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On Time-Frequency Localization: Gabor Frames and the Balian-Low Theorem

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Abstract

This work studies time-frequency localization in the context of signal analysis. More precisely, it focuses on the Fourier transform, the short-time Fourier transform, and Gabor systems as tools to represent signals in both time and frequency. Special attention is given to the limitations imposed by the uncertainty principle and the Balian-Low theorem, which show that a signal representation cannot be simultaneously discrete, orthonormal, and well localized in time and frequency. Finally, the work presents Gabor frames as a natural alternative to orthonormal bases, emphasizing the role of Gaussian windows and the density condition for stable reconstruction.

Resum

Aquest treball estudia la localització temps-freqüència en el context de l'anàlisi de senyals. Més concretament, se centra en la transformada de Fourier, la transformada de Fourier de temps curt i els sistemes de Gabor com a eines per representar senyals tant en el temps com en la freqüència. Es presta una atenció especial a les limitacions imposades pel principi d'incertesa i pel teorema de Balian-Low, que mostren que una representació de senyals no pot ser simultàniament discreta, ortonormal i ben localitzada en temps i freqüència. Finalment, el treball presenta els frames de Gabor com una alternativa natural a les bases ortonormals, destacant el paper de les finestres gaussianes i de la condició de densitat per a una reconstrucció estable.

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Chapter 1

Introduction

A signal is, at first sight, a very familiar object. A sound recorded by a microphone, a piece of music stored in a computer, an image transmitted through a screen, or a radio wave traveling through space can all be understood as signals. Mathematically, however, a signal is usually represented by a function. The variable may be time, space, or some other parameter, and the value of the function measures the intensity, amplitude, pressure, brightness, or energy of the phenomenon under consideration.

Once a signal has been represented as a function, a natural question arises: what kind of information does the function contain, and how can we extract it? Looking only at the values of the function in time gives one possible description. For instance, if $f(t)$ is an audio signal, then the graph of f tells us how the air pressure changes over time. However, this description does not immediately tell us which notes, tones, or frequencies are present in the sound. A chord played on a piano, for example, is not perceived by the ear as a random oscillation, but as a combination of several frequencies. This is where Fourier analysis enters the picture.

The central idea of Fourier analysis is that complicated signals can be studied by decomposing them into simpler oscillations. In the periodic case, these oscillations are the exponentials

$$e^{int}, \quad n \in \mathbb{Z},$$

and a suitable function can be represented through its Fourier series. In the non-periodic case, the same philosophy leads to the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} dt,$$

which measures how much of the frequency ξ is contained in the signal f . Thus,

Fourier analysis changes the point of view: instead of asking what the signal looks like at each time t , we ask which frequencies build it.

This change of perspective is extremely powerful. Many operations that are complicated in the time domain become much simpler in the frequency domain. For instance, differentiation becomes multiplication by the frequency variable, and convolution becomes pointwise multiplication. This is one of the reasons why Fourier methods appear in the study of partial differential equations, quantum mechanics, image processing, audio compression, and many other fields. In this work, however, our interest is not only to use the Fourier transform as a computational tool, but also to understand its limitations when applied to the analysis and reconstruction of signals.

The first important limitation comes from the relationship between time and frequency. If a signal is very concentrated in time, then its Fourier transform cannot also be very concentrated in frequency. Conversely, if the frequency content of a signal is very concentrated, then the signal itself must be spread out in time. This phenomenon is formalized by the Heisenberg uncertainty principle.

Heisenberg uncertainty principle. *If $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$, then*

$$\left(\int_{\mathbb{R}} (t - a)^2 |f(t)|^2 dt \right)^{1/2} \left(\int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \geq \frac{\|f\|_2^2}{4\pi}.$$

This inequality says that the product of the time spread and the frequency spread of a non-zero signal cannot be arbitrarily small. Equality occurs for Gaussian functions, which already suggests that Gaussians will play a privileged role throughout the work.

This principle is not only a mathematical curiosity. It expresses a fundamental trade-off that appears whenever we try to analyze a signal. If we want to know very precisely when something happens, we must sacrifice precision about its frequency. If we want to know very precisely which frequency is present, we must look at the signal over a longer interval of time. In other words, time and frequency cannot both be observed with unlimited resolution.

A second classical result, apparently more optimistic, is the Shannon sampling theorem. In digital signal processing, one cannot store a continuous signal by keeping all of its values. Instead, one records a discrete sequence of samples. A basic question is then: how many samples are necessary in order to reconstruct the whole signal without losing information?

The answer depends on the frequencies contained in the signal. If f is band-limited, meaning that its Fourier transform is supported in an interval $[-\tau, \tau]$, then f contains no frequencies larger than τ . In that case, the Kotelnikov-Shannon-

Whittaker theorem states that f can be recovered from the samples

$$\left\{ f\left(\frac{k}{2\tau}\right) \right\}_{k \in \mathbb{Z}}$$

through the formula

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2\tau}\right) \operatorname{sinc}\left(2\tau\left(t - \frac{k}{2\tau}\right)\right),$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}.$$

Thus, if the signal is band-limited, sampling at rate 2τ is enough for perfect reconstruction. This value is known as the Nyquist rate.

The theorem explains a basic principle of digital audio. The human ear is usually sensitive to frequencies up to approximately 20 kHz. Therefore, in order to represent all audible frequencies, one should sample at least at twice that frequency, that is, at approximately 40 kHz. The standard sampling rate for audio CDs is 44.1 kHz, slightly above this theoretical threshold. This extra margin is not accidental: in practice, ideal mathematical assumptions are never perfectly satisfied, and one needs some room to deal with imperfections.

The Shannon theorem is remarkable because it says that, under the right hypothesis, a continuous signal can be completely reconstructed from a discrete set of data. However, the hypothesis of band-limitedness hides a subtle problem. A non-trivial band-limited signal cannot be compactly supported in time. In fact, such a signal extends to an entire function, and therefore cannot vanish on an interval unless it vanishes identically. This is counterintuitive from a physical point of view. Real sounds, songs, or spoken words start and end. They do not last forever. Thus, perfect band-limitedness is an idealization: useful, but not fully realistic.

This tension between mathematical reconstruction and physical localization is one of the guiding ideas of the present work. On the one hand, Fourier analysis allows us to understand signals through their frequencies and, in ideal situations, to reconstruct them from discrete information. On the other hand, the uncertainty principle tells us that we cannot localize a signal perfectly both in time and in frequency. The rest of the work can be seen as an attempt to understand how far one can go in this direction.

The ordinary Fourier transform gives a global description of a signal. It tells us which frequencies are present, but it does not tell us when they occur. This is not a problem for stationary signals, whose frequency content does not change much over time. But many real signals are not stationary. A melody changes

note by note, speech changes from syllable to syllable, and many physical signals evolve dynamically. In these cases, knowing only the global frequency content is not enough.

A natural idea is to analyze the signal locally. Instead of applying the Fourier transform to the whole function, we multiply the signal by a window function concentrated near a given time, and then compute the Fourier transform of this localized piece. Moving the window along the time axis gives a representation depending on two variables: time and frequency. This is the idea behind the short-time Fourier transform, or STFT.

Given a window $g \in L^2(\mathbb{R})$, the STFT of a function f with respect to g is defined by

$$V_g f(t, \xi) = \int_{\mathbb{R}} f(s) \overline{g(s-t)} e^{-2\pi i \xi s} ds.$$

For each fixed time t , the function $s \mapsto f(s) \overline{g(s-t)}$ is the signal f observed through a window centered at t . Taking its Fourier transform tells us which frequencies are present around that time. Thus, $V_g f(t, \xi)$ measures the contribution of the frequency ξ near the time t .

This construction gives a much richer description than the ordinary Fourier transform. Instead of representing a signal only on the frequency axis, the STFT represents it on the time-frequency plane. Each point (t, ξ) corresponds to a local measurement: what does the signal look like near time t and near frequency ξ ?

The window g is therefore crucial. If g is very narrow in time, the STFT has good time resolution: it can distinguish events that occur close to each other. However, the Fourier transform of such a narrow window is spread out, so the frequency resolution becomes poor. If g is very spread out in time, the frequency resolution improves, but the time resolution worsens. Once again, the uncertainty principle appears.

The Gaussian function is the natural candidate for the window because it is the unique optimizer of the uncertainty principle. If

$$\varphi(t) = 2^{1/4} e^{-\pi t^2},$$

then its Fourier transform is again the same Gaussian:

$$\widehat{\varphi}(\xi) = \varphi(\xi).$$

This symmetry between time and frequency makes the Gaussian the best possible window from the point of view of time-frequency concentration. It is therefore reasonable to ask whether Gaussian windows can be used to build a complete and stable discrete representation of arbitrary signals.

To make this question precise, one has to discretize the time-frequency plane. Instead of considering all shifts and modulations of a window g , one takes only a lattice of them. For $a, b > 0$, the corresponding Gabor system is

$$\mathcal{G}(g, a, b) = \{M_{bk}\tau_{an}g : n, k \in \mathbb{Z}\},$$

where

$$\tau_{an}g(t) = g(t - an), \quad M_{bk}g(t) = e^{2\pi i b k t} g(t).$$

The parameter a measures the spacing in time, and b measures the spacing in frequency. The product ab is the area of a fundamental cell in the time-frequency lattice. Thus, the smaller ab is, the denser the system is.

A first discretization works perfectly if we use the characteristic function of an interval as the window. For example, taking

$$g = \chi_{[-1/2, 1/2)}$$

one obtains an orthonormal basis of $L^2(\mathbb{R})$ by translating g by integers and modulating it by integer frequencies:

$$\{M_k\tau_n g : n, k \in \mathbb{Z}\}.$$

This is a very simple and instructive construction. The real line is divided into intervals of length one, and on each interval the signal is expanded in a Fourier series. In this way, the global signal is reconstructed by combining local Fourier series.

However, this construction is not satisfactory from the point of view of time-frequency localization. The window $\chi_{[-1/2, 1/2)}$ is perfectly localized in time, but it has jump discontinuities. Consequently, its Fourier transform decays slowly and is poorly localized in frequency. In practical terms, the sharp edges of the window introduce artificial frequency components. The method works as a basis, but the window is far from ideal.

It is then natural to ask whether one can replace this rough window by a smoother and better localized one, such as the Gaussian, while preserving the basis property. The Balian-Low theorem gives a negative answer at the critical density.

Balian-Low theorem. *Suppose that $a, b > 0$ and $ab = 1$. If the Gabor system*

$$\{M_{bk}\tau_{an}g : n, k \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$, then

$$\left(\int_{\mathbb{R}} t^2 |g(t)|^2 dt \right) \left(\int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2 d\xi \right) = +\infty.$$

Thus, a window generating an orthonormal Gabor basis at critical density cannot be well localized both in time and in frequency. In particular, the Gaussian cannot generate such a basis.

This is one of the central obstructions studied in the work. The theorem shows that there is an incompatibility between three desirable properties: discreteness, basis structure, and good time-frequency localization. We would like a discrete system because real applications use samples. We would like a basis because it gives unique and non-redundant expansions. And we would like the window to be well localized in both variables because that is precisely the goal of time-frequency analysis. The Balian-Low theorem says that, at the critical lattice density, we cannot have all three at once.

At this point, one could think that Gaussian time-frequency expansions are impossible. But this conclusion would be too pessimistic. The problem is not the Gaussian itself, but the requirement of having an orthonormal basis. Bases are very rigid objects. They give unique decompositions, but they do not tolerate redundancy. In signal processing, however, redundancy is often not a defect. On the contrary, it can make representations more stable, more robust to noise, and more flexible.

This motivates the introduction of frames. A frame in a Hilbert space H is a family $\{e_j : j \in J\}$ for which there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j \in J} |\langle f, e_j \rangle|^2 \leq B\|f\|^2, \quad f \in H.$$

The lower bound ensures that the coefficients do not lose information about the signal. The upper bound ensures that the coefficients do not blow up. Together, they say that the energy of f is faithfully represented by the sequence of frame coefficients.

Every orthonormal basis is a frame with $A = B = 1$, but frames are much more general. Their elements need not be orthogonal, and they may be linearly dependent. This means that a frame may contain redundant information. Still, every element of the Hilbert space can be reconstructed from its frame coefficients, using the frame operator and the associated dual frame.

For Gabor systems, the frame inequalities take the form

$$A\|f\|_2^2 \leq \sum_{n,k \in \mathbb{Z}} |\langle f, M_{bk}\tau_{an}g \rangle|^2 \leq B\|f\|_2^2, \quad f \in L^2(\mathbb{R}).$$

Since

$$\langle f, M_{bk}\tau_{an}g \rangle = V_g f(an, bk),$$

this condition says that sampling the STFT on the lattice $a\mathbb{Z} \times b\mathbb{Z}$ captures the whole energy of the signal in a stable way. This is exactly the kind of result one

wants after the limitations imposed by Balian-Low: not a perfect orthonormal basis, but a stable and possibly redundant time-frequency representation.

The central positive result in this direction is the theorem of Daubechies and Grossmann, together with Seip's density theorem.

Daubechies-Grossmann theorem. *For the normalized Gaussian $G(t) = 2^{1/4}e^{-\pi t^2}$, the Gabor system $\mathcal{G}(G, a, b) = \{M_{bk}\tau_{an}G : n, k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ if and only if $ab < 1$.*

This condition has a clear geometric meaning. The number ab is the area of a cell of the time-frequency lattice. The inequality $ab < 1$ means that the lattice must be denser than the critical lattice. Equivalently, the sampling density must be strictly larger than one. Thus, the obstruction of Balian-Low at the critical density is overcome by allowing redundancy: we sample slightly more densely, and in exchange we recover stability with the Gaussian window.

To study this result, it is useful to reformulate the problem in a different setting. This is done through the Bargmann transform, which allows the STFT with a Gaussian window to be expressed in terms of an entire function. More precisely, if $z = t + i\tilde{\xi}$, then the STFT with Gaussian window is related to the Bargmann transform Bf by the formula

$$V_\phi f(t, -\tilde{\xi}) = e^{\pi i t \tilde{\xi}} e^{-\frac{\pi}{2}(t^2 + \tilde{\xi}^2)} Bf(z).$$

The Bargmann transform maps $L^2(\mathbb{R})$ isometrically onto the Bargmann-Fock space, the space of entire functions F on $\mathbb{C}$ such that

$$\|F\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dz < \infty.$$

Therefore, the problem of deciding whether a Gaussian Gabor system is a frame can be translated into a sampling problem in the Bargmann-Fock space.

Indeed, after identifying the time-frequency plane with the complex plane through $z = t + i\tilde{\xi}$, the lattice $a\mathbb{Z} \times b\mathbb{Z}$ is written as

$$\Gamma = a\mathbb{Z} + ib\mathbb{Z} \subset \mathbb{C}.$$

The Gabor frame inequality becomes the sampling inequality

$$A\|F\|_{\mathcal{F}^2}^2 \leq \sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-\pi|\gamma|^2} \leq B\|F\|_{\mathcal{F}^2}^2.$$

Thus, the question becomes: when does the discrete set Γ sample every function in the Bargmann-Fock space in a stable way?

Seip's density theorem answers this question. In the case of a rectangular lattice, the condition is exactly that the density of the lattice is greater than one, or equivalently that $ab < 1$.

This is the same condition appearing in the Daubechies-Grossmann theorem. Hence, the final part of the work shows how the original problem about Gaussian Gabor frames can be reduced to a geometric density question for entire functions.

In summary, the aim of this work is to understand how signals can be represented simultaneously in time and frequency, what limitations arise from this perspective, and how Gabor frames provide a stable alternative when orthonormal bases are too restrictive.

1.1 Structure of the manuscript

The work is organized in four main chapters after this introduction.

Chapter 2 contains the preliminaries. The aim is not to prove new results, but to fix notation and recall the tools that will be used later. We begin with Hilbert spaces, orthonormal bases, Bessel's inequality and Parseval's identity. These notions are necessary because the natural setting for signals of finite energy is the Hilbert space $L^2(\mathbb{R})$. We then recall the basic facts about Fourier series, the Fourier transform on $L^1(\mathbb{R})$, and its extension to $L^2(\mathbb{R})$ through Plancherel's theorem. Since most of these results are standard and have already appeared in previous courses of the Bachelor's degree, they are mostly stated without proof. Their role is to make the rest of the work self-contained and easier to follow.

Chapter 3 introduces the connection with signal theory. We first present the Heisenberg uncertainty principle, which establishes the fundamental limitation between time concentration and frequency concentration. The proof also shows why Gaussian functions are special: they are precisely the functions for which equality occurs. After that, we discuss the Shannon sampling theorem for band-limited functions. This theorem explains how a signal whose Fourier transform is supported in $[-\tau, \tau]$ can be reconstructed from samples taken at the Nyquist rate 2τ . The example of CD audio illustrates the practical meaning of this theorem. At the same time, the chapter points out the limitations of perfect band-limitedness, preparing the transition to local time-frequency methods.

Chapter 4 is devoted to the short-time Fourier transform. We introduce the STFT as a way of measuring the frequency content of a signal locally in time. We then prove the orthogonality relations and the inversion formula, which show that the STFT contains enough information to reconstruct the original signal. After this, we study the problem of discretizing the time-frequency plane. The characteristic window gives a first example of an orthonormal Gabor basis, but its poor frequency localization motivates the search for smoother windows. This leads to the Balian-Low theorem, which shows that no well-localized window can generate an orthonormal Gabor basis at critical density. This theorem is the main

negative result of the work and explains why bases are too restrictive for the kind of time-frequency representation we want.

Chapter 5 explains how frames overcome this obstruction. We introduce frames in a general Hilbert space, together with the analysis, synthesis and frame operators. These tools allow us to reconstruct signals from redundant families of vectors. We then specialize to Gabor frames and state the Daubechies-Grossmann theorem for Gaussian windows. The rest of the chapter develops the main ideas behind this result. The Bargmann transform identifies the Gaussian STFT with entire functions in the Bargmann-Fock space, and the Gabor frame inequality becomes a sampling inequality for a lattice in the complex plane. Finally, Seip's density theorem gives the density condition that characterizes sampling sequences. In this way, the work ends by connecting the original problem in signal analysis with a geometric theorem about entire functions.

Chapter 2

Preliminaries

In this chapter some essential definitions and results will be provided to make the reading as pleasing and comfortable as possible. We give a brief introduction to Hilbert spaces and some basic results on Fourier analysis. Since the purpose of this section is mainly expository, most results will be stated without proof. Moreover, many of these topics have already been covered in previous courses, so we will not provide proofs here. If the reader is interested in detailed proofs, you can find them in [4].

2.1 Hilbert Spaces

Definition 2.1. A *Hilbert space* is a vector space H on $\mathbb{C}$ with an inner product $\langle x, y \rangle, x, y \in H$, such that the norm defined by $\|x\| = \langle x, x \rangle^{1/2}$ turns H into a complete metric space. With this norm, the space H has the structure of a Banach space.

Closed subspaces play a special role in the following chapters.

Definition 2.2. Given a closed subspace $M \subset H$, we define its *orthogonal space* as:

$$M^\perp = \{x \in H : \langle x, y \rangle = 0 \ \forall y \in M\}.$$

Theorem 2.3. (Projection theorem) Let $M \subset H$ be a closed subspace. For any $x \in H$ there exist unique $y \in M$ and $z \in M^\perp$ such that $x = y + z$. That is, $H = M \oplus M^\perp$.

We define the projection operators $P_M : H \rightarrow M$ and $Q_M : H \rightarrow M^\perp$ respectively as $P(x) = y$ and $Q(x) = z$. These operators are linear, continuous, and satisfy the Pythagorean identity

$$\|x\|^2 = \|P_M(x)\|^2 + \|Q_M(x)\|^2, \quad x \in H.$$

In particular, $\|P_M\| \leq 1$ and $\|Q_M\| \leq 1$.

Definition 2.4. A *Hilbert basis* is a complete orthonormal system $\{e_i\}_{i \in I}$ of elements in H . To be complete means that the closure of its span, $\overline{\langle e_i \rangle_{i \in I}}$, is the whole space H .

We will only deal with separable Hilbert spaces, that is, spaces for which there is a countable orthonormal basis $\{e_n\}_{n=1}^\infty$.

Theorem 2.5. Let $\{e_n\}_{n=1}^\infty$ be a countable orthonormal system, and let $V = \langle e_n \rangle_{n=1}^\infty$ be its span. Given $x \in H$,

- (i) $P_V(x) = \sum_{n=1}^\infty \langle x, e_n \rangle e_n$
- (ii) $\sum_{n=1}^\infty |\langle x, e_n \rangle|^2 \leq \|x\|^2$ (Bessel's inequality)

Corollary 2.6. Let $\{e_n\}_{n=1}^\infty$ be a countable orthonormal system. The following are equivalent:

- (i) $\{e_n\}_{n=1}^\infty$ is a Hilbert basis.
- (ii) If $\langle x, e_n \rangle = 0$ for all $n \geq 1$, then $x = 0$.
- (iii) $\|x\|^2 = \sum_{n=1}^\infty |\langle x, e_n \rangle|^2$ (Parseval's identity).

Notice that when $\{e_n\}_{n=1}^\infty$ is an orthonormal basis, the map

$$\begin{aligned} S: \ell^2(\mathbb{N}) &\longrightarrow H \\ (a_n)_{n=1}^\infty &\longmapsto \sum_{n=1}^\infty a_n e_n \end{aligned}$$

is well defined: given $(a_n)_{n=1}^\infty \in \ell^2(\mathbb{N})$, the partial sums of $S(a_n)$ form a Cauchy sequence, since by orthonormality

$$\left\| \sum_{n=k+1}^m a_n e_n \right\|^2 = \sum_{n=k+1}^m |a_n|^2 \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and by completeness of H , the series converges. On the other hand, the map

$$\begin{aligned} A: H &\longrightarrow \ell^2(\mathbb{N}) \\ x &\longmapsto (\langle x, e_n \rangle)_{n=1}^\infty \end{aligned}$$

is also well defined, since Parseval's identity gives $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2 < \infty$ for any $x \in H$, so $A(x) \in \ell^2(\mathbb{N})$.

Moreover, S is an isometric isomorphism with inverse A . Indeed, the identities $A \circ S = \text{id}_{\ell^2}$ and $S \circ A = \text{id}_H$ follow directly from orthonormality and the fact that every $x \in H$ equals its Fourier series. Isometry follows from

$$\|S((a_n))\|^2 = \left\| \sum_{n=1}^{\infty} a_n e_n \right\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \|(a_n)\|_{\ell^2}^2,$$

again by orthonormality.

2.2 Fourier Series

Consider the Hilbert space

$$L^2([0, 2\pi]) := \{f : [0, 2\pi] \rightarrow \mathbb{C} \mid \|f\|_2 := \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt < \infty\}$$

endowed with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt, \quad f, g \in L^2([0, 2\pi]).$$

The interval $[0, 2\pi]$ is the natural choice here, as it leads to particularly clean formulas. The theory, however, extends to any bounded interval by a straightforward rescaling argument.

In physics and engineering, the space $L^2([0, 2\pi])$ is a model for the 2π -periodic functions with finite energy. A function $f \in L^2([0, 2\pi])$ is considered as a signal, and $\|f\|_2$ is commonly known as the energy of the signal.

By Euler's formula, the complex exponential $e^{int} = \cos(nt) + i \sin(nt)$ is a natural encoding of a pure oscillation of frequency n : its real and imaginary parts are precisely the two fundamental signals at that frequency. It is thus convenient to introduce the shorthand

$$e_n(t) := e^{int},$$

so that e_n stands for the pure n -frequency signal.

Lemma 2.7. *The system $\{e^{int}\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2([0, 2\pi])$.*

Therefore, we can write any $f \in L^2([0, 2\pi])$ as

$$f(t) = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle e_n(t). \tag{2.1}$$

Remark 2.8. For the general space $L^2([0, T])$, an orthonormal basis is given by $\{e^{i\frac{2\pi}{T}nt}\}_{n \in \mathbb{Z}}$.

Definition 2.9. Given $n \in \mathbb{Z}$, the n^{th} Fourier coefficient of a function $f \in L^2([0, 2\pi])$ is

$$\hat{f}(n) = \langle f, e_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt.$$

The series given in (2.1) can make sense for functions which are not necessarily in $L^2([0, 2\pi])$, as long as $\hat{f}(n)$ is well defined.

Definition 2.10. Let $f \in L^1([0, 2\pi])$. We define the Fourier series of f , denoted by $Sf(t)$, as

$$Sf(t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{int}.$$

For functions with additional regularity, other types of convergence may occur, such as pointwise convergence, uniform convergence, or almost everywhere convergence.

Identity (2.1) holds for $f \in L^2([0, 2\pi])$; that means we have convergence in the L^2 norm. One may wonder in which other ways the series converge. In fact, a famous theorem by L. Carleson shows that for $f \in L^2([0, 2\pi])$ the Fourier series converges pointwise a.e. (see [2], p. 135). We will not go any further than this about the convergence of the Fourier series because it is not the goal of this work.

Furthermore, from (2.1), we obtain the *Plancherel identity*, which is nothing other than Parseval's identity for $f \in L^2([0, 2\pi])$:

$$\|f\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2.$$

This shows that the map $f \mapsto (\hat{f}(n))_{n \in \mathbb{Z}}$ is an isometry from $L^2([0, 2\pi])$ into $\ell^2(\mathbb{Z})$, and in particular that the Fourier coefficients completely determine f in L^2 .

We now introduce the convolution operation, which plays a fundamental role in analysis and in many other areas, such as probability theory, signal processing, and physics, and will be essential for the formulation and proof of several results.

Definition 2.11. Given two functions $f, g \in L^1([0, 2\pi])$, extended as 2π -periodic functions on $\mathbb{R}$, their convolution is defined by

$$(f * g)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(x) g(t - x) dx.$$

We obtain interesting bounds on the convolution in the next theorem.

Theorem 2.12. *Let $f, g \in L^1[0, 2\pi]$, extended as 2π -periodic functions in $\mathbb{R}$. Then*

- (i) $f * g \in L^1[0, 2\pi]$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$
- (ii) For all $n \in \mathbb{Z}$, $\widehat{(f * g)}(n) = \hat{f}(n) \hat{g}(n)$.

2.3 Fourier integral

Consider the space of integrable functions in $\mathbb{R}$:

$$L^1(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{C} \mid \|f\|_1 = \int_{\mathbb{R}} |f(t)| dt < \infty\}$$

Definition 2.13. The *Fourier transform* of a function $f \in L^1(\mathbb{R})$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \cdot \xi} dx.$$

Just as in the discrete setting, the exponential $e^{2\pi i x \cdot \xi}$ represents a ξ -pure frequency signal, and the Fourier transform can be seen as the inner product $\langle f, e^{2\pi i \xi \cdot} \rangle$: a measure of how much of that frequency is contained in f . This is, however, only a heuristic interpretation, since the exponential $e^{2\pi i x \cdot \xi}$ belongs to neither $L^1(\mathbb{R})$ nor $L^2(\mathbb{R})$, and so the inner product is not literally well-defined.

We can observe, by taking absolute values, that the Fourier transform is bounded:

$$|\hat{f}(\xi)| \leq \|f\|_1.$$

In particular, $\hat{f}(\xi)$ is well defined. More can be said.

Lemma 2.14. (Riemann-Lebesgue) *If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is uniformly continuous and $\lim_{|\xi| \rightarrow \infty} |\hat{f}(\xi)| = 0$.*

If we consider the Banach space $C_0(\mathbb{R})$, by Lemma 2.14 we can consider the following map:

$$\mathcal{F} : L^1(\mathbb{R}) \rightarrow C_0(\mathbb{R})$$

where $\mathcal{F}$ is the operator that sends f to $\hat{f}$. This operator is commonly called *analysis*. On the other hand, in many cases, we can define the reconstruction operator $\hat{f} \mapsto f$, if it exists, called *synthesis*.

Let us recall how the Fourier transform behaves under the basic operations we will perform often.

Definition 2.15. Let $a \in \mathbb{R}$ and $\lambda > 0$. We define the following operators on $f \in L^1(\mathbb{R})$:

- **Translation:** $\tau_a f(t) = f(t - a)$,
- **Modulation:** $M_a f(t) = e^{2\pi i a t} f(t)$,
- **Dilation:** $D_\lambda f(t) = \frac{1}{\lambda} f\left(\frac{t}{\lambda}\right)$.

These operators preserve the L^1 -norm:

$$\|f\|_1 = \|\tau_a f\|_1 = \|M_a f\|_1 = \|D_\lambda f\|_1.$$

The Fourier transforms of these transformations are easily computed from the definition, yielding the following properties.

Proposition 2.16. Let $f, g \in L^1(\mathbb{R})$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathbb{R}$. We have the following properties:

- (i) *Fourier transform linearity:* $(\alpha f + \beta g)^\wedge(\xi) = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi)$, $\xi \in \mathbb{R}$.
- (ii) *Conjugation:* $(\bar{f})^\wedge(\xi) = \overline{\hat{f}(-\xi)}$, $\xi \in \mathbb{R}$.
- (iii) *Translation:* $(\tau_a f)^\wedge(\xi) = \hat{f}(\xi) e^{-2\pi i a \xi}$, $\xi \in \mathbb{R}$.
- (iv) *Modulation:* $(M_a f)^\wedge(\xi) = \tau_a \hat{f}(\xi) = \hat{f}(\xi - a)$, $\xi \in \mathbb{R}$.
- (v) *Dilation:* $(D_\lambda f)^\wedge(\xi) = \hat{f}(\lambda \xi)$, $\xi \in \mathbb{R}$.

The Fourier integral was first motivated by the study of the heat equation. The differentiation operator, which is difficult to work with, becomes significantly simpler under the Fourier transform: it turns into a multiplication operator. This fundamental property was precisely what made the Fourier transform such a powerful tool for solving the heat equation. The following result formalizes this statement.

Theorem 2.17. Let $f \in L^1(\mathbb{R})$. Then

- (i) If $f, f' \in L^1(\mathbb{R})$, then

$$\hat{f}'(\xi) = 2\pi i \xi \hat{f}(\xi), \quad \xi \in \mathbb{R}.$$

- (ii) If $tf \in L^1(\mathbb{R})$, then $\hat{f}$ is differentiable and

$$(\hat{f})'(\xi) = (-2\pi i t f)^\wedge(\xi), \quad \xi \in \mathbb{R}.$$

(iii) If $f, g \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} f(t)\hat{g}(t) dt = \int_{\mathbb{R}} \hat{f}(t)g(t) dt.$$

Let us now consider an important example, namely the Gaussian function. It will play a central role throughout this work, not only because of its remarkable properties in Fourier analysis, but also because of its optimal behavior in time-frequency localization.

Example 2.18. Consider the Gaussian $G(t) := e^{-\pi t^2}$, which is normalized so that $\|G\|_1 = 1$. If we try to obtain $\hat{G}(\xi)$ by its definition, we run into a complicated integral, so let's try another way, using the properties of Theorem 2.17.

Notice that G satisfies the following differential equation:

$$\begin{cases} G'(t) = -2\pi t G(t) \\ G(0) = 1. \end{cases}$$

Taking Fourier transforms on both sides, we get

$$\widehat{G'}(\xi) = (-2\pi t G(t))^\wedge(\xi)$$

and by applying (i) and (ii) of Theorem 2.17, we obtain

$$(2\pi i \xi) \hat{G}(\xi) = -(i \hat{G})'(\xi).$$

Since $\hat{G}(0) = \int_{\mathbb{R}} G(t) dt = 1$, we see that $\hat{G}$ is also a solution of the system above. Therefore, by uniqueness,

$$\hat{G}(\xi) = e^{-\pi \xi^2} = G(\xi).$$

We would like to see, as we saw in Fourier series, that we can reconstruct the signal f from the set of frequencies $\hat{f}(\xi)$. The following theorem shows when it can be done.

Theorem 2.19. (Inversion formula) Let $f \in L^1(\mathbb{R})$ be such that $\hat{f} \in L^1(\mathbb{R})$. Then

$$f(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i t \xi} d\xi \quad \text{a.e. } t \in \mathbb{R}.$$

Moreover, the equality holds everywhere on the set of continuity points of f .

Corollary 2.20. (Uniqueness theorem) If $f \in L^1(\mathbb{R})$ satisfies $\hat{f}(\xi) = 0$ a.e. $\xi \in \mathbb{R}$, then $f(t) = 0$ a.e. $t \in \mathbb{R}$.

Remark 2.21. Applying the inversion formula twice shows that for $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, $(\widehat{\hat{f}})(\xi) = f(-\xi)$ a.e. $\xi \in \mathbb{R}$.

2.4 Fourier transform in L^2

If we give up on the Fourier transform being pointwise defined everywhere, we can extend its definition to other spaces.

We would like to take advantage of the results seen for $L^1(\mathbb{R})$ by applying them to $L^2(\mathbb{R})$, which is a Hilbert space, something that will be convenient for the Fourier analysis. However, not all $L^2(\mathbb{R})$ functions are in $L^1(\mathbb{R})$, so we have to see first some results.

For functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have the following fundamental identity known as Plancherel's identity.

Theorem 2.22. *If $f \in L^1 \cap L^2(\mathbb{R})$ then $\|f\|_2 = \|\hat{f}\|_2$.*

2.4.1 Construction of the Fourier transform in L^2

We would like to drop the assumption $f \in L^1(\mathbb{R})$ and extend the Fourier transform to $L^2(\mathbb{R})$. The idea is to approximate f by a sequence of functions in $L^1 \cap L^2$ and pass to the limit.

Let $f \in L^2(\mathbb{R})$. Since $L^1 \cap L^2$ is dense in L^2 , there exists a sequence $f_n \in L^1 \cap L^2$ such that $\|f_n - f\|_2 \xrightarrow{n \rightarrow \infty} 0$; a canonical choice is $f_n = f\chi_{[-n,n]}$. For each f_n , the Fourier transform $\hat{f}_n$ is well defined. Moreover, by Theorem 2.22, the sequence $\{\hat{f}_n\}_n$ is also Cauchy in L^2 :

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2 \rightarrow 0.$$

Since $L^2(\mathbb{R})$ is complete, the limit exists in L^2 , and we define

$$\hat{f} := \lim_{n \rightarrow \infty} \hat{f}_n \quad \text{in } L^2(\mathbb{R}).$$

This definition is independent of the choice of the approximating sequence. Indeed, if $\{g_n\}_n \subset L^1 \cap L^2$ is another sequence converging to f in L^2 , then by the isometry of Theorem 2.22,

$$\|\hat{f}_n - \hat{g}_n\|_2 = \|f_n - g_n\|_2 \leq \|f_n - f\|_2 + \|f - g_n\|_2 \xrightarrow{n \rightarrow \infty} 0,$$

so both sequences yield the same limit in L^2 .

2.4.2 Main results in L^2

It is natural in signal theory to work in the space $L^2(\mathbb{R})$, since functions $f \in L^2(\mathbb{R})$ model signals with finite energy.

Theorem 2.23. For $f \in L^2(\mathbb{R})$

$$(i) \hat{f}(\xi) = \lim_{n \rightarrow \infty} \int_{-n}^n f(t) e^{-2\pi i t \xi} dt$$

$$(ii) \|f\|_2 = \|\hat{f}\|_2 \text{ (Plancherel's identity)}$$

(iii) If $g \in L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(t) \hat{g}(t) dt = \int_{\mathbb{R}} \hat{f}(t) g(t) dt$$

and

$$\int_{\mathbb{R}} f(t) \overline{g(t)} dt = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi$$

With the above definition of the Fourier transform in $L^2(\mathbb{R})$, the inversion formula also holds, but in the L^2 -sense.

Theorem 2.24. Let $f \in L^2(\mathbb{R})$. Then $\widehat{\hat{f}}(-t) = f(t)$ as an identity in $L^2(\mathbb{R})$. Equivalently, the equality holds for almost every $t \in \mathbb{R}$. In particular,

$$f(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi t} d\xi = \lim_{n \rightarrow \infty} \int_{-n}^n \hat{f}(\xi) e^{2\pi i \xi t} d\xi,$$

as a limit in $L^2(\mathbb{R})$.

Moreover, the equality holds for all $t \in \mathbb{R}$ where f is continuous.

Remark 2.25. Observe that, unlike the L^1 case, there is no need to have an extra hypothesis that requires $\hat{f} \in L^1$. In the L^2 case, it is enough to assume that $f \in L^2(\mathbb{R})$. Indeed, by Theorem 2.23, if $f \in L^2(\mathbb{R})$, then automatically $\hat{f} \in L^2(\mathbb{R})$, and the inversion formula holds in the L^2 sense.

Chapter 3

Introduction to Signal Theory: Shannon's Theorem and the Curse of Uncertainty

Since Fourier introduced his ideas in the early 19th century, the Fourier transform has proven to be far more than a theoretical construct. It is a powerful analytical tool with deep implications across mathematics, physics and engineering. In this chapter, we discuss two classical results that illustrate this: the *Heisenberg uncertainty principle* and the *Shannon sampling theorem*.

3.1 Heisenberg's uncertainty principle

A natural question is whether a signal can be simultaneously well localized in time and in frequency. The Fourier transform imposes a fundamental limitation: the more concentrated f is in time, the more spread out $\hat{f}$ must be in frequency, and vice versa. The following theorem makes this trade-off precise.

Theorem 3.1. (Heisenberg uncertainty principle) ([8, Theorem 8.44]) *Let $f \in L^2(\mathbb{R})$ and let $a, b \in \mathbb{R}$. Then*

$$\left(\int_{\mathbb{R}} (t - a)^2 |f(t)|^2 dt \right)^{1/2} \left(\int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \geq \frac{\|f\|_2^2}{4\pi}.$$

Moreover, the equality holds if and only if f is a Gaussian of the form $f(t) = ce^{2\pi ibt} e^{-\gamma(t-a)^2}$, for some $c \in \mathbb{C}$ and $\gamma > 0$.

In quantum mechanics, $f(t)$ represents the wave function of a system with finite energy. The position and momentum of the particle are described by corresponding operators, and applying Plancherel to these reveals a fundamental

constraint: the more precisely the position of a particle is known, the less precisely its momentum can be determined, and vice versa. This impossibility is one of the most celebrated results in physics.

With this interpretation, the integral $\int_{\mathbb{R}} (t-a)^2 |f(t)|^2 dt$ is minimized when a is the mean of the particle's position; in that case, the integral represents the variance of the position. Therefore, Heisenberg's uncertainty principle states that the variance of position and momentum cannot both be arbitrarily small, but instead satisfy a non-trivial lower bound.

Proof. Without loss of generality, we assume $a = b = 0$, which is justified by the properties of the Fourier transform under translations and modulations. We can also assume that $tf(t), \xi \hat{f}(\xi) \in L^2(\mathbb{R})$, since otherwise the inequality is trivially satisfied and there is nothing to prove. Under these assumptions, we claim that $f, \hat{f} \in L^1(\mathbb{R})$. Indeed, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_{\mathbb{R}} |f(t)| dt &= \int_{\mathbb{R}} (1+|t|) |f(t)| \frac{dt}{1+|t|} \\ &\leq \left(\int_{\mathbb{R}} (1+|t|)^2 |f(t)|^2 dt \right)^{1/2} \left(\int_{\mathbb{R}} \frac{dt}{(1+|t|)^2} \right)^{1/2}, \end{aligned}$$

where both factors on the right are finite by hypothesis. The same argument applies to $\hat{f}$. Since $\hat{f} \in L^1(\mathbb{R})$, the Fourier inversion formula holds pointwise, and the Riemann-Lebesgue lemma applied to $\hat{f}$ yields that f is continuous and satisfies

$$\lim_{|t| \rightarrow \infty} f(t) = 0.$$

Hence $f \in C_0(\mathbb{R})$.

By the assumption that $\xi \hat{f} \in L^2(\mathbb{R})$, we get, by the differentiation property of the Fourier transform, that $\hat{f}'(\xi) = (2\pi i \xi) \hat{f}(\xi) \in L^2(\mathbb{R})$. We can apply Plancherel to deduce that $f' \in L^2(\mathbb{R})$, since

$$(|f|^2)' = (f \cdot \bar{f})' = f' \cdot \bar{f} + \overline{f' \cdot \bar{f}} = 2 \operatorname{Re}(f \cdot f').$$

Given $c, d \in \mathbb{R}$, such $c < d$, we have, integrating by parts

$$\begin{aligned} 2 \operatorname{Re} \left(\int_c^d t f(t) \overline{f'(t)} dt \right) &= \int_c^d t 2 \operatorname{Re} \left(f(t) \overline{f'(t)} \right) dt \\ &= [t |f(t)|^2]_c^d - \int_c^d |f(t)|^2 dt. \end{aligned} \tag{3.1}$$

It remains to show that the boundary term vanishes as $c \rightarrow -\infty$ and $d \rightarrow +\infty$. Since $tf(t) \in L^2(\mathbb{R})$, we have $t^2 |f(t)|^2 \in L^1(\mathbb{R})$, so there exist sequences $c_n \searrow -\infty$

and $d_n \nearrow +\infty$ along which

$$c_n^2 |f(c_n)|^2 \rightarrow 0 \quad \text{and} \quad d_n^2 |f(d_n)|^2 \rightarrow 0.$$

Since $|t| |f(t)|^2 \leq \frac{t^2 |f(t)|^2}{|t|} \rightarrow 0$, we conclude that $\left[t |f(t)|^2 \right]_{c_n}^{d_n} \rightarrow 0$. Taking the limit and substituting $f'(t) = [(2\pi i \xi) \hat{f}]^\wedge(t)$ on (3.1), we arrive at

$$\int_{\mathbb{R}} |f(t)|^2 dt = -2 \operatorname{Re} \int_{\mathbb{R}} t f(t) \overline{[(2\pi i \xi) \hat{f}]^\wedge(t)} dt = -4\pi \operatorname{Im} \int_{\mathbb{R}} t f(t) \overline{(\xi \hat{f})^\wedge(t)} dt.$$

This proves that the integral

$$\int_{\mathbb{R}} t f(t) \overline{(\xi \hat{f})^\wedge(t)} dt$$

is purely imaginary. Hence, squaring both sides and applying the Cauchy-Schwarz inequality together with Plancherel's identity, we obtain

$$\begin{aligned} \|f\|_2^4 &\leq 16\pi^2 \left| \int_{\mathbb{R}} t f(t) \overline{(\xi \hat{f})^\wedge(t)} dt \right|^2 \\ &\leq 16\pi^2 \left(\int_{\mathbb{R}} t^2 |f(t)|^2 dt \right) \left(\int_{\mathbb{R}} |(\xi \hat{f})^\wedge(t)|^2 dt \right) \\ &= 16\pi^2 \left(\int_{\mathbb{R}} t^2 |f(t)|^2 dt \right) \left(\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi \right). \end{aligned}$$

Notice that the Cauchy-Schwarz inequality holds as an identity when both terms are dependent. In our case, that means that when $tf(t)$ and $f'(t)$ are linearly dependent, we have an identity. This leads to the ODE $tf(t) = \gamma f'(t)$ for some $\gamma \in \mathbb{R}$, whose solutions are the Gaussians $f(t) = Ce^{\gamma t^2}$, for some constant $C \in \mathbb{R}$. The condition $\gamma < 0$ is necessary to ensure $f \in L^2(\mathbb{R})$. This identifies the Gaussian functions as the unique optimizers of the Heisenberg uncertainty principle, further illustrating the central role they play in Fourier analysis. $\square$

3.2 Shannon sampling theorem

In digital signal processing, any continuous signal must be discretized into a sequence of samples before it can be stored or transmitted. A fundamental question then arises: how many samples per unit time are needed to faithfully represent the original signal? The Shannon Sampling Theorem answers this precisely, giving sufficient conditions under which no information is lost and the original signal can be perfectly recovered.

Theorem 3.2. (The Kotelnikov-Shannon-Whittaker theorem) ([8, Theorem 8.41])

Let $f \in L^2(\mathbb{R})$ with $\text{supp } \hat{f} \subset [-\tau, \tau]$. Then f can be completely recovered from its samples $\{f(k/2\tau)\}_{k \in \mathbb{Z}}$ through the cardinal series

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2\tau}\right) \text{sinc}\left[2\tau\left(t - \frac{k}{2\tau}\right)\right], \quad (3.2)$$

where $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. Moreover,

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(t)|^2 dt = \frac{1}{2\tau} \sum_{k \in \mathbb{Z}} \left|f\left(\frac{k}{2\tau}\right)\right|^2. \quad (3.3)$$

In this context, $f(t)$ is a continuous signal of finite energy whose Fourier transform is supported on $[-\tau, \tau]$, meaning it contains no frequencies above τ . Such a signal is called band-limited, a natural assumption in practice. The quantity 2τ is known as the Nyquist rate: sampling below this threshold causes aliasing, a distortion where high frequencies are misrepresented as lower ones, and perfect reconstruction is no longer possible.

The human ear, for instance, is sensitive only to frequencies between 20 Hz and 20 kHz. Since the Nyquist rate requires sampling at least twice the highest frequency present, faithfully capturing this range demands a sampling rate of at least 40 kHz. The CD standard uses 44.1 kHz, leaving a small margin to account for imperfections in practice.

Although the assumption of band-limitedness is mathematically convenient, it has some counterintuitive consequences. Indeed, if $f \in L^2(\mathbb{R})$ is band-limited, then it can be extended to an entire function on $\mathbb{C}$. In particular, such a function can only vanish on a discrete set with no accumulation points, unless it is identically zero. Thus, a non-trivial band-limited signal cannot vanish on any interval and must, in a sense, be non-zero for all time. This stands in contrast with physical intuition, since real-world signals, such as a piece of music, have a beginning and an end. The resolution of this apparent paradox is that band-limitedness is an idealization: in practice, signals are only approximately band-limited, and any perfect reconstruction result necessarily relies on assumptions that cannot be exactly satisfied in reality.

Proof. We see by Plancherel's identity that $\hat{f} \in L^2[-\tau, \tau]$

$$\|\hat{f}\|_2^2 = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{-\tau}^{\tau} |\hat{f}(\xi)|^2 d\xi = \|f\|_2^2 < +\infty.$$

Since $\{e^{i\frac{\pi}{\tau}k\xi}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of $L^2[-\tau, \tau]$ (see Remark 2.8), we can write

$$\hat{f}(\xi) = \sum_{k \in \mathbb{Z}} c_k e^{i\frac{\pi}{\tau}k\xi},$$

where, by the inversion formula,

$$c_k = \langle \hat{f}, e^{i\frac{\pi}{\tau}k\zeta} \rangle = \frac{1}{2\tau} \int_{-\tau}^{\tau} \hat{f}(\zeta) e^{-i\frac{\pi}{\tau}k\zeta} d\zeta = \frac{1}{2\tau} \int_{\mathbb{R}} \hat{f}(\zeta) e^{-2\pi i \frac{k}{2\tau}\zeta} d\zeta = \frac{1}{2\tau} f\left(-\frac{k}{2\tau}\right).$$

Then, again, by the inversion formula,

$$\begin{aligned} f(t) &= \int_{\mathbb{R}} \hat{f}(\zeta) e^{2\pi i \zeta t} d\zeta = \int_{-\tau}^{\tau} \hat{f}(\zeta) e^{2\pi i \zeta t} d\zeta = \sum_{k \in \mathbb{Z}} c_k \int_{-\tau}^{\tau} e^{2\pi i \zeta \left(\frac{k}{2\tau} + t\right)} d\zeta \\ &= \sum_{k \in \mathbb{Z}} \frac{1}{2\tau} f\left(-\frac{k}{2\tau}\right) \int_{-\tau}^{\tau} e^{2\pi i \zeta \left(\frac{k}{2\tau} + t\right)} d\zeta. \end{aligned}$$

Since

$$\begin{aligned} \int_{-\tau}^{\tau} e^{2\pi i \zeta \left(\frac{k}{2\tau} + t\right)} d\zeta &= \left[\frac{e^{2\pi i \zeta \left(\frac{k}{2\tau} + t\right)}}{2\pi i \left(\frac{k}{2\tau} + t\right)} \right]_{\zeta=-\tau}^{\zeta=\tau} \\ &= \frac{e^{2\pi i \tau \left(\frac{k}{2\tau} + t\right)} - e^{-2\pi i \tau \left(\frac{k}{2\tau} + t\right)}}{2\pi i \left(\frac{k}{2\tau} + t\right)} \\ &= \frac{\sin\left(2\pi \tau \left(\frac{k}{2\tau} + t\right)\right)}{\pi \left(\frac{k}{2\tau} + t\right)} = 2\tau \operatorname{sinc}\left(2\tau \left(\frac{k}{2\tau} + t\right)\right), \end{aligned}$$

we get

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(-\frac{k}{2\tau}\right) \operatorname{sinc}\left(2\tau \left(\frac{k}{2\tau} + t\right)\right),$$

which after replacing k by $-k$ gives (3.2). In order to prove (3.3), observe that by Plancherel's identity for Fourier series

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |c_k|^2 &= \frac{1}{(2\tau)^2} \sum_{k \in \mathbb{Z}} \left| f\left(-\frac{k}{2\tau}\right) \right|^2 = \frac{1}{(2\tau)^2} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k}{2\tau}\right) \right|^2 \\ &= \|\hat{f}\|_{L^2[-\tau, \tau]}^2 = \frac{1}{2\tau} \int_{-\tau}^{\tau} |\hat{f}(\zeta)|^2 d\zeta = \frac{1}{2\tau} \|\hat{f}\|_2^2, \end{aligned}$$

where in the second equality we used that $k \mapsto -k$ is a bijection on $\mathbb{Z}$.

Thus, by Plancherel (for $L^2(\mathbb{R})$)

$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \frac{1}{2\tau} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k}{2\tau}\right) \right|^2,$$

as stated. □

Chapter 4

The short-time Fourier transform

The Fourier transform provides a global picture of a signal's frequency content: for each frequency, it integrates the signal over the entire time axis, yielding a single coefficient that captures how much of that frequency is present globally. This representation discards all temporal information; we cannot tell when a particular frequency occurs. In many applications, however, this is precisely what we care about: not just which frequencies are present, but how the frequency content of a signal evolves over time.

A simple example illustrates this loss of temporal information. Consider a signal f that has its content only on a bounded interval $[-a, a]$, modeled by the characteristic function $f = \chi_{[-a, a]}$. Its Fourier transform is

$$\hat{f}(\xi) = \int_{-a}^a e^{-2\pi i \xi s} ds = \frac{\sin(2\pi \xi a)}{\pi \xi} = 2a \operatorname{sinc}(2\xi a),$$

which decays only as $|\xi|^{-1}$, a slow decay that spreads energy across the entire frequency axis. All we are left with is a global average: we can see which frequencies are present, but have lost any trace of when they occur.

A natural first attempt is to localize the analysis by windowing: rather than integrating over all time, we multiply the signal by a compactly supported window function centered at some time t_0 , and then compute the Fourier transform of the result. This is the central idea behind the Short-Time Fourier Transform (STFT). By sliding the window across time, we obtain a two-dimensional representation that describes, at each time instant, the local frequency content of the signal.

One might hope to make the window arbitrarily small to achieve precise localization in both time and frequency simultaneously. This turns out to be fun-

damentally impossible. The uncertainty principle imposes a hard lower bound on the product of the time spread and the frequency spread of any windowed analysis: the more precisely we localize in time, the more spread out the representation becomes in frequency, and vice versa.

With this motivation in mind, we proceed to introduce the Short-Time Fourier Transform in both its continuous and discrete-time formulations, and we conclude the chapter with a celebrated result that makes the uncertainty principle precise in the context of signal expansions: the Balian-Low theorem.

4.1 The short-time Fourier transform (STFT)

Definition 4.1. Let $f, g \in L^2(\mathbb{R})$ and let $g \neq 0$ be a fixed function, called *window function*. The *short-time Fourier transform* (STFT) of f with respect to g is defined as

$$V_g f(t, \xi) = \int_{\mathbb{R}} f(s) \overline{g(s-t)} e^{-2\pi i \xi s} ds,$$

that is, the Fourier transform of the function $f \overline{\tau_t g}$.

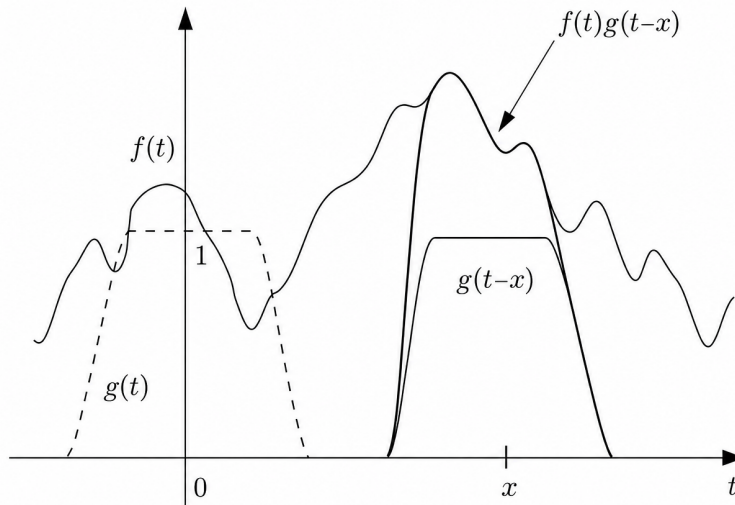


Figure 4.1: The short-time Fourier transform (see [5], p. 39).

Typically, the function g is chosen to be compactly supported and centered around 0. In that case, for each t , the STFT $V_g f(t, \cdot)$ is the Fourier transform of a

segment of f centered around t .

It is often convenient to think in terms of time-frequency shifts in the complex plane. Let $z = t + i\zeta \in \mathbb{C}$ and define

$$\pi(z)g(s) = g(s - t)e^{2\pi i\zeta s} = M_\zeta \tau_t g(s).$$

With this definition,

$$V_g f(t, \zeta) = \int_{\mathbb{R}} f(s) \overline{\pi(z)g(s)} ds = \langle f, M_\zeta \tau_t g \rangle. \quad (4.1)$$

An equivalent form of the STFT is the *fundamental identity of time-frequency analysis*:

$$V_g f(t, \zeta) = e^{-2\pi i t \zeta} V_{\hat{g}} \hat{f}(\zeta, -t), \quad t, \zeta \in \mathbb{R}.$$

It combines both f and $\hat{f}$ into a joint time-frequency representation. The identity tells us that the STFT of f with window g is essentially the same as the STFT of $\hat{f}$ with window $\hat{g}$, up to a phase factor $e^{-2\pi i t \zeta}$ and a swap of variables $(t, \zeta) \mapsto (\zeta, -t)$. In other words, time and frequency play symmetric roles in the STFT, which is not obvious from the definition. The identity follows directly from the properties of the Fourier transform and Plancherel's theorem.

We now present another well-known property of the STFT.

Proposition 4.2. (Covariance property) *Whenever $V_g f$ is defined, we have*

$$V_g(\tau_u M_\eta f)(t, \zeta) = e^{-2\pi i u \zeta} V_g f(t - u, \zeta - \eta), \quad t, \zeta, u, \eta \in \mathbb{R}.$$

The result is obtained by inserting $\tau_t M_\zeta f$ into the definition of the STFT and performing a change of variables.

Example 4.3. When thinking about taking short segments as windows one could consider to simply truncate the signal over the interval of interest using a characteristic function. Let $a > 0$, and consider the window $g = \frac{1}{2a} \chi_{[-a, a]}$. The STFT of a function f with this window is given by

$$V_g f(t, \zeta) = \int_{\mathbb{R}} f(s) \frac{1}{2a} \chi_{[-a, a]}(s - t) e^{-2\pi i \zeta s} ds = \frac{1}{2a} \int_{t-a}^{t+a} f(s) e^{-2\pi i \zeta s} ds,$$

which is the ζ -frequency content of f over the time interval $[t - a, t + a]$, the kind of joint time-frequency information we are after. However, as we saw above, the sharp discontinuities of $\chi_{[-a, a]}$ at $\pm a$ spread energy across all frequencies and introduce artificial oscillations that corrupt the analysis. Therefore, smoother functions, like Schwartz functions or the Gaussian, are better suited in practice.

4.2 Orthogonality relations and inversion formula

Due to its close relationship with the Fourier transform, the short-time Fourier transform shares some analogous properties. In particular, the following theorem on inner products of STFTs can be viewed as a counterpart to Plancherel's theorem and constitutes a fundamental result in the theory.

Theorem 4.4. (Orthogonality theorem) *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then $V_{g_j} f_j \in L^2(\mathbb{R}^2)$ for $j = 1, 2$ and*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}.$$

In particular, taking $g_1 = g_2 = g$ for a window with $\|g\|_2 = 1$,

$$\langle V_g f_1, V_g f_2 \rangle_{L^2(\mathbb{R}^2)} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R})}$$

and $V_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$ is an isometry.

Proof. Assume that $g_1, g_2 \in (L^1 \cap L^\infty)(\mathbb{R}) \subset L^2(\mathbb{R})$. Notice that $f_j \cdot \tau_t \bar{g}_j \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$, $j = 1, 2$, and that

$$V_g f(t, \xi) = \int_{\mathbb{R}} f(s) \overline{g(s-t)} e^{-2\pi i s \xi} ds = (f \cdot \tau_t \bar{g})^\wedge(\xi).$$

Then, by Plancherel on the ξ -integral,

$$\begin{aligned} \langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} (f_1 \cdot \tau_t \bar{g}_1)^\wedge(\xi) \overline{(f_2 \cdot \tau_t \bar{g}_2)^\wedge(\xi)} d\xi dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(s) \overline{g_1(s-t)} \overline{f_2(s)} g_2(s-t) ds dt \\ &= \int_{\mathbb{R}} f_1(s) \overline{f_2(s)} \left(\int_{\mathbb{R}} \overline{g_1(s-t)} g_2(s-t) dt \right) ds \\ &= \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \end{aligned}$$

The extension to a general $g_j \in L^2(\mathbb{R})$ is done by a standard density argument (see [5], p. 329). Fixing a window $g_1 \in (L^1 \cap L^\infty)(\mathbb{R})$, the map $g_2 \mapsto \langle V_{g_1} f_1, V_{g_2} f_2 \rangle$ is a linear functional coinciding with $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on the dense subspace $L^1 \cap L^\infty$. It is therefore bounded and extends to all $g_2 \in L^2(\mathbb{R})$. Analogously, for arbitrary f_1, f_2 and $g_2 \in L^2(\mathbb{R})$, the conjugate linear functional $g_1 \mapsto \langle V_{g_1} f_1, V_{g_2} f_2 \rangle$ equals $\langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}$ on $L^1 \cap L^\infty$ and extends to all L^2 .

The orthogonality relations are therefore established for all $f_j, g_j \in L^2(\mathbb{R})$. $\square$

It follows from this result that f is completely determined by $V_g f$. Moreover, the implication

$$V_g f(t, \xi) = \langle f, M_\xi \tau_t g \rangle = 0 \quad \forall t, \xi \in \mathbb{R} \implies f = 0$$

is equivalent to saying that for each $g \in L^2(\mathbb{R})$ the set $\{M_\xi \tau_t g : t, \xi \in \mathbb{R}\}$ spans a dense subspace in $L^2(\mathbb{R})$. However, this still leaves open the question of how f can be recovered from $V_g f$.

For the formulation of the inversion formula, we briefly recall the notion of a vector-valued integral. Let B be a Banach space and let $g : \mathbb{R}^d \rightarrow B$ be a function taking values in B . An expression of the form

$$\int_{\mathbb{R}^d} g(x) dx$$

is understood in the weak sense: an element $F \in B$ has a representation

$$F = \int_{\mathbb{R}^d} g(x) dx$$

if

$$\langle F, h \rangle = \int_{\mathbb{R}^d} \langle g(x), h \rangle dx$$

for every $h \in B^*$, whenever the scalar integral on the right-hand side is well defined.

In the particular case considered below, the Banach space will be $B = L^2(\mathbb{R})$, and the inversion formula for the STFT will therefore be interpreted in this weak sense.

Theorem 4.5. (Inversion formula for the STFT) *Let $g, \gamma \in L^2(\mathbb{R})$ with $\langle g, \gamma \rangle \neq 0$. Then, for all $f \in L^2(\mathbb{R})$,*

$$f(t) = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) M_\xi \tau_s \gamma(t) d\xi ds, \quad t \in \mathbb{R}. \quad (4.2)$$

In particular, if $g = \gamma$ and $\|g\|_2 = 1$,

$$f(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) M_\xi \tau_s g(t) d\xi ds, \quad t \in \mathbb{R}.$$

Proof. Since $V_g f \in L^2(\mathbb{R}^2)$, the function

$$\tilde{f}(t) := \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) M_\xi \tau_s \gamma(t) d\xi ds, \quad t \in \mathbb{R},$$

is well-defined in $L^2(\mathbb{R})$, in the sense that for all $h \in L^2(\mathbb{R})$, the product

$$\langle \tilde{f}, h \rangle = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) \langle M_{\xi} \tau_s \gamma, h \rangle d\xi ds \quad (4.3)$$

is finite. As a consequence of the Cauchy-Schwarz inequality and the orthogonality theorem we obtain the following bound:

$$\begin{aligned} |\langle \tilde{f}, h \rangle| &= \left| \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) V_{\gamma} h(s, \xi) d\xi ds \right| \\ &\leq \frac{1}{|\langle \gamma, g \rangle|} \|V_g f\|_2 \|V_{\gamma} h\|_2 = \frac{1}{|\langle \gamma, g \rangle|} \|f\|_2 \|g\|_2 \|h\|_2 \|\gamma\|_2. \end{aligned}$$

Therefore, by the Riesz representation theorem (see [7], p. 188), $\tilde{f}$ is given by a function in $L^2(\mathbb{R})$. We shall be done as soon as we show that this function is f , that is, that

$$\langle \tilde{f}, h \rangle = \langle f, h \rangle \quad \forall h \in L^2(\mathbb{R}).$$

From (4.3) and the orthogonality theorem we have

$$\begin{aligned} \langle \tilde{f}, h \rangle &= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) \langle h, M_{\xi} \tau_s \gamma \rangle d\xi ds \\ &= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} V_g f(s, \xi) V_{\gamma} h(s, \xi) d\xi ds \\ &= \frac{1}{\langle \gamma, g \rangle} \langle V_g f, V_{\gamma} h \rangle \\ &= \frac{1}{\langle \gamma, g \rangle} \langle f, h \rangle \langle g, \gamma \rangle = \langle f, h \rangle, \end{aligned}$$

as desired. $\square$

Remark 4.6. The inversion formula shows that f can be expressed as a continuous superposition of time-frequency shifts with the STFT as a weight function. In this way, (4.2) is similar to the inversion formula for the Fourier transform, that is, $f(t) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi t} d\xi$. Nonetheless, in the Fourier transform case, the elementary functions are exponentials $e^{2\pi i t \xi}$, which are not in $L^2(\mathbb{R})$, while in the STFT case, the elementary functions $M_{\xi} \tau_t \gamma$ are particularly nice functions in $L^2(\mathbb{R})$. This is desirable for two reasons: first, the STFT coefficients $V_g f(t, \xi) = \langle f, M_{\xi} \tau_t g \rangle$ are well-defined as L^2 inner products, without appealing to distribution theory; second, the inversion formula is a norm-convergent integral in $L^2(\mathbb{R})$, giving the reconstruction of f a precise and stable meaning.

The inversion formula theorem gives us a clue of what kind of time-frequency analysis is possible despite the uncertainty principle. Suppose that g is concentrated on $T \subset \mathbb{R}$, meaning that the integral of $|g|^2$ outside of T is very small,

and $\hat{g}$ is concentrated on $\Omega \subset \mathbb{R}$. Then, $M_{\xi}\tau_t g$ is concentrated on $t + T$ and its Fourier transform on $\xi + \Omega$. Thus $M_{\xi}\tau_t g$ occupies the cell $(t + T) \times (\xi + \Omega)$ in the time-frequency plane, and the size of $V_g f(t, \xi)$ measures the contribution of this time-frequency atom in the decomposition of f . The uncertainty principle limits the possible time-frequency concentration to $|T||\Omega| \gtrsim 1$.

We can now understand the role played by the window in the properties of the STFT. Good time resolution requires a window with small support, but this comes at the price of a poor frequency resolution due to the uncertainty principle. In the same way, good frequency resolution by means of a band-limited window implies poor resolution in time.

These brief remarks demonstrate the influence of the window on the time-frequency properties of the STFT. In practice, one chooses a window such that both g and $\hat{g}$ decay rapidly, in order to achieve as much time-frequency concentration as possible. The choice of the Gaussian function as a window is thus canonical.

In summary, the time-frequency analysis of a signal f with a normalized window g consists of three main steps:

1. *Analysis*: Given the signal f , we compute its STFT with respect to a suitable window g , that is, $f \mapsto V_g f$.
2. *Processing*: $V_g f(t, \xi)$ is modified to obtain a new function $F(t, \xi)$ through operations such as truncation, separation of signal components, compression, among others.
3. *Synthesis*: The processed signal is reconstructed using the modified inversion formula

$$\tilde{f}(t) = \iint_{\mathbb{R}^2} F(s, \xi) M_{\xi}\tau_s g(t) ds d\xi$$

with respect to an appropriate synthesis window g . Note that different windows may be used for analysis and synthesis.

4.3 Space discretization

So far, we have discussed continuous time-frequency analysis only. In practice, however, this is not really useful. In applications, one would like to work with a finite or countable set of samples to recover a signal; let's see if that is truly possible or not.

Consider again the inversion formula (4.2) for the STFT. This is a continuous approach to f with respect to the uncountable system of elementary functions $\{M_{\xi}\tau_t g : (t, \xi) \in \mathbb{R}^2\}$. Nonetheless, as $L^2(\mathbb{R})$ is a separable Hilbert space, it admits a countable orthonormal basis. Then, a series expansion with respect to

a countable subset of time-frequency shifts should be enough to recover every $f \in L^2(\mathbb{R})$, in the spirit of Shannon's formula for band-limited functions.

Let's begin with a simple case. We choose the characteristic function $g(t) = \chi_{[-1/2, 1/2)}(t)$ as window to discretize the space. Since $g(t - n) = \chi_{[n-1/2, n+1/2)}$, we can split our continuous signal f in little segments the following way:

$$f(t) = \sum_{n \in \mathbb{Z}} f(t)g(t - n), \quad t \in \mathbb{R}. \quad (4.4)$$

Since $f \in L^2(\mathbb{R})$, each localized piece is also in $L^2(\mathbb{R})$, so we also have $f \in L^2([n - 1/2, n + 1/2))$ for all $n \in \mathbb{Z}$. Since we are now working on intervals of length 1, we use the orthonormal basis $\{e^{2\pi ikt}\}_{k \in \mathbb{Z}}$ of $L^2([0, 1])$ (translated to the corresponding interval), rather than the basis $\{e^{ikt}\}_{k \in \mathbb{Z}}$ of $L^2([0, 2\pi])$. Therefore, we can write f as a Fourier series

$$f(t) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi ikt}, \quad t \in [n - 1/2, n + 1/2],$$

where

$$c_k = \langle f, e^{2\pi ikt} \rangle = \int_{n-1/2}^{n+1/2} f(s) e^{-2\pi iks} ds = \int_{\mathbb{R}} f(s) g(s - n) e^{-2\pi iks} ds = V_g f(n, k).$$

Thus, on $[n - 1/2, n + 1/2]$ we have the representation

$$f(t) = \sum_{k \in \mathbb{Z}} V_g f(n, k) e^{2\pi ikt},$$

and, going back to (4.4)

$$f(t) = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} V_g f(n, k) e^{2\pi ikt} g(t - n) = \sum_{n, k \in \mathbb{Z}} V_g f(n, k) M_k \tau_n g(t). \quad (4.5)$$

We can summarize all this in the following proposition.

Proposition 4.7. *Let $g = \chi_{[-\frac{1}{2}, \frac{1}{2})}$. The system $\{M_k \tau_n g\}_{n, k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. In particular, for all $f \in L^2(\mathbb{R})$,*

$$f = \sum_{n, k \in \mathbb{Z}} V_g f(n, k) M_k \tau_n g = \sum_{n, k \in \mathbb{Z}} \langle f, M_k \tau_n g \rangle M_k \tau_n g,$$

and

$$\|f\|_2^2 = \sum_{n, k \in \mathbb{Z}} |V_g f(n, k)|^2.$$

Proof. It is clear that $\{M_k \tau_n g\}_{n,k \in \mathbb{Z}}$ is an orthogonal system. To begin with, $M_k \tau_n g$ is just a translation and a modulation of an L^2 function, so it belongs to $L^2(\mathbb{R})$ and $\|M_k \tau_n g\|_2 = \|g\|_2 = 1$. The orthogonality is also clear. For $n \neq m$, the supports of $M_k \tau_n g$ and $M_j \tau_m g$ are disjoint, hence $\langle M_k \tau_n g, M_j \tau_m g \rangle = 0$. Also, for $n = m$,

$$\langle M_k \tau_n g, M_j \tau_n g \rangle = \int_{n-1/2}^{n+1/2} e^{2\pi i(k-j)t} dt = \delta_{jk}.$$

On the other hand, we have seen in (4.5) that $\{M_k \tau_n g\}_{n,k \in \mathbb{Z}}$ is a generating system. Finally, by Parseval on each interval $[n - 1/2, n + 1/2]$,

$$\|f\|_2^2 = \sum_{n \in \mathbb{Z}} \int_{n-1/2}^{n+1/2} |f(s)|^2 ds = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |V_g f(n, k)|^2.$$

□

4.4 Balian-Low Theorem

Given the previous result, a natural question comes up to mind: can we obtain an orthonormal basis using a more suitable window than the characteristic function, which is far from optimal? As the Heisenberg uncertainty principle suggests, the Gaussian is the optimizer of the time-frequency concentration inequality, and so it is the first candidate one might try. The following theorem, however, dashes any hope of constructing such a basis with a well-localized window.

Theorem 4.8. (Balian-Low) ([1, Theorem 1.1]) *Let $g \in L^2(\mathbb{R})$ be a window function. Discretize $M_{\xi} \tau_t g$ on a lattice $a\mathbb{Z} \times b\mathbb{Z}$, where $a, b > 0$ and $ab = 1$ ($a = b = 1$ is the canonical case).*

If the system $\{M_{bk} \tau_{an} g\}_{n,k \in \mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{R})$, then

$$\left(\int_{\mathbb{R}} t^2 |g(t)|^2 dt \right) \left(\int_{\mathbb{R}} \xi^2 |\hat{g}(\xi)|^2 d\xi \right) = +\infty.$$

Remark 4.9. The Balian-Low theorem is a particular manifestation of the Heisenberg uncertainty principle: if $\{M_{bk} \tau_{an} g\}_{n,k \in \mathbb{Z}}$ is a basis of $L^2(\mathbb{R})$, the window g cannot be simultaneously concentrated in time and frequency.

As we have shown in the introduction of the chapter, if we choose the characteristic function

$$g(t) = \chi_{[-1/2, 1/2)}(t),$$

then its Fourier transform is given by

$$\hat{g}(\xi) = \int_{-1/2}^{1/2} e^{-2\pi i \xi t} dt = \frac{\sin(\pi \xi)}{\pi \xi} = \text{sinc}(\xi).$$

Hence,

$$\int_{\mathbb{R}} |\xi| |\hat{g}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\xi| \left| \frac{\sin(\pi\xi)}{\pi\xi} \right|^2 d\xi = \infty,$$

so the second integral appearing in the Balian-Low theorem diverges. This is consistent with the fact that the characteristic window does generate a basis for $L^2(\mathbb{R})$.

The Gaussian, on the other hand, has the remarkable property that its Fourier transform is again a Gaussian. In particular, if

$$\varphi(t) = 2^{1/4} e^{-\pi t^2},$$

then

$$\hat{\varphi}(\xi) = \varphi(\xi),$$

and consequently both integrals appearing in the Balian-Low theorem are finite:

$$\int_{\mathbb{R}} |t|^2 |\varphi(t)|^2 dt < \infty, \quad \int_{\mathbb{R}} |\xi|^2 |\hat{\varphi}(\xi)|^2 d\xi < \infty.$$

Therefore, the theorem rules out the possibility of constructing a basis for $L^2(\mathbb{R})$ using a Gaussian window.

Nevertheless, the impossibility of obtaining a basis from the Gaussian does not extinguish the hope for a meaningful time-frequency representation. Rather, it suggests that the notion of basis is too restrictive in this setting and that some of the hypotheses must be relaxed. In the following chapter, we will see how the theory of frames provides a natural way to overcome this obstruction, allowing stable and redundant expansions generated by Gaussian windows and, more generally, by suitable Gabor systems.

Proof. Without loss of generality, consider $a = b = 1$; the general case can be proved similarly. Under this assumption, the system takes the form

$$g_{n,k}(t) = M_k T_n g(t) = e^{2\pi i k t} g(t - n).$$

We introduce the position and momentum operators from quantum mechanics, defined by:

$$X(f(t)) = t f(t), \quad P(f(t)) = \frac{1}{2\pi i} f'(t).$$

By Theorem 2.17 (i), Pg is equivalent, looking at the Fourier side, to ξg . For the proof to be non-trivial, we assume that both Xg and Pg belong to $L^2(\mathbb{R})$; otherwise the conclusion holds immediately.

Notice that these operators are self-adjoint: on the one hand, for $f, h \in L^2(\mathbb{R})$, we have

$$\langle Xf, h \rangle = \int_{\mathbb{R}} t f(t) \overline{h(t)} dt = \langle f, Xh \rangle.$$

On the other hand, since $f, h \in L^2(\mathbb{R})$, there exist sequences $a_n \rightarrow -\infty$ and $b_n \rightarrow +\infty$ such that $f(a_n) \rightarrow 0$, $f(b_n) \rightarrow 0$, $h(a_n) \rightarrow 0$, and $h(b_n) \rightarrow 0$. Hence, writing the integral as a limit over finite intervals and integrating by parts, we obtain

$$\begin{aligned}
 \langle Pf, h \rangle &= \lim_{n \rightarrow \infty} \int_{a_n}^{b_n} \frac{1}{2\pi i} f'(t) \overline{h(t)} dt \\
 &= \lim_{n \rightarrow \infty} \left(\frac{1}{2\pi i} \left[f(t) \overline{h(t)} \right]_{a_n}^{b_n} - \int_{a_n}^{b_n} \frac{1}{2\pi i} f(t) \overline{h'(t)} dt \right) \\
 &= - \lim_{n \rightarrow \infty} \int_{a_n}^{b_n} \frac{1}{2\pi i} f(t) \overline{h'(t)} dt \\
 &= \int_{\mathbb{R}} f(t) \overline{\frac{1}{2\pi i} h'(t)} dt \\
 &= \langle f, Ph \rangle.
 \end{aligned}$$

Notice that, by the choice of a_n and b_n , the boundary term satisfies

$$\left[f(t) \overline{h(t)} \right]_{a_n}^{b_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

To proceed, we recall that $\{M_k \tau_n g\}_{n,k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ by hypothesis, so we may expand

$$Xg = \sum_{n,k \in \mathbb{Z}} \langle Xg, M_k \tau_n g \rangle M_k \tau_n g$$

$$Pg = \sum_{n,k \in \mathbb{Z}} \langle Pg, M_k \tau_n g \rangle M_k \tau_n g.$$

Applying linearity and orthonormality of the basis,

$$\langle Xg, Pg \rangle = \sum_{n,k \in \mathbb{Z}} \langle Xg, M_k \tau_n g \rangle \langle M_k \tau_n g, Pg \rangle.$$

To continue with the proof we will first have to state a claim, which is just a computation.

$$\text{Claim: } \langle Xg, M_k \tau_n g \rangle = \langle M_{-k} \tau_{-n} g, Xg \rangle \quad \text{and} \quad \langle M_k \tau_n g, Pg \rangle = \langle Pg, M_{-k} \tau_{-n} g \rangle.$$

Assuming this, and noting that $\{M_{-k} \tau_{-n} g\}_{n,k \in \mathbb{Z}}$ is simply a re-indexing of the same orthonormal basis, we obtain

$$\langle Xg, Pg \rangle = \sum_{n,k \in \mathbb{Z}} \langle Pg, M_{-k} \tau_{-n} g \rangle \langle M_{-k} \tau_{-n} g, Xg \rangle = \langle Pg, Xg \rangle \quad (4.6)$$

We have seen that both operators are self-adjoint. Notice that,

$$\begin{aligned} (PX - XP)g(t) &= P(tg(t)) - X\left(\frac{1}{2\pi i}g'(t)\right) \\ &= \frac{1}{2\pi i}(g(t) + tg'(t)) - \frac{t}{2\pi i}g'(t) = \frac{1}{2\pi i}g(t). \end{aligned}$$

and therefore,

$$\langle (PX - XP)g, g \rangle = \frac{1}{2\pi i} \|g\|_2^2. \quad (4.7)$$

However, since both X and P are self-adjoint, equation (4.6) gives $\langle Xg, Pg \rangle = \langle Pg, Xg \rangle$, which implies

$$\langle PXg, g \rangle = \langle XPg, g \rangle,$$

and hence

$$\langle (PX - XP)g, g \rangle = 0,$$

which contradicts equation (4.7).

It remains to prove the claim. We prove the first identity; the second follows by an analogous argument.

By self-adjointness of X ,

$$\langle Xg, M_k \tau_n g \rangle = \langle g, XM_k \tau_n g \rangle.$$

A direct computation gives

$$\begin{aligned} XM_k \tau_n g(t) &= t e^{2\pi i k t} g(t - n) = n e^{2\pi i k t} g(t - n) + (t - n) e^{2\pi i k t} g(t - n) \\ &= n M_k \tau_n g(t) + M_k \tau_n (Xg)(t), \end{aligned}$$

and therefore

$$\langle Xg, M_k \tau_n g \rangle = n \langle g, M_k \tau_n g \rangle + \langle g, M_k \tau_n (Xg) \rangle.$$

Since $g = M_0 \tau_0 g$, the orthogonality of the system $\{M_k \tau_n g\}_{n,k \in \mathbb{Z}}$ shows that $n \langle g, M_k \tau_n g \rangle$ is always 0, as the scalar product is always zero except for $k = n = 0$, but the factor n makes the term vanish when $n = 0$. Then, finally

$$\begin{aligned} \langle Xg, M_k \tau_n g \rangle &= \langle g, M_k \tau_n (Xg) \rangle = \int_{\mathbb{R}} g(t) e^{-2\pi i k t} (Xg)(t - n) dt \\ &= \int_{\mathbb{R}} g(s + n) e^{-2\pi i k (s+n)} Xg(s) ds = \langle M_{-k} \tau_{-n} g, Xg \rangle. \end{aligned}$$

□

Chapter 5

Gabor frames

So far, we have seen that the system $\{M_{bk}\tau_{an}g\}_{n,k\in\mathbb{Z}}$ ($a, b > 0, ab = 1$), known as the *Gabor system*, cannot simultaneously form a basis and be well localized both in time and frequency, as highlighted by the Balian-Low theorem discussed in the previous chapter. Nevertheless, in practice one would still like to achieve stable reconstruction of signals while preserving, as much as possible, the optimal time-frequency localization provided by the Gaussian function. Indeed, the Gaussian is distinguished by the fact that both the function g and its Fourier transform $\hat{g}$ are simultaneously optimally localized, in the sense that it minimizes the uncertainty principle.

This motivates relaxing the requirement of a basis, what leads naturally to the notion of a *frame*, which allows us to continue developing a robust time-frequency representation framework.

Guided by this motivation, we begin by introducing the notion of frames and the basic operators associated with them. We then state the Daubechies and Grossmann theorem on lattice Gabor frames and gradually develop the necessary tools for its proof. In particular, we introduce the Bargmann-Fock space and establish several auxiliary results that reveal a chain of equivalences connecting the different perspectives involved. Finally, all these ingredients come together in a sketch of the proof of the Daubechies-Grossmann theorem to conclude this work.

5.1 Frames

We would like to write, letting $g_{n,k} = M_{bk}\tau_{an}g$, an expansion of the form

$$f = \sum_{n,k\in\mathbb{Z}} c_{n,k}(f) M_{bk}\tau_{an}g = \sum_{n,k\in\mathbb{Z}} c_{n,k}(f) g_{n,k},$$

where the total energy $\|f\|_2$ can be recovered from the coefficients $c_{n,k}(f)$.

However, as we have already seen, a Gabor system does not in general provide an exact characterization of the energy of a signal f .

Recall that

$$V_g f(an, bk) = \langle f, M_{bk} \tau_{an} g \rangle = \langle f, g_{n,k} \rangle.$$

Therefore, if we want the system $\{g_{n,k}\}_{n,k \in \mathbb{Z}}$ to capture most of the energy of every $f \in L^2(\mathbb{R})$, it is natural to require the existence of a constant $A > 0$ such that

$$A\|f\|_2^2 \leq \sum_{n,k \in \mathbb{Z}} |V_g f(an, bk)|^2 = \sum_{n,k \in \mathbb{Z}} |\langle f, g_{n,k} \rangle|^2.$$

On the other hand, we also want to avoid excessive redundancy in the representation. In particular, we would like the coefficients to remain controlled, which leads to the requirement that there exists $B > 0$ such that

$$\sum_{n,k \in \mathbb{Z}} |\langle f, g_{n,k} \rangle|^2 = \sum_{n,k \in \mathbb{Z}} |V_g f(an, bk)|^2 \leq B\|f\|_2^2.$$

Together, these two inequalities provide a more flexible and robust alternative to orthonormal bases. Indeed, frames allow for redundancy while still ensuring stable reconstruction. For example, duplicating the elements of $\{g_{n,k}\}_{n,k \in \mathbb{Z}}$ still preserves the validity of the inequalities above. This stability under redundancy is one of the key features that makes frames particularly useful in applications.

Definition 5.1. A sequence $\{e_j : j \in J\}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{j \in J} |\langle f, e_j \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in \mathcal{H}. \quad (5.1)$$

Any two constants A, B satisfying (5.1) are called *frame bounds*. If $A = B$, then $\{e_j : j \in J\}$ is called a *tight frame*.

In general, the elements of a frame are neither orthogonal nor linearly independent.

In particular, we are interested in frames of the form

$$\{g_{n,k}\}_{n,k \in \mathbb{Z}} = \{M_{bk} \tau_{an} g\}_{n,k \in \mathbb{Z}},$$

known as *Gabor frames*, in the Hilbert space $L^2(\mathbb{R})$, where g is a fixed window function (typically a Gaussian).

Notice that every orthonormal basis is a frame with $A = B = 1$, although the converse is not true.

Having introduced the notion of a frame, we now study how signals can be reconstructed from their frame coefficients.

5.2 Frame operators

In general, given a subset $\{e_j : j \in J\} \subset \mathcal{H}$, consider the operators defined by the action on $f \in \mathcal{H}$ and the sequence $c = \{c_j\}_j \subset \mathbb{C}$ in the following way:

$$C(f) = \{\langle f, e_j \rangle\}_{j \in J} \quad (\text{analysis operator})$$

$$S(c) = \sum_{j \in J} c_j e_j \quad (\text{synthesis operator})$$

$$T(f) = \sum_{j \in J} \langle f, e_j \rangle e_j \quad (\text{frame operator})$$

In this chapter, unless otherwise specified, $\|\cdot\|$ denotes the operator norm. The following propositions collect some important properties of the analysis, synthesis, and frame operators.

Proposition 5.2. *Suppose that $\{e_j : j \in J\}$ is a frame for $\mathcal{H}$ with frame bounds A, B . Then,*

- a) *C is a bounded operator from $\mathcal{H}$ to $\ell^2(J)$ with closed range.*
- b) *The operators C and S are adjoint to each other; that is, $S = C^*$. Consequently S extends to a bounded operator from ℓ^2 into $\mathcal{H}$ with*

$$\|S(c)\|_{\mathcal{H}} = \left\| \sum_j c_j e_j \right\|_{\mathcal{H}} \leq \sqrt{B} \|c\|_{\ell^2}, \quad c \in \ell^2. \quad (5.2)$$

- c) *The frame operator $T = C^*C = SS^*$ maps $\mathcal{H}$ onto $\mathcal{H}$ and is a positive invertible operator satisfying*

$$A \cdot I \leq T \leq B \cdot I, \quad B^{-1} \cdot I \leq T^{-1} \leq A^{-1} \cdot I.$$

Note: It can be seen that the optimal frame bounds are $B_{opt} = \|T\|$ and $A_{opt} = \|T^{-1}\|^{-1}$.

Proof. (a) It is clear that $C(f) \in \ell^2$ and $\|C\| \leq \sqrt{B}$, by the second inequality in (5.1). That C has closed range is a consequence of the first inequality $\sqrt{A}\|f\|_{\mathcal{H}} \leq \|C(f)\|_{\ell^2}$ in (5.1): if $\{C(f_n)\}_n \rightarrow \alpha$ in ℓ^2 , then $\{f_n\}_n$ is a Cauchy sequence in $\mathcal{H}$. Hence, there exists $f \in \mathcal{H}$ such that $f_n \rightarrow f$.

We now show that $C(f) = \alpha$. Indeed, using the boundedness of C , we have

$$\|C(f) - \alpha\|_{\ell^2} \leq \|C(f) - C(f_n)\|_{\ell^2} + \|C(f_n) - \alpha\|_{\ell^2}$$

and since C is bounded,

$$\|C(f) - C(f_n)\|_{\ell^2} \leq \sqrt{B} \|f - f_n\|_{\mathcal{H}}.$$

Therefore,

$$\|C(f) - \alpha\|_{\ell^2} \leq \sqrt{B} \|f - f_n\|_{\mathcal{H}} + \|C(f_n) - \alpha\|_{\ell^2}.$$

Letting $n \rightarrow \infty$ yields $C(f) = \alpha$.

(b) Let $c = \{c_j\}_j$ be a finite sequence. Then, for every $f \in \mathcal{H}$,

$$\langle C^*(c), f \rangle = \langle c, C(f) \rangle = \sum_j c_j \overline{\langle f, e_j \rangle} = \left\langle \sum_j c_j e_j, f \right\rangle = \langle S(c), f \rangle.$$

Since finite sequences are dense in ℓ^2 and C is bounded with $\|C\| \leq \sqrt{B}$, it follows that S extends uniquely to a bounded operator $S : \ell^2 \rightarrow \mathcal{H}$ satisfying $S = C^*$ and $\|S\| = \|C\| \leq \sqrt{B}$.

(c) By definition, $T = C^*C = SS^*$, and therefore T is self-adjoint:

$$T^* = (C^*C)^* = C^*C = T$$

and positive:

$$\langle Tx, x \rangle = \langle C^*Cx, x \rangle = \langle Cx, Cx \rangle = \|Cx\|_{\ell^2}^2 \geq 0$$

Since,

$$\langle T(f), f \rangle = \sum_j |\langle f, e_j \rangle|^2, \quad (5.3)$$

the frame inequality (5.1) is precisely $A \cdot I \leq T \leq B \cdot I$. Also, T is invertible since

$$A\|f\|_{\mathcal{H}}^2 \leq \langle Tf, f \rangle$$

implies that T is bounded below. Hence T is injective and has closed range. Moreover,

$$(\text{Ran } T)^\perp = \text{Ker}(T^*) = \text{Ker}(T) = \{0\},$$

since T is self-adjoint. Therefore $\text{Ran } T$ is dense. Being both dense and closed, we conclude that $\text{Ran } T = \mathcal{H}$, and hence T is invertible.

Since the previous inequalities are preserved under multiplication by positive commuting operators, we obtain $A \cdot T^{-1} \leq TT^{-1} \leq B \cdot T^{-1}$, as desired. $\square$

Statement (b) shows that $\sum_{j \in I} c_j e_j$ is well defined for arbitrary ℓ^2 -sequences, even though the frame vectors e_j are not orthogonal in general. Proposition 5.2 provides a weak definition of such non-orthogonal series. To understand the convergence properties of the non-orthogonal series $\sum_{j \in I} c_j e_j$ better, we exploit Proposition 5.2 further.

Corollary 5.3. *Let $\{e_j : j \in J\}$ be a frame for $\mathcal{H}$. If $f = \sum_{j \in J} c_j e_j$ for some $c \in \ell^2(J)$, then for every $\varepsilon > 0$ there exists a finite subset $F_0 = F_0(\varepsilon) \subset J$ such that for every finite subset $F \subset J$ with $F_0 \subset F$*

$$\left\| f - \sum_{j \in F} c_j e_j \right\|_{\mathcal{H}} < \varepsilon.$$

We say that the series $\sum_{j \in J} c_j e_j$ converges unconditionally to $f \in \mathcal{H}$.

Proof. Choose a finite set $F_0 \subset J$ such that

$$\sum_{j \notin F_0} |c_j|^2 < \frac{\varepsilon^2}{B}.$$

Let $F \subset J$ be any finite set satisfying $F_0 \subset F$, and define $c_F = c \cdot \chi_F \in \ell^2(J)$. Then, $\sum_{j \in F} c_j e_j = S(c_F)$, and applying (5.2), we obtain

$$\begin{aligned} \left\| f - \sum_{j \in F} c_j e_j \right\|_{\mathcal{H}} &= \|S(c) - S(c_F)\|_{\mathcal{H}} = \|S(c - c_F)\|_{\mathcal{H}} \\ &\leq B^{1/2} \|c - c_F\|_{\ell^2} = B^{1/2} \left(\sum_{j \notin F} |c_j|^2 \right)^{1/2}. \end{aligned}$$

Since $F_0 \subset F$, we have

$$\sum_{j \notin F} |c_j|^2 \leq \sum_{j \notin F_0} |c_j|^2 < \frac{\varepsilon^2}{B},$$

and therefore

$$\left\| f - \sum_{j \in F} c_j e_j \right\|_{\mathcal{H}} < \varepsilon.$$

□

Unconditional convergence is the most natural notion of convergence for non-orthogonal series over general, unstructured index sets, since in this setting there is no canonical ordering and convergence must therefore be independent of how the terms are grouped or summed.

As another consequence of Proposition 5.2 we obtain a reconstruction formula for f from the frame coefficients $\langle f, e_j \rangle$.

Corollary 5.4. *If $\{e_j : j \in J\}$ is a frame with frame bounds $A, B > 0$, then $\{T^{-1}e_j : j \in J\}$ is also a frame (called the dual frame), with frame bounds $A^{-1}, B^{-1} > 0$. Every $f \in \mathcal{H}$ has non-orthogonal expansions*

$$f = \sum_{j \in J} \langle f, T^{-1}e_j \rangle e_j$$

and

$$f = \sum_{j \in J} \langle f, e_j \rangle T^{-1}e_j,$$

where both sums converge unconditionally in $\mathcal{H}$.

Proof. First observe that by the self-adjointness of T and by (5.3)

$$\langle T^{-1}f, f \rangle = \langle f, T^{-1}f \rangle = \langle T(T^{-1}f), T^{-1}f \rangle = \sum_{j \in J} |\langle T^{-1}f, e_j \rangle|^2 = \sum_{j \in J} |\langle f, T^{-1}e_j \rangle|^2.$$

Therefore, Proposition 5.2 (c) implies that

$$B^{-1}\|f\|_{\mathcal{H}}^2 \leq \langle T^{-1}f, f \rangle = \sum_{j \in J} |\langle f, T^{-1}e_j \rangle|^2 \leq A^{-1}\|f\|_{\mathcal{H}}^2.$$

Thus the collection $\{T^{-1}e_j : j \in J\}$ is a frame with frame bounds A^{-1} and B^{-1} .

Using the factorization $I = TT^{-1} = T^{-1}T$, we obtain the series expansions

$$f = T(T^{-1}f) = \sum_{j \in J} \langle T^{-1}f, e_j \rangle e_j = \sum_{j \in J} \langle f, T^{-1}e_j \rangle e_j.$$

and

$$f = T^{-1}(Tf) = \sum_{j \in J} \langle f, e_j \rangle T^{-1}e_j.$$

Because both $\{\langle f, e_j \rangle\}$ and $\{\langle f, T^{-1}e_j \rangle\}$ are in $\ell^2(J)$, both series converge unconditionally by Corollary 5.3. $\square$

Thus, we can represent f in terms of the frame $\{e_j\}_{j \in J}$, but the coefficients are expressed with respect to the dual frame rather than the original one. If $\{e_j\}_{j \in J}$ is an orthonormal basis, then $T = I$, and the dual frame coincides with the original frame.

5.3 Daubechies and Grossmann Theorem

After the Balian-Low theorem, a natural question arises: for the Gaussian window G and a lattice $a\mathbb{Z} \times b\mathbb{Z}$ in the time-frequency plane, when does the system $\{M_{bk}\tau_{an}G\}_{n,k \in \mathbb{Z}}$ form a frame? This is equivalent to asking when sampling on the

lattice allows one to fully encode a signal and recover it via the corresponding dual frame representation, which has been our goal throughout this work.

The following theorem brings light into this darkness.

Theorem 5.5. (Daubechies-Grossmann; Seip) ([5, Theorem 7.5.3]) *Let $G(t) = 2^{1/4}e^{-\pi t^2}$ be the normalized Gaussian, and let $a, b > 0$. The system $\{M_{bk}\tau_{an}G\}_{n,k \in \mathbb{Z}}$ is a frame if and only if $ab < 1$.*

Notice that ab is the area of a cell of the lattice. Therefore, the condition $ab < 1$ tells us that the lattice has to be more densely packed than $\mathbb{Z}^2$.

This density theorem can be reformulated as a density result for sampling lattices in a specific class of entire functions, namely the Bargmann-Fock space. Let us see how this works.

5.3.1 Bargmann-Fock space

Let $\varphi(x) = 2^{1/4}e^{-\pi x^2}$ be the normalized Gaussian, so that $\|\varphi\|_2 = 1$. A direct computation gives

$$\begin{aligned} V_\varphi f(t, \xi) &= 2^{1/4} \int_{\mathbb{R}} f(s) e^{-\pi(s-t)^2} e^{-2\pi i \xi s} ds \\ &= 2^{1/4} \int_{\mathbb{R}} f(s) e^{-\pi s^2} e^{2\pi i t s} e^{-\pi t^2} e^{-2\pi i \xi s} ds \\ &= 2^{1/4} e^{-\pi i t \xi} e^{-\frac{\pi}{2}(t^2 + \xi^2)} \int_{\mathbb{R}} f(s) e^{-\pi s^2} e^{2\pi i s(t - i\xi)} e^{-\frac{\pi}{2}(t - i\xi)^2} ds. \end{aligned} \quad (5.4)$$

Now identify $(t, \xi) \in \mathbb{R}^2$ with the complex variable

$$z = t + i\xi.$$

Then, comparing to (5.4) the following definition is quite natural.

Definition 5.6. The Bargmann transform of a function $f \in L^2(\mathbb{R})$ is the function $\mathcal{B}f$ on $\mathbb{C}$ defined by

$$\mathcal{B}f(z) = 2^{1/4} \int_{\mathbb{R}} f(s) e^{2\pi i s z - \pi s^2 - \frac{\pi}{2} z^2} ds.$$

The importance of the Bargmann transform stems from the following proposition.

Proposition 5.7. *Let $z = t + i\xi$ and let $\varphi(x) = 2^{1/4}e^{-\pi x^2}$. Then*

$$V_\varphi f(t, -\xi) = e^{\pi i t \xi} e^{-\frac{\pi}{2}(t^2 + \xi^2)} (\mathcal{B}f)(z).$$

The identity follows by comparing the last expression in (5.4) with the definition of the Bargmann transform.

The *Bargmann-Fock space*, $\mathcal{F}^2(\mathbb{C})$, is the Hilbert space of all entire functions F on $\mathbb{C}$ satisfying

$$\|F\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dz < \infty.$$

Its inner product is given by

$$\langle F, G \rangle_{\mathcal{F}^2} = \int_{\mathbb{C}} F(z) \overline{G(z)} e^{-\pi|z|^2} dz.$$

A key proposition is the following.

Proposition 5.8. *The Bargmann transform maps $L^2(\mathbb{R})$ into the space of entire functions and satisfies*

$$\|f\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{C}} |\mathcal{B}f(z)|^2 e^{-\pi|z|^2} dz = \|\mathcal{B}f\|_{\mathcal{F}^2}^2.$$

In particular, $\mathcal{B}$ is an isometry from $L^2(\mathbb{R})$ onto $\mathcal{F}^2(\mathbb{C})$.

Proof. The fact that $\mathcal{B}f$ is entire is a direct application of Morera's theorem.

By Theorem 4.4, and using Proposition 5.7

$$\begin{aligned} \|f\|_{L^2}^2 &= \|V_{\varphi}f\|_{L^2(\mathbb{R}^2)}^2 \\ &= \int_{\mathbb{R}^2} \left| e^{\pi i t \xi} \mathcal{B}f(t + i\xi) e^{-\frac{\pi}{2}(t^2 + \xi^2)} \right|^2 dt d\xi \\ &= \int_{\mathbb{R}^2} |\mathcal{B}f(t + i\xi)|^2 e^{-\pi(t^2 + \xi^2)} dt d\xi \\ &= \int_{\mathbb{C}} |\mathcal{B}f(z)|^2 e^{-\pi|z|^2} dz. \end{aligned}$$

□

5.3.2 Gabor systems and sampling sequences

The Bargmann transform allows us to reinterpret Gaussian Gabor systems as sampling sequences for the Bargmann-Fock space.

A key concept in the reformulation of Theorem 5.5 is the notion of a sampling set.

Definition 5.9. A discrete set $\Lambda \subset \mathbb{C}$ is called a *sampling set* for the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C})$ if there exist constants $A, B > 0$ such that

$$A\|F\|_{\mathcal{F}^2}^2 \leq \sum_{\lambda \in \Lambda} |F(\lambda)|^2 e^{-\pi|\lambda|^2} \leq B\|F\|_{\mathcal{F}^2}^2, \quad \forall F \in \mathcal{F}^2(\mathbb{C}).$$

In the particular case in which the sequence is a lattice, we have the following equivalence.

Proposition 5.10. *Let $\Gamma = a\mathbb{Z} + i b\mathbb{Z}$ be a rectangular lattice in the complex plane. The following are equivalent:*

1. *The Gaussian Gabor system*

$$\mathcal{G}(\varphi, a, b) = \{M_{bk}\tau_{an}\varphi\}_{n,k \in \mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$.

2. *The lattice*

$$\Gamma = a\mathbb{Z} + i b\mathbb{Z}$$

is a sampling set for $\mathcal{F}^2(\mathbb{C})$.

Proof. Recall that the Gaussian Gabor system is defined by

$$\mathcal{G}(\varphi, a, b) = \{M_{bk}\tau_{an}\varphi\}_{n,k \in \mathbb{Z}}$$

where φ is the normalized Gaussian. A key observation is that the coefficients of the frame expansion are given by the Short-Time Fourier Transform, as shown in equation (4.1):

$$\langle f, M_{bk}\tau_{an}\varphi \rangle = V_\varphi f(an, bk).$$

Therefore, the frame condition

$$A\|f\|_2^2 \leq \sum_{n,k \in \mathbb{Z}} |\langle f, M_{bk}\tau_{an}\varphi \rangle|^2 \leq B\|f\|_2^2$$

can be rewritten as

$$A\|f\|_2^2 \leq \sum_{n,k \in \mathbb{Z}} |V_\varphi f(an, bk)|^2 \leq B\|f\|_2^2.$$

Thus, the frame property of the Gabor system is equivalent to a discrete sampling condition for the STFT.

From Proposition 5.7, replacing ξ by $-\xi$, we obtain

$$|V_\varphi f(t, \xi)|^2 = e^{-\pi(t^2 + \xi^2)} |\mathcal{B}f(t + i\xi)|^2.$$

Hence

$$|V_\varphi f(an, bk)|^2 = e^{-\pi(a^2 n^2 + b^2 k^2)} |\mathcal{B}f(an + ibk)|^2.$$

Since $\mathcal{B}$ is an isometry by Proposition 5.8, letting $F = \mathcal{B}f$, the frame inequality becomes

$$A\|F\|_{\mathcal{F}^2}^2 \leq \sum_{n,k \in \mathbb{Z}} |F(an + ibk)|^2 e^{-\pi(a^2 n^2 + b^2 k^2)} \leq B\|F\|_{\mathcal{F}^2}^2.$$

Therefore, the lattice $\Gamma = a\mathbb{Z} + i b\mathbb{Z}$ is a sampling set for $\mathcal{F}^2(\mathbb{C})$. □

5.3.3 Seip's density theorem

In a remarkable result, K. Seip gave a geometric characterization of sampling sequences for the Bargmann-Fock space. For the statement we need an elementary definition.

Definition 5.11. A sequence $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ is said to be separated if there exists a constant $\delta > 0$ such that

$$\inf_{n \neq m} |\gamma_n - \gamma_m| \geq \delta.$$

The following theorem, due to Seip and Wallstén [9, 10], gives a complete characterization of sampling sets in the Bargmann-Fock space in terms of their density.

Theorem 5.12 (Sampling Theorem; Seip-Wallstén). A separated sequence $\Gamma \subset \mathbb{C}$ is a sampling set for $\mathcal{F}^2(\mathbb{C})$ if and only if

$$D^-(\Gamma) := \liminf_{r \rightarrow \infty} \inf_{z \in \mathbb{C}} \frac{\#(\Gamma \cap D(z, r))}{\pi r^2} > 1.$$

The quantity $D^-(\Gamma)$ is called the lower Beurling density of Γ . Intuitively, it measures the asymptotic number of points of Γ per unit area. In the particular case where Γ is a lattice, all centers $z \in \mathbb{C}$ play the same role in the limit: changing the center of the disk may affect the number of points near the boundary, but this boundary effect becomes negligible as $r \rightarrow \infty$.

Thus, for a rectangular lattice $\Gamma = a\mathbb{Z} + ib\mathbb{Z}$, the fundamental cell has area ab . Hence, asymptotically, the number of lattice points contained in a region of area A is approximately $A/(ab)$. Applying this to disks of area πr^2 , we obtain

$$\#(\Gamma \cap D(z, r)) \sim \frac{\pi r^2}{ab}.$$

Therefore, the lower Beurling density is

$$D^-(\Gamma) = \liminf_{r \rightarrow \infty} \inf_{z \in \mathbb{C}} \frac{\#(\Gamma \cap D(z, r))}{\pi r^2} = \frac{1}{ab}.$$

Thus, Theorem 5.12 immediately yields the following corollary.

Corollary 5.13. Let $\Gamma = a\mathbb{Z} + ib\mathbb{Z} \subset \mathbb{C}$ be a rectangular lattice. Then Γ is a sampling set for $\mathcal{F}^2(\mathbb{C})$ if and only if $ab < 1$, or equivalently, $D^-(\Gamma) = \frac{1}{ab} > 1$.

This, combined with Proposition 5.10, proves the Daubechies-Grossmann theorem.

5.3.4 Sketch of the proof of Seip's theorem for a lattice

Proof. Assume that $\Gamma = a\mathbb{Z} + ib\mathbb{Z}$. Consider its associated Weierstrass σ -function:

$$\sigma_\Gamma(z) = z \prod_{\gamma \in \Gamma \setminus \{0\}} \left(1 - \frac{z}{\gamma}\right) \exp\left(\frac{z}{\gamma} + \frac{z^2}{2\gamma^2}\right).$$

This is an entire function whose zero set is Γ , and all its zeros are simple. Moreover, for a rectangular lattice, one has the asymptotic estimate

$$|\sigma_\Gamma(z)| \asymp e^{\frac{\pi}{2A}|z|^2} d(z, \Gamma),$$

where

$$A = \text{area}(\Gamma) = ab$$

and

$$d(z, \Gamma) = \inf_{\gamma \in \Gamma} |z - \gamma|.$$

Consequently,

$$|\sigma_\Gamma(z)|^2 e^{-\pi|z|^2} \asymp d(z, \Gamma)^2 e^{-\pi(1-\frac{1}{A})|z|^2}. \quad (5.5)$$

This estimate determines the behavior of the lattice as a sampling set.

We first prove that if Γ is sampling, then $D^-(\Gamma) > 1$. We argue by contraposition, showing that if $D^-(\Gamma) \leq 1$, then Γ is not sampling.

First suppose that $D^-(\Gamma) < 1$, equivalently, $A > 1$. Since

$$1 - \frac{1}{A} > 0,$$

by (5.5), there exists $c > 0$ such that

$$|\sigma_\Gamma(z)|^2 e^{-\pi|z|^2} \lesssim e^{-c|z|^2}.$$

Hence, the Weierstrass σ -function is in $\mathcal{F}^2(\mathbb{C})$.

But σ_Γ vanishes on all of Γ . Thus there exists a nonzero function in $\mathcal{F}^2(\mathbb{C})$ vanishing on the sampling set. Hence no inequality of the form

$$A\|F\|_{\mathcal{F}^2}^2 \leq \sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-\pi|\gamma|^2}$$

can hold. Therefore Γ is not sampling.

Next, consider the critical case $D^-(\Gamma) = 1$ ($A = 1$). Then, again by (5.5)

$$|\sigma_\Gamma(z)|^2 e^{-\pi|z|^2} \asymp d(z, \Gamma)^2.$$

Since this function is not integrable over $\mathbb{C}$, we have $\sigma_\Gamma \notin \mathcal{F}^2(\mathbb{C})$.

Thus, Γ is a uniqueness set for the Bargmann-Fock space. However, it is still not a sampling set. Indeed, if χ_R is a smooth cutoff equal to 1 on $|z| \leq R$ and supported in $|z| \leq 2R$, one may consider

$$F_R(z) = \chi_R(z) \sigma_\Gamma(z).$$

After solving a suitable $\bar{\partial}$ -equation to restore holomorphicity, one constructs functions $G_R \in \mathcal{F}^2(\mathbb{C})$ satisfying

$$\|G_R\|_{\mathcal{F}^2} \asymp 1,$$

while

$$\sum_{\gamma \in \Gamma} |G_R(\gamma)|^2 e^{-\pi|\gamma|^2} \longrightarrow 0.$$

Hence the lower sampling inequality fails, and Γ is not sampling when $A = 1$. Considerable work is required to complete this outline.

Assume now that $D^-(\Gamma) > 1$ and let us see that Γ is a sampling set for the Bargmann-Fock space.

Suppose that $A < 1$, equivalently $D^-(\Gamma) > 1$. In this regime, the estimate shows that the weight becomes exponentially increasing, so that the sigma-function cannot belong to $\mathcal{F}^2(\mathbb{C})$. Consequently, one obtains the sampling inequalities for Γ .

Let

$$Q = [0, a) \times [0, b)$$

be a fundamental cell of the lattice. Then every $z \in \mathbb{C}$ can be written uniquely as

$$z = \gamma(z) + w, \quad \gamma(z) \in \Gamma, \quad w \in Q.$$

Since Q is bounded, this provides a uniform decomposition of $\mathbb{C}$ into translates of a compact set.

By an adapted version of the local mean-value property for entire functions, for every $r > 0$ there exists a constant $C_r > 0$ such that

$$|F(z)|^2 e^{-\pi|z|^2} \leq C_r \int_{B(z, r)} |F(w)|^2 e^{-\pi|w|^2} dm(w).$$

Applying this estimate near each lattice point and summing over Γ (using that the balls $B(\gamma, r)$ have uniformly bounded overlap for fixed r) yields

$$\sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-\pi|\gamma|^2} \leq \sum_{\gamma \in \Gamma} \int_{B(\gamma, r)} |F(\gamma)|^2 e^{-\pi|\gamma|^2} d\gamma \lesssim \|F\|_{\mathcal{F}^2}^2.$$

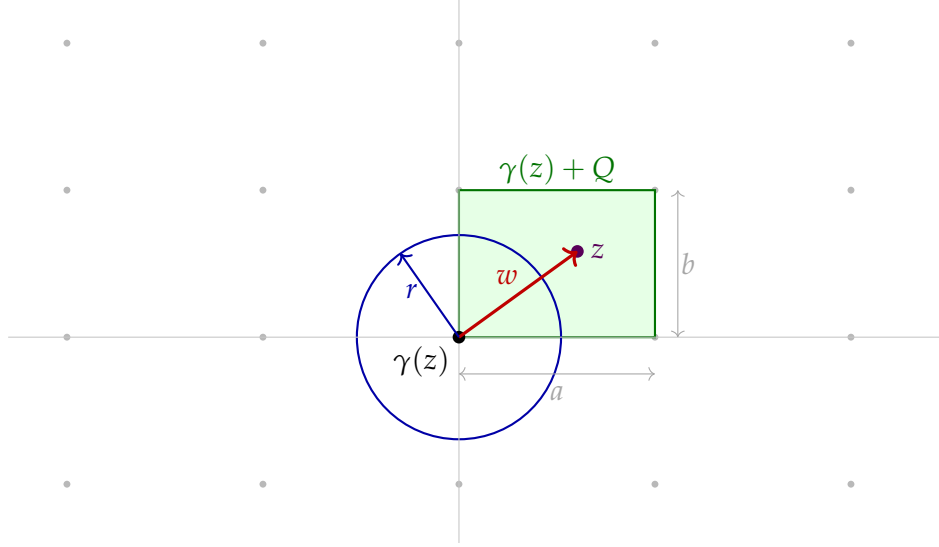


Figure 5.1: Decomposition $z = \gamma(z) + w$ in a rectangular lattice.

This gives the upper sampling inequality. To obtain the reverse inequality, assume, by contradiction, that it fails. Then there exists a sequence $(F_n) \subset \mathcal{F}^2(\mathbb{C})$ such that

$$\|F_n\|_{\mathcal{F}^2} = 1,$$

and

$$\sum_{\gamma \in \Gamma} |F_n(\gamma)|^2 e^{-\pi|\gamma|^2} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Using compactness of normal families and reproducing-kernel estimates, one extracts a subsequence converging locally uniformly to a function

$$F \in \mathcal{F}^2(\mathbb{C}).$$

Since $|F_n(\gamma)|^2 e^{-\pi|\gamma|^2} \leq \sum_{\gamma \in \Gamma} |F_n(\gamma)|^2 e^{-\pi|\gamma|^2}$, the limit satisfies

$$F(\gamma) = 0, \quad \gamma \in \Gamma.$$

Since $A < 1$, the density of the lattice exceeds the critical value, and the only Fock-space function vanishing on Γ is the zero function. Hence

$$F \equiv 0.$$

This contradicts the normalization

$$\|F_n\|_{\mathcal{F}^2} = 1.$$

Therefore the lower sampling inequality holds:

$$\|F\|_{\mathcal{F}^2}^2 \lesssim \sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-\pi|\gamma|^2}.$$

Combining the upper and lower estimates yields

$$A\|F\|_{\mathcal{F}^2}^2 \leq \sum_{\gamma \in \Gamma} |F(\gamma)|^2 e^{-\pi|\gamma|^2} \leq B\|F\|_{\mathcal{F}^2}^2,$$

and therefore Γ is a sampling set. $\square$

We have presented the proof only in the case of rectangular lattices, as this is the setting relevant to the reduction of Daubechies and Grossmann's theorem.

However, Seip's density theorem holds in a much more general context. Rather than being restricted to lattices, it applies to arbitrary separated sequences in $\mathbb{C}$, and therefore to unstructured discrete sets as well. In this sense, it is a considerably stronger result and is of independent interest in the sampling theory of the Bargmann-Fock space.

As a consequence, one obtains a corresponding characterization of Gaussian Gabor frames beyond the lattice setting. Let $\Gamma = \{\gamma_k\}_{k=1}^\infty$ be a separated sequence in $\mathbb{C}$, with $\gamma_k = t_k + i\zeta_k$. Then, the Gaussian Gabor system $\{M_{\zeta_k} \tau_{t_k} G\}_{k=1}^\infty$ is a frame for $L^2(\mathbb{R})$ if and only if Γ is separated and satisfies $D^-(\Gamma) > 1$.

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