$. Thus, the rule $\langle\Gamma,\varphi\rangle$ is validated by all reduced $\mathcal{S}_K$-models. In particular, it is validated by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and therefore it is admissible for $\mathcal{S}$. However, $\langle\Gamma,\varphi\rangle$ is clearly underivable for $\mathcal{S}$.
($\Leftarrow$) A rule $\langle\Gamma,\varphi\rangle$ is admissible for $\mathcal{S}$ iff it is validated by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. It is well-known that the validity of a rule persists in direct products, submatrices and isomorphic copies. Finally, as in the proof of Lemma 4.3.3, we obtain that $\langle\Gamma,\varphi\rangle$ would be validated by any matrix all whose $|Var|$-generated submatrices validate $\langle\Gamma,\varphi\rangle$ and therefore the validity of the rule is inherited by the whole class $\mathbb{UISP}(\{\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*\})$. By iii, this class is equal to $Matr^*(\mathcal{S})$ and therefore by Theorem 2.2.2, the rule is derivable in $\mathcal{S}$.
$iii \Rightarrow iv$: Let $\langle \mathbf{A}, F\rangle$ be a $|Var|$-generated RSI and reduced matrix model of $\mathcal{S}$. By iii, $\langle \mathbf{A}, F\rangle$ is isomorphic to a submatrix of $\prod_{i\in I}\langle \mathbf{B}_i, G_i\rangle$, where each $\langle \mathbf{B}_i, G_i\rangle$ is $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.
However, $\mathbb{SP}(K) \subseteq \mathbb{P}_{\mathbb{SD}}\mathbb{S}(K)$ for any class K of $\mathfrak{L}$-structures, and therefore there is a submatrix $\langle \mathbf{C}, H\rangle$ of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and a subdirect product $\langle \mathbf{B}, G\rangle$ of $\prod_{j\in J}\langle \mathbf{C}, H\rangle$ such that $\langle \mathbf{A}, F\rangle$ is isomorphic to $\langle \mathbf{B}, G\rangle$. By the definition of RSI reduced matrices, $\langle \mathbf{A}, F\rangle$ is isomorphic to $\langle \mathbf{C}, H\rangle$ and therefore it can be embedded in $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.
Finally, in the case $\mathcal{S}$ is finitary, the direction $iv \Rightarrow i$ instantiates Theorem 4.1.3 ii, since matrix embeddings are strict. □

The hereditary analogue of Theorem 4.3.2 is easily obtained as a Corollary:

Corollary 4.3.1 ([32]). *Assume $\mathcal{S}$ is an equivalential logic. The following conditions are equivalent:*

- *i For any extension $\mathcal{S}'$ of $\mathcal{S}$, every rule admissible for $\mathcal{S}'$ is derivable in $\mathcal{S}'$.*
- *ii For any $\mathbb{UISP}$-subclasses K_1, K_2 of $Matr^*(\mathcal{S})$, if $K_1 \subsetneq K_2$, then $\mathbb{H}(K_1) \subsetneq \mathbb{H}(K_2)$.*
- *iii For every $\mathbb{UISP}$-subclass K of $Matr^*(\mathcal{S})$, $Matr^*(\mathcal{S}) \cap \mathbb{H}(K) = K$, i.e. K is a relative atomic subclass of $Matr^*(\mathcal{S})$.*

Proof. $i \Rightarrow ii$: Immediate by Theorem 4.3.2.
$ii \Rightarrow iii$: Let K be a $\mathbb{UISP}$-subclass of $Matr^*(\mathcal{S})$. Observe that $Matr^*(\mathcal{S}) \cap \mathbb{H}(K)$ is also a $\mathbb{UISP}$-subclass of $Matr^*(\mathcal{S})$ and clearly $K \subseteq Matr^*(\mathcal{S}) \cap \mathbb{H}(K)$. Thus, if $K \subsetneq Matr^*(\mathcal{S}) \cap \mathbb{H}(K)$, then by ii: $\mathbb{H}(K) \subsetneq \mathbb{H}\big(Matr^*(\mathcal{S}) \cap \mathbb{H}(K)\big) \subseteq \mathbb{H}(Matr^*(\mathcal{S})) \cap \mathbb{H}(K)$, a contradiction. Thus, $K = Matr^*(\mathcal{S}) \cap \mathbb{H}(K)$.
$iii \Rightarrow i$: Pick any extension $\mathcal{S}'$ of $\mathcal{S}$ and any proper $\mathbb{UISP}$-subclass K of $Matr^*(\mathcal{S}')$. Since, $Matr^*(\mathcal{S}')$ and K are both $UISP$-subclasses of $Matr^*(\mathcal{S})$, then by iii it follows that $Matr^*(\mathcal{S}) \cap \mathbb{H}(K) = K$ and $Matr^*(\mathcal{S}) \cap \mathbb{H}(Matr^*(\mathcal{S}')) = Matr^*(\mathcal{S}')$. Due to these conditions and the fact that K is a proper subclass of $Matr^*(\mathcal{S}')$, we obtain that $\mathbb{H}(K) \subsetneq \mathbb{H}(Matr^*(\mathcal{S}'))$. Hence, by Theorem 4.3.2, every rule (finite or infinite) admissible for $\mathcal{S}'$ is derivable in $\mathcal{S}'$. □

4.4 Admissibility in finitely equivalential logics

The corresponding version of the isomorphism Theorem 4.3.1 for finitely equivalential logics is the following:

Theorem 4.4.1 ([13]). *Assume $\mathcal{S}$ is finitary and finitely equivalential. The map $\mathcal{S}' \mapsto Matr^*(\mathcal{S}')$ is a bijection from the finitary extensions of $\mathcal{S}$ to the universal Horn subclasses of $Matr^*(\mathcal{S})$. Its inverse sends a universal Horn class $K \subseteq Matr^*(\mathcal{S})$ to the logic induced by K.*

Proof. As explained in section 2.6, any extension $\mathcal{S}'$ of $\mathcal{S}$ will still be finitely equivalential. Hence, for any finitary extension $\mathcal{S}'$ of $\mathcal{S}$, $Matr^*(\mathcal{S}')$ is indeed a universal Horn subclass of $Matr^*(\mathcal{S})$, by Theorem 2.6.4. Clearly, as explained in Theorem 4.3.1, the map $\mathcal{S}' \mapsto Matr^*(\mathcal{S}')$ is injective. In order to prove surjectivity, let $K \subseteq Matr^*(\mathcal{S})$ be a universal Horn class. Let $\mathcal{S}_K$ be the logic induced by K. Since K is closed under ultraproducts, the logic $\mathcal{S}_K$ is finitary. Moreover, $\mathcal{S}_K$ is finitely equivalential since it is an extension of $\mathcal{S}$. Clearly, each member of K is a reduced matrix model of $\mathcal{S}_K$ and therefore $K \subseteq Matr^*(\mathcal{S}_K)$. By Theorem 2.6.5 we obtain that $Matr^*(\mathcal{S}_K) = \mathbb{ISPP}_{\mathbb{U}}(K)$ and since K is a universal Horn class, $Matr^*(\mathcal{S}_K) = K$. □

Theorem 4.4.2 ([32]). *Let $\mathcal{S}$ be a finitary and finitely equivalential logic. The following conditions are equivalent:*

- *i* *$\mathcal{S}$ is structurally complete.*
- *ii* $Matr^*(\mathcal{S}) = \mathbb{ISPP}_{\mathbb{U}}(\{\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*\})$.
- *iii* *Every RSI and reduced matrix model of $\mathcal{S}$ can be embedded into an ultrapower of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.*
- *iv* *Every finitely generated RSI and reduced matrix model of $\mathcal{S}$ can be embedded into an ultrapower of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.*

Proof. $i \Rightarrow ii$: By contraposition, let $K = \mathbb{ISPP}_{\mathbb{U}}(\{\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*\})$ and assume that K is properly contained in $Matr^*(\mathcal{S})$. Let $\mathcal{S}_K$ be the logic induced by K. By Theorem 4.4.1, $K = Matr^*(\mathcal{S}_K)$. Thus, $Matr^*(\mathcal{S}_K)$ is properly contained in $Matr^*(\mathcal{S})$ and therefore $\mathcal{S}_K$ is a proper extension of $\mathcal{S}$, which is in fact finitary since $Matr^*(\mathcal{S}_K)$ is closed under ultraproducts. So let $\Gamma' \cup \{\varphi\} \subseteq Fm$ be such that $\Gamma' \vdash_{\mathcal{S}_K} \varphi$ but $\Gamma' \nvdash_{\mathcal{S}} \varphi$. Since $\mathcal{S}_K$ is finitary, there is a finite $\Gamma \subseteq \Gamma'$ such that $\Gamma \vdash_{\mathcal{S}_K} \varphi$. Clearly also, $\Gamma \nvdash_{\mathcal{S}} \varphi$. Consequently, $\langle \Gamma, \varphi \rangle$ is a finite rule of $\mathcal{S}_K$ and therefore it is validated by all reduced

matrix models of $\mathcal{S}_K$. In particular, it is validated by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and hence the rule $\langle \Gamma, \varphi\rangle$ is admissible for $\mathcal{S}$ but clearly underivable in $\mathcal{S}$.
$ii \Rightarrow iii$: Recall that $\mathbb{SP}(K) \subseteq \mathbb{P}_{\mathbb{SD}}\mathbb{S}(K)$ for any class K of similar structures. Let $\langle \mathbf{A}, F\rangle$ be an RSI reduced matrix model of $\mathcal{S}$. By ii we have that $\langle \mathbf{A}, F\rangle \cong \langle \mathbf{B}, G\rangle$, where $\langle \mathbf{B}, G\rangle \in \mathbb{SPP}_{\mathbb{U}}(\{\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*\}) \subseteq \mathbb{P}_{\mathbb{SD}}\mathbb{SP}_{\mathbb{U}}(\{\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*\})$. Thus, $\langle \mathbf{A}, F\rangle$ is isomorphic to a subdirect product of submatrices of ultrapowers of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and since $Matr^*(\mathcal{S})$ is closed under submatrices and $\langle \mathbf{A}, F\rangle$ is RSI, it follows that $\langle \mathbf{A}, F\rangle$ is isomorphic to one of these submatrices. Thus, it is embeddable into an ultrapower of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.
$iii \Rightarrow iv$: Clear.
$iv \Rightarrow i$: This direction instantiates Theorem 4.1.3i, since matrix embeddings are strict. □

As an immediate Corollary of Theorem 2.4.4 we obtain:

Corollary 4.4.1 ([32]). *Assume that $\mathcal{S}$ is finitary and finitely equivalential. If $\mathcal{S}$ is structurally complete, then $Matr^*(\mathcal{S})$ has the joint embedding property.*

Combining Theorems 2.4.5, 4.1.3i and 4.4.2 we obtain:

Corollary 4.4.2 ([32]). *Assume $\mathcal{S}$ is finitary. If every finitely generated RSI and reduced matrix model of $\mathcal{S}$ is locally embeddable into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$, then $\mathcal{S}$ is structurally complete. The converse holds if $\mathcal{S}$ is finitely equivalential with a finite language.*

Theorem 4.4.3 ([32]). *Assume that L is finite and $\mathcal{S}$ is a finitary and finitely equivalential logic of type L, with the s.f.m.p. (with respect to $Matr^*(\mathcal{S})$). The following conditions are equivalent:*

i $\mathcal{S}$ is structurally complete.

ii Every finite RSI reduced matrix model of $\mathcal{S}$ can be embedded into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$.

Proof. $i \Rightarrow ii$: This direction is an immediate consequence of Corollary 4.4.2 since a finite structure is locally embeddable into a structure $\mathcal{A}$ iff it is embeddable into $\mathcal{A}$.
$ii \Rightarrow i$: Let $\langle \Gamma, \varphi\rangle$ be an underivable finite rule. By the s.f.m.p., there is a finite reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\mathcal{S}$, which can be chosen RSI by Lemma 4.1.2. By ii, $\langle \mathbf{A}, F\rangle$ can be embedded into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and therefore the rule is also invalidated by $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. Hence, $\langle \Gamma, \varphi\rangle$ is inadmissible in $\mathcal{S}$. □

Proposition 4.4.1 ([32]). *If $\mathcal{S}$ is finitely equivalential and strongly finite, then every RSI and reduced matrix model of $\mathcal{S}$ is finite.*

Proof. Let K be a finite set of reduced finite matrices which is a matrix semantics for $\mathcal{S}$ (we can choose K to contain only reduced matrices, since a matrix and its reduction validate the same rules). By Theorem 2.5.1, $\mathcal{S}$ is finitary. Hence, since $\mathcal{S}$ is finitary and finitely equivalential and it is complete with respect to $K \subseteq Matr^*(\mathcal{S})$, then by Theorem 2.6.5 we obtain that $Matr^*(\mathcal{S}) = \mathbb{ISPP}_{\mathbb{U}}(K)$. However, the isomorphic closure of a finite set of finite matrices is closed under ultraproducts and therefore $\mathbb{ISPP}_{\mathbb{U}}(K) = \mathbb{ISP}(K)$, while clearly, $\mathbb{ISP}(K) = \mathbb{IP}_{\mathbb{SD}}\mathbb{S}(K)$. Hence, $Matr^*(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}\mathbb{S}(K)$ and therefore, every RSI in $Matr^*(\mathcal{S})$ can be embedded into a member of K and consequently is finite. □

Theorem 4.4.4 ([32]). *Assume that L is finite and $\mathcal{S}$ is a strongly finite, finitary and finitely equivalential logic of type L. If $\mathcal{S}$ is structurally complete, then every rule (finite or infinite) admissible for $\mathcal{S}$ is derivable in $\mathcal{S}$.*

Proof. Since $\mathcal{S}$ is strongly finite, it is finitary and it has the s.f.m.p. (with respect to $Matr^*(\mathcal{S})$). By hypothesis, $\mathcal{S}$ is structurally complete and therefore by Theorem 4.4.3 and Proposition 4.4.1, every RSI and reduced matrix model of $\mathcal{S}$ can be embedded into $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. Hence, by Theorem 4.3.2, every rule (finite or infinite) admissible for $\mathcal{S}$ is derivable in $\mathcal{S}$. □

Theorem 4.4.5 ([32])**.** *Let $\mathcal{S}$ be a finitary and finitely equivalential logic. Then:*

i $\mathcal{S}$ is structurally complete iff for any proper universal Horn subclass K of $Matr^*(\mathcal{S})$, $\mathbb{H}(K) \subsetneq \mathbb{H}(Matr^*(\mathcal{S}))$.

ii $\mathcal{S}$ is hereditarily structurally complete iff $Matr^*(\mathcal{S})$ is primitive (see section 2.4).

Proof. i: ($\Rightarrow$) By contraposition. Let K be a proper universal Horn subclass of $Matr^*(\mathcal{S})$ with $\mathbb{H}(K) = \mathbb{H}(Matr^*(\mathcal{S}))$. Let $\mathcal{S}_K$ be the logic induced by K. By Theorem 4.4.1, $K = Matr^*(\mathcal{S}_K)$. Since $K \subsetneq Matr^*(\mathcal{S})$, then $\mathcal{S}_K$ is a proper finitary extension of $\mathcal{S}$. So let $\langle \Gamma, \varphi\rangle$ be a finite rule such that $\Gamma \vdash_{\mathcal{S}_K} \varphi$ but $\Gamma \nvdash_{\mathcal{S}}$. Then, $\mathcal{S} \leq \mathcal{S} + \langle \Gamma, \varphi\rangle \leq \mathcal{S}_K$ and therefore $Matr^*(\mathcal{S}_K) \subseteq Matr^*(\mathcal{S} + \langle \Gamma, \varphi\rangle) \subseteq Matr^*(\mathcal{S})$. Thus, all of these classes have the same atomic closure. Clearly, $\mathcal{S}$ is also protoalgebraic and hence by Theorem 4.2.2, it follows that $\langle \Gamma, \varphi\rangle$ is a finite rule admissible for $\mathcal{S}$. However, it is underivable in $\mathcal{S}$ and consequently $\mathcal{S}$ is not structurally complete.
($\Leftarrow$) By contraposition. Let $\langle \Gamma, \varphi\rangle$ be a finite rule admissible but underivable in $\mathcal{S}$. Then, $\mathcal{S} + \langle \Gamma, \varphi\rangle$ is a proper finitary extension of $\mathcal{S}$ and consequently $Matr^*(\mathcal{S} + \langle \Gamma, \varphi\rangle)$ is a proper universal Horn subclass of $Matr^*(\mathcal{S})$ (since $\mathcal{S} + \langle \Gamma, \varphi\rangle$ is finitary and finitely equivalential). However, in view of Theorem 4.2.2 (since $\mathcal{S}$ is also protoalgebraic), it follows that $Matr^*(\mathcal{S})$ and $Matr^*(\mathcal{S} + \langle \Gamma, \varphi\rangle)$ have the same atomic closure, i.e. $\mathbb{H}(Matr^*(\mathcal{S})) = \mathbb{H}(Matr^*(\mathcal{S} + \langle \Gamma, \varphi\rangle))$ and we obtained the required.
ii: This is an immediate consequence of i and the definitions. □

In view of Theorems 4.4.5ii, 4.4.1 and Proposition 2.4.1 we obtain:

Proposition 4.4.2 ([32])**.** *If $\mathcal{S}$ is finitary, finitely equivalential and structurally complete, then its finitary extensions form a distributive lattice.*
(The finitary extensions are axiomatic in this case, by Theorem 3.1.3.)

Definition 4.4.1 ([32])**.** Let $\mathcal{S}$ be an equivalential logic with $\Delta(x, y)$ a set of equivalence formulas. Two formulas $\varphi, \psi \in Fm$, are *logically equivalent* in $\mathcal{S}$ if $\vdash_{\mathcal{S}} \Delta(\varphi, \psi)$. The logic $\mathcal{S}$ will be called **locally tabular** if it has only finitely-many logically inequivalent formulas in n fixed variables, for any $n \in \omega$.

If an equivalential logic $\mathcal{S}$ is tabular, then every non-theorem of $\mathcal{S}$ is invalidated by some finite (reduced) matrix model of $\mathcal{S}$ and therefore it is obviously locally tabular. On the other hand, if an equivalential logic $\mathcal{S}$ is locally tabular, then it is finitely equivalential, since in this case it has only finitely-many logically inequivalent formulas in two variables and hence finitely-many formulas in $\Delta(x, y)$ witness this fact.
Recall from section 2.4 that a universal Horn class K is called locally finite, if every finitely generated member of K is finite.

Proposition 4.4.3 ([32])**.** *Assume that $\mathcal{S}$ is equivalential. Then, $\mathcal{S}$ is locally tabular iff $Matr^*(\mathcal{S})$ is locally finite.*

Proof. Fix some $n \in \omega$ and let $\langle \mathbf{A}, F\rangle$ be any n-generated and reduced matrix model of $\mathcal{S}$. Let $X \subseteq Var$ be of cardinality n. Then, there is a bijection from X onto a generating set for $\mathbf{A}$, which extends naturally to a surjective homomorphism $h : \mathbf{Fm} \to \mathbf{A}$. By the *First Homomorphism Theorem* of Universal Algebra, $\mathbf{A} \cong \mathbf{Fm}/ker(h)$. By Proposition 2.2.3 and the fact that $\langle \mathbf{A}, F\rangle$ is reduced, it follows that:

$$ker(h) = h^{-1}[Id_A] = h^{-1}[\Omega^{\mathbf{A}}(F)] = \Omega^{\mathbf{Fm(X)}}\big(h^{-1}[F]\big).$$

The set $G_X = \mathbf{Fm}(X) \cap Cn_{\mathcal{S}}(\varnothing)$ is obviously an $\mathcal{S}$-filter of $\mathbf{Fm}(X)$. Moreover, it is clear that $h^{-1}[F] \in Fi_{\mathcal{S}}\mathbf{Fm}(X)$ and $G_X \subseteq h^{-1}[F]$, since G_X consists of $\mathcal{S}$-theorems with variables amongst the elements of X. Hence, since $\mathcal{S}$ is protoalgebraic, $\Omega^{\mathbf{Fm}(X)}(G_X) \subseteq \Omega^{\mathbf{Fm}(X)}(h^{-1}[F]) = ker(h)$ and therefore the map $\varphi/\Omega^{\mathbf{Fm}(X)}(G_X) \mapsto h(\varphi)$ is a well-defined surjective matrix homomorphism from $\langle \mathbf{Fm}(X), G_X\rangle^*$ onto $\langle \mathbf{A}, F\rangle$.
Finally, for any two subsets X, Y of Var of cardinality n, we know that $\mathbf{Fm}(X) \cong \mathbf{Fm}(Y)$ and therefore the reductions of the matrices $\langle \mathbf{Fm}(X), G_X\rangle$ and $\langle \mathbf{Fm}(Y), G_Y\rangle$ would be isomorphic as well. Consequently, for each $n \in \omega$, we fix an arbitrary $X \subseteq Var$ of cardinality n and we denote by $\langle \mathbf{A}_n, F_n\rangle$ the matrix $\langle \mathbf{Fm}(X), G_X\rangle^*$. Thus, we have proven that every n-generated and reduced matrix model of $\mathcal{S}$ is a homomorphic image of $\langle \mathbf{A}_n, F_n\rangle$. Hence, it is obvious that $Matr^*(\mathcal{S})$ is locally finite iff $\langle \mathbf{A}_n, F_n\rangle$ is finite, for each $n \in \omega$.

Claim: $\mathcal{S}$ is locally tabular iff $\langle \mathbf{A}_n, F_n\rangle$ is finite, for every $n \in \omega$

Let $\Delta(x, y)$ be a set of equivalence formulas for $\mathcal{S}$. Recall that for any algebra $\mathbf{A}$, any $F \in Fi_{\mathcal{S}}\mathbf{A}$ and any $a, b \in A$:

$$\langle a, b\rangle \in \Omega^{\mathbf{A}}(F) \text{ iff } \Delta^{\mathbf{A}}(a, b) \subseteq F.$$

$\Leftarrow$: By contraposition. Assume $\mathcal{S}$ is not locally tabular. Then, there are n fixed variables $x_1, \dots, x_n$ with $n \in \omega$, an infinite set I and infinitely-many formulas $\{\varphi_i : i \in I\} \subseteq Fm$, all whose variables are amongst $x_1, \dots, x_n$, such that $\nvdash_{\mathcal{S}} \Delta(\varphi_i, \varphi_j)$ for all $i, j \in I$ with $i \neq j$. Thus, $\langle \varphi_i, \varphi_j\rangle \notin \Omega(Cn_{\mathcal{S}}(\varnothing))$ for any different $i, j \in I$. Since for any $Y, Z \subseteq Var$ with $|Y| = |Z| = n$, it holds that $\mathbf{Fm}(Y) \cong \mathbf{Fm}(Z)$, we may assume without loss of generality that $X = \{x_1, \dots, x_n\}$. Since $\mathcal{S}$ is equivalential, from Theorem 2.6.2 we know that:

$$\Omega^{\mathbf{Fm}(X)}(G_X) = \big(\mathbf{Fm}(X) \times \mathbf{Fm}(X)\big) \cap \Omega(Cn_{\mathcal{S}}(\varnothing)) \qquad (*)$$

Moreover, since all formulas φ_i have variables amongst $x_1, \dots, x_n$, it is clear that $\varphi_i^{\mathbf{Fm}(X)} = \varphi_i$ for each $i \in I$, by Lemma 4.3.2. Consequently, $\langle \varphi_i, \varphi_j\rangle \notin \Omega^{\mathbf{Fm}(X)}(G_X)$ for all $i, j \in I$ with $i \neq j$, and therefore $\langle \mathbf{A}_n, F_n\rangle$ is infinite.
$\Rightarrow$: By contraposition. Assume that for some $n \in \omega$, $\langle \mathbf{A}_n, F_n\rangle$ is infinite. Then, $\mathbf{A}_n = \mathbf{Fm}(X)/\Omega^{\mathbf{Fm}(X)}(G_X)$ contains infinitely-many classes. So let I be an infinite set and let $\{\varphi_i : i \in I\} \subseteq Fm(X)$ be such that $\langle \varphi_i, \varphi_j\rangle \notin \Omega^{\mathbf{Fm}(X)}(G_X)$, for all $i, j \in I$ with $i \neq j$. By $(*)$, it follows that $\langle \varphi_i, \varphi_j\rangle \notin \Omega(Cn_{\mathcal{S}}(\varnothing))$, for all $i \neq j$ and therefore by equivalentiality, $\nvdash_{\mathcal{S}} \Delta(\varphi_i, \varphi_j)$, for all $i \neq j$. Thus, $\mathcal{S}$ is not locally tabular. □

By combining Proposition 4.4.3 and Theorem 2.4.6, we obtain a partial converse of Theorem 4.2.3:

Theorem 4.4.6 ([32]). *Assume $\mathcal{S}$ is finitary, equivalential and locally tabular. Then, $\mathcal{S}$ is hereditarily structurally complete iff every finite RSI reduced matrix model of $\mathcal{S}$ is weakly projective.*

4.5 Admissibility in truth-equational logics

Recall that a logic $\mathcal{S}$ is truth-equational when truth is equationally definable in $Matr^*(\mathcal{S})$ (if necessary, see section 2.6). The following Lemma is well-known:

Lemma 4.5.1. *Let $\tau(x)$ be a set of of equations and $\langle \mathbf{A}, F\rangle$ be any matrix. Then, τ defines the filter of the matrix $\langle \mathbf{A}/\Omega^{\mathbf{A}}(F), F/\Omega^{\mathbf{A}}(F)$ iff $F = \{a \in A : \tau^{\mathbf{A}}(a) \subseteq \Omega^{\mathbf{A}}(F)\}$.*

As an immediate Corollary of Lemma 4.5.1 and Theorem 2.6.6, we obtain:

Corollary 4.5.1. *Let $\mathcal{S}$ be a logic and $\tau(x)$ a set of equations. Then, τ defines the filters in $Matr^*(\mathcal{S})$ iff for any algebra $\mathbf{A}$ and any $F \in Fi_{\mathcal{S}}\mathbf{A}$, $F = \{a \in A : \tau^{\mathbf{A}}(a) \subseteq \Omega^{\mathbf{A}}(F)\}$.*

Notation: Throughout the forthcoming sections we shall use the notation $\mathbf{B} \subseteq_{SP} \prod\limits_{i \in I} \mathbf{B}_i$ to indicate that the algebra $\mathbf{B}$ is a subdirect product of the family $\{\mathbf{B}_i : i \in I\}$, and similarly for subdirect products of matrices.

Lemma 4.5.2. *Assume that $\mathcal{S}$ is truth-equational with $\tau(x)$ a set of defining equations. If $\langle \mathbf{B}, G\rangle$ is a subdirect product of reduced matrix models of $\mathcal{S}$, then $G = \tau\mathbf{B}$.*

Proof. Assume $\langle \mathbf{B}, G\rangle \subseteq_{SP} \prod\limits_{i \in I}\langle \mathbf{B}_i, G_i\rangle$, where $\{\langle \mathbf{B}_i, G_i\rangle : i \in I\} \subseteq Matr^*(\mathcal{S})$. Since each $\langle \mathbf{B}_i, G_i\rangle$ is a reduced matrix model of $\mathcal{S}$, then $G_i = \tau\mathbf{B}_i$, by Theorem 2.6.6. Moreover, $\langle \mathbf{B}, G\rangle$ is a submatrix of $\prod\limits_{i \in I}\langle \mathbf{B}_i, G_i\rangle$ and therefore, $G = B \cap \prod\limits_{i \in I} G_i$. Observe that:

$$\begin{aligned} b \in \tau\mathbf{B} \text{ iff } & \delta_l^{\mathbf{B}}(b) = \delta_r^{\mathbf{B}}(b), \text{ for all } \langle \delta_l, \delta_r\rangle \in \tau \\ \text{iff } & \delta_l^{\mathbf{B}_i}(b(i)) = \delta_r^{\mathbf{B}_i}(b(i)), \text{ for all } i \in I \text{ and all } \langle \delta_l, \delta_r\rangle \in \tau \\ \text{iff } & b(i) \in \tau\mathbf{B}_i, \text{ for all } i \in I \\ \text{iff } & b \in \prod_{i \in I} \tau\mathbf{B}_i = \tau \prod_{i \in I} \mathbf{B}_i \end{aligned}$$

Thus, $\tau\mathbf{B} = \tau\prod\limits_{i \in I}\mathbf{B}_i = \prod\limits_{i \in I}\tau\mathbf{B}_i = \prod\limits_{i \in I} G_i$. Consequently, $G = B \cap \prod\limits_{i \in I} G_i = \tau\mathbf{B}$. $\square$

Given a logic $\mathcal{S}$, an algebra $\mathbf{A}$ and $X \subseteq A$, the least $\mathcal{S}$-filter of $\mathbf{A}$ containing X is $\bigcap\{F \in Fi_{\mathcal{S}}\mathbf{A} : X \subseteq F\}$ and is denoted by $Fg_{\mathcal{S}}^{\mathbf{A}}(X)$. The least $\mathcal{S}$-filter of $\mathbf{A}$ is the intersection of *all* $\mathcal{S}$-filters of $\mathbf{A}$ and is denoted by $\min Fi_{\mathcal{S}}\mathbf{A}$.

Proposition 4.5.1. *Assume that $\mathcal{S}$ is truth-equational with $\tau(x)$ a set of defining equations. Then:*

i For any algebra $\mathbf{A}$, $Fg_{\mathcal{S}}^{\mathbf{A}}(\tau\mathbf{A}) = \min Fi_{\mathcal{S}}\mathbf{A}$.

ii For any algebra $\mathbf{A} \in Alg^*(\mathcal{S})$, $\tau\mathbf{A} = \min Fi_{\mathcal{S}}\mathbf{A}$.

iii For any algebra $\mathbf{A} \in Alg(\mathcal{S})$, $\tau\mathbf{A} = \min Fi_{\mathcal{S}}\mathbf{A}$.

Proof. *i*: Pick any algebra $\mathbf{A}$ and let $F = \min Fi_{\mathcal{S}}\mathbf{A}$. If $a \in \tau\mathbf{A}$, then $\delta_l^{\mathbf{A}}(a) = \delta_r^{\mathbf{A}}(a)$, for all $\langle \delta_l, \delta_r\rangle \in \tau$. Hence, $\tau^{\mathbf{A}}(a) \subseteq Id_A \subseteq \Omega^{\mathbf{A}}(F)$. By Corollary 4.5.1, it follows that $a \in F$ and therefore $\tau\mathbf{A} \subseteq F$. Thus, $Fg_{\mathcal{S}}^{\mathbf{A}}(\tau\mathbf{A}) = \min Fi_{\mathcal{S}}\mathbf{A}$.
ii: Pick any $\mathbf{A} \in Alg^*(\mathcal{S})$. Then, there is some $F \in Fi_{\mathcal{S}}\mathbf{A}$ such that $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})$. By Theorem 2.6.6, $F = \tau\mathbf{A}$ and in view of item i, $\tau\mathbf{A} = \min Fi_{\mathcal{S}}\mathbf{A}$.
iii: Pick any $\mathbf{A} \in Alg(\mathcal{S})$. We know that $Alg(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}Alg^*(\mathcal{S})$ and therefore $\mathbf{A} \cong$

$\mathbf{B} \subseteq_{SP} \prod_{i \in I} \mathbf{B}_i$ where $\{\mathbf{B}_i : i \in I\} \subseteq Alg^*(\mathcal{S})$. By item ii, for each $i \in I$, $\tau\mathbf{B}_i = \min Fi_{\mathcal{S}}\mathbf{B}_i$ and by Theorem 2.6.6, $\langle \mathbf{B}_i, \tau\mathbf{B}_i\rangle \in Matr^*(\mathcal{S})$. By Lemma 4.5.2, it follows that $\langle \mathbf{B}, \tau\mathbf{B}\rangle$ is a subdirect product of $\{\langle \mathbf{B}_i, \tau\mathbf{B}_i\rangle : i \in I\}$. Consequently, $\tau\mathbf{B} \in Fi_{\mathcal{S}}\mathbf{B}$ and in view of item i, $\tau\mathbf{B} = \min Fi_{\mathcal{S}}\mathbf{B}$. □

Notation: Given a rule $\langle \Gamma, \varphi\rangle$ and a set of equations in one variable $\tau(x)$, we shall denote by $\Diamond^{\tau}_{\Gamma,\varphi}$ the following generalized quasiequation:

$$\Big(\bigwedge_{\langle \delta_l, \delta_r\rangle \in \tau,\ \gamma \in \Gamma} \delta_l(\gamma) \approx \delta_r(\gamma)\Big) \to \Big(\bigwedge_{\langle \delta_l, \delta_r\rangle \in \tau} \delta_l(\varphi) \approx \delta_r(\varphi)\Big).$$

Observe that if $\mathcal{S}$ is truth-equational and $\tau(x), \tau'(x)$ are two sets of defining equations for $\mathcal{S}$, then $\Diamond^{\tau}_{\Gamma,\varphi}$ is valid in $Alg^*(\mathcal{S})$ iff $\Diamond^{\tau'}_{\Gamma,\varphi}$ is valid in $Alg^*(\mathcal{S})$ and therefore the same happens in $Alg(\mathcal{S})$. The reason for this is essentially due to Proposition 4.5.1, since if τ and τ' define the filters in $Matr^*(\mathcal{S})$, then for any algebra in $\mathbf{A} \in Alg^*(\mathcal{S})$ (or in $Alg(\mathcal{S})$), it holds that $\tau\mathbf{A} = \min Fi_{\mathcal{S}}\mathbf{A} = \tau'\mathbf{A}$.

Lemma 4.5.3 ([32]). *Assume $\mathcal{S}$ is truth-equational with $\tau(x)$ a set of defining equations. Let $\langle \mathbf{A}, F\rangle$ and $\langle \mathbf{B}, G\rangle$ be two matrix models of $\mathcal{S}$, where $\langle \mathbf{A}, F\rangle$ is reduced. If $h : \mathbf{B} \to \mathbf{A}$ is an algebraic homomorphism, then:*

i If $\langle \mathbf{B}, G\rangle$ is a subdirect product of reduced matrix models of $\mathcal{S}$ (in particular if $\langle \mathbf{B}, G\rangle$ is reduced), then h is a matrix homomorphism from $\langle \mathbf{B}, G\rangle$ into $\langle \mathbf{A}, F\rangle$.

ii If h is a matrix homomorphism from $\langle \mathbf{B}, G\rangle$ into $\langle \mathbf{A}, F\rangle$ and $ker(h) \subseteq \Omega^{\mathbf{B}}(G)$, then h is strict.

iii Every injective matrix homomorphism from $\langle \mathbf{B}, G\rangle$ into $\langle \mathbf{A}, F\rangle$ is an embedding.

Proof. i: Assume $\langle \mathbf{B}, G\rangle \subseteq_{SP} \prod_{i \in I}\langle \mathbf{B}_i, G_i\rangle$, where $\{\langle \mathbf{B}_i, G_i\rangle : i \in I\} \subseteq Matr^*(\mathcal{S})$. By Lemma 4.5.2, $G = \tau\mathbf{B}$. Since $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})$, $F = \tau\mathbf{A}$. Consequently, if $b \in \tau\mathbf{B}$, then $\delta^{\mathbf{B}}_l(b) = \delta^{\mathbf{B}}_r(b)$, for all $\langle \delta_l, \delta_r\rangle \in \tau$, and since h is an algebraic homomorphism, then $\delta^{\mathbf{A}}_l(h(b)) = h(\delta^{\mathbf{B}}_l(b)) = h(\delta^{\mathbf{B}}_r(b)) = \delta^{\mathbf{A}}_r(h(b))$, for all $\langle \delta_l, \delta_r\rangle \in \tau$. Thus, $h(b) \in \tau\mathbf{A}$.
ii: Since h is a matrix homomorphism, $G \subseteq h^{-1}[F]$. Moreover, since $\langle \mathbf{A}, F\rangle$ is a reduced matrix model of $\mathcal{S}$, $F = \tau\mathbf{A}$. Pick $b \in h^{-1}[\tau\mathbf{A}]$. Then $h(b) \in \tau\mathbf{A}$ and therefore $\delta^{\mathbf{A}}_l(h(b)) = \delta^{\mathbf{A}}_r(h(b))$, for all $\langle \delta_l, \delta_r\rangle \in \tau$. Thus, $h(\delta^{\mathbf{B}}_l(b)) = h(\delta^{\mathbf{B}}_r(b))$, for all $\langle \delta_l, \delta_r\rangle \in \tau$. Consequently, for all $\langle \delta_l, \delta_r\rangle \in \tau$, $\langle \delta^{\mathbf{B}}_l(b), \delta^{\mathbf{B}}_r(b)\rangle \in ker(h) \subseteq \Omega^{\mathbf{B}}(G)$, and therefore $\tau^{\mathbf{B}}(b) \subseteq \Omega^{\mathbf{B}}(G)$. By Corollary 4.5.1, it follows that $b \in G$.
iii: For any injective matrix homomorphism $g : \langle \mathbf{B}, G\rangle \to \langle \mathbf{A}, F\rangle$ it is obvious that $ker(g) = Id_B \subseteq \Omega^{\mathbf{B}}(G)$. Thus, by item ii, g is strict and therefore an embedding. □

Lemma 4.5.4. *Assume that $\mathbf{A}_i$ is a homomorphic image of $\mathbf{B}_i$, for each $i \in I$. Then, every subdirect product of $\{\mathbf{A}_i : i \in I\}$ is a homomorphic image of a subdirect product of $\{\mathbf{B}_i : i \in I\}$.*

Proof. Let $\mathbf{A} \subseteq_{SP} \prod_{i \in I}\mathbf{A}_i$. Since each $\mathbf{A}_i$ is a homomorphic image of $\mathbf{B}_i$, pick surjective homomorphisms $h_i : \mathbf{B}_i \to \mathbf{A}_i$. The map $h : \prod_{i \in I}\mathbf{B}_i \to \prod_{i \in I}\mathbf{A}_i$ defined by $h(b) := \langle h_i(b(i)) : i \in I\rangle$ is a well-defined surjective homomorphism. From Universal Algebra we know that $h^{-1}[A]$ is the universe of a subalgebra of $\prod_{i \in I}\mathbf{B}_i$. Call this subalgebra $\mathbf{B}$. Thus, $B = h^{-1}[A]$ and obviously, $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$.

Claim: $\mathbf{B}$ is a subdirect product of $\{\mathbf{B}_i : i \in I\}$.

Denote $\pi_j^{\mathbf{B}} : \prod_{i\in I} \mathbf{B}_i \to \mathbf{B}_j$ and $\pi_j^{\mathbf{A}} : \prod_{i\in I} \mathbf{A}_i \to \mathbf{A}_j$ each projection map. We show that for each $j \in I$, $\pi_j^{\mathbf{B}} \upharpoonright B$ is onto B_j. Fix some $j \in I$ and pick any $c \in B_j$. Then, $h_j(c) \in A_j$ and since $\mathbf{A}$ is a subdirect product of $\{\mathbf{A}_i : i \in I\}$ it follows that there is some $a \in A$ such that $\pi_j^{\mathbf{A}}(a) = h_j(c)$, i.e. $a(j) = h_j(c)$. However, $B = h^{-1}[A]$ and therefore there is some $b \in B$ with $h(b) = a$. Thus, $h(b)(i) = a(i)$ for all $i \in I$ and by the definition of h, $h(b)(i) = h_i(b(i))$, for all $i \in I$. In particular, $h_j(b(j)) = a(j) = h_j(c)$. Clearly, it is not necessary that $b(j) = c$ and therefore we shall define a function $b' \in \prod_{i\in I} B_i$ by:

$$b'(i) = \begin{cases} b(i) & \text{if } i \neq j \\ c & \text{if } i = j \end{cases}$$

It is obvious that $\langle b, b'\rangle \in ker(h)$ and since $b \in B = h^{-1}[A]$, then $h(b) = h(b') \in A$. Hence, $b' \in B$ and $\pi_j^{\mathbf{B}}(b') = c$. Consequently, $\pi_j^{\mathbf{B}} \upharpoonright B$ is indeed onto B_j. □

Observe that given a rule $\langle \Gamma, \varphi\rangle$, a set of equations $\tau(x)$ and an algebra $\mathbf{A}$, the following equivalence is an immediate consequence of the definitions:

$$\Diamond^{\tau}_{\Gamma,\varphi} \text{ is valid in } \mathbf{A} \ \text{ iff } \ \langle \mathbf{A}, \tau\mathbf{A}\rangle \vDash \langle \Gamma, \varphi\rangle.$$

Theorem 4.5.1. *Let $\mathcal{S}$ be a truth-equational logic with $\tau(x)$ a set of defining equations. The following conditions are equivalent:*

- *i* *$\langle \Gamma, \varphi\rangle$ is admissible for $\mathcal{S}$.*
- *ii* *Every algebra in $Alg^*(\mathcal{S})$ is a homomorphic image of one in $Alg(\mathcal{S})$ in which $\Diamond^{\tau}_{\Gamma,\varphi}$ is valid.*
- *iii* *Every algebra in $Alg(\mathcal{S})$ is a homomorphic image of one in $Alg(\mathcal{S})$ in which $\Diamond^{\tau}_{\Gamma,\varphi}$ is valid.*

As in Theorem 4.1.2, it would be enough to check only the finitely generated algebras in items ii and iii, or the algebras which are (finitely generated) RSI in $Alg^*(\mathcal{S})$, in the case $\mathcal{S}$ is finitary.

Proof. $i \Rightarrow ii$: Pick any $\mathbf{A} \in Alg^*(\mathcal{S})$. So there is $F \in Fi_{\mathcal{S}}\mathbf{A}$ such that $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})$. By Theorem 4.1.2, $\langle \mathbf{A}, F\rangle$ is a homomorphic image of a matrix model $\langle \mathbf{B}, G\rangle$ of $\mathcal{S}+\langle \Gamma, \varphi\rangle$ which is itself a subdirect product of reduced matrix models $\{\langle \mathbf{B}_i, G_i\rangle : i \in I\}$ of $\mathcal{S}$. By Lemma 4.5.2, it follows that $G = \tau\mathbf{B}$. Thus, $\tau\mathbf{B} \in Fi_{\mathcal{S}+\langle\Gamma,\varphi\rangle}\mathbf{B}$, and therefore $\langle \mathbf{B}, \tau\mathbf{B}\rangle \vDash \langle \Gamma, \varphi\rangle$. Consequently, $\Diamond^{\tau}_{\Gamma,\varphi}$ is valid in $\mathbf{B}$ and clearly $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$. Finally, $\mathbf{B} \subseteq_{SP} \prod_{i\in I} \mathbf{B}_i$ and hence $\mathbf{B} \in Alg(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}(Alg^*(\mathcal{S}))$.

$ii \Rightarrow i$: We show that Theorem 4.1.2 iii holds. Let $\langle \mathbf{A}, F\rangle$ be a reduced matrix model of $\mathcal{S}$. Hence, $\mathbf{A} \in Alg^*(\mathcal{S})$. By ii, there is some $\mathbf{B} \in Alg(\mathcal{S})$ in which $\Diamond^{\tau}_{\Gamma,\varphi}$ is valid, i.e. $\langle \mathbf{B}, \tau\mathbf{B}\rangle \vDash \langle \Gamma, \varphi\rangle$, and for some $h \in Hom(\mathbf{B}, \mathbf{A})$, $h[B] = A$. Since $Alg(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}(Alg^*(\mathcal{S}))$, there are $\{\mathbf{B}_i : i \in I\} \subseteq Alg^*(\mathcal{S})$ such that $\mathbf{B}$ is, up to isomorphism, a subdirect product of $\{\mathbf{B}_i : i \in I\}$. By Theorem 2.6.6, $\{\langle \mathbf{B}_i, \tau\mathbf{B}_i\rangle : i \in I\} \subseteq Matr^*(\mathcal{S})$. By Lemma 4.5.2, $\langle \mathbf{B}, \tau\mathbf{B}\rangle$ is, up to isomorphism, a subdirect product of $\{\langle \mathbf{B}_i, \tau\mathbf{B}_i\rangle : i \in I\}$. Finally, since $\langle \mathbf{A}, F\rangle$ is reduced and $h : \mathbf{B} \to \mathbf{A}$ is an algebraic homomorphism which is surjective, then by Lemma 4.5.3 i, it follows that h is in fact a matrix homomorphism from $\langle \mathbf{B}, \tau\mathbf{B}\rangle$ onto $\langle \mathbf{A}, F\rangle$. Obviously, $\langle \mathbf{B}, \tau\mathbf{B}\rangle$ is matrix model of $\mathcal{S} + \langle \Gamma, \varphi\rangle$.

$iii \Rightarrow ii$: Clear since $Alg^*(\mathcal{S}) \subseteq Alg(\mathcal{S})$.
$ii \Rightarrow iii$: Pick any $\mathbf{B} \in Alg(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}(Alg^*(\mathcal{S}))$. Then $\mathbf{B} \cong \mathbf{A} \subseteq_{SP} \prod_{i \in I} \mathbf{A}_i$, where $\{\mathbf{A}_i : i \in I\} \subseteq Alg^*(\mathcal{S})$. By ii, each $\mathbf{A}_i$ is a homomorphic image of some $\mathbf{B}_i \in Alg(\mathcal{S})$ in which $\diamond^{\tau}_{\Gamma,\varphi}$ is valid, i.e. $\langle \mathbf{B}_i, \tau\mathbf{B}_i \rangle \vDash \langle \Gamma, \varphi \rangle$, for each $i \in I$. By Lemma 4.5.4, $\mathbf{A}$ is a homomorphic image of a subdirect product $\mathbf{C}$ of $\{\mathbf{B}_i : i \in I\}$. Clearly, since each $\langle \mathbf{B}_i, \tau\mathbf{B}_i \rangle$ validates $\langle \Gamma, \varphi \rangle$, then so does their direct product and hence so does $\langle \mathbf{C}, \tau\mathbf{C} \rangle$. Thus, $\diamond^{\tau}_{\Gamma,\varphi}$ is valid in $\mathbf{C}$. Finally, we know that $Alg(\mathcal{S})$ is closed under subdirect products and therefore $\mathbf{C} \in Alg(\mathcal{S})$. Obviously, $\mathbf{B}$ is a homomorphic image of $\mathbf{C}$. □

We shall close this section by providing the analogue of Theorem 4.2.3 for truth-equational logics. Observe that if $\mathcal{S}$ is truth-equational and $\langle \mathbf{A}, F \rangle \in Matr^*(\mathcal{S})$, then $\langle \mathbf{A}, F \rangle$ is RSI iff $\mathbf{A}$ is RSI in $Alg^*(\mathcal{S})$ (if necessary see definitions 4.1.1 and 4.1.2).

Theorem 4.5.2. *Assume $\mathcal{S}$ is finitary and truth-equational. If every finitely generated RSI algebra in $Alg^*(\mathcal{S})$ is weakly projective in $Alg(\mathcal{S})$, then $\mathcal{S}$ is hereditarily structurally complete.*

Proof. By Theorem 3.1.3, it is enough to prove that every axiomatic extension of $\mathcal{S}$ is structurally complete. But if $\mathcal{S}'$ is an axiomatic extension of $\mathcal{S}$, then by repeating the arguments of Theorem 4.2.3, we see that all the assumptions of the present Theorem persist in $\mathcal{S}'$ and therefore it is enough to show that $\mathcal{S}$ is structurally complete.
Let $\langle \Gamma, \varphi \rangle$ be a finite rule underivable for $\mathcal{S}$. We show that the rule is inadmissible for $\mathcal{S}$. By Lemma 4.1.2, there is a finitely generated RSI $\langle \mathbf{A}, F \rangle \in Matr^*(\mathcal{S})$ that invalidates $\langle \Gamma, \varphi \rangle$. Clearly, the algebra $\mathbf{A}$ is finitely generated and RSI in $Alg^*(\mathcal{S})$. Since $\langle \mathbf{A}, F \rangle$ is a reduced matrix model of $\mathcal{S}$ and $\mathcal{S}$ is truth-equational, then $F = \tau\mathbf{A}$, where $\tau(x)$ is a set of defining equations for $\mathcal{S}$. Thus, $\langle \mathbf{A}, \tau\mathbf{A} \rangle \nvDash \langle \Gamma, \varphi \rangle$ and therefore $\diamond^{\tau}_{\Gamma,\varphi}$ is invalid in $\mathbf{A}$. In view of Theorem 4.5.1, the following claim will verify that $\langle \Gamma, \varphi \rangle$ is inadmissible for $\mathcal{S}$:

Claim: $\mathbf{A}$ is not a homomorphic image of any algebra in $Alg(\mathcal{S})$, in which $\diamond^{\tau}_{\Gamma,\varphi}$ is valid.

Suppose not and let $\mathbf{B} \in Alg(\mathcal{S})$ be an algebra in which $\diamond^{\tau}_{\Gamma,\varphi}$ is valid and for some $h \in Hom(\mathbf{B}, \mathbf{A})$, $h[B] = A$. By weak projectivity assumption, there is an embedding from $\mathbf{A}$ into $\mathbf{B}$. By Lemma 2.3.1, we know that embeddings preserve invalidity and reflect validity of generalized quasiequations and therefore $\diamond^{\tau}_{\Gamma,\varphi}$ should be invalid in $\mathbf{B}$, a contradiction. □

4.6 Admissibility in weakly algebraizable logics

Recall that a logic is weakly algebraizable when it is both protoalgebraic and truth-equational. Consequently, the results in this case will be obtained by combining theorems from the corresponding previous sections.
The following can be found in the sections 5.3 and 5.4 of [15].

Definition 4.6.1. The **Tarski congruence** of a logic $\mathcal{S}$ is the largest congruence on $\mathbf{Fm}$ compatible with all $T \in Th(\mathcal{S})$ and is denoted by $\widetilde{\Omega}(\mathcal{S})$. In other words,

$$\widetilde{\Omega}(\mathcal{S}) = \bigcap\{\Omega(T) : T \in Th(\mathcal{S})\}.$$

Definition 4.6.2. The **intrinsic variety** of a logic $\mathcal{S}$ is the variety generated by $\mathbf{Fm}/\widetilde{\Omega}(\mathcal{S})$ and is denoted by $\mathbb{V}(\mathcal{S})$, i.e.

$$\mathbb{V}(\mathcal{S}) := \mathbb{V}\{\mathbf{Fm}/\widetilde{\Omega}(\mathcal{S})\}.$$

Proposition 4.6.1. *Let $\mathcal{S}$ be a logic. The following conditions hold:*

i *$Alg(\mathcal{S})$ is closed under isomorphisms and subdirect products.*

ii *$Alg(\mathcal{S})$ contains all trivial algebras.*

iii $\mathbb{V}(Alg^*(\mathcal{S})) = \mathbb{V}(Alg(\mathcal{S})) = \mathbb{V}(\mathcal{S})$.

iv *$\mathbf{Fm}/\widetilde{\Omega}(\mathcal{S})$ is the free algebra in the classes $Alg^*(\mathcal{S})$ and $Alg(\mathcal{S})$ with $|Var|$-many generators.*

v $\mathbb{V}(\mathcal{S}) = \mathbb{H}(Alg(\mathcal{S}))$.

Proof. The first four items can be found in [15]. For item v, observe that from Tarski's HSP-Theorem we know that $\mathbb{V}(Alg(\mathcal{S})) = \mathbb{HSP}(Alg(\mathcal{S}))$ and from Theorem 2.4.2, $\mathbb{HSP}(Alg(\mathcal{S})) = \mathbb{HP}_{\mathbb{SD}}(Alg(\mathcal{S}))$. The latter is essentially $\mathbb{H}(Alg(S))$ by item i and therefore by item iii, $\mathbb{V}(\mathcal{S}) = \mathbb{H}(Alg(\mathcal{S}))$. □

The following Theorem was essentially proved in [32] with the use of Theorem 4.2.2. However, it can be also obtained as a Corollary from Theorem 4.5.1, Proposition 4.6.1 iii and the fact that for a protoalgebraic logic $\mathcal{S}$, $Alg^*(\mathcal{S}) = Alg(\mathcal{S})$, since $Alg(\mathcal{S}) = \mathbb{IP}_{\mathbb{SD}}(Alg^*(\mathcal{S}))$ and $Matr^*(\mathcal{S})$ is closed under subdirect products, as witnessed by Theorem 2.6.1.

Theorem 4.6.1. *Let $\mathcal{S}$ be a weakly algebraizable logic. The following conditions are equivalent:*

i *$\langle\Gamma,\varphi\rangle$ is admissible for $\mathcal{S}$.*

ii $Alg(\mathcal{S}) \subseteq \mathbb{H}(Alg(\mathcal{S} + \langle\Gamma,\varphi\rangle))$.

iii $\mathbb{V}(\mathcal{S}) = \mathbb{V}(\mathcal{S} + \langle\Gamma,\varphi\rangle)$.

iv *Every algebra in $Alg(\mathcal{S})$ is a homomorphic image of one in $Alg(\mathcal{S})$ in which $\Diamond^{\tau}_{\Gamma,\varphi}$ is valid, where $\tau(x)$ is any set of defining equations for $\mathcal{S}$.*

A similar situation happens with the next result. It was proved in [32] but now it is an immediate Corollary of Theorem 4.5.2 since $Alg^*(\mathcal{S}) = Alg(\mathcal{S})$ for a protoalgebraic logic $\mathcal{S}$.

Corollary 4.6.1. *Assume $\mathcal{S}$ is finitary and weakly algebraizable. If every finitely generated RSI algebra in $Alg(\mathcal{S})$ is weakly projective in $Alg(\mathcal{S})$, then $\mathcal{S}$ is hereditarily structurally complete.*

4.7 Admissibility in algebraizable logics

Recall that the equivalent algebraic semantics of an algebraizable logic $\mathcal{S}$ is the largest class K such that $\mathcal{S}$ is algebraizable with respect to K. The following can be found in [15] (Prop. 4.57):

Proposition 4.7.1. *Assume $\mathcal{S}$ is an algebraizable logic with $\tau(x)$ a set of defining equations and with equivalent algebraic semantics the class K. Then:*

$$\langle\mathbf{A},F\rangle \in Matr^*(\mathcal{S}) \text{ iff } \mathbf{A} \in K \text{ and } F = \tau\mathbf{A}.$$

Thus, $K = Alg^*(\mathcal{S}) = Alg(S)$ (since $\mathcal{S}$ is also protoalgebraic).

Consequently, for any algebraizable logic $\mathcal{S}$, $Alg(\mathcal{S})$ is always its equivalent algebraic semantics. However, $Alg(\mathcal{S})$ is not necessarily closed under ultraproducts and therefore it is not necessarily a quasivariety. In any case, if $\mathcal{S}$ is algebraizable, then in view of Lemma 2.6.3 and Proposition 4.7.1, we know that $Alg(\mathcal{S})$ is a generalized quasivariety. Moreover, we know that any extension $\mathcal{S}'$ of an algebraizable logic $\mathcal{S}$, will also be algebraizable with the same sets of defining equations and equivalence formulas as $\mathcal{S}$ (see for example [6]). Recall that our fixed similarity type is L with a fixed and infinite set of variables Var. We shall apply the class operator $\mathbb{U}_{|Var|}$, defined in section 4.3, to classes of algebras. That is, given a class of algebras K we have:

$$\mathbb{U}(K) \coloneqq \{\mathbf{A} : \text{ every } |Var| - \text{generated subalgebra of} \mathbf{A} \text{ belongs to } K\}.$$

A class of algebras K is called a $\mathbb{UISP}$-class if it closed under the operators $\mathbb{U}, \mathbb{I}, \mathbb{S}$ and $\mathbb{P}$. The smallest such class containing K is $\mathbb{UISP}(K)$.
As an immediate Corollary of Theorem 4.3.1 and the fact that algebraizable logics are truth-equational, we obtain:

Theorem 4.7.1. *Assume that $\mathcal{S}$ is algebraizable with $\tau(x)$ a set of defining equations. The map $\mathcal{S}' \mapsto Alg(\mathcal{S}')$ is a bijection from the extensions of $\mathcal{S}$ to the $\mathbb{UISP}$-subclasses of $Alg(\mathcal{S})$. Its inverse sends a $\mathbb{UISP}$-subclass K of $Alg(\mathcal{S})$ to the logic induced by $\{\langle \mathbf{A}, \tau\mathbf{A}\rangle : \mathbf{A} \in K\}$.*

Moreover, combining Theorems 4.3.2 and 4.7.1 it follows that:

Theorem 4.7.2. *Let $\mathcal{S}$ be an algebraizable logic. The following conditions are equivalent:*

i Every rule (finite or infinite) admissible for $\mathcal{S}$ is derivable in $\mathcal{S}$.

ii For any proper $\mathbb{UISP}$-subclass K of $Alg(\mathcal{S})$, $\mathbb{H}(K) \subsetneq \mathbb{V}(\mathcal{S})$.

iii $Alg(\mathcal{S}) = \mathbb{UISP}\{\mathbf{Fm}/\Omega(Cn_{\mathcal{S}}(\varnothing))\}$.

Moreover, these conditions imply the next one:

iv Every $|Var|$-generated RSI algebra in $Alg(\mathcal{S})$ can be embedded into $\mathbf{Fm}/\Omega(Cn_{\mathcal{S}}(\varnothing))$.

If $\mathcal{S}$ is finitary, then all the above conditions are equivalent.

If $\mathcal{S}$ is BP-algebraizable, then it is finitely equivalential and therefore by Theorem 2.6.4, $Matr^*(\mathcal{S})$ is closed under ultraproducts. Consequently, $Alg^*(\mathcal{S})$ is closed under ultraproducts and $Alg^*(\mathcal{S}) = Alg(\mathcal{S})$ due to protoalgebraicity. Thus, if $\mathcal{S}$ is BP-algebraizable, then the equivalent algebraic semantics $Alg(\mathcal{S})$ of $\mathcal{S}$ is a quasivariety. In this case, we can speak of the equivalent quasivariety of $\mathcal{S}$.

Theorem 4.7.3. *Assume that $\mathcal{S}$ is BP-algebraizable with $\tau(x)$ a set of defining equations. The map $\mathcal{S}' \mapsto Alg(\mathcal{S}')$ is a bijection from the finitary extensions of $\mathcal{S}$ to the subquasivarieties of $Alg(\mathcal{S})$. Its inverse sends a quasivariety $Q \subseteq Alg(\mathcal{S})$ to the logic induced by $\{\langle \mathbf{A}, \tau\mathbf{A}\rangle : \mathbf{A} \in Q\}$.*

Proof. As it is well-known, any finitary extension $\mathcal{S}'$ of $\mathcal{S}$ is also finitely algebraizable (with the same sets of defining equations and equivalence formulas as $\mathcal{S}$) and its equivalent quasivariety is $Alg(\mathcal{S}')$. Clearly, the map $\mathcal{S}' \mapsto Alg(\mathcal{S}')$ is injective. In order to prove surjectivity, let Q be a subquasivariety of $Alg(\mathcal{S})$ and let $\mathcal{S}_K$ be the logic induced by $K = \{\langle \mathbf{A}, \tau\mathbf{A}\rangle : \mathbf{A} \in Q\}$. Since Q is a quasivariety, then it is almost immediate that K is a universal Horn class and therefore the logic $\mathcal{S}_K$ is indeed finitary (since it is induced by a class of matrices closed under ultraproducts). Moreover, $\mathcal{S}_K$ is BP-algebraizable since it is a finitary extension of $\mathcal{S}$. So it only remains to prove that Q is indeed the equivalent quasivariety of $\mathcal{S}_K$, i.e. that $Q = Alg(\mathcal{S}_K)$. Observe that for any $\alpha, \beta \in Fm$, $\alpha \approx \beta \dashv\vDash_Q \tau[\rho(\alpha, \beta)]$, since $Q \subseteq Alg(\mathcal{S})$ and $\alpha \approx \beta \dashv\vDash_{Alg(\mathcal{S})} \tau[\rho(\alpha, \beta)]$ (where $\rho(x, y)$ is any set of equivalence formulas for $\mathcal{S}$). Moreover,

$$\begin{aligned} \Gamma \vdash_{\mathcal{S}_K} \varphi &\text{ iff } \langle \mathbf{A}, \tau\mathbf{A}\rangle \vDash \langle \Gamma, \varphi\rangle, \text{ for all } \mathbf{A} \in Q \\ &\text{ iff } \Diamond^{\tau}_{\Gamma,\varphi} \text{ is valid in } Q \\ &\text{ iff } \tau[\Gamma] \vDash_Q \tau(\varphi) \end{aligned}$$

Thus, by Lemma 2.6.2 it follows that $\mathcal{S}_K$ is indeed algebraizable with respect to Q and finally, since Q is itself a quasivariety, then by Lemma 2.6.3, it is the equivalent quasivariety of $\mathcal{S}_K$, i.e. $Q = Alg(\mathcal{S}_K)$. □

Recall that if Q is a quasivariety, then $\mathbb{V}(Q) = \mathbb{H}(Q)$.

Definition 4.7.1 ([2]). A quasivariety Q is called **structurally complete** if each of its proper subquasivarieties generates a proper subvariety of $\mathbb{H}(Q)$. It is called **primitive** if each of its subquasivarieties is structurally complete.

Proposition 4.7.2 ([2]). *Let V be a variety.*

i V is structurally complete iff every proper subquasivariety of V generates a proper subvariety of V.

ii V is primitive iff every subquasivariety of V is a variety.

Proof. i: It is an immediate consequence of definition 4.7.1.
ii: ($\Rightarrow$) Assume V is primitive and let Q be a subquasivariety of V. Clearly $\mathbb{V}(Q)$ is a subquasivariety of V and it includes Q. By assumption, $\mathbb{V}(Q)$ is structurally complete and therefore $Q = \mathbb{V}(Q)$, i.e. Q is indeed a variety.
($\Leftarrow$) Assume that every subquasivariety of V is a variety and let Q be a subquasivariety of V. We show that Q is structurally complete. Let Q' be a subquasivariety of Q and suppose $\mathbb{V}(Q') = \mathbb{V}(Q)$. By assumption, both Q, Q' are varieties and therefore $Q' = \mathbb{V}(Q') = \mathbb{V}(Q) = Q$, i.e. Q is indeed structurally complete. □

The study of structurally complete varieties and quasivarieties (and their hereditary analogue) is by itself a field of research, in principle independent of (and not limited by) the studies on algebraizable logics. Although the study of such (quasi-)varieties is not our present purpose, we shall review some elementary (but of great importance) results on this topic.
Given a quasivariety Q and a set of variables X, we know that Q contains an algebra free in Q over X, which we denote by $\mathbf{F}_Q(X)$. If V is a variety, then it is well-known that $V = \mathbb{V}(\mathbf{F}_V(Var))$ (see for example [10]). For convenience, we denote by $\mathbf{F}_Q$ the free algebra of Q over Var and similarly for a variety V.

Lemma 4.7.1 ([2]). *Let V be a variety and Q a quasivariety. Then,*

$$\mathbb{V}(Q) = V \text{ iff } \mathbf{F}_Q \cong \mathbf{F}_V.$$

Proof. ($\Rightarrow$) Assume that $V = \mathbb{V}(Q) = \mathbb{H}(Q)$. Then $\mathbf{F}_V$ is a homomorphic image of an algebra $\mathbf{A} \in Q$. Let $h : \mathbf{A} \to \mathbf{F}_V$ be a surjective homomorphism. Thus, for every $x \in Var$, there is some $a_x \in A$ such that $h(a_x) = x$. Define $g : Var \to A$ by $g(x) = a_x$. It is clear that

the map $h \circ g : Var \to \mathbf{F}_V$ sends each variable to itself. On the other hand, since $\mathbf{F}_V \in V$, there is a mapping from $\mathbf{F}_V$ to $\mathbf{F}_Q$ that maps each $x \in Var$ to itself. Consequently, by freeness, $\mathbf{F}_V \cong \mathbf{F}_Q$.
($\Leftarrow$) If $\mathbf{F}_V \cong \mathbf{F}_Q$, then $\mathbb{V}(Q) \supseteq \mathbb{V}(\mathbf{F}_V) = V$, while the other inclusion is obvious. Thus, $\mathbb{V}(Q) = V$. $\square$

Theorem 4.7.4 ([2]). *Let Q be a quasivariety. There is a unique subquasivariety Q' of Q which is structurally complete and $\mathbb{V}(Q') = \mathbb{V}(Q)$. In fact, $Q' = \mathbb{Q}(\mathbf{F}_Q)$. Consequently, Q is structurally complete iff $Q = \mathbb{Q}(\mathbf{F}_Q)$.*

Proof. Let $V = \mathbb{V}(Q)$. By Lemma 4.7.1, $\mathbf{F}_V \cong \mathbf{F}_Q$. Let $Q' = \mathbb{Q}(\mathbf{F}_Q)$. Using the fact that any variety is generated by its free algebra with $|Var|$-many generators, we obtain that $\mathbb{V}(Q') = \mathbb{V}(\mathbf{F}_Q) = V = \mathbb{V}(Q)$.
Now let K be any subquasivariety of Q such that $\mathbb{V}(K) = V$. Again by Lemma 4.7.1, $\mathbf{F}_Q \cong \mathbf{F}_K \in K$. Since any proper subquasivariety of $\mathbb{Q}(\mathbf{F}_Q)$ cannot contain $\mathbf{F}_Q$, it follows that $\mathbb{Q}(\mathbf{F}_Q)$ is structurally complete.
Finally, any structurally complete subquasivariety K of Q that generates V, must contain $\mathbf{F}_Q$ and therefore $\mathbb{Q}(\mathbf{F}_Q) \subseteq K$. Thus, $\mathbb{Q}(\mathbf{F}_Q) = K$ since both quasivarieties are structurally complete and therefore $\mathbf{Q}(\mathbf{F}_Q)$ is indeed unique. $\square$

Let $\mathfrak{L}$ be a first-order language with equality. The *existential positive $\mathfrak{L}$-sentences* are, up to logical equivalence, sentences of the form $\exists \bar{x}\Psi$, where Ψ is a disjunction of one or more $\mathfrak{L}$-formulas each of which is a conjunction of one or more *atomic* $\mathfrak{L}$-formulas. (If no variable occurs in Ψ, then no quantifiers are required.)

Theorem 4.7.5 ([2]). *If V is a structurally complete variety, then every positive existential first-order sentence is either true throughout V or false in all non-trivial members of V.*

Theorem 4.7.6 ([34]). *Let $\mathcal{S}$ be a BP-algebraizable logic with $\tau(x)$ a set of defining equations and $\rho(x, y)$ a set of equivalence formulas. The following conditions hold:*

i $\mathcal{S}$ is structurally complete iff $Alg(\mathcal{S})$ is a structurally complete quasivariety.

ii $\mathcal{S}$ is hereditarily structurally complete iff $Alg(\mathcal{S})$ is a primitive quasivariety.

Proof. i: ($\Rightarrow$) By contraposition. Assume $Alg(\mathcal{S})$ is not a structurally complete quasivariety and let Q be a proper subquasivariety of $Alg(\mathcal{S})$ such that $\mathbb{H}(Q) = \mathbb{H}(Alg(\mathcal{S})) = \mathbb{V}(\mathcal{S})$. Let $\mathcal{S}_Q$ be the logic induced by $\{\langle \mathbf{A}, \tau\mathbf{A}\rangle : \mathbf{A} \in Q\}$. Clearly $\mathcal{S}_Q$ is a finitary (since it is induced by a class of matrices closed under ultraproducts) extension of $\mathcal{S}$ and hence it is BP-algebraizable. By Theorem 4.7.3, $Q = Alg(\mathcal{S}_Q)$ and therefore $\mathcal{S}_Q$ is a proper finitary extension of $\mathcal{S}$. So let $\langle \Gamma, \varphi\rangle$ be a finite rule such that $\Gamma \vdash_{\mathcal{S}_Q} \varphi$ but $\Gamma \nvdash \varphi$. Then, $\mathcal{S} \leq \mathcal{S} + \langle \Gamma, \varphi\rangle \leq \mathcal{S}_Q$ and therefore $\mathbb{V}(\mathcal{S}_Q) \subseteq \mathbb{V}(\mathcal{S} + \langle \Gamma, \varphi\rangle) \subseteq \mathbb{V}(\mathcal{S})$. Thus, $\mathbb{V}(\mathcal{S}) = \mathbb{V}(\mathcal{S} + \langle \Gamma, \varphi\rangle)$ and therefore by Theorem 4.6.1 iii, it follows that $\langle \Gamma, \varphi\rangle$ is an admissible rule for $\mathcal{S}$. However, it is clearly underivable in $\mathcal{S}$, and therefore $\mathcal{S}$ is not structurally complete.
($\Leftarrow$) By contraposition. Assume $\mathcal{S}$ is not structurally complete and let $\langle \Gamma, \varphi\rangle$ be a finite rule which is admissible but underivable in $\mathcal{S}$. Thus, $\mathcal{S} + \langle \Gamma, \varphi\rangle$ is a proper finitary extension of $\mathcal{S}$, i.e. $Alg(\mathcal{S} + \langle \Gamma, \varphi\rangle) \subsetneq Alg(\mathcal{S})$ but $\mathbb{V}(\mathcal{S} + \langle \Gamma, \varphi\rangle) = \mathbb{V}(\mathcal{S})$ in view of Theorem 4.6.1 iii and the admissibility of the rule for $\mathcal{S}$. Clearly, since $\mathcal{S} + \langle \Gamma, \varphi\rangle$ is also BP-algebraizable, then $Alg(\mathcal{S} + \langle \Gamma, \varphi\rangle)$ is a quasivariety and consequently $Alg(\mathcal{S})$ is not structurally complete.
ii: This is an immediate consequence of i and the definitions. $\square$

Theorem 4.7.7. *Assume $\mathcal{S}$ is BP-algebraizable. The following conditions are equivalent:*

i *$\mathcal{S}$ is structurally complete.*

ii *For any subquasivariety $Q \subsetneq Alg(\mathcal{S})$, $\mathbb{V}(Q) \subsetneq \mathbb{V}(\mathcal{S})$.*

iii $Alg(\mathcal{S}) = \mathbb{Q}(\mathbf{Fm}/\Omega(Cn_{\mathcal{S}}(\varnothing)))$.

iv *Every RSI algebra in $Alg(\mathcal{S})$ can be embedded into an ultrapower of $\mathbf{Fm}/\Omega(Cn_{\mathcal{S}}(\varnothing))$.*

v *Every finitely generated RSI algebra in $Alg(\mathcal{S})$ can be embedded into an ultrapower of $\mathbf{Fm}/\Omega(Cn_{\mathcal{S}}(\varnothing))$.*

Proof. $i \Rightarrow ii$: This direction instantiates Theorem 4.7.6 i.
$ii \Rightarrow iii$: Assume ii. Then, $Alg(\mathcal{S})$ is a structurally complete quasivariety. By Theorem 4.7.4 and Proposition 4.6.1 iv, $Alg(\mathcal{S}) = \mathbb{Q}(\mathbf{Fm}/\widetilde{\Omega}(\mathcal{S}))$. Due to protoalgebraicity of $\mathcal{S}$, $\widetilde{\Omega}(\mathcal{S}) = \Omega(Cn_{\mathcal{S}}(\varnothing))$. Thus, $Alg(\mathcal{S}) = \mathbb{Q}(\mathbf{Fm}/\Omega(Cn_{\mathcal{S}}(\varnothing)))$.
$iii \Rightarrow iv$: By the definition of RSI algebras and the fact that $\mathbb{SP}(K) \subseteq \mathbb{P}_{\mathbb{SD}}\mathbb{S}(K)$, for any class of algebras K, the required follows immediately.
$iv \Rightarrow v$: Trivial.
$v \Rightarrow i$: Since $\mathcal{S}$ is truth-equational, let $\tau(x)$ be a set of defining equations for $\mathcal{S}$. Then, $\langle \mathbf{A}, F\rangle \in Matr^*(\mathcal{S})$ iff $\mathbf{A} \in Alg^*(\mathcal{S})$ and $F = \tau\mathbf{A}$. Moreover, due to protoalgebraicity of $\mathcal{S}$, $Alg^*(\mathcal{S}) = Alg(\mathcal{S})$ and therefore the algebra $\mathbf{A}$ is RSI in $Alg(\mathcal{S})$ iff $\langle \mathbf{A}, \tau\mathbf{A}\rangle$ is RSI. Thus, item iv implies that every finitely generated RSI and reduced matrix model of $\mathcal{S}$ can be embedded into an ultrapower of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and therefore by Theorem 4.1.3 i, it follows that $\mathcal{S}$ is structurally complete. $\square$

4.8 Overflow rules

For the rest of this section, fix a logic $\mathcal{S}$ with set of variables Var and a first-order language with equality $\mathfrak{L}$ of the type of $Matr^*(\mathcal{S})$. Observe that Var serves also as the set of variables for $\mathfrak{L}$ and the unary designation predicate P belongs to $\mathfrak{L}$. An *existential positive $\mathfrak{L}$-condition* is a (possibly infinitary) formal expression of the form:

$$\exists\bar{x} \bigvee_{i\in I} \bigwedge_{j\in J_i} \Psi_{ij} \qquad (*)$$

where I and all of the J_i are non-empty possibly infinite sets, every Ψ_{ij} is an atomic $\mathfrak{L}$-formula, and $\bar{x}$ is a possibly infinite (and possibly empty) sequence of variables, including all that occur in $(*)$.

Definition 4.8.1 ([32]). An **overflow rule** of a logic $\mathcal{S}$ is a rule of the form $\langle \Gamma, y\rangle$, where $y \in Var$ and the formulas in Γ are of the type of $\mathcal{S}$, none of which contains an occurrence of the variable y.

Lemma 4.8.1 ([32]). *Assume that $\langle \mathbf{A}, F\rangle$ is a non-trivial reduced matrix model of $\mathcal{S}$ and let $\langle \Gamma, y\rangle$ be an overflow rule of $\mathcal{S}$ with $\Gamma \neq \varnothing$. Let $\bar{x}$ be the sequence of variables occuring in Γ (taken in any order). Then,*

$$\exists\bar{x} \bigwedge_{\gamma\in\Gamma} P\gamma \text{ is true in } \langle \mathbf{A}, F\rangle \text{ iff } \langle \mathbf{A}, F\rangle \text{ does not validate } \langle \Gamma, y\rangle.$$

Proof. ($\Rightarrow$) If $\exists\bar{x}\bigwedge_{\gamma\in\Gamma} P\gamma$ is true in $\langle\mathbf{A}, F\rangle$ (where $\langle\mathbf{A}, F\rangle$ is considered as a first-order $\mathfrak{L}$-structure), then there is $\bar{a} \in A$ such that $\gamma^{\mathbf{A}}(\bar{a}) \in F$, for all $\gamma \in \Gamma$. Moreover, since $\langle\mathbf{A}, F\rangle$ is non-trivial, there is some $b \in A \setminus F$. By the definition of an overflow rule, y does not occur in $\bar{x}$ and therefore for any homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ with $h(\bar{x}) = \bar{a}$ (coordinatewise) and $h(y) = b$, we have that $h(\gamma) = \gamma^{\mathbf{A}}(\bar{a}) \in F$, for all $\gamma \in \Gamma$ but $h(y) = b \notin F$, i.e. $\langle\mathbf{A}, F\rangle$ does not validate $\langle\Gamma, y\rangle$.
($\Leftarrow$) If $\langle\mathbf{A}, F\rangle$ does not validate $\langle\Gamma, y\rangle$, then there is a homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h[\Gamma] \subseteq F$ but $h(y) \notin F$. Thus, there is some sequence $\bar{a} \in A$ such that $h(\bar{x}) = \bar{a}$ (coordinatewise) and $\gamma^{\mathbf{A}}(\bar{a}) \in F$, for all $\gamma \in \Gamma$. Consequently, $\exists\bar{x}\bigwedge_{\gamma\in\Gamma} P\gamma$ is true in $\langle\mathbf{A}, F\rangle$. $\square$

Theorem 4.8.1 ([32]). *If every equality-free existential positive $\mathfrak{L}$-condition is true either in every member of $Matr^*(\mathcal{S})$ or in no non-trivial member of $Matr^*(\mathcal{S})$, then every admissible overflow rule of $\mathcal{S}$ is derivable in $\mathcal{S}$.*
The converse holds if $\mathcal{S}$ is equivalential, in which case it applies to all existential positive $\mathfrak{L}$-conditions, not only the equality-free ones.

Proof. We may assume without loss of generality that $\mathcal{S}$ is strongly consistent, so that the matrix $\langle\mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ is non-trivial.
($\Rightarrow$) Let $\langle\Gamma, y\rangle$ be an overflow rule underivable for $\mathcal{S}$. We show that the rule is inadmissible for $\mathcal{S}$. Since if $\Gamma = \varnothing$, the rule is obviously inadmissible for $\mathcal{S}$ (since it adds theorems), we may assume that $\Gamma \neq \varnothing$. By Theorem 2.2.2, there is a reduced matrix model $\langle\mathbf{A}, F\rangle$ of $\mathcal{S}$ that invalidates $\langle\Gamma, y\rangle$, which is of course non-trivial since a trivial matrix validates every rule. By Lemma 4.8.1, it follows that the equality-free existential positive $\mathfrak{L}$-condition $\exists\bar{x}\bigwedge_{\gamma\in\Gamma} P\gamma$ is true in $\langle\mathbf{A}, F\rangle$. By hypothesis, it must be true in every member of $Matr^*(\mathcal{S})$. In particular, it is true in $\langle\mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ and therefore again by Lemma 4.8.1, $\langle\mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$ invalidates $\langle\Gamma, y\rangle$. Consequently, the rule $\langle\Gamma, y\rangle$ is inadmissible for $\mathcal{S}$.
($\Leftarrow$) Assume that $\mathcal{S}$ is equivalential with $\Delta(x, y)$ a set of equivalence formulas. Since $\mathcal{S}$ is strongly consistent, $\Delta \neq \varnothing$. Suppose $\exists\bar{x}\bigvee_{i\in I}\bigwedge_{j\in J_i}\Psi_{ij}$ is an existential positive $\mathfrak{L}$-condition which is true in some non-trivial reduced matrix model $\langle\mathbf{A}, F\rangle$ of $\mathcal{S}$. For convenience, denote for each $i \in I$, $\Phi_i := \bigwedge_{j\in J_i}\Psi_{ij}$. Since $\exists\bar{x}\bigvee_{i\in I}\Phi_i$ is true in $\langle\mathbf{A}, F\rangle$, then there is some $i \in I$ such that $\exists\bar{x}\Phi_i$ is true in $\langle\mathbf{A}, F\rangle$. Thus, it suffices to show that $\exists\bar{x}\Phi_i$ is true in every reduced matrix model of $\mathcal{S}$. Observe that any equational subformula $\alpha = \beta$ of Φ_i can be replaced in Φ_i by $\bigwedge_{\delta\in\Delta} P\delta(\alpha, \beta)$ without affecting the truth of $\exists\bar{x}\Phi_i$ in any reduced matrix model of $\mathcal{S}$. Thus, we may assume that Φ_i is of the form $\bigwedge_{\gamma\in\Gamma} P\gamma$, for some non-empty $\Gamma \subseteq Fm$. Moreover, we may also assume that some $y \in Var$ does not occur in any member of Γ. This can be assumed since Var is an infinite set and therefore by standard cardinality arguments, the set of variables appearing in Γ, can be replaced by a set of cardinality $|Var|$ which is a proper subset of Var, without affecting the truth of $\exists\bar{x}\Phi_i$ in any $\mathfrak{L}$-structure. Now, since $\exists\bar{x}\Phi_i$ is true in the non-trivial reduced matrix model $\langle\mathbf{A}, F\rangle$ and $\langle\Gamma, y\rangle$, $\Gamma \neq \varnothing$, is an overflow rule of $\mathcal{S}$, then by Lemma 4.8.1, $\langle\mathbf{A}, F\rangle$ invalidates $\langle\Gamma, y\rangle$. Thus, $\langle\Gamma, y\rangle$ is underivable in $\mathcal{S}$ and therefore by hypothesis, the rule is inadmissible for $\mathcal{S}$, i.e. it is invalidated by $\langle\mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. Hence, again by Lemma 4.8.1, $\exists\bar{x}\Phi_i$ is true in $\langle\mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$. By well-known model-theoretic results, the truth

of $\exists\bar{x}\Phi_i$ is preserved in homomorphic images and superstructures. Moreover, since $\mathcal{S}$ is also protoalgebraic, then by Proposition 4.2.2, every $|Var|$-generated and reduced matrix model of $\mathcal{S}$ is a homomorphic image of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle$. Finally, since $\mathcal{S}$ is equivalential, then $Matr^*(\mathcal{S})$ is closed under submatrices and $\mathbb{U}$ and therefore, for any matrix $\langle \mathbf{B}, G\rangle$, $\langle \mathbf{B}, G\rangle \in Matr^*(\mathcal{S})$ iff every $|Var|$-generated submatrix of $\langle \mathbf{B}, G\rangle$ belongs to $Matr^*(\mathcal{S})$. Consequently, given any $\langle \mathbf{B}, G\rangle \in Matr^*(\mathcal{S})$, $\exists\bar{x}\Phi_i$ is true in every $|Var|$-generated submatrix of $\langle \mathbf{B}, G\rangle$ (since any such submatrix will be a homomorphic image of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle^*$) and therefore in $\langle \mathbf{B}, G\rangle$ itself. □

Definition 4.8.2 ([32])**.** We say that a logic $\mathcal{S}$ is **overflow complete** if every *finite* overflow rule admissible for $\mathcal{S}$ is derivable in $\mathcal{S}$.

As an immediate consequence of Theorem 4.8.1 we obtain:

Theorem 4.8.2 ([32])**.** *Assume that $\mathcal{S}$ is finitely equivalential. The following conditions are equivalent:*

i $\mathcal{S}$ is overflow complete.

ii Every positive existential $\mathfrak{L}$-sentence is true either in every non-trivial member of $Matr^*(\mathcal{S})$, or in none of them.

The following Theorem is a useful criteria for disproving overflow completeness in equivalential logics. Observe that a matrix is 0-generated only if its similarity type includes a constant symbol (since we exclude empty structures from consideration).

Proposition 4.8.1 ([32])**.** *Assume that $\mathcal{S}$ is equivalential. If $\mathcal{S}$ is overflow complete, then any two non-trivial 0-generated reduced matrix models of $\mathcal{S}$ are isomorphic.*

Proof. Let $\langle \mathbf{A}, F\rangle$ be a 0-generated reduced matrix model of $\mathcal{S}$. So L has a constant symbol, say c. Consequently, the map $x \mapsto c^{\mathbf{A}}$, $x \in Var$ extends to a surjective homomorphism $h : \mathbf{Fm} \to \mathbf{A}$, since $\mathbf{A}$ is 0-generated. Clearly, $h^{-1}[F] \in Th(\mathcal{S})$ and $Cn_{\mathcal{S}}(\varnothing) \subseteq h^{-1}[F]$. Since $\mathcal{S}$ is also protoalgebraic, Ω is monotone on $Th(\mathcal{S})$ and therefore $\Omega(Cn_{\mathcal{S}}(\varnothing)) \subseteq \Omega(h^{-1}[F])$. Moreover, since L has a constant symbol, then the variable-free formulas of $\mathbf{Fm}$ constitute a subalgebra $\mathbf{B}$ of $\mathbf{Fm}$. Let $G = Cn_{\mathcal{S}}(\varnothing) \cap B$. Then, $\langle \mathbf{B}, G\rangle$ is a submatrix of $\langle \mathbf{Fm}, Cn_{\mathcal{S}}(\varnothing)\rangle$ and clearly $\langle \mathbf{B}, G\rangle^* \in Matr^*(\mathcal{S})$. By Theorem 2.6.2 *iii* and Proposition 2.2.3 *ii* it follows that:

$$\Omega^{\mathbf{B}}(G) = (B \times B) \cap \Omega(Cn_{\mathcal{S}}(\varnothing)) \subseteq \Omega(h^{-1}[F]) = h^{-1}[\Omega^{\mathbf{A}}(F)] = ker(h)$$

since $\langle \mathbf{A}, F\rangle$ is reduced. Consequently, the map $\widetilde{h} : \langle \mathbf{B}, G\rangle^* \to \langle \mathbf{A}, F\rangle$, given by $\varphi/\Omega^{\mathbf{B}}(G) \mapsto h(\varphi)$, is a well-defined matrix homomorphism, which is clearly surjective as $\langle \mathbf{A}, F\rangle$ is 0-generated. We show that $\widetilde{h}$ is strict.
For any $\varphi \in B$, the expression $P\varphi$ is an existential positive $\mathfrak{L}$-sentence since φ is variable-free. So let $\varphi \in B$, with $\varphi/\Omega^{\mathbf{B}}(G) \in \widetilde{h}^{-1}[F]$. Then, $\widetilde{h}(\varphi/\Omega^{\mathbf{B}}(G)) \in F$ and therefore $P\varphi$ is true in $\langle \mathbf{A}, F\rangle$. Now, observe that since $\langle \mathbf{A}, F\rangle$ is a non-trivial and reduced matrix model of $\mathcal{S}$, then by overflow completeness of $\mathcal{S}$, it follows that $P\varphi$ is true in all members of $Matr^*(\mathcal{S})$, i.e. φ takes only designated values in all reduced matrix models of $\mathcal{S}$. Thus, by Theorem 2.2.2, φ is an $\mathcal{S}$-theorem, i.e. $\varphi \in Cn_{\mathcal{S}}(\varnothing) \subseteq G$. Hence, $\varphi/\Omega^{\mathbf{B}}(G) \in G/\Omega^{\mathbf{B}}(G)$ and $\widetilde{h}$ is indeed strict. Finally, by Lemma 4.3.1, it follows that $\widetilde{h}$ is an embedding and therefore $\widetilde{h} : \langle \mathbf{B}, G\rangle^* \cong \langle \mathbf{A}, F\rangle$. However, since $\langle \mathbf{B}, G\rangle$ is fixed, we obtain the required. □

In the literature, a notion similar to overflow completeness has been studied; namely *passive structural completeness*. We will see that these two notions are equivalent.

Definition 4.8.3 ([11])**.** Let $\mathcal{S}$ be a logic. A rule $\langle\Gamma,\varphi\rangle$ is **passive** for $\mathcal{S}$ if there is no substitution σ such that $\vdash_{\mathcal{S}} \sigma[\Gamma]$.
The logic $\mathcal{S}$ is called **passively structurally complete** if every *finite* passive rule of $\mathcal{S}$ is derivable in $\mathcal{S}$.

It immediately follows from the definitions that if a logic $\mathcal{S}$ is structurally complete, then it is passively structurally complete. However, these two notions in principle are different. The most well-known example that witnesses this fact is intuitionistic propositional logic (IPL), which is passively structurally complete but not structurally complete (see for example [36] and [11]).
In what follows, we shall summarize some results proven in [11], concerning the inheritance of passive structural completeness in extensions and fragments of a logic. Recall that our fixed similarity type is L.

Definition 4.8.4 ([11])**.** Let $L' \subseteq L$ be a similarity type. The L'-fragment $\mathcal{S}_{L'}$ of $\mathcal{S}$ is a **passive fragment** of $\mathcal{S}$ if every rule passive in $\mathcal{S}_{L'}$ is also passive in $\mathcal{S}$.

Theorem 4.8.3 ([11])**.** *Let $\mathcal{S}$ be a passively structurally complete logic. The following conditions hold:*

i Any extension of $\mathcal{S}$ is passively structurally complete.

ii Any passive fragment of $\mathcal{S}$ is passively structurally complete.

Proof. i: Let $\mathcal{S}'$ be any extension of $\mathcal{S}$ and $\langle\Gamma,\varphi\rangle$ be a finite rule for $\mathcal{S}'$ which is underivable in $\mathcal{S}'$. Then, $\langle\Gamma,\varphi\rangle$ is also underivable in $\mathcal{S}$ and since $\mathcal{S}$ is passively structurally complete, the rule is not passive for $\mathcal{S}$. Then, there is a substitution σ such that $\vdash_{\mathcal{S}} \sigma[\Gamma]$. Since $\mathcal{S} \leq \mathcal{S}'$, then clearly $\vdash_{\mathcal{S}'} \sigma[\Gamma]$ and therefore the rule is not passive for $\mathcal{S}'$.
(It is obvious that the proof can be done for any rule $\langle\Gamma,\varphi\rangle$, not only the finite ones).
ii: This is an immediate consequence of the definitions. □

Proposition 4.8.2. *A logic $\mathcal{S}$ is overflow complete iff it is passively structurally complete.*

Proof. Without loss of generality, we may assume that $\mathcal{S}$ is consistent, i.e. that $Cn_{\mathcal{S}}(\varnothing) \neq Fm$, since otherwise the Proposition holds trivially.
($\Rightarrow$) By contraposition. Assume $\mathcal{S}$ is not passively structurally complete. So there is a finite passive rule $\langle\Gamma,\varphi\rangle$ for $\mathcal{S}$ which is underivable in $\mathcal{S}$. Let y be a variable not occuring in the formulas in Γ (in our case, where Γ is finite, it is obvious that such a variable exists; but even if Γ was infinite, we may reformulate the rule so that such a variable exists, as in the proof of Theorem 4.8.1). Since $\langle\Gamma,\varphi\rangle$ is passive, there is no substitution σ such that $\vdash_{\mathcal{S}} \sigma[\Gamma]$ and therefore the overflow rule $\langle\Gamma, y\rangle$ is trivially admissible for $\mathcal{S}$. Finally, since $\langle\Gamma,\varphi\rangle$ is underivable for $\mathcal{S}$, there is a non-trivial reduced matrix model $\langle\mathbf{A},F\rangle$ of $\mathcal{S}$ and a homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h[\Gamma] \subseteq F$ but $h(\varphi) \notin F$. Clearly, since $\langle\mathbf{A},F\rangle$ is non-trivial, then $F \subsetneq A$. So pick some $a \in A \smallsetminus F$ and define $g : Var \to A$ by:

$$g(x) := \begin{cases} h(x) & \text{if } x \neq y \\ a & \text{if } x = y \end{cases}$$

and extend it naturally to a homomorphism $\widetilde{g} : \mathbf{Fm} \to \mathbf{A}$. Since the variable y does not occur in Γ and $\widetilde{g}, h$ coincide in all variables except y, it follows that $\widetilde{g}(\gamma) = h(\gamma)$ for all

$\gamma \in \Gamma$ and therefore $\widetilde{g}[\Gamma] \subseteq F$. However, obviously $\widetilde{g}(y) = a \notin F$ and hence the rule $\langle \Gamma, y \rangle$ is underivable for $\mathcal{S}$, i.e. $\mathcal{S}$ is not overflow complete.
$(\Leftarrow)$ By contraposition. Assume $\mathcal{S}$ is not overflow complete. So there is a finite overflow rule $\langle \Gamma, y \rangle$ admissible but underivable for $\mathcal{S}$. It is enough to show that $\langle \Gamma, y \rangle$ is passive for $\mathcal{S}$. Recall that none of the formulas in Γ contains an occurrence of y. Thus, if there is a substitution σ such that $\vdash_{\mathcal{S}} \sigma[\Gamma]$, then we can define, as we did above, a substitution $\widetilde{\sigma}$ such that $\widetilde{\sigma}(x) = \sigma(x)$ for all $x \in Var$ with $x \neq y$ and $\widetilde{\sigma}(y) \in Fm \smallsetminus Cn_{\mathcal{S}}(\varnothing)$, since we assumed that $\mathcal{S}$ is consistent. But this contradicts the admissibility of $\langle \Gamma, y \rangle$ for $\mathcal{S}$. Consequently, there is no substitution that turns all formulas in Γ into $\mathcal{S}$-theorems and therefore the rule $\langle \Gamma, y \rangle$ is passive for $\mathcal{S}$. However, it is underivable for $\mathcal{S}$ and hence $\mathcal{S}$ is not passively structurally complete.
(Although the proof could be done even for infinite rules, we restricted ourselves to finite rules in order to emphasize the equivalence of the defined notions.) $\square$

Bibliography

[1] Avron, A. “Implicational F-structures and implicational relevance logics”. In: *Journal of Symbolic Logic* 65 (2000), pp. 788–802.

[2] Bergman, C. “Structural Completeness in Algebra and Logic”. In: *Algebraic Logic (Budapest, 1988), edited by H. Andréka, J. D. Monk and I. Nemeti* Vol. 54 of Colloquia Mathematica Societatis János Bolyai, North-Holland, Amsterdam (1991), pp. 59–73.

[3] Bergman, C. *Universal Algebra-Fundamentals and Selected Topics*. Boca Raton: CRC Press, 2012.

[4] Blok, W. J. and Pigozzi, D. *Abstract algebraic logic and the deduction theorem*. URL: http://orion.math.iastate.edu/dpigozzi/.

[5] Blok, W. J. and Pigozzi, D. “Algebraic semantics for universal Horn logic without equality”. In: *Universal Algebra and Quasigroup Theory (Jadwisin, 1989), edited by A. Romanowska and J. D. H. Smith, vol. 19 of Research and Exposition in Mathematics, Heldermann, Berlin* (1992), pp. 1–56.

[6] Blok, W. J. and Pigozzi, D. *Algebraizable Logics*. Memoirs of the American Mathematical Society, Number 396, Amer. Math. Soc., Providence, 1989.

[7] Blok, W. J. and Pigozzi, D. *Algebraizable Logics*. Vol. 396 of Memoirs of the American Mathematical Society. American Mathematical Society, Providence, 1989.

[8] Blok, W. J. and Pigozzi, D. “Protoalgebraic logics”. In: *Studia Logica* 45 (1986), pp. 337–69.

[9] Blok, W. J. and Raftery, J. G. “Fragments of R-mingle”. In: *Studia Logica* 78 (2004), pp. 59–106.

[10] Burris, S. and Sankappanavar, H. P. *A Course in Universal Algebra*. Vol. 68 of Graduate Texts in Mathematics. Springer, New York, 1981.

[11] Cintula, P. and Metcalfe, G. “Structural completeness in fuzzy logics”. In: *Notre Dame Journal of Formal Logic* 50 (2009), pp. 153–82.

[12] Czelakowski, J. “Equivalential Logics I,II”. In: *Studia Logica* 40 (1981), pp. 227–36, 355–72.

[13] Czelakowski, J. *Protoalgebraic Logics*. Vol. 10. Studia Logica, Kluwer, Dordrecht, 2001.

[14] Czelakowski, J. and Jansana, R. “Weakly algebraizable logics”. In: *Journal of Symbolic Logic* 65 (2000), pp. 641–68.

[15] Font, J. M. *Abstract Algebraic Logic - An Introductory Textbook*. College Publications, London, 2016.

[16] Font, J. M., Jansana, R., and Pigozzi, D. *A survey of abstract algebraic logic*. in Abstract Algebraic Logic, Part II (Barcelona, 1997). Vol. 74 of Studia Logica, Kluwer, Dordrecht, 2003, pp. 13–97.

[17] Gorbunov, V. A. *Algebraic Theory of Quasivarieties*. Consultants Bureau, New York, 1998.

[18] Gorbunov, V. A. and Tumanov, V. I. “Construction of lattices of quasivarieties”. In: *Mathematical Logic and the Theory of Algorithms, vol. 2 of Trudy Instituta Matematiki Sibirsk. Otdel. Akad. Nauk SSSR, Nauka, Novosibirsk* (1982), pp. 12–44.

[19] Grätzer, G. and Lakser, H. “A note on the implicational class generated by a class of structures”. In: *Canadian Mathematical Bulletin* 16 (1973), pp. 603–05.

[20] Herrman, B. “Characterizing equivalential and algebraizable logics by the Leibniz operator”. In: *Studia Logica* 58 (1997), pp. 305–23.

[21] Kogalovskiĭ, S. R. “On Birkhoff’s theorem”. In: *Uspekhi Matematicheskikh Nauk* 20, No. 5 (1965), pp. 206–207.

[22] Lorenzen, P. *Einführung in die operative Logik und Mathematik*. Vol. 78. Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, 1955.

[23] Makinson, D. “A characterization of structural completeness of a structural consequence operation”. In: *Reports on Mathematical Logic* 6 (1976), pp. 99–101.

[24] Maltsev, A. I. *Algebraic Systems*. Vol. 192. Grundlehren der Mathematischen Wissenschaften. Springer,Berlin, 1973.

[25] Maltsev, A. I. “Several remarks on quasivarieties of algebraic systems”. In: *Algebra i Logika* 5 (1966), pp. 3–9.

[26] Pogorzelski, W. A. *Completeness Theory for Propositional Logics, Studies in Universal Logic*. Birkhäuser, Basel, 2008.

[27] Pogorzelski, W. A. “Structural completeness of the propositional calculus”. In: *Bulletin de l’Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques* 19 (1971), pp. 349–51.

[28] Prucnal, T. “On the structural completeness of some pure implicational propositional calculi”. In: *Studia Logica* 30 (1972), pp. 45–52.

[29] Prucnal, T. “Structural completeness of Medvedev’s propositional calculus”. In: *Reports on Mathematical Logic* 6 (1976), pp. 103–05.

[30] Prucnal, T. “Structural completeness of purely implicational intermediate logics”. In: *Foundations of Logic and Linguistics (Salzburg, 1983), edited by G. Dorn and P. Weingartner, Plenum, New York* (1985), pp. 31–41.

[31] Prucnal, T. and Wroński, A. “An algebraic characterization of the notion of structural completeness”. In: *Bulletin of the Section of Logic* 3 (1974), pp. 30–3.

[32] Raftery, J. G. “Admissible Rules and the Leibniz Hierarchy”. In: *Notre Dame Journal of Formal Logic* 57.no. 4 (2016), pp. 569–606.

[33] Raftery, J. G. “The equational definability of truth predicates”. In: *Reports on Mathematical Logic* 41 (2006), pp. 95–149.

[34] Raftery, J. G., Olson, J. S., and Alten, J. G. van. “Structural completeness in substructural logics”. In: *Logic Journal of the IGPL* 16 (2008), pp. 455–95.

[35] Raftery, J. G. and Świrydowicz, K. “Structural Completeness in Relevance Logics”. In: *Studia Logica* 104 (3 2016), pp. 381–387.

[36] Rybakov, V. V. *Admissibility of Logical Inference Rules*. Vol. 136 of Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1997.

[37] Wójcicki, R. “Matrix approach in methodology of sentential calculi”. In: *Studia Logica* 32 (1973), pp. 7–39.

[38] Wójcicki, R. “Theory of Logical Calculi: Basic Theory of Consequence Operations”. In: *Synthese Library vol. 199, Kluwer, Dordrecht* (1988).

[39] Wroński, A. “Reflections and distensions of BCK-algebras”. In: *Mathematica Japonica* 28 (1983), pp. 215–25.