UNIVERSITAT DE BARCELONA

Treball final de grau

GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques i Informàtica
Universitat de Barcelona

KNOT THEORY: On the study of chirality of knots through polynomial invariants

---

Autor: Sergi Justiniano Claramunt

Director: Dr. Javier José Gutiérrez Marín
Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, January 18, 2019

Contents

Abstract

In this project we introduce the theory of knots and specialize in the computation of the knot polynomials. After presenting the Jones polynomial, its two two-variable generalizations are also introduced: the Kauffman and HOMFLY polynomial. Then we study the ability of these polynomials on detecting chirality, obtaining a knot not detected chiral by the HOMFLY polynomial, but detected chiral by the Kauffman polynomial.

Introduction

The main idea of this project is to give a clear and short introduction to the theory of knots and in particular the utility of knot polynomials on detecting chirality of knots. The core idea in Knot Theory is in how many different ways one can embed a circle in $\mathbb{R}^3$. From this simple idea a vast world opens to us: Reidemeister moves, rational knots, invariants, polynomials, etc. Properties of knots can be derived from studying their polynomials, like chirality, the property of a knot to be equivalent to its mirror image.

Knot Theory is a very new branch in mathematics (it was born a couple of hundreds of years ago) so there are new potential bridges to build between subjects. Yet, this branch of topology found its place in physics, biology and chemistry quickly. The way quantum particles interact can be modeled using knots and even the structure of the DNA is a double helix and its interactions can also be modeled using knot surgery.

Such an interdisciplinary field caught my attention early. It is also true that the simple way Knot Theory is seen, through the use of regular diagrams of knots, seem welcoming. It is hidden difficulty and the possibility of new and creative ways to manipulate such simple things was what made me decide to choose this topic among others.

The structure of this project goes as follows: first of all, in the first chapter, we begin with the most basic definitions that arise when trying to understand what a knot means mathematically. After discussing the proper way to define equivalence between knots we show how to project them on the plane and the basic moves we can do in this projection. Invariants arise naturally from the previous sections. Some further discussions on properties of knots and simple ways to create them can be found at the end of the chapter. In particular, we introduce the knot 10_{48} which properties will be discussed in Chapter 3. Chapter 2 is devoted entirely to the understanding of the Jones polynomial , with some simple computations to make the reader get used to the manipulation of diagrams. The last chapter introduces the main generalizations of the Jones polynomial, getting both the Kauffman polynomial and HOMFLY polynomial of the knot 10_{48}. A final test on the ability of these knots on the detection of the chirality of 10_{48} closes Chapter 3. Conclusions and Bibliography are found at the end of this project.

Chapter 1

Mathematical bases

Knots have been useful for humankind since the beginning of times. Our ancestors would need them to tie their cattle or horses, parking their ships or even for religious emblems or events. The mathematical interest only came after Lord Kelvin hypothesis of the structure of atoms based on knots [17]. Even thought it was proved wrong, it motivated for research on the basis of what we consider now to be a big branch in topology, Knot Theory.

1.1 Definition of a knot

First of all we need to visualize our object of study. A knot is nothing more than a tangled string. If we let the ends run free we should be able to play with it. We could, for example, untangle it completely. Hence, to be able to study knottedness we need to trap the knotted part of the string. We do so in two possible ways: sending each end of the string to infinity like in Figure 1.1 or gluing both ends together like in Figure 1.2.

Figure 1.1: Projection of a knot, like a shadow of a real knot in $\mathbb{R}^3$, with both ends coming from infinity. Example of the knot 8_1 with ends coming and going to infinity.

(a) Unknot (b) Trefoil (c) Figure 8 knot

Figure 1.2: The three most basic knots, each of them having zero, three and four crossings respectively. The unknot can be considered to be a simple circle and is used as the unitary element.

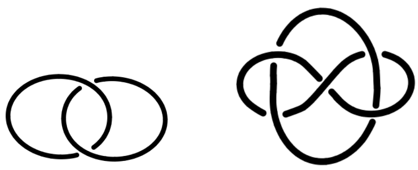
The vast majority of Knot Theory is concerned with the topological properties of loops embedded in $\mathbb{R}^3$ or $\mathbb{S}^3$. We want our objects of study to represent reality as much as possible, but mathematicians only work on the realm of mathematics so we need to define what we understand for a mathematical knot.

Definition 1.1. A *knot* is an embedding of the circle $\mathbb{S}^1$ into $\mathbb{R}^3$ or $\mathbb{S}^3$.

During this project the use of *links* will appear in the next chapters so we introduce what we understand for a *link*:

Definition 1.2. A *link* $\mathbb{L}$ is a finite disjoint union of knots $\mathbb{L} = \mathbb{K}_{\mathbb{1}} \cup \cdots \cup \mathbb{K}$.

Two of the most basic links with only two components can be found in Figure 1.3. Most of the definitions and properties that we discuss for knots can be transferred to links easily. For example, the concept of unknottedness also exists for links: The *unlink* of k components, $\bigcirc^k$, is defined as $\underbrace{\bigcirc \sqcup \ldots \sqcup \bigcirc}_{k}$.



(a) Hopf Link (b) Whitehead link

Figure 1.3: Two of the most basic links with two components. They are so basic they have their own names.

This would be the simplest way to introduce a mathematical knot (or a link): to consider it a closed curve in real space without self-intersections. By using this definition we are imposing the second way of representing knots mentioned before (when both ends of the string are glued together). This very definition already creates some ambiguities that need to be solved. For example, by definition, apart from the already introduced knots in Figure 1.2, the one in Figure 1.4 should also be considered a knot. This type of wild behaviours do not represent a real situation in our world. So we need a first classification of knots that will erase this ambiguity: those knots that behave well, that do not collapse into a singular point, will be called *tame knots* and those that seem to fall into an endless vortex will be referred as *wild knots*, as the one in Figure 1.4.

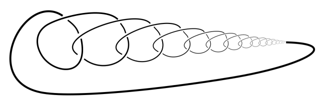


Figure 1.4: Example of a wild knot.
(Source: https://wikipedia.org)

How can we translate this cases into rigorous classifications? One way is to use differential knots. By doing this we remove the situation where a knots collapses into a point because the derivative would respectively collapse in that point. Nevertheless, in this project we follow another procedure: we consider a tame knot to be composed of polygonal curves instead of differential ones, as descrived in detail in [13, Chapter 3]. Since polygonal curves are finite by nature they cannot define wild knots.

For any two distinct points in $\mathbf{R}^3$, p and q let $[p, q]$ denote the line segment joining them. For an ordered set of distinct points, $(p_1, \ldots, p_n)$, the union of the segments $[p_1, p_2], [p_2, p_3], \ldots, [p_{n-1}, p_n]$ and $[p_n, p_1]$ is called a closed polygonal curve. If each segments only intersects two other segments at the endpoints, the curved is said to be simple.

Definition 1.3. A *knot* is a simple closed polygonal curve in R^3.

If the ordered set $(p_1, \ldots, p_n)$ defines a knot and no proper ordered subset defines the same knot , each $\{p_i\}$ is called the vertex i.

A third way to define tame knots is by using the concept of *local flatness*:

Definition 1.4. A point p in a knot $\mathbf{K}$ is *locally flat* if there is some neighbourhood $U \ni p$ such that the pair $(U, U \cap K)$ is homeomorphic to $(B_0(1), B_0(1) \cap X)$ where $B_0(1)$ represents the unit ball in $\mathbb{R}^3$ and X represents the x-axis. A knot is locally flat if each point $p \in \mathbf{K}$ is locally flat.

Definition 1.5. A *tame knot* is a locally flat subset of points homeomorphic to a circle.

The three representations are interchangeable: both polygonal and differential knots have the property of local flatness. The proof is a bit technical but can be found in [4, Chapter 1.11].

Now we have our object of study well defined in $\mathbb{R}^3$. We desire to study which of these objects define the same essential type of knot.

1.2 Equivalence of knots

We want our objects of study to be close to real case. Therefore, we consider two knots to be equivalent if we can deform one into the other like if they were real ropes. Two equivalent knots belong to the same knot type class. A function between our objects that preserves the knot type is what we seek to find. To be more precise, we want to find a morphism with special conditions to be able to create classes of equivalence of knots. Sometimes there will be an abuse of language and we will refer to both a knot and its equivalence class using the same words. No confusion should arise from this.

The deformation we are looking for could not be a homotopy (continuous map $h : X \times [0,1] \to Y$) since it allows self-intersections and we would conclude that all knots are equivalent to the trivial knot. If we impose the condition of bijection we are using an *isotopy*: continuous collections of embeddings from X to Y.

Again, this deformation does not meet all the requirements we impose onto our desired equivalence. The most well-known counter-example is the *bachelors' unknotting* where the knotted part is shrunk continuously into a point, making it disappear, as in Figure 1.5. We can visualize it in real life where the knot is made smaller and smaller by "pulling both ends of the string", which would be the "easiest way" to get rid of it, from there comes the name. Again we would conclude that all our knots were equivalent to the unknot.

To stop this shrinking we only need to make a small modification. We need to make sure that the space surrounding the knot moves continuously along with the knot. To do so, we modify the definition of *isotopy* into an *ambient isotopy*:

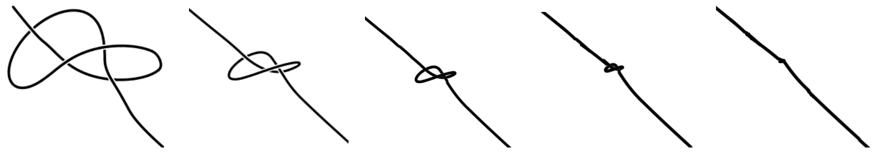


Figure 1.5: From left to right the moves of the bachelors' unknotting are showed, where the knotted part is shrunk to a point by "pulling both ends of the string".

Definition 1.6. *Let X and Y be two different manifolds and f_0 and f_1 be two distinct embeddings from X to Y. A continuous map $h : Y \times [0,1] \longrightarrow Y$ is called an ambient isotopy from f_0 to f_1 if $h(\cdot, 0)$ is the identity, $h(\cdot, t)$ is a homeomorphism from Y to itself and $f_1 = h(\cdot, 1) \circ f_0$.*

This is the equivalence relation that allows to transform one knot into another while preserving all its properties. As mentioned before, from now on knots and their classes will be treated equally and will be referred using their *knot type*. Two knots are the same if they have the same knot type while two knots are distinct if they do not have the same knot type.

A parallel definition can be made by imposing an *orientation preserving homeomorphism* between two knots as the condition of equivalence. These two approaches are equivalent since every orientation preserving homeomorphism is isotopic to the identity map [8].

In low dimension Knot Theory we can work with a more approachable method that has been mentioned before and will give us the same results. Instead of using smooth function relating knots, a polygonal approach is used.

Let K be a knot and Δ be a triangle such that: K does not meet the interior of Δ, K intersects the triangle in one or two sides only and that they share the same vertices in the intersection. If Δ fulfills this three conditions we can define an *elementary move* or Δ-move as K'=$(K - (K \cap \Delta)) \cup (\partial\Delta - K)$, as seen in Figure 1.6. The inverse process is denoted by Δ^{-1}.

It can be proved that the previous definitions are equivalent. Finding an ambient isotopy between K_1 and K_2 is equivalent to finding a finite sequence of Δ-moves that transforms K_1 into K_2 [2, Chapter 1, Section B].

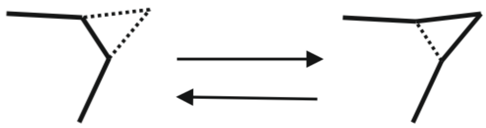


Figure 1.6: Elementary move: The black straight line represents our knot diagram while the doted lines form a triangle with the coincident lines of the knot. Switching the edge on the left for the two edges on the right defines a planar isotopy.

1.3 Knot projections and diagrams

So far we have only used a specific way of representing knots rather than their true form in $\mathbb{R}^3$, like in Figure 1.1. The idea is to project a knot into a plane, like a shadow of the knot. But by doing this, sometimes, two points of the knot project to the same point and we lose information about which parts of the knot passed over other parts, like in Figure 1.7.

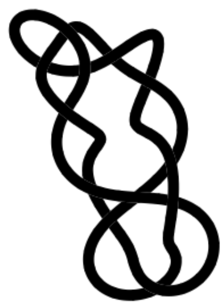


Figure 1.7: Example of a knot without enough information to manipulate it.

To solve this uncertainty, gaps are left in the drawing of the projection to indicate which portions of the knot pass under other parts. Such drawings are called diagrams.

In some diagrams we can face undesired situations like in Figure 1.8: A case where 3 points project to the same point, creating a triple crossing, and a case where a vertex is projected onto another point, creating a tangential point. In a need to erase these scenarios we will define what we consider to be a *nice* diagram.

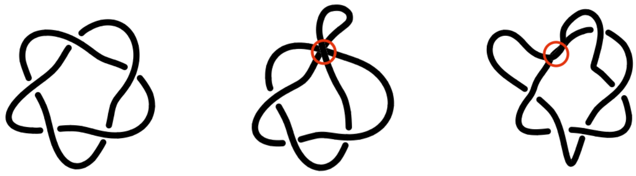


Figure 1.8: Thee diagrams of the 5_1 knot. From left to right: a regular diagram; a diagram that has a triple crossing and a diagram that has a tangential (a vertex is projected on a double crossing). Both specific points where this uncertainties appear are circled in red.

Definition 1.7. *A knot diagram is called regular if no three points of the knot project to the same point, and no vertex projects to the same point as any other point of the knot.*

Regular diagrams are useful because of the following theorems which can be found in [5, Chapter 1, Section 2] and [2, Chapter 1, Section C] respectively:

Theorem 1.8. *Every tame knot has a regular diagram.*

Theorem 1.9. *Two knots are equivalent if they have regular projections and identical diagrams*

From now on every simple curve of a knot diagram will be called a *strand*, for example, for the trefoil we have 3 strands. In each intersection, called a *crossing point* or just *crossing*, the strand that splits the diagram will be called an *overcrossing* while the one that splits in two will be referred as an *undercrossing*.The number of strands is equal to the number of crossings. See Figure 1.9 for an example. The first notation that was invented for classifying knots depends on the number of crossings n and an integer m, like n_m, where m is just a arbitrary label. A complete list of knot with $n \leq 9$ can be found, for instance, in [4, Appendix A].

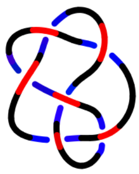


Figure 1.9: Knot 6.1: First knot tabulated to have 6 crossings. The over-crossing parts of the knot have been coloured with red while the under-crossing has been done with blue. Each curve that runs from a blue tag to a blue tag is referred as a strand.

Planar isotopies are the most basic moves on knot diagrams. They do not create or remove any crossing, they just twist, expand or shrink strands of the knot without changing the intersections. It is clear that if two knots are related by a planar isotopy are equivalent.

1.4 Reidemeister moves

Apart from aesthetically reasons, planar diagrams are useful for classifying knots. To do so, we introduce the three Reidemeister moves (commonly called R-moves). Figures 1.10, 1.11 and 1.12 represent the three different moves that Reidemeister defined in 1927 [16]: twist, poke and slide. He proved that all equivalent knots can be related through finite sequences of these moves and planar isotopies. They are basic in Knot Theory since they encapsulate the concept of ambient isotopy of knots in three-dimensional space:

Theorem 1.10. *Two knots are topologically equivalent if and only if their diagrams can be deformed into each other by a finite sequence of Reidemeister moves [15, Chapter 4, Section 1].*

Two links are regular isotopic if they can be related through R-II and R-III only.

Figure 1.10: R1 move: the twist.

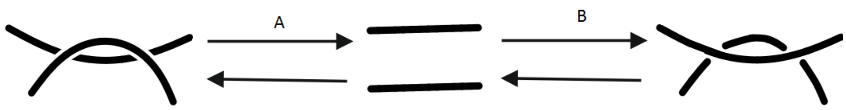
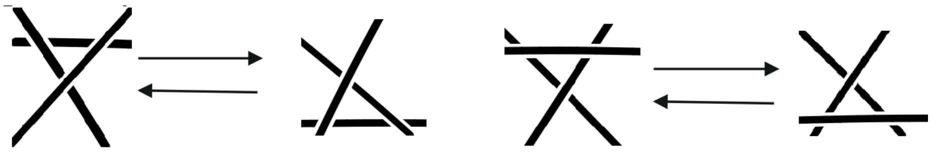


Figure 1.11: R2 move: the poke. Separating two strings. As commented before, a vertex cannot be projected onto a double point. However, through elementary moves, Reidemeister proved that the two final states are equivalent.



(a) R3 move: slide down (b) R3 move: slide up

Figure 1.12: As commented before, three points cannot project to the same point. However, through elementary moves, Reidemeister proved that the two final states are equivalent.

If we tried to go from the trefoil knot to the unknot throughout Reidemeister moves it would take us a while to realize there is no way to do so, since they are not actually equivalent. But what if we did not know beforehand the inequivalence of two knots? What if after trying 1.000 moves we stop but the next step was taking us to a final solution? This very question takes us to the next section: **Knot invariants**.

1.5 Invariants

How can we know two apparently knots are actually different? How do we even know there are more knots apart from the trivial one? Was there another way of untangling the trefoil without cutting any strand and still get the unknot?

As we have commented in the previous section, Reidemeister moves are basic to discern between equivalent knots when we can actually find a finite path of Reidemeister moves from one knot to another. In this sense we could consider the Reidemeister moves to be the most basic invariants. But what if those knots are not be equivalent? How can one prove there is no way to modify one knot into the other by using Reidemeister moves? Our solution will be called *knot invariants*.

Definition 1.11. *A Knot invariant is a function from the isotopy classes of knots to some algebraic structure.*

As seen in Algebraic Topology, we connect our objects of study to algebraic structures usually easier to manipulate, like groups. The key point is that the representative of each knot type class must remain the same throughout Reidemeister moves. Since Reidemeisters moves determine the existence of ambient isotopy between knots we only need to deal with these invariants to categorize knots.

There are really simple to introduce invariants like the link component: number of components needed to create a specific link. Since we cannot "cut and reglue" any of the components they stay the same number throught ambient isotopies. A more interesting numerical invariants are the crossing number, the minimal number of crossing among all the diagrams of a knot, and the unknotting number, minimal number of crossings that must change from over to undercrossing or vice versa to obtain the unknot. Invariants can even be defined so that a knot fulfills specific properties, like tricorolability. This invariant is not that useful since it is a binary invariant: you either fulfill the property or not. In this specific case, a knot is said to be tricolorable if each strand can be colored with one of the 3 colors available following these simple rules:

- At least two colors must be used.

- When there is a crossing, the three incident branches have either all the same color or all different colors.

Since the unknot (in any possible diagram) cannot be colored using more than 1 color it is not tricolorable. In Figure 1.13 we see a representation of the trifoil fulfilling the previous mentioned rules. Then we can state now that, at least, there exist a knot different than the unknot.

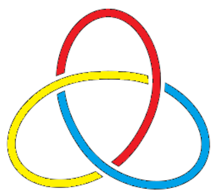


Figure 1.13: Colored trefoil: in each crossing the 3 colors meet.

These seems to be easy to state invariants and can be very useful to classify knots with few crossing because they are well understood. However, finding the exact value or even boundings can be difficult for a knot with high amount of crossings. In contrast, polynomial invariants (from each knot you derive a, usually, Laurent polynomial) can be computed following a short algorithm. That is the reason why they have been studied so much in Knot Theory. They classify pretty well basic knots and depending on the complexity of the polynomial can encapsulate basic properties of the knots studied.

1.6 Symmetries, properties and generation of knots

One of the main goals in Knot Theory is the total classification of knots. One may try to find not equivalences between knots with different crossings or also a knot and its mirror image. The image mirror of a knot K is the knot that we obtain after applying a reflection in the origin ($h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$), usually denoted with K^*. One can obtain such a knot by simply changing each crossing: every over-strand will now be an under-strand and vice versa, like in Figure 1.14. If K is **not** ambient isotopic (equivalent) to its mirror image we say that K is *cheiral*. If $K = K^*$ we say that the knot is *acheiral*.

For example, the left trifoil and the right trifoil are cheiral (that's the reason why there are two different trefoil knots). They are inequivalent but we will need the tools from next chapter to prove it.

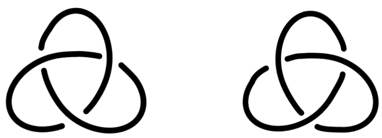
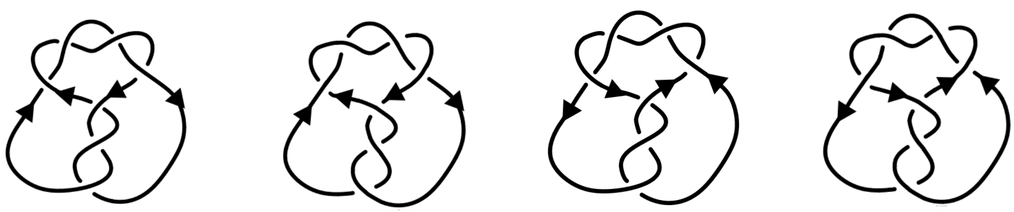


Figure 1.14: From left to right: Right and left trefoil. There is no physical way to untangle one into the other without cutting.

A knot is a closed curved so we can assign an orientation to that curve. The knot will inherit that specific orientation, marked as usual with an arrow on the curve. Every knot has two possible orientations, so from K we can create its *reverse knot* $-K$. A knot K such that $K = -K$ is called reversible. The composition of an mirror image plus a reverse, denoted with $-K^*$, is called the *inverse* of K. Figure 1.15 gives an example of these operations.



(a) Oriented Knot K (b) K^*: Mirror image of K (c) $-K$: Reverse image of K (d) $-K^*$: Inverse image of K

Figure 1.15: Four knots related through symmetries operations: mirror image, reverse image and the combination of these two, the inverse image.

As if knots were numbers we can also define an operation of addition of knots, the *connected sum* (#), from which the definition of *prime knots* arises. For this operation to be well-defined we need two oriented knots: we make a "small cut" on both knots and reglue the free ends like in Figure 1.16.

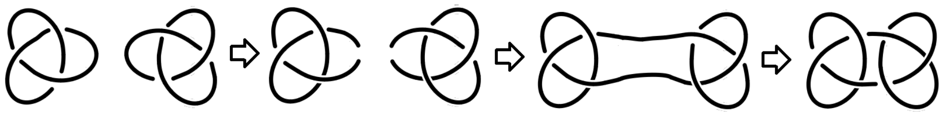


Figure 1.16: From left to right: surgery operation using two right trefoils ($3_1\#3_1$). Their connected sum is obtained by joining the lose ends of a cut strand. The result knot is known as the *granny knot*.

A similar knot from the one seen in Figure 1.16 can be created by changing one of the right trefoil knot 3_1 for its mirror image 3_1^*, creating the *square knot* ($3_1\#3_1^*$).

1.7 Tangles and Conway notation

A simple way to deconstruct a knot into basic units is done through the study of *tangles*, introduced by Conway in [3].

Definition 1.12. *A tangle is a region of the projection plane delimited by a circumference that only intersect the knot in four different points.*

We label these intersections with the four compass directions *NW*, *NE*, *SW*, *SE* and we talk the *NW-SE* as the primary diagonal. Two tangles are equivalent if they can be related thought a finite sequence of Reidemeister moves that keep the four endpoints fixed. We usually represent a general tangle, *L*, as in Figure 1.17.

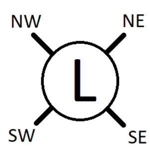


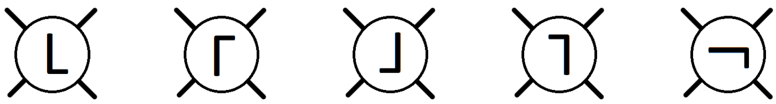
Figure 1.17: General tangle *L* with the four ends label with the four compass directions.

The most simple tangles are shown in Figure 1.18. Each tangle could be visualized with a circumference around or without, both notations are used.

(a) $+1$ (b) -1 (c) 0 (d) ∞

Figure 1.18: The four basic tangles with one or zero crossings.

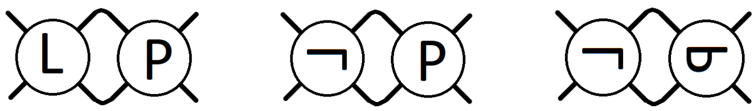
Conway introduced different operations onto these objects by rotation and reflection as depicted in Figure 1.19. In subfigure (a) we have the original tangle labeled L. In (b), the first operation, which is a reflection over the *W-E* axis (horizontal axis). In a similar way (c) has been created, using the *N-S* axis (vertical axis). Tangle (d) is a composition of the two previous reflections. Finally, tangle (e), which is labeled as $-L$, is a reflection over the NW-SE axis (primary diagonal reflection).



(a) (b) (c) (d) (e)

Figure 1.19: A tangle labeled L and its 4 reflections.

Conway also introduced three binary operations: from two tangles L and P we can take their *sum* $(L+P)$; *multiplication* (LP) and *ramification* (L,P), all visualized in Figure 1.20. These three operations are not independent since $LP = (-L) + P$ and $L,P = (-L) + (-P)$.



(a) Sum (b) Multiplication (c) Ramification

Figure 1.20: Binary tangle operations.

The n integer tangles are defined as $n = \overbrace{1 + \cdots + 1}^{n}$ and $\bar{n} = \overbrace{(-1) + \cdots + (-1)}^{n}$. For instance, the tangle 4 is represented in Figure 1.21 .

Figure 1.21: Integer tangle with four positive half twists.

Rational tangles are then defined as those created using integer tangles and the three binary operations (sum, multiplication and ramification). A knot can be created by taking the *numerator* or *denominator* of a tangle. These operations close the four ends of the tangle in different ways depicted in Figure 1.22. The *numerator* of a tangle is obtained by joining the NW end with the NE end and the SW point with the SE end, respectively. The *denominator* is created by joining the pair of ends NW with SW and NE with SE respectively. A knot created (in on of these two ways) from a rational tangle is called a rational knot. Lots of times in Conway notation the operation of numerator is done without mentioning since it is compulsory to obtain a knot.

Knots not created through integer tangles and their operations also exist, for example the knot 8_5. However, Conway proved, for example, that all knots and links up to ten crossings are either rational or are obtained by inserting rational tangles into a small number of planar graphs [12].

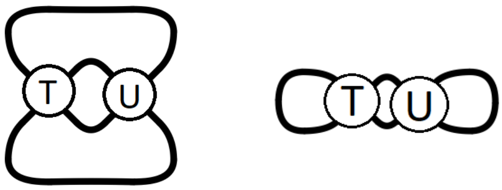


Figure 1.22: From left to right: Numerator and denominator of the sum $T + U$.

For instance, and following the notation introduced, the $41, 3, 2$ (knot 10_{48} in the old nomenclature) appears in Figure 1.23. It is important to notice that the Conway operations are not distributive so we need to read the operations from left to right. In our example case we would have $41, 3, 2 = ((4 * 0 + 1) * 0) + (3 * 0) + (2 * 0)$.

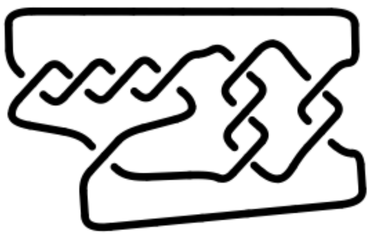


Figure 1.23: Knot 10_{48} obtained through tangle operation as 41,3,2

Chapter 2

Jones Polynomial

2.1 Introduction

In the search of new invariants, J.W. Alexander discovered the first polynomial invariant for links [1]. It had a complex way to compute through algebra before Conway gave its skein relation. However, in Knot Theory, one of the most studied polynomial invariants which allowed further developments is the Jones polynomial. At spring of 1984, Vaughan Jones introduced a Laurent polynomial (with integer coefficient) invariant of links, $V_L(t)$, arising from von Neumann algebras.

Most of the excitement coming from this new discovery, for which V. Jones received the Fields medal in 1990, was due to its origin. It made his appearance while studying operator algebras, useful in the formalization of *Quantum Field Theory*. It was not only stronger than its predecessor but it had clearly created a new path between Mathematics and Physics [11]. Lots of paper have been written about the Jones polynomial, allowing mathematicians to prove old conjectures [14] but yet being discovered more than 30 years ago there are still open questions related with how it encapsulates intrinsic properties of links. For example, it is still unknown whether there exist a non-trivial knot with trivial Jones polynomial.

This association of a polynomial to a link can be made by using link diagrams. The key is for the polynomial to remain unchanged when performing Reidemeister moves, defined the same for all the possible diagrams of a link. Therefore, if two diagrams have different Jones polynomials they cannot be equivalent.

2.2 Rules of bracket polynomial

To be able to understand the potential of the Jones polynomial we first need to take one step back and understand its simpler form, the *Kauffman bracket polynomial* [9]. It is defined as a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients. An easy introduction to this new polynomial goes as follow:

We assign a *bracket* $\langle L \rangle$ to every unoriented link diagram L. Our polynomial initially has three variables A, B and C satisfying the following three rules, (2.1), (2.2) and (2.3).

$$\text{Rule 1} : \langle \bigcirc \rangle = 1. \tag{2.1}$$

This is the normalization equation: the bracket polynomial of the 0 crossing representation of the unknot is the unit polynomial.

$$\text{Rule 2} = \begin{cases} \langle L_+ \rangle = A \langle L_\infty \rangle + B \langle L_0 \rangle . \\ \langle L_- \rangle = A \langle L_0 \rangle + B \langle L_\infty \rangle . \end{cases} \tag{2.2}$$

These rules represent the linear equation relating the bracket polynomials of diagrams that are equal except inside a small circle in which they look as in Figure 1.18.

Even thought we have used two equations in (2.2) they are indeed the same due to symmetry over rotation of 90 degrees. If we rotate the diagram L_+ we obtain L_-. Same result between L_0.

$$\text{Rule 3} : \langle L \sqcup \bigcirc \rangle = C \langle L \rangle . \tag{2.3}$$

Removing a crossingless loop from the rest of the diagram will be at cost of multiplying the bracket polynomial of what's left by a constant.

When we impose (2.2) and (2.3) to preserve the *Kauffman bracket* when using Reidemeister moves II and III we obtain the following relation between our variables A, B and C:

$$A^2 + ABC + B^2 = 0,$$

$$AB = 1,$$

which can be solved making $B = A^{-1}$ and $C = -A^2 - A^{-2}$.

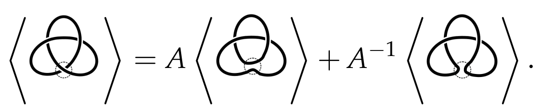
After applying these changes to Rule (2.2) and (2.3) we obtain:

$$\text{Rule 1} : \langle \bigcirc \rangle = 1. \tag{2.4}$$

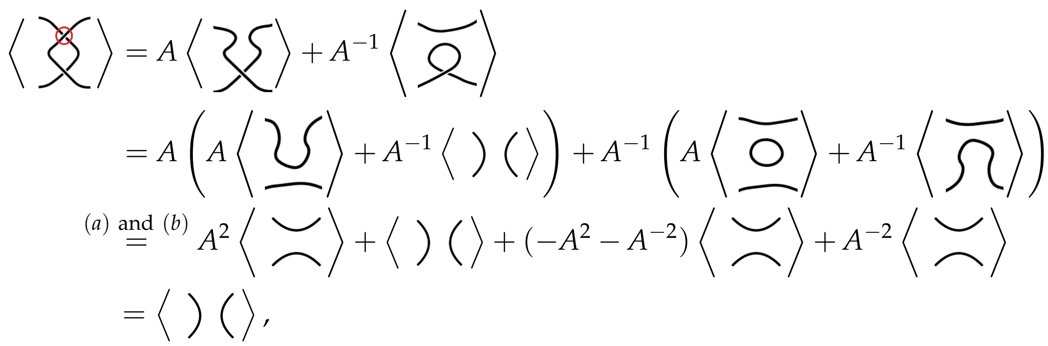
$$\text{Rule 2} : \begin{cases} \langle L_+ \rangle = A \langle L_\infty \rangle + A^{-1} \langle L_0 \rangle , \\ \langle L_- \rangle = A \langle L_0 \rangle + A^{-1} \langle L_\infty \rangle . \end{cases} \tag{2.5}$$

$$\text{Rule 3} : \langle L \sqcup \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle . \tag{2.6}$$

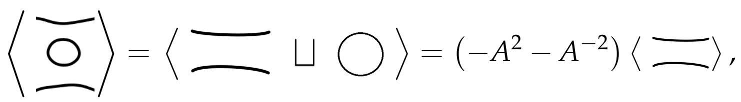
For example, by (2.5) the Trefoil knot can be related to the Hopf link and the unknot,

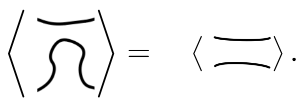


We want the Kauffman Bracket polynomial to be invariant under the Reidemeister-moves since they are equivalent to ambient isotopies. We are imposing the polynomial to be the same when, for example, *poking* two strands, like in Figure 1.11. We will show this property:



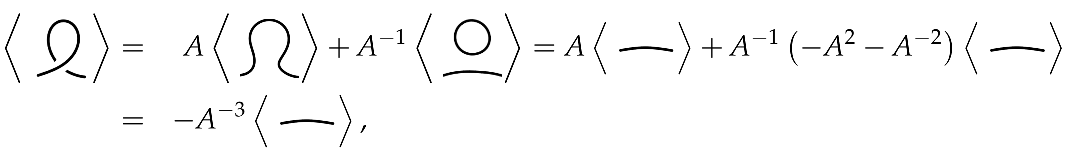
where we have used that

(a) 

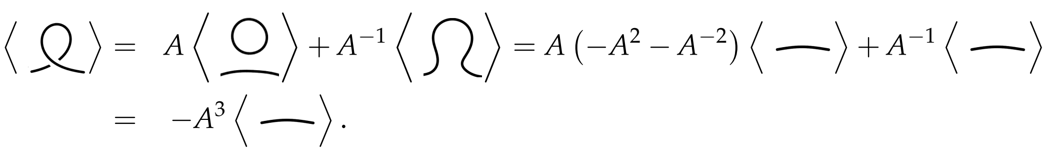
(b) 

A similar result would be found at modifying . The proof of the invariance of the R III move takes the same path but more cases need to be studied.

However, when we apply a Type I Reidemeiester we obtain the following:



(2.7)



So we have proved that the Kauffman bracket is not an invariant. Only a small modification is needed.

2.3 Writhe and invariance of Jones polynomial

To be able to compensate the term $-A^{\pm 3}$ that appears in the R-1 moves we will introduce the concept of *writhe*.

First of all we need to impose an orientation on our knot. Therefore, every crossing will take one of the forms shown in Figure 2.1 and Figure 2.2.

Figure 2.1: Positive writhe = +1.

Figure 2.2: Negative writhe = -1.

Now that we have imposed an orientation these two diagrams are not equivalent upon rotation so we can distinguish them in our knot. We define the total writhe of a knot K to be the sum of each writhe on each crossing, and will be denoted using *w(K)*.

Definition 2.1. *The writhe of a diagram is the sum of the number of times crossings look like Figure 2.1 in our knot, minus the amount of times we have a crossing looking like Figure 2.2.*

An example of the computation of the writhe for a specific knot is given by the Figure 2.3.

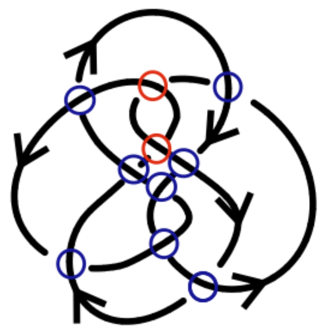


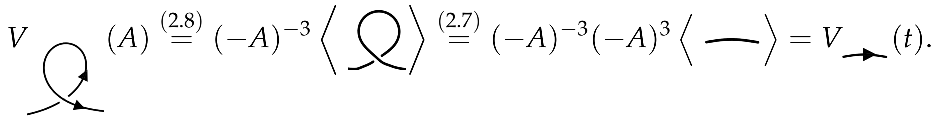
Figure 2.3: Oriented knot K which labeled positive and negative writhe with red and blue circles respectively. The total writhe is the sum of them counting the sign, $w(K) = -6$.

Now we are in a position to introduce the Jones polynomial the way L. Kauffman saw it:

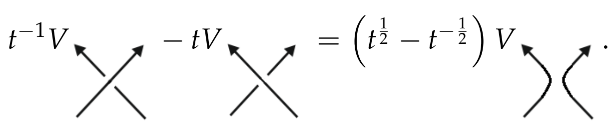
$$V_L(A) = (-A)^{-3w(L)} \langle L \rangle \,. \tag{2.8}$$

The original Jones polynomial is defined over a variable t. The relation between the Kauffman version and the Jones is $A^2 = t^{-\frac{1}{2}}$. Anyway, we use both variables when talking about the Jones polynomial indistinctly.

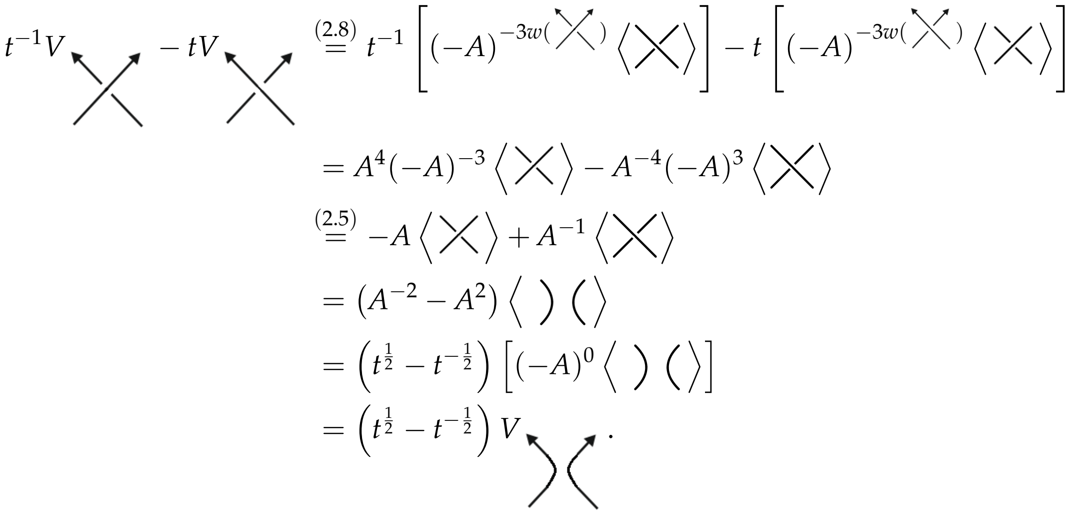
By correcting the Kauffman bracket polynomial by a factor of $(-A)^{-3w(L)}$ we make the Jones polynomial invariant under the three Reidemeister moves: $\langle \cdot \rangle$ was already invariant over R-II and R-III and these two moves do not alter the writhe of a diagram. Now, we compute again (2.7) with the writhe correction (noting that the twist that we are studying implies a positive unitary writhe):



An equivalent definition of the Jones polynomial can be taken if one imposes the polynomial to be the only Laurent polynomial invariant over regular isotopies of the plane, normalized by (2.4) and determined by the *skein relation*

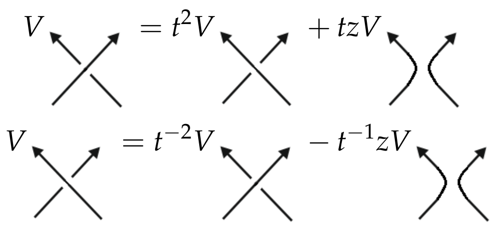
 (2.9)

Proof.



□

We can isolate each L_+ and L_- on (2.9) which gives us these two equivalent equations:

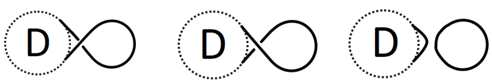


(2.10)

(2.11)

where $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. They will come in hand during the computations and proofs.

To be able to prove some properties of the Jones polynomial we first need to compute the Jones polynomial of a family of disjoint unknots. To do so we imagine a knot diagram D where we have applied a R-I move on one of its strands (positive and negative writhe), like in Figure 2.4.



(a) D^+ (b) D^- (c) $D \sqcup \bigcirc$

Figure 2.4: Three diagrams related by the Jones skein relation

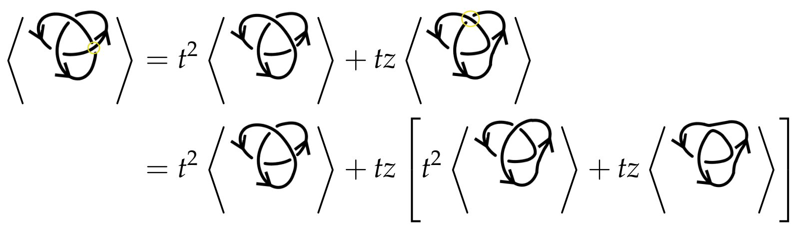
The three diagrams in Figure 2.4 are related by (2.9), with the extra condition that $D^+ \sim D^- \sim D$ which implies that their Jones polynomial are equal ($V_{D^+} = V_{D^+} = V_D$). All these conditions together implies that

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{\mathrm{D} \sqcup \bigcirc} = (t^{-1} - t)V_{\mathrm{D}} \to V_{\mathrm{D} \sqcup \bigcirc} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})V_{\mathrm{D}}. \tag{2.12}$$

We have seen the case for an unlink of 1 component. By induction on k,

$$V_{\mathrm{D} \sqcup \bigcirc^{k}}(t) = (-1)^k (t^{\frac{1}{2}} + t^{-\frac{1}{2}})^k V_{\mathrm{D}}(t). \tag{2.13}$$

The basic algorithm to compute any Jones polynomial is to apply knot surgery step by step until we are left with different copies of the unlink. For example, as we can see in Figure 2.5 for the right trefoil we obtain a certain decomposition which allows to compute its Jones polynomial in an easy way:



$$\begin{aligned} &\overset{(2.13)}{=} t^2 + tz\left[t^2\left[-(t^{\frac{1}{2}} + t^{-\frac{1}{2}})\right] + tz\right] \\ &= t^2 - t^3(t^{\frac{1}{2}} - t^{-\frac{1}{2}})(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) + t^2(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2 \\ &= t + t^3 - t^4. \end{aligned}$$

It is important to notice that every knot can be expressed as a series on the Jones polynomial of the unlink with polynomials on t as coefficients. After applying (2.13) we can express every Jones polynomial only using t (which is the way it is defined).

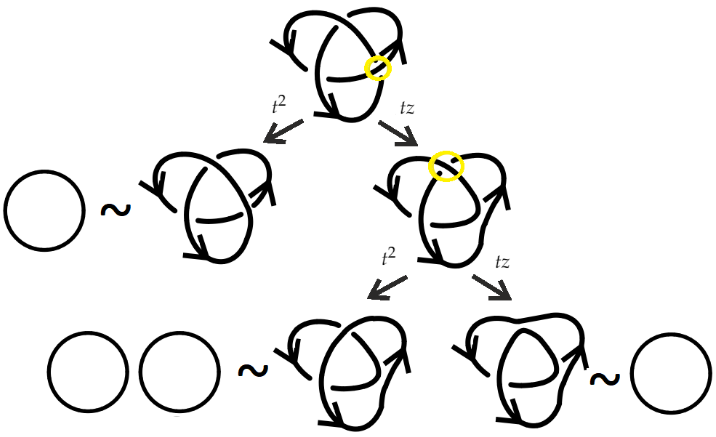


Figure 2.5: Decomposition of the right trefoil using the spanning tree of its Jones polynomial. On each yellow circled crossing equation (2.10) has been applied.

2.4 Main theorems and applications

In Chapter 1 we introduced a couple of ways to create a new knot from two previous given knots using the connected sum. The Jones polynomial preserves this operation, as can be seen in Theorem 2.2. Also in Chapter 1 we introduced the concept of *chirality*. The mirror image of a given knot can be obtained by switching every crossing in the knot. The Theorem 2.3 gives us an easy way to compute the Jones polynomial of the mirror image of a knot by simply exchanging t for t^{-1}. Both theorems can be found in [15].

Theorem 2.2. *Let L_1 and L_2 be two links:*

$$V_{L_1 \# L_2}(t) = V_{L_1}(t) V_{L_2}(t).$$

Proof. Suppose we consider the link L_2 to be a single point in L_1. From what has already been seen, there exist a decomposition of V_{L_1} as a linear combination of $V_{\bigcirc^k}$. If we decompose only the link L_1 leaving all the crossings from L_2 unchanged, we can write:

$$\begin{aligned} V_{L_1 \# L_2}(t) &= f_1(t) V_{L_2}(t) + f_2(t) V_{L_2 \sqcup \bigcirc} + \ldots + f_k(t) V_{L_2 \sqcup \bigcirc^{k-1}} \\ &\overset{(2.13)}{=} V_{L_2}(t) \left(f_1(t) + f_2(t)(-1)(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) + \ldots + f_k(t)(-1)^{k-1}(t^{\frac{1}{2}} + t^{-\frac{1}{2}})^{k-1} \right) \\ &= V_{L_2}(t) V_{L_1}(t). \end{aligned}$$

□

Theorem 2.3. *Let L denote a link and L^* its mirror image, then it is true that*

$$V_L(t) = V_{L^*}(t^{-1}).$$

Proof. L^* is constructed simply by changing all the crossings of L from over-crossings to under-crossings and vice versa. From equations (2.10) and (2.11) we realize that the *Jones trees* spanning from L and L^* follow the same equation after a change in the crossings and a change of variables $t^2 \iff t^{-2}$ and $tz \iff -t^{-1}z$ which implies a change of t for t^{-1}. □

Thus, if the Jones polynomial of a knot is not palindromic (like in the right trefoil) we can state that it is chiral. The converse is not always true.

Chapter 3

HOMFLY and Kauffman polynomials on chirality detection

The Jones polynomial can give us important information about the topological properties of a knot. For example, just by inverting t we can test the chirality of a given knot. After its discovery it was the strongest polynomial invariant for classifying knots. Its unexpected origin raised interest among researchers and not long after, *two* two-variable polynomials were presented: the HOMFLY and the Kauffman polynomial ([6] and [10]). HOMFLY takes its name from the six independent researchers, in four different groups, who published their results together in 1985. The Kauffman polynomial was introduced 5 years later (which is distinct from the Kauffman bracket polynomial). Both were conceived as generalized forms of the Jones polynomial, taking it as a particular case. However, they are not dependent from each other since they fail to detect, in different situations, some knots. From now on they will be referred as H- and K-polynomial.

In this chapter a short introduction of both H- and K-polynomials will be presented with an extra study on the ability of detecting chirality. Our study will be done on the knot $41, 3, 2(10_{48})$ introduced at the end of Chapter 1, showing that it is detected chiral by the K-polynomial but not by the H-polynomial. The question whether there exists a chiral knot not detected by the K-polynomial but detected by the H-polynomial remains open.

3.1 HOMFLY polynomial

The H-polynomial will be referred using the letter P with its diagram as subindex. It can be constructed giving a similar skein relation to the one in the

Jones polynomial.

Definition 3.1. *The H-polynomial P is the only homogeneous two-variable polynomial on oriented links satisfying*

(1) $v^{-1}P_{L+}(v,z) - vP_{L-}(v,z) = zP_{L_0}(v,z)$,

(2) $P_L(z,v) = 1$ *if L is the unlink of one component.*

As the Jones polynomial, the H-polynomial preserves the connected sum and the H-polynomial of the mirror image can be expressed as

$$P_{L^*}(v,z) = P_L(v^{-1},z). \tag{3.1}$$

The split union of two links L and L' can be expressed as

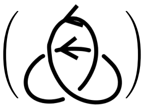
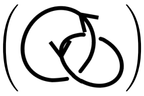
$$P_{L\sqcup L'} = \delta P_L P_{L'}, \tag{3.2}$$

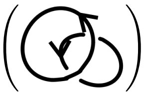
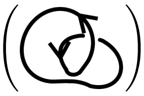
where $\delta = \frac{v^{-1}-v}{z}$.

The Jones polynomial can be seen as a particular case of the H-polynomial after the change of variable

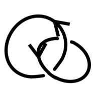
$$V_L(t) = P_L(t^{-1}, t^{-\frac{1}{2}} - t^{\frac{1}{2}}).j$$

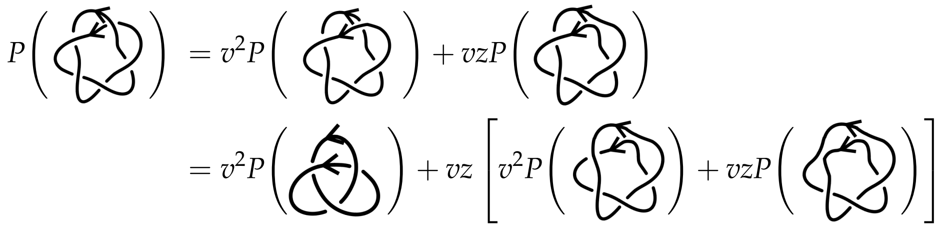
As basic examples we compute the H-polynomial of the right trefoil (equation (3.3)) and the knot 5_1 (equation (3.4)). In the following equations the notation is simplified: $P_L(v,z) = P(L)$.

$P\Big($  $\Big) = v^2 P\Big($  $\Big) + vzP\Big($  $\Big)$

$= v^2 + vz\Big[v^2P\Big($  $\Big) + vzP\Big($  $\Big)\Big]$

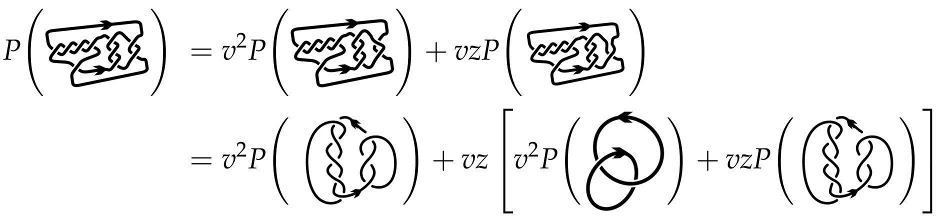
$$\overset{(3.2)}{=} v^2 + vz(v^2\delta + vz)$$
$$= 2v^2 - v^4 + v^2z^2. \tag{3.3}$$

While decomposing the knot 5_1 we use that  $\sim$  and  $\sim$ .



$$\overset{(3.3)}{=} v^2(2v^2 - v^4 + v^2z^2) + v^3z(v^2\delta + vz) + v^2z^2(2v^2 - v^4 + v^2z^2)$$
$$= 3v^4 - 2v^6 + (4v^4 - v^6)z^2 + v^4z^4. \tag{3.4}$$

We recover the knot presented at the end of Chapter 1 and proceed to decompose it, as we can see in Figure 3.1. This decomposition is useful because the diagram  is the connected sum of $3_1\#5_1^*$, so we can express $P(3_1\#5_1^*) = P(3_1)P(5_1^*)$. To compute the H-polynomial of 5_1^* we can use the property of the mirror image and the result of (3.4). Then the H-polynomial computation of the knot 10_{48} is given in (3.5).



$$= v^2(2v^2 - v^4 + v^2z^2)(-2v^{-6} + 3v^{-4} + z^2(4v^{-4} - v^{-6}) + v^{-4}z^4)$$
$$+ vz[v^2(v^{-2}\delta - v^{-1}z) + vz(2v^2 - v^4 + v^2z^2)(-2v^{-6} + 3v^{-4}$$
$$+ z^2(4v^{-4} - v^{-6})]$$
$$= (-4v^{-2} + 9 - 4v^2) + (-8v^{-2} + 20 - 8v^2)z^2$$
$$+ (-5v^{-2} + 18 - 5v^2)z^4 + (-v^{-2} + 7 - v^2)z^6 + z^8. \tag{3.5}$$

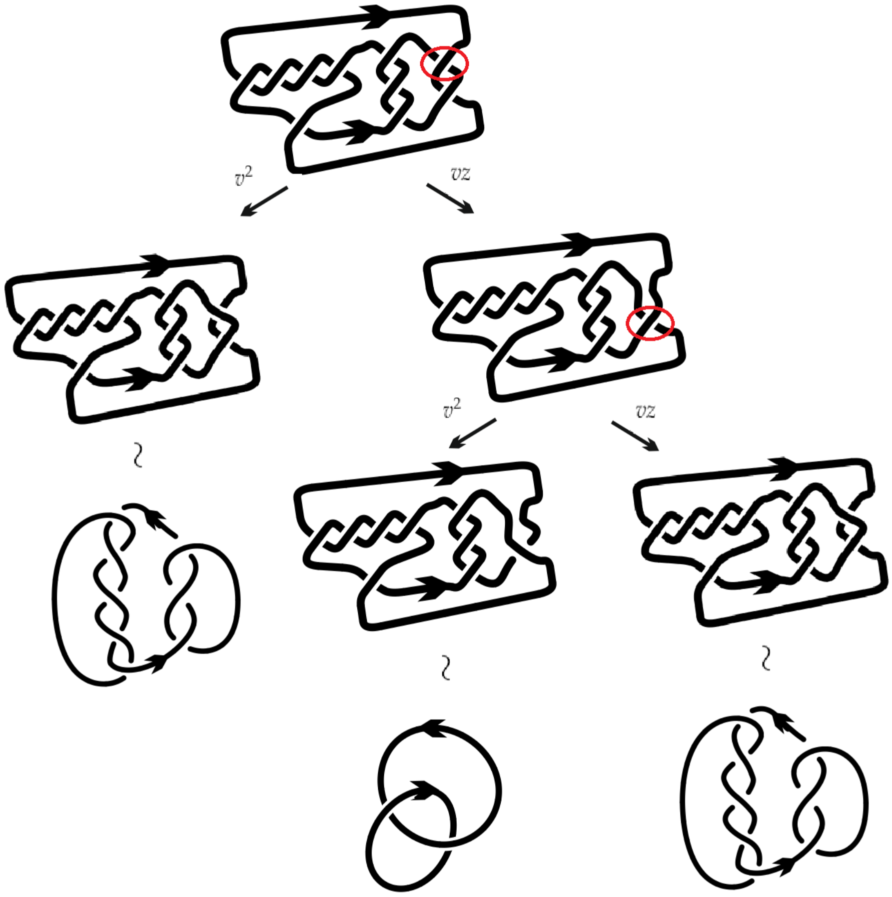


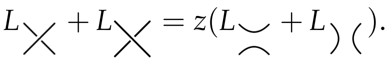
Figure 3.1: Decomposition of the knot 41,3,2 following the skein relation of the HOMFLY polynomial. The red circles mark the crossing on which we act.

3.2 Kauffman polynomial

Besides the H-polynomial, the other main generalization of the Jones polynomial is the Kauffman polynomial $F_K(a,z)$, discovered by Kauffman and published in 1990 [10]. The same way Kauffman created the Kauffman bracket polynomial, that extends to the Jones through the use of the writhe of the knot (2.8); he also created the $L_K(a,z)$ polynomial that extends to the Kauffman polynomial $F_K(a,z)$ through the use of the writhe. This L polynomial is defined for unoriented links and follows these properties:

(i) If two knots are regular isotopic their polynomial L is the same.

(ii) $L_{\bigcirc} = 1$.

(iii) 

(iv) 

Definition 3.2. *F is the only homogeneous two-variable polynomial on semioriented links satisfying*

$$F_K(a,z) = \alpha^{-w(K)} L_K(a,z).$$

When we talk about semioriented links it refers to the fact that to compute L_K we do not need an orientation over the knot but we do need it when calculating the writhe of a knot. .

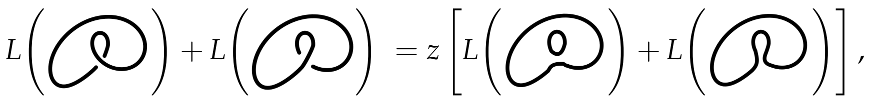
In a similar way to the H-polynomial, the K-polynomial of the mirror image of a knot is obtained by inverting a:

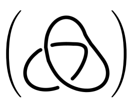
$$F_{K*}(a,z) = F_K(a^{-1},z). \tag{3.6}$$

As in the H-polynomial, the Jones polynomial is a particular case of the K-polynomial $F_K(a,z)$:

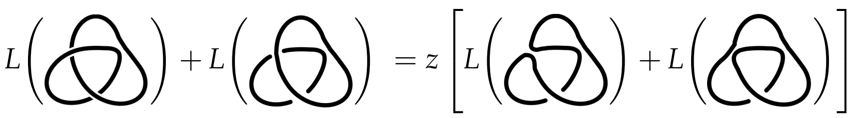
$$V_K(t) = F_K(-t^{-\frac{3}{4}}, t^{-\frac{1}{4}} + t^{\frac{1}{4}})$$

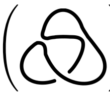
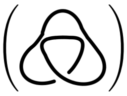
In the following equations the notation is simplified as $L_K(a,z) = L(K)$. As an example, we compute the K-polynomial of the trefoil. We need to do some computations before, the K-polynomial of the unlink of 2-components and of the Hopf Link. With respect to the unlink of 2-components, applying the property (iii),

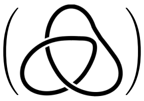


which implies that L  $= \frac{a+a^{-1}}{z} - 1$.

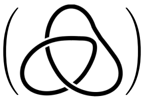
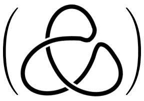
With respect to the K-polynomial of the Hopf link, applying again the third property (iii):



where L  $= a^{-1}$ and L  $= a$. So we get

L  $= (a^{-1}+a)z - (a^{-1}+a)z^{-1} + 1.$

And finally, the K-polynomial for the Trefoil:

L  $+ L$  $= z\,\Big[L$  $+ L$  $\Big],$

L  $+ a = z\left[(a^{-1}+a)z - (a^{-1}+a)z^{-1} + 1 + a^{-2}\right],$

L  $= (-a^{-1}-2a) + (a^{-2}+1)z + (a^{-1}+a)z^2,$

which implies by Definition 3.2 (in the case of the right trefoil, $w = 3$) that

F  $= a^{-3}L$  $= (-a^{-4}-2a^{-2}) + (a^{-5}+a^{-3})z + (a^{-4}+a^{2})z^2.$

By a similar decomposition of 10_{48} on Figure 3.1 we can decompose it following property (ii) and find its K-polynomial like we have just done for the Trefoil. The recurrence in this case is much longer than the case for the H-polynomial so we give the final result:

$$
\begin{aligned}
F(10_{48}) = {} & (4a^{-2}+9+4a^{2})z^{0} + (a^{-5}-3a^{-3}-9a^{-1}-7a+2a^{5})z^{1} \\
& + (a^{-4}-13a^{-2}-27-11a^{2}+2a^{4})z^{2} \\
& + (-3a^{-5}+8a^{-3}+21a^{-1}+12a-a^{3}-3a^{5})z^{3} \\
& + (-5a^{-4}+18a^{-2}+37+9a^{2}-5a^{4})z^{4} \\
& + (a^{-5}-9a^{-3}-11a^{-1}-5a-3a^{3}+a^{5})z^{5} \\
& + (2a^{-4}-11a^{-2}-20-5a^{2}+2a^{4})z^{6} + (3a^{-3}+a^{-1}+2a^{3})z^{7} \\
& + (3a^{-2}+5+2a^{2})z^{8} + (a^{-1}+a)z^{9}. \qquad (3.7)
\end{aligned}
$$

3.3 Testing chirality

In this section we will discuss the ability of detecting chirality of the two generalitzacions of the Jones polynomial introduced on the knot 10_{48}. As we mentioned previously, to test the ability to detect chirality, we only need to check if each pair of coefficient after inverting a remains unchanged [(3.1) and (3.6)].

The H-polynomial of 10_{48} is

$$\begin{aligned} P(10_{48}) = &(-4v^{-2} + 9 - 4v^2) + (-8v^{-2} + 20 - 8v^2)z^2 \\ &+ (-5v^{-2} + 18 - 5v^2)z^4 + (-v^{-2} + 7 - v^2)z^6 + z^8, \end{aligned}$$

where we can see that the change $v \sim v^{-1}$ leaves the polynomial unchanged, so the H-polynomial cannot detect the chirality of 10_{48}. Since the H-polynomial is a generalizations of the Jones polynomial we know that the Jones polynomial cannot detect chilarity of this knot either.

The K-polynomial of 10_{48} is

$$\begin{aligned} F(10_{48}) = &(4a^{-2} + 9 + 4a^2) + (a^{-5} - 3a^{-3} - 9a^{-1} - 7a + 2a^5)z \\ &+ (a^{-4} - 13a^{-2} - 27 - 11a^2 + 2a^4)z^2 \\ &+ (-3a^{-5} + 8a^{-3} + 21a^{-1} + 12a - a^3 - 3a^5)z^3 \\ &+ (-5a^{-4} + 18a^{-2} + 37 + 9a^2 - 5a^4)z^4 \\ &+ (a^{-5} - 9a^{-3} - 11a^{-1} - 5a - 3a^3 + a^5)z^5 \\ &+ (2a^{-4} - 11a^{-2} - 20 - 5a^2 + 2a^4)z^6 + (3a^{-3} + a^{-1} + 2a^3)z^7 \\ &+ (3a^{-2} + 5 + 2a^2)z^8 + (a^{-1} + a)z^9, \end{aligned}$$

where, by inspection on the term with z, we can see that the coefficient of a^{-5} and a^5 are different so the K-polynomial detects chirality of 10_{48}.

The knot $10_{48} = [41, 3, 2]$ is the beginning of a family of knots $[(k+2)1, 3, k]$ which chirality cannot be detected by the H-polynomial but it does by the K-polynomial, as stated in [7, Chapter 2, Section 10].

Chapter 4

Conclusions

The two main objectives of this project have been accomplished. The first one was to give short and clear presentation of Knot Theory as well as useful tools in the study of knots. We looked into the important ideas that makes knots what they are and how one can study the properties of a given knot.

The second one, which can be viewed as an extension of the first one, is related to the concept of knot polynomials. The main polynomial, the Jones polynomial, is described in detail, giving guided computations and proofs of the main properties. The study of these invariants continues in Chapter 3, where the real application of them takes place.

From a personal point of view this project, has meant a big conceptual challenge. From the first scratches on the surface of Knot Theory to a final definition of the goal of this project many topics haven been checked: the use of braids as representation of knots, the study of knots through the study of the surfaces they spawn, invariants of many types and a deep research into a wide variety of knot polynomials. Among them, the study of knots using polynomials appeared direct, efficient and, not as powerful as other approaches, but attainable for an undergraduate student.

Further study on the ability of chirality detection may take place in the future because as stated in [7, Chapter 2, Section 10], the knot 10_48 is the beginning of a family of knots whose chirality is undetected by the H-polynomial but detected by the K-polynomial.

Bibliography

[1] J.W. Alexander, *Topological invariants of knots and links*, Transactions of the American Mathematical Society, **30**, no. 2, (1928), 275–306.

[2] G. Burde and H. Zieschang, *Knots*, Berlin: Walter de Gruyter , (2003).

[3] J.H. Conway, *An Enumeration of Knots and Links, and Some of Their Algebraic Properties*, in J. Leech, *Computational Problems in Abstract Algebra*, Oxford, England: Pergamon Press, (1970), 329—358.

[4] P.R. Cromwell, *Knots and links*, Cambridge: Cambridge University Press, (2004).

[5] R.H. Crowell and R.H. Fox, *Introduction to knot theory*, New York: Springer-Verlag, (1963).

[6] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millett and A. Ocneanu *A New polynomial invariant of knots and links*, Bulletin (New Series) of the American Mathematical Society, **12**, no. 2, (1985), 239–246.

[7] S. Jablan and R. Sazdanovic, *LinKnot. Knot theory by computer*, Singapore: World Scientific, (2007).

[8] B. Jiang, S. Wang and Y. Wu, *Homeomorphisms of 3-manifolds and the realization of Nielsen Number*, Communications in Analysis and Geometry, **9**, no. 4, (2001), 825–877.

[9] L.H. Kauffman, *State models and the Jones polynomial*, Topology, **26**, no. 3, (1987), 395–407.

[10] L.H. Kauffman, *An invariant of regular isotopy*, Transactions of the American Mathematical Society, **318**, no. 2, (1990), 417–471.

[11] L.H. Kauffman, *Knots and physics*, World Scientific, (1991).

[12] L.H. Kauffman and S. Lambropoulou, *On the classification of rational knots*, L'Enseignement Mathématiques, **49**, (2003), 357–410.

[13] C. Livingston, *Knot theory*, Washington, D.C. : Mathematical Association of America, (1993).

[14] K. Murasugi, *Jones polynomials and classical conjetures in knot theory*, Topology, **26**, no. 2, (1987), 187–194.

[15] K. Murasugi, *Knot theory and its applications*, Boston: Birkhäuser, (1996).

[16] K. Reidemeister, *Knotten und Gruppen*, Abh. Math. Sem. Univ. Hamburg **5**, (1927), 7–23.

[17] W. Thomson, *On Vortex Atoms*, Proceedings of the Royal Society of Edinburgh, **6**, (1869), 94–105.