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WEAK-TYPE WEIGHTS AND NORMABLE LORENTZ SPACES

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ABSTRACT. We show that the Lorentz space $\Lambda^1(w)$ is a Banach space if and only if the Hardy-Littlewood maximal operator M satisfies a certain weak-type estimate. We also consider the case of general measures. Finally, we study some properties of several indices associated to these spaces.

1. INTRODUCTION

We are going to study weighted Lorentz spaces of functions defined in $\mathbb{R}^n$ as follows (for standard notation we refer to [BS] and [GR]): If u is a weight in $\mathbb{R}^n$, w is a weight in $\mathbb{R}^+$, f_u^* denotes the decreasing rearrangement of f with respect to the measure $u(x)\,dx$ and $0 < p < \infty$, we define

$$\Lambda_u^p(w) = \left\{ f; \|f\|_{\Lambda_u^p(w)} = \left(\int_0^\infty \left(f_u^*(t) \right)^p w(t)\,dt \right)^{1/p} < \infty \right\}.$$

If $u \equiv 1$, we will only write $\Lambda^p(w)$. Classical examples are obtained by choosing $w(t) = t^{(p/q)-1}$. In this case $\Lambda^p(w) = L^{q,p}$. A classical result of G.G. Lorentz (see [Lo]) shows that $\|\cdot\|_{\Lambda^1(w)}$ is a norm, if and only if, w is a decreasing function. The problem of finding conditions on w so that $\Lambda^p(w)$ is a Banach space (that is, there exists a norm equivalent to $\|\cdot\|_{\Lambda^p(w)}$) was solved, for $p > 1$, by E. Sawyer ([Sa]). This condition is that the Hardy-Littlewood maximal operator is bounded on $\Lambda^p(w)$. The weights for which this holds were first characterized by M.A. Ariño and B. Muckenhoupt ([AM]), and it is known as the B_p condition: there exists $C > 0$ such that, for all $r > 0$,

$$r^p \int_r^\infty \frac{w(x)}{x^p}\,dx \le C \int_0^r w(x)\,dx. \tag{1}$$

It is clear that (1) is not the right condition for $p = 1$, since with $w \equiv 1$, we have that $\Lambda^1(w) = L^1$ is a Banach space, but w does not satisfy (1). It is well known that the weighted strong-type and weak-type boundedness of the Hardy-Littlewood maximal operator coincide for $p > 1$. This motivates to consider the same kind of weak-type estimates for the spaces $\Lambda_u^p(w)$. To this end we recall the following definition:

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Definition 1.1 ([CS1]). Let u and w be weights as above and $0 < p < \infty$. We define

$$\Lambda_u^{p,\infty}(w) = \left\{ f; \|f\|_{\Lambda_u^{p,\infty}(w)} = \sup_{y>0} y\left(\int_0^{\lambda_f^u(y)} w(t)\,dt\right)^{1/p} < \infty \right\},$$

where λ_f^u is the distribution function of f with respect to the measure $u(x)\,dx$.

Observe that $\|f\|_{\Lambda_u^{p,\infty}(w)} = \sup_{x>0} f_u^*(x)(\int_0^x w(t)\,dt)^{1/p}$. Also, for the weight $w(t) = t^{(p/q)-1}$, we have that $\Lambda_u^{p,\infty}(w) = L_u^{q,\infty}$. Our main result is the fact that in order to fully characterize when $\Lambda^p(w)$ is a Banach space, for the whole range $1 \leq p < \infty$, we should replace the boundedness of M in $\Lambda^p(w)$ (as in [Sa]) by the a priori weaker condition on the maximal operator $M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w)$. We give the details in section 2. In section 3 we study the case of a general measure and in section 4 we define an index associated to $\Lambda^p(w)$ and show several characterizations of this index in terms of the B_p condition.

2. Weak-type weights

Let us recall the following characterization of the weak-type boundedness of M for $p > 1$ (see Theorem 3.9 in [CS3]):

Theorem 2.1. *If $p > 1$, then $M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w)$, if and only if, there exists $C > 0$ such that for all $r > 0$*

$$(2) \qquad \left(\int_0^r \left(\frac{1}{x}\int_o^x w(t)\,dt\right)^{-p'} w(x)\,dx\right)^{1/p'} \left(\int_0^r w(t)\,dt\right)^{1/p} \leq Cr.$$

It is easy to show that (2) is equivalent to (1) (see [Sa]) and hence, it is also equivalent to the strong-type boundedness $M : \Lambda^p(w) \longrightarrow \Lambda^p(w)$. The case $p = 1$ of (2) motivates the following definition:

Definition 2.2. We say that $w \in B_{1,\infty}$ if there exists $C > 0$ such that for all $0 < s \leq r < \infty$

$$\frac{1}{r}\int_0^r w(t)\,dt \leq \frac{C}{s}\int_0^s w(t)\,dt.$$

If we set $W(x) = \int_0^x w(t)\,dt$, then $w \in B_{1,\infty}$ is equivalent to saying that W is quasi-concave (see [KPS]). We now prove our main theorem (which is the weak-type version of the main theorem in [AM] and the end point case $p = 1$ of [Sa]). Recall that $Sf(x) = x^{-1}\int_0^x f(t)\,dt$ is the Hardy operator, $f^{**}(t) = Sf^*(t)$ and $L^1_{\text{dec}}(w)$ is the set of all positive decreasing functions in $L^1(w)$.

Theorem 2.3. *The following conditions are equivalent:*

(i) *$\Lambda^1(w)$ is a Banach space.*
(ii) *$w \in B_{1,\infty}$.*
(iii) *$M : \Lambda^1(w) \longrightarrow \Lambda^{1,\infty}(w)$ is bounded.*
(iv) *For all $r > 0$, $\|rt^{-1}\chi_{\{t\geq r\}}\|_{L^{1,\infty}(w)} \leq C\|\chi_{(0,r)}\|_{L^1(w)}$.*
(v) *$S : L^1_{\text{dec}}(w) \longrightarrow L^{1,\infty}(w)$.*

Proof. Let us first show the equivalence of (ii), (iii) and (v). By definition, $M : \Lambda^1(w) \longrightarrow \Lambda^{1,\infty}(w)$ is bounded, if and only if

$$\sup_{y>0} y \int_0^{\lambda_{Mf}(y)} w(t)\,dt \leq C\|f\|_{\Lambda^1(w)} = C\|f^*\|_{L^1(w)}. \tag{3}$$

Now since $(Mf)^* \approx f^{**}$ and $\lambda_{Mf} = \lambda_{(Mf)^*}(y)$ (see [BS]), we obtain that (3) is equivalent to

$$\sup_{y>0} y \int_0^{\lambda_{Sf}(y)} w(t)\,dt \leq C\|f\|_{L^1(w)}, \tag{4}$$

for all $f \in L^1_{\text{dec}}(w)$. Therefore (4) is the same as saying that

$$S : L^1_{\text{dec}}(w) \longrightarrow \Lambda^{1,\infty}(w).$$

Now it is easy to observe that if f is a decreasing function, then $\|Sf\|_{\Lambda^{1,\infty}(w)} = \|Sf\|_{L^{1,\infty}(w)}$ (which proves (v)). But Theorem 3.3-(b) in [CS2] shows that this is equivalent to $w \in B_{1,\infty}$ (take $w_0 = w_1 = w$, $p_0 = p_1 = 1$ and $k(x,t) = x^{-1}\chi_{(0,x)}(t)$ in that theorem). Assume now (i) holds and let us show (ii). If $\Lambda^1(w)$ is a Banach space, then there exists an equivalent norm on $\Lambda^1(w)$, $\|\cdot\| \approx \|\cdot\|_{\Lambda^1(w)}$ and hence, there exists $C > 0$ such that for all $N \in \mathbb{N}$ and $g_1, \cdots, g_N \in \Lambda^1(w)$, $\|g_1 + \cdots + g_N\|_{\Lambda^1(w)} \leq C(\|g_1\|_{\Lambda^1(w)} + \cdots + \|g_N\|_{\Lambda^1(w)})$. Suppose first $r = 2^k s$, with $k = 1, 2, \cdots$. Set $f = \chi_{(0,2^k s)}$ and $f_j = \chi_{(js,(j+1)s)}$, $j = 0, 1, \cdots, 2^k - 1$. Let F and F_j be functions defined in $\mathbb{R}^n$ such that $F^* = f$, $F_j^* = f_j^* = \chi_{(0,s)}$ and $F = \sum_j F_j$. Now,

$$W(2^k s) = \|F\|_{\Lambda^1(w)} \leq C \sum_{j=0}^{2^k-1} \|F_j\|_{\Lambda^1(w)} = C2^k W(s).$$

Finally, for a general $r > s$, let $k = 1, 2, \cdots$ be such that $2^{k-1}s \leq r < 2^k s$. Then

$$\frac{1}{r}\int_0^r w(t)\,dt \leq \frac{1}{2^{k-1}s} W(2^k s) \leq \frac{C2^k}{2^{k-1}s} W(s) = \frac{2C}{s}\int_0^s w(t)\,dt.$$

To see the converse, we use Theorem 1.1 of [KPS, §II] and obtain that there exists a decreasing weight v so that if $V(x) = \int_0^x v(t)\,dt$, then $W \approx V$, and it is now easy to show that $\Lambda^1(v)$ is a Banach space and $\Lambda^1(w) = \Lambda^1(v)$. To finish, we observe that

$$\left\|rt^{-1}\chi_{\{t\geq r\}}\right\|_{L^{1,\infty}(w)} = \sup_{y\leq 1} y \int_r^{r/y} w(t)\,dt,$$

and hence, (iv) is equivalent to

$$\sup_{y\leq 1} \frac{y}{r}\int_0^{r/y} w(t)\,dt \leq C\frac{1}{r}\int_0^r w(t)\,dt,$$

which is equivalent to (ii). □

Remarks 2.4. (a) The equivalence between (i) and (ii) was already known, as a consequence of the study of the p-convexity of a rearrangement invariant space (we want to thank Professor Y. Raynaud for pointing out this to us).

(b) It is now clear that, by (iii), $B_1 \subset B_{1,\infty}$. It is also true that $B_{1,\infty} \subset B_p$, for $p > 1$. In fact, if $r > 0$, we have just shown that for all $k = 1, 2, \cdots$, $W(2^k r) \le C 2^k W(r)$, and hence,

$$\int_r^\infty \frac{w(t)}{t^p}\,dt = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{w(t)}{t^p}\,dt \le \frac{1}{r^p}\sum_{k=0}^\infty \frac{1}{2^{kp}} \int_0^{2^{k+1}r} w(t)\,dt$$
$$\le \frac{1}{r^p}\sum_{k=0}^\infty \frac{C2^{k+1}}{2^{kp}} \int_0^r w(t)\,dt \le \frac{C}{r^p}\left(\sum_{k=0}^\infty \frac{1}{2^{k(p-1)}}\right)\int_0^r w(t)\,dt.$$

(c) If we set $\|f\|_* = \sup_{y>0} f^{**}(y)W(y)$, then $\|\cdot\|_*$ is a norm, and if $w \in B_{1,\infty}$, then $\|f\|_* \le C\|f\|_{\Lambda^1(w)}$. Sometimes the converse inequality is also true, and hence $\|\cdot\|_*$ is the equivalent norm in $\Lambda^1(w)$. This happens, for example, if $w \equiv 1$ (see [Sj] for related results). The equivalence is not true in general: if $w(t) = t^{-1/2}$, then $\Lambda^1(w)$ is a Banach space, but if $f^*(t) = t^{-1/2}$, then $f \notin \Lambda^1(w)$ and $\|f\|_* < \infty$. However, if $y^{-1}W(y)$ is equivalent to an increasing function, then $\|f\|_{\Lambda^1(w)} \le C\|f\|_*$. It is also easy to show that this condition is not necessary.

(d) For some related results see [Ne,§6].

3. Extensions to general measures

We want to study geometric properties of the spaces $\Lambda^p_u(w)$, for general weights. For example, it was proved in [CS1] that $\|\cdot\|_{\Lambda^p_u(w)}$ is a quasinorm, if and only if W satisfies a certain doubling condition ($W \in \Delta_2$). Also it is an easy exercise to show that these spaces are always complete (as long as $W(x) > 0$ if $x > 0$). We now ask the same question we did for $\Lambda^p(w)$: under which conditions are they Banach spaces? We observe that Theorem 2.3 does not hold in general, since for $w \equiv 1$, $\Lambda^p_u(w) = L^p_u$, which is a Banach space if $p \ge 1$, but the Hardy-Littlewood maximal operator is not always bounded $M : L^p_u \longrightarrow L^{p,\infty}_u$. However, we can show that, in many cases, this property only depends on the weight w:

Theorem 3.1. *If $u \notin L^1$ (or du is an infinite non-atomic measure) and $p > 0$, then $\Lambda^p_u(w)$ is a Banach space, if and only if $\Lambda^p(w)$ is a Banach space.*

Proof. By a symmetric argument, it suffices to consider the case $\Lambda^p(w)$ Banach: we want to show that there exists $C > 0$ so that for simple functions $S_1, \cdots, S_N$,

$$\left\|\sum_{j=1}^N S_j\right\|_{\Lambda^p_u(w)} \le C\sum_{j=1}^N \|S_j\|_{\Lambda^p_u(w)}, \tag{5}$$

since the Monotone Convergence Theorem implies (5) for arbitrary functions, and it is easy to show that this implies the existence of an equivalent norm. We will only consider the case $N = 2$ (the case $N > 2$ is completely analogous). So, let $S_j = \sum_k \lambda^j_k \chi_{A^j_k}$, $j = 1, 2$, and the sets A^j_k are pairwise disjoint. Then,

$$S_1 + S_2 = \sum_k \lambda^1_k \chi_{A^1_k \setminus \bigcup_j A^2_j} + \sum_k \lambda^2_k \chi_{A^2_k \setminus \bigcup_j A^1_j} + \sum_{k,j} (\lambda^1_k + \lambda^2_j)\chi_{A^1_k \cap A^2_j}.$$

We choose a family of disjoint sets $\{E_k^1, E_j^2, I_{k,j} : k, j\}$ such that

$$|E_k^1| = u(A_k^1 \setminus \bigcup_j A_j^2),$$
$$|E_k^2| = u(A_k^2 \setminus \bigcup_j A_j^1),$$
$$|I_{k,j}| = u(A_k^1 \cap A_j^2),$$

where, as usual, for a set $A \subset \mathbb{R}^n$, $u(A) = \int_A u(x)\,dx$. Define also,

$$C_1 = \sum_k \lambda_k^1 \chi_{E_k^1 \cup (\bigcup_j I_{k,j})},$$
$$C_2 = \sum_k \lambda_k^2 \chi_{E_k^2 \cup (\bigcup_j I_{j,k})},$$

so that $C_1^* = (S_1)_u^*$, $C_2^* = (S_2)_u^*$ and $(C_1 + C_2)^* = (S_1 + S_2)_u^*$. Finally,

$$\begin{aligned}\|S_1 + S_2\|_{\Lambda_u^p(w)} &= \|C_1 + C_2\|_{\Lambda^p(w)} \leq C(\|C_1\|_{\Lambda^p(w)} + \|C_2\|_{\Lambda^p(w)}) \\ &= C(\|S_1\|_{\Lambda_u^p(w)} + \|S_2\|_{\Lambda_u^p(w)}) \cdot \square\end{aligned}$$

This result allows us to extend (2.6) of [Hu]:

Corollary 3.2. *Let $u \notin L^1$ (or du an infinite non-atomic measure) let $p > 0$, and suppose $\Lambda_u^p(w)$ is a Banach space. Then $p \geq 1$.*

Proof. From the previous theorem we see that it is enough to consider $u \equiv 1$. Assume now that $0 < p < 1$. As in [Hu], we are going to find a sequence of functions $\{f_j\}_j$, satisfying

$$\|f_j\|_{\Lambda^p(w)} = 1 \quad \text{but} \quad \frac{1}{N}\left\| \sum_{k=1}^N f_k \right\|_{\Lambda^p(w)} \underset{N\to\infty}{\longrightarrow} \infty.$$

Choose $A_k \subset \mathbb{R}^n$, $\cdots \subset A_{k+1} \subset A_k$, such that $W(|A_k|) = 2^{-pk}$. Let $f_k = 2^k \chi_{A_k}$. Then,

$$\|f_k\|_{\Lambda^p(w)} = 2^k \left(\int_0^\infty \left((\chi_{A_k})^* \right)^p w(t)\,dt \right)^{1/p} = 2^k W(|A_k|)^{1/p} = 1.$$

But, if for N fixed, we set $A_{N+1}=\emptyset$, then,

$$
\begin{aligned}
\frac{1}{N}\left\|\sum_{k=1}^{N} f_k\right\|_{\Lambda^p(w)} &= \frac{1}{N}\left\|\sum_{k=1}^{N}\left(\sum_{j=1}^{k} 2^j\right)\chi_{(A_k\setminus A_{k+1})}\right\|_{\Lambda^p(w)} \\
&= \frac{1}{N}\left[\int_0^\infty \left(\left[\sum_{k=1}^{N}\left(\sum_{j=1}^{k} 2^j\right)\chi_{(A_k\setminus A_{k+1})}\right]^*\right)^p w(t)\,dt\right]^{1/p} \\
&= \frac{1}{N}\left[\int_0^\infty \left(\sum_{k=1}^{N}\left(\sum_{j=1}^{k} 2^j\right)\chi_{[|A_{k+1}|,|A_k|)}\right)^p w(t)\,dt\right]^{1/p} \\
&= \frac{1}{N}\left(\sum_{k=1}^{N}\left(\sum_{j=1}^{k} 2^j\right)^p \int_{|A_{k+1}|}^{|A_k|} w(t)\,dt\right)^{1/p} \\
&\geq \frac{1}{N}\left(\sum_{k=1}^{N}(2^{k+1}-2)^p\left(2^{-kp}-2^{-(k+1)p}\right)\right)^{1/p} \\
&\geq C_p\frac{1}{N}N^{1/p} \underset{N\to\infty}{\longrightarrow} \infty \cdot \square
\end{aligned}
$$

Remark 3.3. The above result is not true for general measures. In fact, if $du=\delta_0$, the Dirac delta at the origin, then $\|f\|_{\Lambda^p_u(w)}=|f(0)|(\int_0^1 w(t)\,dt)^{1/p}$. Hence, $\Lambda^p_u(w)\approx\mathbb{R}$, for all $p>0$.

4. Indices

Given $\Lambda^p(w)$ we define the following invariant index:

Definition 4.1. Given $k=1,2,\cdots$ set

$$D_k=\sup_{t>0}\frac{W(2^k t)}{W(t)},$$

and

$$p(w)=\inf\left\{p>0;\text{for some } C>0, D_k\leq C2^{kp}, k=1,2,\cdots\right\}.$$

We now prove the fundamental property of $p(w)$:

Theorem 4.2.

$$D_k^{1/k}\underset{k\to\infty}{\longrightarrow}2^{p(w)}.$$

Proof. Set $a_k=\log D_k$. Then $a_{j+k}\leq a_j+a_k$ and therefore there exists $l\geq 0$ such that $a_k/k\longrightarrow l$, as $k\to\infty$ (this is an easy exercise, see for example [GuR]). Thus, $D_k^{1/k}\underset{k\to\infty}{\longrightarrow}D=e^l$. So, we need to show that $D=2^{p(w)}$. If $p>p(w)$, then $D_k\leq C2^{kp}$ and hence $D\leq 2^p$; i.e., $D\leq 2^{p(w)}$. Conversely, if $D<2^{p(w)}$, choose $0<p<p(w)$ such that $D<2^p<2^{p(w)}$. Let $k_0\in\mathbb{N}$ such that $D_k^{1/k}\leq 2^p$, if $k\geq k_0$. Let $C=\max\{1,D_k/2^{kp}:k=1,\cdots,k_0-1\}$. Then $D_k\leq C2^{kp}$, $k\in\mathbb{N}$, and we reach the contradiction $p(w)\leq p<p(w)$. $\square$

Remarks 4.3. (i) As a consequence of this result, it turns out that if two weights w_0, w_1 satisfy $W_0 \approx W_1$, then $p(w_0) = p(w_1)$.

(ii) It is not true, in general, that $D_k = D^k$. For this, it suffices to consider the weight $w(t) = \chi_{(0,1)}(t) + 2\chi_{[1,\infty)}(t)$. Here, $D_k = 2^{k+1} - 1$.

(iii) The definition of $p(w)$ is closely related to the condition on the weights R_p in [Ne] (notice that $R_1 = B_{1,\infty}$).

The following result will be needed later. Recall that $B_{1,\infty} \subset \bigcap_{p>1} B_p$.

Lemma 4.4. *There exists* $w \in \bigcap_{p>1} B_p \setminus B_{1,\infty}$.

Proof. We choose a decreasing sequence $\{p_k\}_k$, with $\lim_k p_k = 1$ ($p_k \neq 1$, $k = 1, 2, \cdots$). Consider also $\{a_k\}_k$, $a_k > 0$ and $\sum_k a_k = 1$. Let us define

$$w(t) = \begin{cases} \sum_{k=1}^{\infty} a_k p_k t^{p_k - 1} & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1. \end{cases}$$

It is clear that the series converges uniformly on $[0,1]$. Now,

$$W(t) = \begin{cases} \sum_{k=1}^{\infty} a_k t^{p_k} & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t > 1. \end{cases}$$

Thus, $t^{-1}W(t) \longrightarrow 0$, as $t \to 0$, and therefore $w \notin B_{1,\infty}$. To prove that $w \in B_p$, $p > 1$, it suffices to show that if $p \neq p_k$, there exists $C_p > 0$ such that condition (1) holds, for $0 < r < 1$. Now,

$$\int_r^{\infty} \frac{w(t)}{t^p}\, dt = \sum_{k=1}^{\infty} a_k p_k \int_r^1 t^{p_k - 1 - p}\, dt = \sum_{k=1}^{\infty} \frac{a_k p_k}{p_k - p}(1 - r^{p_k - p})$$

and

$$r^{-p} \int_0^r w(t)\, dt = \sum_{k=1}^{\infty} a_k r^{p_k - p}.$$

Thus, if we let $S_p = \sup_k p_k/(p_k - p) > 0$, $I_p = \inf_k p_k/(p_k - p) < 0$ and $D_p = \operatorname{card}\{k : p_k > p\}$, and choose k_0 so that $p_{k_0} < p$, then with $C_p = D_p S_p / a_{k_0} - I_p$, we obtain the result. □

Theorem 4.5. (a) *If* $1 \leq p < \infty$, *then* $w \in B_p$ *if and only if* $p(w) < p$.

(b) *If* $w \in B_{1,\infty}$, *then* $p(w) \leq 1$, *and the converse is not true in general.*

(c) *If* $\Lambda^{p(w)}(w)$ *is a Banach space, then* $p(w) = 1$, *and the converse is not true in general.*

Proof. (a) If $w \in B_p$ then (see [AM]) there exists $q < p$ so that $w \in B_q$. Now

$$\frac{1}{2^{kq} t^q} \int_t^{2^k t} w(s)\, ds \leq \int_t^{2^k t} \frac{w(s)}{s^q}\, ds \leq \frac{C}{t^q} \int_0^t w(s)\, ds,$$

and hence,

$$\int_0^{2^k t} w(s)\, ds \leq (1 + C 2^{kq}) \int_0^t w(s)\, ds,$$

which implies $D_k \le C2^{kq}$. Conversely, if $D_k \le C2^{kq}$, with $q < p$, then for $r > 0$,

$$\int_r^\infty \frac{w(t)}{t^p}\,dt = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{w(t)}{t^p}\,dt \le \frac{1}{r^p}\sum_{k=0}^\infty \frac{1}{2^{kp}} \int_0^{2^{k+1}r} w(t)\,dt$$
$$\le \frac{1}{r^p}\sum_{k=0}^\infty \frac{D_{k+1}}{2^{kp}} \int_0^r w(t)\,dt \le \frac{2^q C}{r^p}\left(\sum_{k=0}^\infty \frac{1}{2^{k(p-q)}}\right)\int_0^r w(t)\,dt.$$

(b) This is clear by definition. Now if $w \in \bigcap_{p>1} B_p \setminus B_{1,\infty}$, as in Lemma 4.4, then $p(w) \le 1$ (in fact $p(w) = 1$).

(c) By Corollary 3.2, we have that $p(w) \ge 1$. If $p(w) > 1$, then $w \in B_{p(w)}$ and by (a) we obtain a contradiction. Now, with w as in (b), we show that the converse does not hold. □

Remarks 4.6. (i) An equivalent result to part (a) of the previous theorem was proved by Raynaud ([Ra]), using more complicated arguments.

(ii) If $w(t) = t^\alpha$, with $-1 < \alpha \le 0$, then $p(w) = 1 + \alpha$, and so $p(w) \le 1$ is the best we can say in (b) of the theorem.

(iii) It is easy to show that $D_k \le C2^{kp}$ is equivalent to saying that W is a p-quasi-concave function.

(iv) There are other indices (Simonenko, Matuszewska-Orlicz, etc.) for which it is possible to show the equivalence with $p(w)$. In particular $p(w) = \beta_W^a = \sigma_W^a = B_W^a = Q_W^a$ (see [Ma] for the definitions).

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