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THESIS

BSc in Mathematics

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RENORMALIZATION IN COMPLEX DYNAMICS

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Abstract

Renormalization theory is a powerful technique both in mathematics and physics. In particular, it is essential in the study of the MLC conjecture, which wonders whether or not the Mandelbrot set is locally connected. The main purpose of this project is to comprehend the Straightening Theorem with the utmost rigour, that is an essential result behind the aforementioned technique. In order to approach our aim, we proceed to concentrate on quasiconformal geometry and give some basic background concerning dynamical systems. Besides, we explore some applications of the cornerstone of this thesis within the framework of complex dynamics. Lastly, we outline the notion of renormalization when dealing with percolation theory in physics.

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Contents

Introduction

In 1870, the German mathematician and logician [2]Ernst Schröder was the first to consider iteration of complex functions in a dynamical framework. He tackled solving equations with iterative algorithms and such insight became a boost that gave rise to Newton's method algorithm used to approximate roots of certain functions. This beginning was followed by ups and downs that finished at the same time World War I came to an end. The pioneering work of [3]Gaston Julia and [4]Pierre Fatou while striving to gain *Le Grand Prix des Sciences Mathématiques de 1918* was the spark of a theory that, as the icing on the cake, brought some sparkle to mathematicians' eyes when computers revealed around 1980 the beautiful objects that both geniuses could only imagine with their minds.

Since then, a great theory has been developed within the context of complex dynamics and it still has ambitious questions to solve. Possibly the most famous unsolved problem in this branch of mathematics is a conjecture known as MLC. It wonders whether or not the [5]Mandelbrot set is locally connected. But, what exactly is this set and how could we deal with such a conjecture?

The dynamics of a holomorphic mapping from a certain space to itself is concerned with the fate of the iterates of such a function at some chosen point of our space. If we focus on the complex plane, we mean to study the behaviour of the sequence

$$z, f(z), f^2(z), f^3(z), \ldots$$

for a given holomorphic map $f : U \subset \mathbb{C} \to U$ and every $z \in U$, where $f^n(z)$ denotes the n-th iterate of f at z. This string of values is called the orbit of z and can escape to infinity while increasing the number of iterates. If we consider the quadratic family $\{Q_c(z) := z^2 + c\}_{c \in \mathbb{C}}$, then we can define K_c as the set of points whose iterates over Q_c do not diverge to infinity and introduce the Mandelbrot set $\mathcal{M} := \{c \in \mathbb{C} \mid K_c \text{ is connected}\}$.

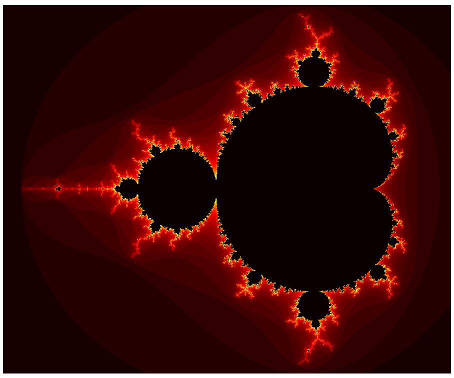


Figure 1: The Mandelbrot set.

The importance of the Mandelbrot set lies in the fact that $\mathcal{M}$ is universal, in the sense that we can

[2]Friedrich Wilhelm Karl Ernst Schröder: 1841 – 1902
[3]Gaston Maurice Julia: 1893 – 1978
[4]Pierre Joseph Louis Fatou: 1878 – 1929
[5]Benoît Mandelbrot: 1924 – 2010

find homeomorphic copies of $\mathcal{M}$ when looking at parameter spaces of certain families of holomorphic maps seemingly unrelated to the quadratic one. The MLC conjecture, on the other hand, would allow a complete topological description of $\mathcal{M}$ and in particular, a parametrization of its boundary, a striking fact when bearing in mind that this is an object of Hausdorff dimension 2. Up to now, many properties regarding the Mandelbrot set have been proven. For instance, we know that it is compact and connected. At the moment of proving local connectedness of $\mathcal{M}$, we must check it for each $c \in \mathcal{M}$. So far, many cases have been settled and the technique to do it involves a theory known as renormalization.

To understand the philosophy behind such a tool, we must set a map $f : X \to X$ whose dynamics on some space X want to be studied. If we restrict f to a subset $Y \subset X$ such that the orbit of every $x \in Y$ returns to Y after $n(x)$ iterates, we can define an induced map $g : Y \to Y$ defined as $g(x) := f^{n(x)}(x)$. In general, g may look quite different from f. Nevertheless, for certain dynamical systems such an inducing procedure produces a map of the same class as the original one. In this case, we can rescale our system after each step, so that all the dynamics occur at a fixed spatial scale. In essence, this is what we know as renormalization theory.

The technique of renormalization has proven useful in many areas of science to explain small scale phenomena, by the procedure described above. The theory takes different particular forms depending on the field, it being chemistry, physics, mathematics, etc. Our main goal is to give a complete and self-contained proof of "The Straightening Theorem", a result that lays at the foundation of each and every renormalization argument in this thesis in holomorphic dynamics. It broadly asserts that a local dynamical system, under certain hypothesis, is tied to a polynomial by means of a conjugacy (i.e. a local change of variables) which exhibits a certain degree of regularity beyond it being a homeomorphism. Namely, the conjugacies are "quasiconformal", that is they deform angles between curves but in a controlled way.

The Straightening Theorem uses a variety of mathematical tools which form an important part of this project. We start by concentrating on quasiconformal geometry, which is a fully-fledged theory within complex analysis. At the beginning, we introduce almost complex structures to meet the survey of quasiconformal maps through the use of pullbacks. Exploring in greater depth, we impose less rigidity to such maps and characterize the concept of quasiregularity, whose flexibility will play an important role when dealing with surgery in the proof of the cornerstone of this thesis. We include also the theory of quasisymmetric maps (a one-dimensional version of quasiconformality), the optimal condition for boundary extensions of quasiconformal functions and a crucial tool to perform surgery.

We continue in Chapter 2 with some background on dynamical systems and in special holomorphic ones. In particular, the local theory of fixed points and the conformal changes of coordinates that appear within, play an important role in our constructions.

The main goal of the third chapter is to prove the Straightening Theorem, by means of quasiconformal surgery and using the concepts presented in the previous chapters.

A direct consequence of this result is the presence of copies of polynomial Julia sets in the dynamical plane of holomorphic functions a priori unrelated to polynomials, but also the self-similarity of the polynomial Julia sets themselves. But strikingly, this phenomenon is also latent in the parameter planes: not only small Mandelbrot set copies appear at all scales in the parameter space of quadratic polynomials, but also Mandelbrot sets appear in many other families of holomorphic functions. Both phenomena are explained by renormalization and, more precisely, by the parameter version of the Straightening Theorem, which we state (but do not prove) in this chapter. Its proof relies on the saying formulated by Adrien Douady:

"You first plough in the dynamical plane and then harvest in the parameter space"

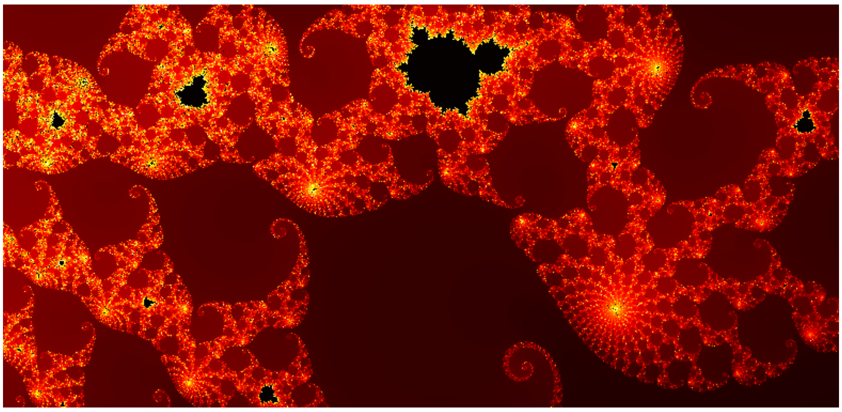


Figure 2: Copies of the Mandelbrot set within itself.

In the next chapter, we aim to prove some results on cubic polynomials in terms of the behaviour of its critical points (i.e. zeros of the derivative). We show them for a generic family and subsequently we see some particular cases. Furthermore, in this section, we provide some plots to give support to our theoretical results. More precisely, we expect to find some $c \in \mathcal{M}$ so that K_c is homeomorphic to infinitely many connected components of the filled Julia set of a certain class of cubic polynomials, which is characterized by the long-term behaviour of the critical points. Taking advantage of the fact that we asserted the Straightening Theorem for parameters, we venture to benefit from the result that has just been explained to extrapolate to a certain parameter space of a family of cubic polynomials under similar hypothesis.

In addition, we introduce the MLC conjecture and some progress in the context of renormalization. So far, some mathematicians have between their eyebrows the remaining cases to validate the conjecture. It is noteworthy that several authors who made important breakthroughs on this issue were awarded a Fields Medal, such as [6]J. C. Yoccoz (1994), [7]C. T. McMullen (1998) and [8]Artur Avila (2014).

Finally, we emphasize that the concept of renormalization arises in many forms through mathematics and physics, namely percolation theory shapes the content of this latter chapter. Since applied science is not always totally accurate and could proceed in a mathematically dubious way, we will only give an outline. We choose percolation instead of another topic because it does not need specific physics concepts. In the same way that renormalization yielded Field Medals, percolation appears in the work of critical phenomena by [9]Kenneth G. Wilson, for which he was awarded the Nobel Prize in Physics. Moreover, [10]Stanislav Smirnov, current Professor at the University of Geneva, received a Fields Medal in 2010 "for the proof of conformal invariance of percolation and the planar Ising model in statistical physics".

For ease of reading, two appendices have been added. The first one contains some general results which are cited when necessary while the other appendix contains the Python code used to obtain the images in this work.

[6]Jean-Christophe Yoccoz: 1957 – 2016
[7]Curtis Tracy McMullen: 1958 – present
[8]Artur Avila Cordeiro de Melo: 1979 – present
[9]Kenneth Geddes Wilson: 1936 – 2013
[10]Stanislav Konstantínovich Smirnov: 1970 – present

Chapter 1

Quasiconformal geometry

In this first chapter, we aim to concentrate on quasiconformal mappings, whose flexibility in comparison with the rigidity of conformal ones allows to tackle the notion of quasiconformal surgery. In order to understand the underlying concepts, we introduce almost complex structures and pullbacks.

When dealing with quasiconformal maps, we give both an analytic characterization and a geometric one. In a sense, this kind of mappings deform angles between curves but in a controlled way. The main result concerning quasiregularity is the Integrability Theorem. As the name suggests, given a certain almost complex structure, it asserts the existence of a quasiconformal mapping so that such a structure can be written as the pullback of the standard almost complex structure.

Then, we impose even less rigidity and we move on to quasiregular mappings. Finally, we give some results of quasisymmetric maps, the optimal condition for boundary extensions of quasiconformal functions.

It is noteworthy that this kind of mappings are essential for the proof of the Straightening Theorem in the third chapter, where surgery has a key role.

1.1 Beltrami coefficient, ellipses and dilatation: the linear case

Let $\mathbb{C}_{\mathbb{R}}$ denote the complex plane, viewed as the two-dimensional oriented Euclidean $\mathbb{R}$-vector space with the orthonormal positively oriented standard basis $\{1, i\}$. Taking into account that we can write $z = x + iy$ and $\overline{z} = x - iy$, we can use $(z, \overline{z})$ as coordinates instead of (x, y). Then, any $\mathbb{R}$-linear map $L : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ can be written as $L(z) = az + b\overline{z}$, with $a, b, z \in \mathbb{C}$. From now on, we will consider only the ones that are invertible and orientation preserving, i.e. such that $\det(L) > 0$. Let us find this relation more explicitly, computing L in the standard basis $\{1, i\}$:

$$\begin{cases} a = a_1 + a_2 i \\ b = b_1 + b_2 i \end{cases} \implies \begin{cases} L(1) = a + b = (a_1 + b_1) + (a_2 + b_2)i \\ L(i) = (a - b)i = (b_2 - a_2) + (a_1 - b_1)i \end{cases} \implies$$

$$\implies L = \begin{pmatrix} a_1 + b_1 & b_2 - a_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} \implies \det(L) = |a|^2 - |b|^2 > 0 \implies |a| > |b|$$

Definition 1.1.1. (Beltrami coefficient) *The value $\mu(L) := b/a$ is called the [1]Beltrami coefficient or complex dilatation of L.*

Remark 1.1.2. We will use the expression $\mu(L) = (|b/a|)e^{i2\theta}$, where $\theta \in \mathbb{R}/(\pi\mathbb{Z})$ is half its argument. Besides, $\mu(L) \in \mathbb{D}$ when L preserves the orientation and L is holomorphic if and only if $b = 0$ (i.e. $\mu(L) = 0$).[2]

[1]Eugenio Beltrami: 1835 – 1900
[2]$\overline{z}$ is not holomorphic.

We can write $L = R \circ S$, where $S(z) := |a|(z + |\mu| e^{i2\theta}\overline{z})$ is an $\mathbb{R}$-linear and self-adjoint map (see Section A.1), and $R(z) := e^{i\alpha} z$ is a rotation of an angle $\alpha := \mathrm{Arg}(a)$. Furthermore, $v_m := e^{i\theta}$ and $v_M := e^{i(\theta+\pi/2)}$ are eigenvectors of S corresponding to the eigenvalues $\lambda_m := |a|(1 + |\mu|)$ and $\lambda_M := |a|(1 - |\mu|)$ respectively. This define an ellipse $E(L)$ that is mapped by S to $\mathbb{S}^1$ and then mapped to itself by the rotation R. Its half axes are λ_m^{-1} and λ_M^{-1}.

Definition 1.1.3. (Dilatation) *We define the dilatation $K(L)$ of L as the ratio of the major axis $2M$ to the minor axis $2m$:*

$$K(L) := \frac{M}{m} = \frac{1 + |\mu|}{1 - |\mu|} = \frac{|a| + |b|}{|a| - |b|}$$

Remark 1.1.4. From the last definition, we have $|\mu| = (M - m)/(M + m)$. Then, given an ellipse E, we compute the Beltrami coefficient by $\mu(E) = \frac{M-m}{M+m} e^{i2\theta}$, where θ is the argument in $[0, \pi)$ of the minor axis.

Definition 1.1.5. (Standard conformal structure) *We define the standard conformal structure (also called standard complex structure) of $\mathbb{C}_{\mathbb{R}}$, denoted by σ_0, as $\mathbb{C}_{\mathbb{R}}$ seen as a $\mathbb{C}$-vector space with the standard complex scalar multiplication.*

We are going to deduce $\sigma(L)$, an analogous definition for the domain of L as deduced from the equality $L(E(L)) = \mathbb{S}^1$:

Definition 1.1.6. (Conformal structure) *The conformal structure of $L^{-1}(\mathbb{S}^1)$ (induced by L) is known as the complex plane $\mathbb{C}_{\mathbb{R}}$ endowed with the operation $c * z = \mathrm{Re}(c)z + \mathrm{Im}(c)J(z)$ for all $c, z \in \mathbb{C}$, where J is the $\mathbb{R}$-linear map defined as $J(z) := L^{-1}(iL(z))$. This operation replaces the standard complex product and makes $\mathbb{C}_{\mathbb{R}}$ into a $\mathbb{C}$-vector space. We denote this conformal structure by $\sigma(L)$.*

A geometric view of this definition can be obtained by introducing the notion of conjugate diameter.

Definition 1.1.7. (Conjugate diameter) *Given a diameter in an ellipse, the conjugate diameter is defined as the set of midpoints of the cords parallel to the given diameter. Equivalently, it is the one parallel to the line tangent to the ellipse at the point of intersection with the given diameter.*

Remark 1.1.8. Notice that we can read into this that z and $J(z)$ are on conjugate diameters, turning from z in the positive direction, for all point in the ellipse. Further, $J(J(z)) = -z$, which can also be obtained imposing $i^2 * z = i * i * z$. In an informal way, we can understand Definition 1.1.6 as a kind of deformation of the complex plane that keep constant the product by a real scalar, but does not by an imaginary one; as if a sheaf of circles with centre $z = 0$ that cover the whole plane were deformed to a sheaf of identical ellipses up to scaling.

Remark 1.1.9. With this idea, we can go further: if we define ellipses in $\mathbb{C}_{\mathbb{R}}$ up to scaling, then each of these families determines a conformal structure σ in $\mathbb{C}_{\mathbb{R}}$.

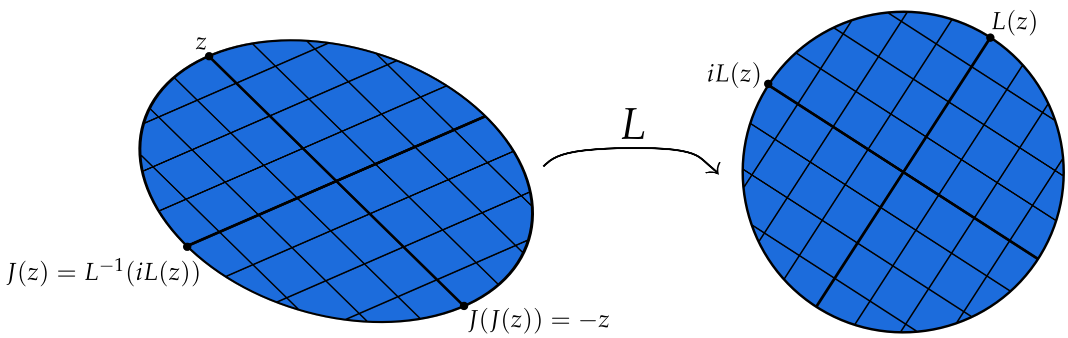


Figure 1.1: Geometric interpretation of the mapping J.

We finish this section considering inverses and compositions. It is easy to compute how Beltrami coefficients and dilatations change under this kind of mappings, which are also $\mathbb{R}$-linear maps. Indeed,

$$L^{-1}(\omega) = \frac{1}{|a|^2 - |b|^2}(\overline{a}\omega - b\overline{\omega})$$

and

$$\mu(L^{-1}) = \frac{-b}{\overline{a}} = \frac{-b}{ae^{-i2\mathrm{Arg}(a)}} = -\mu(L)e^{i2\mathrm{Arg}(a)},$$

that gives rise to

$$K(L^{-1}) = K(L).$$

Now, if we consider $L_j(z) = a_j z + b_j \overline{z}$ ($j \in \{1, 2\}$) with Beltrami coefficient μ_j and dilatation K_j,

$$(L_1 \circ L_2)(z) = (a_1 a_2 + b_1 \overline{b_2})z + (a_1 b_2 + b_1 \overline{a_2})\overline{z}$$

and[3]

$$\mu(L_1 \circ L_2) = \frac{b_2 + \mu_1 \overline{a_2}}{a_2 + \mu_1 \overline{b_2}} = \frac{\mu_2 + \mu_1 e^{-i2\mathrm{Arg}(a_2)}}{1 + \mu_1 \mu_2 e^{-i2\mathrm{Arg}(b_2)}}.$$

Moreover, by the definition of dilatation, $K(L_1 \circ L_2) \leq K_1 K_2$.

1.2 Almost complex structures and pullbacks

Definition 1.2.1. (Tangent bundle) *Let $U \subset \mathbb{C}$ be an open subset. The collection of the tangent spaces over points $u \in U$ seen as copies of $\mathbb{C}_{\mathbb{R}}$ is said to be the tangent bundle over U and is denoted by $TU = \bigcup_{u \in U} T_u U$, where $T_u U$ is the tangent space of U at u.*

Definition 1.2.2. (Almost complex structure) *An almost complex structure σ on U is a measurable field of infinitesimal ellipses $E \subset TU$. This means an ellipse $E_u \subset T_u U$ defined up to scaling for almost every point $u \in U$, such that the map $u \mapsto \mu(u) := \mu(E_u)$ from U to $\mathbb{D}$ is Lebesgue measurable.*

Remark 1.2.3. Notice that any measurable function $\mu : U \to \mathbb{D}$ defines an almost complex structure.

For any $u \in U$, the conformal structure defined by E_u on $T_u U$ is denoted by $\sigma(u)$ and is a $\mathbb{C}$-vector space.

Definition 1.2.4. (Dilatation of an almost complex structure) *The dilatation of σ is the essential supremum $K(\sigma) := \sup_{u \in U} K(u) \in [1, \infty]$, where $K(u) := K(E_u) = \frac{1+|\mu(u)|}{1-|\mu(u)|}$.*

From now on, given two open subsets $U, V \subset \mathbb{C}$, $D^+(U, V)$ will denote the set of continuous functions $f : U \to V$ such that:

- f is orientation preserving;
- f is $\mathbb{R}$-differentiable almost everywhere;
- $D_u f : T_u U \to T_{f(u)} V$ is non-singular almost everywhere.

For this kind of functions, we can write $D_u f = \partial_z f(u) dz + \partial_{\overline{z}} f(u) d\overline{z}$, that defines an infinitesimal ellipse in $T_u U$ for all $u \in U$ such that f is differentiable. What we obtain is a field of infinitesimal ellipses in $T_u U$ with a measurable Beltrami coefficient $\mu_f(u) = \partial_{\overline{z}} f(u) / \partial_z f(u)$ (quotient of measurable functions) and dilatation $K_f(u) := K(D_u f)$, which defines an almost complex structure on U that we will denote by σ_f. Notice that the [4]Cauchy-[5]Riemann equation $\partial_{\overline{z}} f(u) = 0$ implies that an analytic map leads to a zero Beltrami coefficient, so that the ellipse is a circle.

[3] $\overline{a} = ae^{-i2\mathrm{Arg}(a)}$

[4] Augustin Louis Cauchy: 1789 – 1857

[5] Georg Friedrich Bernhard Riemann: 1826 – 1866

Definition 1.2.5. (Pullback) *We say that σ_f is the pullback of σ_0 by f and μ_f is the pullback of $\mu_0 \equiv 0$ by f, and we denote them by $\mu_f(u) = f^*\mu_0(u)$ and $\sigma_f(u) = f^*\sigma_0(u)$ for almost every $u \in U$.*

Remark 1.2.6. Since $D_u f$ is not necessarily continuous with respect to u, neither do the field of infinitesimal ellipses nor μ_f.

We define $D_0^+(U,V)$ as the functions in $D^+(U,V)$ that are absolutely continuous with respect to the Lebesgue measure (see Definition A.2.6). This allows us to generalize the pullback when we do not necessarily have a non-standard almost complex structure. Indeed, given $f \in D_0^+(U,V)$ and an almost complex structure σ on V with ellipses $E_v \subset T_v V$ defined for almost every $v \in V$, the ellipses $E'_u := (D_u f)^{-1}(E_{f(u)})$ are well defined for almost every $u \in U$. Actually, they are not well defined for these u such that $E_{f(u)}$ is not defined and $D_u f$ does not exist or is singular, which consists of a set of measure zero because of the absolute continuity of f with respect to the Lebesgue measure.

This gives us such a generalization of the pullback and we will write $(U,\mu_1) \xrightarrow{f} (V,\mu_2)$ to indicate simultaneously $f : U \to V$ and $f^*\mu_2 = \mu_1$.

The composition can be deduced easily. Let $g \in D^+(V,W)$ and $f \in D_0^+(U,V)$, where $U,V,W \subset \mathbb{C}$ are open subsets. Then,

$$f^*\mu_g = f^*(g^*\mu_0) = (g \circ f)^*\mu_0 = \mu_{g\circ f}.$$

Using results seen above,

$$K_{g\circ f} \leq K_f K_g$$

and

$$\begin{aligned}\mu_{g\circ f}(u) = f^*\mu_g(u) = \frac{\partial_{\bar z} f(u) + \mu_g(f(u))\overline{\partial_z f(u)}}{\partial_z f(u) + \mu_g(f(u))\overline{\partial_{\bar z} f(u)}} = \\ = \frac{\mu_f(u) + \mu_g(f(u))e^{-i2\mathrm{Arg}(\partial_z f(u))}}{1 + \mu_f(u)\mu_g(f(u))e^{-i2\mathrm{Arg}(\partial_{\bar z} f(u))}}.\end{aligned} \tag{1.1}$$

Remark 1.2.7. If f is holomorphic, the Cauchy-Riemann equation $\partial_{\bar z} f(u) = 0$ implies that $f^*\mu_g(u) = \mu_g(f(u))\frac{\overline{\partial_z f(u)}}{\partial_z f(u)}$. So, if f is holomorphic, then $|\mu_g(f(u))| = |f^*\mu_g(u)|$. Observe that a bounded Beltrami coefficient in V leads to a bounded Beltrami coefficient in U by the same bound.

Definition 1.2.8. (Invariant almost complex structure) *Let $U \subset \mathbb{C}$ be an open subset, $f \in D_0^+(U,U)$ and σ an almost complex structure on U with Beltrami coefficient μ. Then, μ (or σ) is said to be f-invariant if and only if $f^*\mu(u) = \mu(u)$ (or $f^*\sigma = \sigma$) for almost every $u \in U$. Equivalently, we say that f is holomorphic with respect to μ or σ.*

Proposition 1.2.9. *Let $f \in D_0^+(U,V)$, $F \in D_0^+(V,V)$ and $G \in D_0^+(U,U)$ with the following commutative diagram:*

$$\begin{tikzcd} U \arrow[r, "G"] \arrow[d, "f"'] & U \arrow[d, "f"] \\ V \arrow[r, "F"] & V \end{tikzcd}$$

Let σ be an F-invariant almost complex structure on V with Beltrami coefficient μ. Then, $f^*\sigma$ is G-invariant.

Proof.

$$G^*(f^*\mu) = (f \circ G)^*\mu = (F \circ f)^*\mu = f^*(F^*\mu) = f^*\mu$$

□

Remark 1.2.10. If f^{-1} is absolutely continuous with respect to the Lebesgue measure, then it makes sense to push forward; $f_* := (f^{-1})^*$.

Now, we will introduce the concept of pullback and symmetries for orientation reversing mappings (e.g. antiholomorphic functions: see Remark A.3.4). In these definitions it is implicit that $U, V \subset \mathbb{C}$ are open subsets and $f \in D_0^-(U, V)$, where D_0^- is defined analogously to D_0^+ but for orientation reversing maps. Besides, we will use the antiholomorphic (thus, orientation reversing) functions $c(z) := \overline{z}$ and $\tau(z) := 1/\overline{z}$.

Definition 1.2.11. (Pullback for orientation reversing maps) *The pullback of μ under f is defined as $f^{\circledast}\mu := \overline{f}^*\overline{\mu}$. In particular, $f^{\circledast}\mu_0 := \mu_{\overline{f}}$ and $K_f := K_{\overline{f}}$.*

Let us show the excuse that makes sense to define it in this manner. Let $a, b \in \mathbb{C}$ such that $|b| > |a|$ and let $A(z) := az + b\overline{z}$ be an $\mathbb{R}$-linear orientation reversing map. Given an ellipse E such that $A(E) = \mathbb{S}^1$, we define $\mu(A) := \mu(E)$ and $K(A) := K(E)$, and the orientation preserving map $L(z) := \overline{A(z)} = \overline{b}z + \overline{a}\overline{z}$ $\left(|\overline{b}| = |b| > |a| = |\overline{a}|\right)$. As $L(E) = \overline{A(E)} = \mathbb{S}^1$, it implies

$$\mu(A) := \mu(E) = \mu(L) = \frac{\overline{a}}{\overline{b}} = \overline{\left(\frac{a}{b}\right)}$$

and

$$K(A) := K(E) = K(L) = \frac{|\overline{b}| + |\overline{a}|}{|\overline{b}| - |\overline{a}|} = \frac{|b| + |a|}{|b| - |a|}.$$

Now, if we consider two open sets $U, V \subset \mathbb{C}$, $f \in D^-(U, V)$ and $E_u := (\overline{D_u f})^{-1}(\mathbb{S}^1) \subset T_u U$ for almost all $u \in U$, then

$$\mu(E_u) = \overline{\left(\frac{\partial_z f(u)}{\partial_{\overline{z}} f(u)}\right)} = \frac{\overline{\partial_z f(u)}}{\overline{\partial_{\overline{z}} f(u)}} = \frac{\partial_{\overline{z}} \overline{f}(u)}{\partial_z \overline{f}(u)} = \mu_{\overline{f}}(u)$$

and

$$K(E_u) = K_{\overline{f}}(u),$$

where the third equality is obtained from

$$\begin{cases} \partial_z = \partial_z x \partial_x + \partial_z y \partial_y = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\overline{z}} = \partial_{\overline{z}} x \partial_x + \partial_{\overline{z}} y \partial_y = \frac{1}{2}(\partial_x + i\partial_y) \end{cases} \implies \begin{cases} \overline{\partial_z f} = \partial_{\overline{z}} \overline{f} \\ \overline{\partial_{\overline{z}} f} = \partial_z \overline{f} \end{cases}.$$

Here finish the explanation that leads to the definition of pullback of the standard form by an orientation reversing map. Now, for the general case, we must consider the composition of two orientation reserving linear maps $A_j(z) := a_j z + b_j \overline{z}$ ($a_j, b_j \in \mathbb{C}$, $|b_j| > |a_j|$ and $j \in \{1, 2\}$). Thus,

$$(A_1 \circ A_2)(z) = (a_1 a_2 + b_1 \overline{b_2})z + (a_1 b_2 + b_1 \overline{a_2})\overline{z}$$

is orientation preserving and, if we denote $\mu_1 := \mu(A_1) = \overline{(a_1/b_1)}$,

$$\mu(A_1 \circ A_2) = \frac{a_1 b_2 + b_1 \overline{a_2}}{a_1 a_2 + b_1 \overline{b_2}} = \frac{\frac{a_1}{b_1} b_2 + \overline{a_2}}{\frac{a_1}{b_1} a_2 + \overline{b_2}} = \frac{\overline{\mu_1} b_2 + \overline{a_2}}{\overline{\mu_1} a_2 + \overline{b_2}}.$$

Besides, $\mu(\overline{E_1}) = \mu(A_1 \circ c) = \overline{\mu_1}$. The second equality is given by the last formula and the first arises from the following equality:

$$\overline{E_1} = \overline{A_1^{-1}(\mathbb{S}^1)} = (c \circ A_1^{-1})(\mathbb{S}^1) = (c^{-1} \circ A_1^{-1})(\mathbb{S}^1) = (A_1 \circ c)^{-1}(\mathbb{S}^1)$$

Here is another result that will be useful when we work with the Jacobian matrix right now:

$$\begin{aligned}(\overline{A_2})^{-1}(\overline{E_1}) &= (c \circ A_2)^{-1}(c \circ A_1^{-1}(\mathbb{S}^1)) = (A_2^{-1} \circ c^{-1} \circ c \circ A_1^{-1})(\mathbb{S}^1) = (A_2^{-1} \circ A_1^{-1})(\mathbb{S}^1) = \\ &= A_2^{-1}(E_1)\end{aligned}$$

Now, if we consider $f \in D^-(U,V)$ and denote by $\mu(v)$ the Beltrami coefficient corresponding to a field of infinitesimal ellipses $(E_v)_{v\in V}$, we obtain

$$(D_u f)^{-1}(E_{f(u)}) = (\overline{D_u F})^{-1}(\overline{E_{f(u)}}) = (D_u \overline{f})^{-1}(\overline{E_{f(u)}}),$$

that with the above arguments justify the general pullback definition for reversing maps. Observe that the second equality follows directly from the definition of conjugate and Jacobian matrix of $f := u + iv$ (where u and v are mappings $\mathbb{C} \mapsto \mathbb{R}$):

$$\overline{D_u f} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \begin{pmatrix} \partial_x u & \partial_y u \\ -\partial_x v & -\partial_y v \end{pmatrix} = D_u \overline{f}$$

Finally, we give a couple of formulas that arise from the definition and can be deduced using (1.1).

$$f^{\circledast}\mu(u) = \overline{f}^{*}\overline{\mu}(u) = \frac{\overline{\partial_z f(u)} + \overline{\mu(f(u))}\partial_{\overline{z}} f(u)}{\overline{\partial_{\overline{z}} f(u)} + \overline{\mu(f(u))}\partial_z f(u)}$$

If f is antiholomorphic, $\partial_z f = \overline{\partial_{\overline{z}}\overline{f}} = 0$ implies

$$f^{\circledast}\mu(u) = \overline{\mu(f(u))}\frac{\partial_{\overline{z}} f(u)}{\overline{\partial_{\overline{z}} f(u)}}.$$

1.3 Quasiconformal mappings

Although there are different equivalent definitions for quasiconformal mappings (e.g. using distributional derivatives, that are beyond the scope of this thesis), we are going to face our problem using the concept of absolute continuity on lines.

Definition 1.3.1. (Absolute continuity on an interval) *Let $f : I \subset \mathbb{R} \to \mathbb{C}$ be a continuous function defined on an interval. We say that f is absolutely continuous on I if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_j |f(b_j) - f(a_j)| < \epsilon$ for every finite sequence of pairwise disjoint subintervals (a_j, b_j) with $a_j, b_j \in I$ and such that $\sum_j |b_j - a_j| < \delta$.*

Definition 1.3.2. (Absolute continuity on lines) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a continuous function. We say that f is absolutely continuous on lines (ACL) if for any family of parallel lines in any disc whose closure is contained in U, f is absolutely continuous on almost all of them.*

The absolute continuity on an interval I of f implies that f is of local bounded variation, that lead to the differentiability of f at almost every point of I (see [LV, III.2.7]). Because of that, if $f \in \mathrm{ACL}(U)$, there exist partial derivatives in the ordinary sense almost everywhere in U. In spite of the fact that this existence does not necessarily implies differentiability of f, the following theorem makes it true under an extra condition.

Theorem 1.3.3. *Given a continuous open mapping $f : U \to V$, if it has partial derivatives $\partial_x f$ and $\partial_y f$ in the ordinary sense almost everywhere, then it is $\mathbb{R}$-differentiable almost everywhere.*

A proof can be found in [Ahl, pp. 17-18, Lemma 1].

Remark 1.3.4. It suffices f to be absolutely continuous along almost every horizontal and almost every vertical line in every rectangle whose closure is contained in U to ensure the existence almost everywhere of $\partial_x f$ and $\partial_y f$. As well, $\partial_z f$ and $\partial_{\overline{z}} f$ are defined almost everywhere. Furthermore, because of [LV, III.3.3], the Jacobian

$$\mathrm{Jac}\, f = |\partial_z f|^2 - |\partial_{\overline{z}} f|^2 = \frac{i}{2}(\partial_x f\, \overline{\partial_y f} - \overline{\partial_x f}\, \partial_y f)$$

is in $L^1_{\mathrm{loc}} := \{f : U \to \mathbb{C} \text{ measurable } \mid f_{|K} \in L^1(K) \text{ for all compact set } K \subset U\}$.

Definition 1.3.5. (Analytic definition of K-quasiconformal mapping) *A mapping $\phi : U \to V$ between two open sets $U, V \subset \mathbb{C}$ is said to be K-quasiconformal with $K \geq 1$ if and only if:*

1. *ϕ is a homeomorphism;*
2. *ϕ is ACL;*
3. *$|\partial_{\bar{z}}\phi| \leq k|\partial_z\phi|$ almost everywhere, where $k := \frac{K-1}{K+1}$.*

Remark 1.3.6. Since ϕ is ACL, using the third point of the definition and Remark 1.3.4, it follows

$$|\partial_{\bar{z}}\phi|^2 \leq k^2|\partial_z\phi|^2 \underset{k\leq 1}{\leq} |\partial_z\phi|^2 = \frac{|\partial_z\phi|^2 - k^2|\partial_z\phi|^2}{1-k^2} \leq \frac{|\partial_z\phi|^2 - |\partial_{\bar{z}}\phi|^2}{1-k^2} = \frac{\operatorname{Jac}\phi}{1-k^2}.$$

and both partial derivatives are in L^2_{loc}.

Remark 1.3.7. Every $\mathcal{C}^1$ homeomorphism ϕ such that $|\partial_{\bar{z}}\phi| \leq k|\partial_z\phi|$ for some $k < 1$ is quasiconformal. Furthermore, if ϕ is a diffeomorphism between compact sets, then ϕ is quasiconformal.

Let us look into the characterization of almost complex structures and pullbacks by a quasiconformal mapping ϕ. Since every homeomorphism is an open map, ϕ is $\mathbb{R}$-differentiable almost everywhere due to Definition 1.3.5 and Theorem 1.3.3. Moreover, the following two results assure that it has a non-singular differential defined almost everywhere (if a proof is desired, see [Ahl, p. 16] and [Ahl, p. 22], respectively).

Theorem 1.3.8. *Given a quasiconformal map ϕ, it maps sets of measure zero to sets of measure zero and*

$$m(\phi(B)) = \int_B \operatorname{Jac}\phi \, dm$$

for every measurable set B, where m denotes the Lebesgue measure.

Corollary 1.3.9. *If ϕ is quasiconformal, then $\partial_z\phi \neq 0$ and $\operatorname{Jac}\phi(z) > 0$ almost everywhere.*

In that case, remembering the Beltrami coefficient expression $\mu_\phi = \partial_{\bar{z}}\phi/\partial_z\phi$, the third point of the last definition gives us the restriction $|\mu_\phi| \leq k < 1$ almost everywhere and hence $||\mu_\phi||_\infty < 1$. So, if we consider the almost complex structure $\phi^*\sigma_0$, we obtain the following:

$$|\mu_\phi| \leq \frac{K-1}{K+1} \underset{K\geq 1}{\Longrightarrow} |\mu_\phi|K + |\mu_\phi| \leq K-1 \implies 1+|\mu_\phi| \leq K(1-|\mu_\phi|) \underset{||\mu||_\infty = k<1}{\Longrightarrow}$$

$$\implies K_\phi := \sup_{u\in U} \frac{1+|\mu_\phi(u)|}{1-|\mu_\phi(u)|} = K$$

Since $||\mu_\phi||_\infty < 1$ implies $K_\phi < \infty$, the last equality shows that K is bounded as the dilatation. Furthermore, K indicates the degree of conformality of ϕ. Indeed, the larger K is, the further the ellipses are from being circles, i.e. the further ϕ is from being conformal.
Notice that we are considering the minimum K (therefore the minimum k) such that ϕ is K-quasiconformal.

We are going to set another equivalent definition of quasiconformal mapping, but now from a geometric point of view.

Definition 1.3.10. (Jordan curve) *A Jordan curve is a plane curve which is topologically equivalent to (a homeomorphic image of) the unit circle, i.e. it is simple and closed.*

Definition 1.3.11. (Jordan domain) *A Jordan domain is the interior of a Jordan curve.*

Definition 1.3.12. (Quadrilateral) *A quadrilateral $Q = Q(z_1, z_2, z_3, z_4)$ is a Jordan domain in $\mathbb{C}$ such that (z_1, z_2, z_3, z_4) is an ordered sequence of boundary points agreeing with the positive orientation of Q. These points are called vertices of Q.*

As it is seen in [LV, p. 15], there exists a conformal map φ such that Q is mapped onto a rectangle, unique up to similarity and mapping vertices to vertices. Thus, the following value is well defined.

Definition 1.3.13. (Conformal modulus of a quadrilateral) *The conformal modulus of Q is defined as* $\operatorname{mod} Q(z_1, z_2, z_3, z_4) := \frac{|\varphi(z_1)-\varphi(z_2)|}{|\varphi(z_2)-\varphi(z_3)|}$.

Definition 1.3.14. (Conformally equivalent quadrilaterals) *Two quadrilaterals Q and Q' are conformally equivalent if and only if they have the same modulus.*

Definition 1.3.15. (Dilatation of a quadrilateral under an orientation preserving homeomorphism) *Given an orientation preserving homeomorphism $\phi : U \to V$ between two domains in $\mathbb{C}$, it sends a quadrilateral Q with $\overline{Q} \subset U$ to another $\phi(Q)$ such that $\overline{\phi(Q)} \subset V$. The dilatation of Q under ϕ is defined as $\frac{\operatorname{mod} \phi(Q)}{\operatorname{mod} Q}$ and the maximal dilatation of ϕ is $K_\phi := \sup_{\overline{Q} \subset U} \frac{\operatorname{mod} \phi(Q)}{\operatorname{mod} Q}$.*

Remark 1.3.16. Since $\operatorname{mod} Q(z_2, z_3, z_4, z_1) = [\operatorname{mod} Q(z_1, z_2, z_3, z_4)]^{-1}$, by rearranging vertices we can assure that $K_\phi \geq 1$.

Definition 1.3.17. (First geometric definition of K-quasiconformal mapping) *Given two domains $U, V \subset \mathbb{C}$ and $K \geq 1$, a mapping $\phi : U \to V$ is K-quasiconformal if and only if ϕ is an orientation preserving homeomorphism such that*

$$\frac{1}{K} \operatorname{mod} Q \leq \operatorname{mod} \phi(Q) \leq K \operatorname{mod} Q$$

for all quadrilaterals Q with $\overline{Q} \subset U$.

Remark 1.3.18. Dividing by $\operatorname{mod} Q$ and considering Remark 1.3.16, the last condition can be written as $K_\phi \leq K$.

A proof that shows the equivalence between both definitions of K-quasiconformal mapping given so far (Definition 1.3.5 and Definition 1.3.17) can be found in [Ahl, pp. 20-21].

We can write the last definition in terms of annuli instead of quadrilaterals. In a similar way as we asserted for quadrilaterals, given an open annulus $A \subset \mathbb{C}$ (i.e. a doubly connected domain), there exists a conformal mapping φ so that it maps A onto a standard annulus $\mathbb{A}_{r,R} := \{z \in \mathbb{C} \mid 0 \leq r < |z| < R \leq \infty\}$ (unique up to multiplication by a real constant).

Definition 1.3.19. (Conformal modulus of an annulus) *The conformal modulus of A is defined as*

$$\mathit{mod}\, A := \operatorname{mod} \mathbb{A}_{r,R} := \begin{cases} \frac{1}{2\pi} \log \frac{R}{r}, & \text{if } r > 0 \text{ and } R < \infty \\ \infty, & \text{if } r = 0 \text{ or } R = \infty \end{cases}.$$

Definition 1.3.20. (Second geometric definition of K-quasiconformal mapping) *Given two domains $U, V \subset \mathbb{C}$ and $K \geq 1$, a mapping $\phi : U \to V$ is K-quasiconformal if and only if ϕ is an orientation preserving homeomorphism such that*

$$\frac{1}{K} \operatorname{mod} A \leq \operatorname{mod} \phi(A) \leq K \operatorname{mod} A$$

for all annuli A with $\overline{A} \subset U$.

The equivalence between both geometric definitions is straightforward taking into account that the exponential $z \mapsto e^z$ maps the rectangle $(0, \log R, \log R + 2\pi i, 2\pi i)$ to the standard annulus $\mathbb{A}_{1,R}$ and identifying points on the horizontal edges of the rectangle. The modulus of both identified objects (rectangle and annulus) are the same.

Remark 1.3.21. (Basic properties of quasiconformal mappings) It is trivial from the geometric definitions that given a K-quasiconformal map, its inverse is K-quasiconformal (see Remark 1.3.16). Besides, any composition on the left or right with a conformal mapping is K-quasiconformal; in other words, the class K-quasiconformal mappings is invariant under conformal mappings. Indeed, it also follows from the definition since in Definition 1.3.20 the quotient R/r is unique. Also, the composition of a couple of quasiconformal mappings with constants K_1 and K_2 is K_1K_2-quasiconformal.

Theorem 1.3.22. (Weyl's Lemma) *If ϕ is 1-quasiconformal, then ϕ is conformal. Equivalently, if ϕ is quasiconformal and $\partial_{\bar{z}}\phi = 0$ almost everywhere, then ϕ is conformal.*

It can be found a short proof in terms of distributions theory in [Hub, p. 115, Theorem 4.1.6].

Remark 1.3.23. The converse is not true in general. For instance, consider the exponential map e^z, which is conformal on $\mathbb{C}$, and recall that a quasiconformal mapping must be a homeomorphism.

The following result is given in [Hub, Propositions 4.2.7 and 4.9.9] and will be useful further on.

Theorem 1.3.24. (Quasiconformal removability of quasiarcs) *Let $\Gamma \subset \mathbb{C}$ be a quasiarc (the image of a straight line under a quasiconformal mapping) and let $\phi : U \to V$ be a homeomorphism. If ϕ is K-quasiconformal on $U \setminus \Gamma$, then it is K-quasiconformal on U. Equivalently, we say that Γ is quasiconformally removable. In particular, points, lines and smooth arcs are quasiconformally removable.*

Despite the fact that we are going to focus on the complex plane, let us briefly see that we can generalize these concepts on Riemann surfaces (e.g the Riemann sphere $\hat{\mathbb{C}}$).

Definition 1.3.25. (Surface) *A surface S is a Hausdorff connected topological space with a collection of maps (φ_j, U_S^j) such that:*

1. *The sets U_S^j form an open cover of S;*
2. *Every φ_j is a homeomorphism of U_S^j onto $\varphi_j(U_S^j)$, where $\varphi_j(U_S^j)$ is an open subset of $\mathbb{C}$.*

Each homeomorphism is referred as chart and the collection of charts is called atlas.

Definition 1.3.26. (Riemann surface) *A surface S is a Riemann surface if, in addition, the transition maps*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_S^i \cap U_S^j) \to \mathbb{C}$$

are conformal whenever $U_S^j \cap U_S^i \neq \emptyset$.

Theorem 1.3.27. (Uniformization Theorem) *Every simply connected Riemann surface is conformally equivalent to $\mathbb{D}$, $\mathbb{C}$ or $\hat{\mathbb{C}}$.*

Remark 1.3.28. (Riemann Mapping Theorem - Uniformization Theorem on the plane) In particular, if U is a non-trivial simply connected subset of $\mathbb{C}$ (neither $\emptyset$ nor $\mathbb{C}$), then it is conformally equivalent to $\mathbb{D}$ (see [Mar, p. 347]). We will refer to Uniformization Theorem irrespective for the result concerning to Riemann surfaces and the one of the plane. The corresponding isomorphisms are usually called Riemann maps.

Definition 1.3.29. (Quasiconformal mapping between Riemann surfaces) *We say that a homeomorphism $\phi : S \to S'$ between Riemann surfaces is quasiconformal if there exists some $K \geq 1$ such that ϕ is locally K-quasiconformal when expressed in charts.*

The following definition is based on [BCM, pp. 139-140].

Definition 1.3.30. (Beltrami form) *A collection $\mu = \{\mu_j\}_j$ of measurable functions $\mu_j : U_S^j \to \mathbb{C}$ associated to charts (U_S^j, φ_j) of a Riemann surface S is a Beltrami form on S if given two overlapping charts φ_1 and φ_2 with conformal transition map $h := \varphi_2 \circ \varphi_1^{-1}$ it is satisfied the transformation rule*

$$\mu_1(z_1) = \mu_2(z_2)\frac{\overline{\partial_z h(z_1)}}{\partial_z h(z_1)}, \tag{1.2}$$

where $z_1 = \varphi_1(s)$ and $z_2 = \varphi_2(s)$ with $s \in S$.

Remark 1.3.31. Notice that $|\mu_1(z)| = |\mu_2(z)|$, so that $||\mu||_\infty$ is well defined.

Definition 1.3.32. (Pullback of a Beltrami form) *Given a quasiconformal mapping $\phi : S \to S'$ between two Riemann surfaces and a Beltrami form μ' on S', the Beltrami form $\phi^*\mu'$ on S is defined such that the previous pullback definition is fulfilled when expressed in charts.*

1.4 Measurable Riemann Mapping Theorem

Up to now, we have seen how a quasiconformal map $\phi : U \to \mathbb{C}$ defined on an open set prompts an almost complex structure σ_ϕ on U with a Beltrami coefficient $\mu_\phi = \partial_{\bar{z}}\phi/\partial_z\phi$ defined almost everywhere such that $||\mu_\phi||_\infty = k < 1$. Now we want to find a kind of reciprocal. This means, given a Beltrami coefficient μ on a certain domain $U \subset \mathbb{C}$, how must be (if it exists) a quasiconformal mapping $\phi : U \to \mathbb{C}$ to induce μ almost everywhere, i.e. such that

$$\partial_{\bar{z}}\phi(z) = \mu(z)\partial_z\phi(z) \qquad \text{(Beltrami equation)}$$

for almost every $z \in U$. It is said that ϕ is an integrating map that integrate μ. The main result behind this issue is the theorem which gives this section its name and is also known as Integrability Theorem.

Theorem 1.4.1. (Local Integrability Theorem) *Let $U \subset \mathbb{C}$ be an open set conformally equivalent to $X \in \{\mathbb{D}, \mathbb{C}\}$ and consider an almost complex structure on U with Beltrami coefficient μ such that $||\mu||_\infty = k < 1$. Then, there exists a quasiconformal homeomorphism $\phi : U \to X$ so that*

$$\mu = \phi^*\mu_0$$

almost everywhere. Moreover, ϕ is unique up to post-composition with automorphisms of X.

Although this result is a cornerstone in quasiconformal theory, the proof requires a solid background of theory of distributions that is not explained in this thesis. Even so, the particular case of a real analytic Beltrami coefficient can be faced quite easily by means of first integrals as it is done in [Hub, pp. 149-153]. Then, it is used immediately after to obtain the general case stated here (Theorem 1.4.1) giving prominence to distributions theory.

Theorem 1.4.2. (Global Integrability Theorem) *Let S be a simply connected Riemann surface and σ be an almost complex structure on S with measurable Beltrami form μ such that $||\mu||_\infty = k < 1$. Then, there exists a quasiconformal homeomorphism $\psi : S \to X$ so that*

$$\mu = \psi^*\mu_0,$$

where $X \in \{\mathbb{D}, \mathbb{C}, \hat{\mathbb{C}}\}$. If S is isomorphic to X, then ψ is unique up to post-composition with automorphisms of X.

Proof.

- Existence.
  - Suppose $X \in \{\mathbb{D}, \mathbb{C}\}$. By the Uniformization Theorem, there exists a conformal equivalence $\phi : S \to X$ so that for every chart $\varphi : U_S \subset S \to U \subset \mathbb{C}$ in a given atlas for S, $\phi \circ \varphi^{-1} : U \to X$ is conformal. Considering the whole atlas and recalling that the conformal equivalence is actually a biholomorphism, we can push forward the Beltrami form μ on S to a Beltrami form μ' on X, which is well defined remembering the transformation rule (1.2) for overlapping charts. Moreover, by Remark 1.2.7 $||\mu'||_\infty = ||\mu||_\infty = k < 1$ and the Local Integrability Theorem provides a quasiconformal mapping $\xi : X \to X$ such that $\mu' = \xi^*\mu_0$. Hence, the quasiconformal map $\xi \circ \phi : S \to X$ leads to $\mu = (\xi \circ \phi)^*\mu_0$. In fact, being more accurate (see Definition 1.3.32) the last equality should be written in terms of the chart: $(\xi \circ \phi \circ \varphi^{-1})^*\mu_0 = (\phi \circ \varphi^{-1})^*\xi^*\mu_0 = (\phi \circ \varphi^{-1})^*\mu' = \mu$.
  - Assume $X = \hat{\mathbb{C}}$. Let $\{\varphi_j : U_S^j \to U^j\}$ be a finite atlas for S such that $U^j \subset \mathbb{C}$ is conformally equivalent to $\mathbb{D}$, i.e. such that every U^j is a simply connected open subset of the complex plane that is not trivial. Note that it can be considered a finite atlas due to the compactness of $\hat{\mathbb{C}}$. By the definition of Beltrami form (see Definition 1.3.30), μ induces compatible Beltrami coefficients μ_j with essential supremum $||\mu_j||_\infty = k < 1$ on U^j. Applying the Local

Integrability Theorem we have K_j-quasiconformal maps $\phi_j : U^j \to \mathbb{D}$ such that $\mu_j = \phi_j^* \mu_0$. Since the maps $\phi_j \circ \varphi_j \circ \varphi_i^{-1} \circ \phi_i^{-1}$ are quasiconformal (composition of the quasiconformal maps ϕ_j and ϕ_i^{-1}, and the conformal transition $\varphi_j \circ \varphi_i^{-1}$) and

$$\begin{aligned}(\phi_j \circ \varphi_j \circ \varphi_i^{-1} \circ \phi_i^{-1})^* \mu_0 &= (\phi_i^{-1})^* (\varphi_j \circ \varphi_i^{-1})^* \phi_j^* \mu_0 = (\phi_i^{-1})^* (\varphi_j \circ \varphi_i^{-1})^* \mu_{\phi_j} = \\ &= (\phi_i^{-1})^* (\varphi_j \circ \varphi_i^{-1})^* \mu_{\phi_i} = (\phi_i^{-1})^* \mu_{\phi_i} = (\phi_i)_* \mu_{\phi_i} = \mu_0,\end{aligned}$$

they are conformal wherever are well defined due to Weyl's Lemma (see Theorem 1.3.22). Therefore, $\{\phi_j \circ \varphi_j : U_S^j \to \mathbb{D}\}$ is a new atlas for S with conformal transition maps. Applying the Uniformization Theorem, we have a conformal isomorphism $\psi : S \to \hat{\mathbb{C}}$, i.e. such that $\psi_j := \xi_j \circ \psi \circ \varphi_j^{-1} \circ \phi_j^{-1} : \mathbb{D} \to V_j$ is conformal, where $\xi_j : \psi(U_S^j) \to V_j$ is a chart on $\hat{\mathbb{C}}$. Since ϕ_j is K_j-quasiconformal and ψ_j is conformal, the composition $\phi_j \circ \psi_j : U_j \to V_j$ is K_j-quasiconformal. Thus, defining $K := \max_j K_j$, $\phi_j \circ \psi_j$ is K-quasiconformal for all j and hence $\psi : S \to \hat{\mathbb{C}}$ is quasiconformal. Besides, $\mu = \psi^* \mu_0$. Indeed, recalling the notion of pullback of a Beltrami form (see Definition 1.3.32),

$$(\xi_j \circ \psi \circ \varphi_j^{-1})^* \mu_0 = (\psi_j \circ \phi_j)^* \mu_0 = \phi_j^* \psi_j^* \mu_0 = \phi_j^* \mu_0 = \mu_j,$$

where the third equality follows from the conformality of ψ_j. This finishes the existence because μ_j was defined as the induced Beltrami coefficient by μ (that is actually μ).

- Uniqueness. Let ψ_1 and ψ_2 be two different integrating quasiconformal maps. Then, $\psi_1^* \mu_0 = \psi_2^* \mu_0$ almost everywhere. Since $(\psi_1 \circ \psi_2^{-1})^* \mu_0 = \mu_0$ almost everywhere, by Weyl's Lemma $\psi_1 \circ \psi_2^{-1}$ is conformal. Besides, $\psi_1 = (\psi_1 \circ \psi_2^{-1}) \circ \psi_2$ and we are done.

$\square$

Remark 1.4.3. The total uniqueness can be achieved by giving a normalization of the integrating map ϕ, namely, the quasiconformal mapping obtained via the Integrability Theorem. This is done when we set two or three distinct points $s_1, s_2, s_3 \in S$ so that:

- If $X = \mathbb{D}$: $\phi(s_1) = 0$ and $\phi(s_2) \in \mathbb{R}^+$;
- If $X = \mathbb{C}$: $\phi(s_1) = 0$ and $\phi(s_2) = 1$;
- If $X = \hat{\mathbb{C}}$: $\phi(s_1) = 0$, $\phi(s_2) = 1$ and $\phi(s_3) = \infty$.

This remark is justified by some results that we will not prove here and give us the form of the automorphisms for each case of X:

- Every automorphism of $\mathbb{D}$ is of the form $f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, where $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$;
- Every automorphism of $\mathbb{C}$ is of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$;
- Every automorphism of $\hat{\mathbb{C}}$ is of the form $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$.

1.5 Quasiregular mappings

The main purpose of this section is to introduce the concept of quasiregular map, that is more general than the quasiconformal one. Let us give three equivalent definitions.

Definition 1.5.1. (Quasiregular mapping) *Let $U \subset \mathbb{C}$ be an open set and $K < \infty$. Then:*

1. *A mapping $g : U \to \mathbb{C}$ is K-quasiregular if and only if $g = f \circ \phi$ for some K-quasiconformal map $\phi : U \to \phi(U)$ and for some holomorphic map $f : \phi(U) \to g(U)$.*

2. *A continuous mapping $g : U \to \mathbb{C}$ is K-quasiregular if and only if g is locally K-quasiconformal except at a discrete set of points.*

3. *A mapping $g : U \to \mathbb{C}$ is K-quasiregular if and only if for every $z \in U$ there exist neighbourhoods of z and $g(z)$ denoted by N_z and $N_{g(z)}$ respectively, a K-quasiconformal mapping $\psi : N_z \to \mathbb{D}$ and a conformal mapping $\varphi : N_{g(z)} \to \mathbb{D}$ such that $(\varphi \circ g \circ \psi^{-1})(z) = z^d$, for some $d \geq 1$.*

Remark 1.5.2. From the first definition, g is locally K-quasiconformal except at the discrete set of points that are preimages by ϕ of critical points of f.

Remark 1.5.3. From the second definition it is obvious that every K-quasiconformal mapping is also K-quasiregular.

Theorem 1.5.4. *The three definitions of K-quasiregular mapping are equivalent.*

Proof.

- $(1) \implies (2)$. It follows immediately from Remark 1.5.2.

- $(2) \implies (1)$. Let Ω be the set of points for which g is not K-quasiconformal. By Proposition A.4.3, we can cover the open set $U \setminus \Omega$ with a countable collection of sets on which g is K-quasiconformal. By Remark 1.3.6, $\partial_z g$ and $\partial_{\bar{z}} g$ are well defined almost everywhere in each set and are in L^2_{loc}. Hence, the partial derivatives are well defined almost everywhere in U because a countable union of sets of measure zero is a set of measure zero. Furthermore, $||\mu||_\infty \leq k = \frac{K-1}{K+1} < 1$ in U, where $\mu = \partial_{\bar{z}} g / \partial_z g$.

  – If U is simply connected, by the Local Integrability Theorem, there exists a K-quasiconformal homeomorphism $\phi : U \to \mathbb{D}$ (or onto $\mathbb{C}$) integrating μ.

  – If U is not simply connected, we define

  $$\hat{\mu}(z) := \begin{cases} \mu(z), & z \in U \\ 0, & z \in \mathbb{C} \setminus U \end{cases}.$$

  Applying the Global Integrability Theorem, there exists a K-quasiconformal homeomorphism $\hat{\phi} : \mathbb{C} \to \mathbb{C}$ integrating $\hat{\mu}$ and we consider $\phi := \hat{\phi}_{|U} : U \to \hat{\phi}(U)$.

  Now, $f := g \circ \phi^{-1}$ is locally K-quasiconformal, except at the discrete set $\phi(\Omega)$ (where it is continuous) and $f^*\mu_0 = (\phi^{-1})^* g^* \mu_0 = (\phi^{-1})^* \mu = \mu_0$, so by Weyl's Lemma f is locally conformal except at $\phi(\Omega)$. Since f is continuous at this discrete set of points, applying Corollary A.3.5 f is holomorphic and we have finished because $g = f \circ \phi$.

- $(3) \implies (2)$. Let z be a point of U and let us argue depending on its d value.

  – If $d = 1$ for the picked point, $(\varphi \circ g \circ \psi^{-1})(z) = z$. Then $g = \varphi^{-1} \circ \psi$, where φ^{-1} is locally conformal by Corollary A.3.11. Therefore, g is locally K-quasiconformal.

  – If $d > 1$ on N_z, then z is the only point in this neighbourhood where g is not locally invertible due to the equality $\varphi \circ g \circ \psi^{-1} = P_d$, where $P_d(z) := z^d$. For the other points, $g = \varphi^{-1} \circ P_d \circ \psi$ is locally K-quasiconformal because $\varphi^{-1} \circ P_d$ is locally conformal and ψ is K-quasiconformal.

- $(1) \implies (3)$. Suppose $g = f \circ \phi : U \to \mathbb{C}$ satisfying the first definition.

  – If $\phi(z)$ is not a critical point of f, i.e. $f'(\phi(z)) \neq 0$, applying Corollary A.3.11 we obtain two neighbourhoods $N_{\phi(z)}$ and $N_{g(z)} \equiv N_{f(\phi(z))}$ of $\phi(z)$ and $g(z)$ respectively such that $f : N_{\phi(z)} \to N_{g(z)}$ is invertible with a conformal inverse $\varphi := f^{-1}$. Then, denoting $N_z =$

$\phi^{-1}(N_{\phi(z)})$, the following diagram commutes and $(\varphi \circ g \circ \phi^{-1})(z) = (\varphi \circ f \circ \phi \circ \phi^{-1})(z) = z^1 = z$.

$$\begin{tikzcd} N_z \arrow[r, "g"] \arrow[d, "\phi"'] & N_{g(z)} \arrow[d, "\varphi:=f^{-1}"] \\ N_{\phi(z)} \arrow[r, "P_1(z)=z^1"] & N_{\phi(z)} \end{tikzcd}$$

If $N_{\phi(z)} \subset \mathbb{D}$ we are done and otherwise we just must apply the Uniformization Theorem to the open proper subset $N_{\phi(z)} \subset \mathbb{C}$, which would be conformally equivalent to $\mathbb{D}$.

– If $f'(\phi(z)) = 0$, as the zeros of a holomorphic function are isolated, there exist neighbourhoods $N_{\phi(z)}$ and $N_{g(z)}$ of $\phi(z)$ and $g(z)$ respectively such that $f : N_{\phi(z)} \to N_{g(z)}$ is a branched covering of degree $d > 1$, ramified only at $\phi(z)$. This means that $f : N_{\phi(z)} \setminus \{\phi(z)\} \to N_{g(z)} \setminus \{g(z)\}$ is a homeomorphism (in fact it is biholomorphic) and the multiplicity of $\phi(z)$ as a preimage of $g(z)$ by f^{-1} is exactly d. Arguing as in Proposition A.3.20 and considering biholomorphisms given by the Uniformization Theorem as charts, say $\varphi : N_{g(z)} \to \mathbb{D}$ and $\widetilde{\varphi} : N_{\phi(z)} \to \mathbb{D}$ such that $\varphi(g(z)) = \widetilde{\varphi}(\phi(z)) = 0$, the following diagram commutes:

$$\begin{tikzcd} N_z \arrow[r, "g"] \arrow[d, "\phi"'] \arrow[dd, bend right=60, "\widetilde{\varphi}\circ\phi"'] & N_{g(z)} \arrow[d, "z"] \arrow[dd, bend left=60, "\varphi"] \\ N_{\phi(z)} \arrow[r, "f"] \arrow[d, "\widetilde{\varphi}"'] & N_{g(z)} \arrow[d, "\varphi"] \\ \mathbb{D} \arrow[r, "P_d(z)=z^d"] & \mathbb{D} \end{tikzcd}$$

We have finished because $\widetilde{\varphi} \circ \phi$ is K-quasiconformal ($\widetilde{\varphi}$ is conformal and ϕ is K-quasiconformal) and φ is conformal.

□

Proposition 1.5.5. (Properties of quasiregular mappings) *Let U, V be open subsets of $\mathbb{C}$.*

1. *If $g_1 : U \to V$ and $g_2 : V \to \mathbb{C}$ are K_1-quasiregular and K_2-quasiregular respectively, then $g_2 \circ g_1$ is K_1K_2-quasiregular.*
2. *A mapping $g : U \to \mathbb{C}$ is holomorphic if and only if g is 1-quasiregular.*
3. *If $f : U \to V$ is holomorphic and $\phi : V \to \mathbb{C}$ is K-quasiconformal, then $\phi \circ f$ is K-quasiregular.*
4. *If in the third definition of quasiregular mapping φ is K'-quasiconformal instead of conformal, then g is KK'-quasiregular.*
5. *If $g : U \to \mathbb{C}$ is quasiconformally conjugate to a holomorphic mapping $f : U \to \mathbb{C}$ (see Definition 2.1.1), then g is quasiregular.*

Proof.

1. It is trivial considering the composition of quasiconformal maps (see Remark 1.3.21) and the second definition of quasiregular map.

2. From the first definition of quasiregular mapping, if g is holomorphic then it is also 1-quasiregular. Again from the first definition, we can write $g = f \circ \phi$, where f is holomorphic and ϕ is 1-quasiconformal. But by Weyl's Lemma ϕ is conformal and therefore g is holomorphic.

3. It follows from the two first properties and Remark 1.5.3.

4. Locally we can write $g = \varphi^{-1} \circ P_d \circ \psi$ with $P_d(z) := z^d$, so that g is KK'-quasiregular bearing in mind the second definition.

5. We can write $g = h \circ f \circ h^{-1}$, where h is the K-quasiconformal conjugacy. Then, g is K^2-quasiregular because $f \circ h^{-1}$ is K-quasiregular and h is K-quasiconformal (and so K-quasiregular).

□

Proposition 1.5.6. (Variant of Weyl's Lemma) *If $g : U \subset \mathbb{C} \to \mathbb{C}$ is a quasiregular map defined on the open set U and $g^*\mu_0 = \mu_0$ almost everywhere in U, then g is holomorphic.*

Proof. By the first definition of quasiregular mapping, we can write $g = f \circ \phi$ for some quasiconformal map ϕ and some holomorphic map f. Hence, as f is holomorphic, $\mu_0 = g^*\mu_0 = \phi^* f^* \mu_0 = \phi^*\mu_0$. Applying Weyl's Lemma (see Theorem 1.3.22) we know that ϕ is then conformal. Therefore, g is holomorphic. □

Lemma 1.5.7. *Given a quasiregular map g, it and each of its inverse branches send sets of measure zero to sets of measure zero.*

Proof. We can write $g = f \circ \phi$, where f is holomorphic and ϕ is quasiconformal. Then, the lemma follows from Theorem 1.3.8, the fact that an injective holomorphic map is quasiconformal and that $\{z \mid f'(z) = 0\}$ has measure zero. □

Remark 1.5.8. The pullback of a Beltrami form defined almost everywhere by a quasiregular map is well defined almost everywhere and we write $(S_1, \mu_1) \xrightarrow{g} (S_2, \mu_2)$ to denote that g is a quasiregular map between the Riemann surfaces S_1 and S_2 so that $g^*\mu_2 = \mu_1$, where μ_j is a Beltrami form on S_j $(j = 1, 2)$.

Lemma 1.5.9. (Key Lemma)

1. *Given a quasiregular map $g : S \to S$ on a Riemann surface S isomorphic to $X \in \{\mathbb{C}, \hat{\mathbb{C}}\}$ and a g-invariant Beltrami form μ on S with $||\mu||_\infty := k < 1$, there exists a holomorphic map $f : X \to X$ such that g and f are quasiconformally conjugate.*

2. *Let $g : S' \to S$ be a quasiregular map from an open subset $S' \subset S$ to a Riemann surface S isomorphic to $\mathbb{D}$. Let μ be a g-invariant Beltrami form on S such that $||\mu||_\infty := k < 1$. Then, there exists a holomorphic map $f : D' \to \mathbb{D}$ where $D' \subset \mathbb{D}$ is open and such that g and f are quasiconformally conjugate.*

Proof. In both cases, consider the quasiconformal mapping given by the Integrability Theorem $\phi : S \to Y$ with $Y \in \{\mathbb{D}, \mathbb{C}, \hat{\mathbb{C}}\}$ and define $f := \phi \circ g \circ \phi^{-1}$. By the second definition of quasiregular mapping, f is quasiregular. Observe that the following diagrams commute:

1.

$$\begin{array}{ccc} (S,\mu) & \xrightarrow{g} & (S,\mu) \\ \downarrow \phi & & \downarrow \phi \\ (X,\mu_0) & \xrightarrow{f} & (X,\mu_0) \end{array}$$

2.

$$\begin{array}{ccc} (S',\mu) & \xrightarrow{g} & (S,\mu) \\ \downarrow \phi & & \downarrow \phi \\ (D' := \phi(S'),\mu_0) & \xrightarrow{f} & (\mathbb{D},\mu_0) \end{array}$$

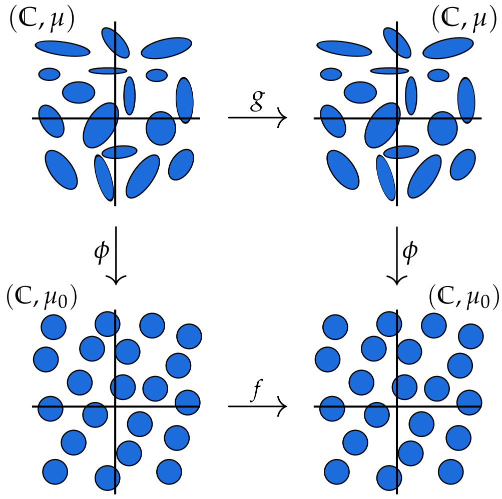


Figure 1.2: Sketch for the Key Lemma when $S = \mathbb{C}$.

As f is quasiregular and $f^*\mu_0 = \mu_0$ almost everywhere, by Proposition 1.5.6 f is holomorphic. □

1.6 Quasisymmetries: surgery

Definition 1.6.1. (Quasisymmetry) *A quasisymmetry is a homeomorphism $h : \mathbb{S}^1 \to h(\mathbb{S}^1) \subset \mathbb{C}$ so that there exists a strictly increasing homeomorphism $\lambda : [0,+\infty) \to [0,+\infty)$ such that*

$$\frac{1}{\lambda\left(\frac{|z_2-z_3|}{|z_1-z_2|}\right)} \leq \frac{|h(z_1)-h(z_2)|}{|h(z_2)-h(z_3)|} \leq \lambda\left(\frac{|z_2-z_3|}{|z_1-z_2|}\right)$$

for all $z_1, z_2, z_3 \in \mathbb{S}^1$.

Remark 1.6.2. One of the inequalities follows from the other one by interchanging the points z_1 and z_3.

Proposition 1.6.3. (Inverse and composition of quasisymmetries)

1. *Given two quasisymmetries $h_j : \mathbb{S}^1 \to h_j(\mathbb{S}^1)$ ($j = 1,2$) such that $h_1(\mathbb{S}^1) = \mathbb{S}^1$, the composition $h_2 \circ h_1$ is also quasisymmetric.*

2. *The inverse function h^{-1} of a quasisymmetry $h : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric.*

Proof.

1. Let $\lambda_j : [0,+\infty) \to [0,+\infty)$ be strictly increasing functions such that

$$\frac{1}{\lambda_j\left(\frac{|z_2-z_3|}{|z_1-z_2|}\right)} \leq \frac{|h_j(z_1)-h_j(z_2)|}{|h_j(z_2)-h_j(z_3)|} \leq \lambda_j\left(\frac{|z_2-z_3|}{|z_1-z_2|}\right)$$

for all $z_1, z_2, z_3 \in \mathbb{S}^1$. Since $h_1(\mathbb{S}^1) = \mathbb{S}^1$,

$$\frac{|(h_2\circ h_1)(z_1)-(h_2\circ h_1)(z_2)|}{|(h_2\circ h_1)(z_2)-(h_2\circ h_1)(z_3)|} \leq \lambda_2\left(\frac{|h_1(z_1)-h_1(z_2)|}{|h_1(z_2)-h_1(z_3)|}\right) \leq (\lambda_2\circ\lambda_1)\left(\frac{|z_1-z_2|}{|z_2-z_3|}\right),$$

where $\lambda_2 \circ \lambda_1$ is strictly increasing because λ_1 and λ_2 are too. The last inequality follows from the fact that h_1 is quasisymmetric and λ_2 is increasing.

2. Now, we are going to use the lower bound of the definition. Taking into account that since h is a homeomorphism (also h^{-1}), every $z = h^{-1}(\omega) \in \mathbb{S}^1$ is identified with some $\omega = h(z) \in \mathbb{S}^1$ and vice versa. Then, as h is quasisymmetric,

$$\frac{1}{\lambda\left(\frac{|h^{-1}(\omega_2)-h^{-1}(\omega_3)|}{|h^{-1}(\omega_1)-h^{-1}(\omega_2)|}\right)} = \frac{1}{\lambda\left(\frac{|z_2-z_3|}{|z_1-z_2|}\right)} \leq \frac{|h(z_1)-h(z_2)|}{|h(z_2)-h(z_3)|} = \frac{|\omega_1-\omega_2|}{|\omega_2-\omega_3|} \implies$$

$$\implies \frac{1}{\frac{|\omega_1-\omega_2|}{|\omega_2-\omega_3|}} \leq \lambda\left(\frac{|h^{-1}(\omega_2)-h^{-1}(\omega_3)|}{|h^{-1}(\omega_1)-h^{-1}(\omega_2)|}\right).$$

As λ^{-1} is an strictly increasing homeomorphism like λ,

$$\left(\frac{1}{\lambda^{-1}\left(\left(\frac{|\omega_1-\omega_2|}{|\omega_2-\omega_3|}\right)^{-1}\right)}\right)^{-1} = \lambda^{-1}\left(\frac{1}{\frac{|\omega_1-\omega_2|}{|\omega_2-\omega_3|}}\right) \leq \frac{|h^{-1}(\omega_2)-h^{-1}(\omega_3)|}{|h^{-1}(\omega_1)-h^{-1}(\omega_2)|}$$

and h^{-1} is quasisymmetric with the strictly increasing homeomorphism $(\lambda^{-1}(1/t))^{-1}$, for every $t > 0$.

□

Definition 1.6.4. (Quasicircle) *A quasicircle is a Jordan curve $\gamma \subset \mathbb{C}$ so that there exists $C > 0$ such that* $\operatorname{diam}\gamma(z_1, z_2) \leq C|z_1 - z_2|$ *for all $z_1, z_2 \in \gamma$, where $\gamma(z_1, z_2)$ is the arc of smallest diameter of γ joining z_1 and z_2.*

Definition 1.6.5. (Quasiannulus) *A quasiannulus is an annulus bounded by two quasicircles.*

Proposition 1.6.6. *If* h *is a quasisymmetry, then $h(\mathbb{S}^1)$ is a quasicircle.*

Proof. We will follow the notations of Definition 1.6.1 and Definition 1.6.4, and we may take into account the identification $\omega = h(z)$, where $\omega \in h(\mathbb{S}^1)$ and $z \in \mathbb{S}^1$. Let us fix two arbitrary points $\omega_1 = h(z_1)$ and $\omega_3 = h(z_3)$ and let $z_2 \in h^{-1}(\gamma(\omega_1, \omega_3))$. Define $z_4 \in \mathbb{S}^1$ as the unique point such that $z_4 \neq z_1$ and $|z_3 - z_1| = |z_3 - z_4|$. Then, $|z_2 - z_3| \leq |z_3 - z_1| = |z_3 - z_4|$ and hence

$$\frac{|z_2-z_3|}{|z_3-z_4|} = \frac{|z_2-z_3|}{|z_3-z_1|} \leq 1 = \frac{|z_4-z_3|}{|z_3-z_1|}.$$

Since λ is strictly increasing,

$$\frac{|\omega_2-\omega_3|}{|\omega_1-\omega_3|} = \frac{|h(z_2)-h(z_3)|}{|h(z_1)-h(z_3)|} = \frac{|h(z_4)-h(z_3)|}{|h(z_3)-h(z_1)|}\frac{|h(z_2)-h(z_3)|}{|h(z_3)-h(z_4)|} \leq (\lambda(1))^2 = \frac{C}{2},$$

where $C := 2(\lambda(1))^2 > 0$. Now, supposing that $\operatorname{diam}\gamma(\omega_1, \omega_3) = |\omega_5 - \omega_6|$ for some $\omega_5, \omega_6 \in \gamma(\omega_1, \omega_3)$, we obtain the required condition:

$$\frac{\operatorname{diam}\gamma(\omega_1,\omega_3)}{|\omega_1-\omega_3|} = \frac{|\omega_5-\omega_6|}{|\omega_1-\omega_3|} \leq \frac{|\omega_5-\omega_3|+|\omega_6-\omega_3|}{|\omega_1-\omega_3|} \leq C$$

□

Remark 1.6.7. It makes sense to refer to quasisymmetries between quasicircles with an analogous definition of Definition 1.6.1. Besides, in some results that we shall see, this proves that a quasisymmetric extension leads to a quasicircle as a boundary.

Let us set the following notation from now on. For any $0 < r < 1 < R$:

$$\begin{aligned}
\mathbb{A}_{r,R} &:= \{z \in \mathbb{C} \mid r < |z| < R\} \\
\mathbb{A}_r &:= \mathbb{A}_{r,1} = \{z \in \mathbb{C} \mid r < |z| < 1\} \\
\mathbb{D}_r &:= \{z \in \mathbb{C} \mid |z| < r\} \\
\mathbb{S}^1_r &:= \{z \in \mathbb{C} \mid |z| = r\}
\end{aligned}$$

We recall a result also used above when we defined quasiconformal mappings in terms of annuli.

Proposition 1.6.8. *Given an annulus A, there exists a conformal isomorphism $\phi : \mathbb{A}_r \to A$ for some $r \in (0,1)$. Besides, r is uniquely determined by the modulus of A.*

The following is an important extension of the Riemann mapping theorem whose statement can be found in [Mar, p. 350] and its name relates to the Greek mathematician [6]C. Carathéodory.

Theorem 1.6.9. (Carathéodory's Theorem) *Given a conformal isomorphism $f : \mathbb{D} \to G$, where $G \subset \mathbb{C}$ is a bounded domain, f has a continuous and injective extension to $\overline{D}$ if and only if ∂G is a Jordan curve.*

Theorem 1.6.10. (Extension of conformal maps on $\mathbb{D}$) *Let $G \subset \mathbb{C}$ be a Jordan domain with associated Jordan curve $\gamma := \partial G$ and let $f : \mathbb{D} \to G$ be a conformal isomorphism. Then:*

1. *f can be extended conformally to some disc of radius $r > 1$ if and only if γ is analytic.*

2. *f has a quasisymmetric extension to $\mathbb{S}^1$ if and only if γ is a quasicircle.*

Proof. We will prove the first part. The second one is shown in [Pom, pp. 109-110] and is known as the Douady-Earle Theorem or the Berling-Ahlfors Theorem, who proved it geometrically and analytically, respectively.

1. Assume $\gamma : \mathbb{S}^1 \to \partial G$ analytic, i.e. such that we can define $\phi : \mathbb{A}_{1/\rho,\rho} \to \mathbb{C}$ for some $\rho > 0$ such that it is conformal and $\phi_{|\mathbb{S}^1} = \gamma$. Then, there exists $r > 1$ such that $\phi^{-1} \circ f : \mathbb{A}_{1/r} \to \mathbb{A}_{1/\rho}$ is conformal. Observe that $\lim_{|z| \to 1} |(\phi^{-1} \circ f)(z)| = 1$, so by [Pom, p. 4, Reflection Principle] $\phi^{-1} \circ f$ can be extended conformally to $\phi^{-1} \circ f : \mathbb{A}_{1/r,r} \to \mathbb{A}_{1/\rho,\rho}$ and hence $f = \phi \circ (\phi^{-1} \circ f)$ is conformal in $\mathbb{A}_{1/r,r}$. Finally, f has been extended so that it is conformal in $\mathbb{D}_r$. The converse is trivial.

□

Corollary 1.6.11. (Extension of conformal maps on annuli) *Consider a conformal equivalence $f : \mathbb{A}_r \to A$ mapped onto a Jordan annulus A and let γ and Γ be the inner and outer boundaries of A, respectively. Then, f has a quasisymmetric extension to $\mathbb{S}^1$ ($\mathbb{S}^1_r$ resp.) if and only if Γ (γ resp.) is a quasicircle.*

Proof. It is going to be proved only for the outer boundary. For the inner one, it is quite similar, but considering the exterior domain of γ in $\hat{\mathbb{C}}$. Let us denote by G^Γ the Jordan domain of the Jordan curve Γ and consider the conformal equivalence $\phi : \mathbb{D} \to G^\Gamma$ given by the Uniformization Theorem. By Carathéodory's Theorem (Theorem 1.6.9), it extends continuously to the boundary as $\phi : \overline{\mathbb{D}} \to \overline{G^\Gamma}$ and the restriction $\phi_{|\mathbb{S}^1}$ has the properties given in the second part of the last theorem. Moreover, $\phi^{-1} \circ f : \mathbb{A}_r \to \phi^{-1}(A)$ is conformal (composition of conformal mappings) and applying [Pom, p. 4, Reflection Principle] it can be extended to an analytic map $(\phi^{-1} \circ f)_{|\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$, so that the composition $\phi_{|\mathbb{S}^1} \circ (\phi^{-1} \circ f)_{|\mathbb{S}^1} : \mathbb{S}^1 \to \Gamma$ is an extension of f with the same properties of $\phi_{|\mathbb{S}^1}$ because $(\phi^{-1} \circ f)_{|\mathbb{S}^1}$ is analytic. □

[6]Constantin Carathéodory: 1873 – 1950

The following proposition ensure the existence of a quasiconformal extension of two quasisymmetries that correspond to the boundaries of a standard annuli. We give only the statement, but a proof can be found in [BF, pp. 83-85], where is introduced the [7]Beurling-[8]Ahlfors extension of a real quasisymmetry to the upper half plane.

Proposition 1.6.12. (Extension of quasisymmetric maps between boundaries of standard annuli) *If $f_1 : \mathbb{S}^1 \to \mathbb{S}^1$ and $f_2 : \mathbb{S}^1_{r_1} \to \mathbb{S}^1_{r_2}$ are two orientation preserving quasisymmetric homeomorphisms for some $0 < r_1, r_2 < 1$, then there exists an extension $f : \overline{\mathbb{A}_{r_1}} \to \overline{\mathbb{A}_{r_2}}$ that is quasiconformal in the interior.*

Proposition 1.6.13. (Extension of quasisymmetric maps between boundaries of quasiannuli) *Let A_j ($j = 1, 2$) be quasiannuli bounded by the quasicircles γ_j (inner boundary) and Γ_j (outer boundary), and let $f_\gamma : \gamma_1 \to \gamma_2$ and $f_\Gamma : \Gamma_1 \to \Gamma_2$ be quasisymmetric maps. Then, there exists a quasiconformal map $f : \overline{A_1} \to \overline{A_2}$ extending f_γ and f_Γ.*

Proof. Consider a conformal isomorphism $\phi_j : \mathbb{A}_{r_j} \to A_j$, where r_j are uniquely determined by the moduli of the annuli. By Corollary 1.6.11, both ϕ_j can be extended continuously to the boundaries through quasisymmetries $\phi_j^\Gamma : \mathbb{S}^1 \to \Gamma_j$ and $\phi_j^\gamma : \mathbb{S}^1_{r_j} \to \gamma_j$. Taking into account Proposition 1.6.3 and Remark 1.6.7,

$$\begin{aligned} \varphi^\gamma &:= (\phi_2^\gamma)^{-1} \circ f_\gamma \circ \phi_1^\gamma : \mathbb{S}^1_{r_1} \to \mathbb{S}^1_{r_2} \\ \varphi^\Gamma &:= (\phi_2^\Gamma)^{-1} \circ f_\Gamma \circ \phi_1^\Gamma : \mathbb{S}^1 \to \mathbb{S}^1 \end{aligned}$$

are quasisymmetries and, by Proposition 1.6.12, there exists a mapping $\varphi : \overline{\mathbb{A}_{r_1}} \to \overline{\mathbb{A}_{r_2}}$ that is quasiconformal in the interior. It follows from Remark 1.3.21 that $f := \phi_2 \circ \varphi \circ (\phi_1)^{-1} : \overline{A_1} \to \overline{A_2}$ is quasiconformal. Moreover, it is an extension of f_γ and f_Γ. Indeed,

$$\begin{aligned} f_{|\gamma_1} &= \phi_2^\gamma \circ \varphi^\gamma \circ (\phi_1^\gamma)^{-1} = f_\gamma \\ f_{|\Gamma_1} &= \phi_2^\Gamma \circ \varphi^\Gamma \circ (\phi_1^\Gamma)^{-1} = f_\Gamma. \end{aligned}$$

□

[7] Arne Carl-August Beurling: 1905 – 1986
[8] Lars Valerian Ahlfors: 1907 – 1996

Chapter 2

Complex dynamics

The main purpose of this chapter is to introduce some concepts in complex dynamics. This means to study how is the behaviour of the iterates of a given point z under a holomorphic function f, i.e. $f^n(z)$ for $n \geq 0$. As a point of departure, we introduce some notions and results concerning general holomorphic functions over Riemann surfaces. In addition, we shall place a strong focus on polynomial dynamics, their filled Julia sets and the basin of attraction of infinity.

Taking advantage of the fact that infinity is a superattracting fixed point of polynomials, we introduce notions of local theory to prompt the Böttcher Theorem. This result is crucial to conjugate polynomials of degree d to the monomial z^d in a neighbourhood of infinity. This enables us to define equipotentials and external rays, which, in turn, endow such neighbourhood of infinity with the notion of escaping velocity of a point. More precisely, we introduce the Green's function of the filled Julia set to quantify this escaping rate.

We end defining the parameter space of the quadratic family $Q_c(z) := z^2 + c$, that is the Mandelbrot set.

Henceforth, we assume $f : S \to S$ holomorphic, where S is a Riemann surface. It can be shown that the only three cases with nontrivial dynamics are the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$ and the punctured plane $\mathbb{C}^*$ (see [Mil, §2, §5 and §6]).

Definition 2.0.1. (Orbit) *Given $z_0 \in S$, the (forward) orbit of z_0 under f is $\mathcal{O}(z_0) = \mathcal{O}^+(z_0) = \{z_0, z_1 := f(z_0), \dots, z_n := f^n(z_0), \dots\}$.*

When dealing with dynamical systems, we have some important points labelled according to their behaviour when iterated.

Definition 2.0.2. (Fixed, periodic, preperiodic and converging points) *Under f, the point z_0 (or its orbit) is said to be*

1. *fixed if $f(z_0) = z_0$;*
2. *p-periodic with period $p > 1$ if $f^p(z_0) = z_0$ and moreover $f^q(z_0) = z_0$ implies $q \geq p$;*
3. *preperiodic if $f^k(z_0)$ is periodic for some $k > 0$, but z_0 is not;*
4. *converging if $f^{np}(z_0) \xrightarrow{n} z^*$ for some $p \geq 1$ and some $z^* \in S$.*

Remark 2.0.3. A p-periodic orbit is also called a p-cycle. In the converging case, z^* is p-periodic if p is minimal with respect to this property.

2.1 Conjugacies and equivalences

Sometimes two different dynamical systems are comparable somehow. In this case, the behaviour of the orbits in both cases are analogous. Since this is not an exclusive property of holomorphic maps, we tackle the following concepts on general topological spaces.

Definition 2.1.1. (Conjugation) *We say that two continuous maps between topological spaces, say $f : X \to X$ and $g : Y \to Y$, are topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

We denote it by $f \underset{top}{\overset{h}{\sim}} g$ or $f \overset{h}{\sim} g$ if we want to specify the conjugation, and just $f \underset{top}{\sim} g$ otherwise. Analogously, depending on the conditions in h, we say $\mathcal{C}^r$-conjugate, linearly conjugate, affine conjugate, etc. If $X, Y \subset \mathbb{C}$, we can also have conformal and quasiconformal conjugation. We say that f and g are semi-conjugate if h is just continuous.

Remark 2.1.2. Conjugate maps preserve orbits and have the same dynamical behaviour because if $f \overset{h}{\sim} g$, then $f^n \overset{h}{\sim} g^n$ for all $n \in \mathbb{N}$.

Definition 2.1.3. (Conjugacy invariants) *A property or a quantity associated to a dynamical system which is preserved under a topological conjugacy is called a topological invariant. It is known in like manner for the other cases of h.*

Lemma 2.1.4. *If $X, Y \subset \mathbb{C}$, f and g are holomorphic, and $h : X \to Y$ is a complex $\mathcal{C}^1$-conjugacy between f and g, then a p-periodic point x_0 is mapped to a p-periodic $y_0 := h(x_0)$ and $(f^p)'(x_0) = (g^p)'(y_0)$.*

Proof. If x_0 is a p-periodic point of f, it follows from Remark 2.1.2 that $y_0 := h(x_0)$ is a p-periodic point of g. Since

$$(h \circ h^{-1})(y) = y \implies h'(h^{-1}(y)) \cdot (h^{-1})'(y) = 1,$$

and $g^p = h \circ f^p \circ h^{-1}$, again by the chain rule we obtain the desired property:

$$\begin{aligned}(g^p)'(y_0) &= h'(f^p(h^{-1}(y_0))) \cdot (f^p)'(h^{-1}(y_0)) \cdot (h^{-1})'(y_0) = h'(f^p(x_0)) \cdot (f^p)'(x_0) \cdot (h^{-1})'(y_0) = \\ &= h'(x_0) \cdot (f^p)'(x_0) \cdot (h^{-1})'(y_0) = h'(h^{-1}(y_0)) \cdot (f^p)'(x_0) \cdot (h^{-1})'(y_0) = (f^p)'(x_0)\end{aligned}$$

□

Definition 2.1.5. (Invariant sets) *A set $U \subset X$ is invariant (or forward invariant) under $f : X \to X$ if $f(U) \subset U$. It is backward invariant if $f^{-1}(U) \subset U$ and totally invariant if $f(U) = U = f^{-1}(U)$.*

Lemma 2.1.6. *Topological conjugacies between f and g map forward (respectively backwards and totally) sets under f to forward (respectively backwards and totally) sets under g.*

Proof. Assume that $U \subset X$ is f-invariant. We want to see that $h(U)$ is g-invariant. Given some $y \in h(U)$, we can write $y = h(x)$ for some $x \in U$. Then, since $g \circ h = h \circ f$,

$$g(y) = g(h(x)) = h(f(x))$$

belongs to $h(U)$ because $f(x) \in U$.

On the other hand, if $f^{-1}(U) \subset U$, the equality $g = h \circ f \circ h^{-1}$ leads to

$$g^{-1}(y) = g^{-1}(h(x)) = (h \circ f^{-1} \circ h^{-1})(h(x)) = h(f^{-1}(x)),$$

so that $g^{-1}(y)$ is in $h(U)$ because $f^{-1}(x) \in U$. □

Definition 2.1.7. (Equivalent mappings) *We say that f and g are topologically equivalent if there exist two homeomorphisms $h_1, h_2 : X \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h_1"'] & X \arrow[d, "h_2"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

2.2 Holomorphic dynamics: the phase space

2.2.1 Basic concepts

Definition 2.2.1. (Normal family of holomorphic maps) *Given a domain $U \subset \hat{\mathbb{C}}$ and a family $\mathcal{F}$ of holomorphic maps $U \to \hat{\mathbb{C}}$, we say that $\mathcal{F}$ is a normal family in U if any infinite sequence of elements of $\mathcal{F}$ contains a subsequence which converges uniformly on compact sets of U.*

Definition 2.2.2. (Fatou and Julia sets) *Let $f : S \to S$ be a holomorphic mapping.*

- $F_f := \{z \in S \mid \{f^n\}_n$ *is normal in a neighbourhood of* $z\}$ *is the Fatou set;*
- $J_f := S \setminus F_f$ *is the Julia set.*

Definition 2.2.3. (Filled Julia set) *If f is a polynomial, then we define the filled Julia set $K_f := \{s \in S \mid f^n(s) \overset{n}{\not\to} \infty\}$.*

Remark 2.2.4. F_f is open and J_f is closed and both are totally invariant. Actually, J_f is the smallest closed set totally invariant under f. Informally, F_f and J_f contain the stable orbits and the chaotic ones, respectively, and both are preserved under topological conjugacies.

Proposition 2.2.5. *Given a holomorphic mapping $f : S \to S$, $J_f = J_{f^p}$ for all $p \in \mathbb{N}$. Equivalently, $F_f = F_{f^p}$.*

Proof. Let $z_0 \in F_f$. Then, $\{f^n\}_n$ is normal in a neighbourhood of z_0. Since $\{f^{np}\}_n \subset \{f^n\}_n$, $\{f^{np}\}_n$ is normal in such a neighbourhood and hence $z_0 \in F_{f^p}$.
Let us see the reciprocal, i.e. consider $z_0 \in F_{f^p}$. Then, $\{f^{np}\}_n$ is normal in a neighbourhood of z_0 and, therefore, $\{f^{np+q}\}_n$ is normal in that neighbourhood (where $q \in \{0, 1, \ldots, p-1\}$) because f is uniformly continuous on compact sets. Now, for any infinite subsequence $\{f^{n_k}\}_k \subset \{f^n\}_n$ there exists an infinite subsequence $\{f^{n_{k_i}}\}_i$ contained in $\{f^{np+q}\}_n$ for some q and then with a uniformly convergent subsequence because $\{f^{np+q}\}_n$ is normal. □

Definition 2.2.6. (Multiplier) *Given a p-cycle $\mathcal{O}(z_0) = \{z_0, z_1, \ldots, z_{p-1}\}$, the multiplier of the cycle is defined as $\lambda = (f^p)'(z_i) = f'(z_0)f'(z_1)\ldots f'(z_{p-1})$ for all $i \in \{0, \ldots, p-1\}$. If $\mathcal{O}(z_0)$ includes ∞, then λ is defined after a change of variables that removes ∞ from $\mathcal{O}(z_0)$.*

Lemma 2.2.7. *Multipliers are preserved by C^1-conjugacies.*

Proof. It follows immediately from Lemma 2.1.4. □

Definition 2.2.8. (Classification of periodic points) *A periodic point is*

1. *attracting if $|\lambda| < 1$ (superattracting if $\lambda = 0$);*
2. *repelling if $|\lambda| > 1$;*
3. *neutral or indifferent if $|\lambda| = 1$. It is*

(a) parabolic or rationally indifferent if $\lambda = e^{2\pi i p/q}$ ($p/q \in \mathbb{Q}$) is a q-th root of unity;

(b) irrationally indifferent if $\lambda = e^{2\pi i\theta}$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$).

Definition 2.2.9. (Critical point and value) *We say that $z_0 \in S$ is a critical point if $f'(z_0) = 0$. The image $f(z_0)$ is called critical value.*

Definition 2.2.10. (Basin of attraction) *Given an attracting point $z_0 \in S$, we define its basin of attraction $\mathcal{A}(z_0) = \mathcal{A}_f(z_0) := \{z \in S \mid f^n(z) \xrightarrow{n} z_0\}$. The immediate basin of attraction of z_0 is the connected component of $\mathcal{A}(z_0)$ containing z_0. Analogously, if $\mathcal{O}(z_0)$ is a p-cycle of f, its basin of attraction is defined as $\mathcal{A}(\mathcal{O}(z_0)) := \{z \in S \mid f^{np}(z) \xrightarrow{n} z_i \in \mathcal{O}(z_0)\}$. Its immediate basin of attraction is the union of the connected components containing z_i, where $i \in \{0, 1, 2, \ldots, p-1\}$.*

2.2.2 Local theory of fixed points

Below are shown some important results of local theory of fixed points. Our aim now is to describe the local dynamics of a holomorphic function f in a neighbourhood of a p-cycle through conformal changes of coordinates which conjugate f with a polynomial. Taking into account that $J_{f^p} = J_f$ (see Proposition 2.2.5), we can consider f^p and study simply a fixed point. Besides, considering a translation if necessary, we may also assume that it is the origin. Denoting by $\lambda = f'(0)$ the corresponding multiplier, we write

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \ldots .$$

Attracting and repelling fixed points ($|\lambda| \notin \{0, 1\}$)

The following result was shown by [1]G. Kœnigs in 1884. Although for this thesis it is more important the result concerning to superattracting fixed points, we have decided to show this one because it is also frequently used in quasiconformal surgery, which will be the technique whereby we will prove the Straightening Theorem in the next chapter. Additionally, its proof gives an idea of the one for superattracting fixed points.

Theorem 2.2.11. (Kœnigs' linearization) *If the multiplier λ satisfies $|\lambda| \notin \{0, 1\}$, then there exists a local holomorphic change of coordinate $\omega = \phi(z)$ with $\phi(0) = 0$, so that $\phi \circ f \circ \phi^{-1}$ is the linear map $\omega \mapsto \lambda\omega$ for all ω in some neighbourhood of the origin. Moreover, the conjugacy ϕ is called a linearizing map of f at the fixed point and is unique up to multiplication by a non-zero constant.*

Proof. Let us denote by $m_\lambda(\omega) := \lambda\omega$ the linear map of the statement.

- Uniqueness. Let ϕ and ψ be two such maps. They are conformal conjugacies, so their inverses are conformal and the composition $\psi \circ \phi^{-1}$ too. This and the fact that $\phi(0) = \psi(0) = 0$ implies that $(\psi \circ \phi^{-1})(\omega) = b_1\omega + b_2\omega^2 + b_3\omega^3 + \ldots$. Furthermore,

$$\begin{cases} f = \phi^{-1} \circ m_\lambda \circ \phi \\ f = \psi^{-1} \circ m_\lambda \circ \psi \end{cases} \Longrightarrow m_\lambda \circ \psi \circ \phi^{-1} = \psi \circ \phi^{-1} \circ m_\lambda.$$

  Comparing coefficients term by term, necessarily $\lambda b_n = b_n \lambda^n$ for all $n \geq 1$, which implies that $|b_n||\lambda| = |b_n||\lambda|^n$, so $b_n = 0$ for all $n \geq 2$ because $|\lambda| \neq 0, 1$. Finally, $(\psi \circ \phi^{-1})(\omega) = b_1\omega$ and $\psi = b_1\phi$.

- Existence.

  - Suppose $0 < |\lambda| < 1$. There exists $c \in \mathbb{R}_{<1}$ such that $c^2 < |\lambda| < c$. Moreover, $f(0) = 0$ implies that there exists $r > 0$ such that $|f(z)| \leq c|z|$ for all $z \in \mathbb{D}_r := \{z \in \mathbb{C} \mid 0 \leq |z| < r\}$. Then, for every $z_0 \in \mathbb{D}_r$, $|z_n| = |f^n(z_0)| \leq |z_0|c^n \leq rc^n$.

[1]Gabriel Xavier Paul Kœnigs: 1858 – 1931

The Taylor's Theorem give us a bound for the remainder of the Taylor series ($|R_n(z)| := |f(z) - P_n(z)| \leq k_n|z|^{n+1}$, where $k_n > 0$ and P_n is the Taylor polynomial of f with degree n). In our case, $|f(z) - \lambda z| \leq k|z|^2$ for every $z \in \mathbb{D}_r$. Thus, $|z_{n+1} - \lambda z_n| = |f(z_n) - \lambda z_n| \leq k|z_n|^2 \leq kr^2c^{2n}$.

Defining $\phi_n(z) := f^n(z)/\lambda^n$ for all $z_0 \in \mathbb{D}_r$,

$$|\phi_{n+1}(z_0) - \phi_n(z_0)| = \frac{1}{|\lambda|^{n+1}}|z_{n+1} - \lambda z_n| \leq \frac{kr^2}{|\lambda|}\left(\frac{c^2}{|\lambda|}\right)^n \xrightarrow{n} 0 \implies \phi_n \overset{n}{\rightrightarrows} \phi$$

and ϕ is holomorphic (ϕ_n are holomorphic). It is easy to check the conjugacy:

$$\phi(f(z)) = \lim_n \frac{f^{n+1}(z)}{\lambda^n} = \lambda \lim_n \frac{f^{n+1}(z)}{\lambda^{n+1}} = \lambda\phi(z)$$

We will finish this case if we see that $\phi'(0) \neq 0$, because this implies that ϕ is a local conformal diffeomorphism (i.e. locally biholomorphic). But this is trivial, because the nontrivial monomial of minimum degree of ϕ_n is z and $\phi_n \overset{n}{\rightrightarrows} \phi$, so $\phi'(0) = 1$ (see Theorem A.3.15).

– Consider $|\lambda| > 1$. Since $\lambda \neq 0$, the inverse f^{-1} is well defined and holomorphic with an attractive fixed point in the origin of multiplier λ^{-1}. Denoting $m_\mu(z) := \mu z$ and bearing in mind that $0 < |\lambda^{-1}| < 1$, we reduce to the previous case and we have the following:

$$\phi \circ f^{-1} = m_{\lambda^{-1}} \circ \phi \underset{m^{-1}_{\lambda^{-1}} = m_\lambda}{\implies} \phi \circ f = m_\lambda \circ \phi$$

□

Corollary 2.2.12. *Suppose that f is globally defined and has an attracting fixed point z_0 with $0 < |\lambda| < 1$, and let $\mathcal{A}$ be its basin of attraction. Then there exists a holomorphic map $\phi : \mathcal{A} \to \mathbb{C}$ with $\phi(z_0) = 0$, so that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{A} \\ \downarrow \phi & & \downarrow \phi \\ \mathbb{C} & \xrightarrow{\omega \mapsto \lambda\omega} & \mathbb{C} \end{array}$$

Furthermore, ϕ is biholomorphic in a neighbourhood of z_0 (where it determines a local conformal conjugacy). The map ϕ is unique up to multiplication by a non-zero constant.

Proof. It suffices to consider the biholomorphism $\phi : N_{z_0} \to \mathbb{C}$ defined on a neighbourhood of z_0 given by Kœnigs' linearization and spread it by the dynamics of f. This means extending ϕ so that $\phi(z) = \phi(f^k(z))/\lambda^k$ for some k such that $f^k(z) \in N_{z_0}$. □

Superattracting fixed points ($\lambda = 0$)

A few years later, [2]L. Böttcher proved in 1904 a result concerning to the superattracting case.

Theorem 2.2.13. (Böttcher coordinates) *Let $f(z) = a_n z^n + a_{n+1}z^{n+1} + \dots$, where $n \geq 2$ and $a_n \neq 0$. Then there exists a local holomorphic change of coordinate $\omega = \phi(z)$ which conjugates f to the n-th power map $\omega \mapsto \omega^n$ throughout some neighbourhood of $\phi(0) = 0$. Besides, ϕ is unique up to multiplication by an $(n-1)$-st root of unity.*

Proof.

[2]Lucjan Emil Böttcher: 1872 – 1937

- Uniqueness. Given two such conformal conjugacies, ϕ and ψ, their inverses are also conformal and the composition $\psi \circ \phi^{-1}$ too. Furthermore, denoting $P_n(z) := z^n$, we have $((\psi \circ \phi^{-1})(z))^n = (\psi \circ \phi^{-1})(z^n)$. Indeed,

$$\begin{cases} f = \phi^{-1} \circ P_n \circ \phi \\ f = \psi^{-1} \circ P_n \circ \psi \end{cases} \Longrightarrow P_n \circ \psi \circ \phi^{-1} = \psi \circ \phi^{-1} \circ P_n.$$

  Therefore, by Lemma A.3.19, necessarily $(\psi \circ \phi^{-1})(z) = \xi z$ with $\xi^{n-1} = 1$. Finally, $\psi = \xi\phi$ as it should be.

- Existence. Considering $c \in \mathbb{C}$ such that $c^{n-1} = a_n \neq 0$, we obtain $cf(z/c) = z^n + \widetilde{a_{n+1}} z^{n+1} + \widetilde{a_{n+2}} z^{n+2} + \dots$, so we may assume $f(z) = z^n(1 + g(z))$, where $g(z) = b_1 z + b_2 z^2 + \dots$. This is possible because the linear conjugation $z \mapsto cz$ does not change the dynamics.

  Let $r \in (0, 1/2)$ be a fixed value such that $|g(z)| < 1/2$, that exists because g is continuous and $g(0) = 0$. Then,

$$|f(z)| = |1 + g(z)||z|^n \le \frac{3}{2}|z|^n \le \frac{3}{2}\frac{1}{2^{n-1}}|z| \le \frac{3}{4}|z|$$

  and

$$|f(z)| \ge |1 - |g(z)|||z|^n \ge \frac{|z|^n}{2}$$

  for $z \in \mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$. Thus, $f(\mathbb{D}_r) \subset \mathbb{D}_r$ and $|f(z)|/|z^n| \ge 1/2$.

  In this case, we can set a holomorphic root

$$m_1(z) := \left(\frac{f(z)}{z^n}\right)^{1/n}$$

  considering an appropriate $r > 0$ so that $f(\mathbb{D}_r)$ does not surround the origin. Moreover, since $f(\mathbb{D}_r) \subset \mathbb{D}_r$, we can recursively use the same argument and consider the roots

$$m_2(z) := \left(\frac{f^2(z)}{(f(z))^n}\right)^{1/n^2},\ m_3(z) := \left(\frac{f^3(z)}{(f^2(z))^n}\right)^{1/n^3},\ \dots, m_k(z) := \left(\frac{f^k(z)}{(f^{k-1}(z))^n}\right)^{1/n^k},\ \dots,$$

  so that we can choose a holomorphic n^k-th root denoted by $(\cdot)^{1/n^k}$ and define

$$\phi_k(z) := z\left(\frac{f^k(z)}{z^{n^k}}\right)^{1/n^k} = z\prod_{j=1}^{k} m_j(z).$$

  for $k > 0$.

  As $\phi_k(f(z)) = (f^{k+1}(z))^{1/n^k} = (\phi_{k+1}(z))^n$ for every $k \in \mathbb{N}$, if $\phi_k : \mathbb{D}_r \to \mathbb{D}_{\sqrt[n^k]{r}}$ converges uniformly to a limit function $\phi : \mathbb{D}_r \to \mathbb{D}$ $\left(\text{i.e. } \phi_k \overset{k}{\rightrightarrows} \phi\right)$, then the required identity $\phi(f(z)) = (\phi(z))^n$ will be satisfied and the conjugation ϕ will be holomorphic because $(\phi_k)_k$ do. Actually, ϕ will be conformal because ϕ_k are conformal in a neighbourhood of the origin (see Theorem A.3.15) and we can assume that r is chosen so that ϕ_k is a biholomorphism. Indeed, defining $\phi_0(z) := z$, we know that $\phi_{k+1}(z) = \phi_k(z)m_{k+1}(z)$, so that $m_{k+1}(0) = 1$, $\phi_k(0) = 0$ and $\phi_0'(0) = 1$ leads to the equality $\phi_{k+1}'(0) = \phi_k'(0) = 1 \neq 0$ for all $m \ge 0$.

  Let us see the aforementioned convergence. We will denote by $\ln(\cdot)$ the real logarithm and by $\log(\cdot)$ the complex logarithm branch such that

$$F(Z) := \log f\left(e^Z\right) = \log(e^{nZ}(1+g)) = nZ + \log(1+g) = nZ + \left(g - \frac{g^2}{2} + \frac{g^3}{3} - \dots\right),$$

where $F : \mathbb{H}_r \to \mathbb{H}_r$ ($\mathbb{H}_r := \{z \in \mathbb{C} \mid \operatorname{Re}(z) < \ln r\}$) is a well-defined holomorphic function and $g := g\left(e^Z\right)$.

Notice that

$$\begin{aligned}
F^2(Z) &= \log f\left(e^{\log f(e^Z)}\right) = \log f^2\left(e^Z\right)\\
F^3(Z) &= \log f^2\left(e^{\log f(e^Z)}\right) = \log f^3\left(e^Z\right)\\
&\ \vdots\\
F^k(Z) &= \log f^{k-1}\left(e^{\log f(e^Z)}\right) = \log f^k\left(e^Z\right)
\end{aligned} \tag{2.1}$$

and

$$|F(Z) - nZ| = |\log(1+g)| = \sqrt{(\ln|1+g|)^2 + (\arg(1+g))^2} \underset{|g|<1/2}{\leq} \ln\frac{3}{2} < 1, \tag{2.2}$$

so, by Lemma A.3.14,

$$\begin{aligned}
\left|\phi_{k+1}\left(e^Z\right) - \phi_k\left(e^Z\right)\right| &= \left|e^{\log\phi_{k+1}(e^Z)} - e^{\log\phi_k(e^Z)}\right| \leq \left|\log\phi_{k+1}\left(e^Z\right) - \log\phi_k\left(e^Z\right)\right| \underset{(2.1)}{=}\\
&= \left|\frac{F^{k+1}(Z)}{n^{k+1}} - \frac{F^k(Z)}{n^k}\right| = \frac{\left|F^{k+1}(Z) - nF^k(Z)\right|}{n^{k+1}} \underset{(2.2)}{\leq} \frac{1}{n^{k+1}} \xrightarrow[n\geq 2]{k} 0.
\end{aligned}$$

□

The Böttcher mapping ϕ can be extended applying the following result (see [Mil, Theorem 9.3] for a proof).

Theorem 2.2.14. (Extended Böttcher coordinates) *Let ϕ^{-1} be the local inverse of the the Böttcher map defined on $\mathbb{D}_\epsilon$ for some $\epsilon > 0$. Then, there exists a unique open disk $\mathbb{D}_r$ of maximal radius $0 < r \leq 1$ such that ϕ^{-1} extends holomorphically to a conformal equivalence $\phi^{-1} : \mathbb{D}_r \to \phi^{-1}(\mathbb{D}_r) \subset \mathcal{A}$, where $\mathcal{A}$ is the immediate basin of attraction of the superattracting fixed point.*

- *If $r = 1$, then $\phi^{-1}(\mathbb{D}_r) = \mathcal{A}$ and the superattracting fixed point is the only critical point in $\mathcal{A}$.*
- *If $r < 1$, then there is at least one other critical point in $\mathcal{A}$ lying on the boundary of $\phi^{-1}(\mathbb{D}_r)$.*

2.3 Polynomial dynamics

Let us introduce the concept of fixed point at infinity.

Definition 2.3.1. (Superattracting fixed infinity) *Given a holomorphic mapping f defined in a neighbourhood of ∞, we say that it has a superattracting fixed point at infinity if $g(\omega) := 1/f(1/\omega)$ has a super attracting fixed point of the same degree at zero. Equivalently, if one of the following limits exists and does not vanish*

$$\lim_{\omega\to 0}\frac{g(\omega)}{\omega^d} \neq 0 \qquad\qquad \lim_{z\to\infty}\frac{f(z)}{z^d} \neq 0,$$

where d is the degree of ∞ as superattracting fixed point.

Lemma 2.3.2. *An entire map with a superattracting fixed point at infinity of certain degree d is a polynomial of degree d.*

Proof. It follows trivially from the characterization of this superattracting fixed point in terms of limits. □

Corollary 2.3.3. (Böttcher coordinates at infinity) *The Böttcher coordinates can be also considered when ∞ is the superattracting fixed point.*

Proof. Keeping the notation of Definition 2.3.1, we can apply the Böttcher Theorem (see Theorem 2.2.13) to g. Thus, if we denote the corresponding conjugation by ϕ, then $\phi(g(\omega)) = (\phi(\omega))^d$ for $|\omega| < r$ for some $r > 0$. Therefore, $\varphi(z) := 1/\phi(1/z)$ is a conjugation such that $f(\varphi(z)) = (\varphi(z))^d$ for $|z| > 1/r$. The two diagrams that commute are the following:

$$\begin{tikzcd} \mathbb{D}_r \arrow[r, "g"] \arrow[d, "\phi"] & \mathbb{D}_r \arrow[d, "\phi"] \\ \mathbb{D} \arrow[r, "\omega\mapsto\omega^n"] & \mathbb{D} \end{tikzcd} \qquad \begin{tikzcd} \hat{\mathbb{C}} \setminus \overline{\mathbb{D}_{1/r}} \arrow[r, "f"] \arrow[d, "\varphi"] & \hat{\mathbb{C}} \setminus \overline{\mathbb{D}_{1/r}} \arrow[d, "\varphi"] \\ \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \arrow[r, "\omega\mapsto\omega^n"] & \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \end{tikzcd}$$

□

Remark 2.3.4. By Theorem 2.2.14 and following the notation in Corollary 2.3.3, if all the critical points are in K_f, then we can extend the Böttcher map φ to a conformal equivalence $\varphi : \hat{\mathbb{C}} \setminus K_f = \mathcal{A}(\infty) \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. If at least one of them escapes to infinity, then φ is defined on an open neighbourhood U of infinity and $\varphi(U) = \hat{\mathbb{C}} \setminus \mathbb{D}_r$ for some $r > 1$. Notice that this radius is the inverse value of the one in Theorem 2.2.14. Furthermore, in this case there is a critical point in ∂U.

2.3.1 Equipotentials and external rays

Definition 2.3.5. (Green's function of the filled Julia set) *Given a polynomial f of degree $d \geq 2$, the mapping $g_f : \mathbb{C} \to \mathbb{R}_{\geq 0}$ defined as $g_f(z) := \lim_n \frac{1}{d^n} \max\{0, \log(|f^n(z)|)\}$ is known as the Green's function of K_f.*

Remark 2.3.6. g_f is a real continuous harmonic function. If more information about Green's functions is desired, we refer to [Bru, 9.5].

Remark 2.3.7. The map g_f measures the escape rate to infinity of the orbit of the given point. Let G_f be the maximal escape rate of the critical points, i.e. $G_f := \max\{g_f(z) \mid f'(z) = 0\}$. Recalling Remark 2.3.4, we can consider the Böttcher map φ defined on $U_f := \{z \in \mathbb{C} \mid g_f(z) > G_f\}$. Then, $g_f(z) = \log(|\varphi(z)|)$. Indeed, since

$$\begin{aligned} \varphi(f(z)) = (\varphi(z))^d \implies |\varphi(z)| = |\varphi(f(z))|^{1/d} \implies \\ \implies |\varphi(z)| = |\varphi(f(z))|^{1/d} = |\varphi(f^2(z))|^{1/d^2} = \cdots = |\varphi(f^n(z))|^{1/d^n} = \ldots, \end{aligned}$$

from $f^n(z) \xrightarrow{n} \infty$ and $\varphi(z) \xrightarrow{z\to\infty} z$ it follows that $|\varphi(f^n(z))|^{1/d^n} \xrightarrow{n} \lim_n |f^n(z)|^{1/d^n} = e^{g_f(z)}$ as required. We can write $\varphi : U_f \to \mathbb{C} \setminus \overline{\mathbb{D}_{e^{G_f}}}$. Note that if the orbit of each critical point is bounded, then $U_f = \mathbb{C} \setminus K_f$ and $\varphi(\hat{\mathbb{C}} \setminus K_f) = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, so K_f is connected. Furthermore, we can write

$$g_f(z) = \begin{cases} 0, & z \in K_f \\ \log(|\varphi(z)|), & z \in U_f \\ \frac{1}{d^n} g_f(f^n(z)), & f^n(z) \in U_f \end{cases}.$$

Proposition 2.3.8. *The map $(f, z) \mapsto g_f(z)$ is a continuous function $Pol_d \times \mathbb{C} \to R_{\geq 0}$, where $Pol_d := \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f \text{ is a polynomial of degree } d\}$.*

For a proof we refer to [DH2, p. 62, Proposition 8.1].

Definition 2.3.9. (Equipotential) *Given some $\eta > 0$, the set $g_f^{-1}(\eta)$ is called the equipotential of value η.*

Remark 2.3.10. The Green's function g_f has no critical points in U_f due to the conformal equivalence φ, so any equipotential of value $\eta > G_f$ is a simple closed curve surrounding K_f. Moreover, if all the critical points of f have a bounded orbit, then all the equipotentials are such curves.

Definition 2.3.11. (External ray) *Let $\varphi : U_f \to \mathbb{C} \setminus \overline{\mathbb{D}_{e^{G_f}}}$ be as above. We define the external ray of argument $\theta \in \mathbb{R}/\mathbb{Z}$ as $R_f(\theta) = \varphi^{-1}(\{e^{\eta+2\pi i\theta}\}_{\eta > G_f})$.*

Remark 2.3.12. Because of the Böttcher conjugacy, we know that $f(R_f(\theta)) = R_f(d\theta)$. Recall that $d \geq 2$ is the degree of the polynomial f.

Definition 2.3.13. (Landing external ray) *Suppose that the orbit of each critical point is bounded, so that φ is defined on $\mathbb{C} \setminus K_f$. An external ray is said to land if $\varphi^{-1}(e^{\eta+2\pi i\theta})$ has a limit as η tends to 0.*

2.3.2 Filled Julia set

At the beginning of this chapter, we dealt with a definition of the Julia set of a holomorphic function in terms of normal families. Even so, since from now on polynomials will play an important role in the thesis, we are going to introduce an equivalent definition inferred from the notion of filled Julia set. If more details are desired regarding Julia sets and some of its distinct definitions, we highly recommend [Dev2, Chapter 3]. Given that we are going to focus on polynomials, we will usually use P instead of f, which has been used so far to denote a generic holomorphic map.

Consider a polynomial of degree $d \geq 2$,

$$P(z) := a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

where $a_d \neq 0$. It has $d-1$ finite critical points. If we state that ∞ is a critical point if 0 is a critical point of $1/P(1/z)$, the equality $(1/P(1/z))' = P'(1/z)/(zP(1/z))^2$ implies that ∞ is a critical point of multiplicity $d-1$.

Also, $P(\infty) = \infty = P^{-1}(\infty)$ is a superattracting fixed point (see Lemma 2.3.2) and the Riemann sphere $\hat{\mathbb{C}}$ can be splitted into the connected basin of attraction of infinity

$$\mathcal{A}(\infty) = \mathcal{A}_P(\infty) := \{z \in \hat{\mathbb{C}} \mid P^n(z) \xrightarrow{n} \infty\}$$

and its complement. Let us recall the definition of filled Julia set given above in Definition 2.2.3.

Definition 2.3.14. (Filled Julia set) *The complement of $\mathcal{A}_P(\infty)$ is known as filled Julia set and denoted by $K_P := \hat{\mathbb{C}} \setminus \mathcal{A}_P(\infty)$.*

Definition 2.3.15. (Julia set) *The Julia set of P is defined as the common boundary of K_P and $\mathcal{A}_P(\infty)$, i.e. $J_P := \partial\mathcal{A}_P(\infty) = \partial K_P$.*

Remark 2.3.16. K_P is totally invariant, compact and such that all bounded Fatou components are simply connected (see [Mil, Lemma 9.4]). Since $\hat{\mathbb{C}} \setminus K_P = \mathcal{A}_P(\infty)$ is connected, K_P is connected if and only if J_P is connected (see [CKL]).

Now, we are going to characterize the connectivity of K_P (and hence also J_P) in terms of the finite critical points.

Theorem 2.3.17. (Connectivity of K_P and J_P)

1. *K_P is connected if and only if it contains all the finite critical points of P. In this case, $P_{|\hat{\mathbb{C}} \setminus K_P}$ is conformally conjugate to $z \mapsto z^d$ on $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.*

2. *If at least one finite critical point of P belongs to $\mathcal{A}(\infty)$, then K_P and J_P have uncountably many connected components.*

Proof.

- Let us see the reciprocal of the first assertion. Assume K_P contains all the finite critical points of P. By Theorem 2.2.14 and keeping the notation in Corollary 2.3.3, we can extend the Böttcher map φ to a conformal equivalence $\varphi : \hat{\mathbb{C}} \setminus K_P = \mathcal{A}(\infty) \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Since $\overline{\varphi^{-1}(\mathbb{A}_{1+\epsilon})}$ is compact, non-empty and connected, and it contains J_P for all $\epsilon > 0$, the intersection $J_P = \bigcap_{\epsilon>0} \overline{\varphi^{-1}(\mathbb{A}_{1+\epsilon})}$ is also connected (see Proposition A.4.5). Recall that this is equivalent to the connectedness of K_P.

- Now, we want to see the second statement and the other implication of the first one. Suppose that there is at least one finite critical point in $\mathcal{A}(\infty)$. Then, by Theorem 2.2.14, there exists some $r > 1$ such that φ^{-1} extends to a conformal isomorphism $\varphi^{-1} : \mathbb{C} \setminus \overline{\mathbb{D}_r} \to \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r}) \subset \mathcal{A}(\infty)$, where $\partial\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})$ is a compact subset of $\mathbb{C} \setminus K_P$ that contains at least one critical point of P, say ω.

  Let $v = P(\omega)$ be the critical value of ω. Via the conjugation φ^{-1} we obtain $|\varphi(v)| = r^d > r$, so $v \in U$. Consider the external ray $\varphi^{-1}(R)$ in $\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})$, where $R := \varphi^{-1}([1, \infty)\varphi(v))$, and the d distinct external rays $\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r}) \cap P^{-1}(R)$ corresponding to the d distinct conjugated rays in $\mathbb{C} \setminus \mathbb{D}_r$. Each of these rays, that we denote by $(R_j)_{j=1,\ldots,d}$, land at some solution z to the equation $P(z) = v$. Since two of these, say R_1 and R_2, land at ω, $\{\omega\} \cup R_1 \cup R_2 \subset \overline{\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})}$ will cut the plane into two open connected sets, which are denoted by V_0 and V_1.

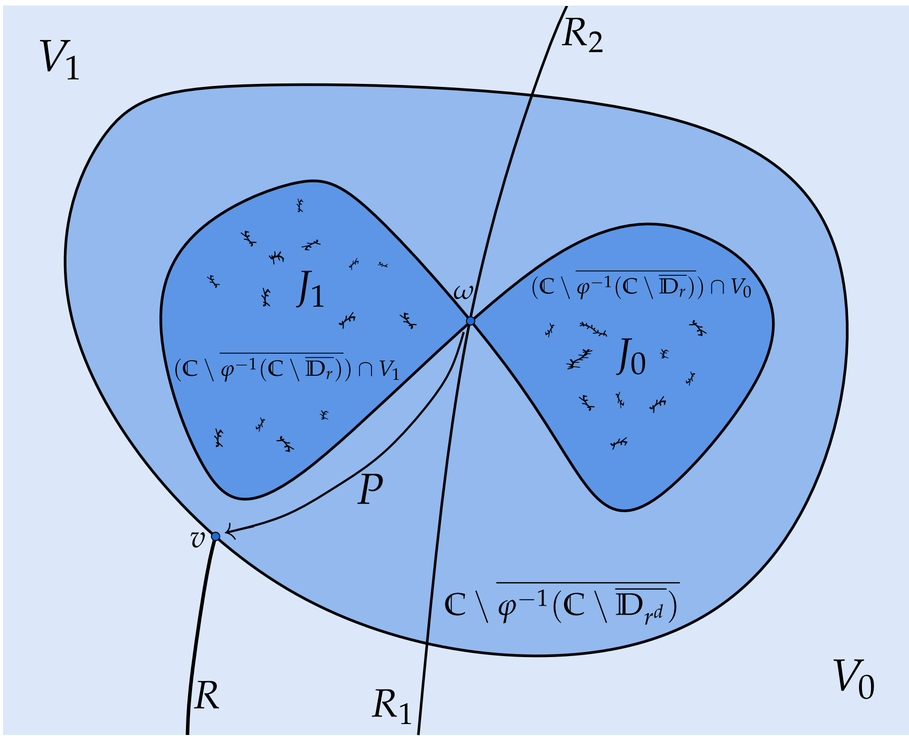


Figure 2.1: Sketch for the proof of Theorem 2.3.17.

Let $z' \in \partial P(V_k)$ for $k \in \{1, 2\}$. Then, by Proposition A.4.6, there is a sequence $\{z_j\}_j \subset V_k$ so that $P(z_j) \xrightarrow{j} z'$. Since z_j are bounded, by Bolzano-Weierstrass Theorem (see Theorem A.4.7) it contains a subsequence which converges to some $z^* \in \mathbb{C}$. Since $P(V_k)$ is open (because P is open) and $z' = P(z^*) \in \partial P(V_k)$, $z^* \notin V_k$ and $z^* \in \partial V_k = \{\omega\} \cup R_1 \cup R_2$. Therefore, $z' \in R$ and, by Proposition A.4.8, $K_P \subset \mathbb{C} \setminus R \subset P(V_k)$.

Denoting $J_0 = J_P \cap V_0$ and $J_1 = J_P \cap V_1$, we obtain $J_P = P(J_0) = P(J_1)$, where J_0 and J_1 are disjoint compact sets with $J_0 \cup J_1 = P(J_1)$, where J_0 and J_1 are disjoint compact sets with $J_0 \cup J_1 = J_P$.

Inductively, recalling that the preimage of an intersection of sets is the intersection of the preimage of each set, we can split J_P in 2^{s+1} disjoint compact sets:

$$\begin{aligned}
&J_{k_0}\\
&J_{k_0k_1} = J_{k_0} \cap P^{-1}(J_{k_1})\\
&J_{k_0k_1k_2} = J_{k_0} \cap P^{-1}(J_{k_1}) \cap P^{-2}(J_{k_2})\\
&\quad\vdots\\
&J_{k_0\dots k_s} = J_{k_0} \cap P^{-1}(J_{k_1}) \cap \cdots \cap P^{-s}(J_{k_s})\\
&\quad\vdots
\end{aligned}$$

Here $(k_1, \dots, k_s) \in \{0,1\}^s$. Notice that $P(J_{k_0\dots k_s}) = J_{k_1\dots k_s}$ and $J_{k_0} \supset J_{k_0k_1} \supset J_{k_0k_1k_2} \supset \dots$. If we denote by $J_{k_0k_1k_2\dots}$ the intersection of the last nested sequence of compact sets, it follows that $J_{k_0k_1k_2\dots}$ is compact and non-vacuous and we are done because $(J_{k_0k_1k_2\dots})_{(k_0,k_1,k_2,\dots)}$ is an uncountable family of disjoint non-vacuous sets such that $J_P = \bigcup_{(k_0,k_1,k_2,\dots)} J_{k_0k_1k_2\dots}$. Recall that the connectedness of J_P and K_P are equivalent, so that we are done.

□

Remark 2.3.18. There is an extra statement that can be added to this characterization: *If all the critical points of P are in $\mathcal{A}_P(\infty)$, then K_P is totally disconnected. In this case, $J_P = K_P$ and it is a Cantor set.* However, we will not go into detail here.

2.3.3 Mandelbrot set

Let us look at the case of degree 2. Given a polynomial $P(z) := a_2z^2 + a_1z + a_0$ with $a_2 \neq 0$, the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "Q_c"] \arrow[d, "L"'] & \mathbb{C} \arrow[d, "L"] \\
\mathbb{C} \arrow[r, "P"] & \mathbb{C}
\end{tikzcd}$$

commutes, where $L(z) := \frac{z}{a_2} - \frac{a_1}{2a_2}$ and $Q_c(z) := z^2 + c$ with $c := a_0 - \frac{a_1^2}{2a_2} + \frac{a_1}{a_2}$. Therefore, the dynamics of polynomials of degree 2 can be represented by the family

$$\{Q_c(z) := z^2 + c \mid c \in \mathbb{C}\}.$$

Furthermore, the representative is unique. Indeed, given two conformally conjugate maps Q_{c_1} and Q_{c_2}, there must exist a conformal conjugacy $h : \mathbb{C} \to \mathbb{C}$ between them. Since the automorphisms of $\mathbb{C}$ are affine maps of the form $h(z) := az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$, then

$$\begin{cases} (h \circ Q_{c_1})(z) = a(z^2 + c_1) + b = az^2 + ac_1 + b \\ (Q_{c_2} \circ h)(z) = (az+b)^2 + c_2 = a^2z^2 + 2abz + b^2 + c_2 \end{cases}$$

leads to

$$(h \circ Q_{c_1})(z) = (Q_{c_2} \circ h)(z) \implies \begin{cases} a = 1 \\ b = 0 \\ c_1 = c_2 \end{cases}.$$

According to the notation established when we introduced the filled Julia set, in this particular case we will denote $K_c := K_{Q_c}$, $A_c := A_{Q_c}(\infty)$ and $J_c := J_{Q_c}$.

Definition 2.3.19. (Mandelbrot set) *The Mandelbrot set $\mathcal{M}$ is defined as*

$$\mathcal{M} := \{c \in \mathbb{C} \mid K_c \textit{ is connected}\} = \{c \in \mathbb{C} \mid J_c \textit{ is connected}\}.$$

Remark 2.3.20. In virtue of the characterization of the connectedness of Julia sets (see *Theorem* 2.3.17), we can write $\mathcal{M} := \{c \in \mathbb{C} \mid Q_c^n(0) \overset{n}{\not\to} \infty\}$. Obviously, this equality allows us to display $\mathcal{M}$ by means of simple computational tools.

The following theorem can be found in [BF, p. 128, Theorem 3.58].

Theorem 2.3.21. *$\mathcal{M}$ is compact, connected and contained in $\mathbb{D}_2 := \{c \in \mathbb{C} \mid |c| \leq 2\}$. Moreover, $\hat{\mathbb{C}} \setminus \mathcal{M}$ is connected.*

As shown in Figure 2.2, different values assigned to the parameter c lead to a diversity of Julia sets. Notice that we are plotting the filled Julia set so that we can appreciate its common boundary with the basin of attraction of infinity.

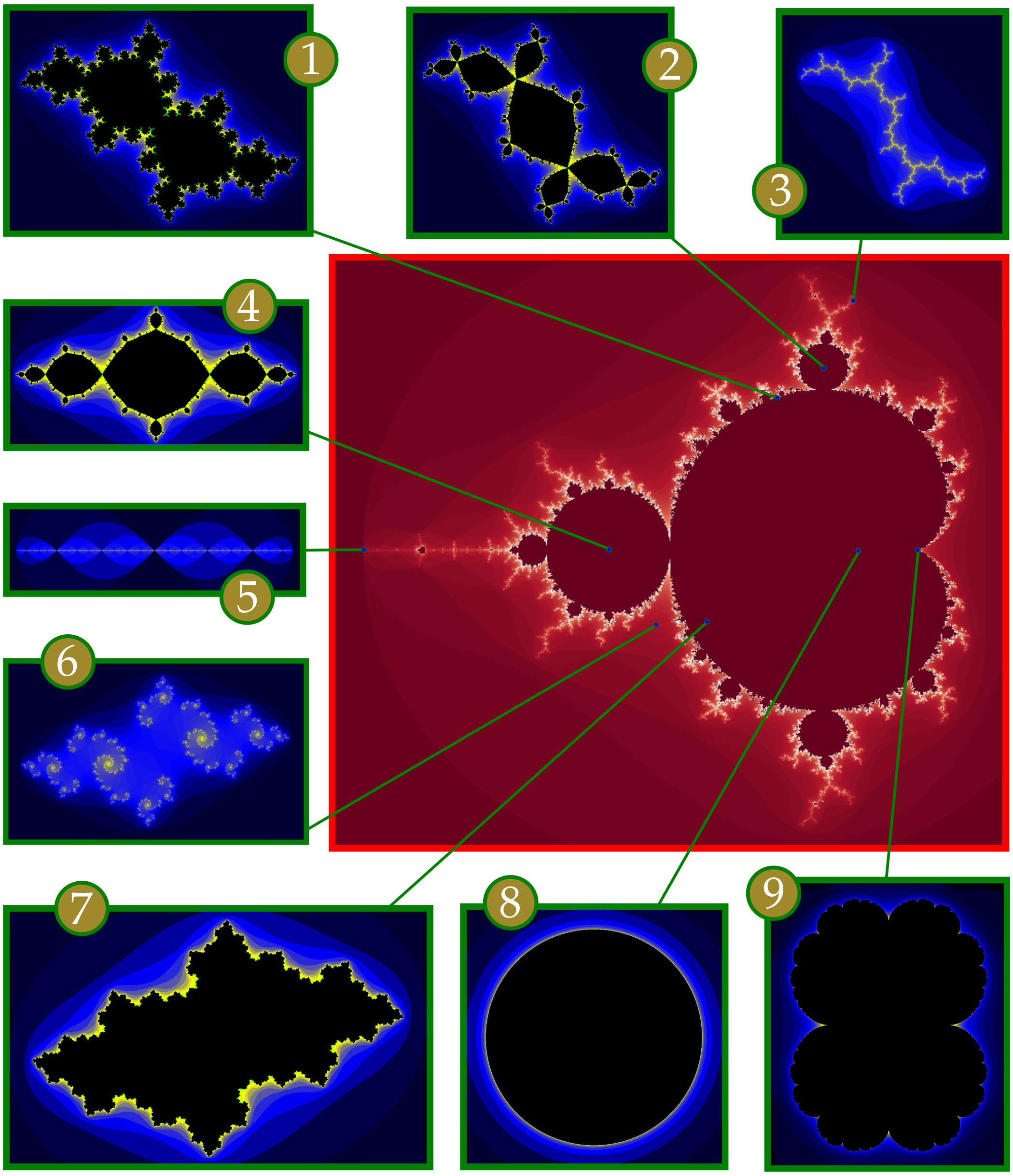


Figure 2.2: The different values of c are as follows: (1) $-0.390540870218 + 0.586787907347i$; (2) $-0.12256 + 0.74486i$; (3) i; (4) -1; (5) -2; (6) $-0.8 - 0.3i$; (7) $-0.6 - 0.3i$; (8) 0; (9) 0.25.

Chapter 3

The Straightening Theorem

The main goal of this chapter is to address the Straightening Theorem, which is the cornerstone of this thesis. Besides, it is an essential result behind the renormalization theory within the framework of complex dynamics.

To achieve our purpose, we first give the concept of polynomial-mapping, which is the class of functions that take part in this theorem. Also, we introduce the concept of hybrid equivalence, which is a crucial notion whereby we warrant an important consequence of the Straightening Theorem. This is particularly the fact that we find copies of polynomial Julia sets in the dynamical plane of holomorphic functions seemingly unrelated to polynomials. Furthermore, this phenomenon is also latent in the parameter planes: not only small Mandelbrot set copies appear at all scales in the parameter space of quadratic polynomials, but also Mandelbrot sets appear in many other families of holomorphic functions. This fact is explained by the parameter version of the Straightening theorem, which we also state (but do not prove) in this chapter. In both cases, we provide some examples displayed with Python to highlight the aforementioned occurrence.

With respect to the proof of the result concerning the dynamical plane, we develop it with the utmost rigour. To do it, we shall use quasiconformal surgery with the support of the tools given so far such as almost complex structures, pullbacks, quasiregularity, etc.

First of all, let us introduce all the concepts behind polynomial-like mappings and hybrid equivalences.

Definition 3.0.1. (Proper map) *A holomorphic map $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is said to be proper if the preimage of every compact subset in V is compact in U.*

Remark 3.0.2. If $f : U \to V$ is proper, the cardinality of the inverse image of every point in V is finite. It follows easily from the fact that zeros of holomorphic functions are isolated and that a discrete compact set is finite.

Definition 3.0.3. (Polynomial-like mapping) *Let U and V be open sets of $\mathbb{C}$ conformally equivalent to $\mathbb{D}$ such that $\overline{U} \subset V$ and let $f : U \to V$ be a proper map such that every point in V has exactly d preimages in U when counted with multiplicity. The triple $(f; U, V)$ is called a polynomial-like mapping of degree d.*

Remark 3.0.4. (Analytic boundaries on polynomial-like mappings) Notice that without loss of generality we can assume that the boundaries of U and V are analytic curves. Indeed, given a polynomial-like map $f : U \to V$ such that $\overline{U} \subset V$, we can consider an open simply connected set V' with analytic boundary such that $\overline{V'} \subset V$ and $U \subset V'$. Denoting $U' := f^{-1}(V')$, $f : U' \to V'$ fulfils the condition on the boundaries. It follows that f is also well defined on the boundary $\partial U'$.

Definition 3.0.5. (Quadratic-like mapping) *A polynomial-like map of degree 2 is called a quadratic-like mapping.*

Definition 3.0.6. (Covering map) *A covering map is a map $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ such that for all $\omega \in V$ there exists a neighbourhood N_ω of ω for which $f^{-1}(N_\omega)$ is a disjoint union of open sets $\{U_i\}_{i\in\mathbb{N}}$ on which $f : U_i \to N_\omega$ is a homeomorphism. The cardinal of $f^{-1}(\omega)$ (counted with multiplicity) is independent of the choice of $\omega \in V$. If finite, it is called degree of f.*

Definition 3.0.7. (Branched covering map) *A map is a branched covering map if it is a covering everywhere except for a nowhere-dense set known as the branch set. If z is a point of a branched set, we say that the branched covering map is ramified at z and with degree d, where d is the maximal cardinal of the preimage of a point in a neighbourhood of $f(z)$ when counted with multiplicity.*

A useful equality for proper maps, whose proof can be found in [Ste], is the Riemann-[1]Hurwitz formula.

Theorem 3.0.8. (Riemann-Hurwitz formula) *Let U and V be domains on the Riemann sphere of finite connectivity m and n, i.e. so that ∂U and ∂V have m and n different connected components, respectively. Let $f : U \to V$ be a proper map of degree k with r critical points when counted with multiplicity. Then,*

$$m - 2 = k(n-2) + r.$$

Remark 3.0.9. Recalling the Riemann-Hurwitz formula (see Theorem 3.0.8), a polynomial-like map of degree d is a branched covering map with $d-1$ critical points in U counted with multiplicity.

Example 3.0.10. Given a polynomial P of degree d, we can choose $r > 0$ such that $V := \{z \in \mathbb{C} \mid |z| < r\}$ contains all the critical values of P and so that the triple $\left(P_{|f^{-1}(V)}; f^{-1}(V), V\right)$ is a polynomial-like of degree d.

Definition 3.0.11. (Filled Julia set and Julia set of a polynomial-like mapping) *Given a polynomial-like map $(f; U, V)$, its filled Julia set is defined as $K_f := \cap_{n>0} f^{-n}(V) = \cap_{n\geq 0} f^{-n}(U)$ and $J_f := \partial K_f$ is known as the Julia set of f.*

Definition 3.0.12. (Hybrid equivalence) *Given two polynomial-like mappings f and g of degree d, we say that they are hybrid equivalent if there exist neighbourhoods $U_f \supset K_f$ and $U_g \supset K_g$, and there exists a quasiconformal conjugacy $\phi : U_f \to U_g$ between f and g such that $\partial_{\bar{z}}\phi = 0$ almost everywhere on K_f.*

Remark 3.0.13. By Weyl's Lemma, this implies that ϕ is conformal in the interior of K_f when it is non-empty.

Remark 3.0.14. Hybrid equivalence is an equivalence relation. Moreover, defining analogously a topological, a quasiconformal and a holomorphic equivalence for polynomial-like mappings, and denoting them by $\sim$, we obtain the following:

$$f \underset{\text{hol}}{\sim} g \implies f \underset{\text{hyb}}{\sim} g \implies f \underset{\text{qc}}{\sim} g \implies f \underset{\text{top}}{\sim} g$$

Theorem 3.0.15. (Hybrid classes are affine classes in the connected case) *Given two polynomials of the same degree and with connected Julia sets, if they are hybrid equivalent then they are affine conjugate.*

This result can be found in [DH1, p. 303, Corollary 2].

Definition 3.0.16. (Fundamental domain of a map) *A fundamental domain of a map $f : U \subset \mathbb{C} \to U$ is a subset $V \subset U$ such that each orbit passes through V at most once.*

We are now ready to state the central theorem in this project, the Straightening Theorem. This result explains why the dynamical space of some holomorphic maps contain copies of polynomial Julia sets. Moreover, we know that these copies are related by a hybrid equivalence, so that they are quasiconformally equivalent and their interiors are conformally equivalent.

[1]Adolf Hurwitz: 1859 – 1919

Theorem 3.0.17. (The Straightening Theorem) *Every polynomial-like mapping $(f;U,V)$ of degree d is hybrid equivalent to a polynomial P of degree d. Moreover, if K_f is connected, then P is unique up to affine conjugation.*

Proof.

- Proof of the existence. Consider a fixed value $\rho > 1$ and $\mathbb{D}_{\rho^d}$. By the Uniformization Theorem, there exist Riemann maps $\phi_1 : \hat{\mathbb{C}} \setminus \overline{V} \to \mathbb{D}$ and $\phi_2 : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}_{\rho^d}} \to \mathbb{D}$ such that $\phi_1(\infty) = \phi_2(\infty) = 0$. Define $R := \phi_2^{-1} \circ \phi_1 : \hat{\mathbb{C}} \setminus \overline{V} \to \hat{\mathbb{C}} \setminus \overline{\mathbb{D}_{\rho^d}}$, that is a biholomorphism such that $R(\infty) = \infty$.

  Since ∂V is analytic by Remark 3.0.4, applying Theorem 1.6.10 R extends continuously to the boundary as an analytic map $\psi_1 : \partial V = \partial\left(\hat{\mathbb{C}} \setminus \overline{V}\right) \to \mathbb{S}^1_{\rho^d} = \partial\left(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}_{\rho^d}}\right)$. Choose a mapping $\psi_2 : \partial U \to \mathbb{S}^1_\rho$ so that $\psi_1(f(z)) = (\psi_2(z))^d$ for all $z \in \partial U$.

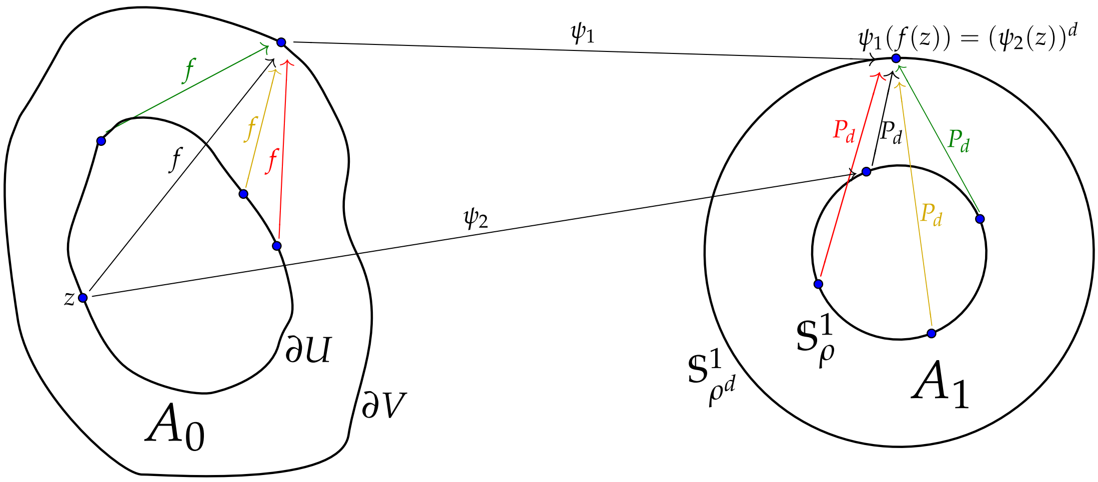


Figure 3.1: Boundary mappings for $d = 4$, where $P_d(z) := z^d$.

  This choice is justified by setting $\psi_2(z) := [\psi_1(f(z))]^{1/d}$ in terms of a fixed d-th root $(\cdot)^{1/d}$, so every preimage of z is identified with a different d-th root throughout the required equality so that the continuity of the chosen root is fulfilled on the whole boundary ∂U.

  Define $A_0 := V \setminus U$ and $A_1 := \mathbb{A}_{\rho,\rho^d}$. By Proposition 1.6.13, we can extend continuously ψ_1 and ψ_2 to a mapping $\psi : \overline{A_0} \to \overline{A_1}$ that is quasiconformal in the interior of A_0. Now consider the following map $F : \mathbb{C} \to \mathbb{C}$:

$$F(z) := \begin{cases} f(z), & \text{if } z \in U \\ R^{-1}\left((\psi(z))^d\right), & \text{if } z \in V \setminus U \\ R^{-1}\left((R(z))^d\right), & \text{if } z \in \mathbb{C} \setminus V \end{cases}$$

  This mapping is quasiregular. Indeed, denoting $P_d(z) := z^d$:

  - For $z \in U$, F is holomorphic and then 1-quasiregular.
  - For $z \in V \setminus \overline{U}$, the quasiregularity of F is obtained from the first definition of quasiregular map taking into account that $P_d \circ \psi$ is locally quasiconformal (P_d is locally conformal and ψ is quasiconformal) and R^{-1} is holomorphic (in fact conformal).
  - For $z \in \mathbb{C} \setminus \overline{V}$, F is holomorphic (i.e. 1-quasiregular) because R^{-1} and $P_d \circ R$ are holomorphic.

– For $z \in \partial U \cup \partial V$, we will use the regularity at $\mathbb{C} \setminus (\partial U \cup \partial V)$. Since F is regular at $\mathbb{C} \setminus (\partial U \cup \partial V)$, it is there locally quasiconformal except at a discrete set of points. As the zeros of a holomorphic function are isolated, Remark 1.5.2 ensures that F is locally quasiconformal at $\mathbb{C} \setminus (\partial U \cup \partial V)$ except at a set of isolated points. Because of that, given a point $z \in \partial U \cup \partial V$, there exists a neighbourhood N_z such that F is quasiconformal at $N_z \setminus \Gamma$, where $\Gamma := N_z \cap (\partial U \cup \partial V)$. By Theorem 1.3.24, F is quasiconformal at N_z and hence quasiregular.

Let us define the Beltrami coefficient

$$\mu(z) := \begin{cases} \psi^* \mu_0(z), & \text{if } z \in A_0 \\ \mu_0(z), & \text{if } z \in \mathbb{C} \setminus V \end{cases} = \begin{cases} \psi^* \mu_0(z), & \text{if } z \in A_0 \\ R^* \mu_0(z), & \text{if } z \in \mathbb{C} \setminus V \end{cases},$$

where the second equality follows from the conformality of R. It is easy to check that μ is F-invariant. We need to see first where the points are mapped by F:

$$R^{-1} \circ P_d \circ \psi : \overline{V \setminus U} \xrightarrow{\psi} \overline{\mathbb{A}_{\rho,\rho^d}} \xrightarrow{P_d} \mathbb{C} \setminus \mathbb{D}_{\rho^d} \xrightarrow{R^{-1}} \mathbb{C} \setminus V \tag{3.1}$$

$$R^{-1} \circ P_d \circ R : \mathbb{C} \setminus V \xrightarrow{R} \mathbb{C} \setminus \mathbb{D}_{\rho^d} \xrightarrow{P_d} \mathbb{C} \setminus \mathbb{D}_{\rho^{d^2}} \subset \mathbb{C} \setminus \mathbb{D}_{\rho^d} \xrightarrow{R^{-1}} \mathbb{C} \setminus V \tag{3.2}$$

Therefore, the F-invariance follows almost immediately:

– In A_0:

$$\begin{aligned} F^* \mu &= (R^{-1} \circ P_d \circ \psi)^* \mu = (R^{-1} \circ P_d \circ \psi)^* R^* \mu_0 = (R \circ R^{-1} \circ P_d \circ \psi)^* \mu_0 = \\ &= \psi^* P_d^* \mu_0 \underset{P_d \text{ hol}}{=} \psi^* \mu_0 = \mu \end{aligned}$$

– In $\mathbb{C} \setminus V$:

$$\begin{aligned} F^* \mu &= (R^{-1} \circ P_d \circ R)^* \mu = (R^{-1} \circ P_d \circ R)^* R^* \mu_0 = (R \circ R^{-1} \circ P_d \circ R)^* \mu_0 = \\ &= R^* P_d^* \mu_0 \underset{P_d \text{ hol}}{=} R^* \mu_0 = \mu \end{aligned}$$

We want to spread this Beltrami coefficient through all the complex plane keeping the F-invariance everywhere. An useful property of A_0 that we will use is that it is a fundamental domain of F (see Definition 3.0.16). Indeed,

$$z \in A_0 \underset{(3.1)}{\Longrightarrow} F(z) \in \mathbb{C} \setminus V \underset{(3.2)}{\Longrightarrow} F^n(z) \in \mathbb{C} \setminus V,$$

for all $n \geq 1$. Hence, we define

$$\mu(z) := \begin{cases} \psi^* \mu_0(z), & \text{if } z \in A_0 \\ (f^n)^* \mu(z), & \text{if } z \in A_n \\ \mu_0(z), & \text{elsewhere} \end{cases},$$

where $A_n := \{z \in U \mid f^n(z) \in A_0\}$. Notice that $K_f = U \setminus \cup_{n>0} A_n$ and A_n are disjoint for different values of n. Indeed, if $n_1 \neq n_2$ and there exists $z_0 \in A_{n_1} \cap A_{n_2}$, then $f_{n_1}(z_0), f_{n_2}(z_0) \in A_0$, which is not possible because we have seen that A_0 is a fundamental domain. It is direct that μ is still F-invariant as a consequence of the inclusions $f(K_f) \subset K_f$ and $f(A_n) \subset A_{n-1}$ for $n \geq 1$:

– Suppose $z \notin U$. It has been seen right above, before the spread of the Beltrami coefficient.

– If $z \in U$, we have the following possibilities:

  ∗ $z \in K_f$.

$$F^* \mu(z) = f^* \mu_0(z) \underset{f \text{ hol}}{=} \mu_0(z) = \mu(z)$$

* $z \in A_1$.

$$F^*\mu(z) = f^*\mu(z) = \mu(z)$$

* $z \in A_n$ $(n > 1)$.

$$F^*\mu(z) = f^*\mu(z) = f^*(f^{n-1})^*\mu(z) = (f^{n-1} \circ f)^*\mu(z) = (f^n)^*\mu(z) = \mu(z)$$

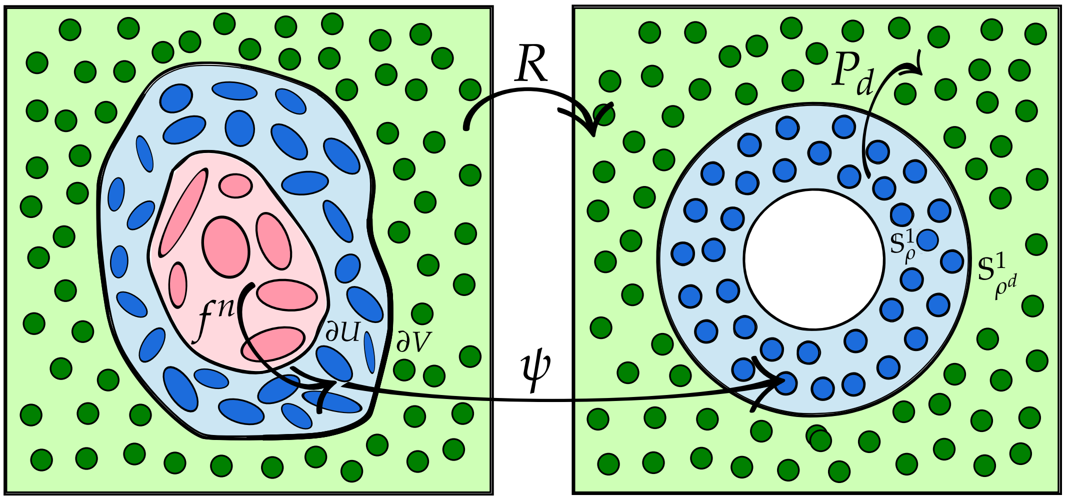


Figure 3.2: Beltrami coefficient through the mappings that define F.

Since f is holomorphic, by Remark 1.2.7 the dilatation of this almost complex structure is given by the dilatation on A_0, i.e. when $\mu(z) = \psi^*\mu_0(z)$. Thus, since ψ is quasiconformal, the dilatation is bounded by definition as we saw below the Corollary 1.3.9. Therefore, we can apply the Integrability Theorem, so there exists a quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$ so that $\mu = \phi^*\mu_0$ and we consider a normalization so that $\phi(\infty) = \infty$. Moreover, $\mu_0 = \phi^*\mu_0$ on K_f, so

$$\partial_{\bar{z}}\phi = 0 \tag{3.3}$$

is fulfilled on K_f.

Inspired by the Key Lemma (see Lemma 1.5.9 and its proof), the mapping $P := \phi \circ F \circ \phi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic. By the definition of F, it is conformally conjugate to z^d in a neighbourhood of ∞ and, at the same time, it is conjugate to P by ϕ. Because of that and since $\phi(\infty) = \infty$, ∞ is a superattracting fixed point of P of degree d. Hence, by Lemma 2.3.2 P is a polynomial of degree d. Finally, by (3.3) ϕ is a hybrid equivalence between f and P.

- Proof of the uniqueness. Assume K_f connected and that f is hybrid equivalent to two polynomials P_1 and P_2, so that their Julia set is also connected. By Remark 3.0.14, P_1 and P_2 are hybrid equivalent to each other and, by Theorem 3.0.15, they are affine conjugate.

□

Example 3.0.18. It can be proven that there is a copy of the Julia set of the quadratic map for $c \approx 0.122 + 0.745i$ in the dynamical space of the one with $c \approx -1.758 + 0.014i$. This Julia set is called Douady rabbit and it is named for the French mathematician [2]Adrien Douady.

[2]Adrien Douady: 1935 – 2006

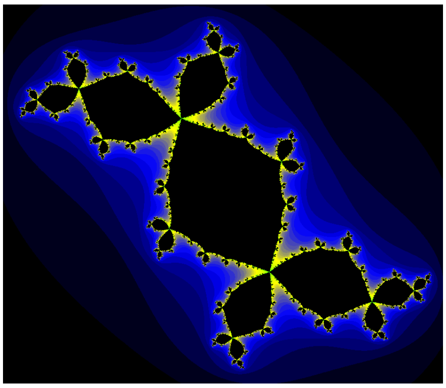


Figure 3.3: Julia set of Q_c with $c = -0.12256 + 0.74486i$.

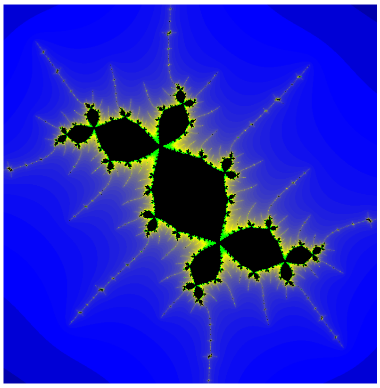


Figure 3.4: Julia set of Q_c with $c = -1.75778 + 0.0137961i$.

Example 3.0.19. More Douady rabbits can be found when looking at the dynamical plane of rational functions. The following figures show some examples and each caption indicates the corresponding map, where $:= 0.00848556 + 0.0547416i$ and $b := 0.1043 + 0.054743i$.

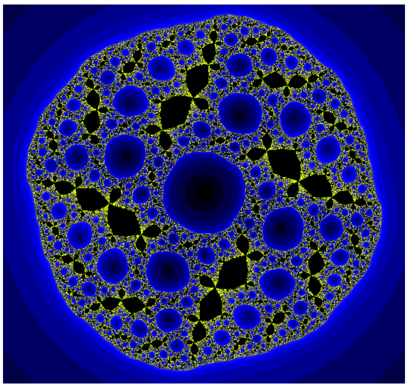


Figure 3.5: $z^2 + a/z^2$.

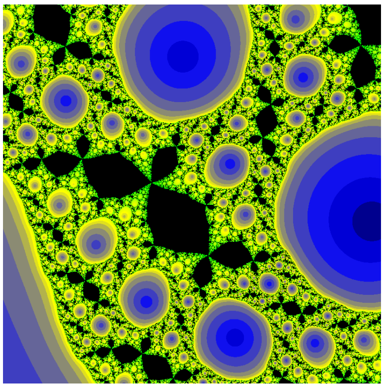


Figure 3.6: $z^2 + a/z^2$ (zoom).

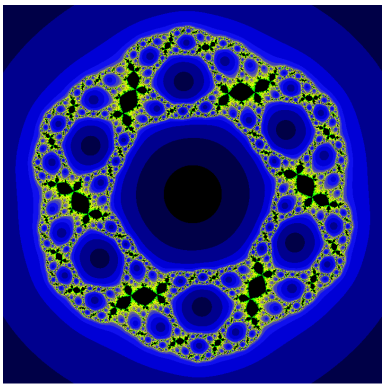


Figure 3.7: $z^3 + b/z^3$.

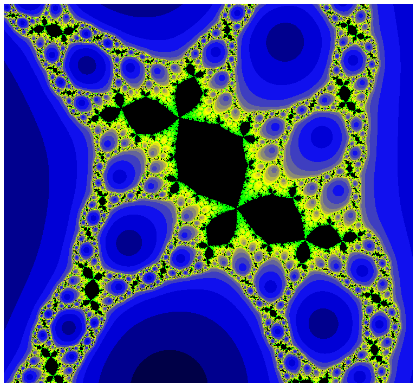


Figure 3.8: $z^3 + b/z^3$ (zoom).

Now, we are going to present a parameter version of the Straightening Theorem corresponding to quadratic-like mappings. As formulated by Adrien Douady:

"You first plough in the dynamical space and then harvest in the parameter plane"

Definition 3.0.20. (Holomorphic family of polynomial-like mappings) *Let S be a Riemann surface, $\mathcal{F} := \{f_s : U_s \to V_s\}$ be a family of polynomial-like maps and*

$$\mathcal{U} := \{(s,z) \mid s \in S \text{ and } z \in U_s\};$$
$$\mathcal{V} := \{(s,z) \mid s \in S \text{ and } z \in V_s\};$$
$$f(s,z) := (s, f_s(z)).$$

We say that $\mathcal{F}$ is a holomorphic family of polynomial-like maps if the following properties are satisfied:

1. *$\mathcal{U}$ and $\mathcal{V}$ are homeomorphic to $S \times \mathbb{D}$.*
2. *The projection $\overline{\mathcal{U}} \subset \mathcal{V} \to S$ defined as $(s,z) \mapsto s$ is proper.*
3. *The map $f : \mathcal{U} \to \mathcal{V}$ is holomorphic and proper.*

Remark 3.0.21. Under this definition, the degree of every polynomial-like map in $\mathcal{F}$ is the same. We associate it to $\mathcal{F}$ and, by the Straightening Theorem, for each $s \in S$ f_s is hybrid equivalent to a polynomial of this degree. Let us consider the case of degree 2 for a given surface Λ conformally equivalent to $\mathbb{D}$. In this case, $\mathcal{F} := \{(f_\lambda; U_\lambda, V_\lambda) \mid \lambda \in \Lambda\}$ is a family of quadratic-like maps. Define

$$\mathcal{M}_{\mathcal{F}} := \{\lambda \in \Lambda \mid K_{f_\lambda} \text{ is connected}\},$$

so that for each $\lambda \in \Lambda$ the map f_λ is hybrid equivalent to a unique quadratic polynomial $Q_c := z^2 + c$. Then, we can define

$$\begin{aligned} c : \mathcal{M}_{\mathcal{F}} &\longrightarrow \mathcal{M} \\ \lambda &\longmapsto c(\lambda) \end{aligned}$$

in this way.

The statement of the Theorem is as follows. The proof goes beyond the scope of this project.

Theorem 3.0.22. (Homeomorphic copies of the Mandelbrot set) *Let $\mathcal{F} := \{(f_\lambda; U_\lambda, V_\lambda) \mid \lambda \in \Lambda\}$ be a holomorphic family of polynomial-like maps of degree 2, where Λ is homeomorphic to $\mathbb{D}$. Consider a closed set $K \subset \Lambda$ homeomorphic to $\overline{\mathbb{D}}$ such that $\mathcal{M}_{\mathcal{F}} \subset K$. Assume that $\mathcal{M}_{\mathcal{F}}$ is compact. Let ω_λ be the critical point of f_λ and suppose that the vector $f_\lambda(\omega_\lambda) - \omega_\lambda$ turns once around 0 as λ turns once around ∂K. Then, the map $c : \mathcal{M}_{\mathcal{F}} \to \mathcal{M}$ is a homeomorphism, and it is holomorphic in the interior of $\mathcal{M}_{\mathcal{F}}$.*

Remark 3.0.23. Analogously to the version in the dynamical plane, we can find holomorphic copies of the Mandelbrot set in other parameter spaces and also within itself (see Figures 3.9 and 3.10).

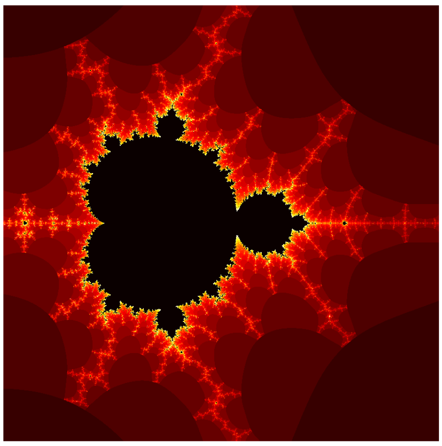


Figure 3.9: Copy of the Mandelbrot set in a neighbourhood $\lambda = \pi$ in the parameter space of $\lambda \cos z$.

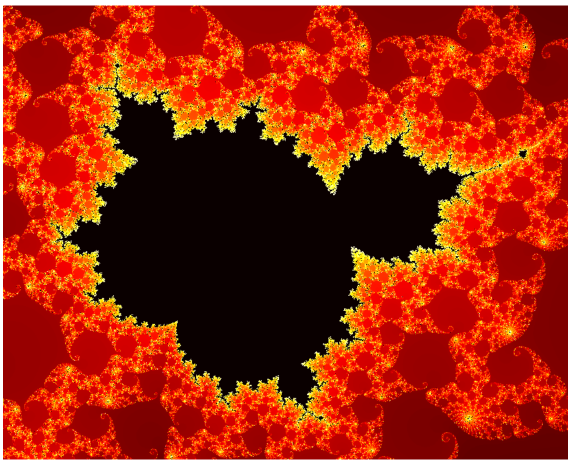


Figure 3.10: Copy of the Mandelbrot set in a neighbourhood of $c = 0.27215 + 0.00542i$ within itself.

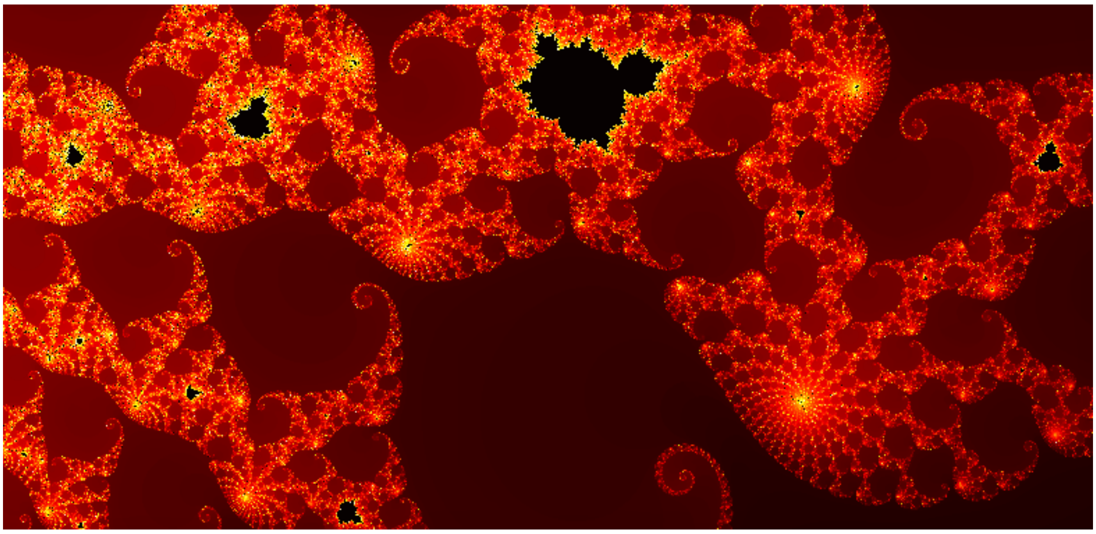


Figure 3.11: Several copies of the Mandelbrot set in a region within itself.

Chapter 4

Renormalization in complex dynamics

Renormalization theory plays a leading role in one of the major objectives of holomorphic dynamics: the proof of the MLC conjecture. It emerged from the work of Adrien Douady and [1]John H. Hubbard.

The importance of $\mathcal{M}$ lies in its universality. If proven true, its local connectivity would allow a complete topological description of $\mathcal{M}$. But, what do we mean when saying universal? As we saw in the last chapter, the Straightening Theorem shows that we can find copies of Julia sets of polynomials in the dynamical space of other holomorphic mappings, even if they are not polynomials. It arose from this fact an analogous version for the parameter space, which ensure that we can find homeomorphic copies of the Mandelbrot set when dealing with parameter spaces of certain families of holomorphic maps of degree 2. This is why we say that $\mathcal{M}$ is universal, since it describes local bifurcations of almost every family of quadratic maps.

The main purpose of this chapter is to apply the Straightening Theorem in the space of cubic polynomials, so that we can find the presence of the aforementioned copies. When focusing on the dynamical plane, we expect to find some $c \in \mathcal{M}$ so that the filled Julia set $K_c := K_{Q_c}$ is homeomorphic to infinitely many connected components of the filled Julia set of a certain class of cubic polynomials. With respect to this kind of polynomials, we focus on the ones with a critical point in the basin of attraction of ∞ and such that the other one fulfils a condition stricter than the boundedness of its orbit. In order to go beyond, we proceed to stretch out this result within a parameter space of a certain class of one-parameter cubic polynomials. Actually, we prove that we can find a copy of the Mandelbrot set in a neighbourhood of a localized value in the parameter space of such mappings approaching the proof with the support of the previous result in the dynamical space.

Before this exploration in the space of cubic polynomials, we present the famous conjecture mentioned above.

4.1 MLC conjecture

Definition 4.1.1. (Locally connected space) *If we consider a topological space X, we say that it is locally connected at $x \in X$ if every neighbourhood of x contains a connected open set U such that $x \in U$. The set X is locally connected if and only if it is locally connected at every point $x \in X$.*

Conjecture 4.1.2. (MLC conjecture) *The Mandelbrot set $\mathcal{M}$ is locally connected.*

As we have already stated, renormalization is the chief technique currently used to face MLC, so that we should give insights into its meaning when we look at the quadratic family.

Definition 4.1.3. (Once renormalizable mapping) *Given some $c \in \mathbb{C}$, we say that Q_c is (once) renormalizable if there exist two open sets $U, V \subset \mathbb{C}$ conformally equivalent to $\mathbb{D}$ and $n \in \mathbb{N}$ such that $0 \in U$ and*

[1]John Hamal Hubbard: 1945 – present

$(Q_c^n; U, V)$ is a quadratic-like map with connected Julia set. The couple (U, V) is called renormalization (or n-renormalization) of Q_c^n.

Remark 4.1.4. Notice that it is equivalent to say that $Q^{nk}(0) \in U$ for all $k \geq 0$.

The following result shows the uniqueness of a renormalization for a fixed $n \in \mathbb{N}$ when dealing with the classes of maps with the same filled Julia set (see [McM1, Theorem 7.0]).

Theorem 4.1.5. *Any two renormalizations of Q_c^n have the same filled Julia set.*

Definition 4.1.6. (Level of renormalization) *Each element of the set $\{n \geq 1 \mid Q_c^n$ is renormalizable$\}$ is known as a level of renormalization of Q_c.*

Definition 4.1.7. (k-times renormalizable mapping) *We say that Q_c is k-times renormalizable for some $k > 0$ if there exist n_i-renormalizations for some $n_1 < n_2 < \cdots < n_k$. In other words, if Q_c has k different levels of renormalization.*

Definition 4.1.8. (Finitely and infinitely renormalizable) *If Q_c is k-times renormalizable for every $k > 0$, we say that it is infinitely renormalizable. If Q_c is renormalizable but not infinitely renormalizable, then we say that Q_c is finitely renormalizable.*

Remark 4.1.9. Although we will not go into detail in this section, it is noteworthy that a map can be defined sending Q_c to a renormalization, i.e. to some iterate of Q_c such that it meets Definition 4.1.3. This operator $\mathrm{Ren} : f \to \mathrm{Ren}(f)$, which acts on the space of quadratic-like maps, is known as *renormalization operator* and it defines a dynamical system in itself. In general, an operator can be defined with an additional rescaling.

An important progress on the MLC conjecture was authored by J. C. Yoccoz Theorem.

Theorem 4.1.10. (Yoccoz Theorem) *If Q_c is not infinitely renormalizable, then the Mandelbrot set $\mathcal{M}$ is locally connected at c.*

An outline of the proof can be found in [McM2, Theorem 5.1]. To deal with it, Yoccoz established the notion of puzzles in the dynamical plane (similarly, parapuzzles in the parameter space), which marked a turning point in the search of suitable polynomial-like mappings. Broadly speaking, it consists of fixing an invariant set Γ of one or more cycles of periodic rays with their landing points, so that the connected components of $\mathbb{C} \setminus f^{-n}(\Gamma)$ $(n \geq 0)$ are either nested or disjoint. Such components are known as puzzle pieces of a certain level n, and they are mapped to puzzle pieces of level $n - 1$. These pieces are usually bounded by means of equipotentials. Finally, the local connectedness is proven through estimating moduli of annuli formed by consecutive puzzle pieces and taking into account that the boundaries of such pieces are escaping points.

In 1995, Yunping Jiang published a paper where he proved that values of $c \in \mathcal{M}$ where Q_c is infinitely renormalizable and such that $\mathcal{M}$ is locally connected at c are dense on the boundary of the Mandelbrot set $\partial\mathcal{M}$ (see [Jia1, Main Theorem]).

To date, some mathematicians have between their eyebrows the remaining cases, all among the infinitely renormalizable ones. Several authors who made important breakthroughs in this matter were awarded a Fields Medal, such as J. C. Yoccoz (1994), Curtis T. McMullen (1998) and Artur Avila (2014).

Example 4.1.11. (Feigenbaum quadratic polynomial) One of the remaining cases is the [2]Feigenbaum polynomial, which is Q_{c_F} for $c_F = -1.40115519....$ Such a parameter is the limit of a certain decreasing and bounded sequence of parameters where period doubling bifurcations occur (see [HJ, p. 102]) and it has been calculated numerically. The peculiarity of this example is that an appropriate restriction of $Q_{c_F}^2$ is a quadratic-like map topologically conjugate to Q_{c_F}, so that $Q_{c_F}^{2n}$ gives a renormalization for all $n \geq 1$. In this case, the renormalization operator maps Q_{c_F} to itself.

[2]Mitchell Jay Feigenbaum: 1944 – present

4.2 Renormalization in the space of cubic polynomials

Now is the time to study certain class of cubic polynomials as we mentioned in the introduction of this chapter. To tackle the following results, we use the notations of Remark 2.3.4, where we introduced the Böttcher map for polynomials. Recall that since ∞ is a superattracting point, the Böttcher coordinates give a conjugacy in a neighbourhood of ∞ between a polynomial of degree d and z^d. In our case, we are in the space of cubic polynomials, so that $d = 3$. Moreover, we saw that such a neighbourhood is the immediate basin of attraction of ∞ if the Julia set of our mapping is connected. Otherwise, it is an open set whose boundary contains the critical point that escapes faster to infinity. Such a conjugacy is denoted by φ and it maps the aforementioned open neighbourhood of ∞ to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}_r}$ for some $r \geq 1$. Also, we highly recommend to take a look at the proof of Theorem 2.3.17 to familiarize yourselves with the arguments used in this section.

Proposition 4.2.1. *Let P be a cubic polynomial with critical points $\omega_1 \in \mathcal{A}_P(\infty)$ and $\omega_2 \in K_P$. Suppose that the orbit of ω_2 under P is contained in a connected component of $f^{-1}(V)$, where $V := \{z \in \mathbb{C} \mid |\varphi(z)| < |\varphi(v)|\}$ and $v := f(\omega_1)$. Then, there is some $c \in \mathcal{M}$ and there exist infinitely many connected components of K_P such that all of these components are homeomorphic to $K_c := K_{z^2+c}$.*

Proof. By the Böttcher Theorem (see Theorem 2.2.14) and following the notation in Corollary 2.3.3, there exists $r > 1$ such that φ^{-1} extends to a conformal isomorphism $\varphi^{-1} : \mathbb{C} \setminus \overline{\mathbb{D}_r} \to \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r}) \subset \mathcal{A}_P(\infty)$, where $\omega_1 \in \partial\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})$. Denoting by v the critical value of ω_1, it is in $\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})$ because $|\varphi(v)| = r^3 > r$ and then $\varphi(v) \in \mathbb{C} \setminus \overline{\mathbb{D}_r}$.

Consider the external ray $\varphi^{-1}(R)$ with $R := \varphi^{-1}([1, \infty)\varphi(v))$ and the 3 distinct external rays in $P^{-1}(R) \cap \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})$ corresponding to the 3 roots obtained when applying the inverse of the Böttcher coordinates. Each of these 3 rays lands at some solution z to the equation $P(z) = v$ and two of them, say R_1 and R_2, land at ω_1, so that $\{\omega_1\} \cup R_1 \cup R_2 \subset \overline{\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})}$ cut the plane into two open connected sets. If we denote them by V_0 and V_1, then $K_P \subset \mathbb{C} \setminus R \cup R_1 \cup R_2 \cup \{\omega_1\} \subset P(V_i)$ for $i \in \{0, 1\}$ as we saw in the proof of Theorem 2.3.17.

Assume $\omega_2 \in V_0$ and define the open sets $U_0 := (\mathbb{C} \setminus \overline{\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})}) \cap V_0$, $U_1 := (\mathbb{C} \setminus \overline{\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_r})}) \cap V_1$ and $V := \mathbb{C} \setminus \overline{\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}_{r^3}})}$. Notice that $\omega_2 \in U_0$ and that V is the open set bounded by the equipotential of value $3 \log r$.

Since $(z \mapsto z^3)(\mathbb{S}_r) = \mathbb{S}_{r^3}$, the Böttcher map leads to the equality $P(U_0) = P(U_1) = V$. Recalling the Riemann-Hurwitz formula (see Theorem 3.0.8), we know that each point in V has exactly two preimages in U_0 and one in U_1. Therefore, defining $f := P_{|U_0}$, the triple $(f; U_0, V)$ is a polynomial-like map of degree 2. Now, by the Straightening Theorem, $K_f \underset{\text{top}}{\sim} K_c$ for some $c \in \mathbb{C}$. Besides, by hypothesis the orbit of ω_2 under P (and f) remains in U_0 and hence $c \in \mathcal{M}$, i.e. K_f is connected.

Notice that $K_f \subset K_P$ and let $\mathcal{C}_0$ be the connected component of K_P that contains K_f. We are going to see that $\mathcal{C}_0 = K_f$. Since $K_f \subset \mathcal{C}_0$, then $P^n(\mathcal{C}_0) \cap U_0 \neq \emptyset$ for all $n \in \mathbb{Z}_{\geq 0}$. Recalling that $P^n(\mathcal{C}_0)$ is connected for all $n \in \mathbb{Z}_{\geq 0}$ (see Proposition A.4.10), necessarily $P^n(\mathcal{C}_0) \subset U_0$ because $\mathcal{C}_0 \subset K_P \subset U_0 \cup U_1$, so that $\mathcal{C}_0 = K_f$. We have found a connected component of K_P, denoted by $\mathcal{C}_0$, that fulfils our statement.

Consider $g := P_{|U_1}$. Since $\omega_1, \omega_2 \notin U_1$, $g'(z) \neq 0$ for all $z \in U_1$ and, by Corollary A.3.10, $g : U_1 \to V$ is a biholomorphism, so $\mathcal{C}_1 := g^{-1}(\mathcal{C}_0)$ is another connected component of K_P that meets our statement.

Let us study the preimage $f^{-1}(\mathcal{C}_1)$. It is clear that $f^{-1}(\mathcal{C}_1) \cap \mathcal{C}_0 = \emptyset$ due to the definition of $\mathcal{C}_0$ and $g^{-1}(\mathcal{C}_1) \cap \mathcal{C}_1 = \emptyset$ because $g(g^{-1}(\mathcal{C}_1) \cap \mathcal{C}_1) = \mathcal{C}_1 \cap \mathcal{C}_0 = \emptyset$. Taking into account that $\omega_1 \notin U_0$ and $\omega_2 \in \mathcal{C}_0$, we know that $\omega_1, \omega_2 \notin f^{-1}(\mathcal{C}_1)$. Thus, by the Riemann-Hurwitz formula (Theorem 3.0.8) $f^{-1}(\mathcal{C}_1) = \mathcal{C}_2^1 \cup \mathcal{C}_2^2$, where $\mathcal{C}_2^1$ and $\mathcal{C}_2^2$ are different connected components of K_P, since $P^2(\mathcal{C}_2) = P^2(\mathcal{C}_3) = P(\mathcal{C}_1) = \mathcal{C}_0 := K_f \subset K_P$. Moreover, $\mathcal{C}_2^1$ and $\mathcal{C}_2^2$ are conformally equivalent to $\mathcal{C}_1$ (and so to $\mathcal{C}_0$) because $\omega_1, \omega_2 \notin \mathcal{C}_2^1 \cup \mathcal{C}_2^2$ and then $P'(z) \neq 0$ in a neighbourhood of these connected components, so we can consider again Corollary A.3.10. Analogously, we obtain a different connected component of K_P, which is $\mathcal{C}_2^3 := g^{-1}(\mathcal{C}_1)$.

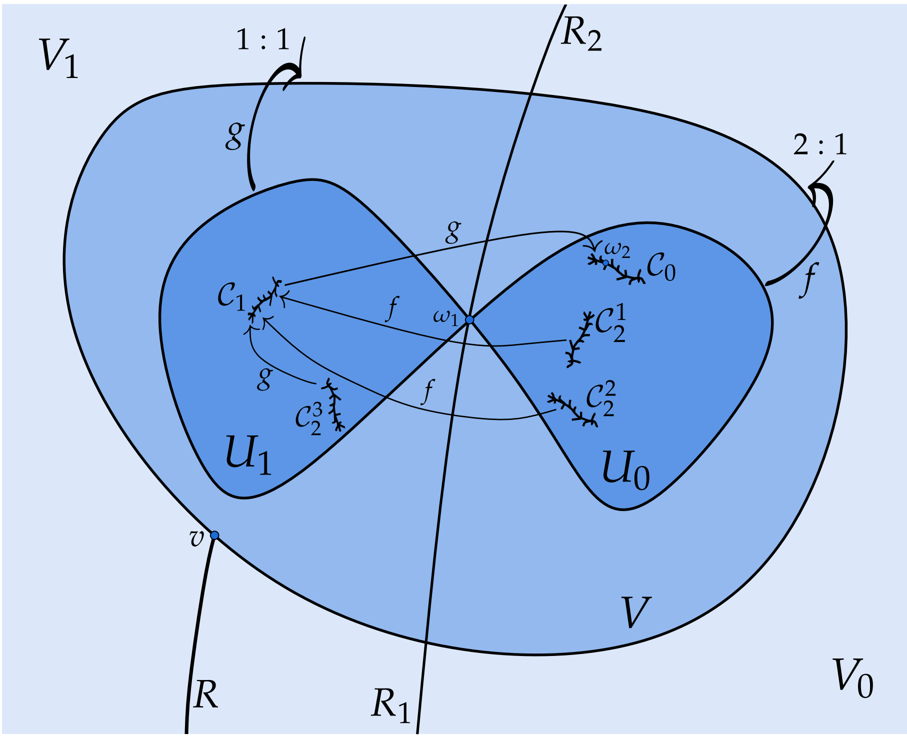
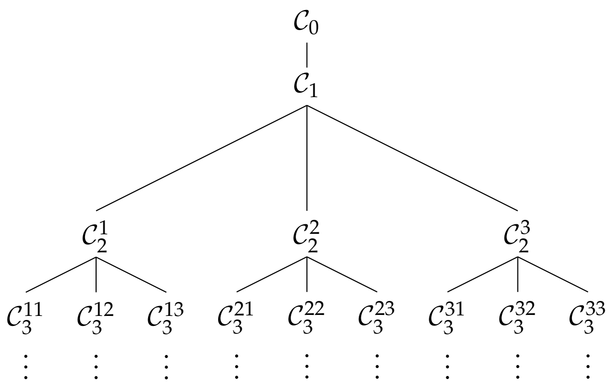


Figure 4.1: Sketch for the proof of Proposition 4.2.1.

Recursively, we can find infinitely many connected components that fulfils the statement:



Notice that the k-th generation is given by $P^{-k}(\mathcal{C}_0)$ and it consists of 3^{k-1} connected components. We are done because all these sets are different and thus we have found infinitely many connected components $\left(1+\sum_{k\geq 1} 3^{k-1}\right)$. Indeed:

1. Analogously to the justification given that $\mathcal{C}_2^i \neq \mathcal{C}_2^j$ for all $i, j \in \{1,2,3\}$ such that $i \neq j$, it follows that $\mathcal{C}_k^{i_1\ldots i_{k-1}} \neq \mathcal{C}_k^{i_1\ldots i_{k-2}j_{k-1}}$ for all $k \in \mathbb{Z}_{\geq 3}$ and all $i_1, \ldots, i_{k-1}, j_{k-1} \in \{1,2,3\}$ such that $i_{k-1} \neq j_{k-1}$.

2. Assume $\mathcal{C}_k^{i_1\ldots i_{k-1}} = \mathcal{C}_k^{j_1\ldots j_{k-1}}$ for some $k \in \mathbb{Z}_{\geq 3}$ and $i_1, \ldots, i_{k-1}, j_1, \ldots, j_{k-1} \in \{1,2,3\}$ such that $i_s \neq j_s$ for some $s \in \{1, \ldots, k-2\}$. Defining $t := \min\{s \in \{1, \ldots, k-2\} \mid i_s \neq j_s\}$, the equality $P^{k+1-t}(\mathcal{C}_k^{i_1\ldots i_{k-1}}) = P^{k+1-t}(\mathcal{C}_k^{j_1\ldots j_{k-1}})$ leads to a contradiction because of the first case studied right above.

3. Suppose that there exist $0 \leq k_1 < k_2$ with $k_2 > 2$ such that $\mathcal{C}_{k_1} = \mathcal{C}_{k_2}$, where $\mathcal{C}_{k_1} := \mathcal{C}_{k_1}^{i_1\ldots i_{k_1-1}}$

and $\mathcal{C}_{k_2} := \mathcal{C}_{k_2}^{j_1 \dots j_{k_2-1}}$ for some $i_1, \dots, i_{k_1-1}, j_1, \dots, j_{k_2-1} \in \{1,2,3\}$. Then, $P^{k_2-2}(\mathcal{C}_{k_1}) = P^{k_2-2}(\mathcal{C}_{k_2})$, which is a contradiction because $P^{k_2-2}(\mathcal{C}_{k_1}) \in \{\mathcal{C}_0, \mathcal{C}_1\}$ and $P^{k_2-2}(\mathcal{C}_{k_2}) \in \{\mathcal{C}_2^i\}_{i=1,2,3}$.

□

Remark 4.2.2. As the proof of Proposition 4.2.1 shows, it is equivalent to require the orbit of ω_2 under P to be contained either in a connected component of $f^{-1}(\{z \in \mathbb{C} \mid |\varphi(z)| < |\varphi(v)|\})$ or in a fixed connected component of K_P.

Example 4.2.3. For cubic polynomials of the form $z^2(z - \frac{3}{2}a)$, the origin is always a fixed superattracting point. The other critical point is the parameter a. If a escapes to infinity, we can apply Proposition 4.2.1. For instance, this is what happens when $a = 16/15$, i.e. when considering $z^3 - 1.6z^2$. Moreover, since the critical point $\omega_2 = 0$ is fixed by the polynomial (and hence by the corresponding quadratic-like map of Proposition 4.2.1), we know that there are infinitely many components homeomorphic to the filled Julia set of $Q_0(z) = z^2$, i.e. $K_0 = \overline{\mathbb{D}}$.

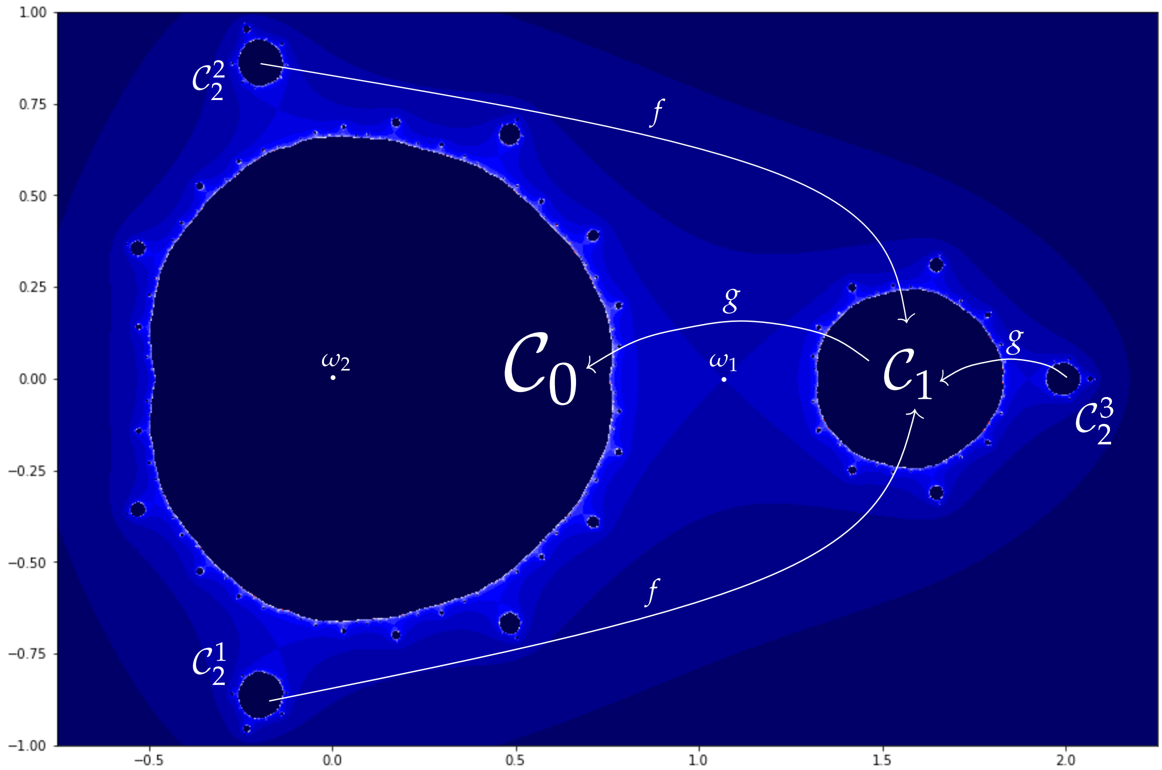


Figure 4.2: Filled Julia set of the cubic polynomial $z^3 - 1.6z^2$.

A one-parameter family of cubic polynomials

Now, we are going to study the parameter space of certain cubic polynomial families.

Theorem 4.2.4. *Let $\Omega \subset \mathbb{C}$ be a compact set homeomorphic to $\overline{\mathbb{D}}$ and consider a family of cubic polynomials $P_\lambda(z) := a_3(\lambda)z^3 + a_2(\lambda)z^2 + a_1(\lambda)z + a_0(\lambda)$, where a_i $(i \in \{0,1,2,3\})$ are holomorphic and $a_3(\lambda) \neq 0$ and $(a_2(\lambda))^2 - 3a_1(\lambda)a_3(\lambda)$ is injective in a neighbourhood of Ω. Let $\omega_1(\lambda)$ and $\omega_2(\lambda)$ be the critical points of P_λ and assume that*

- *both critical points are different and $|P_\lambda(\omega_1(\lambda))| \xrightarrow{n} \infty$ for all $\lambda \in \Omega$;*
- *there exists exactly one $\lambda_0 \in \Omega$ so that $P_{\lambda_0}(\omega_2(\lambda_0)) = \omega_2(\lambda_0)$;*

- $|P_\lambda(\omega_2(\lambda))| \xrightarrow{n} \infty$ *for all* $\lambda \in \partial\Omega$;
- $g_i(\lambda) := g_{P_\lambda}(\omega_i(\lambda))$ *($i = 1,2$) are harmonic, where* g_{P_λ} *is the Green's function of* P_λ *for all* $\lambda \in \Omega$.

Then, there exists a connected component of $\{\lambda \in \Omega \mid |P_\lambda^n(\omega_2(\lambda))| \overset{n}{\not\to} \infty\}$ *contained in* Ω *and homeomorphic to the Mandelbrot set* $\mathcal{M}$.

Proof. Observe that the critical points of P_λ are given by

$$\frac{-a_2(\lambda) \pm \sqrt{(a_2(\lambda))^2 - 3a_1(\lambda)a_3(\lambda)}}{3a_3(\lambda)},$$

so they are holomorphic with respect to $\lambda \in \Omega$. Notice that the square root is holomorphic because $\omega_1(\lambda) \neq \omega_2(\lambda)$ for all $\lambda \in \Omega$ and the injective map $(a_2(\lambda))^2 - 3a_1(\lambda)a_3(\lambda)$ sends Ω to a simply connected set that does not contain the origin.

We claim to find two sets Λ and K, and a family of quadratic-like maps as in Theorem 3.0.22.

Define $\delta := \frac{1}{2}\min\{g_1(\lambda) \mid \lambda \in \Omega\} > 0$, which is well-defined due to the compactness of Ω. Let us consider on the interior of Ω the level curves of g_2 with value δ, i.e. the set $\Delta := \{\lambda \in \operatorname{int}\Omega \mid g_2(\lambda) = \delta\}$.

Let W be the connected component of $\operatorname{int}\Omega \setminus \Delta$ that contains λ_0. If W is simply connected, then we define $K := \overline{W}$. Otherwise, we consider

$$\widetilde{W} := W \cup \left(\bigcup_{\sigma \subset \overline{W}} \operatorname{int}\sigma \right),$$

where σ is a simple closed curve in W. In this case, $\widetilde{W}$ is simply connected and we set $K := \overline{\widetilde{W}}$.

Observe that $g_2(\lambda) \leq \delta < g_1(\lambda)$ for all $\lambda \in K$. Indeed, since $g_2(\lambda) \leq \delta$ for all $\lambda \in \partial K$, the Maximum Principle for harmonic functions ensures our requirement. Moreover, $g_2(\lambda) > 0$ for all $\lambda \in \partial K$. The boundary ∂K is contained in $\Delta \cup \mathbb{S}^1$, where Δ is the aforementioned set of level curves where $g_2(\lambda) = \delta$, so that ∂K is a curve. Note that ∂K is a simple closed curve that divide Ω into two connected components and K is homeomorphic to $\overline{\mathbb{D}}$.

Due to the continuity of g_1 and g_2, we can define an open neighbourhood Λ of K homeomorphic to $\mathbb{D}$ so that $g_2(\lambda) < g_1(\lambda)$ and $g_1(\lambda) > 0$ for all $\lambda \in \Lambda$, and $g_2(\lambda) > 0$ for all $\lambda \in \Lambda \setminus K$.

Proceeding as in Theorem 2.3.17 and Proposition 4.2.1 for each $\lambda \in \Lambda$, it exists an open set U_λ so that $(P_\lambda; U_\lambda, V_\lambda := P_\lambda(U_\lambda))$ is a quadratic-like mapping, where U_λ is the connected component of the eight-figure. Recall that $\omega_2(\lambda) \in U_\lambda$ and $\omega_1(\lambda) \in \partial U_\lambda$. Let us denote by

$$\mathcal{F} := \{(P_\lambda; U_\lambda, V_\lambda) \mid \lambda \in \Lambda\}$$

the family of quadratic-like mappings and consider the connectedness locus

$$\mathcal{M}_\mathcal{F} := \{\lambda \in \Lambda \mid (P_\lambda^n(\omega_2(\lambda)))_{n \geq 0} \subset U_\lambda\},$$

which is contained in K because of the chosen Λ.

If we define the holomorphic map $f(\lambda) := P_\lambda(\omega_2(\lambda)) - \omega_2(\lambda)$ for $\lambda \in \Lambda$, since $\lambda_0 \in \operatorname{int}\partial K$ and $f(\lambda_0) = P_{\lambda_0}(\omega_2(\lambda_0)) - \omega_2(\lambda_0) = 0$, then $0 \in \operatorname{int} f(\partial K)$. Moreover, by the uniqueness of λ_0 as zero of f, we can apply the Argument Principle (see [Con1, p. 123, Theorem 3.4]) so that $f(\lambda)$ turns once around 0 as λ turns once around ∂K.

It is clear that $\mathcal{M}_\mathcal{F} \subset \{\lambda \in \Omega \mid |P_\lambda^n(\omega_2(\lambda))| \overset{n}{\not\to} \infty\}$, so we can denote by $\mathcal{C}_{\mathcal{M}_\mathcal{F}}$ the connected component of $\{\lambda \in \mathbb{C} \mid |P_\lambda^n(\omega_2(\lambda))| \overset{n}{\not\to} \infty\}$ that contains $\mathcal{M}_\mathcal{F}$. It only remains to check that $\mathcal{M}_\mathcal{F}$ is precisely $\mathcal{C}_{\mathcal{M}_\mathcal{F}}$.

Let $\widetilde{U}_\lambda$ be the other connected component of the eight-figure, so that $\widetilde{U}_\lambda$ and U_λ are disjoint for all $\lambda \in \Lambda$. Define

$$\mathcal{U} := \{(\lambda, z) \mid \lambda \in \Lambda \text{ and } z \in U_\lambda\} \subset \mathbb{C}^2$$

and

$$\widetilde{\mathcal{U}} := \{(\lambda, z) \mid \lambda \in \Lambda \text{ and } z \in \widetilde{U}_\lambda\} \subset \mathbb{C}^2,$$

so that $\mathcal{U} \cap \widetilde{\mathcal{U}} = \emptyset$, and consider the mappings

$$\begin{aligned} F_n : \Lambda &\longrightarrow \mathbb{C}^2 \\ \lambda &\longmapsto (\lambda, P_\lambda^n(\lambda)) \end{aligned}$$

for all $n \in \mathbb{N}$. We know that F_n is continuous, $F_n(\mathcal{C}_{\mathcal{M}_\mathcal{F}}) \subset \mathcal{U} \cup \widetilde{\mathcal{U}}$ and $F_n(\mathcal{M}_\mathcal{F}) \subset \mathcal{U}$ for all $n \in \mathbb{N}$. Since $\mathcal{C}_{\mathcal{M}_\mathcal{F}}$ is connected, we know that $F_n(\mathcal{C}_{\mathcal{M}_\mathcal{F}})$ is connected for all $n \in N$ (see Proposition A.4.10) and necessarily $F_n(\mathcal{C}_{\mathcal{M}_\mathcal{F}}) \subset \mathcal{U}$, so that $\mathcal{C}_{\mathcal{M}_\mathcal{F}} = \mathcal{M}_\mathcal{F}$ as required.

□

Once we have proven the general result, we proceed to deal with the particular cubic family $P_\lambda(z) := \lambda z^2(2z-3) + 4 + \lambda$.

Proposition 4.2.5. *Consider the polynomial $P_\lambda(z) := \lambda z^2(2z-3) + 4 + \lambda$ and the set $S_\infty(\lambda) := \{z \in \mathbb{C} \mid |z| \geq \max(2, |4+\lambda|)\}$. For each $(\lambda, z) \in \{\lambda' \in \mathbb{C} \mid |4+\lambda'| < 3\} \times \mathbb{C}$, if there exists some $n_0 \in \mathbb{N}$ such that $P_\lambda^{n_0}(z) \in S_\infty(\lambda)$, then $|P_\lambda^n(z)| \xrightarrow{n} \infty$.*

Proof. Assume $(\lambda, z) \in \{\lambda' \in \mathbb{C} \mid |4+\lambda'| < 3\} \times \mathbb{C}$ and $P_\lambda^{n_0}(z) \in S_\infty(\lambda)$ for some $n_0 \in \mathbb{N}$. Bearing in mind that $|4+\lambda| > 3$ implies $1 < |\lambda| < 7$, we have that $|\lambda z(2z-3)| = |\lambda||z|(2|z|-3) > 2$ for all $z \in S_\infty(\lambda)$.

Let us prove by induction that $|P_\lambda^{n_0+n}(z)| \geq [|\lambda||P_\lambda^{n_0}(z)|(2|P_\lambda^{n_0}(z)|-3)-1]^n |P_\lambda^{n_0}(z)|$ for all $n \geq 1$. First,

$$\begin{aligned} |P_\lambda^{n_0+1}(z)| = |\lambda(P_\lambda^{n_0}(z))^2(2P_\lambda^{n_0}(z)-3)+4+\lambda| \geq |\lambda||P_\lambda^{n_0}(z)|^2(2|P_\lambda^{n_0}(z)|-3) - |4+\lambda| \geq \\ \geq [|\lambda||P_\lambda^{n_0}(z)|(2|P_\lambda^{n_0}(z)|-3)-1]|P_\lambda^{n_0}(z)|. \end{aligned}$$

Notice that if $z \in S_\infty(\lambda)$, then $P_\lambda^k(z) \in S_\infty(\lambda)$ for all $k \geq 0$ because $|P_\lambda^{k+1}(z)| > |P_\lambda^k(z)|$. Now, if we suppose that $|P_\lambda^{n_0+n}(z)| \geq [|\lambda||P_\lambda^{n_0}(z)|(2|P_\lambda^{n_0}(z)|-3)-1]^n |P_\lambda^{n_0}(z)|$, then

$$\begin{aligned} |P_\lambda^{n_0+n+1}(z)| \geq [|\lambda||P_\lambda^{n_0+n}(z)|(2|P_\lambda^{n_0+n}(z)|-3)-1]|P_\lambda^{n_0+n}(z)| \geq \\ \geq [|\lambda||P_\lambda^{n_0}(z)|(2|P_\lambda^{n_0}(z)|-3)-1]^{n+1}|P_\lambda^{n_0}(z)|. \end{aligned}$$

Since $|\lambda||P_\lambda^{n_0}(z)|(2|P_\lambda^{n_0}(z)|-3) > 2$, we obtain $\lim_n |P_\lambda^n(z)| = \lim_k |P_\lambda^{n_0+k}(z)| = \infty$ as desired. □

Proposition 4.2.6. *Given the family of polynomials $P_\lambda(z) := \lambda z^2(2z-3)+4+\lambda$, the connected component of $\{\lambda \in \mathbb{C} \mid |P_\lambda^n(0)| \overset{n}{\not\to} \infty\}$ that contains $\lambda_0 = -4$ is homeomorphic to the Mandelbrot set $\mathcal{M}$.*

Proof. It suffices to show that P_λ fulfils the hypothesis of Theorem 4.2.4. Define $\Omega := \{\lambda \in \mathbb{C} \mid |4+\lambda| < 5/2\}$. We have that $2\lambda \neq 0$ for all $\lambda \in \Omega$ and that $9\lambda^2$ is injective on a neighbourhood of Ω, so that the coefficients conditions are satisfied. The critical points of P_λ are 0 and 1 for all $\lambda \in \Omega$. Besides, $P_\lambda(0) = 4+\lambda$ and $P_\lambda(1) = 4$. Notice that 0 is a fixed point of P_λ if and only if $\lambda = -4$, which is a point of the interior of Ω. Finally, applying Proposition 4.2.5 we know that $P(0) = 4+\lambda \in S_\infty(\lambda)$ for all $\lambda \in \partial\Omega$ and $P(1) = 4 \in S_\infty(\lambda)$ for all $\lambda \in \Omega$, so we are done.

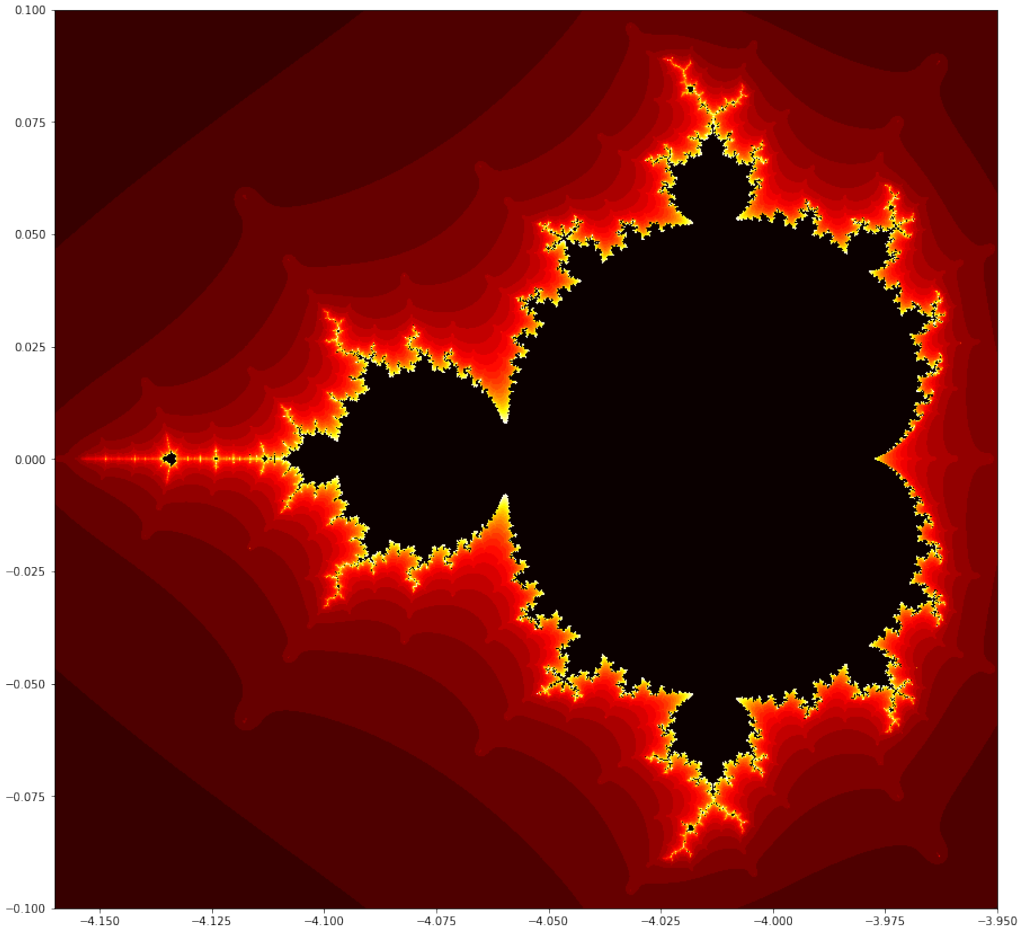


Figure 4.3: Homeomorphic copy of the Mandelbrot set in a parameter space of the family of cubic polynomials $P_\lambda(z) := \lambda z^2(2z-3)+4+\lambda$.

□

Chapter 5

Renormalization in other contexts

The concept of renormalization is also treated in applied sciences such as physics, chemistry or network theory, but from a slightly different point of view and approaching it in a different way. For instance, sometimes physicists use it as a technique to solve divergences of certain values. Although in this writing we will try to be as rigorous as we can, we must bear in mind that applied sciences are not always totally accurate and could proceed in a mathematically dubious way.

To fix ideas, we outline here what is known as percolation theory. It appears, among other contexts, in statistical physics to deal with phase transitions of the matter, magnetization, etc. As a curiosity, it appears in the work of critical phenomena by Kenneth G. Wilson, for which he was awarded the Nobel Prize in Physics. Moreover, Stanislav Smirnov, current Professor at the University of Geneva, received a Fields Medal in 2010 "for the proof of conformal invariance of percolation and the planar Ising model in statistical physics".

5.1 Percolation

Consider a square lattice with some occupied sites (coloured sites) as in Figure 5.1.

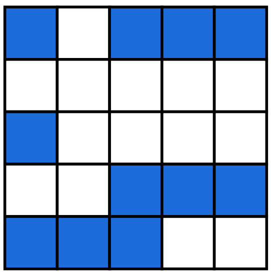


Figure 5.1: Square lattice of (normalized) linear size $L = 5$. There are two 1-clusters, one 3-cluster and one 6-cluster. The last one is a percolating cluster.

Remark 5.1.1. Given a square lattice of certain linear size (i.e. with sides of such length), we can resize (normalize) it so that the linear sizes becomes some $L \in \mathbb{Z}_{>0}$. Notice that we can identify an infinite square lattice, i.e. the limit $L \to \infty$, with $\mathbb{Z}^2$.

Definition 5.1.2. (Cluster) *An s-cluster is a set of s nearest neighbouring occupied sites.*

Definition 5.1.3. (Percolating cluster) *We say that a cluster is percolating if it connects two opposite sides of the lattice. In an infinite lattice, the definition is analogous, but bearing in mind that such a cluster must go to infinity on both sides. We say that it is an infinite cluster.*

Remark 5.1.4. It has been proved by the Kolmogorov Zero-One Law (see [Ros, p. 37, Theorem 3.5.1]) that, in our case, the probability that an infinite cluster exists is either zero or one and moreover, if it exists, it is unique. We will not go into details because the theory behind this result corresponds to graphs and stochastic processes and is beyond the scope of this thesis.

Definition 5.1.5. (Occupation probability) *By hypothesis, every site can be occupied with a certain fixed probability $p \in [0,1]$, which is called the occupation probability.*

It is noteworthy that we are focusing on the specific lattice $\mathbb{Z}^2$, but there are other examples:

- $\mathbb{L}^d := L\mathbb{Z}^d$, where $d \in \mathbb{N}$ is known as the dimension of the lattice;
- The Bethe lattice introduced by [1]Hans A. Bethe, where each site has exactly z neighbouring sites, such that each branch gives rise to $z-1$ other branches;
- The triangular lattice, that is also two dimensional as the square one;
- Other three dimensional lattices that we get by adding new sites to $\mathbb{L}^3$: body centred cubic (bcc), face centred cubic (fcc), diamond, etc. This kind of lattices are quite common in crystallography.

Furthermore, we are also dealing with sites that are connected (not exactly in the topological sense, but simply in the fact that the lattice is a single piece), so that we only focus on the occupation of each site and not on its connectedness. This is called site percolation. On the other hand, if we considered a graph with all the vertices seen as occupied sites and distinguished between open and closed edges, we would be working on the so-called bond percolation.

Let us give some important magnitudes in percolation theory and set some notation.

First of all, $n_s(p)$ will symbolize the probability of an arbitrary site (occupied or not) being a particular (fixed) site in an s-cluster, so that $sn_s(p)$ will be the probability of an arbitrary site being in an s-cluster. We will denote by $\theta(p)$ the probability that a site belongs to the percolating infinite cluster.

Excluding the infinite cluster, $S(p)$ will be the average cluster size, $\xi(p)$ the typical radius of the largest finite cluster and $s_\xi(p)$ the characteristic cluster size. The last magnitude quantity is usually given by the value of s such that $n_s(p)$ has decreased by a factor $1/e$, so that $n_{s_\xi}(p)/n_1(p) = 1/e$.

5.1.1 Critical occupation probability

Looking at the behaviour of $\theta : [0,1] \to [0,1]$, it is seen in [Gri, p. 154] that it is non-decreasing. Furthermore, we know that $\theta(0) = 0$ and $\theta(1) = 1$, so necessarily there exists some $p_c \in [0,1]$ such that

$$\theta(p) \begin{cases} = 0, & p < p_c \\ > 0, & p > p_c \end{cases}.$$

This singular value is called critical occupation probability or percolation threshold. Besides, it is also known that $\theta(p)$ is continuous on $[0,1] \setminus \{p_c\}$ and in the case of $\mathbb{Z}^2$ the continuity holds at p_c (see [BS, 1.3.1. Percolation in slabs]). Moreover, in our case, $0 < p_c < 1$ as a particular case of [Gri, Theorem 3.2] and $\theta(p_c) = 0$ due to [Gri, Theorem 9.1]. The large but finite clusters corresponding to $p = p_c$ are called incipient infinite clusters and sometimes we call them percolating clusters although their finitude. Henceforth, we will focus on $p \geq p_c$.

Given a square window of linear size $l \in \mathbb{Z}$ in a percolating cluster, we define the mass of such a cluster as its number of sites in that window and we denote it by $M(p;l)$. The associated density is $\theta(p;l) := M(p;l)/l^2$ and $\theta(p) = \lim_{l\to\infty} \theta(p;l)$.

It has been shown numerically (see, for instance, [CM, p. 56, Figure 1.25(b)]) that $M(p_c;l) \propto l^D$, where D is known as the fractal dimension of the incipient cluster. In $\mathbb{Z}^2$, $D = 91/48 < 2$, so that

$$\theta(p_c;l) = \frac{M(p_c;l)}{l^2} \propto l^{-5/48} \implies \theta(p_c) = \lim_{l\to\infty} \theta(p_c;l) = 0$$

[1]Hans Albrecht Bethe: 1906 – 2005

as required. Statistically, an incipient infinite cluster looks the same on all length scales. However, its density decreases with increasing l, irrespective of where on the fractal the window is placed.

5.1.2 Universal critical exponents

It has been shown (in some cases numerically) that, when approaching p_c, the magnitudes introduced above have an asymptotic behaviour given by a power of $(p - p_c)$ or s and characterized by a certain exponent. To summarize this, we give the following table from [CM, p. 81] in terms of the dimension of the lattice.

Exponent	Quantity	$d = 1$	$d = 2$	$d = 3$	$d \geq 6$	Bethe
β	$\theta(p) \propto (p - p_c)^{\beta}$	0	5/36	0.4181(8)	1	1
γ	$S(p) \propto \lvert p - p_c\rvert^{-\gamma}$	1	43/18	1.793(3)	1	1
ν	$\xi(p) \propto \lvert p - p_c\rvert^{-\nu}$	1	4/3	0.8765(16)	1/2	1/2
σ	$s_{\xi}(p) \propto \lvert p - p_c\rvert^{-1/\sigma}$	1	36/91	0.4522(8)	1/2	1/2
τ	$n_s(p) \propto s^{-\tau}\mathcal{G}(s/s_{\xi})$	2	187/91	2.18906(6)	5/2	5/2
D	$s_{\xi} \propto \xi^{D}$	1	91/48	2.523(6)	4	4

Table 5.1: Critical exponents. The brackets in the column $d = 3$ give the uncertainty on the last digit(s). For the cases $d = 4$ and $d = 5$ we refer to [CM, p. 81].

Notice that there can be found relations between some of the exponents: $\gamma = (3 - \tau)/\sigma$, $\beta = (\tau - 2)/\sigma$ and $D = 1/(\sigma\nu)$. The function $\mathcal{G}$ is known as the scaling function for the cluster number density and, given the critical exponent τ, it can be found by a procedure known as data collapse. Writing $\mathcal{G}(s/s_{\xi}) \propto s^{\tau} n_s(p)$ and assuming we have a set $\{n_s(p_1), n_s(p_2), n_s(p_3), \dots\}$ for occupation probabilities $p_1, p_2, p_3, \dots$, we must plot $s^{\tau} n_s(p)$ versus the rescaled variable s/s_{ξ}. In accordance with the given relation of proportionality, it is obtained the graph of the scaling function $\mathcal{G}$. We must remark that precisely this proportionality approximation is valid when $s \gg 1$.

These so-called critical exponents do not depend on the lattice details, but only on its dimensionality. Hence, we can say that they are universal. Recall that this is not a property fulfilled by p_c because it depends on the underlying lattice details. The critical exponents are valid for infinite lattices, that is where we are working $(\mathbb{Z}^2)$.

5.1.3 Real-space renormalization group

In general, it is usual to solve a problem finding its characteristic scale and dividing it into smaller uncorrelated "subproblems" of such size so that we can deal with them in an easier way than the initial system. In our case, ξ could be considered as an indicator of the characteristic scale because of its definition. For instance, if we see a mountain range far away, it looks pretty smooth. Otherwise, if you are in the middle of the mountain range, then you appreciate its roughness. Sometimes, ξ is called correlation length. Since it diverges at p_c, the explained approach becomes unworkable.

For this reason, we are going to take advantage of the self-similarity emerging at $p = p_c$ explained above. Imagine that we rescale our system so that all length scales are reduced by a factor $b > 1$, implying in particular the transformation $\xi \mapsto \xi/b$. The fixed points of this transformation, that are 0 and ∞, correspond to the self-similar configurations of our system. Indeed, $\xi = 0$ matches up with $p = 0$ and $p = 1$, that are the trivial configurations with no finite clusters. On the other hand, $\xi = \infty$ corresponds to $p = p_c$ (see the exponent ν in Table 5.1). Thus, the equation $\xi = \xi/b$ leads to scale invariance, which is indeed the requirement for self-similarity. The correlation length is given in Figure 5.2.

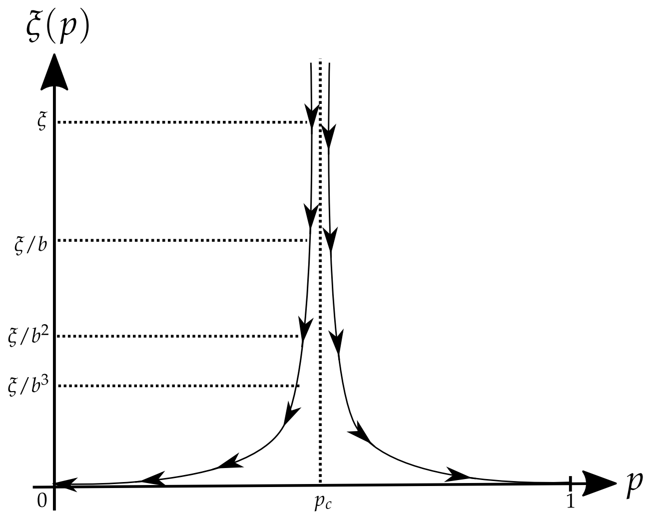


Figure 5.2: Correlation length ξ as a function of the occupation probability p.

When doing the same transformation for $p \neq p_c$, we obtain a sequence of decreasing occupation probabilities for $p < p_c$ and increasing when $p > p_c$, so that the limit values are 0 and 1, respectively. This leads to a flow that is symbolized by arrows in Figure 5.2. Let $T_b : [0,1] \to [0,1]$, $p \mapsto T_b(p)$, be the rescaling transformation giving the new occupation probability in the rescaled system with the smaller correlation length ξ/b. Through the critical exponent relation and imposing $\xi = \xi/b$, we obtain $|p - p_c|^{-\nu}/b = |T_b(p) - p_c|^{-\nu}$ for $p \to p_c$, so that

$$\nu = \frac{\log b}{\log\left(\frac{|T_b(p)-p_c|}{|p-p_c|}\right)} = \frac{\log b}{\log\left(\frac{|T_b(p)-T_b(p_c)|}{|p-p_c|}\right)} = \frac{\log b}{\log\left(\left.\frac{dT_b}{dp}\right|_{p_c}\right)}$$

for $p \to p_c$.

This seems a great idea to find p_c and ν, that are the unknowns of our problem, but we actually do not know how to find T_b easily. We usually approximate T_b by R_b, which is given by the second step of the following procedure:

1. Divide the lattice into blocks of linear size $b \in \mathbb{Z}$ such that each block contains a fixed quantity of sites.

2. Apply $R_b(p)$, defined so that the block is occupied if and only if it contains more occupied sites than empty ones (majority rule). Another possibility is to occupy the block if and only if a cluster spans the block in specified directions (spanning-cluster rule).

3. Rescale all lengths by the factor b to restore the original lattice spacings.

In the literature, R_b is usually called real-space renormalization group transformation. Nevertheless, its name arises from the original problems in quantum field theory and in our case it does not have a well-defined inverse, so mathematically it is not a group.

Assuming this definitions for R_b, we are doing approximations that obviate microscopic connections of our lattice, but we are not going to explain it here. We must just take into account that now the non-trivial fixed point of R_b, denoted by p', does not match with p_c, but it is a good approximation.

As an example, let us find p_c and ν when $b = 2$ and using the vertical spanning-cluster rule. The cases that a 2×2 block becomes occupied, are the following:

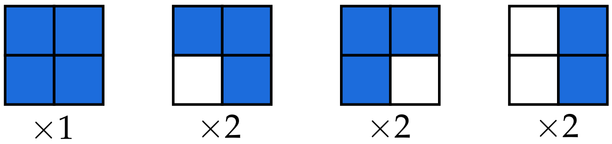


Figure 5.3: Occupied blocks for $b = 2$.

Hence, bear in mind that each configuration can be given in different ways (obtained rotating them), $R_b(p) = p^4 + 4p^3(1-p) + 2p^2(1-p)^2 = 2p^2 - p^4$, $p_c \approx p' = (-1+\sqrt{5})/2$ and $\nu \approx 1.635$.

Although this example can be computed analytically, in general we must iterate the renormalization operator until we find the desired fixed point.

5.2 Applications of percolation and other practical cases of renormalization

As we have seen, macroscopic properties (the lattice may be some body) can be found from microscopic ones (the small sites could be seen as atoms). Now, we are going to introduce some brief examples to understand how can we deal with Section 5.1.

Example 5.2.1. A macroscopic region can be divided in groups of atoms positions, so that we can study the electrical conduction through the surface in terms of the micro-details. For instance, following the model in Figure 5.4 with the majority rule, $R(p) = p^3 + 3p^2(1-p)$ and then $p_c \approx p' = 1/2$. Therefore, we can know if a region is electrical conductive according to its occupation probability, i.e. the fraction of the surface that is occupied by atoms. Roughly:

- If $p \lesssim 1/2$, the probability of being conductive at the macroscopic level is 0.
- If $p \approx 1/2$, then the macroscopic probability is $1/2$.
- Otherwise, the probability of conduction is 1.

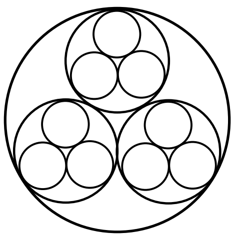


Figure 5.4: Chosen grouping of atoms.

Example 5.2.2. Another simple application consists of taking advantage of the fact that the density of the incipient cluster decreases with length scale. For instance, consider a 3-dimensional oil reservoir with porous fractal material and three samples of volumes 0.001 m^3, 0.008 m^3 and 0.064 m^3 with densities of oil 250 kg m^{-3}, 177 kg m^{-3} and 125 kg m^{-3}, respectively. How much oil will be obtained from an oil field of 10^3 km^3?

The graph $\log \rho_{\text{oil}}$ versus $\log l$ is a straight line with slope $D_{\text{oil}} - d = -0.5$, so that

$$\frac{\rho_{\text{oil}}(l_2)}{\rho_{\text{oil}}(l_1)} = \left(\frac{l_2}{l_1}\right)^{D_{\text{oil}}-d},$$

where in our case $l_1 = \sqrt[3]{0.001 \text{ m}^3} = 0.1$ m, $l_2 = \sqrt[3]{10^3 \text{ km}^3} = 10^4$ m and $\rho_{\text{oil}}(l_1) = 250$ kg m^{-3}. Thus,

$$\rho_{\text{oil}}(l_2) \approx 0.79 \text{ kg m}^{-3}$$

and

$$M_{\text{oil}}(l_2) = l_2^3 \rho_{\text{oil}}(l_2) \approx 7.9 \times 10^{11} \text{ kg}.$$

We highly recommend to have a look at the Ising Model, which is constructed analogously to our explanation to introduce the percolation theory, but going into details about thermodynamics and statistical mechanics concepts that do not belong to the content of this thesis. It can be found in [CM, 2. Ising Model], where it is studied the point of phase transition. In general, the brilliant insight of the renormalization group allows us to know macroscopic features in a similar way to the one followed in our brief and pedagogical example. We recall that renormalization can be found in not daily situations, as it can be the quantum field theory, mechanical statistics or perturbation theory.

Conclusions

As a result of this dissertation, we proceed to evaluate the scope of the project by providing an overview of the achieved goals.

The contents of this thesis are encompassed within the context of dynamical systems. As some other branches of mathematics, this one can be characterized by its multidisciplinarity, not only within the different fields of this theoretical science, but with regard to other disciplines as physics. In my case, since I study both undergraduate degrees, exploring in depth outcomes within this framework could enrich me even more.

With respect to our particular study, before getting started, we aimed to plunge into the technique of renormalization through an essential result behind such a theory, the Straightening Theorem. Bearing in mind that most of the necessary knowledge had not been studied before, we required to develop a background concerning the necessary tools. Once comprehended and given a complete and self-contained proof of the theorem, we proceed to look at new challenges.

Taking in advantage the deep effort done to deal with the cornerstone of this project, our purpose was to apply it and try to achieve some results in the space of cubic polynomials as general as possible. This research started by thinking of specific polynomials and developing some intuitions with the support of the displays obtained with Python. Once some findings appeared, we were about to generalize as much as we could. To date, we decided to continue researching on these results in order to improve and generalize them as much as possible.

To conclude, we desired to see a different point of view of renormalization and benefit from the interdisciplinarity of the chosen subject matter. Thereby, we attained our objective by superficially outlining the basic theory of percolation.

Appendix A

Preliminaries and tools

In this first appendix, we give some general results which are cited during the project when necessary. We classify them by topics to facilitate reading. However, we highly recommend to focus on reading the thesis straightly and only consult this appendix when some result is mentioned.

A.1 Algebra

First of all, we present some basic elemental notions of self-adjoint linear mappings to give support to Section 1.1.

Definition A.1.1. (Self-adjoint linear mapping) *Let V be a complex vector space with an inner product $\langle , \rangle$ and assume $S : V \to V$ is linear. If there is a function $S^* : V \to V$ such that*

$$\langle Sx, y \rangle = \langle x, S^* y \rangle$$

for all $x, y \in V$, then S^* is called the adjoint of S. If $S = S^*$, this linear function is said tot be self-adjoint.

Proposition A.1.2. *Let $S : V \to V$ be linear. Then S^* exists, is unique and is linear.*

Proposition A.1.3. *Given a self-adjoint linear transformation $S : V \to V$, it has real eigenvalues and eigenvectors of different eigenvalues are orthogonal.*

A.2 Functional analysis

Let us recall the notion of uniform convergence.

Definition A.2.1. (Uniform convergence) *Given mappings $f, f_n : A \subset \mathbb{C} \to \mathbb{C}$ for $n = 1, 2, 3, \ldots$, we say that f_n converges uniformly to f if for all $\epsilon > 0$ there exists $\widetilde{n}_\epsilon > 0$ such that, for all $n \geq \widetilde{n}_\epsilon$, $|f_n(z) - f(z)| < \epsilon$ for every $z \in A$. Equivalently, if $\lim_{n\to\infty} ||f - f_n||_\infty = 0$. We denote it by $f_n \overset{n}{\rightrightarrows} f$.*

Since when dealing with quasiconformal geometry we mentioned some concepts of measure theory, we give some elemental definitions to keep them in mind.

Definition A.2.2. (Lebesgue measure) *Given a subset $A \subset \mathbb{C}$, the Lebesgue measure is defined as*

$$m(A) := \inf \left\{ \sum_{n=1}^{\infty} Area(A_n) \mid \{A_n\}_n \text{ is a sequence of open sets in } \mathbb{C} \right\}$$

and it is an outer measure (see [Cer, p. 56]).

Definition A.2.3. (Lebesgue measurable set) *We say that $E \subset \mathbb{C}$ is Lebesgue measurable if $m(A) = m(A \cap E) + m(A \cap E^c)$ for all $A \subset \mathbb{C}$. We denote by $\Sigma(m)$ the set of the Lebesgue measurable sets, that is a σ-algebra, and we we say that $(\mathbb{C}, \Sigma(m))$ is a measurable space.*

Definition A.2.4. (Lebesgue measurable function) *A function $\mu : (\mathbb{C}, \Sigma(m)) \to (\mathbb{C}, \Sigma(m))$ is said to be Lebesgue measurable if $\mu^{-1}(B) \in \Sigma(m)$ for all $B \in \Sigma(m)$.*

Remark A.2.5. Given a function between a couple of subsets in $\mathbb{C}$, Definition A.2.4 is analogous with the corresponding induced σ-algebra for each set.

Definition A.2.6. (Absolute continuity with respect to the measure) *A map $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is absolutely continuous with respect to the Lebesgue measure m if and only if $m(A) = 0 \implies m(f^{-1}(A)) = 0$, for all $A \subset V$ Lebesgue measurable.*

In Section A.3 and Section A.4, there are some results that have been needed in the development of this thesis.

A.3 Holomorphic functions

From now on, we will refer by region or domain to an open connected subset of $\mathbb{C}$.

Definition A.3.1. (Conformal map) *A function $f : U \subset \mathbb{C} \to \mathbb{C}$ defined on an open set is said to be conformal at $z_0 \in U$ if and only if it preserves angles and orientation between curves through z_0. We say that it is conformal if it is conformal at every point $z_0 \in U$.*

In the literature, there is another definition that is equivalent to the last one. We give it as the following theorem.

Theorem A.3.2. (Analytic characterization of conformal mappings) *A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is conformal at z_0 if and only if it is holomorphic at $z_0 \in U$ and $f'(z_0) \neq 0$.*

During this thesis, we say conformal map to refer to both (the definition and its analytic characterization) without specifying what of these concepts we are dealing with. Now, let us see more explicitly this notion.

Given $\gamma \subset U$, an arc with equation $z = z(t)$ for $t \in [0,1]$, and a holomorphic function $f : U \to \mathbb{C}$, consider the arc $\widetilde{\gamma}$ with equation $\omega = \omega(t) := f(z(t))$. Moreover, assume $z_0 := z(t_0)$ such that $z'(t_0) \neq 0$ and $f'(z_0) \neq 0$. Then, by the chain rule,

$$\omega'(t_0) = f'(z_0)z'(t_0) \neq 0$$

and it exists a tangent of $\widetilde{\gamma}$ at $\omega_0 = f(z_0)$. This implies a sum relation with the respective arguments, so that

$$\arg \omega'(t_0) - \arg z'(t_0) = \arg f'(z_0) \tag{A.1}$$

does not depend on the arc γ (i.e. on $z'(t_0)$) and f preserves the angle between two different arcs at z_0 (f preserves the orientation).

Definition A.3.3. (Antiholomorphic function) *A function $f : U \subset \mathbb{C} \to \mathbb{C}$ defined on an open subset is said to be antiholomorphic if its derivative with respect to $\overline{z}$ exists in a neighbourhood of every point in that set. Equivalently, if it can be expanded in a power series in $\overline{z}$ in a neighborhood of each point in its domain.*

Remark A.3.4. Every antiholomorphic function is orientation reversing. Indeed, $f(\overline{z})$ is holomorphic in $\overline{U}$ and, as the conjugate $c(z) := \overline{z}$ is orientation reversing, necessarily f is it too.

An important property of the holomorphic functions within the continuous ones is Morera's Theorem (see [Con1, p. 86]). It follows this corollary:

Corollary A.3.5. *If $f : U \subset \mathbb{C} \to \mathbb{C}$ is holomorphic in $U \setminus \{p\}$ and continuous in p, where U is open and $p \in U$, then f is holomorphic in the whole domain.*

Remark A.3.6. Notice that it can be generalized when we have a discrete set of points instead of a point p.

Definition A.3.7. (Order of a zero) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $z_0 \in U$ such that $f(z_0) = 0$. We say that f has a zero of order n at z_0 if and only if there exists a holomorphic function $g : U \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^n g(z)$ for all $z \in U$.*

Definition A.3.8. (Locally injective) *We say that a function $f : U \subset \mathbb{C} \to \mathbb{C}$ is locally injective at $z_0 \in U$ if and only if there is a neighbourhood $V \subset U$ of z_0 such that $f_{|V}$ is injective. f is called locally injective iff it is locally injective at each $z \in U$.*

Theorem A.3.9. (Inverse function theorem for holomorphic maps) *Let $f : U \subset \mathbb{C} \to \mathbb{C}$ be holomorphic with $f'(z_0) \neq 0$ for some $z_0 \in U$. Then, there is a neighbourhood $V \subset U$ of z_0 such that $f : V \to f(V)$ is a bijection and its inverse f^{-1} is holomorphic with $(f^{-1})'(z) = 1/f'(f^{-1}(z))$.*

Proof. Considering $\tilde{f} : \{(x, y) \in \mathbb{R}^2 \mid x + iy \in U\} \to \mathbb{R}^2$ defined as $\tilde{f}(x, y) := (\mathrm{Re}(f(x + iy)), \mathrm{Im}(f(x + iy)))$ and (x_0, y_0), where $z_0 = x_0 + iy_0$,

$$\mathrm{Jac}\, \tilde{f} = \mathrm{Jac}\, f = |\partial_z f|^2 - |\partial_{\bar{z}} f|^2 \underset{f \text{ hol}}{=} |\partial_z f|^2.$$

Since $\mathrm{Jac}\, \tilde{f}(x_0, y_0) = |f'(z_0)|^2 \neq 0$, by the real version of the inverse function theorem we obtain a $\mathbb{R}$-differentiable inverse, whose Jacobian matrix is given by the inverse of the one of f. From the matrix equality

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

when $a^2 + b^2 \neq 1$ and the fact that f is holomorphic, it follows that f^{-1} is also holomorphic. Actually, $f : V \to f(V)$ and $f^{-1} : f(V) \to V$ are conformal.

Finally,

$$(f \circ f^{-1})(z) = z \implies f'(f^{-1}(z)) \cdot (f^{-1})'(z) = 1 \implies (f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}$$

□

Corollary A.3.10. *If $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is holomorphic and bijective, then the inverse $f^{-1} : V \to U$ is also holomorphic.*

Proof. It suffices to see that f' never vanishes, so that we can apply Theorem A.3.9. Suppose that $f'(z_0) = 0$ for some $z_0 \in U$ and write $f(z) = f(z_0) + \sum_{n \geq n_0} a_n (z - z_0)^n$, where $n_0 \geq 2$, $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}_{\geq n_0}$ and $a_{n_0} \neq 0$.

Since $(f(z) - f(z_0))/(z - z_0)^{n_0}$ is a nonzero holomorphic function in a neighbourhood of z_0, it admits a n_0-th root, i.e. a holomorphic function h such that $(h(z))^{n_0} = (f(z) - f(z_0))/(z - z_0)$. Thus, $f(z) = f(z_0) + [(z - z_0)h(z)]^{n_0}$, where $(z - z_0)h(z)$ is a holomorphic function defined in a neighbourhood of z_0 and hence, by Theorem A.3, its image contains a neighbourhood of 0. Therefore, we can find some $r > 0$ so that it contains $z = r$ and $z = re^{2\pi i/n_0}$, that are sent to the same point when computing f. □

Corollary A.3.11. *In other words, a holomorphic function $f : U \subset \mathbb{C} \to \mathbb{C}$ is conformal if and only if it is locally injective. In that case, f^{-1} makes sense and it is locally conformal. Moreover, when the conformal inverse makes sense, we obtain the following relation:*

$$(f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}$$

Remark A.3.12. It cannot be generalized to a global injectivity. Notice that although being injective and holomorphic implies conformality, the exponential function $z \mapsto e^z$ is a holomorphic function with a non-zero derivative and is not injective due to its periodicity.

Definition A.3.13. (Biholomorphic map) *A biholomorphic function is a bijective holomorphic function whose inverse is also holomorphic.*

Lemma A.3.14. $|e^{z_2} - e^{z_1}| < |z_2 - z_1|$ *for every* $z_1, z_2 \in \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$.

Proof. Let $[z_1, z_2]$ denote the straight line segment from z_1 to z_2. Then,

$$|e^{z_2} - e^{z_1}| = \left| \int_{[z_1,z_2]} e^z dz \right| \leq |z_2 - z_1| \sup_{z \in [z_1,z_2]} |e^z| < |z_2 - z_1|$$

because $|e^z| = e^{\operatorname{Re}(z)}$ and $\operatorname{Re}(z) < 0$ for $z \in [z_1, z_2] \subset \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$. $\square$

Theorem A.3.15. (Weierstrass - Uniform convergence of holomorphic functions) *Let* $(f_n)_n$ *be a sequence of holomorphic functions* $f_n : U \to \mathbb{C}$ *defined on an open set* $U \subset \mathbb{C}$. *If* $f_n \overset{n}{\rightrightarrows} f$, *then* $f : U \to \mathbb{C}$ *is holomorphic and* $f_n' \overset{n}{\rightrightarrows} f'$.

See [Con1, p. 151, 2.1 Theorem] for a proof.

Theorem A.3.16. (Open Mapping Theorem) *Let* $U \subset \mathbb{C}$ *be a domain and assume* $f : U \to \mathbb{C}$ *is analytic and non constant. Then* f *is open, i.e. maps open sets in* U *to open sets in* $\mathbb{C}$.

A proof can be found in [Con1, p. 99, 7.5. Open Mapping Theorem].

The following theorem and the corresponding corollary are really useful (e.g. when we spread a real analytic function through the complex plane) and are written in detail in [Con1, pp. 78-79, 3.7 Theorem and 3.8 Corollary].

Theorem A.3.17. (Identity Theorem) *Let* $U \subset \mathbb{C}$ *be a domain and let* $f : U \to \mathbb{C}$ *be an analytic function. Then the following statements are equivalent:*

(a) $f \equiv 0$;

(b) There exists some $z_0 \in U$ *such that* $f^{(n)}(z_0) = 0$ *for each* $n \geq 0$;

(c) The set $\{z \in U \mid f(z) = 0\}$ *has a limit point in* U.

Corollary A.3.18. *Given two analytic functions,* f *and* g, *on a domain* U, $f \equiv g$ *if and only if* $\{z \in U \mid f(z) = g(z)\}$ *has a limit point in* U.

Lemma A.3.19. *Let* $f : U \to \mathbb{C}$ *be an analytic function defined on a neighbourhood of the origin. Under this constraint, the nontrivial solutions of the functional equation* $(f(z))^n = f(z^n)$ *for some* $n \geq 2$ *are* $f(z) = \xi z^m$, *where* ξ *is a* $(n-1)$*-th root of unity and* $m \in \mathbb{N}$.

Proof. Assume $f(0) \neq 0$. Let us iterate the functional equation $k \in \mathbb{N}$ times:

$$\begin{aligned} f\left(z^{n^2}\right) &= (f(z^n))^n = (f(z))^{n^2} \\ f\left(z^{n^3}\right) &= (f(z^n))^{n^2} = (f(z))^{n^3} \\ &\vdots \\ f\left(z^{n^k}\right) &= (f(z^n))^{n^{k-1}} = (f(z))^{n^k} \end{aligned}$$

Then, for all $z \in \mathbb{D}$, $(f(z))^{n^k} \xrightarrow{k} f(0) \neq 0$. This implies $|f(z)| = 1$ for every $z \in \mathbb{D}$, because otherwise $(f(z))^{n^k} \xrightarrow{k} 0$ if $|f(z)| < 1$ and $|(f(z))|^{n^k} \xrightarrow{k} \infty$ if $|f(z)| > 1$. Therefore, $f(\mathbb{D}) \subset \mathbb{S}^1$ and, by f must be constant ($f \equiv \xi \in \mathbb{S}^1$). Using the functional equation of the statement, $\xi^n = \xi$ and f must be a $(n-1)$-th root of unity.

If $f(0) \neq 0$, let m be the order of 0 as a zero of f (i.e. $f(z) = a_m z^m + a_{m+1}z^{m+1} + \dots$ for some coefficients in $\mathbb{C}$ such that $a_m \neq 0$) and define the holomorphic function

$$g(z) := \begin{cases} \frac{f(z)}{z^m}, & \text{if } z \neq 0 \\ a_m, & \text{if } z = 0 \end{cases}.$$

It trivially satisfies the functional equation for $z \neq 0$ and, applying Corollary A.3.18 to the analytic function $(g(z))^n - g(z^n)$, it is fulfilled also for $z = 0$. As $g(0) \neq 0$, we can use the result already seen and write $g(z) = \xi$, that finally implies $f(z) = \xi z^m$. □

Proposition A.3.20. (Local change of coordinates on ramified branched coverings) *Given a holomorphic map f between Riemann surfaces X and Y, there is a unique integer m such that there are local coordinates near p and $f(p)$ with f having the form $z \mapsto z^m$.*

Proof. We can choose two charts $\psi : N_p \subset X \to N_0 \subset \mathbb{C}$ and $\phi : N_{f(p)} \subset Y \to N_0' \subset \mathbb{C}$ such that $\psi(p) = 0$ and $\phi(f(p)) = 0$, so that $P := \psi \circ f \circ \phi^{-1}$ is analytic. Here N_p and $N_{f(p)}$ denote open neighbourhoods of p and $f(p)$ respectively, and N_0 and N_0' are neighbourhoods of 0. Since $P(0) = 0$, we can write $P(\omega) = \omega^m g(\omega)$ with $g(\omega) = a_0 + a_1\omega + a_2\omega^2 + \dots$ and choose a holomorphic branch r of the m-th root so that $(r(\omega))^m = g(\omega)$. Let us consider $\nu(\omega) := \omega r(\omega)$, that satisfy $\nu'(0) \neq 0$ because $r(0) \neq 0$ and hence is locally invertible. Finally, setting $\widetilde{\phi} := \nu \circ \phi$, we obtain

$$(\psi \circ f \circ \widetilde{\phi}^{-1})(z) = (\psi \circ f \circ \phi^{-1} \circ \nu^{-1})(z) = (P \circ \nu)(z) = z^m.$$

□

A.4 Topology

Definition A.4.1. (Topology) *Let X be a set and let τ be a family of subsets of X. Then τ is called a topology on X if $\emptyset, X \in \tau$ and if any union of elements of τ and any intersection of finitely many elements of τ is an element of τ.*

Definition A.4.2. (Base) *A base of a topological space (X, τ) is a subset $\beta \subset \tau$ such that any element of τ can be written as an union of elements of β.*

Proposition A.4.3. *Let $U = \cup_{\lambda \in \Lambda} U_\lambda \subset \mathbb{C}$ be open, where Λ is an index set and $\{U_\lambda\}_\lambda$ is a collection of open sets. Then, there exists a countable subcollection $(U_i)_{i=1}^\infty$ so that $U = \cup_{i=1}^\infty U_i$.*

Proof. Taking into account the density of $\mathbb{Q}$ in $\mathbb{R}$, it can be shown that the collection of open disks $D(x, r) \subset \mathbb{C}$ with rational center x and rational radius $r > 0$ is a countable base of $\mathbb{C}$. We denote it by $\{D_n\}_{n \in \mathbb{N}}$. For every $x \in U$, consider $\lambda(x) \in \Lambda$ such that $x \in U_{\lambda(x)}$. Then, take $n(x) \in \mathbb{N}$ such that $x \in D_{n(x)} \subset U_{\lambda(x)}$. For each $n \in \mathcal{N} := \{n_x \mid x \in U\}$, pick exactly one $\lambda(n) = \lambda(n_x) \in \{\lambda(x) \mid x \in U\}$. Then, $\{U_{\lambda(n)} \mid n \in \mathcal{N}\}$ is the required countable subcollection. □

Remark A.4.4. This is a particular case of the Lindelöf Theorem, that assures that there is a countable subcover of each open cover of a subset of a space whose topology has a countable base (i.e. a second-countable or completely separable space).

Proposition A.4.5. *Given a nest of non-empty connected compact subsets $\dots \subset K_3 \subset K_2 \subset K_1 \subset \mathbb{C}$, the intersection $K := \bigcap_{n \geq 1} K_n$ is non-empty and connected.*

Proof. Let $K_0 \subset \mathbb{C}$ be a compact neighbourhood of K_1. Notice that $K \neq \emptyset$. Indeed, in that case we could obtain a finite subcover of the open cover $\{K_0 \setminus K_j\}_{j \geq 1}$ of K and thus $K_j = \emptyset$ for some $j \geq 1$.

Suppose that K is not connected, so that we can write $K = C_1 \cup C_2$, where C_1 and C_2 are two disjoint non-empty closed sets. Let U_1 and U_2 be two disjoint open sets such that $C_i \subset U_i$ for $i \in \{1,2\}$ and set $\widetilde{K}_j := K_j \setminus (U_1 \cup U_2)$ for $j \geq 1$, which consists of a nested sequence of compact sets with empty intersection. Hence, $\widetilde{K}_j = \emptyset$ for some j and then $K_j \subset U_1 \cup U_2$. But this is a contradiction with the connectedness of K_j because $K_j \cap U_1 \neq \emptyset \neq K_j \cap U_2$ since $K_j \cap C_1 \neq \emptyset \neq K_j \cap C_2$. $\square$

Proposition A.4.6. *Let $\Omega \subset \mathbb{C}$ and $z \in \mathbb{C}$. Then, $z \in \partial\Omega$ if and only if there are two sequences $(\omega_n)_n \subset \Omega$ and $(\omega'_n)_n \subset \Omega^c = \mathbb{C} \setminus \Omega$ such that $z = \lim_n \omega_n = \lim_n \omega'_n$.*

Proof.

- $\Longleftarrow$. It is a consequence of the characterization of closed sets in terms of convergent sequences.
- $\Longrightarrow$. Given $z \in \partial\Omega = \overline{\Omega} \cap \overline{\Omega^c}$, let U be an open set such that $z \in U$. Then, $U \cap \Omega \neq \emptyset \neq U \cap \Omega^c$. Indeed, if for example $U \cap \Omega = \emptyset$, then $\Omega \subset \overline{\Omega} \subset U^c$ because U^c is closed. Hence, this contradicts $z \in U$. Defining $U_n := \{z' \in \mathbb{C} \mid |z - z'| < 1/n\}$ and choosing $\omega_n \in U_n \cap \Omega$ and $\omega'_n \in U_n \cap \Omega^c$ for $n \in \mathbb{Z}_{n\geq 1}$, it follows that $(\omega_n)_n \subset \Omega$ and $(\omega'_n)_n \subset \Omega^c$ satisfy $z = \lim_n \omega_n = \lim_n \omega'_n$.

$\square$

Theorem A.4.7. (Bolzano - Weierstrass Theorem) *Each bounded sequence in $\mathbb{C}$ contains a convergent subsequence.*

This theorem was shown by Bernard Bolzano[1] (see [Bol]) and it was later also independently deduced by Karl Weierstrass[2].

Proposition A.4.8. *Let $R \subset \mathbb{C}$ be a half of a line proceeding from an initial point (i.e. a ray) and consider a non-vacuous open set $U \subset \mathbb{C}$ such that $\partial U \subset R$. Then, $\mathbb{C} \setminus R \subset U$.*

Proof. Assume $U \neq \mathbb{C}$ because otherwise we are done.

Since U is open, it cannot be contained in R. From the fact that U is non-vacuous, it follows that $U \cap \mathbb{C} \setminus R \neq \emptyset$. If we show that $\mathbb{C} \setminus U \subset R$ we are done. We know that $\partial U \subset R$ by hypothesis, so it suffices to see that $\mathbb{C} \setminus \overline{U} \subset R$. Suppose that $(\mathbb{C} \setminus \overline{U}) \cap (\mathbb{C} \setminus R) \neq \emptyset$. Then, since U, $\mathbb{C} \setminus \overline{U}$ and $\mathbb{C} \setminus R$ are open, the intersections $(\mathbb{C} \setminus \overline{U}) \cap (\mathbb{C} \setminus R)$ and $U \cap (\mathbb{C} \setminus R)$ will be also open. It must be fulfilled that $\mathbb{C} \setminus R = [(\mathbb{C} \setminus \overline{U}) \cap (\mathbb{C} \setminus R)] \cup [U \cap (\mathbb{C} \setminus R)]$ because otherwise $\partial\{[(\mathbb{C} \setminus \overline{U}) \cap (\mathbb{C} \setminus R)] \cup [U \cap (\mathbb{C} \setminus R)]\} \cap (\mathbb{C} \setminus R) \neq \emptyset$, which is not possible because $\partial U \subset R$. However, this is in contradiction with the connectedness of $\mathbb{C} \setminus R$, because we have $\mathbb{C} \setminus R$ as an union of two non-vacuous, open and disjoint sets. $\square$

Lemma A.4.9. *Given a continuous function $f : U \subset \mathbb{C} \to \mathbb{C}$ and an open set $V \subset \mathbb{C}$, the preimage $f^{-1}(V) \subset U$ is also open.*

Proof. Let $z \in f^{-1}(V)$. Since $f(z) \in V$ and V is open, there exists some $\epsilon > 0$ such that $\{\omega \in \mathbb{C} \mid |\omega - f(z)| < \epsilon\} \subset V$ and, by the continuity of f, there exists some $\delta > 0$ such that $f(\{\omega \in \mathbb{C} \mid |\omega - z| < \delta\}) \subset \{\omega \in \mathbb{C} \mid |\omega - f(z)| < \epsilon\}$. Hence, $\{\omega \in \mathbb{C} \mid |\omega - z| < \delta\} \subset f^{-1}(V)$ and we are done. $\square$

Proposition A.4.10. *If $C \subset \mathbb{C}$ is connected and $f : C \to \mathbb{C}$ is a continuous mapping, then $f(C)$ is connected.*

Proof. We are going to prove it by contrapositive.

Suppose that $f(C)$ is not connected, i.e. that there exist two disjoint open sets $U, V \subset \mathbb{C}$ such that $U \cap f(C) \neq \emptyset \neq V \cap f(C)$ and $f(C) \subset U \cup V$. Then C is not connected because $f^{-1}(U), f^{-1}(V) \subset \mathbb{C}$ are open (see Lemma A.4.9), $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $C \subset f^{-1}(U) \cup f^{-1}(V)$. Indeed:

[1]Bernard Bolzano: 1781-1848

[2]Karl Theodor Wilhelm Weierstrass: 1815-1897

- If there exists some $z \in f^{-1}(U) \cap f^{-1}(V)$, then $f(z) \in U \cap V = \emptyset$, which is a contradiction.
- If $z \in C$, then $f(z) \in f(C) \subset U \cup V$ and therefore $z \in f^{-1}(U) \cup f^{-1}(V)$.

$\square$

Appendix B

Python code

Throughout the thesis, some dynamical planes and parameter spaces have been displayed to have a deeper appreciation of these notions. Also, they have been useful to stimulate the skill and the intuition behind some conjectures that became or not actual results. Although we have attached several plots, we just give the code of the Mandelbrot set and one of the corresponding Julia sets of the quadratic family (dendrite: $c = i$). The code behind the other pictures (e.g. the study of cubic polynomials in the fourth chapter) is similar, but changing the iterated function and adjusting the inputs to the particular case.

Both the dynamical plane and the parameter space have been achieved with the integer escape time algorithm, which is a level-set method. In the case of some Julia sets as the one given in Figure 4.2, we have chosen a large enough escaping bound to approximate better the equipotentials, since when we tend to infinity the Böttcher map of a polynomial of degree d behaves as z^d and the equipotentials shall be roughly treated as circles. Moreover, in this specific case, we chose an equipotential that contains some iterated of the escaping critical point to get the eight-figure in our output.

As for the choice of Python as programming language, it is not the best option to deal with iterations. However, it meets our needs and is convenient to get eye-catching pictures.

B.1 Dynamical plane

```python
from tqdm import tqdm
import numpy as np
import matplotlib.pyplot as plt
from colormap import Colormap

###################################################
######################## INPUT ########################

# Window of the plane that want to be plotted and resolution
xmin, xmax = -1.5, 1.5
ymin, ymax = -1.3, 1.3
xres, yres = 5000, 5000

# Maximum number of iterations
iter = 200

# Value of the constant for quadratic polynomials
c = complex(0.0, 1.0)
```

```
######################## INPUT ########################
#######################################################

# Set the corresponding grid of points
xrange = np.arange(xmin, xmax, (xmax - xmin) / float(xres))
yrange = np.arange(ymin, ymax, (ymax - ymin) / float(yres))

# Matrix with the iteration at which each point escapes
color = np.zeros((len(yrange), len(xrange)))

xcolor = 0
ycolor = yres - 1
# Compute the escaping time
for y in tqdm(yrange):
    for x in xrange:
        z = complex(x, y)

        for i in range(0, iter, 1):
            z = z*z + c

            if abs(z) > max(2, abs(c)):
                color[ycolor, xcolor] = i
                break

        xcolor += 1

    ycolor -= 1
    xcolor = 0

# cmap: we colour the cells according to its escaping time
fig = plt.figure(figsize=(15,15))
plt.axis('off')
plt.imshow(color, cmap="bone", extent=[xmin, xmax, ymin, ymax])
```

B.2 Parameter space

```
from tqdm import tqdm
import numpy as np
import matplotlib.pyplot as plt
from colormap import Colormap

#######################################################
######################## INPUT ########################

# Window of the plane that want to be plotted and resolution
xmin, xmax = -2.2, 0.8
ymin, ymax = -1.5, 1.5
xres, yres = 5000, 5000
```

```python
# Maximum number of iterations
iter = 200

####################### INPUT #######################
#####################################################

# Set the corresponding grid of points
xrange = np.arange(xmin, xmax, (xmax - xmin) / float(xres))
yrange = np.arange(ymin, ymax, (ymax - ymin) / float(yres))

# Matrix with the iteration at which each point escapes
color = np.zeros((len(yrange), len(xrange)))

xcolor = 0
ycolor = yres - 1
# Compute the escaping time
for y in tqdm(yrange):
    for x in xrange:
        # z is the critical point
        z = complex(0.0, 0.0)
        c = complex(x, y)

        for i in range(0, iter, 1):
            z = z*z + c

            if abs(z) > max(2, abs(c)):
                color[ycolor, xcolor] = i
                break

        xcolor += 1

    ycolor -= 1
    xcolor = 0

# cmap: we colour the cells according to its escaping time
fig = plt.figure(figsize=(15,15))
plt.axis('off')
plt.imshow(color, cmap="hot", extent=[xmin, xmax, ymin, ymax])
```

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