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AN INTRODUCTION TO RIEMANN SURFACES

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Introduction

In this document we give a first view to Riemann surface theory. Starting from definition and examples in chapter I, in the next chapter one sees the relation between the well known oriented smooth surfaces and this new object, with the result that any oriented smooth surfaces is equivalent to a Riemann surface. In order to prove this result, almost-complex structures and isothermal coordinates (between others) are explained, and the key point is the existence of these isothermal coordinates for a smooth surface as we will see.

Finally in Chapter III we stablish the relation between Riemann surfaces and algebraic curves. First we construct a Riemann surface from a polynomial, which is relatively easy, and then we give and prove the Main Theorem for Riemann surfaces using Hilbert space techniques and some tools like the Riesz Representation Theorem. This Main Theorem is the key to prove the existence of meromorphic functions on a Riemann surface and the fact that any compact Riemann surface arises from a polynomial.

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Contents

Chapter 1

Basic definitions and examples

1.1 Riemann surfaces

Definition 1.1. *A function $f : \Omega \to \mathbb{C}$ is holomorphic in an open set $\Omega \subset \mathbb{C}$ if for any point $z \in \Omega$ the following limit exists*

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}.$$

Definition 1.2. *A Riemann surface is:*

- *A Hausdorff topological space X.*
- *A collection of open sets $U_\alpha \subset X$ that covers X (α runs over some index set I).*
- *Maps $\phi_\alpha : U_\alpha \to \widetilde{U_\alpha} \subset \mathbb{C}$ which are homeomorphisms with the property that for all α, β the composite map $\phi_\alpha \circ \phi_\beta^{-1}$ is holomorphic on its domain of definition $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta)$.*
- *An atlas containing all possible charts consistent (i.e. the transition maps are holomorphic) with $\{U_\alpha, \phi_\alpha\}$, i.e a maximal atlas.*

Just recall that an atlas for a topological space M is a collection of charts $\{U_\alpha, \phi_\alpha\}$ on M such that $\bigcup U_\alpha = M$

This basic definition is inspired by the definition of differential manifolds, in fact if the word holomorphic is replaced by smooth what results is the definition of smooth surface that it is well known and familiar.

It is clear that any open set in $\mathbb{C}$ is a Riemann surface, like the unit disc or the upper half-plane that are equivalent via the map

$$w \to \frac{w - i}{w + i}.$$

These are very simple and we show some more.

1.2 The Riemann sphere

Consider the Riemann sphere S^2. As a topological space it is the one-point compactification (or Alexandroff compactification) of the complex plane $\mathbb{C} \cup \{\infty\}$, therefore open sets in S^2 are either open sets in $\mathbb{C}$ or $\{\infty\} \cup (\mathbb{C} - K)$ where $K \subset \mathbb{C}$ is compact.

Consider now

$$U_0 = \{z \in \mathbb{C} : |z| < 2\}, U_1 = \{z \in \mathbb{C} : |z| > 1/2\} \cup \{\infty\}.$$

Let the map ϕ_0 be the identity (so $\widetilde{U_0} = U_0$) and let ϕ_1 be $\phi_1(\infty) = 0$ and $\phi_1(z) = 1/z$. Hence the maps $\phi_0 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_0^{-1}$ are both

$$z \to 1/z$$

from $\{z \in \mathbb{C} : 1/2 < |z| < 2\}$ to itself. This map is holomorphic and satisfies the definition.

1.3 Quotients

Consider $2\pi\mathbb{Z} \subset \mathbb{C}$ under addition and form $\mathbb{C}/2\pi\mathbb{Z}$ with the standard quotient topology, this is clearly homeomorphic to the cylinder $S^1 \times \mathbb{R}$. Now we show how to make $\mathbb{C}/2\pi\mathbb{Z}$ into a Riemann surface.

For each $z \in \mathbb{C}$ consider the disc D_z centered in z of radius $1/2$, if $z_1, z_2 \in D_z$ and if $z_1 = z_2 + 2\pi n$ for $n \in \mathbb{Z}$ then $n = 0$ and $z_1 = z_2$. This actually means that the projection $\pi : \mathbb{C} \to \mathbb{C}/2\pi\mathbb{Z}$ maps D_z with its image bijectively, hence the local inverse of the projection (π^{-1}) can be used as charts $\phi_z : \pi(D_z) \to D_z$ for the surface, and the overlap maps between charts will be

$$z \to z + 2\pi n$$

for a suited $n \in \mathbb{Z}$, which is holomorphic.

The case of the torus is similar. Consider a lattice Λ in $\mathbb{C}$ of the form

$$\Lambda = \mathbb{Z} \oplus \mathbb{Z}\lambda,$$

where $\lambda \in \mathbb{C}$ has positive imaginary part. The same process can be repeated however with a new election of the radius of D_z

$$2r < \min_{m,n} |n + \lambda m|,$$

where (m, n) runs over $\mathbb{Z} \times \mathbb{Z} \backslash \{(0, 0)\}$. One can show then, that $\mathbb{C}/\Lambda$ is a Riemann surface, and homeomorphic to the well known torus $S^1 \times S^1$. This example, as well as the Riemann sphere are examples of compact Riemann surfaces.

Chapter 2

Almost-Complex structures and isothermal coordinates

One can construct a smooth surface from a Riemann surface easily, if the overlap maps between charts are holomophic, are therefore smooth (C^∞). However starting from a smooth surface and build a Riemann surface is more complicated and it is the aim of this chapter. We will prove that any smooth oriented surface supports some structure of Riemann surface.

2.1 Smooth surfaces and metric tensor

Definition 2.1. *Let M be a differential manifold and $p \in M$, a Riemann metric g is a differential field tensor*

$$p \to g_p : T_pM \times T_pM \to \mathbb{R}$$

which is symmetric and positive-definite. This means g_p symmetric for all $p \in M$ and $g_p(v_p, v_p) > 0$, for all $v_p \in T_pM(v_p \neq 0)$ for all $p \in M$.

Definition 2.2. *A partition of unity on a differential manifold M is a collection $\{f_i : i \in I\}$ of C^∞ functions on M such that*

- *The collection of supports $\{supp\ f_i : i \in I\}$ is locally finite.*
- $\sum_{i \in I} f_i(p) = 1$ *for all $p \in M$, and $f_i(p) \geq 0$ for all $p \in M$ and $i \in I$.*

A partition of unity $\{f_i : i \in I\}$ is *subordinate* to the cover $\{U_\alpha : \alpha \in A\}$ if for all $i \in I$, there exists $\alpha \in A$ such that supp $f_i \subset U_{\alpha(i)}$.

At the end we will apply these definitions to smooth surfaces, however these will be general as far as possible.

We present now the theorem of existence of partitions of unity (see [1] page 10).

Theorem 2.1. *Let M be a differential manifold and $\{U_\alpha : \alpha \in A\}$ an open cover of M. Then there exists a countable partition of unity $\{f_i : i = 1, 2, ...\}$ subordinate to the cover $\{U_\alpha\}$ with supp f_i compact for each i.*

Definition 2.3. *Let $f : M \to N$ be a differentiable map between manifolds and $p \in M$, if K is a metric field tensor in N, the pull-back of K is defined by*

$$f^*K|_p(v_1, ..., v_k) = K|_{f(p)}(df(p)(v_1), ..., df(p)(v_k)).$$

Proposition 2.1. *All differentiable manifold M admits a Riemann metric.*

Proof. Consider a partition of unity $\{f_i : i \in I\}$ subordinate to an open cover $\{U_\alpha : \alpha \in A\}$.
For each U_α, the chart $\phi_\alpha : U_\alpha \to \widetilde{U}_\alpha$ is a diffeomorphism between U_α and an open set of $\mathbb{R}^n$, then consider the standard metric in $\mathbb{R}^n$ and apply the pull-back to obtain g_α which is a metric tensor in U_α. Finally consider

$$g = \sum_{i \in I} f_i \cdot g_{\alpha(i)}$$

which is a Riemann metric tensor □

2.2 Almost-Complex structures

Definition 2.4. *Let M be a smooth surface, an almost-complex structure is a smooth, linear map $J : TM \to TM$ with the property $J^2 = -id$, and for any $p \in M$, $J_p : T_pM \to T_pM$*

Let M be a smooth surface and $p \in M$, as we have shown above M admits a metric g. For any $v_p \in T_pM$, the metric gives the orthogonal tangent vectors to v_p. If we restrict in having the same module as v_p only two orthogonal tangent vectors to v_p remain. From these two we pick the one that with the original v_p defines a positive orientation (i.e $det(v_p, J(v_p)) > 0$). This is the idea in order to construct J from g.

Now imagine we choose coordinates with the standard metric $g = id$ in a surface $M = (M, U_\alpha, \phi_\alpha)$, in the sense of taking g_α the pull-back of the identity on every open set $\phi_\alpha(U_\alpha) \subset \mathbb{R}^2$. In these coordinates we can write $v_p = (x, y) \in T_pM$ and $J(v_p) = (x', y')$ where the conditions adobe imply

$$xx' + yy' = 0 \text{ and } x^2 + y^2 = x'^2 + y'^2,$$

solving this system results in two possible solution for J

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Applying the last condition (i.e. $det(v_p, J(v_p)) > 0$) results in $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This solution, obtained from the standard metric, will be called J_e from now on.

To sum up, and in order to make more comprehensible the next step, what we have done is schematically

$$v_i \to id^{i,j} v_i v_j = 0 \to v_j = (J_e)^i_j v_i$$

plus the step of the modules.

One can do the general case with a similar equation system or with the idea that once the metric acts on a vector, this form can be thought as a vector in a new manifold where the metric is the identity, and apply the first solution.

$$v_i \to g^{i,j} v_i v_j = 0 \Leftrightarrow id^{\alpha,j} v_\alpha v_j = 0 \text{ where } v_\alpha = id_{\alpha,l} g^{i,l} v_i \Rightarrow$$
$$id^{\alpha,j} id_{\alpha,l} g^{i,l} v_i v_j = 0 \to v_j = (J_e)^\alpha_j id_{\alpha,l} g^{i,l} v_i \Rightarrow J^i_j = (J_e)^\alpha_j id_{\alpha,l} g^{i,l}$$

which in matrix form means $J = J_e g$. It is missing the step of the modules in this process however from the definition of J we have

$$det(J^2) = det(-id) \Leftrightarrow det(J)^2 = 1,$$

then if $J = \lambda J_e g$, where λ is a smooth function

$$det(J)^2 = \lambda^4 det(g)^2 = 1 \Rightarrow \lambda = \frac{1}{+\sqrt{det(g)}}.$$

We picked the positive solution because it maintains the orientation, otherwise it is changed.

Finally, given a metric $g^{i,j}$ the almost-complex structure associated is

$$J^i_j = \frac{1}{\sqrt{g}} (J_e)^\alpha_j id_{\alpha,l} g^{i,l}$$

which in matrix form is

$$J = \frac{1}{\sqrt{eg - f^2}} \begin{pmatrix} -f & -g \\ e & f \end{pmatrix}, \text{ where the metric has the form } \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

and e, f, g are smooth functions that fulfill the conditions of a metric tensor.

Definition 2.5. *Let $M = (M, U_\alpha, \phi_\alpha)$ be a differential manifold, the local inverse of ϕ_α, $\varphi : U \to M$ is a local parameterization.*

Since the charts of a smooth manifold are diffeomorphisms there is not any problem with this definition.

Definition 2.6. *Let M be a differential manifold, isotherm coordinates are local coordinates (or local chart coordinates) $\phi : U \subset M \to (x_1, x_2, ..., x_n) \in \widetilde{U}$ such that the metric locally has the form*

$$ds^2 = e^\rho(dx_1^2 + dx_n^2 + ...dx_n^2)$$

where ρ is a smooth function.

Consider coordinates (x, y) with respect to which the metric has locally the form

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2,$$

the existence of isotherm coordinates is equivalent to the existence of a change of coordinates

$$\begin{aligned} f :&\widetilde{U} \subset \mathbb{R}^2 \to \widetilde{V} \subset \mathbb{R}^2 \\ &(x, y) \to (u(x, y), v(x, y)) = u + iv \end{aligned}$$

where the metric locally is defined as $ds^2 = e^\rho(du^2 + dv^2)$, $\rho \in C^\infty$
Introducing

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \ , \ dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$du = u_x dx + u_y dy \ , \ dv = v_x dx + v_y dy$$

and the complex coordinates $z = x + iy$, $\bar{z} = x - iy$

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \ , \ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$
$$dz = dx + idy \ , \ d\bar{z} = dx - idy$$

it is straightforward to express ds^2 in these complex coordinates

$$ds^2 = e^\rho(du^2 + dv^2) = e^\rho|f_z dz + f_{\bar{z}} d\bar{z}|^2 = e^\rho|f_z|^2|dz + \frac{f_z}{f_{\bar{z}}}d\bar{z}|^2.$$

On the other hand

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2 = \lambda|dz + \mu d\bar{z}|^2$$

where $\lambda = \frac{1}{4}(E + G + 2\sqrt{EG - F^2})$ and $\mu = (E - G + 2iF)/4\lambda$ are smooth functions. Notice that

$$\begin{aligned}|\mu|^2 &= \frac{(E-G)^2 + 4F^2}{(E+G+2\sqrt{EG-F^2})^2} = \frac{(E+G-2\sqrt{EG-F^2})(E+G+2\sqrt{EG-F^2})}{(E+G+2\sqrt{EG-F^2})^2} \\ &= \frac{E+G-2\sqrt{EG-F^2}}{E+G+2\sqrt{EG-F^2}} < 1.\end{aligned}$$

Hence the existence of isotherm coordinates is equivalent to the existence of a solution for the so-called Beltrami equation

$$\frac{\partial f}{\partial z} = \mu \frac{\partial f}{\partial \bar{z}},$$

on a neighbourhood B of 0 and $f_z(0) \neq 0$.

2.3 Beltrami equation

Theorem 2.2. *Let μ be a smooth function defined on a neighbourhood of $0 \in \mathbb{C}$ with $|\mu| < 1$. There is a solution to the differential equation*

$$\frac{\partial f}{\partial \bar{z}} + \mu \frac{\partial f}{\partial z} = 0$$

on a possibly smaller neighbourhood, with $f_z(0) \neq 0$.

The proof begins with a few simplifications:

- We can assume $\mu(0) = 0$ with a linear change of coordinates.
- Making a dilatation $z = \epsilon\tilde{z}$ transforms μ to $\tilde{\mu}(\tilde{z}) = \mu(\epsilon\tilde{z})$ and implies that B can be the unit disc, plus for any ϵ the modulus of μ and its derivatives μ_z, $\mu_{\bar{z}}$ are bounded by ϵ over the disc.
- Consider μ and multiply it by a cut-off function equal to 1 on the disc of radius 1/2. The problem is to find the solution in a subset of the disc so this change does no affect the problem and implies assuming that μ has compact support. In addition, with adjustment of constants, μ and its derivatives are still bounded by a constant.

Therefore now μ is defined on all of $\mathbb{C}$ and supported in the unit disc, in addition $|\mu|$ and its derivatives are very small.

Consider the original equation with the transformation $f = z + \phi$, we get

$$\phi_{\bar{z}} + \mu\phi_z = -\mu. \tag{2.3.1}$$

As $\mu = 0$ outside the unit disc $\Rightarrow \phi_{\bar{z}} = 0$ and hence ϕ is holomorphic outside the unit disc. On the other hand

$$|f_z| = |1 + \phi_z| = 0 \Leftrightarrow \phi_z = -1. \tag{2.3.2}$$

Therefore if

$$|\phi_z(0)| < 1 \Rightarrow |f_z(0)| \neq 0.$$

To solve (2.3.1) we introduce the following operator

$$(Tu)(z) = \frac{1}{2\pi i}\int \frac{u(w)}{w-z} d\mu_w \tag{2.3.3}$$

where $d\mu_w = dw \wedge d\bar{w} = -2idx \wedge dy$ is the ordinary Lebesgue measure on the plane respect w. The integrant is singular however if $u \in C_C^\infty$ (space of smooth functions of compact support) we have

$$|u(w) - u(z)| \leq C|w - z|$$

with $C \in \mathbb{R}$, the integral is well defined.

$$\begin{aligned}
F(\epsilon) &= \frac{1}{2\pi i}\int_{|w-z|>\epsilon} \frac{u(w)}{w-z} d\mu_w \Rightarrow \\
|F(\epsilon) - F(\epsilon')| &\leq \frac{1}{2\pi}\int_{\epsilon'<|z-w|<\epsilon} \frac{|u(w)|}{|z-w|} d\mu_w \\
&= \frac{1}{2\pi}\int_{\epsilon'<|z-w|<\epsilon} \frac{|u(w) - u(z) + u(z)|}{|z-w|} d\mu_w \\
&\leq \frac{1}{2\pi}\int_{\epsilon'<|z-w|<\epsilon} \frac{|u(w) - u(z)|}{|z-w|} d\mu_w + \frac{|u(z)|}{2\pi}\int_{\epsilon'<|z-w|<\epsilon} \frac{1}{|z-w|} d\mu_w \\
&\leq C\frac{\epsilon^2 - \epsilon'^2}{2} + |u(z)|(\epsilon - \epsilon') \to 0 \text{ when } \epsilon, \epsilon' \to 0.
\end{aligned} \tag{2.3.4}$$

Therefore $\lim_{\epsilon \to 0} F(\epsilon)$ exists.

Lemma 2.1. *(see [3] page 34) If* $f \in C_C^1$

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{w}}(w) \frac{d\mu_w}{w-z}. \tag{2.3.5}$$

Proposition 2.2. *For $u \in C_C^\infty$, $\partial Tu/\partial \bar{z} = u$.*

Proof. Recall

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Now we compute $\partial_x(Tu(z))$

$$\begin{aligned}\partial_x(Tu(z)) &= \lim_{h\to 0}\frac{Tu(z+h)-Tu(z)}{h}\\ &= \lim_{h\to 0}\frac{1}{2\pi i}\int\frac{1}{h}\left(\frac{u(w)}{w-z-h}-\frac{u(w)}{w-z}\right)d\mu_w.\end{aligned}$$

Introducing the change $w' = w - h$ implies

$$\begin{aligned}\partial_x(Tu(z)) &= \lim_{h\to 0}\frac{1}{2\pi i}\int\frac{1}{h}\left(\frac{u(w+h)}{w-z}-\frac{u(w)}{w-z}\right)d\mu_w\\ &= \frac{1}{2\pi i}\int\frac{\partial u}{\partial x}(w)\frac{1}{w-z}d\mu_w.\end{aligned}$$

Computing $\partial_y(Tu(z))$ is similar and results in

$$\partial_y(Tu(z)) = \frac{1}{2\pi i}\int\frac{\partial u}{\partial y}(w)\frac{1}{w-z}d\mu_w.$$

Finally

$$\begin{aligned}\partial_{\bar{z}}(Tu(z)) &= \frac{1}{2\pi i}\int\frac{1}{2}\left(\frac{\partial u}{\partial x}(w)+i\frac{\partial u}{\partial y}(w)\right)\frac{1}{w-z}d\mu_w\\ &= \frac{1}{2\pi i}\int\frac{\partial u}{\partial \bar{w}}(w)\frac{1}{w-z}d\mu_w = u(z).\end{aligned}$$

□

Given this proposition, one looks for a solution in the form $\phi = Tv$ and the equation becomes

$$v - \mu S v = -\mu \Leftrightarrow (1-\mu S)v = -\mu \tag{2.3.6}$$

where $Sv = -\dfrac{\partial Tv}{\partial z}$.

The idea now is to consider a small perturbation of the identity $1-\mu S$. Therefore we construct a solution of the form of a Neumann series.

$$v = v_0 + v_1 + ..., \tag{2.3.7}$$

where $v_0 = -\mu$ and $v_i = \mu S(v_{i-1})$ for $i > 0$. Since μ has compact support v_i will be a smooth function with compact support for each i as well. The aim now is to prove the convergence of the series.

This analysis takes place in Hölder spaces. For a function ψ on $\mathbb{C}$ the Hölder norm is

$$[\psi]_\alpha = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|^\alpha}. \tag{2.3.8}$$

Theorem 2.3. *There is a constant K_α such that for any $u \in C_C^\infty$*

$$[Su]_\alpha \leq K_\alpha [u]_\alpha$$

Notice that S is scale-invariant, i.e it conmutes with the map

$$z \to \lambda z.$$

It suffices to establish the bound for $|(Su)(z_1) - (Su)(z_2)|$ when $|z_1 - z_2| = 1$. Since the problem is invariant under rotations and translations, it suffices to consider $z_1 = 1$ and $z_2 = 0$ and therefore the aim is to show

$$|(Su)(1) - (Su)(0)| \leq K_\alpha [u]_\alpha. \tag{2.3.9}$$

The proof needs two lemmas.

Consider differentiating the equation $Tu(z)$ (2.3.4) formally inside the integral sign, we get

$$\frac{\partial Tu}{\partial z}(z) = \frac{1}{2\pi i} \int \frac{u(w)}{(w - z)^2} d\mu_w. \tag{2.3.10}$$

This formula makes no sense because the integrant will not be generally an integrable function.

Lemma 2.2. *Consider $u \in C_C^\infty$ and $u(z) = 0$ for a given $z \in \mathbb{C}$. Then*

$$\frac{u(w)}{(w - z)^2}$$

is an integrable function of w, and the formula (2.3.10) is true for this point z.

Proof. Without loss of generality one can assume $z = 0$. Since u is smooth, $|u(w)| \leq C|w|$ for a suitable constant, and this means using a similar argument that the one used in (2.3.4) that $u(w)/w^2$ is integrable near 0. Since u has compact support, this function is integrable all over $\mathbb{C}$.

Now that the expression is integrable we show that the derivative is what we expect.

$$
\begin{aligned}
(Tu)(z) - (Tu)(0) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \left(\frac{u(w)}{w-z} - \frac{u(w)}{w}\right) d\mu_w \\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{u(w+z)-u(w)}{w} d\mu_w \\
\Rightarrow \left(\frac{\partial Tu}{\partial z}\right)(0) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial u(w)}{\partial w}\frac{1}{w} d\mu_w
\end{aligned}
$$

The right-hand expression above can be write as the limit of the integral I_ϵ over the complement of a disc of radius ϵ. Applying Stokes theorem on

$$
d\left(\frac{u}{w}d\bar{w}\right) = \frac{\partial u}{\partial w}\frac{1}{w}dw \wedge d\bar{w} - \frac{u(w)}{w^2}dw \wedge d\bar{w}
$$

results in

$$
I_\epsilon = \frac{1}{2\pi i}\left(\int_{|w|>\epsilon} \frac{u(w)}{w^2}d\mu_w + \int_{|w|=\epsilon} \frac{u}{w}d\bar{w}\right).
$$

The second term tends to 0 with ϵ because $u(0) = 0$ and finally we get the result desired. $\square$

Lemma 2.3. *Equation* $|(Su)(1) - (Su)(0)| \leq K_\alpha [u]_\alpha$ *is true in the case* $u(0) = u(1) = 0$

Proof. Let Δ_0, Δ_1 be the discs of radii 1/2 centered on 0,1 respectively, and let $\Omega = \mathbb{C}\backslash(\Delta_0 \cup \Delta_1)$. Then applying the previous result,

$$
\begin{aligned}
-2\pi i(Su)(1) &= \int_{\Delta_0} \frac{u(w)}{(w-1)^2}d\mu_w + \int_{\Delta_1} \frac{u(w)}{(w-1)^2}d\mu_w + \int_{\Omega} \frac{u(w)}{(w-1)^2}d\mu_w \\
&= I(1,\Delta_0) + I(1,\Delta_1) + I(1,\Omega), \\
-2\pi i(Su)(0) &= \int_{\Delta_0} \frac{u(w)}{w^2}d\mu_w + \int_{\Delta_1} \frac{u(w)}{w^2}d\mu_w + \int_{\Omega} \frac{u(w)}{w^2}d\mu_w \\
&= I(0,\Delta_0) + I(0,\Delta_1) + I(0,\Omega).
\end{aligned}
$$

Since $u(0) = 0$ we have $|u(w)| \leq [u]_\alpha |w|^\alpha$,

$$
|I(0,\Delta_0)| \leq 2\pi[u]_\alpha \int_0^{1/2} r^{\alpha-1}dr = 2\pi[u]_\alpha \alpha^{-1}2^{-\alpha}.
$$

One can estimate the same for $|I(1, \Delta_1)|$ by symmetry. The case of $|I(1, \Delta_1)|$ and $|I(1, \Delta_1)|$ is identical with a change of variable in the integral. So, it suffices to prove $|I(0, \Omega) - I(1, \Omega)| \leq K'_\alpha [u]_\alpha$.

$$|I(0, \Omega) - I(1, \Omega)| \leq \int_\Omega |u(w)| \left| \frac{1}{w^2} - \frac{1}{(w-1)^2} \right| d\mu_w.$$

Notice that $|w - 1| \leq (1/3)|w|$ on Ω, therefore

$$\begin{aligned} \left| \frac{1}{w^2} - \frac{1}{(w-1)^2} \right| &= \left| \frac{2w-1}{w^2(w-1)^2} \right| \\ &\leq \left| \frac{1}{w(w-1)^2} \right| + \left| \frac{1}{w^2(w-1)} \right| \leq 12|w|^{-3}. \end{aligned}$$

Finally

$$|I(0, \Omega) - I(1, \Omega)| \leq 12 \cdot 2\pi [u]_\alpha \int_{1/2}^\infty r^{\alpha-2} dr = 24\pi [u]_\alpha (1-\alpha)^{-1} 2^{\alpha-1}.$$

□

In order to finish the proof of the theorem 2.3 we need an argument to extend the previous result to arbitrary functions. Actually it suffices to prove that

$$|Su(1) - Su(0)| \leq K$$

for $u \in C_C^\infty$ with $[u]_\alpha = 1$.

Notice that from lemma 2.1, if $g \in C_C^\infty$

$$T\left(\frac{\partial g}{\partial \bar{z}}\right) = g.$$

Consider $g_0 \in C_C^\infty$ supported in the unit disc and equal to $\bar{z}$ on the 1/2 disc Δ_0. Set $u_0 = \partial g_0 / \partial \bar{z}$. Then $u_0 \in C_C^\infty$ is equal to 1 on Δ_0 and is supported in the unit disc. Furthermore $Su_0 = -\partial g_0 / \partial z$, so Su_0 is in C_C^∞ and $[Su_0]_\alpha$ is finite. Let

$$C = \max([u_o]_\alpha, [Su_0]_\alpha).$$

Introduce a parameter $\lambda \leq 0$ and define

$$u_{0,\lambda}(z) = u_0(\lambda^{-1} z).$$

Then from the scalar-invariant property of S

$$[u_{0,\lambda}]_\alpha, [Su_{0,\lambda}]_\alpha \leq C\lambda^{-\alpha}.$$

$u_{0,\lambda}$ is equal to 1 on the disc of radius $\lambda/2$ about 0. Let $u_1(z) = u_0(z-1)$, so, by translation invariance,

$$[u_1]_\alpha, [Su_1]_\alpha \leq C$$

and $u_1(1) = 1$, $u_1(0) = 0$. Now write

$$u = \widetilde{u} + u(0)u_{0,\lambda} + (u(1) - u(0))u_1.$$

Choose $\lambda > 2$ so $u_{0,\lambda}(1) = 1$. This implies that $\widetilde{u}(0) = \widetilde{u}(1) = 0$. Therefore

$$\begin{aligned}|Su(1) - Su(0)| &\leq |S\widetilde{u}(1) - S\widetilde{u}(0)| + |u(0)|[Su_{0,\lambda}]_\alpha + |u(1) - u(0)|[Su_1]_\alpha \\ &\leq |S\widetilde{u}(1) - S\widetilde{u}(0)| + C|u(0)|\lambda^{-\alpha} + C|u(1) - u(0)|.\end{aligned}$$

Now applying lemma 2.3 for $\widetilde{u}$ and having in mind the assumption that $[u]_\alpha = 1$

$$|Su(1) - Su(0)| \leq K'[\widetilde{u}]_\alpha + C|u(0)|\lambda^{-\alpha} + C.$$

On the other side

$$\begin{aligned}\widetilde{u} &= u - u(0)u_{0,\lambda} - (u(1) - u(0))u_1 \\ \Rightarrow [\widetilde{u}]_\alpha &\leq [u]_\alpha + |u(0)|[u_{0,\lambda}]_\alpha + [u_1]_\alpha \\ &\leq 1 + |u(0)|C\lambda^{-\alpha} + C.\end{aligned}$$

And finally

$$|Su(1) - Su(0)| \leq K' + (C + K')|u(0)|\lambda^{-\alpha} + C + 1.$$

Despite not knowing $|u(0)|$ there is no restriction in λ and it can be as large as needed.

This completes the proof of Theorem 2.2.

Now recall v_i are supported in the unit disc and $v_{i+1} = \mu S v_i$. Since $v(2) = 0$, applying lemma 2.2 we have

$$(Sv_i)(2) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{v_i(w)}{(w-2)^2} d\mu_w.$$

Notice $|w - 2| \geq 1$ in the unit disc, therefore $|S(v_i)(2)| \leq c||v_i||_\infty$. We also have $|w - 2| \leq 3$ in the same region, and with the definition of $[\ ]_\alpha$ gives us

$$\sup_D |Sv_i| \leq c||v_i||_\infty + 3^\alpha [Sv_i]_\alpha \leq c||v_i||_\infty + K[v_i]_\alpha.$$

Remember the hypothesis that $||\mu||_\infty, [\mu]_\alpha \le \epsilon$ where ϵ is as small as pleased. This implies

$$||v_{i+1}||_\infty \le \epsilon(c'||v_i||_\infty + K[v_i]_\alpha).$$

On the other hand

$$\frac{f(x)g(x)-f(y)g(y)}{|x-y|^\alpha} = f(x)\left(\frac{g(x)-g(y)}{|x-y|^\alpha}\right) + g(y)\left(\frac{f(x)-f(y)}{|x-y|^\alpha}\right)$$
$$\Rightarrow [fg]_\alpha \le |f|_\infty [g]_\alpha + [f]_\alpha |g|_\infty.$$

Therefore

$$[v_{i+1}]_\alpha \le [\mu]_\alpha(\sup_D |Sv_i|) + ||\mu||_\infty [Sv_i]_\alpha \le \epsilon(c'||v_i||_\infty + 2K[v_i]_\alpha).$$

Finally let $||v_i||_{0,\alpha} \equiv ||v_i||_\infty + [v_i]_\alpha$ implies $||v_{i+1}||_{0,\alpha} \le \epsilon c''||v_i||_{0,\alpha}$ where c'' depends on α. So when $\epsilon < 1/c''$ the sum $\phi = \sum v_i$ converges in the Hölder norm $||\ ||_{,\alpha}$. Consequently $\sum Tv_i$ and its derivates also converge in this norm.

In addition we give the sketch of an argument to make ϕ smooth.

Consider the action μS on $C^{1,\alpha}$, the functions supported on the disc with norm $||v||_{1,\alpha} = ||v||_\infty + [v]_\alpha + [\nabla v]_\alpha$. Since ∇ conmutes with S, $\mu S : C^{1,\alpha} \to C^{1,\alpha}$ has a small operator norm if the derivates of μ are suitably small. Then one can do the whole construction to obtain a $C^{2,\alpha}$ solution. Finally one can apply this as many times as pleased.

This ends the proof of Theorem 2.1 and hence the existence of isothermal coordinates for any smooth surface M.

Definition 2.7. *A local parameterization $\varphi : U \to M$ is compatible with J if*

$$d\varphi(u)(\sqrt{-1}v) = J(\varphi(u)) \cdot d\varphi(u)(v)$$

for all $u \in U$, for all $v \in T\mathbb{R}^2 \sim \mathbb{R}^2$.

Notice that the almost-complex structure associated to isothermal coordinates is $J_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (because of the local form of the metric). And

$$\begin{aligned} J_e :&\mathbb{C} \to \mathbb{C} \\ &z \to J_e(z) = iz \end{aligned}$$

Theorem 2.4. *Let $\varphi : U \to M$, $\psi : V \to M$ local parameterizations compatible with J and $W = \varphi(U) \cap \psi(V) \neq \emptyset$, then*

$$\rho = \psi^{-1} \circ \varphi : \varphi^{-1}(W) \to \phi^{-1}(W)$$

and its inverse are holomorphic.

Proof. We compute $d\rho$ locally in isothermal coordinates

$$\begin{aligned} d\rho(w)(z) &= d\psi^{-1}(\varphi(w))d\varphi(w)(z) = d\psi^{-1}(\varphi(w))d\varphi(w)(x+iy) \\ &= d\psi^{-1}(\varphi(w))d\varphi(w)(x) + id\psi^{-1}(\varphi(w))d\varphi(w)(y) \end{aligned}$$

which satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where u and v are real smooth functions such that $\rho = u + iv$. One can do the same for the inverse. $\square$

Chapter 3

Algebraic curves and the Main Theorem

In this section we will be talking about tensors and differential forms, for a brief introduction see [1] (chapter 2).

3.1 Affine and projective curves

Let $P(z, w)$ be a polynomial in two complex variables and

$$X = \{(z, w) : P(z, w) = 0\}$$

its set of zeros. Assume P has the following property: for each $(z_0, w_0) \in X$, at least one of P_z, P_w does not vanish. Now we show that X is a Riemann surface.

Proposition 3.1. *Suppose $(z_0, w_0) \in X$ and $P_w \neq 0$ at (z_0, w_0). Then there is a disc D_1 centred at $z_0 \in \mathbb{C}$, a disc D_2 centred at $w_0 \in \mathbb{C}$ and a holomorphic map $\phi : D_1 \to D_2$ with $\phi(z_0) = w_0$ such that*

$$X \cap (D_1 \times D_2) = \{(z, \phi(z)) : z \in D_1\}$$

The proof can be found at [2] (page 5). Notice this proposition follows from an analogue of the implicit function theorem for holomorphic functions.

Suppose $P_w \neq 0$ in $(z_0, w_0) \in X$, then there is a holomorphic map $\phi : D_1 \to D_2$ such that $X \cap (D_1 \times D_2) = \{(z, \phi(z)) : z \in D_1\}$. The coordinate charts will have the form $U_\alpha = X \cap (D_1 \times D_2)$, $\widetilde{U_\alpha} = D_1$ and ψ_α the restriction of the projection from $D_1 \times D_2 \to D_1$. The same can be applied if $P_z \neq 0$ for $(z_1, w_1) \in X$ obtaining a map $g : B_2 \to B_1$ such that $X \cap (B_1 \times B_2) = \{(g(w), w) : w \in B_2\}$. Notice that the overlap map between two of the first kind or two of the second kind is the identity. And the overlap map between one of the first and one of the second kind wiil be

$$z \to (z, \phi(z)) \to \phi(z)$$

which is holomorphic. Therefore from a polynomial in two complex variables we obtain a non-compact Riemann surface.

We extend now this idea to compact Riemann surfaces.

Proposition 3.2. *Suppose $p(Z_0, Z_1, Z_2)$ is a homogeneous polynomial of degree $d \geq 1$ such that the only solution of the equations*

$$\frac{\partial p}{\partial Z_0} = \frac{\partial p}{\partial Z_1} = \frac{\partial p}{\partial Z_2} = 0$$

is $Z_0 = Z_1 = Z_2 = 0$. Then the topological subspace defined by $p = 0$ in $\mathbb{C}P^2$ admits a structure of compact Riemann surface.

Proof. From Euler's identity

$$\sum_{i=0}^{2} Z_i \frac{\partial p}{\partial Z_i} = dp$$

we deduce that Z_i does not divide p, if $p = q(Z_1, Z_2)Z_0$ then $\dfrac{\partial p}{\partial Z_0} = q(Z_1, Z_2)$ does vanish at some $(0, Z_1, Z_2) \neq 0$ which contradicts the hypothesis. Consider $[Z_0, Z_1, Z_2] \in \mathbb{C}P^2$ which lies in the zero set of p, at least one of them is not zero, so we can suppose $Z_0 = 1$ and since the only solution to

$$\frac{\partial p}{\partial Z_0} = \frac{\partial p}{\partial Z_1} = \frac{\partial p}{\partial Z_2} = 0$$

is $Z_0 = Z_1 = Z_2 = 0$, one of the partial derivates is not zero. Then by Euler's identity, having one partial derivate different to zero and the other two equal to zero is not possible. Hence, without loss of generality, $\partial p / \partial Z_2 \neq 0$. Here enters the discussion above with the notation $P(z, w) = p(z, w, 1)$, where $\partial P / \partial w \neq 0$ at the point in question. $\square$

Now we will just announce a property related with the proposition above. Consider $\mathbb{C}^3$ as a $\mathbb{C}$-vectorial space, now $\mathbb{C}P^2 = \mathbb{P}(\mathbb{C}^3)$ is the projectivization of $\mathbb{C}^3$. This is

$$\mathbb{P}(\mathbb{C}^3) = (\mathbb{C}^3 \backslash \{0\}) / \mathbb{C}^*.$$

For any $\alpha, \beta \in (\mathbb{C}^3)^* = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^3, \mathbb{C})$, $U_\beta \equiv \mathbb{C}P^2 \backslash \{\beta = 0\}$. In a fixed coordinates, the map

$$\frac{\alpha}{\beta} : U_\beta \to \mathbb{C}$$
$$[x, y, z] \to \frac{\alpha(x, y, z)}{\beta(x, y, z)}$$

has the property that

$$\frac{\alpha}{\beta} : U_\beta \cap \{p = 0\} \to \mathbb{C}$$

is holomorphic for all $\alpha, \beta \in (\mathbb{C}^3)^*$. Furthermore, these maps determine the Riemann surface structure of the topological space $\{p = 0\}$.

Despite of Proposition 3.2 having a strong hypothesys about the curve: that the only solution of the equations

$$\frac{\partial p}{\partial Z_0} = \frac{\partial p}{\partial Z_1} = \frac{\partial p}{\partial Z_2} = 0$$

is $Z_0 = Z_1 = Z_2 = 0$, the result can be extended, however we will not prove it here. The problem is related to curves, such $w^2 - z^2(1 - z)$, with self-intersection. This singular point, where two branches of the curve cross, does not have a neighbourhood homeomorphic to $\mathbb{C}$. The solution to this problem is to give the Riemann surface structure to a very similar object where the branches are separated, however not to the original curve.

The other way around, that is, from a Riemann surface obtain a polynomial such that X could be construct with the argument above is more involved, and will be based on the following section.

3.2 The Main Theorem

First some notation, definitions and properties that will be useful. Let X be a Riemann surface, let $f : X \to \mathbb{C}$ (i.e. $f \in \Omega^0_X \equiv \Omega^0$) and $z, \bar{z}$ be complex coordinates

$$\partial f \equiv \frac{\partial f}{\partial z} dz, \ \bar{\partial} f \equiv \frac{\partial f}{\partial \bar{z}} d\bar{z},$$

therefore

$$df = \partial f + \bar{\partial} f.$$

Definition 3.1. *On a Riemann surface, the Laplace operator is defined as*

$$\Delta = 2i\bar{\partial}\partial : \Omega^0 \to \Omega^2$$

In holomorphic coordinates $z = x + iy$,

$$\Delta f = -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy.$$

Definition 3.2. *Let f, g be real-valued functions, with at least one of them having compact support. The Dirichlet inner product is defined as*

$$\langle f, g\rangle_D = -\int_{\mathbb{C}} \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right) dxdy.$$

The definition above is equivalent to

$$\langle f, g\rangle_D = 2i \int_X \partial f \wedge \bar{\partial} g.$$

Theorem 3.1. *Let X be a compact connected Riemann surface and let ρ be a smooth 2-form in X. There is a smooth solution f to the equation $\Delta f = \rho$ if and only if $\int_X \rho = 0$, and the solution is unique up to the addition of a constant.*

Suppose f a solution to the equation, so that

$$\int_X \Delta f = 2i \int_X \bar{\partial}\partial f = 2i \int_X d(\partial f) = 0$$

where $\partial\partial f = 0$ and Stokes' Theorem has been used.

The uniqueness up to constant is equivalent to the assertion that the solutions of the equation $\Delta f = 0$ are constants. Consider a point in $x \in X$ where f is maximal, which exists by compactness, and then apply the maximum principle for harmonic functions, which says that if f is a harmonic function then f cannot exhibit a local maximum within its domain of definition. Therefore f is constant or there exists other points arbitrarily close to x at which f takes larger values, which is a contradiction.

The remaining part is to prove that if $\int_X \rho = 0$ then there exists a solution f.

Let $C^\infty(X)/\mathbb{R}$ be the vector space obtained by dividing out by the constant functions.

Proposition 3.3. *$C^\infty(X)/\mathbb{R}$ with the Dirichlet norm and inner product is a pre-Hilbert space*

The proof is straightforward from the definition of pre-Hilbert space.

Consider ρ a 2-form and ϕ, ψ functions on X, then

$$\int_X \psi(\rho - \Delta\phi) = \int_X \psi\rho - \int_X \psi\Delta\phi = \int_X \psi\rho - \int_X \nabla\phi\nabla\psi = \int_X \psi\rho - \langle\phi, \psi\rangle_D,$$

therefore the original problem $\Delta\phi = \rho$ is equivalent to

$$\int_X \psi(\rho - \Delta\phi) = 0 \Leftrightarrow \int_X \psi\rho = \langle\phi, \psi\rangle_D$$

for all functions ψ. Thus, if $\int_X \rho = 0$, the operator

$$\begin{aligned} \hat{\rho} :& C^\infty(X)/\mathbb{R} \to \mathbb{R} \\ \psi \to& \int_X \rho\psi \end{aligned}$$

is well defined and the problem is to find ϕ such that $\hat{\rho}(\psi) = \langle\phi, \psi\rangle_D$ for all ψ

Theorem 3.2. ***Riesz Representation Theorem.*** *Let H be a real Hilbert space and let $\sigma : H \to \mathbb{R}$ be a linear map such that $|\sigma(x)| \leq C||x||$ for a constant C and for all $x \in H$. Then there is a $z \in H$ such that*

$$\sigma(x) = \langle z, x\rangle$$

for all $x \in H$

The strategy is now clear, we want to fit the problem into Theorem 3.2. Since $C^\infty(X)/\mathbb{R}$ is pre-Hilbertian simply consider its abstract completion under the Dirichlet norm. A point of H is an equivalence class of Cauchy sequences $(\psi_i) \in C^\infty(X)/\mathbb{R}$ under the equivalence relation

$$(\psi_i) \sim (\psi_i') \Leftrightarrow ||\psi_i - \psi_i'||_D \to 0.$$

In order to show that $\hat{\rho}$ is bounded we need one lemma.

Lemma 3.1. *(see [2] page 123). Let Ω be a bounded, convex, open set of $\mathbb{R}^2$, A its area and d its diameter. Let ψ be a smooth function on an open set containing $\bar{\Omega}$ and let $\bar{\psi}$ be its average*

$$\bar{\psi} = \frac{1}{A}\int_\Omega \psi d\mu$$

where $d\mu$ is the Lebesgue measure on $\mathbb{R}^2$. Then we have

$$\int_\Omega |\psi(x) - \bar{\psi}|^2 d\mu_x \leq \left(\frac{d^3\pi}{A}\right)^2 \int_\Omega |\nabla\psi|^2 d\mu = \left(\frac{d^3\pi}{A}\right)^2 ||\nabla\psi||_{L^2}^2.$$

This lemma is a 2-dimensional analogue of the Poincare unequality. The idea is to control the function having information only about its derivative.

Theorem 3.3. *The functional $\hat{\rho}$ is bounded (i.e $|\sigma(x)| \leq C||x||$ for a constant C and for all $x \in H$)*

Proof. First, consider ρ supported in a single coordinate chart $\phi : U \to \Omega \subset \mathbb{C}$. Since the volume form (Lebesgue measure in this case) gives an isomorphism between the space of smooth functions and the space of smooth 2-form, consider ρ as a function of integral 0 supported in Ω. Likewise any function ψ on X can be seen, thanks to that chart, as a function on $\Omega \in \mathbb{C}$. Therefore

$$\hat{\rho} = \int_\Omega \rho\psi d\mu = \int_\Omega \rho(\psi - \bar{\psi})d\mu$$

since $\int_\Omega \rho d\mu = 0$ and $\bar{\psi} \in \mathbb{R}$ of course. Then applying Chauchy-Schwarz inequality

$$|\hat{\rho}| = \left|\int_\Omega \rho(\psi - \bar{\psi})d\mu\right| \leq ||\rho||_{L^2(\Omega)}||\psi - \bar{\psi}||_{L^2(\Omega)}$$
$$\Rightarrow |\hat{\rho}| \leq \frac{d^3\pi}{A}||\rho||_{L^2(\Omega)}||\nabla\psi||_{L^2(\Omega)} \leq C||\nabla\psi||_{L^2(X)} = C||\psi||_D.$$

For the general case, we use the Poincare duality. In particular the isomorphism between $\mathbb{R} = H_0(X)$ and $H^2(X)$, and that is the integration over X. Therefore this map sends ρ to 0, since the map is an isomorphism and for any smooth 1-form θ the map sends $d\theta$ to 0, $\rho = d\theta$ for some 1-form $d\theta$. Consider a finite cover $\{U_\alpha\}$ for X, and let g_i be a partition of unity subordinate to this cover (recall from Chapter 2 that any Riemann surface can be seen as a smooth surface). Put $\rho_\alpha = d(g_\alpha\theta)$ and then every ρ_α is supported in $\widetilde{U_\alpha}$ and

$$\int_X \rho_\alpha = \int_X d(g_\alpha\theta) = 0.$$

since X is compact and has no boundaries. On the other hand

$$\rho = d\theta = \sum d((g_\alpha)\theta) = \sum \rho_\alpha$$

Then one can apply the previous argument to show that every $\hat{\rho_\alpha}$ is bounded and finally the finite sum of bounded linear maps is also bounded. □

In order to apply Theorem 3.2 it is needed to extend $\hat{\rho}$ from the pre-Hilbert space $C^\infty(X)/\mathbb{R}$ to its natural extension mentioned above. The extension that will be called $\hat{\rho}$ as well, must be bounded. Consider a Cauchy sequence $(\psi_i) \in C^\infty(X)/\mathbb{R}$.

$$\forall \epsilon > 0,\ \exists n \in N\ \forall i, j > n\ ||\psi_i - \psi_j||_D < \epsilon$$

then the sequence $\hat{\rho}(\psi_i)$ is Cauchy in $\mathbb{R}$ (since $\hat{\rho}$ is linear)

$$|\hat{\rho}(\psi_i) - \hat{\rho}(\psi_j)| \leq |\hat{\rho}(\psi_i - \psi_j)| < C||\psi_i - \psi_j||_D < C\epsilon$$

for a constant C. Therefore the extension can be defined as $\hat{\rho}_H(\{\psi_i\}) = \lim \hat{\rho}_{PH}(\psi_i)$, which is bounded.

$$|\lim \hat{\rho}(\psi_i)| \leq \lim |\hat{\rho}(\psi_i)| < \lim\ C||\psi_i||_D = C||\psi||_D$$

where $\lim\ \psi_i = \psi \in H$

At this point one can apply the Riesz Representation Theorem obtaining a function $\phi \in H$ such that $\hat{\rho}(\psi) = \langle \phi, \psi \rangle_D$ for all ψ. ϕ is called a weak solution to the problem. So the last step to prove Theorem 3.1 is the following.

Theorem 3.4. *If ρ is a smooth 2-form on X of integral zero, a weak solution ϕ in H is smooth, i.e. lies in $C^\infty(X)/\mathbb{R} \subset H$.*

Consider the weak solution ϕ. Since $\phi \in H$, there is a Cauchy sequence such $\phi_i \to \phi$, $\phi_i \in \mathbb{C}^\infty(X)/\mathbb{R}$, that converges with the Dirichlet norm. We now show that ϕ is (up to a constant) locally in L^2 (i.e represented by a L^2 function in any local coordinate chart). Consider any fixed chart, identified with $\Omega \subset \mathbb{C}$. Then we add suitable constants to ϕ_i to make its integral over Ω equal to 0, so now using Lemma 3.1 we have

$$||\phi_i - \phi_j||_{L^2_\Omega} \leq C||\phi_i - \phi_j||_D.$$

Hence, since (ϕ_i) is Cauchy with the Dirichlet norm, it is as well Cauchy in L^2_Ω, and since this space is a Hilbert space (which in particular means it is complete) it results that ϕ converges to an L^2 limit. We now show that the sequence ϕ_i converges locally in L^2 over all X.

Let A be the set of points in $x \in X$ with the property that there is a coordinate chart around x that ϕ_i converges to ϕ in L^2. We can say that A is open and non-empty because what we have seen above. Since X is connected, the complement of A is not open unless it is the empty set. So either $A = X$ or there is a point y which is in the clousure of A but not in A. However in the second case we could find a coordinate neighbourhood $y \in \Omega'$ and a sequence of real numbers c'_i such that $\phi_i - c_i$ converges in L^2 over Ω'. Now there is a point $x \in \Omega' \cap A$ and on a small neighbourhood of x, ϕ_i and $\phi_i - c'_i$ converge. This means c'_i tends to 0, so $y \in A$ after all.

In order to proceed with the proof of Theorem 3.4 two lemmas are needed.

Definition 3.3. *The Newton potential in two dimension is defined as*

$$K(x) = \frac{1}{2\pi} \log |x|.$$

Recall for any smooth function f of compact support in $\mathbb{C}$, the convolution

$$K * f(x) = \int K(y) f(x-y) d\mu_y$$

is defined and it is smooth

Lemma 3.2. *If σ has compact support in $\mathbb{R}^2$ then $K * (\Delta\sigma) = \sigma$.*

Proof. If $K * (\Delta\sigma)(0) = \sigma(0)$ is true, then applying translation invariance, we obtain the result. Consider

$$K * (\Delta\sigma)(0) = \int K(y)(\Delta\sigma)_y d\mu_y,$$

since $\Delta \log(y)$ vanishes on $\mathbb{C}/\{0\}$, and with the second Green's identity, the expression above can be written as

$$\lim_{\epsilon \to 0} \int_{U=\{|y| \leq \epsilon\}} K(y)(\Delta\sigma)_y d\mu_y = \lim_{\epsilon \to 0} \int_{\partial U} \left(K(y) \frac{\partial \sigma}{\partial n} - \sigma \frac{\partial K(y)}{\partial n} \right)$$

which is equal to $\sigma(0)$. □

Lemma 3.3. *If f has compact support, then $\Delta(K * f) = f$.*

Proof. Let us compute $\Delta(K * f)(x)$

$$(K * f)(x) = \int K(y) f(x-y) d\mu_y \Rightarrow \Delta(K * f)(x) = \int K(y) \Delta_x f(x-y)$$

where Δ_x is the Laplacian respect to x. Then applying the lemma above, one obtains the result. □

Lemma 3.4. *Let ψ be a smooth harmonic function on a neighbourhood of a closed disc, then*

$$\int_0^{2\pi} \psi(r,\theta) d\theta = 2\pi\psi(0).$$

This well-known property is called the mean value property of harmonic functions which we will not prove here.

Proposition 3.4. *Let ψ be a smooth function on $\mathbb{C}$, and suppose that $\Delta\psi$ is supported in a compact set $J \subset \mathbb{C}$. Let $\beta(r)$ be a smooth function on $\mathbb{R}$ such that $\beta(r)$ is constant for small r, vanishing for $r \leq \epsilon$, and such that*

$$2\pi \int_0^\infty r\beta(r)dr = 1$$

Then $B * \psi = \psi$ outside the ϵ-neighbourhood of J, where $B(z) = \beta(|z|)$.

Proof. We will prove the property for $z = 0$ and then by translation invariance it is generalized.

$$\int B(-z)\psi(z)d\mu_z = \int_0^\infty \int_0^{2\pi} r\beta(r)\psi(r,\theta)drd\theta = \psi(0)2\pi \int_0^\infty r\beta(r)dr = \psi(0).$$

$\square$

For the last steps of the proof we need a version of Weyl's Lemma.

Proposition 3.5. *Let $\Omega \subset \mathbb{C}$ be a bounded open set and let ρ be a smooth 2-form on Ω. Suppose ϕ is a L^2 function on Ω with the property that, for any smooth function χ of compact support in Ω,*

$$\int_\Omega \Delta\chi\phi = \int_\Omega \chi\rho.$$

Then ϕ is smooth and satisfies the equation $\Delta\phi = \rho$

Proof. Since smoothness is a local property, it is sufficient to prove that ϕ is smooth in a smaller neighbourhood Ω'. Then consider $\rho' = \rho$ in a neighbourhood of $\bar{\Omega}$ and of compact support in Ω. If a smooth solution ϕ' exists (i.e $\Delta\phi' = \rho'$) in Ω, then $\Delta(\phi - \phi') = 0$. Therefore if $\psi = \phi - \phi'$ is a smooth solution of $\Delta\psi = 0$, ϕ will be smooth as well.

Since for any $\phi \in L^2$, $B * \phi$ is smooth, the aim is to establish $B * \phi = \phi$ in Ω', which is equivalent to prove that for any test function χ of compact support in Ω'

$$\langle \chi, \phi - B\phi \rangle = 0.$$

Consider $h = K * (\chi - B * \chi) = K * \chi - K * B * \chi$. Now applying the lemma 3.4 we get $\Delta(K * \chi) = \chi$, in particular out of the support of χ, $\Delta(K * \chi) = 0$. Then we apply Proposition 3.4 and we get $B * K * \chi = K * \chi$ outside a neighbourhood of the support of χ. Therefore h has compact support contained in Ω and can be used as test function.

Since χ and $B * \chi$ have compact support, applying lemma 3.4 we get

$$\Delta h = \Delta(K * (\chi - B * \chi)) = \chi - B * \chi.$$

If $\Delta\phi = 0 \Leftrightarrow \langle h, \Delta\phi\rangle = 0 \Leftrightarrow \langle \Delta h, \phi\rangle = 0$:

$$\langle \Delta h, \phi\rangle = \langle \chi - B * \chi, \phi\rangle = \langle \chi, \phi - B * \chi\rangle = 0.$$

□

Finally, we point out that as a consequence of the Main Theorem, the following can be proved.

Theorem 3.5. *Any compact connected Riemann surface arises from a polynomial.*

That means for any compact connected Riemann surface X, it exists an irreductible polynomial in two complex variables $P(z, w)$ such that we can apply the ideas on Section 3.1 and recover X.

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