UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques
i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

PERIODIC POINTS AND HOLOMORPHIC FUNCTIONS

Autora: Rut Martínez Madrid

Directora: Dra. Anna Jové Campabadal
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 15 de gener de 2026

Abstract

The main result presented in this project is the existence of repelling periodic points of any minimal period for entire transcendental functions with a bounded set of singular values (i.e. functions in class $\mathcal{B}$), following the approach of A. M. Benini [3]. We show this is not true for functions outside the class $\mathcal{B}$.

To that end, we first study the different behaviours of holomorphic functions around fixed points and the existence of conjugacies and normal forms, in particular (but not restricted to) repelling fixed points. We also present some specific tools which are fundamental in this proof: hyperbolic geometry and the mapping structure of maps of class $\mathcal{B}$.

Resum

El resultat principal presentat en aquest projecte és l'existència de punts periòdics repulsors de qualsevol període mínim per a funcions enteres transcendents tals que el seu conjunt de valors singulars és acotat, també conegudes com a funcions dins de la classe $\mathcal{B}$. Seguim l'enfocament donat per A. M. Benini a [3], i també mostrem que això no és cert per a funcions enteres transcendents fora de la classe $\mathcal{B}$.

Amb aquesta finalitat, primer estudiem els diferents comportaments de les funcions holomorfes al voltant de punts fixos i l'existència de conjugacions i formes normals. En particular, però sense restringir-nos-hi, estudiem punts fixos repulsors. També presentem algunes eines específiques que són fonamentals per aquesta prova: la geometria hiperbòlica i l'estructura de les funcions pertanyents a la classe $\mathcal{B}$.

2020 Mathematics Subject Classification. 37F10, 37F20, 30D05

Contents

Introduction

In this project we are concerned about the iteration of holomorphic functions. Indeed, given a holomorphic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ and a point $z_0 \in \mathbb{C}$, we are interested in the *orbit* of z_0, that is, in the set defined by

$$\{z_0, f(z_0), f^2(z_0), \dots\},$$

where $f^0 = id$ and $f^n = f \circ f^{n-1}$ for $n \geq 1$. The area of mathematics that studies the iteration of holomorphic maps is known as *complex* (or *holomorphic*) *dynamics*.

When studying the orbits $\{f^n(z)\}_n$ for a holomorphic map f, we encounter two possible situations: either all iterates $f^n(z)$ are distinct and the orbit is infinite, or two iterates eventually coincide and the orbit is finite. Finite orbits

$$\{z_0, f(z_0), \cdots, f^{p-1}(z_0)\}$$

can be *preperiodic*, if $f^p(z_0) \neq z_0$, or *periodic*, if $f^p(z_0) = z_0$. This last particular type of orbit is of special interest due to its simplicity, and its points are referred to as *periodic points*.

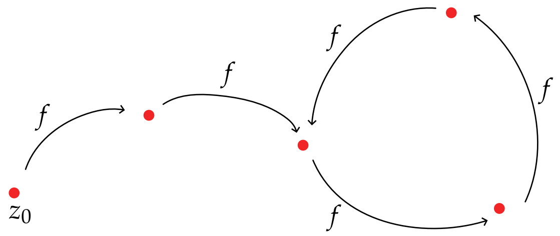


Figure 1: Schematic representation of a preperiodic orbit.

Periodic points are characterized by their period and multiplier. On the one hand, if z_0 is a periodic point of f, then there exists a positive integer n such that $f^n(z_0) = z_0$. Such n is called a *period* of z_0, and the smallest positive integer with this property is called the *minimal period* of z_0. In particular, periodic points of period 1 are called *fixed points*. On the other hand, the *multiplier* of a periodic point z_0 of minimal period n is defined as

$$\lambda := (f^n)'(z_0).$$

Since periodic points of period n are fixed points of f^n, the study of periodic points is reduced to fixed points. Fixed points are classified according to its multiplier into *attracting*, *neutral*, or *repelling*, depending on whether $|\lambda|$ is less than, equal to, or greater than 1, respectively. In particular, attracting fixed points with multiplier $\lambda = 0$ are called *superattracting*, and neutral fixed points can be further classified into *parabolic*, *Siegel* or *Cremer* points.

The previous terminology is not merely conventional; rather, the modulus of the multiplier actually determines the dynamics around the fixed point. Formally, this is proven in the studies conducted at the end of the 19th century and throughout the 20th century mainly by E. Schr̈oder, G. Koenigs, L. Leau, L. Bötcher, P. Fatou, G. Julia, C. L. Siegel and H. Cremer.

Schröder was the first one interested in the local dynamics around fixed points. Aware that when iterating a function f, it is usually difficult to find an explicit formula for the iterates f^n, Schröder [21] defined the notion of *conjugacy*. Two holomorphic functions f and ψ are *conjugate* in an open set $U \subseteq \mathbb{C}$ if there exists a biholomorphic map (i.e. holomorphic and bijective) φ such that the following diagram commutes.

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\varphi"'] & U \arrow[d, "\varphi"] \\
\varphi(U) \arrow[r, "\psi"] & \varphi(U)
\end{tikzcd}$$

Then, φ is a conjugacy between f and ψ and can be thought as a change of variables. Therefore, the dynamical systems induced by f and ψ are equivalent, which reduces the iteration of f to the iteration of ψ. This is particularly useful if ψ is a function that is easy to iterate.

From this perspective, two fundamental problems arise. The first problem is known as the *Linearization Problem*, and concerns the conditions under which f is conjugate to the linear map $\psi(z) = \lambda z$ in a neighbourhood of a fixed point z_0 with multiplier λ. This is equivalent to ask whether the *Schröder functional equation*

$$\varphi \circ f = \lambda \cdot \varphi$$

has a solution in a neighbourhood of z_0. The second problem asks whether, if f is not locally linearizable around the fixed point z_0, there exists another normal form ψ sufficiently simple to locally understand the behaviour of z_0.

Following the work of Schröder, Koenigs developed a robust local theory describing the dynamics under iteration of attracting fixed points with multiplier $\lambda \neq 0$. In 1884, he proved in [16] that if z_0 is either an attracting fixed point with multiplier $0 < |\lambda| < 1$, or a repelling fixed point, then there exists a biholomorphic function φ which solves the Schröder equation in a neighbourhood of z_0. Hence, the conjugacy φ provides a local linearization of f in a neighbourhood of z_0 when the multiplier satisfies $|\lambda| \neq 0$ and $|\lambda| \neq 1$.

At this point, the remaining cases to study were superattracting and neutral fixed points. In 1897, Leau, who was a student of Koenigs, gave a detailed study regarding the dynamics around neutral fixed points whose multiplier λ is a root of unity, that is, around *parabolic* fixed points. In his thesis [17], he showed that holomorphic functions are not linearizable around parabolic fixed points, and anticipated a version of the *Flower Theorem*, which was later refined by Fatou and Julia in 1918-1920. The works of both Fatou and Julia were inspired by Leau's work, even though Fatou took an entirely fresh approach in [12].

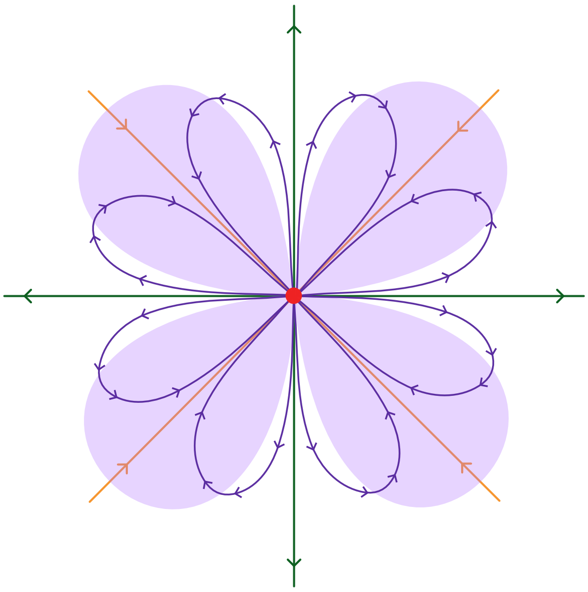
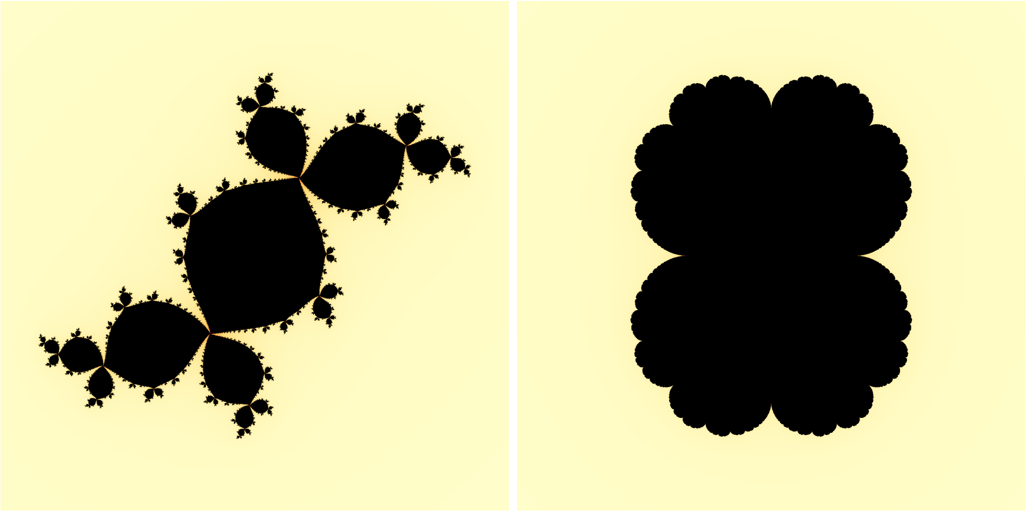


Figure 2: The dynamics around a neutral fixed point whose multiplier is a 4-th root of unity.

In 1904, L. Bötcher proved that around *superattracting* fixed points, holomorphic functions are obviously not linearizable, but they are locally conjugate to a map $h(z) = z^d$. Later studies by H. Cremer (1932) and C. L. Siegel (1942) provided a deeper understanding of the dynamics of neutral fixed points whose multiplier is not a root of unity, showing that they can be either linearizable (*Siegel points*) or non-linearizable (*Cremer points*), depending on the arithmetic properties of the multiplier.

The previous results, in some sense, provided a complete description of fixed points (and by extension, of periodic points), which historically led to a loss of interest in the field. However, the discovery of the Mandelbrot set in 1980 generated renewed attention to complex dynamics. As a result of the research conducted by Fatou and Julia, combined with the development of computer graphics, it became possible to plot basins of attraction, parabolic basins, and Siegel disks (i.e. the domains of normality associated respectively to attracting, parabolic and Siegel points), which enables us to observe fractal behaviour on their boundaries. In fact, the study of these boundaries (among other related topics) is currently one of the most active research areas in complex dynamics.



(a) $f(z) = z^2 - 0.12 + 0.7i$.

(b) $f(z) = z^2 + \frac{1}{4}$.

Figure 3: In black, the basin of attraction of the period 3 attracting cycle for the map $f(z) = z^2 - 0.12 + 0.7i$, and the parabolic basin of attraction of the parabolic fixed point $z = \frac{1}{2}$ for the map $f(z) = z^2 + \frac{1}{4}$.

At this point, a natural question to ask is whether fixed points (or, in general, periodic points) exist, so that we can apply the previous theory. For rational maps this is straightforward. If f is a rational map, then all the iterates f^n are also rational maps; thus, we can write

$$f^n(z) = \frac{P_n(z)}{Q_n(z)}$$

for some polynomials P_n, Q_n. Finding a periodic point is equivalent to solving the equation $P_n(z) = zQ_n(z)$, which has at least one solution by the Fundamental Theorem of Algebra. Hence, for rational maps, the existence of periodic points of (not necessarily minimal) period n is guaranteed for every $n \geq 1$.

In contrast to rational functions, the dynamics of an entire transcendental map is much more complicated because, by definition, it possesses an essential singularity at ∞, which can lead to a richer dynamical behaviour. In particular, contrary to rational functions, not all entire transcendental maps have fixed points. For instance, since $e^z \neq 0$ for all $z \in \mathbb{C}$, the function $f(z) = e^z + z$ has no fixed points.

Nevertheless, every entire transcendental map must have at least a periodic point of period 2. This result was proven by Fatou on *Sur l'itération des fonctions transcendantes entières* [13], and then generalised by Rosenbloom in 1948. Rosenbloom proved in [20] that all entire transcendental maps have infinitely many periodic points of (not necessarily minimal) period n for all $n \geq 2$.

Following this result, two natural questions arise regarding entire transcendental maps. Can we ensure the existence of periodic points of any *minimal* period $n \geq 2$? Under which conditions do fixed points exist?

The first question was addressed by W. Bergweiler in the paper *Periodic points of entire functions: proof of a conjecture of Baker* [6]. Bergweiler proved that entire transcendental functions have infinitely many repelling periodic points of minimal period n for all $n \geq 2$, which had already been conjectured by I. N. Baker in 1967 [14].

The existence of fixed points for entire transcendental functions is closely related to the distribution of *singular values* (i.e. such values for which inverse branches of the function are not well-defined). In fact, A. Eremenko and M. Lyubich proved in [11] that entire transcendental functions with a finite number of singular values have infinitely many fixed points. Actually, the crucial assumption is not that there are finitely many singular values, but that the singular values do not accumulate at infinity. This enables us to understand the mapping structure around ∞ in terms of *tracts* and *fundamental domains*, so that we can study the dynamics near the essential singularity.

The main goal of this project is to present a proof of the following more general result, proven by A. M. Benini in [3].

Main Theorem. (Benini, [3]**)** *Any entire transcendental function with a bounded set of singular values has infinitely many repelling periodic points of minimal period n, for all $n \geq 1$.*

Entire transcendental functions with a bounded set of singular values form the Eremenko-Lyubich class, also known as class $\mathcal{B}$. This class is special because for any $f \in \mathcal{B}$, there exists a punctured neighbourhood Ω of infinity in which f has no singular values. The preimages of Ω under f are simply connected unbounded sets called *tracts*, which accumulate only at infinity and do not contain any neighbourhood of ∞. This enables us to properly partition each tract into infinitely many fundamental domains in which we may find periodic points of every minimal period. We closely follow the proof presented by Benini and show that all but finitely many fundamental domains contain at least one periodic point of minimal period n for each $n \geq 1$.

Structure of the project. In Chapter 1 we give some preliminary results and definitions that are used throughout the project and needed to prove the Main Theorem. We recall some basic notions of Complex Analysis, including conformal maps and holomorphic functions of the unit disk. Then, we delve into transcendental maps, covering maps, singular values, and the Eremenko-Lyubich class.

In Chapter 2 we give an overview of hyperbolic geometry by studying the hyperbolic metric in the unit disk and on other proper simply connected domains. We also give some distortion estimates for univalent maps.

Chapter 3 is devoted to provide a precise description of the local dynamics of holomorphic functions around fixed points.

In Chapter 4 we study the existence of periodic points and the mapping structure of maps of class $\mathcal{B}$ near infinity. Finally, we provide a proof for the Main Theorem, using all the tools previously developed.

Appendix A contains the Python code used for plotting the different basins of attraction and parabolic basins used throughout the project.

Notation. Throughout this thesis, we denote by $\mathbb{C}$ and $\hat{\mathbb{C}}$ the complex plane and the Riemann sphere, respectively. Also, $\mathbb{D} = D(0,1)$ denotes the unit disk and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ the punctured unit disk.

Chapter 1

Background on Complex Analysis

The main goal of this chapter is to provide all the necessary tools of Complex Analysis needed to study the local behaviour of fixed points (Chapter 3) and to determine their existence for entire transcendental functions (Chapter 4).

To that end, we start by recalling some basic notions of Complex Analysis, stating all concepts and results that are used throughout this project. For more background on holomorphic functions we refer to [7, 9, 10]. Then, in Sections 1.3 and 1.4, we delve into Transcendental Maps, thereby completing all the foundational material needed further in the project.

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ be an open set. A function $f : \Omega \longrightarrow \mathbb{C}$ is **holomorphic** at $z_0 \in \Omega$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, this limit is denoted $f'(z_0)$.

We say that f is holomorphic in Ω if it is holomorphic at all points of Ω, and we denote by $H(\Omega)$ the set of holomorphic functions in Ω. If f is holomorphic in all $\mathbb{C}$, we say that f is **entire**.

Theorem 1.2. **(Maximum Modulus Principle)** *Let $\Omega \subseteq \mathbb{C}$ be a connected domain and $f \in H(\Omega)$. Then, the maximum of $|f|$ is attained in $\partial\Omega$. If the maximum of $|f|$ is attained in Ω, then f is constant.*

Theorem 1.3. **(Morera's Theorem)** *Let $\Omega \subseteq \mathbb{C}$ be open and f be a continuous function in Ω such that it is holomorphic in all Ω except for one point $z_0 \in \Omega$. Then, f is holomorphic in all Ω.*

We say that f has an **isolated singularity** at z_0 if f is not holomorphic at z_0 and there exists a neighbourhood U of z_0 such that $f \in H(U \setminus \{z_0\})$. Those singularities can be classified into *removable*, *poles* or *essential* singularities, as defined below.

Definition 1.4. Let z_0 be an isolated singularity of f. We say that z_0 is a **removable singularity** of f if $\lim_{z \to z_0} (z - z_0) f(z) = 0$, whereas if $\lim_{z \to z_0} |f(z)| = \infty$, z_0 is called a **pole**. Otherwise, z_0 is an **essential singularity** of f.

Recall that we can define the Laurent expansion of a holomorphic function near its isolated singularities, allowing us to fully characterize the type of isolated singularity in terms of the coefficients of the expansion. Indeed, let z_0 be an isolated singularity of f. Then, the Laurent expansion of f around z_0 is

$$f(z) = \sum_{n=-\infty}^{n=\infty} c_n (z - z_0)^n.$$

Consider now the coefficients c_{-n} for $n \geq 1$. If all such coefficients are 0, then z_0 is a removable singularity. If not, if there are only finitely many coefficients different from 0, the singularity is a pole, and otherwise, it is an essential singularity.

1.1 Conformal Mappings

We say that a holomorphic function f is **conformal** if it is bijective. We can also say that f is *biholomorphic*. If f is holomorphic and injective we say it is **univalent**.

A **Möbius transformation** is a function $M : \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$M(z) = \frac{az + b}{cz + d},$$

with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Observe that $M(z)$ can be extended to $\widehat{\mathbb{C}}$ by defining $M(\frac{-d}{c}) = \infty$, $M(\infty) = \frac{a}{c}$ if $c \neq 0$, and $M(\infty) = \infty$ if $c = 0$. The inverse map of M is given by

$$M^{-1}(z) = \frac{dz - b}{-cz + a},$$

which is again a Möbius transformation. In particular, M is bijective, and hence a conformal automorphism of $\widehat{\mathbb{C}}$. In fact, the Möbius transformations are precisely the conformal automorphisms of $\widehat{\mathbb{C}}$.

Some particular Möbius transformations are translations, dilations, rotations, and the inversion map. In fact, any Möbius transformation can be written as a composition of these elementary functions.

Recall that straight lines in $\mathbb{C}$ can be thought as circles in $\widehat{\mathbb{C}}$ which pass through ∞. The following result is useful for characterizing the conformal automorphisms of $\mathbb{D}$.

Proposition 1.5. *Möbius transformations map circles in* $\widehat{\mathbb{C}}$ *onto circles in* $\widehat{\mathbb{C}}$.

Among conformal maps, we are interested in studying automorphisms of simply connected domains. Any simply connected domain $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to the unit disk $\mathbb{D}$, that is, there exists a conformal map $\phi : \Omega \longrightarrow \mathbb{D}$. This result is known as the Riemann Mapping Theorem.

Theorem 1.6. (Riemann Mapping Theorem) *Let* $\Omega \subsetneq \mathbb{C}$ *be a simply connected domain and* $z_0 \in \Omega$. *Then, there exists a unique conformal map* $\phi : \Omega \longrightarrow \mathbb{D}$ *such that* $\phi(z_0) = 0$ *and* $\phi'(z_0) > 0$.

The Riemann Mapping Theorem allows us to reduce the study of automorphisms of simply connected domains to automorphisms of $\mathbb{C}$ and of $\mathbb{D}$. We start by characterizing the automorphisms of $\mathbb{C}$.

Theorem 1.7. (Automorphisms of $\mathbb{C}$) *Let* $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ *be an entire conformal map. Then, there exist* $a, b \in \mathbb{C}$, $a \neq 0$, *such that* $\phi(z) = az + b$. *Moreover, all polynomials of degree* 1 *are entire conformal maps.*

Notice that the automorphisms of $\mathbb{C}$ are precisely the Möbius transformations which fix infinity. Indeed, any Möbius transformation

$$M(z) = \frac{az+b}{cz+d}$$

fixes ∞ if and only if $M(\infty) = \frac{a}{c} = \infty$, that is, if $c = 0$. Thus,

$$M(z) = \frac{a}{d}z + \frac{b}{d} = a'z + b'$$

for some coefficients a', $b' \in \mathbb{C}$. Also, since $ad - bc \neq 0$, the coefficient $a' = \frac{a}{d} \neq 0$ and hence M is a polynomial of degree 1.

1.2 Holomorphic functions of the unit disk

In view of the distinguished role of the unit disk in conformal mappings, in this section we focus on holomorphic functions of the unit disk. We give two classical results that will be used throughout the whole project.

Lemma 1.8. (Schwarz Lemma) *Let* $f : \mathbb{D} \longrightarrow \mathbb{D}$ *holomorphic such that* $f(0) = 0$. *Then,*

(a) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$ *and* $|f'(0)| \leq 1$,

(b) if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ *such that* $|f(z_0)| = |z_0|$ *or* $|f'(0)| = 1$, *then* f *is a rotation on the origin, that is, there exists* $\theta \in \mathbb{R}$ *so that* $f(z) = e^{i\theta} z$ *for all* $z \in \mathbb{D}$.

Proof. Consider the function $g(z) = \frac{f(z)}{z}$ defined in $\mathbb{D} \setminus \{0\}$. Notice that since f is holomorphic in $\mathbb{D}$, and in particular in $z = 0$,

$$\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{f(z)}{z} = f'(0) \; .$$

Therefore, $g(z)$ has a removable singularity in $z = 0$ and it can be extended to a function $\widetilde{g}(z)$ defined in $\mathbb{D}$ as

$$\widetilde{g}(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \in \mathbb{D} \setminus \{0\} \\ f'(0) & \text{if } z = 0 \; . \end{cases}$$

Moreover, $\widetilde{g}$ is continuous in $\mathbb{D}$, and since f is holomorphic in $\mathbb{D}$, then $\widetilde{g}$ is holomorphic in $\mathbb{D} \setminus \{0\}$. By Morera's Theorem 1.3 it follows that $\widetilde{g} \in H(\mathbb{D})$.

(a) By the Maximum Modulus Principle (MMP) 1.2,

$$\sup_{|z|<1} |\widetilde{g}(z)| = \sup_{0<r<1} \sup_{|z| \leq r} |g(z)| = \lim_{r \to 1^-} \sup_{|z|=r} |g(z)| = \lim_{r \to 1^-} \sup_{|z|=r} \left| \frac{f(z)}{z} \right| =$$
$$= \lim_{r \to 1^-} \frac{1}{r} \sup_{|z|=r} |f(z)| \leq \lim_{r \to 1^-} \frac{1}{r} = 1.$$

Therefore, $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|f'(0)| \leq 1$.

(b) The hypothesis in (b) is equivalent to assume that there exists $z_0 \in \mathbb{D}$ so that $|\widetilde{g}(z_0)| = 1$. By (a), $|\widetilde{g}(z)| \leq |\widetilde{g}(z_0)|$ for all $z \in \mathbb{D}$. Since the maximum is attained in the interior of $\mathbb{D}$, by the MMP it follows that $\widetilde{g} \equiv \widetilde{g}(z_0)$.
Since $|\widetilde{g}(z_0)| = 1$, there exists $\theta \in \mathbb{R}$ so that $\widetilde{g}(z_0) = e^{i\theta}$. Hence, $\widetilde{g} \equiv e^{i\theta}$, and consequently, $f(z) = e^{i\theta} z$ for all $z \in \mathbb{D}$. □

As a consequence of Schwarz Lemma we can characterize the group of automorphisms of $\mathbb{D}$, $Aut(\mathbb{D})$. In fact, automorphisms of the disk are exactly the Möbius transformations which fix $\partial\mathbb{D}$.

Theorem 1.9. (Automorphisms of $\mathbb{D}$) *Let $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic map. Then, ϕ is conformal if, and only if,*

$$\phi(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}$$

for some $\theta \in [0, 2\pi)$ and $a \in \mathbb{D}$.

Proof. First, we prove that given any $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$, the map

$$\phi(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}$$

is conformal. Observe that ϕ is a Möbius transformation, because since $a \in \mathbb{D}$,

$$a\bar{a} - 1 = |a|^2 - 1 \neq 0.$$

Therefore, ϕ is a conformal automorphism of $\widehat{\mathbb{C}}$ and it is enough to check that $\phi(\mathbb{D}) = \mathbb{D}$. Since $\phi(0) = e^{i\theta}a \in \mathbb{D}$, it suffices to show that $\phi(\partial\mathbb{D}) = \partial\mathbb{D}$. Take a point $w \in \partial\mathbb{D}$. Then, there exists $\alpha \in [0, 2\pi)$ such that $w = e^{i\alpha}$ and

$$|\phi(w)| = \left|\phi(e^{i\alpha})\right| = \left|e^{i\theta} \frac{a - e^{i\alpha}}{1 - \bar{a}e^{i\alpha}}\right| = \left|\frac{a - e^{i\alpha}}{\bar{a} - e^{-i\alpha}}\right| = 1,$$

where the last equality holds because the numerator and denominator are complex conjugates. Thus, $\phi(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$, and by Proposition 1.5 it follows that $\phi(\partial\mathbb{D}) = \partial\mathbb{D}$. Therefore, $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ is a conformal map.

Now suppose $\phi : \mathbb{D} \longrightarrow \mathbb{D}$ is a conformal map and take $a = \phi^{-1}(0) \in \mathbb{D}$. Since we have proven that the map $\varphi_a : \mathbb{D} \longrightarrow \mathbb{D}$,

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

is conformal, the map $g := \phi \circ \varphi_a : \mathbb{D} \longrightarrow \mathbb{D}$ is also an automorphism of $\mathbb{D}$. Also,

$$g(0) = \phi(\varphi_a(0)) = \phi(a) = 0,$$

and by the Schwarz Lemma 1.8 it follows that g is a rotation and there exists $\theta \in [0, 2\pi)$ such that $g(z) = e^{i\theta}z$. Since $\varphi_a^{-1} = \varphi_a$ we conclude that $\phi = e^{i\theta}\varphi_a$. This completes the proof. $\square$

1.3 Entire Transcendental Maps

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. Then, we say that f is a **transcendental** map if it has an essential singularity at ∞. Alternatively, any entire function which is not a polynomial is transcendental. This section is devoted to give some properties and results on entire transcendental functions, which are needed on Chapter 4.

Observe that if f is an entire transcendental function, then $g(z) := f(\frac{1}{z})$ has an essential singularity at $z = 0$. Hence, we can understand the behaviour of an entire transcendental map near ∞ by studying the behaviour of holomorphic functions near an essential singularity. Next theorem shows that the image of a neighbourhood of an essential singularity cannot omit an open disk.

Theorem 1.10. (Cassorati-Weierstrass) *Let z_0 be an essential singularity of a holomorphic function f and consider U a neighbourhood of z_0. Then, $\overline{f(U)} = \mathbb{C}$.*

Proof. To see that $f(U)$ is dense in $\mathbb{C}$, we argue by contradiction. Suppose that $f(U)$ omits a disk $D(c, \varepsilon)$, i.e, $f(U) \subseteq \mathbb{C} \setminus D(c, \varepsilon)$. We want to see that then z_0 is either a removable singularity or a pole of f.
Consider the map $h(z) = \frac{1}{f(z)-c} \in H(U \setminus \{z_0\})$ and observe that it is bounded because $|f(z) - c| \geq \varepsilon$ for all $z \in U$. Then,

$$\lim_{z \to z_0} (z - z_0)h(z) = 0,$$

and therefore, z_0 is a removable singularity of h. Hence, there exists $\alpha \in \mathbb{C}$ such that

$$\alpha = \lim_{z \to z_0} h(z) = \lim_{z \to z_0} \frac{1}{f(z) - c}.$$

- If $\alpha = 0$, then $\lim\limits_{z \to z_0} |f(z) - c| = \infty$ and therefore $\lim\limits_{z \to z_0} |f(z)| = \infty$, which implies that z_0 is a pole of f.
- If $\alpha \neq 0$, then $\lim\limits_{z \to z_0} f(z) = c + \dfrac{1}{\alpha}$ and thus z_0 is a removable singularity of f.

In both cases we obtain a contradiction with the assumption that z_0 is an essential singularity of f. □

The Cassorati-Weierstrass Theorem ensures that if a holomorphic function has an essential singularity, then the image of any neighbourhood of the singularity is dense in $\mathbb{C}$. However, E. Picard proved a much stronger result: for any neighbourhood of an essential singularity, its image attains all complex values infinitely many often, except for at most one value.

Theorem 1.11. **(Picard's Great Theorem,** [9, Theorem 7.4.2]**)** *Let f be a holomorphic function with an essential singularity at $z_0 \in \mathbb{C}$. Then, there exists at most $a \in \mathbb{C}$ such that for all $w \neq a$, the equation $f(z) = w$ has infinitely many solutions in any neighbourhood of z_0.*

Hence, if f is an entire transcendental function, f attains every complex number infinitely many times, with at most one exception. We say that f is an ∞-to-1 map.

Corollary 1.12. **(Picard's Little Theorem,** [10, Theorem 16.4.2]**)** *Let f be an entire function. If there exist $a, b \in \mathbb{C}$ different such that $f(z) \neq a, b$ for all $z \in \mathbb{C}$, then f is constant.*

1.4 Covering Maps and Singular Values

In this section we introduce the notion of covering map and singular value, two concepts that are required in Chapter 4 in order to properly study entire transcendental maps of class $\mathcal{B}$. The content of this section can be found in [22].

1.4.1 Covering Maps

Definition 1.13. A map $f : X \longrightarrow Y$ is a **covering map** if f is continuous and for each $y \in Y$ there exists a neighbourhood $U \subseteq Y$ of y so that $f^{-1}(U)$ is a disjoint union of open sets V_α, each of which satisfies that $f|_{V_\alpha} : V_\alpha \longrightarrow U$ is a homeomorphism.

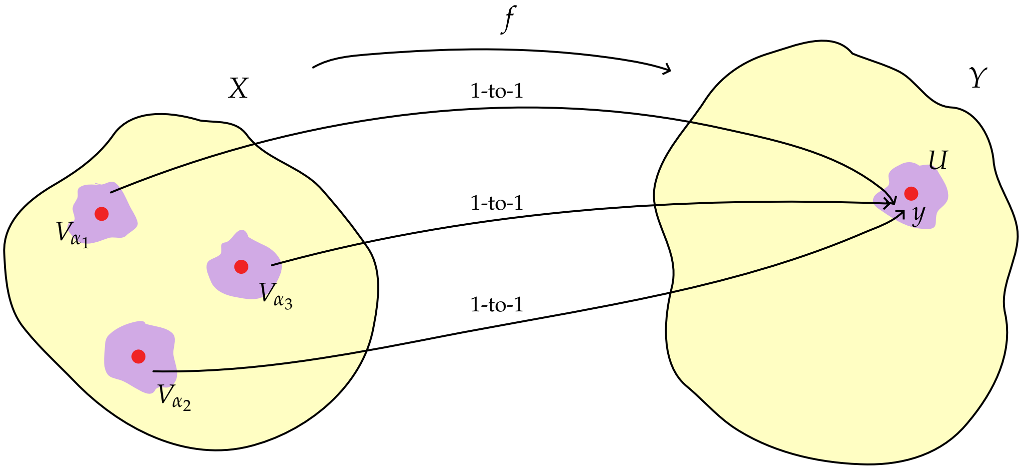


Figure 1.1: On red, a point $y \in Y$ and its preimages; on purple, a neighbourhood U of y and its preimages. Each preimage V_α maps homeomorphically to U.

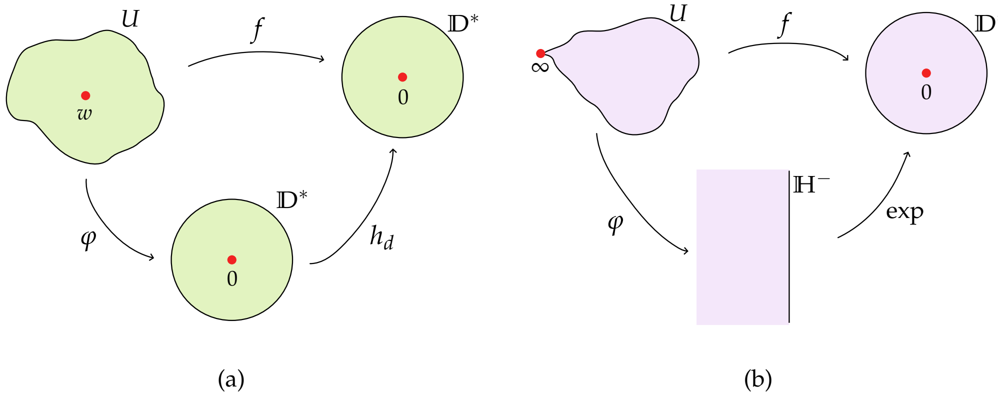
For instance, consider the map $f(z) = z^2$. Then, f is a covering map of $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. Indeed, for every $w = \rho e^{i\theta} \in \mathbb{D}^*$ there exist exactly 2 points $z_1, z_2 \in \mathbb{D}$, $z_1 = \sqrt{\rho} e^{i\theta}, z_2 = \sqrt{\rho} e^{i(\pi+\theta)}$ such that $f(z_1) = f(z_2) = w$. Therefore, there exists $U \in \mathbb{D}^*$ such that $f^{-1}(U) = V_1 \cup V_2$, with $V_1 \cap V_2 = \emptyset$, $z_1 \in V_1$, $z_2 \in V_2$ and $f|_{V_j} : V_j \longrightarrow U$ is a homeomorphism for $j = 1, 2$. Moreover, the sets V_1 and V_2 are identical up to a rotation of angle π about the origin. We say that f is a 2-to-1 map.

The following lemma characterizes the coverings of $\mathbb{D}^*$ and is one of the main tools used in Section 4.1 for studying the mapping structure of maps of class $\mathcal{B}$

Lemma 1.14. (Coverings of $\mathbb{D}^*$, [5, Lemma 2.2]) *Let $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ and $U \subseteq \widehat{\mathbb{C}}$. Suppose $f : U \longrightarrow \mathbb{D}^*$ is a holomorphic covering. Then,*

(a) either U and $\mathbb{D}^*$ are biholomorphic and there exists a conformal map $\varphi : U \longrightarrow \mathbb{D}^*$ such that $f = h_d \circ \varphi$ for some $d \geq 1$, where $h_d(w) = w^d$,

(b) or U is simply connected and there exists a conformal map $\varphi : U \longrightarrow \mathbb{H}^-$ such that $f = \exp \circ \varphi$, where $\mathbb{H}^- = \{z \in \mathbb{C}; \quad Re(z) < 0\}$.



(a) (b)

Figure 1.2: Let $f : U \longrightarrow \mathbb{D}^*$ be a covering map. Then, by Lemma 1.14, we can distinguish 2 cases: either $U \cong \mathbb{D}^*$ and $f = h_d \circ \varphi$ (case (a)), or $U \cong \mathbb{D}$ and $f = \exp \circ \varphi$ (case (b)).

1.4.2 Singular Values

Definition 1.15. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. The **set of singular values** of f is denoted $S(f)$ and is defined as the smallest closed set $S(f)$ such that

$$f : \mathbb{C} \setminus f^{-1}(S(f)) \longrightarrow \mathbb{C} \setminus S(f)$$

is a covering map.

Thus, $S(f)$ is precisely the set of points in $\mathbb{C}$ where it is not possible to locally define all inverse branches of f. We later give a more precise characterization of $S(f)$, in terms of critical and asymptotic values.

Definition 1.16. A point $w \in \mathbb{C}$ is a **critical value** of f if there exists $z \in \mathbb{C}$ so that $w = f(z)$ and $f'(z) = 0$. Then, z is called a **critical point** of f.
A point $w \in \mathbb{C}$ is an **asymptotic value** of f if there exists a continuous curve $\gamma : [0, \infty) \longrightarrow \mathbb{C}$ such that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$.

We now give some examples of functions with critical values and asymptotic values.

Example 1.17. Consider the map $f(z) = z^2$. Then, $f'(z) = 2z = 0$ if, and only if, $z = 0$. Therefore, $f(0) = 0$ is a critical value of f.

Example 1.18. Consider now the map $f(z) = e^z$ and the continuous curve

$$\gamma : [0, \infty) \longrightarrow \mathbb{C}, \quad \gamma(t) = -t.$$

Observe that $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = \lim_{t\to\infty} e^{-t} = 0$. Therefore, 0 is an asymptotic value of f. Moreover, $z = 0$ is the only singular value of f. Indeed, since $f'(z) = e^z \neq 0$ for all $z \in \mathbb{C}$, f has no critical values. Also, consider $\gamma(t)$ a curve such that $\gamma(t) \to \infty$ as $t \to \infty$.

- If $\operatorname{Re}(\gamma(t)) \to -\infty$, then $|f(\gamma(t))| = \left|e^{\gamma(t)}\right| \to 0$ as $t \to \infty$.
- If $\operatorname{Re}(\gamma(t)) \to \infty$, then $|f(\gamma(t))| = \left|e^{\gamma(t)}\right| \to \infty$ as $t \to \infty$.
- If $\operatorname{Re}(\gamma(t)) \to c \in \mathbb{C}$ as $t \to \infty$, then $\operatorname{Im}(\gamma(t)) \to \infty$ as $t \to \infty$ and $e^{\gamma(t)}$ does not converge, as $t \to \infty$.

Hence, $z = 0$ is the only singular value of the exponential map.

Observe that it is not possible to locally define all inverse branches of f around critical and asymptotic values. Thus, critical values, asymptotic values, and their accumulation points are singular values of f. The converse is also true, as showed in the following theorem.

Theorem 1.19. *The set of singular values of an entire function f is*

$$S(f) = \overline{\{w \in \mathbb{C};\ w \text{ is a critical value of } f \text{ or an asymptotic value of } f\}}.$$

Proof. Denote by CV and AV the set of critical values of f and asymptotic values of f, respectively. Then, we need to show

$$S(f) = \overline{CV \cup AV}.$$

It is clear that $CV \cup AV \subseteq S(f)$ because critical values and asymptotic values are points of $\mathbb{C}$ where it is not possible to locally define all inverse branches of f. Moreover, since $S(f)$ is closed, we conclude that $\overline{CV \cup AV} \subseteq S(f)$.

It remains to see that $S(f) \subseteq \overline{CV \cup AV}$. Let $w_0 \in S(f)$ and assume $w_0 \notin \overline{CV \cup AV}$. Then, there exists a neighbourhood U of w_0 which does not contain critical nor asymptotic values.

Since U does not contain any critical value, $f'(z) \neq 0$ for all $z \in f^{-1}(U)$. Then, by the Inverse Function Theorem, there exist V_z, $W_{f(z)}$ neighbourhoods of z and $f(z)$ respectively such that $f_{|V_z} : V_z \longrightarrow W_{f(z)}$ is biholomorphic. Thus, for all $w \in U$, f admits a locally well-defined inverse branch.

Now we need to see that these inverse branches of f are well-defined in all U. Indeed, fix some local inverse branch of f, $g_z : W_z \longrightarrow \mathbb{C}$, for some $W_z \subseteq U$ such that $f \circ g_z = id$. We want to see that g_z can be analytically continued along any path inside U. Suppose this is not true. Then, there exists $\alpha : [0, \infty) \longrightarrow U$ a path such that either it hits a critical value (which cannot happen because $U \cap CV = \emptyset$), or $g(\alpha(t)) \to \infty$ as $t \to \infty$, which also cannot happen. Indeed, the curve $\gamma(t) := (g \circ \alpha)(t) \to \infty$ as $t \to \infty$ and

$$\lim_{t\to\infty} f(\gamma(t)) = \lim_{t\to\infty} \alpha(t) \in U.$$

But this cannot happen because U contains no asymptotic values. Therefore, all inverse branches of f are well-defined in all U, and hence $w \notin S(f)$, contradicting the assumption. □

1.4.3 The Eremenko-Lyubich Class

A special class of entire transcendental functions consists of those for which $S(f)$ is bounded. In this case, f acts as a covering map in a punctured neighbourhood of ∞, a fact that plays a particularly important role in dynamics, as we shall see in Chapter 4.

Definition 1.20. Let f be a transcendental entire function. If $S(f)$ is bounded, then we say that f is in the **Eremenko-Lyubich** class, also known as class $\mathcal{B}$.

We claim that $\mathcal{B}$ is closed under composition, that is, if $f = g \circ h$, then $S(f) = g(S(h)) \cup S(g)$. Indeed, let w be a singular value of f. There are two possibilities, either w is a critical value, or an asymptotic value.

First suppose that w is a critical value of f. Then, there exists $z \in \mathbb{C}$ so that $f(z) = g(h(z)) = w$ and $f'(z) = g'(h(z))h'(z) = 0$. From the second equation we deduce that one of the factors must be zero. If $g'(h(z)) = 0$, then $h(z)$ is a critical point of g and $w = g(h(z))$ is a critical value of g. Hence, $w \in S(g)$. If not, then $h'(z) = 0$ and z is a critical point of h. Therefore, $h(z)$ is a critical value of h, and thus, $w = g(h(z)) \in g(S(h))$.

Now suppose that w is an asymptotic value of f. Then, there exists a continuous curve $\gamma : [0, \infty) \longrightarrow \mathbb{C}$ such that $\lim_{t\to\infty} \gamma(t) = \infty$ and

$$\lim_{t\to\infty} f(\gamma(t)) = \lim_{t\to\infty} g(h(\gamma(t))) = w.$$

If $\lim_{t\to\infty} h(\gamma(t)) = \infty$, due to the fact that $\gamma \circ h$ is also a continuous curve, w is an asymptotic value of g. Thus, $w \in S(g)$. If not, then there exists $w' \in \mathbb{C}$ so that $\lim_{t\to\infty} h(\gamma(t)) = w'$, and therefore w' is an asymptotic value of h, so $w' \in S(h)$. Hence, $w = g(w') \in g(S(h))$.

We deduce that class $\mathcal{B}$ is closed under composition and

$$S(f^n) = S(f) \cup f(S(f)) \cup \cdots \cup f^{n-1}(S(f)).$$

1.5 Banach Fixed Point Theorem

Although the Banach Fixed Point Theorem is not specific to Complex Analysis, we include it here as a general tool that will be used for the proof of the Main Theorem in Section 4.3.

Theorem 1.21. (Banach Fixed Point Theorem) *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a contractive map, i.e. $d(T(x), T(y)) \leq k \cdot d(x, y)$ for some $k \in (0, 1)$. Then, T has a unique fixed point in X.*

Chapter 2

Hyperbolic Geometry

When working with the unit disk $\mathbb{D}$ and holomorphic functions, it is often useful to consider the hyperbolic metric instead of the Euclidean metric. The hyperbolic metric induces the same topology as the Euclidean one, but it offers several advantages over it; namely holomorphic maps act as contractions for the hyperbolic metric, as shown by the Schwarz-Pick Lemma.

This chapter is meant to be an overview of Hyperbolic Geometry and to present some of the tools that are needed in Section 4.3. The content presented in this section can be found in [1, 2, 7, 15].

2.1 The Hyperbolic Metric in the Unit Disk

The hyperbolic metric in the unit disk $\mathbb{D}$ is defined as the unique metric in $\mathbb{D}$ whose isometries are precisely the conformal automorphisms of $\mathbb{D}$. In order to make this precise, we shall work with the density of the metric, denoted by $\rho(t)\,|dt|$, and ask that it must be invariant under conformal automorphisms of the unit disk, that is, under Möbius transformations fixing $\partial\mathbb{D}$. This means that, if M is an automorphism of the disk, then

$$\rho(M(z))\,\left|M'(z)\right| = \rho(z),\ z \in \mathbb{D}. \tag{2.1}$$

Let M be a Möbius transformation fixing $\partial\mathbb{D}$. Then, by Theorem 1.9, there exist $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$ such that $M(z) = e^{i\theta}\dfrac{z+a}{1+\bar{a}z}$. The derivative of $M(z)$ is

$$M'(z) = e^{i\theta}\frac{1-|a|^2}{(1+\bar{a}z)^2},$$

and therefore

$$|M'(z)| = \frac{1-|a|^2}{|1+\bar{a}z|^2}.$$

From Equation (2.1) it follows that

$$\rho(z) = \rho\left(e^{i\theta}\frac{z+a}{1+\bar{a}z}\right) \cdot \frac{1-|a|^2}{|1+\bar{a}z|^2}.$$

Setting $z = 0$, we obtain $\rho(e^{i\theta}a) = \dfrac{\rho(0)}{1-|a|^2}$. By convention, we let $\rho(0) = 1$, and write

$$\rho(a) = \frac{1}{1-|a|^2}. \tag{2.2}$$

Notice that this expression only depends on the modulus of a. Now we can define the hyperbolic distance in $\mathbb{D}$.

Definition 2.1. Let γ be a path from p to q in $\mathbb{D}$. Then the **hyperbolic length** or **ρ-length** of γ is

$$\rho(\gamma) = \int_\gamma \rho(t)\,|dt|.$$

The **hyperbolic distance** or **ρ-distance** from p to q is

$$\rho(p,q) = \inf_{\gamma \in \Gamma_{p,q}} \rho(\gamma),$$

where $\Gamma_{p,q} = \{\gamma \subset \mathbb{D};\ \gamma \text{ is a path between } p \text{ and } q\}$.

It is easy to prove that ρ is actually a distance. Moreover, by applying Equation (2.2) to Definition 2.1 we obtain an explicit formula for the hyperbolic length of any path γ in $\mathbb{D}$. Indeed, first consider $\gamma_0(t) = tp$ for $0 \leq t \leq 1$ a path between 0 and p. Then,

$$\rho(\gamma_0) = \int_{\gamma_0} \rho(t)\,|dt| = \int_0^1 \frac{1}{1-t^2\,|p|^2}\,|p|\,dt = \frac{1}{2}\log\frac{1+|p|}{1-|p|}.$$

The path γ_0 has the shortest hyperbolic length among all paths between 0 and p (see [15, Section 2.1.1]), and therefore the hyperbolic distance between 0 and p is

$$\rho(0,p) = \frac{1}{2}\log\frac{1+|p|}{1-|p|}.$$

Let $p, q \in \mathbb{D}$. We are interested in obtaining a formula for the hyperbolic distance between p and q. Since hyperbolic density is invariant under Möbius transformations, $\rho(\gamma) = \rho(M(\gamma))$ for any γ path in $\mathbb{D}$ and any Möbius transformation $M \in Aut(\mathbb{D})$.

Consider the Möbius transformation $M(z) = \frac{z-p}{1-\overline{p}z}$, so that $M(p) = 0$, and set $s = M(q) = \frac{q-p}{1-\overline{p}q}$. Then,

$$\rho(p,q) = \inf_{\gamma\in\Gamma_{p,q}} \rho(\gamma) = \inf_{\gamma\in\Gamma_{p,q}} \rho(M(\gamma)) = \inf_{\gamma\in\Gamma_{0,s}} \rho(\gamma) = \rho(0,s) = \frac{1}{2}\log\frac{1+|s|}{1-|s|}.$$

Therefore, the hyperbolic distance between any two points $p, q \in \mathbb{D}$ is

$$\rho(p,q) = \frac{1}{2}\log\frac{|1-\overline{p}q| + |q-p|}{|1-\overline{p}q| - |q-p|}. \tag{2.3}$$

Remark 2.2. Observe that if $p \to \partial\mathbb{D}$, then $\rho(p,q) \to \infty$ for any $q \in \mathbb{D}$. Indeed, if $|p| \to 1^-$, then

$$|1-\overline{p}q| - |q-p| \longrightarrow |1-q| - |q-1| = 0,$$

and therefore $\rho(p,q) \to \infty$.

In fact, it can be showed that the unit disk with the hyperbolic distance, $(\mathbb{D}, \rho)$, is a complete metric space, as shown in the following proposition.

Proposition 2.3. *The space $(\mathbb{D}, \rho)$ is a complete metric space.*

Proof. First observe that all Cauchy sequences with respect to ρ are also Cauchy sequences with respect to the Euclidean metric. Indeed, given p, $q \in \mathbb{D}$ and γ the shortest path in $\mathbb{D}$ joining p and q,

$$\rho(p,q) = \rho(\gamma) = \int_\gamma \rho(t)\,|dt| = \int_\gamma \frac{1}{1-|t|^2}\,|dt| \geq \int_\gamma 1\,|dt| = |q-p|\,.$$

Therefore, let $\{z_n\}_n$ be a Cauchy sequence with respect to ρ. Then, given any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all n, $m \geq n_0$, $\rho(z_n, z_m) < \varepsilon$. By the inequality above,

$$|z_n - z_m| \leq \rho(z_n, z_m) < \varepsilon,$$

and hence $\{z_n\}_n$ is a Cauchy sequence with respect to the Euclidean metric.

Take $R = \max\{\rho(z_{n_0}, z_1), \cdots, \rho(z_{n_0}, z_{n_0-1}), \varepsilon\}$. Then, $\rho(z_{n_0}, z_m) \leq R$ for all $m \in \mathbb{N}$ and we conclude that the sequence $\{z_n\}_n$ is bounded with respect to the hyperbolic metric. By Remark 2.2, it follows that there exists a compact subset $K \subseteq \mathbb{D}$ such that $\{z_n\}_n \subseteq K$. Since $\{z_n\}_n$ is also a Cauchy sequence with respect to the Euclidean metric and K is compact, there exists $z \in K$ so that $z_n \to z$ with respect to the Euclidean metric.

It remains to see that $z_n \to z$ also with respect to the hyperbolic metric. Indeed, since $|z_n - z| \to 0$ as $n \to \infty$,

$$\rho(z_n, z) = \frac{1}{2}\log\frac{|1-\overline{z_n}z| + |z - z_n|}{|1-\overline{z_n}z| - |z - z_n|} \longrightarrow \frac{1}{2}\log\frac{|1-\overline{z_n}z|}{|1-\overline{z_n}z|} = \frac{1}{2}\log(1) = 0.$$

We conclude that $(\mathbb{D}, \rho)$ is complete. □

Next we see that any holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{D}$ is a contraction with respect to the hyperbolic metric. Moreover, we see that f is a hyperbolic isometry if and only if f is an automorphism of the disk. This is known as Schwarz-Pick Lemma.

Theorem 2.4. (Schwarz-Pick Lemma) *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function. Then,*

(a) f is an infinitesimal contraction. That is,

$$\rho(f(z))\,|f'(z)| \leq \rho(z) \quad \textit{for all } z \in \mathbb{D}. \tag{2.4}$$

(b) f is a global contraction. That is,

$$\rho(f(z), f(w)) \leq \rho(z, w) \quad \textit{for all } z, w \in \mathbb{D}. \tag{2.5}$$

Proof. Let $z \in \mathbb{D}$ and consider the Möbius transformations

$$M_1(w) = \frac{z - w}{1 - \bar{z}w}\ , \text{and } M_2(w) = \frac{f(z) - w}{1 - \overline{f(z)}w},$$

which send z and $f(z)$ to 0, respectively. Now consider $g = M_2 \circ f \circ M_1^{-1}$, and observe that

$$g(0) = M_2(f(M_1^{-1}(0)) = M_2(f(z)) = 0.$$

Now we compute the derivative of g, and by the chain rule we obtain that

$$g'(w) = \frac{|f(z)|^2 - 1}{(1 - \overline{f(z)}f(\frac{z-w}{1-\bar{z}w}))^2} \cdot f'\left(\frac{z - w}{1 - \bar{z}w}\right) \cdot \frac{|z|^2 - 1}{(1 - \bar{z}w)^2}.$$

Therefore,

$$g'(0) = f'(z)\frac{|z|^2 - 1}{|f(z)|^2 - 1}.$$

By the Schwarz Lemma 1.8 it follows that

$$|g'(0)| = |f'(z)|\,\frac{|z|^2 - 1}{|f(z)|^2 - 1} \leq 1, \quad \text{for all } z \in \mathbb{D}.$$

Since $\rho(z) = \dfrac{1}{1-|z|^2}$ and $\rho(f(z)) = \dfrac{1}{1-|f(z)|^2}$, we obtain that

$$\left|f'(z)\right| \frac{\rho(f(z))}{\rho(z)} = \left|f'(z)\right| \frac{|z|^2-1}{|f(z)|^2-1} \leq 1, \qquad \text{for all } z \in \mathbb{D},$$

and therefore the inequality (2.4) holds and f is an infinitesimal contraction.
It remains to show that f is a global contraction. Let $z, w \in \mathbb{D}$ and let γ be the shortest path between z and w. Then, since $f(\gamma)$ is a path between $f(z)$ and $f(w)$,

$$\begin{aligned} \rho(f(z), f(w)) \leq \rho(f(\gamma)) = \int_{f(\gamma)} \rho(t)\,|dt| = \int_\gamma \rho(f(t))\,\left|f'(t)\right|\,|dt| \leq \int_\gamma \rho(t)\,|dt| = \\ = \rho(\gamma) = \rho(z,w), \end{aligned}$$

where the last inequality is due to Equation (2.4). ☐

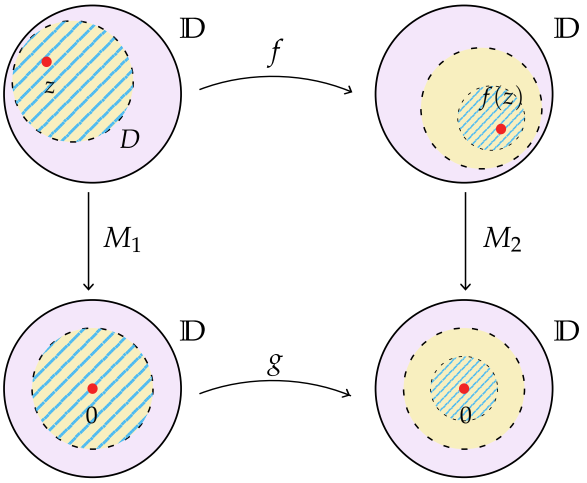


Figure 2.1: On the top-left disk we consider a point $z \in \mathbb{D}$ and a hyperbolic disk $D(z,r) \subseteq \mathbb{D}$. In yellow, hyperbolic disks of radius r centred at the corresponding image of z. In blue, the corresponding images of D.

Observe that if there exists $z \in \mathbb{D}$ such that $\rho(f(z))\,|f'(z)| = \rho(z)$, then f is a conformal self-map of $\mathbb{D}$. Indeed, if such z exists, the function g defined in the proof of the previous lemma satisfies $|g'(0)| = 1$. Then, by the Schwarz Lemma 1.8, g is a rotation, and therefore f is conformal. Thus, f is an automorphism of the unit disk and the equality holds for all $z \in \mathbb{D}$. This is also true if Equation (2.5) is an equality for some $z, w \in \mathbb{D}$, as shown in the following proposition.

Proposition 2.5. *Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function and let $z, w \in \mathbb{D}$ be such that $\rho(f(z), f(w)) = \rho(z,w)$. Then f is conformal and the equality holds for all $z, w \in \mathbb{D}$.*

Proof. Using Equation (2.3) for the hyperbolic distance between any two points, we obtain that $\rho(f(z), f(w)) = \rho(z,w)$ if, and only if,

$$\left|\frac{z-w}{1-\bar{z}w}\right| = \left|\frac{f(z)-f(w)}{1-\overline{f(z)}f(w)}\right|. \tag{2.6}$$

Consider the auxiliar functions M_1, M_2, g as in the proof of the previous theorem. Then, since $M_2 \circ f = g \circ M_1$, the equality (2.6) is equivalent to

$$|M_1(w)| = |M_2(f(w))| = |g(M_1(w))|.$$

By the Schwarz Lemma 1.8 it follows that g is a rotation. Therefore, we conclude that f is conformal and $\rho(f(z), f(w)) = \rho(z,w)$ for all $z, w \in \mathbb{D}$. □

In summary, we have defined the hyperbolic metric on the unit disk as the metric such that all automorphisms of the unit disk are isometries. The previous lemma states that there are no other hyperbolic isometries besides automorphisms of $\mathbb{D}$, and therefore, hyperbolic isometries are precisely the conformal automorphisms of the unit disk.

2.2 Hyperbolic Geometry for Simply Connected Domains

Up to now we have only defined the hyperbolic metric in $\mathbb{D}$, but it can be defined in any proper simply connected domain of $\mathbb{C}$, by pulling back ρ by conformal maps. From this point on, we denote the hyperbolic metric in $\mathbb{D}$ as $\rho_{\mathbb{D}}$.

Definition 2.6. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $f : \Omega \longrightarrow \mathbb{D}$ a conformal map. Then, the **hyperbolic metric** $\rho_\Omega(z)\,|dz|$ of Ω is

$$\rho_\Omega(z) := \rho_{\mathbb{D}}(f(z))\,|f'(z)|.$$

Proposition 2.7. *The hyperbolic metric ρ_Ω is independent of the choice of the conformal map f.*

Proof. Let $f : \Omega \longrightarrow \mathbb{D}$ be a conformal map. Since all conformal self-maps of $\mathbb{D}$ are Möbius transformations fixing $\partial\mathbb{D}$, any other conformal map from Ω to $\mathbb{D}$ is of the form $g = M \circ f$, where M is an automorphism of $\mathbb{D}$. Then,

$$\rho_{\mathbb{D}}(g(z))\,|g'(z)| = \rho_{\mathbb{D}}(M(f(z)))\,|M'(f(z))|\,|f'(z)| = \rho_{\mathbb{D}}(f(z))\,|f'(z))|,$$

where last equality is due to Equation (2.1). □

Therefore, ρ_Ω is well-defined and only depends on Ω. As an example we compute the hyperbolic metric of the right-half plane $\mathbb{H} = \{z \in \mathbb{C} : \ \mathrm{Re}(z) > 0\}$. The map $f : \mathbb{H} \longrightarrow \mathbb{D}$, $f(z) = \frac{z-1}{z+1}$ is conformal. Therefore,

$$\rho_{\mathbb{H}}(z) = \frac{1}{1 - \left|\frac{z-1}{z+1}\right|^2} \cdot \left| \frac{2}{(z+1)^2} \right| = \frac{1}{2\mathrm{Re}(z)}. \tag{2.7}$$

The hyperbolic length of a path $\gamma \subset \Omega$ and the hyperbolic distance of two points in Ω are defined analogously in any simply connected domain.

Definition 2.8. Let γ be a path from p to q in Ω. Then the **hyperbolic length** or **ρ_Ω-length** of γ is

$$\rho_\Omega(\gamma) = \int_\gamma \rho_\Omega(t)\, |dt|.$$

The **hyperbolic distance** or **ρ_Ω-distance** from p to q is

$$\rho_\Omega(p,q) = \inf_{\gamma \in \Gamma^\Omega_{p,q}} \rho_\Omega(\gamma),$$

where $\Gamma^\Omega_{p,q} = \{\gamma \subset \Omega;\ \gamma \text{ is a path between } p \text{ and } q\}$.

It is clear that $\rho_\Omega(p,q)$ defines a distance in Ω. Moreover, given any conformal map $f : \Omega \longrightarrow \mathbb{D}$,

$$\rho_\Omega(\gamma) = \int_\gamma \rho_{\Omega(z)}\, |dz| = \int_\gamma \rho_{\mathbb{D}}(f(z))\, |f'(z)|\, |dz| = \int_{f(\gamma)} \rho_{\mathbb{D}}(z)\, |dz| = \rho_{\mathbb{D}}(f(\gamma)),$$

and, therefore, we can obtain an equivalent definition for the hyperbolic distance,

$$\rho_\Omega(z,w) = \inf_{\gamma \in \Gamma^\Omega_{z,w}} \rho_\Omega(\gamma) = \inf_{\gamma \in \Gamma^\Omega_{z,w}} \rho_{\mathbb{D}}(f(\gamma)) = \rho_{\mathbb{D}}(f(z), f(w)). \tag{2.8}$$

This definition allows us deduce that (Ω, ρ_Ω) is a complete metric space, directly from the completeness of $(\mathbb{D}, \rho_{\mathbb{D}})$.

Proposition 2.9. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Then, (Ω, ρ_Ω) is a complete metric space.*

Proof. We need to see that any Cauchy sequence of Ω converges in Ω. Let $\{z_n\}_n \subseteq \Omega$ be a Cauchy sequence. Then, for all $\varepsilon > 0$, there exists $n_0 \geq 1$ such that $\rho_\Omega(z_n, z_m) < \varepsilon$ for all $n, m \geq n_0$. Consider a conformal map $f : \Omega \longrightarrow \mathbb{D}$. Then, since $\rho_\Omega(z_n, z_m) = \rho_{\mathbb{D}}(f(z_n), f(z_m))$, the sequence $\{w_n := f(z_n)\}_n \subseteq \mathbb{D}$ is a Cauchy sequence, and therefore is convergent in $\mathbb{D}$, i.e., there exists $w \in \mathbb{D}$ such that $\rho_{\mathbb{D}}(w_n, w) \to 0$ as $n \to \infty$. Since f is conformal, there exists $z \in \Omega$ such that $w = f(z)$, and thus, $\rho_\Omega(z_n, z) = \rho_{\mathbb{D}}(w_n, w) \to 0$ as $n \to \infty$. Hence, $\{z_n\}_n$ is convergent in Ω. □

Proposition 2.10. (Hyperbolic isometries) *Let* $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$ *be simply connected domains and* $f : \Omega_1 \longrightarrow \Omega_2$ *conformal. Then,* f *is a hyperbolic isometry, that is,*

(i) $\rho_{\Omega_2}(f(z))\,|f'(z)| = \rho_{\Omega_1}(z)$ *for all* $z \in \Omega_1$,

(ii) $\rho_{\Omega_2}(f(z), f(w)) = \rho_{\Omega_1}(z, w)$ *for all* $z, w \in \Omega_1$.

Proof. Consider $\phi_2 : \Omega_2 \longrightarrow \mathbb{D}$ a conformal map and set $\phi_1 = \phi_2 \circ f : \Omega_1 \longrightarrow \mathbb{D}$.

$$\begin{tikzcd} \Omega_1 \arrow[r, "f"] \arrow[dr, "\phi_1"'] & \Omega_2 \arrow[d, "\phi_2"] \\ & \mathbb{D} \end{tikzcd}$$

Since f and ϕ_2 are conformal, ϕ_1 is also conformal. Take $z \in \Omega_1$. Then, $f(z) \in \Omega_2$, and by definition,

$$\rho_{\Omega_2}(f(z)) = \rho_{\mathbb{D}}(\phi_2(f(z)))\,\left|\phi_2'(f(z))\right|.$$

Therefore, since $\phi_2 \circ f = \phi_1$, we obtain that

$$\rho_{\Omega_2}(f(z))\,\left|f'(z)\right| = \rho_{\mathbb{D}}(\phi_2(f(z)))\,\left|\phi_2'(f(z))\right|\,\left|f'(z)\right| = \rho_{\mathbb{D}}(\phi_1(z))\,\left|\phi_1'(z)\right| = \rho_{\Omega_1}(z).$$

The second equality of the proposition follows from the definition of hyperbolic distance. Indeed,

$$\rho_{\Omega_2}(f(z), f(w)) = \rho_{\mathbb{D}}(\phi_2(f(z)), \phi_2(f(w))) = \rho_{\mathbb{D}}(\phi_1(z), \phi_1(w)) = \rho_{\Omega_1}(z, w),$$

as desired. □

In the general case of simply connected domains there is also a version of the Schwarz-Pick Lemma, which states that any holomorphic map $f : \Omega_1 \longrightarrow \Omega_2$ with Ω_1, Ω_2 simply connected domains is a hyperbolic contraction.

Theorem 2.11. (Schwarz-Pick Lemma for Simply Connected Domains) *Let* $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$ *be simply connected and* $f : \Omega_1 \longrightarrow \Omega_2$ *holomorphic. Then, either*

(a) f *is a strict hyperbolic contraction, that is,*

  (i) $\rho_{\Omega_2}(f(z))\,|f'(z)| < \rho_{\Omega_1}(z)$ *for all* $z \in \Omega_1$ *and,*

  (ii) $\rho_{\Omega_2}(f(z), f(w)) < \rho_{\Omega_1}(z, w)$ *for all* $z, w \in \Omega_1$,

(b) or f *is a hyperbolic isometry, that is,*

  (i) $\rho_{\Omega_2}(f(z))\,|f'(z)| = \rho_{\Omega_1}(z)$ *for all* $z \in \Omega_1$ *and,*

  (ii) $\rho_{\Omega_2}(f(z), f(w)) = \rho_{\Omega_1}(z, w)$.

Proof. Notice that because of Theorem 2.10, we only need to show that if f is not bijective, then f is a strict hyperbolic contraction.
Take $z_0 \in \Omega_1$ and set $w_0 = f(z_0)$. Consider two conformal maps $h : \mathbb{D} \longrightarrow \Omega_1$, $g : \Omega_1 \longrightarrow \Omega_2$ such that $h(0) = z_0$ and $g(z_0) = w_0$ (the existence of these maps follows from the Riemann Mapping Theorem 1.6). Now consider the map

$$k = h^{-1} \circ g^{-1} \circ f \circ h : \mathbb{D} \longrightarrow \mathbb{D},$$

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "h"] \arrow[rrrr, bend left, "k"] & \Omega_1 \arrow[r, "f"] & \Omega_2 \arrow[r, "g^{-1}"] & \Omega_1 \arrow[r, "h^{-1}"] & \mathbb{D} \\
0 \arrow[r, mapsto] & z_0 \arrow[r, mapsto] & w_0 \arrow[r, mapsto] & z_0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

Observe that $k(0) = 0$. Since k is a composition of holomorphic functions, it is holomorphic. However, k cannot be conformal, because since h and g are conformal, then f would be conformal. Hence, by the Schwarz Lemma 1.8 we conclude that $|k'(0)| < 1$. By the chain rule, we obtain that

$$\begin{aligned}
k'(0) &= (h^{-1})'(g^{-1} \circ f \circ h(0)) \cdot (g^{-1})'(f \circ h(0)) \cdot f'(h(0)) \cdot h'(0) = \\
&= (h^{-1})(z_0) \cdot (g^{-1})'(w_0) \cdot f'(z_0) \cdot h'(0) = \frac{f'(z_0)}{g'(z_0)} < 1.
\end{aligned}$$

Therefore, $|f'(z_0)| < |g'(z_0)|$. Notice that g is a conformal isometry (by Theorem 2.10) and $f(z_0) = g(z_0)$. Hence,

$$\rho_{\Omega_2}(f(z_0))\,|f'(z_0)| < \rho_{\Omega_2}(g(z_0))\,|g'(z_0)| = \rho_{\Omega_1}(z_0).$$

This completes the proof of part (i).
Now let $z, w \in \Omega_1$ and γ be the shortest path joining z and w. Then,

$$\begin{aligned}
\rho_{\Omega_1}(z,w) &= \int_\gamma \rho_{\Omega_1}(t)\,|dt| > \int_\gamma \rho_{\Omega_2}(f(t))\,|f'(t)|\,|dt| = \\
&= \int_{f(\gamma)} \rho_{\Omega_2}(t)\,|dt| \geq \rho_{\Omega_2}(f(z), f(w)),
\end{aligned}$$

since $f(\gamma)$ is a path joining $f(z)$ and $f(w)$. □

Next we apply this theorem to compare the hyperbolic metric of two simply connected domains $\Omega_1 \subseteq \Omega_2$. This result is needed to prove the Main Theorem in Section 4.3.

Corollary 2.12. (Comparison Principle) *Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be simply connected domains. If $\Omega_1 \subseteq \Omega_2$, then $\rho_{\Omega_2}(z) \leq \rho_{\Omega_1}(z)$ for all $z \in \Omega_1$. Furthermore, if $\Omega_1 \neq \Omega_2$, then $\rho_{\Omega_2}(z) < \rho_{\Omega_1}(z)$ for all $z \in \Omega_1$.*

Proof. Consider the inclusion map $f : \Omega_1 \longrightarrow \Omega_2$, $f(z) = z$. Then, $|f'(z)| = 1$ for all $z \in \Omega_1$, and by Theorem 2.11, we obtain that $\rho_{\Omega_2}(z) \leq \rho_{\Omega_1}(z)$ for all $z \in \Omega_1$. Now suppose that there exists $z_0 \in \Omega_1$ such that $\rho_{\Omega_2}(z_0) = \rho_{\Omega_1}(z_0)$. Then, also by Theorem 2.11, the equality holds for all $z \in \Omega_1$ and f is conformal. Since f is the identity map, we conclude that $\Omega_1 = \Omega_2$. □

2.3 Distortion Estimates for Univalent Maps

This section aims to find a lower bound for ρ_Ω. To that end, we first need Koebe's $\frac{1}{4}$-Theorem.

Theorem 2.13. (Koebe's $\frac{1}{4}$-Theorem, [7, Theorem 1.3]**)** *Let $g : \mathbb{D} \longrightarrow \mathbb{C}$ be a univalent map in $\mathbb{D}$ such that $g(0) = 0$ and $g'(0) = 1$. Then,*

$$D\left(0, \frac{1}{4}\right) \subseteq g(\mathbb{D}).$$

Corollary 2.14. *Let $f : \mathbb{D} \longrightarrow \mathbb{C}$ be a univalent map. Then,*

$$D\left(f(0), \frac{|f'(0)|}{4}\right) \subseteq f(\mathbb{D}).$$

Proof. Notice that since f is injective, $f'(0) \neq 0$. Then, consider the conformal maps $\phi_1(z) = z - f(0)$, $\phi_2(z) = \frac{z}{f'(0)}$ and let $g = \phi_2 \circ \phi_1 \circ f$. Observe that $g(0) = 0$ and $g'(0) = 1$. Then, by Koebe's $\frac{1}{4}$ Theorem 2.13, $D(0, \frac{1}{4}) \subseteq g(\mathbb{D})$. Since ϕ_1 and ϕ_2 are biholomorphic and they translate and dilate $f(\mathbb{D})$ (as shown on Figure 2.2), composing with $\phi_1^{-1} \circ \phi_2^{-1}$ we obtain that

$$D\left(f(0), \frac{|f'(0)|}{4}\right) \subseteq f(\mathbb{D}).$$

□

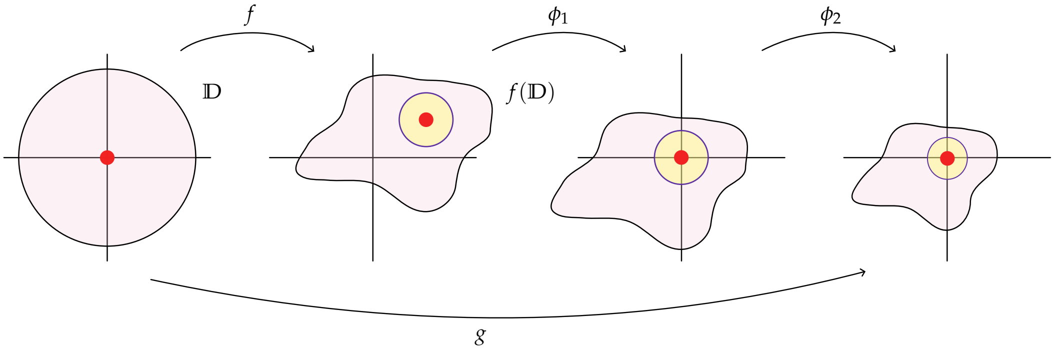


Figure 2.2: The function f composed with a translation ϕ_1 and a dilation ϕ_2 such that $g = \phi_2 \circ \phi_1 \circ f$ fixes 0 and $g'(0) = 1$. In red, the images of 0, and in yellow, the disk $D(0, \frac{1}{4}) \subseteq g(\mathbb{D})$ and its preimages.

Proposition 2.15. (Bound on Hyperbolic Density) *Let* $\Omega \subseteq \mathbb{C}$ *be a simply connected domain. Then for all* $z \in \Omega$,

$$\frac{1}{4\, d(z, \partial\Omega)} \leq \rho_\Omega(z).$$

Proof. Let $z \in \Omega$ and consider a conformal map $f : \mathbb{D} \longrightarrow \Omega$ with $f(0) = z$. Then, by Corollary 2.14, it follows that $D(z, \frac{|f'(0)|}{4}) \subseteq f(\mathbb{D}) = \Omega$, and therefore,

$$d(z, \partial\Omega) \geq \frac{|f'(0)|}{4}.$$

Observe that $f^{-1} : \Omega \longrightarrow \mathbb{D}$ is a conformal map such that $f^{-1}(z) = 0$. Then,

$$\rho_\Omega(z) = \rho_\mathbb{D}(f^{-1}(z)) \left|(f^{-1})'(z)\right| = \rho_\mathbb{D}(0) \frac{1}{|f'(0)|}.$$

Hence,

$$1 = \rho_\mathbb{D}(0) = \rho_\Omega(z) \left|f'(0)\right| \leq 4\rho_\Omega(z) d(z, \partial\Omega). \qquad \square$$

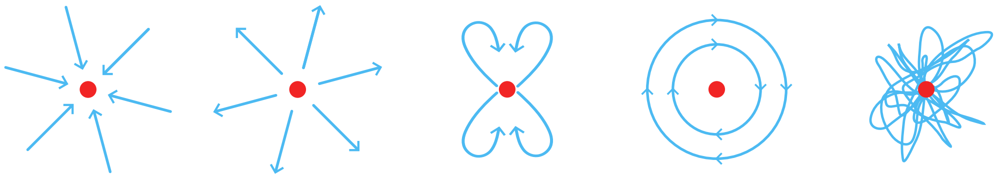
Chapter 3

Local Fixed Point Theory

Let f be a holomorphic map. We will write $f^0 = id$ and $f^n = f \circ f^{n-1}$. The **orbit** of any given point z_0 is the set of points $\mathcal{O}(z_0) = \{z_0, f(z_0), f^2(z_0), \cdots\}$ and we usually write $z_n = f^n(z_0)$.

Definition 3.1. Let f be holomorphic. We say that z_0 is a **periodic point** of f if there exists $n \geq 1$ such that $f^n(z_0) = z_0$, where n is a **period** of z_0. The smallest n with such property is called the **minimal period** of z_0.
A periodic point of period 1 is called a **fixed point**.
If z_0 is a periodic point of minimal period n of f, $\lambda := (f^n)'(z_0)$ is called the **multiplier** of z_0.

We can classify periodic points according to its multiplier. Let z_0 be a periodic point of minimal period n of f and $\lambda = (f^n)'(z_0)$. If $|\lambda| < 1$, then we say that z_0 is an **attracting** periodic point. Attracting periodic points of multiplier $\lambda = 0$ are called **superattracting**, otherwise, are called **geometrically attracting**. If $|\lambda| > 1$, then we say that z_0 is a **repelling** periodic point. If $|\lambda| = 1$, the dynamics around the fixed point depend on the properties of z_0, and are further classified into **parabolic**, **Siegel** or **Cremer** points.



(a) Attracting (b) Repelling (c) Parabolic (d) Siegel (e) Cremer

Figure 3.1: Intuitive representation of the local dynamics around each different type of fixed point.

We devote the rest of the chapter to providing a precise description of the dynamics around fixed points of the different types, and to providing the existence of conjugacies and normal forms. We follow the exposition given by J. Milnor in [18, Chapters 8-11].

Notice that periodic points of period p of f are fixed points of f^p. Thus, we will reduce to the study of fixed points.

Remark 3.2. Observe that given a holomorphic map f with a fixed point, we can choose a local uniformizing parameter z for expressing our map so that the fixed point corresponds to $z = 0$. In fact, suppose that $f(w)$ has a fixed point in $w = a$, that is $f(a) = a$, and consider the change of variables $z = C(w) = w - a$. Then, $\widetilde{f} = C \circ f \circ C^{-1}$ satisfies

$$\widetilde{f}(z) = (C \circ f \circ C^{-1})(z) = f(z + a) - a$$

Therefore, $\widetilde{f}(0) = f(a) - a = a - a = 0$. This will be useful in some of the next results.

3.1 Attracting Fixed Points

We first start by giving a topological definition of attracting fixed points. Intuitively, z_0 is an attracting fixed point if all orbits starting sufficiently close to z_0 eventually converge to z_0.

Definition 3.3. A fixed point z_0 of a map f is **topologically attracting** if there exists a neighbourhood U of z_0 such that the iterates f^n are well-defined throughout U and the sequence $\{f^n|_U\}_n$ converges uniformly to the constant map z_0.

Now we give a characterization of attracting fixed points in terms of their multiplier. This will be useful in deriving important results, such as the Koenigs Linearization Theorem.

Lemma 3.4. *A fixed point of a holomorphic map is topologically attracting if, and only if, its multiplier satisfies* $|\lambda| < 1$.

Proof. Following Remark 3.2 we can assume that the fixed point is $z_0 = 0$.
Suppose that $|\lambda| < 1$. Then, the Taylor expansion of f around 0 yields

$$f(z) = 0 + f'(z)z + O(z^2) = \lambda z + O(z^2).$$

This means that there exist constants $C_0 > 0$, $r_0 > 0$, such that $|f(z) - \lambda z| \leq C_0 \left|z^2\right|$ for $|z| < r_0$.
We have that $|f(z)| - |\lambda z| \leq |f(z) - \lambda z| \leq C_0 \left|z^2\right|$. Therefore,

$$|f(z)| \leq |\lambda z| + C_0 \left|z^2\right| = (|\lambda| + C_0 |z|) |z| .$$

Now choose C and r such that $|\lambda| < C < 1$ and $0 < r \leq r_0$, so that $|\lambda| + C_0 r < C$. Then, if $|z| < r$,

$$|f(z)| \leq (|\lambda| + C |z|) |z| \leq (|\lambda| + Cr) |z| \leq C |z| .$$

Therefore, $\left|f^2(z)\right| = |f(f(z))| \leq C |f(z)| \leq C^2 |z|$, and by induction, for all $n \geq 1$ it holds that $|f^n(z)| \leq C^n |z|$, which tends to zero uniformly as $n \to \infty$. This alredy implies that $z_0 = 0$ is topologically attracting.

Now suppose that 0 is topologically attracting. Then, for any disk $D_\varepsilon = D(0, \varepsilon)$ small enough, there exists $n \in \mathbb{N}$ so that $f^n(D_\varepsilon) \subseteq D_\varepsilon$. Now let $\phi : D_\varepsilon \longrightarrow \mathbb{D}$ be the Riemann map with $\phi(0) = 0$ and $\phi'(0) > 0$. Define $h = \phi \circ f^n \circ \phi^{-1}$, i.e.

$$\begin{tikzcd}
D_\varepsilon \arrow[r, "f^n"] \arrow[d, "\phi"'] & D_\varepsilon \arrow[d, "\phi"] \\
\mathbb{D} \arrow[r, "h"'] & \mathbb{D}
\end{tikzcd}$$

Since 0 is a fixed point of f and $\phi(0) = 0$, then $h(0) = 0$. By the Schwarz Lemma 1.8, it follows that $|h'(0)| \leq 1$.

If $|h'(0)| = 1$, h would be a rotation, and given that ϕ is a conformal map, this would imply that f^n is a rotation, which would contradict the hypothesis that f is topologically attracting. Therefore, $|h'(0)| < 1$. Observe that $|h'(0)| < 1$ implies that $|\lambda| < 1$. Indeed,

$$h'(0) = \phi'(f^n(\phi^{-1}(0))) \cdot (f^n)'(\phi^{-1}(0)) \cdot (\phi^{-1})'(0) = (f^n)'(0),$$

and by the chain rule, the derivative of f^n is

$$(f^n)'(z) = f'(f^{n-1}(z)) \cdots f'(f(z)) f'(z).$$

Since 0 is a fixed point of f,

$$(f^n)'(0) = f'(0) \overset{n)}{\cdots} f'(0) = \lambda^n.$$

$\square$

Definition 3.5. Let z_0 be an attracting fixed point of a holomorphic function f. The **basin of attraction** of z_0 is $\mathcal{A} = \mathcal{A}(z_0) = \{z \in S;\ \lim\limits_{n\to\infty} f^n(z) = z_0\}$. The **immediate basin** $\mathcal{A}_0$ is the connected component of $\mathcal{A}$ which contains z_0.

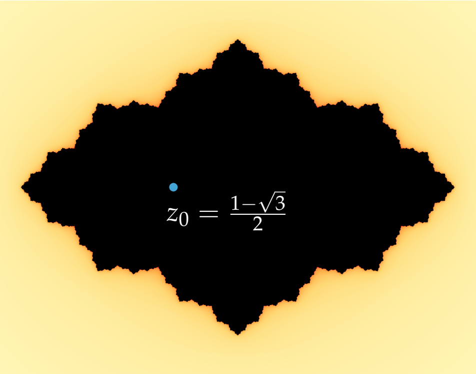


Figure 3.2: In black, the basin of attraction of $z_0 = \frac{1-\sqrt{3}}{2}$, the attracting fixed point of $z^2 - \frac{1}{2}$, with multiplier $\lambda = 1 - \sqrt{3}$. The fixed point is painted in blue.

Now we prove that if f has a geometrically attracting periodic point z_0 of period p, then f^p behaves like $z \mapsto \lambda z$ near z_0. Since z_0 is a fixed point of f^p, it suffices to reduce to the case of fixed points.

Theorem 3.6. (Koenigs Linearization Theorem) *Let z_0 be a fixed point of f with multiplier λ. If $0 < |\lambda| < 1$, then there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(z_0) = 0$, such that $\phi \circ f \circ \phi^{-1}$ is the linear map $m_\lambda(w) = \lambda w$, for w in a neighbourhood of the origin.*
Furthermore, ϕ is unique up to multiplication by a nonzero constant.

Proof. By Remark 3.2, it suffices to consider the case $z_0 = 0$. We first address the existence of ϕ. Let $C \in \mathbb{R}$ so that $C^2 < |\lambda| < C$. As shown in the proof of Lemma 3.4, there exist $C_0 > 0$ and a neighbourhood of zero, $D_r = D(0,r)$, such that $|f(z)| \leq C|z|$ and $|f(z) - \lambda z| \leq C_0 \left|z^2\right|$ for all $z \in D_r$. Thus, for all $z_0 \in D_r$, its orbit $\mathcal{O}(z_0) = \{z_n = z_n(z_0)\}_n$ converges geometrically to zero with $|z_n| \leq rC^n$. Therefore, $|z_{n+1} - \lambda z_n| \leq C_0 r^2 C^{2n}$. Since $\frac{C^2}{|\lambda|} < 1$, setting $k := \frac{C_0 r^2}{|\lambda|}$ and $w_n := \frac{z_n}{\lambda^n}$, it follows that

$$|w_{n+1} - w_n| = |z_{n+1} - \lambda z_n| \frac{1}{|\lambda^{n+1}|} \leq C_0 r^2 C^{2n} \frac{1}{|\lambda^{n+1}|} = k \left(\frac{C^2}{|\lambda|}\right)^n \longrightarrow 0$$

uniformly when $n \to \infty$. Therefore, the functions $z_0 \mapsto w_n(z_0)$ converge uniformly to $\phi(z_0) := \lim\limits_{n\to\infty} \dfrac{z_n(z_0)}{\lambda^n}$ in D_r. Observe that

$$\phi(f(z)) = \lim_{n\to\infty} \frac{f(z_n)}{\lambda^n} = \lim_{n\to\infty} \frac{z_{n+1}}{\lambda^n} = \lim_{n\to\infty} \lambda \frac{z_{n+1}}{\lambda^{n+1}} = \lambda\phi(z).$$

Hence, $(\phi \circ f \circ \phi^{-1})(z) = \lambda z$ and $\phi(0) = \lim\limits_{n\to\infty} \dfrac{z_n(0)}{\lambda^n} = 0$.
To see that ϕ is a local conformal isomorphism notice that the map $w_n : D_r \longrightarrow D_r$, $w_n(z_0) = \frac{z_n(z_0)}{\lambda^n}$ has derivative

$$w_n'(z) = \frac{1}{\lambda^n}(z_n'(z_0)) = \frac{1}{\lambda^n} f'(z_{n-1}) \cdots f'(z_1) f'(z_0),$$

and therefore,

$$w_n'(0) = \frac{1}{\lambda^n} f'(0) \overset{n)}{\cdots} f'(0) = \frac{1}{\lambda^n}\lambda^n = 1.$$

By Schwarz Lemma 1.8, it follows that ϕ is conformal.

It remains to show that ϕ is unique up to multiplication by a nonzero constant.
Let U be a neighbourhood of z_0 and ϕ, ψ be two holomorphic maps such that $(\phi \circ f \circ \phi^{-1})(w) = m_\lambda(w)$ and $(\psi \circ f \circ \psi^{-1})(w) = m_\lambda(w)$. Then, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{C} & f(U) \arrow[l, "\psi"'] \arrow[r, "\phi"] & \mathbb{C} \\
\mathbb{C} \arrow[u, "m_\lambda"] & U \arrow[l, "\psi"] \arrow[r, "\phi"'] \arrow[u, "f"'] & \mathbb{C} \arrow[u, "m_\lambda"']
\end{tikzcd}$$

Hence, $m_\lambda \circ (\psi \circ \phi^{-1}) = (\psi \circ \phi^{-1}) \circ m_\lambda$ and $\psi \circ \phi^{-1}$ commutes with m_λ. Since ϕ and ψ are holomorphic, it follows that $\psi \circ \phi^{-1}$ is holomorphic and can be expanded as a power series near 0,

$$(\psi \circ \phi^{-1})(w) = a_0 + a_1 w + a_2 w^2 + \cdots = \sum_{n\geq 0} a_n w^n.$$

Notice that $(\psi \circ \phi^{-1})(0) = \psi(0) = 0$ and therefore $a_0 = 0$. Now, using that $m_\lambda \circ (\psi \circ \phi^{-1}) = (\psi \circ \phi^{-1}) \circ m_\lambda$, we obtain that

$$\lambda \sum_{n\geq 1} a_n w^n = \sum_{n\geq 1} a_n (\lambda w)^n.$$

Comparing coefficients, we see that $\lambda a_n = a_n \lambda^n$ for all $n \geq 1$. Since $|\lambda| \neq 0, 1$, this implies that $a_n = 0$ for all $n \geq 2$, and $(\psi \circ \phi^{-1})(w) = a_1 w$, with $a_1 \neq 0$. We conclude that $\psi(w) = a_1 \phi(w)$, as desired. This ends the proof of the theorem. $\square$

The function ϕ is called the **Koenigs function** of z_0, and it can be extended to the whole basin of attraction $\mathcal{A}$ of z_0 as follows.

Corollary 3.7. (Global Linearization) *Let z_0 be an attracting fixed point of f, with basin of attraction $\mathcal{A}$ and muliplier λ such that $0 < |\lambda| < 1$. Then, there exists a holomorphic map ϕ such that $\phi(z_0) = 0$ and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "\phi"'] & \mathcal{A} \arrow[d, "\phi"] \\
\mathbb{C} \arrow[r, "m_\lambda"'] & \mathbb{C}
\end{tikzcd}$$

Furthermore, ϕ is unique up to multiplication by a nonzero constant.

Proof. By Koenigs Linearization Theorem 3.6, there exists U a neighbourhood if z_0 and $\widetilde{\phi}$ the Koenigs function of z_0, such that the following diagram commutes.

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\widetilde{\phi}"'] & f(U) \arrow[d, "\widetilde{\phi}"] \\
\mathbb{C} \arrow[r, "m_\lambda"'] & \mathbb{C}
\end{tikzcd}$$

The idea of the proof is to take $z \in \mathcal{A}$, iterate it until its orbit meets U, then apply $\widetilde{\phi}$ and go backwards. Let $z \in \mathcal{A}$. Since z_0 is attracting, there exists $k \geq 0$ such that $f^k(z) \in U$ and $f^j(z) \notin U$ for all $j < k$. Then, $\lambda\widetilde{\phi}(f^k(z)) = \widetilde{\phi}(f^{k+1}(z))$ and we can define

$$\phi(z) = \lambda^{-k}\widetilde{\phi}(f^k).$$

The map ϕ is well-defined and $\phi(z_0) = \widetilde{\phi}(z_0) = 0$. It remains to see that $\phi \circ f = m_\lambda \circ \phi$ for all $z \in \mathcal{A}$. Take $z \in \mathcal{A}$ and let $k \geq 0$ be the minimum integer such that $f^k(z) \in U$ and observe that $\widetilde{\phi}(f(f^k(z))) = \lambda\widetilde{\phi}(f^k(z))$. Then,

$$\phi(f(z)) = \lambda^{-k}\widetilde{\phi}(f^k(f(z))) = \lambda^{-k}\widetilde{\phi}(f(f^k(z))) = \lambda^{-k+1}\widetilde{\phi}(f^k(z)),$$

and also,

$$m_\lambda(\phi(z)) = \lambda \cdot \lambda^{-k}\widetilde{\phi}(f^k(z)) = \lambda^{-k+1}\widetilde{\phi}(f^k(z)).$$

Therefore, $\phi \circ f = m_\lambda \circ \phi$, and ϕ is an extension of $\widetilde{\phi}$ to the whole basin of attraction of z_0. The proof of the uniqueness of ϕ is the same as in Theorem 3.6. □

Lemma 3.8. *Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ and $z_0 \in \widehat{\mathbb{C}}$ a geometrically attracting fixed point of f with basin of attraction $\mathcal{A}$. Let ϕ be the Koenigs function of z_0, and let ψ be its local inverse around 0, i.e. $\psi(0) = z_0$ and $\phi \circ \psi \equiv id$. Then, there exists a maximal $r > 0$ such that ψ is conformal in $D(0, r)$. Furthermore, ψ extends homeomorphically over the boundary of D_r, and the image $\psi(\partial D_r) \subset \mathcal{A}_0$ necessarily contains a critical point of f.*

Proof. Since the Koenigs function ϕ is conformal in a neighbourhood of z_0 (Theorem 3.6), then there exists $\varepsilon > 0$ such that

$$\psi_\varepsilon : D_\varepsilon \longrightarrow \mathcal{A}_0$$

is a well defined holomorphic inverse branch of ϕ, with $\psi_\varepsilon(0) = z_0$ and $\psi_\varepsilon \circ \phi = id$. Suppose that ψ_ε can be extended to $\mathbb{C}$. Then, $\psi : \mathbb{C} \longrightarrow \mathcal{A}_0$ is the inverse of ϕ, and is entire and conformal. Thus, $\phi : \mathcal{A}_0 \longrightarrow \mathbb{C}$ can be extended to an entire conformal map $\phi : \mathbb{C} \longrightarrow \mathbb{C}$. Indeed, by the Riemann Mapping Theorem 1.6, if a simply connected domain $\mathcal{A}_0$ is biholomorphic to $\mathbb{C}$, then $\mathcal{A}_0 = \mathbb{C}$. By Theorem 1.7, it follows that $\phi(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$. Since $\phi(f(z)) = \lambda\phi(z)$, then, $af(z) + b = \lambda(az + b)$ for all $z \in \mathbb{C}$, and therefore

$$f(z) = \lambda z + \frac{\lambda b - b}{a}$$

is a rational map of degree 1, which contradicts the hypothesis. Hence, there exists a maximal $r > 0$ such that ψ is conformal in $D(0, r)$.
The proof of the remaining part of the lemma can be found in [18, Lemma 8.5]. □

3.1.1 Superattracting Fixed Points

In the case of superattracting fixed points, we obtain results analogous to those for geometrically attracting fixed points. Assume that 0 is a superattracting fixed point of a holomorphic function f. Then, locally, we can write

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \text{(higher order terms)}$$

with $n \geq 2$ and $a_n \neq 0$. We have seen in Theorem 3.6 that around geometrically attracting fixed points, f is conjugated to the map $m_\lambda(w) = \lambda w$. In contrast, around superattracting fixed points, f is conjugated to the map $h(w) = w^n$; this is known as Böttcher's Theorem.

Theorem 3.9. (Böttcher's Theorem, [18, Theorem 9.1]**)** *Let z_0 be a supperattracting fixed point of f. Then, locally around z_0, we can write*

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + (\textit{higher order terms}),$$

with $a_n \neq 0$, and there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(z_0) = 0$, such that $\phi \circ f \circ \phi^{-1}$ is the power map $h(w) = w^n$ for all w in some neighbourhood of the origin.
Furthermore, ϕ is unique up to multiplication by an $(n-1)$-th root of unity.

Contrary to the case of Koenigs function, ϕ cannot always be extended to the whole basin of attraction $\mathcal{A}$. Indeed, by Böttcher's Theorem 3.9, there exists a neighbourhood U of the fixed point z_0, and ϕ the Bötcher's function, such that $\phi \circ f = h \circ \phi$ in U, where $h(w) = w^n$. Let $z \in \mathcal{A}$. Then, there exists $k \geq 0$ such that $f^k(z) \in U$. Therefore, $(\phi(f^k(z)))^n = \phi(f^{k+1}(z))$, and in order to properly define ϕ in $\mathcal{A}$, it would involve computing some n-th root, which is not always possible. However, the function $|\phi|$ does extend to $\mathcal{A}$ and satisfies $|\phi(f(z_0))| = |\phi(z_0)|^n$.

We now show that the existence of an attracting periodic orbit of any period implies the existence of a critical point in its basin.

Theorem 3.10. *Let f be a rational map of degree $d \geq 2$. Then, the immediate basin of every attracting periodic orbit contains at least one critical point.*

Proof. Let p be the period of the attracting periodic orbit $\{z_1, ..., z_p\}$, with $f(z_j) = z_{j+1}$, taking the indices to be integers modulo p.

- If $p = 1$, then z_1 is a fixed point. If it is geometrically attracting, then the result follows from Lemma 3.8, while if it is superattracting, $f'(z_1) = 0$ and therefore z_1 is itself a critical point in its basin.

- Now we consider the case $p > 1$. It is clear that $f(\mathcal{A}_0(z_j)) \subset \mathcal{A}_0(z_{j+1})$. Suppose that $\mathcal{A}_0(z_j)$ does not contain any critical point of f for all j. Then, by the chain rule, $f^p(\mathcal{A}_0(z_j))$ would not contain any critical point of f, which is impossible because z_j is a fixed point of f^p. $\square$

From this theorem follows that the number of attracting periodic orbits is finite, less than or equal to the number of critical points of f.

Remark 3.11. If f is transcendental, then the immediate basin of every attracting periodic orbit contains at least one singular value (either critical or asymptotic). See, for example, [19, Theorem 8.2.1].

3.2 Repelling Fixed Points

As in the case of attracting fixed points, we begin by giving a topological definition of repelling fixed points. Intuitively, a fixed point z_0 is repelling if all orbits starting sufficiently close to z_0 eventually escape from any neighbourhood of z_0.

Definition 3.12. A fixed point z_0 of a map f is **topologically repelling** if there exists a neighbourhood U of z_0 such that for all $z \in U$, $z \neq z_0$, there exists $n \geq 1$ so that $f^n(z) \notin U$. We say that U is a **forward isolating neighbourhood** of z_0.

Now we give a characterization of repelling fixed points in terms of their multiplier. This will be useful for deriving some results analogous to those in the attracting case.

Lemma 3.13. *A fixed point of a holomorphic map is topologically repelling if, and only if, its multiplier satisfies* $|\lambda| > 1$.

Proof. Let z_0 be the fixed point and suppose that $|\lambda| > 1$. Then,

$$f(z) = z_0 + \lambda\,|z - z_0| + O(|z - z_0|^2)$$

in a neighbourhood of z_0. Therefore, for any $\mu \in \mathbb{R}$ so that $1 < \mu \leq |\lambda|$, there exists $\delta > 0$ such that $|f(z) - z_0| \geq \mu\,|z - z_0|$ if $|z - z_0| \leq \delta$. By induction, $\left|f^k(z) - z_0\right| \geq \mu^k\,|z - z_0|$ if $|f^m(z) - z_0| \leq \delta$ for all $m \in \{1, 2, \cdots, k-1\}$. However, since $\mu^k \to \infty$ when $k \to \infty$, for $z \neq z_0$ this inequality cannot be hold for all k. Hence, there exists $n \geq 1$ so that $|f^n(z) - z_0| > \delta$, which means that z_0 is topologically repelling. In this case $D(z_0, \delta)$ is a forward isolating neighbourhood of z_0.

Now suppose that z_0 is topologically repelling. Since z_0 cannot be simultaneously attracting and repelling, $|\lambda| \geq 1$. Now we choose a compact forward isolating neighbourhood N of z_0 small enough so that $f|_N : N \longrightarrow f(N)$ is a homeomorphism and $f(N)$ is compact. Set

$$N_k = N \cap f^{-1}(N) \cap \cdots \cap f^{-k}(N).$$

The set N_k is a compact neighbourhood of z_0 consisting of points whose first k images lie in N. Notice that $N_0 = N$ and $N_{k+1} \subseteq N_k$ for all $k \geq 0$. Since N is a forward isolating neighbourhood, $\bigcap_{k\geq 0} N_k = \{z_0\}$ and by compacity, $\lim_{k\to\infty} \operatorname{diam}(N_k) = 0$. Thus, for k large, $N_{k-1} \subseteq f(N)$. Observe that, by construction,

$$f(N_k) = N_{k-1} \cap f(N),$$

and therefore, for k large, $f(N_k) = N_{k-1}$. Let U_k be the connected component of the interior of N_k which contains z_0. Then $f^{-1} : U_{k-1} \longrightarrow U_k$ is conformal and $U_k \subsetneq U_{k-1}$. By the Schwarz Lemma, it follows that $\left|\lambda^{-1}\right| < 1$, which proves that $|\lambda| > 1$. □

Observe that the Koenigs Linearization Theorem 3.6 can be also applied to repelling fixed points. Indeed, if $|\lambda| > 1$, then by the Inverse Function Theorem, there exists F a well defined branch of f^{-1} around z_0. Notice that

$$F'(z_0) = \frac{1}{f'(z_0)} = \lambda^{-1},$$

with $0 < \left|\lambda^{-1}\right| < 1$. The result follows from applying the same argument used in the proof of the theorem to F. Consequently, the following diagram commutes.

$$\begin{tikzcd}
f(U) \arrow[r, "F"] \arrow[d, "\phi"'] & U \arrow[d, "\phi"] \\
\mathbb{C} \arrow[r, "m_{\frac{1}{\lambda}}"'] & \mathbb{C}
\end{tikzcd}$$

Therefore,

$$(\phi \circ f \circ \phi^{-1})(z) = (\phi \circ F \circ \phi^{-1})^{-1}(z) = m_{\frac{1}{\lambda}}^{-1}(z) = m_{\lambda}(z) = \lambda z$$

and f is locally conjugated to the map m_λ in a neighbourhood of z_0. In this case, if $f : S \longrightarrow S$, we can extend the inverse of the Koenigs function, ϕ^{-1}, to a map $\mathbb{C} \longrightarrow S$.

3.3 Parabolic Fixed Points

Let f be a holomorphic map and z_0 a fixed point of f with multiplier λ, such that $|\lambda| = 1$. If λ is a root of unity, i.e. there exists $q \in \mathbb{Z}$ so that $\lambda^q = 1$, then we say that z_0 is a **parabolic** fixed point of f.

We first consider the case $\lambda = 1$ and we suppose $z_0 = 0$. Then, we can write

$$f(z) = z + az^{n+1} + \text{(higher order terms)},$$

where $a \neq 0$, $n \geq 1$ and $n+1$ is called the **multiplicity** of the parabolic fixed point. Observe that $n + 1 \geq 2$.

Definition 3.14. We say that v is a **repelling vector** for f at the origin if $nav^n = 1$ and is an **attracting vector** for f if $nav^n = -1$.

Notice that f has n attracting vectors and n repelling vectors, which are equally spaced and alternate with each other. We denote those vectors by $v_0, \cdots, v_{2n-1}$, where even indices correspond to repelling vectors and odd indices for attracting vectors. In particular, the angle between a vector of a type and one of its neighbours of the opposite type is $\frac{\pi}{n}$.

Definition 3.15. Let z_0 be a fixed point of f of multiplicity $n + 1 \geq 2$, and let v_j be an attracting vector at z_0. Consider some neighbourhood N of z_0 where f is injective. Then, an open set $\mathcal{P} \subset N$ is an **attracting petal** for f for v_j at z_0 if:

1. $f(\mathcal{P}) \subset \mathcal{P}$, and

2. an orbit $\{z_k\}_k$ under f converges to z_0 from the direction v_j if, and only if, there exists $m \in \mathbb{N}$ so that $f(z_k) \in \mathcal{P}$ for $k \geq m$.

An open subset $\mathcal{P}' \subset f(N)$ is a **repelling petal** for f for a repelling vector v_k if $\mathcal{P}'$ is an attracting petal for f^{-1}.

Theorem 3.16. (Leau-Fatou Flower Theorem) *Let z_0 be a parabolic fixed point of multiplicity $n+1 \geq 2$. Then, within any neighbourhood of z_0, there exist n attracting and n repelling simple connected petals $\mathcal{P}_j$, $0 \leq j < 2n$, which can be chosen so that $\{z_0\} \cup \mathcal{P}_0 \cup \cdots \cup \mathcal{P}_{2n-1}$ is an open neighbourhood of z_0.*
Furthermore, each $\mathcal{P}_j$ only intersects $\mathcal{P}_{j-1}$ and $\mathcal{P}_{j+1}$.

Proof. It suffices to prove the case $z_0 = 0$. Consider $w = \varphi(z) = \frac{-1}{naz^n}$ and for each attracting or repelling vector v_j define $\Delta_j = \{re^{i\theta}v_j;\, r > 0, |\theta| < \frac{\pi}{n}\}$. Observe that φ maps Δ_j conformally onto $\mathbb{C} \setminus \mathbb{R}_+$ (if j even) or $\mathbb{C} \setminus \mathbb{R}_-$ (if j odd) .
Then, there exists a uniquely defined branch ψ_j of φ^{-1} with $\psi_j(\mathbb{C} \setminus \mathbb{R}_{(-1)^j}) \cong \Delta_j$. Consider now $F_j(w) = (\varphi \circ f \circ \psi_j)(w)$. Since

$$F_j(w) = w + 1 + o(1),$$

(see [18, proof of Lemma 10.1]), there exists $R > 0$ so that $\left|F_j(w) - w - 1\right| < \frac{1}{2}$ for $|w| > R$. Denote

$$W_R = \{w = u + iv \in \mathbb{C};\, u + |v| > 2R\} \text{ and } -W_R = \{w \in \mathbb{C};\, -w \in W_R\}.$$

Then, $F_j(W_R) \subset W_R$ for j odd, and $F_j^{-1}(-W_R) \subset -W_R$ for j even. Indeed, if $w = u + iv \in W_R$, then $F_j(w) = (u+1) + iv$ and $(u+1) + |v| > u + |v| > 2R$, so $F_j(w) \in W_R$. Analogously, if $w = u + iv \in -W_R$, then $-F_j^{-1}(w) = -(u-1) - iv$ and $-(u-1) + |v| > -u + |v| > 2R$ because $-w \in W_R$, so $F_j^{-1}(w) \in -W_R$. It can be shown that $\mathcal{P}_j = \psi_j(W_R)$ for j odd is an attracting petal and $\mathcal{P}_j = \psi_j(-W_R)$ for j even is a repelling petal (see [18, Theorem 10.7]). ☐

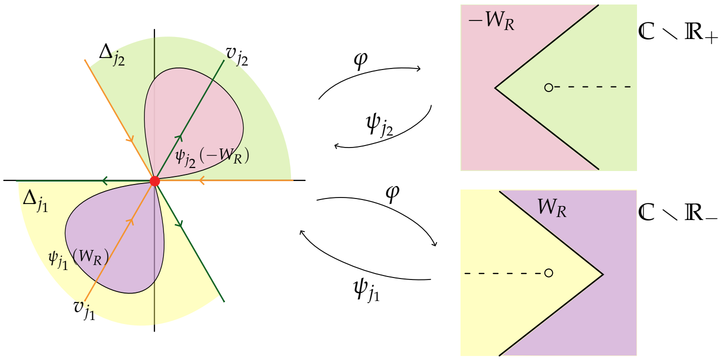


Figure 3.3: The mappings φ for the sets Δ_j and the construction of the corresponding petals. The indices j_1 and j_2 are odd and even respectively.

Observe that in the proof we have chosen R large enough so that $|F(w) - w - 1| < \frac{1}{2}$ for $|w| > R$. Thus, $\mathrm{Re}(\varphi(f(z))) > \mathrm{Re}(\varphi(z)) + \frac{1}{2}$ for $|z|$ sufficiently small. We conclude that there are no periodic cycles contained in a neighbourhood of the fixed point, other than the fixed point itself.

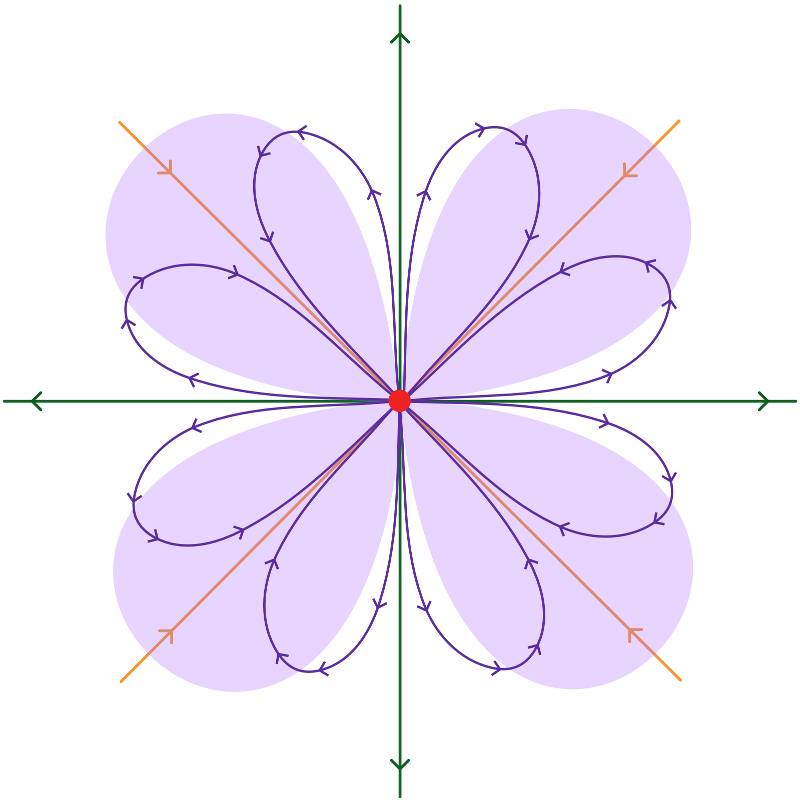
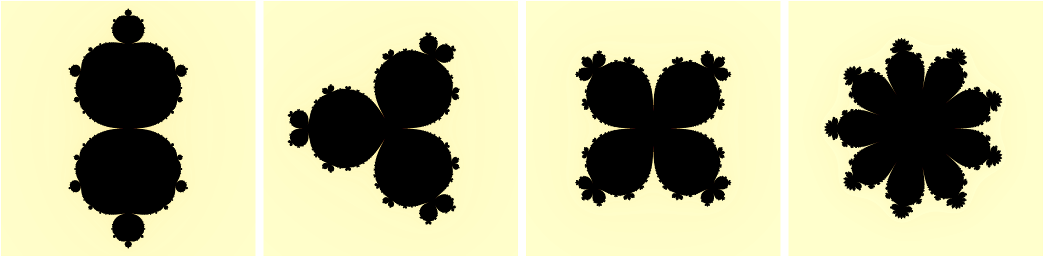


Figure 3.4: Dynamics near a parabolic fixed point of multiplicity $n+1 = 5$.

Let z_0 be a parabolic fixed point of a holomorphic function f. Then, we can define the **parabolic basin of attraction** of z_0 as the open set consisting of orbits which converge to z_0. The fixed point z_0 lies in the boundary of the basin.

For instance, consider the function $f(z) = z + z^n$. Then, for all $n \geq 0$, the point $z = 0$ is a parabolic fixed point with multiplier $\lambda = 1$. Figure 3.5 shows some parabolic basins of $z = 0$ for f, for different values of n.



(a) n = 3 (b) n = 4 (c) n = 5 (d) n = 10

Figure 3.5: In black, the parabolic basin of attraction of the parabolic fixed point $z = 0$ for the map $f(z) = z + z^n$. Results are shown for different values of n.

3.4 Siegel and Cremer Points

Let z_0 be a fixed point of f with multiplier λ so that $|\lambda| = 1$ and it is not a root of unity. Then there exists $\xi \in \mathbb{R} \setminus \mathbb{Q}$ so that $\lambda = e^{2\pi i \xi}$. We say that z_0 is an **irrationally neutral** fixed point. We will classify those fixed points according to whether f is locally linearizable around z_0 or not.

Definition 3.17. Let z_0 be a fixed point of a holomorphic function f with multiplier λ. We say that f is **locally linearizable** around z_0 if there exists a local change of coordinate $z = h(w)$ so that $f(h(w)) = h(\lambda w)$ in a neighbourhood of z_0. An irrationally neutral fixed point is a **Siegel point** if f is locally linearizable around it, and its maximal domain of linearization is called a **Siegel disk**. Otherwise, the fixed point is a **Cremer point**.

Theorem 3.18. *A holomorphic function f is locally linearizable around z_0 if and only if the sequence of iterates $\{f^n\}_n$ is uniformly bounded in some neighbourhood of z_0.*

Proof. We may assume that $z_0 = 0$. First, suppose that f is locally linearizable and consider the map $m_\lambda(z) = \lambda z$. Then, there exist U a neighbourhood of z_0 and h a local change of coordinate such that $f = h \circ m_\lambda \circ h^{-1}$ in U. Therefore, $f^n(z) = h(\lambda^n h^{-1}(z))$ is bounded, since it corresponds to iterated rotations of a small disk.

Now suppose that $\{f^n\}_n$ is uniformly bounded in some neighbourhood U of z_0. Then there exists $M > 0$ so that $|f^n(z)| \leq M$ for all $z \in U$ and $n \geq 1$. For each $n \geq 1$ define

$$\varphi_n(z) = \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} f^j(z).$$

The sequence $\{\varphi_n\}_n$ is uniformly bounded, hence it contains a convergent subsequence that converges uniformly to a function φ. Since

$$\varphi_n(f(z)) = \frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}f^{j+1}(z) = \frac{n+1}{n}\lambda\varphi_{n+1}(z) - \frac{\lambda z}{n},$$

letting $n \to \infty$, we conclude $\varphi \circ f = \lambda\varphi$ and f is locally linearizable. $\square$

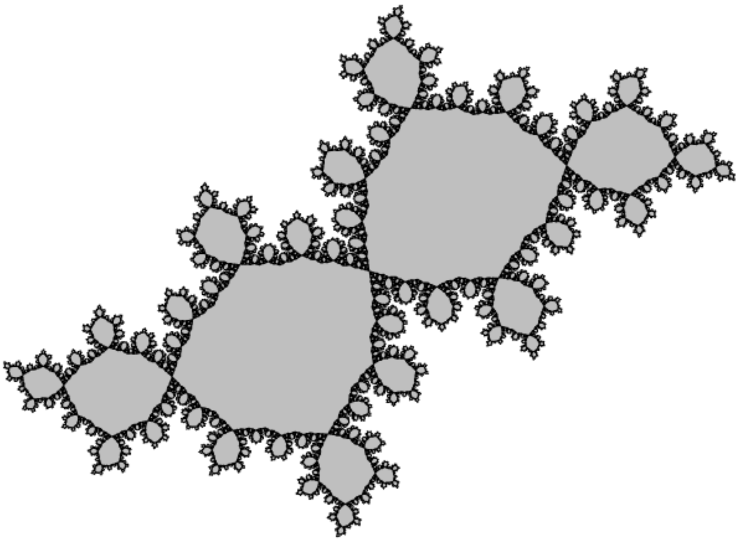


Figure 3.6: The Siegel disk of $z = 0$, the Siegel fixed point of $f(z) = e^{2\pi i \frac{\sqrt{5}-1}{2}}z + z^2$. [8]

Chapter 4

Repelling Fixed Points and Transcendental Functions

In this chapter we study the existence of periodic points depending on some analytic features of f. We give some general results about the existence of periodic points of holomorphic functions, and then we focus on the Main Theorem given by Benini on [3] about the existence of periodic points on entire transcendental functions with a bounded set of singular values.

It is clear that if f is a rational map, then for any $n \geq 1$ it has a periodic point of (not necessarily minimal) period n. Indeed, for any $n \geq 1$, f^n is also a rational map, so we can write

$$f^n(z) = \frac{P_n(z)}{Q_n(z)},$$

with P_n, Q_n polynomials. Since a periodic point of period n of f is a point z_0 so that $f^n(z_0) = z_0$, finding such a periodic point is equivalent to solving the equation

$$P_n(z) = z\, Q_n(z).$$

The existence of a solution z_0 follows from the Fundamental Theorem of Algebra.

However, this is not true for transcendental functions. For example, consider $f(z) = e^z + z$. Since $e^z \neq 0$ for all $z \in \mathbb{C}$, f has no fixed points. Nevertheless, any entire transcendental function must have at least one periodic point of period 2, as shows the following proposition, proven by Fatou on [13].

Proposition 4.1. *Let f be an entire transcendental function. Then f has a periodic orbit of period 2.*

Proof. Let f be an entire transcendental function with no periodic points of period 2, and consider the auxiliary function

$$h(z) = \frac{f^2(z) - z}{f(z) - z}.$$

Notice that $h(z)$ omits the values 0 and 1. Thus, by Picard's Little Theorem 1.12, h is a constant function, $h(z) \equiv K$, $K \neq 0, 1$. Therefore,

$$f^2(z) - z = K\,(f(z) - z), \qquad \text{for all } z \in \mathbb{C}.$$

Deriving this expression we obtain

$$f'(f(z))f'(z) - 1 = Kf'(z) - K.$$

Thus,

$$f'(z)(f'(f(z)) - K) = 1 - K,$$

and since $K \neq 0, 1$, we conclude that $f'(z)$ omits the values 0 and K. By applying Picard's Little Theorem again, we obtain that f' is constant, hence there exist $a, b \in \mathbb{C}$ so that $f(z) = az + b$, which contradicts the fact that f is transcendental. $\square$

Moreover, W. Bergweiler proved in [6] that any entire transcendental function has infinitely many repelling periodic points of minimal period n for all $n \geq 2$. The following theorem claims that if we reduce to functions with a bounded set of singular values (also known as class $\mathcal{B}$), then the result is also true for fixed points.

Main Theorem. (Benini, [3]**)** *Any entire transcendental function with a bounded set of singular values has infinitely many repelling periodic points of minimal period n, for all $n \geq 1$.*

Before presenting the proof of this theorem, we dedicate Sections 4.1 and 4.2 to give some preliminary results needed for the proof concerning the mapping structure of maps of class $\mathcal{B}$ near ∞. Necessary background material on complex analysis for this results can be found on Chapter 1 and Chapter 2. The proof itself is given in Section 4.3 and closely follows the argument given by Benini in [3].

4.1 Mapping Structure of Maps of Class $\mathcal{B}$ near ∞

This section is devoted to give some preliminary results that are needed to prove the Main Theorem, concerning the mapping structure of maps of class $\mathcal{B}$

near infinity. The Emerenko-Lyubich class $\mathcal{B}$ is special because for any $f \in \mathcal{B}$, there exists a punctured neighbourhood of infinity without singular values, and hence, we can analyse precisely the behaviour of f near infinity, as showed in this section. Indeed, given f an entire transcendental function of class $\mathcal{B}$, and D an open disk (big enough) containing all singular values of f, we show that if f is a covering, then all connected components of $f^{-1}(\mathbb{C} \setminus D)$ are **tracts**, that is, simply connected unbounded sets.

Let f be an entire transcendental function with a bounded set of singular values, and let D be an open disk containing $S(f)$. Set $\Omega := \mathbb{C} \setminus \overline{D}$. Since $S(f) \subseteq \overline{D}$ and $f : \mathbb{C} \setminus f^{-1}(S(f)) \longrightarrow \mathbb{C} \setminus S(f)$ is a covering map, then f is a covering map from $\mathbb{C} \setminus f^{-1}(\overline{D}) = f^{-1}(\Omega)$ to $\mathbb{C} \setminus \overline{D} = \Omega$. In particular, for any connected component U of $f^{-1}(\Omega)$, $f : U \longrightarrow \Omega$ is a covering map.

Notice that Ω can be seen as a disk with a puncture at ∞. Since Ω is conformally equivalent to $\mathbb{D}^*$, there exists a conformal map $\widetilde{\phi} : \Omega \longrightarrow \mathbb{D}^*$, and the map $\widetilde{g} := \widetilde{\phi} \circ f : U \longrightarrow \mathbb{D}^*$ is a holomorphic covering.

$$\begin{tikzcd}
U \arrow[d, "f"'] \arrow[dr, "\widetilde{g}"] & \\
\Omega \arrow[r, "\widetilde{\phi}"] & \mathbb{D}^*
\end{tikzcd}$$

Thus, we can apply Lemma 1.14 and we obtain that either $U \cong \mathbb{D}^*$ or U is simply connected. Suppose that $U \cong \mathbb{D}^*$ and w is the puncture at U. By continuity, $f(w) = \infty$ and if $w \neq \infty$, then f has a pole at w, which cannot happen because f is entire. Therefore, we obtain that $f(\infty) = \infty$. Since f is transcendental, ∞ is an essential singularity of f, and by Cassorati-Weierstrass Theorem 1.10, $\overline{f(U)} = \mathbb{C}$, which contradicts the fact that $f : U \longrightarrow \Omega$ is a covering.

Therefore, U is simply connected, and by Lemma 1.14, there exists $\widetilde{\varphi} : U \longrightarrow \mathbb{H}^-$ such that $\widetilde{g} = \exp \circ \widetilde{\varphi}$. Since f is continuous and the puncture of Ω is at infinity, U must be unbounded. Hence, all connected components of $f^{-1}(\Omega)$ are tracts that accumulate only at ∞ and do not contain any neighbourhood of infinity.

Observe that we have obtained the following commutative diagram.

$$\begin{tikzcd}
U \arrow[r, "\widetilde{\varphi}"] \arrow[d, "f"'] \arrow[dr, "\widetilde{g}"] & \mathbb{H}^- \arrow[d, "\exp"] \\
\Omega \arrow[r, "\widetilde{\phi}"'] & \mathbb{D}^*
\end{tikzcd}$$

However, for the proof of the Main Theorem we want to use that there exist conformal functions $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \Omega$ and $\varphi : U \longrightarrow \mathbb{H}$ such that $f = \phi \circ \exp \circ \varphi$. We can construct these functions by considering the conformal maps

$$\begin{aligned} \phi_0 \colon \mathbb{D}^* &\longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}, & \qquad \varphi_0 \colon \mathbb{H}^- &\longrightarrow \mathbb{H} \\ z &\longmapsto \frac{1}{z} & z &\longmapsto -z \end{aligned}$$

and defining $\phi := \widetilde{\phi}^{-1} \circ \phi_0^{-1}$ and $\varphi := \varphi_0 \circ \widetilde{\varphi}$. Then, the following diagram commutes.

$$\begin{tikzcd}
U \arrow[r, "\widetilde{\varphi}"] \arrow[rr, bend left, "\varphi"] \arrow[d, "f"'] \arrow[dr, "\widetilde{g}"] & \mathbb{H}^- \arrow[r, "\varphi_0"] \arrow[d, "\exp"] & \mathbb{H} \arrow[d, "\exp"] \\
\Omega \arrow[r, "\widetilde{\phi}"'] & \mathbb{D}^* \arrow[r, "\phi_0"'] & \mathbb{C} \setminus \overline{\mathbb{D}} \arrow[ll, bend left, "\phi"]
\end{tikzcd}$$

In summary, we have shown that if f is transcendental, then there exist $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \Omega$ and $\varphi : U \longrightarrow \mathbb{H}$ such that

$$f = \phi \circ \exp \circ \varphi.$$

Moreover, ϕ can actually be expressed as a composition of a dilation and a translation. Therefore, there exist $R > 0$, $\theta \in [0, 2\pi)$ and $w \in \mathbb{C}$ such that $\phi(z) = Re^{i\theta} z + w$.

Remark 4.2. The map $g := \phi \circ \exp$ is $2\pi i$ periodic. Indeed, for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$

$$g(z + 2\pi i k) = Re^{i\theta} e^{z + 2\pi i k} + w = Re^{i\theta} e^{z} e^{2\pi i k} + w = Re^{i\theta} e^{z} + w = g(z).$$

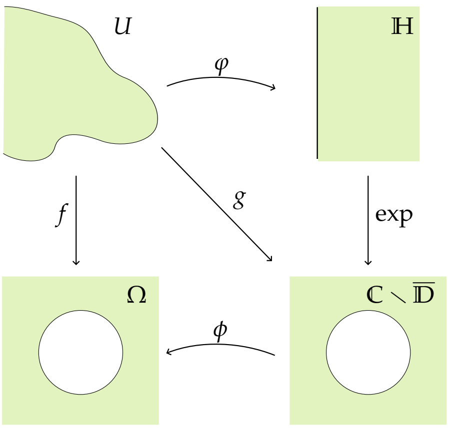


Figure 4.1: Commutative diagram which shows that if f is transcendental, then for every connected component U of $f^{-1}(\Omega)$, there exist conformal maps $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \Omega$ and $\varphi : U \longrightarrow \mathbb{H}$ such that $f = \phi \circ \exp \circ \varphi$.

Notice that tracts have disjoint closures. Indeed, it is clear that tracts cannot intersect because they are different connected components of $f^{-1}(\Omega)$. We have to see that the closures are also disjoint. Since D is an open disk containing $S(f)$, there exists $D' \subsetneq D$ a smaller open disk that also contains all singular values of f. Set $\Omega' = \mathbb{C} \smallsetminus \overline{D'}$, Then, as seen on Figure 4.2, in this case tracts of $f^{-1}(\Omega')$ get larger and contain the closure of the tracts of $f^{-1}(\Omega)$. Since tracts of $f^{-1}(\Omega')$ are disjoint, the closure of tracts of $f^{-1}(\Omega)$ is also disjoint.

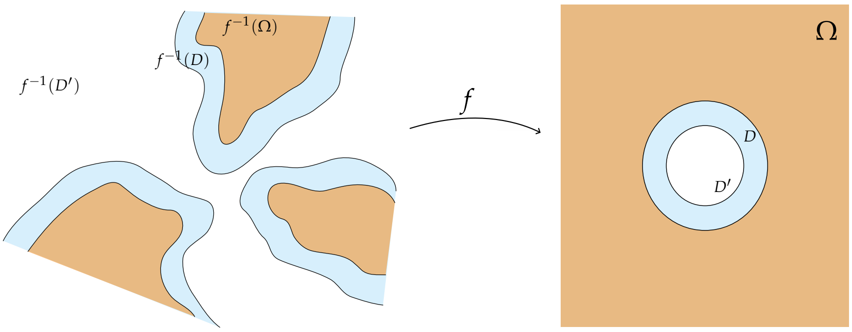


Figure 4.2: In orange the exterior of a disk D and its preimage, in blue the exterior of a smaller disk $D' \subseteq D$ and its preimage. This figure shows that the closure of tracts of $f^{-1}(\Omega)$ is contained in the tracts of $f^{-1}(\Omega')$.

Remark 4.3. There exists an open disk D containing $S(f)$ large enough such that D is not contained in any tract given by the preimage of $\Omega = \mathbb{C} \smallsetminus \overline{D}$. Indeed, suppose that D' is an open disk containing $S(f)$, $\Omega' = \mathbb{C} \smallsetminus \overline{D'}$, and that there exists T' a tract of $f^{-1}(\Omega')$ such that $D' \subseteq T'$. Then, any open disk D such that $D' \subseteq D$ will contain $S(f)$, and as seen in Figure 4.3, as D gets larger, the tracts of $f^{-1}(\Omega)$ shrink. Thus, at some point, D will not be entirely contained in any tract.

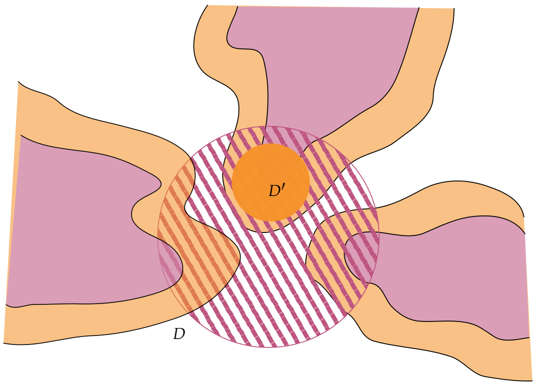


Figure 4.3: In orange, a disc D' entirely contained in a tract of $f^{-1}(\Omega')$. In purple, a bigger disk D such that it is not contained in any tract of $f^{-1}(\Omega)$.

Let $\mathcal{T}$ denote the union of all tracts of $f^{-1}(\Omega)$. Next lemma shows that if we choose D big enough, then there exists a curve joining $\overline{D}$ to infinity without intersecting $\overline{\mathcal{T}}$. Notice that this is trivial if $f^{-1}(\Omega)$ has a finite number of tracts (see Figure 4.4).

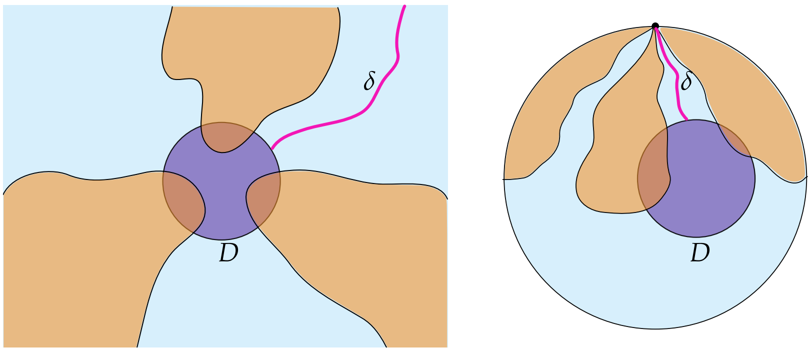


Figure 4.4: Case where $f^{-1}(\Omega)$ has a finite number of tracts. On the left seen in $\mathbb{C}$, and on the right, on the Riemann sphere $\widehat{\mathbb{C}}$.

Lemma 4.4. ([4, Lemma 2.1]) *There exists a simple curve $\delta \subset \Omega \setminus \overline{\mathcal{T}}$ that connects $\overline{D}$ to infinity.*

Since f is transcendental, f is an ∞-to-1 map, and therefore the preimages of δ partition each tract into infinitely many fundamental domains F_i. We say F is a **fundamental domain** of $\Omega \setminus \delta$ for f if $f : F \longrightarrow \Omega \setminus \delta$ is a 1-to-1 map.

Since tracts only accumulate at infinity and each fundamental domain is contained in one tract, the domains $\{F_i\}_i$ also do not accumulate on any compact set. Hence, only a finite number of fundamental domains intersect $\overline{D}$.

Example 4.5. For instance, consider the map $f(z) = e^z$. As seen in Example 1.18, f has only one singular value, $z = 0$. Therefore, we can consider the disk $D = D(0,2)$. The exponential map sends vertical lines to circles centred at 0, and the right-half plane with respect to this line, to the exterior of the disk. Thus, setting $\Omega = \mathbb{C} \setminus \overline{D}$, we obtain that

$$T = \{z \in \mathbb{C}; \quad \mathrm{Re}(z) > \ln(2)\}$$

is the only tract of f. Then, we can consider δ to be a ray of angle $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, which connects $\overline{D}$ with infinity (see Figure 4.5). The preimages of δ are the lines $\{z = (2\pi k + \theta)i\,; \quad k \in \mathbb{Z}\} \cap T$ and each horizontal band

$$F_k := \{z \in T; \quad 2\pi(k-1) + \theta < \mathrm{Im}(z) < 2\pi k + \theta\} \subseteq T$$

is a fundamental domain. Thus, the height of these fundamental domains is 2π.

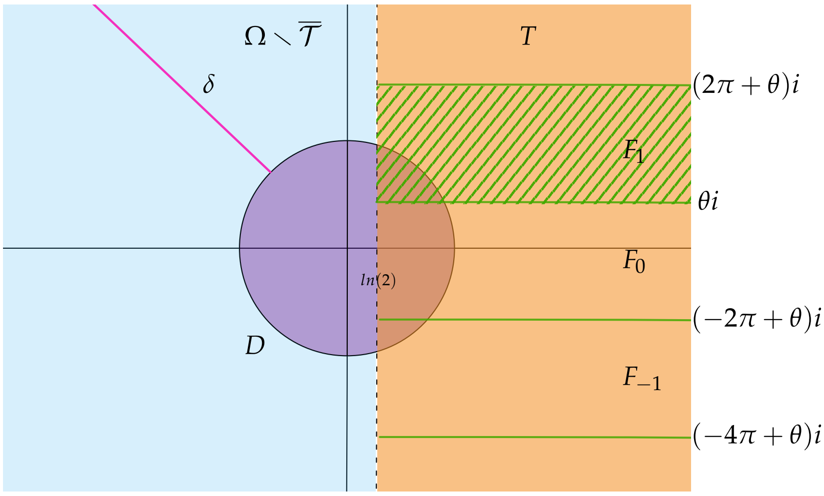


Figure 4.5: Tracts and fundamental domains of e^z.

4.2 Hyperbolic Metric on Tracts

Notice that all tracts, fundamental domains and the domain $\Omega \setminus \delta$ are simply connected domains. Therefore, they admit a hyperbolic metric. Now we want to see that for any fundamental domain F that does not intersect D, $\rho_{\Omega\setminus\delta}$ is bounded by ρ_F in F. First, we show this is true in a neighbourhood of infinity.

Lemma 4.6. *Let F be a fundamental domain of Ω for f. Then, there exist a constant $\kappa_F < 1$ and $U_F \subseteq \widehat{\mathbb{C}}$ a neighbourhood of infinity such that*

$$\frac{\rho_{\Omega\setminus\delta}(z)}{\rho_F(z)} \leq \kappa_F < 1, \quad \textit{for all } z \in F \cap U_F.$$

Proof. Consider T the tract that contains F. Since f is entire transcendental and $\Omega \cong \mathbb{D}^*$, then there exist conformal maps $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \Omega$ and $\varphi : T \longrightarrow \mathbb{H}$ such that $f = \phi \circ \exp \circ \varphi$. Let $\widetilde{F} \subseteq T$ be another fundamental domain of Ω for f different from F. Then, since $\phi \circ \exp$ is $2\pi i$ periodic (as seen in Remark 4.2), $\varphi(\widetilde{F}) = \varphi(F) + 2\pi i k$ for some $k \in \mathbb{Z}$. In particular, all vertical segments of $\varphi(F)$ have size smaller or equal to 2π (see Figure 4.6). Hence, $\operatorname{dist}(z, \partial\varphi(F)) \leq \pi$ for every $z \in F$, where dist denotes the Euclidean distance. Since $\varphi(F)$ is simply connected, it admits a hyperbolic metric, and by Proposition 2.15, it follows that

$$\rho_{\varphi(F)}(z) \geq \frac{1}{4\operatorname{dist}(z, \partial\varphi(F))} \geq \frac{1}{4\pi} \quad \text{for all } z \in \varphi(F).$$

As shown in Theorem 2.10, since φ is conformal, it is an isometry of the hyperbolic metric. Also, the hyperbolic metric of $\mathbb{H}$ is $\rho_{\mathbb{H}}(z) = \frac{1}{2\operatorname{Re}(z)}$ (see Equation 2.7).

Thus, using that $\varphi(T) = \mathbb{H}$ and the fact that $\varphi'(z) \neq 0$ for all $z \in F$ (because φ is conformal), we obtain that for all $z \in F$,

$$\frac{\rho_T(z)}{\rho_F(z)} = \frac{\rho_{\varphi(T)}(\varphi(z))\,|\varphi'(z)|}{\rho_{\varphi(F)}(\varphi(z))\,|\varphi'(z)|} = \frac{\rho_{\mathbb{H}}(\varphi(z))}{\rho_{\varphi(F)}(\varphi(z))} \leq \frac{\frac{1}{2\mathrm{Re}(z)}}{\frac{1}{4\pi}} = \frac{2\pi}{\mathrm{Re}(\varphi(z))}. \tag{4.1}$$

Notice that $|\varphi(z)| \to \infty$ as $|z| \to \infty$ and $\varphi(F)$ has vertical size $2\pi i$. Then, $\mathrm{Re}(\varphi(z)) \to \infty$ as $|z| \to \infty$, and thus, the quotient $\frac{\rho_T(z)}{\rho_F(z)}$ can be made arbitrarily small. Furthermore, there exists a neighbourhood of infinity $\widetilde{U}_F \subseteq \widehat{\mathbb{C}}$ such that for any neighbourhood of infinity $U_F \subseteq \widetilde{U}_F$, $T \subseteq \Omega \smallsetminus \delta$ in U_F. Thus, by the Comparison Principle 2.12 it follows that $\frac{\rho_{\Omega\smallsetminus\delta}(z)}{\rho_T(z)} < 1$ in $\widetilde{U}_F$. Now choose $\kappa_F < 1$ and take $U_F \subseteq \widetilde{U}_F$ a neighbourhood of ∞ small enough such that $\mathrm{Re}(\varphi(z)) \geq \frac{2\pi}{\kappa_F}$ for $z \in U_F \cap F$. Then, using Equation 4.1 and the fact that $\rho_{\Omega\smallsetminus\delta}(z) \leq \rho_T(z)$ in U_F, we conclude

$$\frac{\rho_{\Omega\smallsetminus\delta}(z)}{\rho_F(z)} \leq \frac{\rho_T(z)}{\rho_F(z)} \leq \frac{2\pi}{\mathrm{Re}(\varphi(z))} \leq \kappa_F \quad \text{for all } z \in F \cap U_F.$$

This completes the proof of the lemma. □

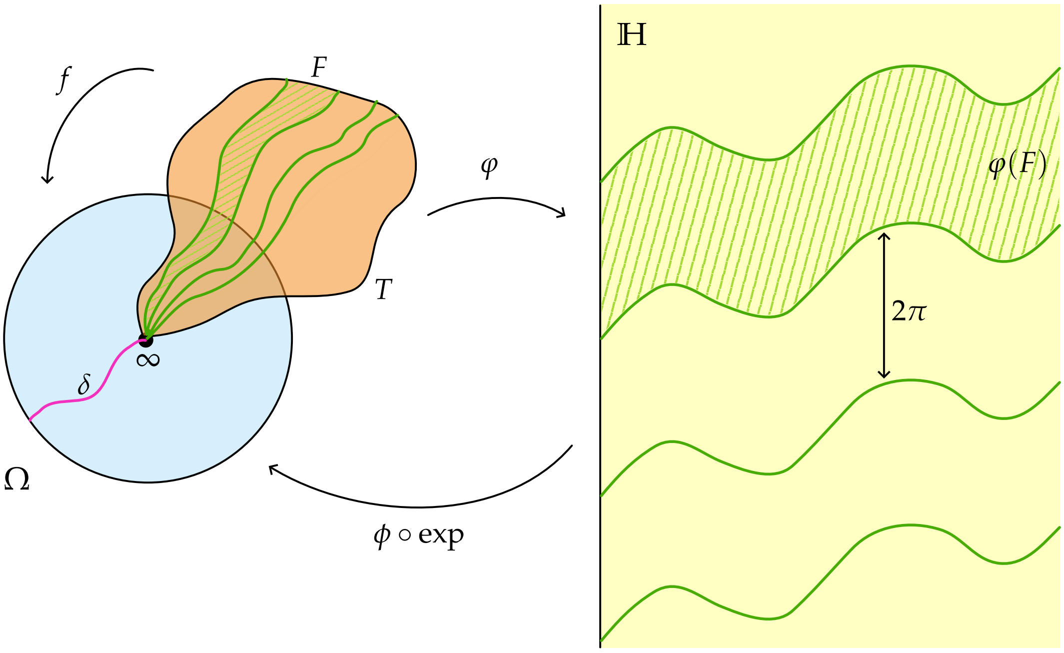


Figure 4.6: The construction of fundamental domains used for the proof of this lemma.

Notice that by restricting U_F, κ_F can be taken to be arbitrarily small. As a corollary of this result, we obtain that this inequality is satisfied in all F.

Corollary 4.7. *Let F be a fundamental domain which does not intersect D. Then, there exists $\kappa_F < 1$ such that*

$$\frac{\rho_{\Omega \setminus \delta}(z)}{\rho_F(z)} \leq \kappa_F < 1 \quad \textit{for all } z \in F.$$

Proof. Notice that F is compactly contained in $\Omega \setminus \delta$ except in a neighbourhood of infinity U_F. By the Comparison Principle 2.12, there exists $\kappa_F^1 < 1$ such that $\frac{\rho_{\Omega \setminus \delta}(z)}{\rho_F(z)} \leq \kappa_F^1 < 1$ for $z \in F \setminus U_F$. By Lemma 4.6, there exists $\kappa_F^2 < 1$ such that $\frac{\rho_{\Omega \setminus \delta}(z)}{\rho_F(z)} \leq \kappa_F^2 < 1$ for $z \in F \cap U_F$. Take $\kappa_F := \max\{\kappa_F^1, \kappa_F^2\} < 1$. Then,

$$\frac{\rho_{\Omega \setminus \delta}(z)}{\rho_F(z)} \leq \kappa_F < 1 \quad \text{for all } z \in F. \qquad \square$$

4.3 Proof of the Main Theorem

Let f be an entire transcendental function with a bounded set of singular values. We want to prove that f has infinitely many repelling periodic points of minimal period n, for any $n \geq 1$.

Let D be an open disk large enough so that it contains $S(f)$, and set $\Omega := \mathbb{C} \setminus \overline{D}$. As discussed in Section 4.1, f behaves essentially as the exponential map and the preimages of Ω by f are tracts. By Lemma 4.4, there exists a simple curve $\delta \subseteq \Omega \setminus \overline{\mathcal{T}}$ connecting $\overline{D}$ to ∞, and the preimages of δ partition each tract into infinitely many fundamental domains.

First we show that f has infinitely many fixed points. Since only a finite number of the fundamental domains of Ω by f intersects D, it is enough to prove that f has a repelling fixed point w_F in every fundamental domain F which does not intersect D. Since F is a fundamental domain, $f : F \longrightarrow \Omega \setminus \delta$ is injective and surjective, and thus, there exists $\psi_F : \Omega \setminus \delta \longrightarrow F$ inverse branch of f, which is injective.

Now our goal is to see that ψ_F is strictly contracting so that we can use that (F, ρ_F) is a complete metric space and apply the Banach Fixed Point Theorem 1.21 to obtain that ψ_F has an attracting fixed point in F.

Since F does not intersect D, by Corollary 4.7, there exists $\kappa_F < 1$ such that $\frac{\rho_{\Omega \setminus \delta}(z)}{\rho_F(z)} \leq \kappa_F$ in F. Also, since ψ_F is conformal, it is a hyperbolic isometry between $\Omega \setminus \delta$ and F, that is, for all $z, w \in F \subseteq \Omega \setminus \delta$,

$$\rho_F(\psi_F(z), \psi_F(w)) = \rho_{\Omega \setminus \delta}(z, w) \leq \kappa_F(z, w).$$

Hence, ψ_F is strictly contractive, and by the Banach Fixed Point Theorem, it follows that there exists $w_F \in F$ an attracting fixed point of ψ_F. Since ψ_F is an inverse branch of f, w_F is actually a repelling fixed point of f.

Now let $n \geq 2$. We want to show that f has infinitely many repelling periodic points of minimal period n. Consider $s = (F_1, \cdots, F_n)$ a sequence of n different fundamental domains which do not intersect D. Clearly, there are infinitely many different choices of these domains. For every $i = 1, \cdots, n$ let $\psi_{F_i} : \Omega \setminus \delta \longrightarrow F_i$ be the inverse branch of f from $\Omega \setminus \delta$ to F_i. Then, define the inverse branch of f^n.

$$\psi_s := \psi_{F_1} \circ \psi_{F_2} \circ \cdots \psi_{F_n}.$$

Since $F_i \subseteq \Omega \setminus \delta$ for all $i = 1, \cdots, n$ and it is a composition of conformal maps, $\psi_s : \Omega \setminus \delta \longrightarrow F_1$ is well-defined and conformal. By Corollary 4.7 and using that ψ_s is conformal, it follows the existence of $\kappa_{F_1} < 1$ such that

$$\rho_{F_1}(\psi_s(z), \psi_s(w)) \leq \kappa_{F_1} \rho_{F_1}(z, w) \quad \text{for all } z, w \in F_1.$$

By the same argument given in the case of fixed points, there exists $w_s \in F_1$ an attracting fixed point of ψ_s, which by definition of ψ_s, is a repelling fixed point of f^n.

The next step is to show that n is the minimal period of w_s. Indeed, by construction, $f^{i-1}(w_s) \in F_i$ for all $i = 1, \cdots, n$, and since the fundamental domains chosen are all different, and hence disjoint, we can ensure that w_s has minimal period n.

Finally, since we can choose infinitely many different fundamental domains to be F_1, there are infinitely many different repelling points of minimal period n. □

Remark 4.8. Moreover, if we choose n different fundamental domains $\{F_1, \cdots, F_n\}$ and let s and l be two different finite sequences of this domains, $s = (F_1, \cdots, F_n)$ and $l = (F_{\sigma(1)}, \cdots, F_{\sigma(n)})$ for some permutation $\sigma \neq id$, then, the corresponding periodic points w_s, w_l are different. Indeed, since $s \neq l$, there exists $i \in \{1, \cdots, n\}$ such that $\sigma(i) \neq i$, and therefore $f^{i-1}(w_s) \neq f^{i-1}(w_l)$. Hence, $w_s \neq w_l$ and we conclude that for any two different sequences of different fundamental domains, we obtain two different repelling periodic points of f whose orbits lie in such fundamental domains.

Bibliography

[1] James W. Anderson. *Hyperbolic geometry*. Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, second edition, 2005.

[2] A. F. Beardon and D. Minda. The hyperbolic metric and geometric function theory. In *Quasiconformal mappings and their applications*, pages 9–56. Narosa, New Delhi, 2007.

[3] Anna Miriam Benini. A note on repelling periodic points for meromorphic functions with a bounded set of singular values. *Rev. Mat. Iberoam.*, 32(1):267–274, 2016.

[4] Anna Miriam Benini and Núria Fagella. A separation theorem for entire transcendental maps. *Proc. Lond. Math. Soc. (3)*, 110(2):291–324, 2015.

[5] Anna Miriam Benini and Núria Fagella. Singular values and bounded Siegel disks. *Math. Proc. Cambridge Philos. Soc.*, 165(2):249–265, 2018.

[6] Walter Bergweiler. Periodic points of entire functions: proof of a conjecture of Baker. *Complex Variables Theory Appl.*, 17(1-2):57–72, 1991.

[7] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[8] Arnaud Chéritat. Galerie II: Dynamique holomorphe et analyse complexe. `https://www.math.univ-toulouse.fr/~cheritat/GalII/galery.html`, 2009. Last consultation: 02-01-2026.

[9] John B. Conway. *Functions of one complex variable*, volume 11 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1978.

[10] John B. Conway. *Functions of one complex variable. II*, volume 159 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[11] A. È. Erëmenko and M. Yu. Lyubich. Structural stability in certain classes of families of entire functions. *Funktsional. Anal. i Prilozhen.*, 19(4):86–87, 1985.

[12] P. Fatou. Sur les èquations fonctionelles. *Bulletin de la S. M. F.*, 47:161–271, 1919.

[13] P. Fatou. Sur l'itération des fonctions transcendantes Entières. *Acta Math.*, 47(4):337–370, 1926.

[14] Walter K. Hayman and Eleanor F. Lingham. *Research problems in function theory*. Problem Books in Mathematics. Springer, Cham, anniversary edition, 2019.

[15] Linda Keen and Nikola Lakic. *Hyperbolic geometry from a local viewpoint*, volume 68 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2007.

[16] G. Koenigs. Recherches sur les intégrales de certaines équations fonctionnelles. *Ann. Sci. École Norm. Sup. (3)*, 1:3–41, 1884.

[17] Léopold Leau. étude sur les équations fonctionnelles à une ou à plusieurs variables. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.*, 11(3):E25–E110, 1897.

[18] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[19] Cándido Piñeiro. *Dynamics of entire functions—the fractal sets of Julia and Mandelbrot*. De Gruyter, Berlin, [2025] ©2025.

[20] Paul Charles Rosenbloom. L'itération des fonctions entières. *C. R. Acad. Sci. Paris*, 227:382–383, 1948.

[21] Ernst Schröder. Ueber iterirte Functionen. *Math. Ann.*, 3(2):296–322, 1870.

[22] David J. Sixsmith. Dynamics in the Eremenko-Lyubich class. *Conform. Geom. Dyn.*, 22:185–224, 2018.

Appendix A

Code for Visualizing Dynamical Spaces

Throughout this project we have shown pictures of basins of attraction and of parabolic basins. In this appendix we give the code used for plotting one of the basins of attraction (corresponding to the map $f(z) = z^2 - 0.12 + 0.7i$). The code behind the other pictures is analogous, changing the iterated function.

```python
import math
import numpy as np
import matplotlib.cm as cm
import matplotlib.pyplot as plt

def p(z, c):
    # p(z) = z^2 + c
    return z * z + c

c = complex(-0.12, 0.7)
itermax = 130
R = 2 # guaranteed escape radius for quadratic maps z^2+c
xmin, xmax = -1.5, 1.5
ymin, ymax = -1.5, 1.5
height, width = 1601, 1601 # number of pixels
hx = (xmax - xmin)/(width-1)
hy = (ymax - ymin)/(height-1)

# 3D matrix for RGB
img = np.zeros((height, width, 3), dtype=np.uint8)

z = complex(-1.5, 1.5)
```

```
for i in range(height):
    y = -1.5 + i*hy
    for j in range(width):
        x = -1.5 + j*hx
        z = complex(x, y)
        iter = 0
        w = z
        while iter < itermax:
            w = p(w, c)
            iter += 1
            if abs(w) > R:
                # the orbit of z escapes to infinity
                # paint in a palet of colors dependig on the convergence time
                nu = iter + 1 - math.log(math.log(abs(w))) / math.log(2)
                t = min(nu / itermax, 1)
                color_rgb = cm.YlOrRd(t)[:3]
                img[i, j] = tuple(int(255*c) for c in color_rgb)
                break
        if iter == itermax:
            # the orbit of z either converges to the attracting cycle or lies in the boundary of the basin of attraction
            # paint in black
            img[i, j] = (0, 0, 0)
plt.imshow(img)
plt.axis(’off’)
plt.imsave("basin_attraction_c=-0.12+0.7i.png", img)
```