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ABSTRACT. From an infinite dimensional version of a generalization, due to Peleg, of the Knaster-Kuratowski-Mazurkiewicz's theorem, we obtain a generalization of Browder's fixed point theorem, for multi-valued mappings from the product of a finite family of non-empty compact convex sets (each in a Hausdorff topological vector space) into each of its factors. By applying this thorem, we deduce some Ky Fan type inequalities, from one of which a generalization of the Ky Fan's intersection theorem on sets with convex sections is obtained.

I. INTRODUCTION

The famous Knaster-Kuratowski-Mazurkiewicz's theorem is a fundamental result of nonlinear analysis. More than twenty years ago, Peleg, in a paper on the existence of equilibrium points in many-person games [10], gave an interesting generalization of this theorem, concerning closed subsets of a product of simplexes. But it seems that this result of Peleg has not been exploited in nonlinear analysis. The purpose of this paper is to obtain, from Peleg's theorem, some generalizations of fundamental results in this field.

In Section II, we give an infinite dimensional version of Peleg's theorem which is parallel to the Ky Fan's extension of Knaster-Kuratowski-Mazurkiewicz's theorem.

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The main result in Section III is a generalization of Browder's fixed point theorem, involving multi-valued mappings from the product of a finite family of non-empty compact convex sets (each in a Hausdorff topological vector space) into each of its factors, obtained from the basic lemma of the preceding section.

Section IV is devoted to Ky Fan type inequalities, which, analogously to the classical case, are shown to be equivalent to their corresponding Browder type theorems of Section III.

Finally, in Section V, a result of the preceding section is used to derive a generalization of the Ky Fan's intersection theorem for sets with convex sections.

We shall denote by "*co*" the usual convex hull operator in a vector space. Given a multi-valued mapping between two sets, $T : A \longrightarrow 2^B$, by $T^{-1}(y)$, $y \in B$, we represent the sets $\{x \in A |\ y \in T(x)\}$. If x is an element of a cartesian product $X_1 \times \ldots \times X_m$, its $k-$th component will be denoted by x^k, whereas $x^{\hat{k}}$ will represent the element in $X_{\hat{k}} = X_1 \times \ldots \times X_{k-1} \times X_{k+1} \times \ldots \times X_m$ obtained from x by deleting its $k-$th component; in this way, $(x^{\hat{k}}, x^k)$ may be regarded as identical to x and, more generally, an element of the form $(x^{\hat{k}}, y^k)$, with $y^k \in X_k$, will be interpreted as that point of X obtained from x by replacing its $k-th$ component by y^k.

II. THE FUNDAMENTAL LEMMA

Our starting point is the following generalization, due to Peleg [10], of the famous Knaster-Kuratowski-Mazurkiewicz's theorem:

Lemma 1 [10]. For $k = 1, \ldots, m$ let N_k be a non-empty finite set and

$$S_k = \left\{\alpha^k : N_k \longrightarrow R \mid \alpha^k(i) \geq 0 \quad \text{for all} \quad i \in N_k, \sum_{i \in N_k} \alpha^k(i) = 1\right\}$$

(the simplex the coordinates of whose points are indexed by the members of N_k). If C_i^k, $i \in N_k$, $k = 1, \ldots, m$, are closed subsets of $S = S_1 \times \ldots \times S_m$, such that for each $Q \subset N_k$, $k = 1, \ldots, m$,

$$\bigcup_{j \in Q} C_j^k \supset \{(\alpha^1, \ldots, \alpha^m) \in S \mid \alpha^k(i) = 0 \quad \text{for all} \quad i \in N_k \backslash Q\},$$

then

$$\bigcap_{k=1}^{m} \bigcap_{i \in N_k} C_i^k \neq \emptyset.$$

When $m = 1$, the preceding lemma becomes the classical Knaster-Kuratowski-Mazurkiewicz's theorem. In the same way as Ky Fan obtained a generalization of the latter for the infinite dimensional case [4], our fundamental lemma states the infinite dimensional version of Lemma 1:

Lemma 2. For $k = 1, \ldots, m$ let X_k be an arbitrary set in a Hausdorff topological vector space Y_k and let $F_k : X_k \longrightarrow 2^Y$ be a mapping taking closed values in the product space $Y = Y_1 \times \ldots \times Y_m$ such that the following conditions are satisfied.

(i) If, for each $k = 1, \ldots, m$, N_k is a non-empty finite subset of X_k, then

$$co\, N_1 \times \ldots \times co\, N_m \subset \bigcap_{k=1}^{m} \bigcup_{x^k \in N_k} F_k(x^k) .$$

(ii) There exists $k \in \{1, \ldots, m\}$ and $x^k \in X_k$ such that $F_k(x^k)$ is compact.

Then

$$\bigcap_{k=1}^{m} \bigcap_{x^k \in X_k} F_k(x^k) \neq \emptyset .$$

Proof. For each k, let N_k denote a non-empty finite subset of X_k and let S_k be the simplex the coordinates of whose points are indexed by the elements of N_k. Define $q : S_1 , \times \ldots \times S_m \longrightarrow Y_1 \times \ldots \times Y_m$ by

$$q(\alpha^1, \ldots, \alpha^m) = \Big(\sum_{x^1 \in N_1} \alpha^1(x^1) x^1, \ldots, \sum_{x^m \in N_m} \alpha^m(x^m) x^m \Big) .$$

Regarding each simplex as a subset of an Euclidean space and considering the corresponding product topology on $S_1 \times \ldots \times S_m$, the function q is continuous. Hence, the sets $C^k_{x^k} = q^{-1}\big(F_k(x^k)\big)$, $x^k \in N_k$, $k = 1, \ldots, m$ are closed. Let $Q \subset N_k$ for some k.

Using (i), we obtain

$$\begin{aligned} \bigcup_{x^k \in Q} C^k_{x^k} &= q^{-1}\Big(\bigcup_{x^k \in Q} F_k(x^k) \Big) \supset \\ &\supset q^{-1}(co\, N_1 \times \ldots \times co\, Q \times \ldots \times co\, N_m) = \\ &= \Big\{ (\alpha^1, \ldots, \alpha^m) \in S \mid \sum_{x^k \in N_k} \alpha^k(x^k) x^k \in co\, Q \Big\} \supset \\ &\supset \Big\{ (\alpha^1, \ldots, \alpha^m) \in S \mid \alpha^k(x^k) = 0 \quad \text{for all} \quad x^k \in N_k \backslash Q \Big\} , \end{aligned}$$

whence, by Lemma 1,

$$q^{-1}\Big(\bigcap_{k=1}^{m}\bigcap_{x^k\in N_k} F_k(x^k)\Big)=\bigcap_{k=1}^{m}\bigcap_{x^k\in N_k} C^k_{x^k}\neq\emptyset\,.$$

Therefore, $\bigcap_{k=1}^m\bigcap_{x^k\in N_k} F_k(x^k)\neq\emptyset$. Hence, using (ii) and the finite intersection property of compact sets one easily gets $\bigcap_{k=1}^m\bigcap_{x^k\in X_k} F_k(x^k)\neq\emptyset$. Q.E.D.

A more general version of Lemma 2 can be easily obtained by adapting the proof of Lemma 1 in [2], replacing the assumption that the mappings F_k are closed-valued and that one of them has some compact image by the following weaker hypotheses:

(i) For each $k = 1,\ldots,m$ and every $x^k\in X_k$, the intersection of $F_k(x^k)$ with any product of finite dimensional subspaces is closed,

(ii) If, for each $k=1,\ldots,m$, D_k is a finite dimensional subspace of Y_k, then

$$\overline{\bigcap_{k=1}^{m}\bigcap_{x^k\in X_k\cap D_k}} F_k(x^k)\cap(D_1\times\ldots\times D_m)=\bigcap_{k=1}^{m}\bigcap_{x^k\in X_k\cap D_k} F_k(x^k)\cap(D_1\times\ldots\times D_m)$$

and

(iii) There exists $k\in\{1,\ldots,m\}$ and $x^k\in X_k$ such that $\overline{F_k(x^k)}$ is compact.

Since we shall not use this stronger version of Lemma 2, we omit the proof.

III. A GENERALIZATION OF BROWDER'S FIXED POINT THEOREM

Throughout this section, we shall denote by X_k, $k=1,\ldots,m$, a non-empty compact convex subset of a Hausdorff topological vector space Y_k and by X the product $X_1\times\ldots\times X_m$. Based on Lemma 2, we shall first establish a generalization of Browder's fixed point theorem:

Theorem 3. For $k=1,\ldots,m$ let $T_k : X\longrightarrow 2^{X_k}$ be such that, for each $x\in X, T_k(x)$ is convex (or empty) and, for each $x^k\in X_k$, $T_k^{-1}(x^k)$ is open in X. If for each $x\in X$ there exists $k=k(x)\in\{1,\ldots,m\}$ for which $T_k(x)\neq\emptyset$, then there exists $\bar{x}=(\bar{x}^1,\ldots,\bar{x}^m)\in X$ and $k\in\{1,\ldots,m\}$ such that $\bar{x}^k\in T_k(\bar{x})$.

Proof. For each k, let $F_k : X_k\longrightarrow 2^Y$ be the mapping defined by $F_k(x^k) = X\backslash T_k^{-1}(x^k)$. Clearly, F_k is compact-valued. We have

$$\bigcap_{k=1}^{m}\bigcap_{x^k\in X_k} F_k(x^k)=X\backslash\bigcup_{k=1}^{m}\bigcup_{x^k\in X_k} T_k^{-1}(x^k)=\emptyset\,,$$

since any $x \in X$ has some non-empty image $T_k(x)$. Therefore, by Lemma 2, there exist non-empty finite subsets $N_k \subset X_k$, $k = 1, \ldots, m$, such that

$$co\, N_1 \times \ldots \times co\, N_m \not\subset \bigcap_{k=1}^{m} \bigcup_{x^k \in N_k} F_k(x^k) .$$

Take $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^m) \in co\, N_1 \times \ldots \times co\, N_m \setminus \bigcap_{k=1}^m \bigcup_{x^k \in N_k} F_k(x^k)$.
For some $k \in \{1, \ldots, m\}$ we have $\bar{x} \notin F_k(x^k)$ for all $x^k \in N_k$ or, equivalently, $N_k \subset T_k(\bar{x})$. Hence, by the convexity of $T_k(\bar{x})$, we obtain

$$\bar{x}^k \in co\, N_k \subset T_k(\bar{x}) .$$

Q.E.D.

Browder's fixed point theorem corresponds to the particular case $m = 1$ in the preceding theorem. Although Theorem 3 is not a fixed point theorem, from it one easily obtains the following fixed point result, which is also a generalization of Browder's one.

Corollary 4. Let $T : X \longrightarrow 2^X$ be such that, for each $x \in X$, the sets $\{y^k \in X_k \,|\, (x^{\hat{k}}, y^k) \in T(x)\}$, $k = 1, \ldots, m$, are convex (or empty) and at least one of them is non-empty and, for each $k = 1, \ldots, m$ and $y^k \in X_k$ the set $\{x \in X , | (x^{\hat{k}}, y^k) \in T(x)\}$ is open in X. Then T has a fixed point.

Proof. The mappings $T_k : X \longrightarrow 2^{X_k}$ defined by $T_k(x) = \{y^k \in X_k \,|\, (x^{\hat{k}}, y^k) \in T(x)\}$ satisfy the hypotheses of Theorem 3, whence there exist $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^m) \in X$ and $k \in \{1, \ldots, m\}$ such that $\bar{x}^k \in T_k(\bar{x})$. But this means that $(\bar{x}^{\hat{k}}, \bar{x}^k) \in T(\bar{x})$, i.e., that $\bar{x}$ is a fixed point of T. Q.E.D.

The mapping T satisfies the assumptions of Corollary 4 when, for each $x \in X$, the image $T(x)$ is multiconvex (in the sense of [7], i.e., when its sections in each component are convex; see also [1]) and contains some point differing from x in at most one component and, for each $k \in \{1, \ldots, m\}$ and each $y^k \in X_k$, the set $\{(x, y^{\hat{k}}) \in X \times X_{\hat{k}} \,|\, (y^{\hat{k}}, y^k) \in T(x)\}$ is open in X.

From Theorem 3, we get the following generalization of Lemma 4 in [4]:

Corollary 5. Let

$A_k \subset X_k \times X$, $k = 1, \ldots, m$, be such that the following conditions are satisfied:

(i) For each $k = 1, \dots, m$ and $y^k \in X_k$, the set $\{x \in X \mid (y^k, x) \in A_k\}$ is closed in X.

(ii) For any $x = (x^1, \dots, x^m) \in X$ and each $k = 1, \dots, m$, $(x^k, x) \in A_k$.

(iii) For any $x \in X$, the sets $\{y^k \in X_k \mid (y^k, x) \notin A_k\}$, $k = 1, \dots, m$, are convex (or empty).

Then there exists $\bar{x} \in X$ such that $X_k \times \{\bar{x}\} \subset A_k$ for each $k = 1, \dots, m$.

Proof. For each k define $T_k : X \longrightarrow 2^{X_k}$ by $T_k(x) = \{y^k \in X_k \mid (y^k, x) \notin A_k\}$. By assumptions (iii) and (i), the images $T_k(x)$, $x \in X$, are convex (or empty) and the inverse images $T_k^{-1}(y^k)$, $y^k \in X_k$, are open in X. On the other hand, by (ii), we have $x^k \notin T_k(x)$ for every $x = (x^1, \dots, x^m) \in X$ and each $k \in \{1, \dots, m\}$. Therefore, by Theorem 3, there exists $\bar{x} \in X$ such that $T_k(\bar{x}) = \emptyset$ for each k. But the emptiness of $T_k(\bar{x})$ is clearly equivalent to the inclusion $X_k \times \{\bar{x}\} \subset A_k$. Q.E.D.

Assumption (i) in the preceding corollary holds when the sets A_k are closed in $X_k \times X$.

Corollary 6. Let $A \subset X \times X$ be such that the following conditions are satisfied:

(i) For each $k = 1, \dots, m$ and $y^k \in X_k$, the set $\{x \in X \mid (x^{\hat{k}}, y^k, x) \in A\}$ is closed in X.

(ii) For any $x \in A$, $(x, x) \in A$.

(iii) For any $x \in X$, the sets $\{y^k \in X_k \mid (x^{\hat{k}}, y^k, x) \notin A\}$, $k = 1, \dots, m$, are convex (or empty).

Then there exists $\bar{x} \in X$ such that $\left(\bigcup_{k=1}^m (\{\bar{x}^{\hat{k}}\} \times X_k)\right) \times \{\bar{x}\} \subset A$.

Proof. For each k, let $A_k = \{(y^k, x) \in X_k \times X \mid (x^{\hat{k}}, y^k, x) \in A\}$. These sets satisfy the hypotheses of Corollary 5, whence there exists $\bar{x} \in X$ such that $X_k \times \{\bar{x}\} \subset A_k$ for each k. But the set of these inclusions is equivalent to $\left(\bigcup_{k=1}^m (\{\bar{x}^{\hat{k}}\} \times X_k)\right) \times \{\bar{x}\}$.
Q.E.D.

Alternatively, Corollary 6 could have been proved from Corollary 4, using the mapping $T : X \longrightarrow 2^X$ defined by $T(x) = \{y \in X \mid (y, x) \notin A\}$.

Assumption (i) in Corollary 6 is satisfied when A is closed, while condition (iii) holds when the sets $\{y \in X \mid (y, x) \notin A\}$, $x \in X$, are multiconvex.

We observe that the set $\bigcup_{k=1}^{m}(\{\bar{x}^{\hat{k}}\} \times X_k)$, appearing in the conclusion of Corollary 6, consists of those points in X which differ from $\bar{x}$ in at most one component.

IV. KY FAN TYPE INEQUALITIES

Throughout this section, as in the preceding one, X_k, $k = 1, \ldots, m$, will denote a non-empty compact convex subset of a Hausdorff topological vector space Y_k and X will represent their product $X_1 \times \ldots \times X_m$. Our first result generalizes Lemma 1.3 in [8], which is in turn a generalization of the famous Ky Fan's minimax inequality [6]:

Theorem 7. Let $f_k : X \times X_k \longrightarrow R$, $G_k : X \longrightarrow 2^{X_k}$, $k = 1, \ldots, m$, be such that $f_k(\cdot, x^k)$ is lower semicontinuous and $G_k^{-1}(x^k)$ is open in X for every $x^k \in X_k$ and $f_k(x, \cdot)$ is quasiconcave and $G_k(x)$ is convex for every $x \in X$. Then there exists $\bar{x} \in X$ such that

$$\sup_{y^k \in G_k(\bar{x})} f_k(\bar{x}, y^k) \leq \sup_{x^k \in G_k(x)} f_k(x, x^k)$$

for each $k = 1, \ldots, m$.

Proof. For each k, let $\mu_k = \sup_{x^k \in G_k(x)} f_k(x, x^k)$ and $A_k = \{(y^k, x) \in X_k \times X \mid f_k(x, y^k) \leq \mu_k$ or $y^k \notin G_k(x)\}$. The sets A_k satisfy the conditions of Corollary 5, whence there exists $\bar{x} \in X$ such that $X_k \times \{\bar{x}\} \subset A_k$ for each k. But it is easy to check that the inclusion $X_k \times \{\bar{x}\} \subset A_k$ is equivalent to the inequality

$$\sup_{y^k \in G_k(\bar{x})} f_k(\bar{x}, y^k) \leq \sup_{x^k \in G_k(x)} f_k(x, x^k).$$

Q.E.D.

The assumptions of the preceding theorem can be replaced by the following weaker conditions:

(i) For each $k = 1, \ldots, m$ and $y^k \in X_k$, the set $\{x \in X \mid f_k(x, y^k) \leq \mu_k\} \cup (X \setminus G_k^{-1}(y^k))$ (with μ_k as in the proof of Theorem 7) is closed in X.

(ii) For any $x \in X$, the sets $\{y^k \in G_k(x) \mid f_k(x, y^k) > \mu_k\}$, $k = 1, \ldots, m$, are convex (or empty).

Indeed, these conditions are equivalent to (i) and (iii) of Corollary 5, respectively, for the sets A_k defined in the preceding proof.

Taking $G_k(x) \equiv X_k$, $k = 1, \ldots, m$, in Theorem 7, one obtains a version of Ky Fan's minimax inequality with several functions; the classical inequality corresponds to the case $m = 1$.

We have derived Theorem 7 from Corollary 5 and the latter from Theorem 3. In the converse direction, Theorem 3 can be easily deduced from Theorem 7, by applying it to the functions $f_k : X \times X_k \longrightarrow R$ defined by $f_k(x, y^k) = 1$ if $y^k \in T_k(x)$. Otherwise, and to the mappings $G_k : X \longrightarrow 2^{X_k}$ given by $G_k(x) \equiv X_k$.

Corollary 8. Let $f : X \times X \longrightarrow R$ and $G : X \longrightarrow 2^X$ be such that, for each $k = 1, \ldots, m$, the function $x \in X \longmapsto f(x, x^{\hat{k}}, y^k) \in R$ is lower semicontinuous and the set $\{x \in X \mid (x^{\hat{k}}, y^k) \in G(x)\}$ is open in X for any $y^k \in X_k$ and $f(x, x^{\hat{k}}, \cdot)$ is quasiconcave and the set $\{y^k \in X_k \mid (x^{\hat{k}}, y^k) \in G(x)\}$ is convex for any $x \in X$. Then there exists $\bar{x} \in X$ such that

$$\sup_{1 \leq k \leq m} \sup_{(\bar{x}^{\hat{k}}, y^k) \in G(x)} f(\bar{x}, \bar{x}^{\hat{k}}, y^k) \leq \sup_{x \in G(x)} f(x, x) .$$

Proof. Apply Theorem 7 to the functions $f_k : X \times X_k \longrightarrow R$ defined by $f_k(x, y^k) = f(x, x^{\hat{k}}, y^k)$ and the mappings $G_k : X \longrightarrow 2^{X_k}$ given by $G_k(x) = \{y^k \in X_k \mid (x^{\hat{k}}, y^k) \in G(x)\}$, observing that, for each k and $x = (x^1, \ldots, x^m) \in X$, one has $f_k(x, x^k) = f(x, x)$ and equivalency between the conditions $x^k \in G_k(x)$ and $x \in G(x)$. Q.E.D.

According to the observation we have made after Theorem 7, the assumptions we have made on f and G in the preceding corollary can be replaced by the following weaker conditions

(i) For each $k = 1, \ldots, m$ and $y^k \in X_k$, the set $\{x \in X \mid f(x, x^{\hat{k}}, y^k) \leq \sup_{x \in G(x)} f(x, x)$ or $(x^{\hat{k}}, y^k) \notin G(x)\}$ is closed in X.

(ii) For any $x \in X$, the sets $\{y^k \in X_k \mid (x^{\hat{k}}, y^k) \in G(x),\ f_k(x, y^k) > \sup_{x \in G(x)} f(x, x)\}$, $k = 1, \ldots, m$, are convex (or empty).

Taking $G(x) \equiv X$ in Corollary 8, one obtains a version of Ky Fan's minimax inequality valid for the case when the usual quasiconcavity assumption on the functions $f(x, \cdot)$ is relaxed to multiquasiconcavity (i.e., quasiconcavity in each component [1]), which can be expressed in the following way:

$$\sup_{y \sim \bar{x}} f(\bar{x}, y) \leq \sup_{x \in X} f(x, x) ,$$

the relation $y \sim \bar{x}$ meaning that y differs from $\bar{x}$ in at most one component. Again when $m = 1$, this coincides with the usual Ky Fan's inequality.

An alternative proof of Corollary 8 can be obtained from Corollary 6 applied to $A = \{(y,x) \in X \times X \mid f(x,y) \le \sup_{x \in G(x)} f(x,x) \text{ or } y \notin G(x)\}$. We have already observed that Corollary 6 is an immediate consequence of Corollary 4. Conversely, Corollary 4 can be easily derived from Corollary 8, by applying it to the function $f : X \times X \longrightarrow R$ defined by $f(x,y) = 1$ if $y \in T(x)$, 0 if $y \notin T(x)$ and to the mapping $G : X \longrightarrow 2^X$ given by $G(x) \equiv X$.

Corollary 9. Let $g_k : X \times X_k \longrightarrow R$, $k = 1, \ldots, m$, be continuous and such that $g_k(x, \cdot)$ is quasiconvex for every $x \in X$. Then there exists

$$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^m) \in X \quad \text{such that} \quad g_k(\bar{x}, \bar{x}^k) = \min_{y^k \in X_k} g_k(\bar{x}, y^k) \quad \text{for each} \quad k = 1, \ldots, m .$$

Proof. It follows from Theorem 7 applied to the functions $f_k : X \times X_k \longrightarrow R$ defined by $f_k(x, y^k) = g_k(x, x^k) - g_k(x, y^k)$ and the mappings $G_k : X \longrightarrow 2^{X_k}$ given by $G_k(x) \equiv X_k$, observing that $\sup_{y^k \in X_k} f_k(\bar{x}, y^k) = g_k(\bar{x}, \bar{x}^k) - \inf_{y^k \in X_k} g_k(x, y^k)$ and $f_k(x, x^k) = 0$ for any $x = (x^1, \ldots, x^m) \in X$. Q.E.D.

When one takes $m = 1$ in Corollary 9, it becomes Corollary 1 in [6]. On the other hand, the Nash Theorem on the existence of equilibrium points in n-person games [9] is also an immediate consequence of Corollary 9. Indeed, if $f_k : X \longrightarrow R$, $k = 1, \ldots, m$, are continuous multiquasiconcave functions, then applying Corollary 9 to the functions $g_k : X \times X_k \longrightarrow R$ defined by $g_k(x, y^k) = -f(x^{\hat{k}}, y^k)$ one gets the existence of $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^m) \in X$ such that $f_k(\bar{x}) = \max_{y^k \in X_k} f_k(\bar{x}^{\hat{k}}, y^k)$ for each $k = 1, \ldots, m$.

Corollary 10. Let $g : X \times X \longrightarrow R$ be continuous and such that, for each $k = 1, \ldots, m$, the function $g(x, x^{\hat{k}}, \cdot)$ is quasiconvex for any $x \in X$. Then there exists $\bar{x} \in X$ such that $g(\bar{x}, \bar{x}) = \min_{y^k \in X_k} g(\bar{x}, \bar{x}^{\hat{k}}, y^k)$ for each $k = 1, \ldots, m$.

Proof. For each k define $g_k : X \times X_k \longrightarrow R$ by $g_k(x, y^k) = g(x, x^{\hat{k}}, y^k)$. These functions satisfy the assumptions of Corollary 9, whence there exists $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^m) \in X$ such that

$$g(\bar{x}, \bar{x}) = g_k(\bar{x}, \bar{x}^k) = \min_{y^k \in X_k} g_k(\bar{x}, y^k) = \min_{y^k \in X_k} g(\bar{x}, \bar{x}^{\hat{k}}, y^k)$$

for each k. Q.E.D.

The preceding corollary could have been deduced alternatively from Corollary 8, applied to $f : X \times X \longrightarrow R$ defined by $f(x,y) = g(x,x) - g(x,y)$ and $G : X \longrightarrow 2^X$ given by $G(x) \equiv X$.

Comparing Corollary 10 with Corollary 1 in [6], we observe that the quasiconvexity assumption on the functions $g(x,\cdot)$ in the latter has been relaxed to a weak form of multiquasiconvexity in Corollary 10, but the conclusion in this corollary, which can be expressed as $g(\bar{x},\bar{x}) = \min_{y\sim\bar{x}} g(\bar{x},y)$ ($y \sim \bar{x}$ denoting that y differs from $\bar{x}$ in at most one component), is also weaker than that of Corollary in [6] (where $y \in X$ appears instead of $y \sim \bar{x}$).

V. A GEOMETRIC THEOREM ON SETS WITH CONVEX SECTIONS

In this section, $X_{k,i}$, $i = 1,\ldots,n_k$ $(n_k \geq 2)$, $k = 1,\ldots,m$ denote non-empty compact convex sets, each in a Hausdorff topological vector space, $X_k = X_{k,1} \times \ldots \times X_{k,n_k}$, $k = 1,\ldots,m$ and $X = X_1 \times \ldots \times X_m$. Our first result generalizes Theorem 7 in [6] (which corresponds to the particular case $m = 1$).

Theorem 11. Let $f_{k,i} : X \longrightarrow R$, $i = 1,\ldots,n_k$, $k = 1,\ldots,m$ be such that, for any $x^{k,i} \in X_{k,i}$, the function $(x^{\hat{k}}, x^{k,\hat{i}}) \in X_{\hat{k}} \times X_{k,\hat{i}} \longmapsto f_{k,i}(x^{\hat{k}}, x^{k,\hat{i}}, x^{k,i}) \in R$ is lower semicontinuous and, for any $(x^{\hat{k}}, x^{k,\hat{i}}) \in X_{\hat{k}} \times X_{k,\hat{i}}$, the function $x^{k,i} \in X_{k,i} \longmapsto f_{k,i}\,(x^{\hat{k}}, x^{k,\hat{i}}, x^{k,i}) \in R$ is quasiconcave.
Let $t_{k,i} \in R$, $i = 1,\ldots,n_k$, $k = 1,\ldots,m$. If for each $x = \big((x^{1,1},\ldots,x^{1,n_1}),\ldots,(x^{m,1},\ldots,x^{m,n_m})\big) \in X$, there exists $k = k(x) \in \{1,\ldots,m\}$ such that for every $i = 1,\ldots,n_k$, there is a point $\tilde{y}^{k,i} \in X_{k,i}$ for which $f_{k,i}(x^{\hat{k}}, x^{k,\hat{i}}, \tilde{y}^{k,i}) > t_{k,i}$, then there exist $\bar{k} \in \{1,\ldots,m\}$ and $\bar{x} \in X$ such that $f_{\bar{k},i}(\bar{x}) > t_{\bar{k},i}$, $i = 1,\ldots,n_{\bar{k}}$.

Proof. For each k and $i \in \{1,\ldots,n_k\}$, define $f_k : X \times X_k \longrightarrow R$ by $f_k(x,y^k) = \min_{1\leq i\leq n_k}\{f_{k,i}(x^{\hat{k}}, x^{k,\hat{i}}, y^{k,i}) - t_{k,i}\}$ for $X = \big((x^{1,1},\ldots,x^{1,n_1}),\ldots,(x^{m,1},\ldots,x^{m,n_m})\big) \in X$ and $y^k = (y^{k,1},\ldots,y^{k,n_k}) \in X_k$. These functions satisfy the assumptions of Theorem 7, whence there exists $\bar{x} \in X$ such that $\sup_{y^k\in X_k} f_k(\bar{x},y^k) \leq \sup_{x\in X} f_k(x,x^k)$, $k = 1,\ldots,m$. On the other hand, for $\bar{k} = k(\bar{x})$ we have $\sup_{y^{\bar{k}}\in X_{\bar{k}}} f_{\bar{k}}(\bar{x},y^{\bar{k}}) \geq f_{\bar{k}}(\bar{x},\tilde{y}^{\bar{k}}) > 0$, with $\tilde{y}^{\bar{k}} = (\tilde{y}^{\bar{k},1},\ldots,\tilde{y}^{\bar{k},n_{\bar{k}}})$ as in the statement. Combining these results, we get the existence of $\bar{x} = (\bar{x}^1,\ldots,\bar{x}^m) \in X$ such that $f_k(\bar{x},\bar{x}^k) > 0$. But, according to the definition of the f_k's, this is equivalent to the set of inequalities $f_{\bar{k},i}(\bar{x}) > t_{\bar{k},i}$, $i = 1,\ldots,n_{\bar{k}}$. Q.E.D.

As a consequence of the preceding theorem, we next obtain a generalization of an intersection theorem for sets with convex sections, due to Ky Fan [4]:

Theorem 12. Let $E_{k,i} \subset X$, $i = 1,\ldots,n_k$, $k = 1,\ldots,m$ be such that, for any $x^{k,i} \in X_{k,i}$, the section $E_{k,i}(x^{k,i}) = \{(x^{\hat{k}}, x^{k,\hat{i}}) \in X_{\hat{k}} \times X_{k,\hat{i}} \,|\, (x^{\hat{k}}, x^{k,\hat{i}}, x^{k,i}) \in E_{k,i}\}$ is open in $X_{\hat{k}} \times X_{k,\hat{i}}$ and, for any $(x^{\hat{k}}, x^{k,\hat{i}}) \in X_{\hat{k}} \times X_{k,\hat{i}}$, the section $E_{k,i}^{-1}(x^{\hat{k}}, x^{k,\hat{i}}) = \{x^{k,i} \in X_{k,i} \,|\, (x^{\hat{k}}, x^{k,\hat{i}}, x^{k,i}) \in E_{k,i}\}$ is convex (or empty). If for each $x = \big((x^{1,1},\ldots,x^{1,n_1}),\ldots,(x^{m,1},\ldots,x^{m,n_m})\big) \in X$, there exists $k = k(x) \in \{1,\ldots,m\}$ such that $E_{k,i}^{-1}(x^{\hat{k}}, x^{k,\hat{i}}) \neq \emptyset$, $i = 1,\ldots,n_k$, then

$$\bigcup_{k=1}^{m} \bigcap_{i=1}^{n_k} E_{k,i} \neq \emptyset \,.$$

Proof. For each k and $i \in \{1,\ldots,n_k\}$ let $f_{k,i} : X \longrightarrow R$ be the characteristic function of $E_{k,i}$, i.e., $f_{k,i}(x) = 1$ if $x \in E_{k,i}$, 0 if $x \notin E_{k,i}$. Letting $t_{k,i} = 0$, the assumptions of Theorem 11 are satisfied, whence there exists $\bar{k} \in \{1,\ldots,m\}$ and $\bar{x} \in X$ such that $f_{\bar{k},i}(\bar{x}) > 0$, $i = 1,\ldots,n_{\bar{k}}$. But, in view of the definition of the $f_{k,i}$'s, the inequality $f_{\bar{k},i}(\bar{x}) > 0$ is equivalent to $\bar{x} \in E_{\bar{k},i}$. Therefore, we have

$$\bar{x} \in \bigcap_{i=1}^{n_{\bar{k}}} E_{\bar{k},i} \subset \bigcup_{k=1}^{m} \bigcap_{i=1}^{n_k} E_{k,i} \,.$$

Q.E.D.

The above mentioned theorem of Ky Fan corresponds to the case $m = 1$ in the preceding theorem. As observed by Ky Fan for this particular case, [5], [6], in the same way that Theorem 12 follows from Theorem 11, the latter can be easily derived from Theorem 12, by considering the sets $E_{k,i} = \{x \in X \,|\, f_{k,i}(x) > t_{k,i}\}$.

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