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A life insurance model with asymmetric time preferences ☆

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ABSTRACT

We build a life insurance model in the tradition of Richard (1975) and Pliska and Ye (2007). Two agents purchase life insurance by continuously paying two premiums. At the random time of death of an agent, the life insurance payment is added to the household wealth to be used by the other agent. We allow for the agents to discount future utilities at different rates, which implies that the household has inconsistent time preferences. To solve the model, we employ the equilibrium of Ekeland and Lazrak (2010), and we derive a new dynamic programming equation which is designed to find this equilibrium for our model. The most important contribution of the paper is to combine the issue of inconsistent time preferences with the presence of several agents. We also investigate the sensitivity of the behaviors of the agents to the parameters of the model by using numeric analysis. We find, among other things, that while the purchase of life insurance of one agent increases in her own discount rate, it decreases in the discount rate of the other agent.

1. Introduction

Over the last 15 years, many papers have been published in continuous time life insurance models. Much of the work has been based on the seminal paper by Pliska and Ye (2007), although the basic framework goes back to Yaari (1965) and Richard (1975). The models in this literature typically set up three separate decisions for an agent to make. First, there's the trade-off between consumption in the present and consumption in the future (through saving in the present). Secondly, there's the investment of savings into either an asset with a fixed return or an asset with a random return. Thirdly, there's the expenditure on an insurance premium which, at the time of the agent's death, results in a sum of money being paid out to his descendants.

The papers that are part of the literature on the life insurance model, can generally be classified into two categories. On the one hand, there are those papers that expand on the basic models by Richard (1975) and Pliska and Ye (2007), by allowing the agent to have inconsistent time preferences. On the other hand, there are those papers that expand on the basic models by having a household that consists of several agents, each of which may purchase her own life insurance.

In this paper, we construct a life insurance model which combines the two features of, on the one hand, households with inconsistent time preferences and, on the other hand, households with multiple agents. Specifically, we assume that the household consists of two agents, and that each agent has his own unique discount rate. This implies that although each agent has time preferences that are consistent, the household taken as a whole does not. Thus, we will derive a cooperative solution where the decision rules determine expenditures on consumption, investment and premium for a household that has inconsistent time preferences because the members of the household discount future utilities at different rates. This type of model seems to not have been investigated previously in the literature on life insurance models.

The assumption of asymmetric discount rates can be motivated in terms of increased realism. It also allows us to investigate new questions. People do indeed differ in their time preferences, that is, in how they evaluate an event in the near future relative to an identical event in the distant future. When two people with different time preferences manage a portfolio jointly, they will differ in how they see the trade-off between consumption and saving, and some kind of compromise must be reached. The problem becomes more complicated if they also have the option of buying life insurance. The purchase of life insurance by an agent imposes on her a cost in the immediate present, but the benefits for her descendants will be enjoyed over an extended period of time in the future. It is reasonable to suspect that if the descendants are patient, the agent will want to purchase a large life insurance (regardless of whether the agent herself is patient or not). This is indeed one of the results of the present paper. However, investigating

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this question requires that we can isolate the effects of the time preferences of one agent from those of another. This can only be done in a model with asymmetric discount rates.

We will also assume that the agents have constant but different mortality rates. The assumption of constant mortality rates is made for reasons of tractability. This means that the probability of dying at a particular distance of time into the future is the same regardless of what is the current time. This simplifies the task of deriving properties of the solution. Constant mortality rates are nonstandard in the literature. The most common approach is that no specific mortality rate is assumed in the analytic section of the paper, and that the mortality rate given by Gompertz law is assumed in the numeric section. According to Gompertz law, the mortality rate at time t is $e^{\frac{t-m}{n}}/n$, where n and m are constants. This implies that the behavior of an agent is affected by his age. One exception to this is found in Bayraktar and Young (2013), where the assumption is that the members of the household have the same constant discount and mortality rates. The present study extends their setup by allowing asymmetric discount and mortality rates.

It is well known that when an agent or a group of agents acting together have inconsistent time preferences, a commitment solution is not an intertemporal equilibrium. That is, a decision rule which maximizes the intertemporal utility function at a particular point in time cannot be maintained if the agents are free to change their decision rule at a later point in time. This raises the issue of which type of solution is to be derived, a commitment solution or an equilibrium solution.[1] In this paper, we use the solution defined in Ekeland and Lazrak (2010), which can be regarded as an intertemporal equilibrium. In the appendix, we derive a dynamic programming equation which can be used to find such a solution to the particular model of this paper.

The main contributions of the present paper are as follows. First, we derive a dynamic programing equation for finding the above mentioned intertemporal cooperative equilibrium in a stochastic model with two agents that may have different but constant discount rates of future utilities. Secondly, we employ this dynamic programing equation to a previously uninvestigated problem, namely that of the purchase of life insurance by two agents with different but constant discount rates. Thirdly, we find through numeric analysis that while the purchase of life insurance of one agent increases in her own discount rate (a result already found by previous authors, see for example Pliska and Ye (2007)), it decreases in the discount rate of the other agent. This third results contrasts with the purchase of consumption which, in our model, turns out to be the same for both agents regardless of the discount rates. Fourthly, we also investigate how the agents react to changes in their mortality rate. Although this already been done in several other multiple agent models (with symmetric discount rates), the present model allows for a more detailed analysis by decomposing the impact of the mortality rates into three separate "channels".

The rest of this paper is organized as follows. In Section 2 we provide a literature review on life insurance models. In Section 3 we formally describe the model and in Section 4 we try to solve the model analytically. We shall find that a fully analytic solution can be obtained only in special cases, which motivates the numeric solutions of Section 5. In Section 6, we provide some concluding remarks.

2. Literature review

We've already mentioned the early contribution of Richard (1975). He builds upon the model of Merton (1971), in which an agent withdraws money for consumption expenditures from his budget over a fixed time interval $[0, T]$, while simultaneously distributing his savings amongst one asset with a fixed return and one or several assets with random return. He added to this model that the agent dies at a random time $\tau \in [0, T]$, turning the problem into one of random terminal time. He also added a running wage income and gave the agent the option of purchasing life insurance by continuously paying a premium that is withdrawn from the budget, hence turning the model into a life insurance model. At the time of death, the descendants receive a payment which is a function of the rate at which the premium is being paid at that time. The agent is made interested in purchasing life insurance by the introduction of a bequest function which yields utility at the time of death. The bequest function takes as an argument the current wealth plus the insurance payment, i.e. the total amount of money that the deceased agent leaves behind to her descendants. The agent's intertemporal utility function[2] contains both the bequest and her own running utility which, as in Merton's model, depends only on current consumption.

The wealth dynamics of Richard's model have been the basis for much of the literature on life insurance models, with only slight variations added by later authors. As for the intertemporal utility function, there's been more variation. An important extension comes from Pliska and Ye (2007). They removed the assumption of τ being bounded to the interval $[0, T]$, thus allowing the agent to live beyond time T. This solved a problem inherent in Richard's model, first pointed out by Leung (1994), namely that when τ is bounded to be smaller than or equal to T, the model does not have an interior solution that lasts until T. Instead, Pliska and Ye interpret T as a time of retirement. The agent may live beyond this time, but she no longer receives a wage income and makes no more decisions with respect to consumption, investment and life insurance. In fact, their intertemporal utility function is constructed in such a way that, if the agent is still alive at time T, a terminal function is activated and the problem ends. This terminal function takes wealth as an argument, but not an insurance payment. That is, it is assumed that the agent only purchases life insurance up to the point of his retirement. If he dies after retirement, no insurance is paid out because no premium is being paid anymore.

One downside of the model of Pliska and Ye is that the terminal function is assumed to be independent of the probability distribution of τ. This is problematic because since the model effectively ends at T, but the agent may still be alive, it is reasonable to assume that the terminal function represents the intertemporal utility function at time T, the sum of utilities obtained over the interval $[T, \tau]$, discounted back to T. But if this is so, then the expected length of that interval should affect the value of that discounted sum of utilities. That is, the terminal function should depend on the probability distribution of τ. It should also, perhaps, depend on the other parameters of the model, such as the discount rate. This is typically not the case in papers that have used this type of intertemporal utility function. Instead, the terminal function is assumed to depend on a separate parameter, and it is not imposed that this parameter has any particular relationship with the other parameters of the model.

One way around this issue is to assume that there is no terminal time T at which the problem ends (T is infinite), and that the agent continues to consume, invest and purchase life insurance until his death, whenever it may come. In other words, there is no retirement. Thus, some authors have tackled the life insurance problem by assuming an intertemporal utility function with an infinite planning horizon. Models of this kind can be found in Bruhn and Steffensen (2011), Bayraktar and Young (2013) and Koo and Lim (2021). Of course, assuming that the agent may potentially live forever is unrealistic, and one can still specify a maximum age, beyond which the agent cannot survive, by using a truncated probability distribution

[1] A third type of solution, sometimes called a myopic solution, has also been analyzed. See, for example, Marín-Solano and Navas (2010).

[2] This is defined as the sum of present and future utilities, as discounted back to the present.

Table 1
Literature summary.

Author(s)	Utility F.	Time Pref.	Mortality R.	Int. Utility F.
Richard (1975)	Power	Consistent	Not Specified	Richard
Pliska and Ye (2007)	Power	Consistent	Linear	Pliska and Ye
Huang and Milevsky (2008)	Power	Consistent	Gompertz Law	Other
Kwak et al. (2011)	Power	Consistent	Linear	Multiple Agents
Bruhn and Steffensen (2011)	Power	Consistent	Gompertz Law	Multiple Agents
Pirvu and Zhang (2012)	Power	Consistent	Gompertz Law	Pliska and Ye
Bayraktar and Young (2013)	Exponential	Consistent	Constant	Multiple Agents
Marín-Solano et al. (2013)	Exponential	Inconsistent	Gompertz Law	Pliska and Ye
de Paz et al. (2014)	Power and Exp.	Inconsistent	Gompertz Law	Pliska and Ye
Purcal et al. (2018)	Power	Inconsistent	Other	Richard
Chen and Li (2020)	Logarithm	Inconsistent	Gompertz Law	Pliska and Ye
Wei et al. (2020)	Power	Consistent	Gompertz Law	Multiple Agents
Purcal et al. (2021)	Power	Inconsistent	Other	Richard
Koo and Lim (2021)	Power	Inconsistent	Constant	Other
This Paper	Power	Inconsistent	Constant	Multiple Agents

for τ. However, problems arise here too. When time approaches the maximum time of death, the agent knows for sure that she will die soon, and the incentive to buy ever more life insurance explodes.[3] Moreover, using probability distributions over the whole interval $[0, \infty)$ can be useful for reasons of tractability, as we shall see in the present paper.

For the above reasons, the present paper uses an infinite planning horizon and assumes that the time of death is distributed over the whole interval $[0, \infty)$.

In the literature on life insurance models with more than one agent, an important contribution is Bruhn and Steffensen (2011). They developed a procedure for finding the (fully) cooperative solution for a household with an arbitrary number of agents. Each agent has her own wage income and probability distribution over her time of death, and each agent purchases her own life insurance. When one agent dies, the life insurance that she has purchased is added to the household's budget, to be used by the remaining agents until, eventually, all agents have died. It is also assumed that, unlike in the models of Richard (1975) and Pliska and Ye (2007), there is no utility from a bequest. Instead, the incentive to purchase life insurance arises from the fact that it is in the interest of each agent that every other agent is insured. Given that the household acts as a whole, and that each agent can influence the decision rules decided upon, it follows that the agents can prompt each other to purchase life insurance. Of course, when all agents except one have died, the survivor has no incentive to purchase any life insurance.

Another life insurance model with more than one agent is that of Bayraktar and Young (2013). They model a household consisting of two agents, assuming that there is just a single premium and a single quantity of consumption that both agents benefit from. They also solve a model in which life insurance is paid for with a single lump sum payment, rather than the continuously paid premium that is standard. On the other hand, Kwak et al. (2011) assume that the household consists of one parent and one child, and that only the parent receives a wage income and may purchase life insurance. An interesting aspect of this model is that the parent and the child have power utility functions with different risk aversions. Another recent contribution is Wei et al. (2020). They consider a household with two agents whose respective times of death are positively correlated. This is a realistic assumption for a household that consists of two people of similar age. Interestingly, they find that the stronger the correlation, the less life insurance is purchased by each agent. This is probably because after the first agent has died, the probability that the second agent will die soon increases (due to the positive correlation), which gives the second agent less time to transform the increase in wealth brought forth by the insurance payment into utility. Hence, the value of the life insurance payment in terms of the expected intertemporal utility that it yields is lower when the times of death are correlated.

All the multiple agent models described above assume that the household has consistent time preferences. That is, all members of the household are assumed to have exponential discount functions with the same discount rate. Hence, the issue of inconsistent time preferences is not dealt with. Regarding life insurance models with inconsistent time preferences, Marín-Solano et al. (2013) found the intertemporal equilibrium for a single agent with an arbitrary discount function. That paper can be regarded as an extension of the model in Marín-Solano and Navas (2010), which contained consumption and investment decisions but not life insurance. One of their findings is that the amount of life insurance that is purchased increases in the agent's risk aversion. Similar models were solved by de Paz et al. (2014) and Chen and Li (2020). All of those papers use the intertemporal utility function of Pliska and Ye (2007). As for the intertemporal utility function of Richard's model, intertemporal equilibria under inconsistent time preferences were derived in Purcal et al. (2021) and Koo and Lim (2021). The latter of these also investigated the effects of taxation on the purchase of life insurance. Several papers find that inconsistent time preferences reduce the purchase of life insurance.

Table 1 summarizes essential aspects of some of the papers mentioned above. The columns, starting from the left, give the author(s), the type of utility function, the time preferences (consistent or inconsistent), the mortality rate and the intertemporal utility function.

3. The model

Our model contains a household with two agents. Each agent $i \in \{1, 2\}$ has a time of death $\tau_i \in [0, \infty]$ that follows an exponential distribution with mortality rate (or intensity) λ_i[4]:

$$\mathbb{P}(\tau_i < t) = 1 - e^{-\lambda_i t}. \tag{1}$$

[3] From the point of view of a purchaser of life insurance, this may not be an unreasonable assumption. After all, a person who suspects that she will die soon does indeed have incentive to purchase a very large life insurance. But of course, from the perspective of the insurance company, the incentive runs in the opposite direction, which puts a limit on the ability of the agent to purchase ever more insurance when his time of death approaches.

[4] We shall sometimes let λ_{-i} refer to the mortality rate of the agent who does *not* have the index i. That is, if $i = 1$ then $\lambda_{-i} = \lambda_2$ and vice versa. We shall use this convention for other variables and parameters too.

Table 2
Potential household states.

	$t < \tau_1$	$t > \tau_1$
$t < \tau_2$	Both agents are alive	Only agent 2 is alive
$t > \tau_2$	Only agent 1 is alive	Both agents are dead

We assume that the agents share control over the household budget, and that they both add to the budget through their individual wage incomes. At each point in time, each agent withdraws money from the budget x_t to use for her own individual consumption. Moreover, the two members of the household jointly decide how much of their wealth to allocate to investment in a risky asset and in a safe asset. There are no constraints on the investments; the household may freely borrow in one asset in order to invest more than its whole net worth in the other asset.

Each agent purchases her own life insurance by continuously paying a premium. Thus, money is withdrawn from the budget to pay for two separate premiums, one for each agent. At time $\bar{\tau} = \min\{\tau_1, \tau_2\}$, one of the agents dies and the other will be the sole member of the household until she too dies. If the agent who dies first is paying a premium, the insurance payment will be added to the household's wealth. Hence, the wealth process jumps at $\bar{\tau}$, with the jump being equal to the size of the insurance payment. Specifically, if agent i dies first, so that $\bar{\tau} = \tau_i$, we have

$$x_{\bar{\tau}} = x_{\bar{\tau}-} + \frac{q_{i,\bar{\tau}-}}{\eta_i} \ .$$

Here, $q_{i,t}$ is the premium of agent i and $q_{i,t}/\eta_i$ is the amount that is insured at time t. The parameter η_i is called the premium-insurance ratio of wealth i. Thus, the process x_t is right-continuous and its value at $\bar{\tau}$ is determined by the jump that is initiated "just prior" to $\bar{\tau}$. The event that the agents die at the same time, i.e. that $\tau_1 = \tau_2$, has zero probability and can therefore be ignored.[5]

It is useful to think of the "state of the household" at any time t to be one of four potential states. In the initial state, which lasts for $t \in [0, \bar{\tau})$, both agents are alive. We will refer to this as state A. Then there are two states in which one agent is alive and the other is dead. Which one of these states that is realized depends on which agent dies first. We will refer to these as states B. Finally, there's the state in which both agents are dead. In this state, there are no decisions to be made and hence no problem to be solved. See Table 2 for a summary of the potential states.

Let's first look at state B, in which only one agent is alive. If only agent i is alive at time t, she has the intertemporal utility function

$$J_{i,B}(x_t, t; c_{i,B}, \epsilon_B) = \mathbb{E}\left[\int_t^{\tau_i} e^{-\rho_i(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} ds \,\middle|\, \tau_i > t \ , \ x_t \right] \ . \tag{2}$$

The household's wealth x_t follows the stochastic differential equation

$$dx_t = (r + \epsilon_t(\mu - r))x_t dt - c_{i,t} dt + I_i dt + \epsilon_t \sigma x_t dz_t \ , \tag{3}$$

where $t \in [\bar{\tau}, \tau_i]$. Here, ρ_i is the discount rate and γ is the relative risk aversion. The variables $c_{i,t}$ and I_i are the agent's rate of consumption and wage income, and ϵ_t is the fraction of wealth that is invested in the risky asset. We make the common assumption that the money price of consumption is always equal to one, so that $c_{i,t}$ is both the rate at which the agent consumes and the rate at which money is spent on consumption. We let $c_{i,B}$ and ϵ_B denote the decision rules adopted with respect to these two variables in state B. The remaining parameters are r, μ and σ and they determine the return of the safe asset, the expected return of the risky asset and the volatility of the risky asset.[6] There is no purchase of life insurance in state B because there is no descendant who can benefit from it. Hence, there is no variable $q_{i,t}$. Note that the expectation operator $\mathbb{E}$ appears in (2) for two reasons: the randomness associated with the Brownian motion z_t and the randomness of τ_i, which follows the distribution of (1).

The agent wants to maximize (2) subject to (3) and the initial condition (x_t, t) by choosing an appropriate decision rule out of all admissible $(c_{i,B}, \epsilon_B)$. Let $(c^*_{i,B}, \epsilon^*_B)$ denote the decision rule that solves this problem. We can then write the value function of agent i in state B as

$$V_{i,B}(x_t, t) = J_{i,B}(x_t, t; c^*_{i,B}, \epsilon^*_B) \ . \tag{4}$$

The value function is simply the sum of expected discounted utilities obtained over the interval $[t, \tau_i)$, discounted back to time t (the present) when the decision rule $(c^*_{i,B}, \epsilon^*_B)$ is employed.

Next, let's look at state A, in which both agents are alive. In this state, the intertemporal utility function of agent i at time t is

$$\begin{aligned} & J_{i,A}(x_t, t; c_{1,A}, c_{2,A}, q_{1,A}, q_{2,A}, \epsilon_A) \\ &= \mathbb{E}\left[\int_t^{\bar{\tau}} e^{-\rho_i(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} ds + e^{-\rho_i(\bar{\tau}-t)} V_{i,B}(x_{\bar{\tau}-} + q_{i,\bar{\tau}-}/\eta_i), \bar{\tau}) 1_{\tau_{-i}=\bar{\tau}} \,\middle|\, \bar{\tau}_i > t \ , \ x_t \right] \end{aligned} \tag{5}$$

and the household's wealth follows the stochastic differential equation

$$dx_t = (r + \epsilon_t(\mu - r))x_t dt - (c_{1,t} + c_{2,t} + q_{1,t} + q_{2,t})dt + (I_1 + I_2)dt + \epsilon_t \sigma x_t dz_t \ , \tag{6}$$

where $t \in [0, \bar{\tau})$. The intertemporal utility function (5) can be understood as follows. The first term in the expectation gives the sum of future utilities that agent i receives during state A (that is, until one agent dies), discounted back to t. The second term is the agent's value function in state B, also discounted back to t. This term is the utility that agent i would receive if the other agent dies first, and it depends on the household's total wealth at the time of death, which includes the money paid out by the insurance company. The term is multiplied by the binary random variable $1_{\tau_{-i}=\bar{\tau}}$,

[5] The case of simultaneous deaths is analyzed in Wei et al. (2020).

[6] We are assuming that the price of a share in the risky asset follows a geometric Brownian motion where μ and σ are the drift and volatility coefficients. See Chang (2004) or Merton (1971).

which is equal to 1 if agent i is the one who dies last and zero otherwise. Hence, when choosing a decision rule in state A, agent i takes into account how her future self will behave in state B.

The wealth process of state A, equation (6), contains two consumption variables and two wage incomes. It also contains two premiums because both agents have the option of purchasing life insurance.

The two members of the household determine their individual decision rules jointly. We are looking for a cooperative solution. This implies that the relevant objective function in state A is the intertemporal utility function of the household, which is defined as

$$\begin{aligned} &J_A(x_t, t; c_{1,A}, c_{2,A}, q_{1,A}, q_{2,A}, \epsilon_A) \\ &\equiv J_{1,A}(x_t, t; c_{1,A}, c_{2,A}, q_{1,A}, q_{2,A}, \epsilon_A) + J_{2,A}(x_t, t; c_{1,A}, c_{2,A}, q_{1,A}, q_{2,A}, \epsilon_A) , \end{aligned} \tag{7}$$

that is, as the sum of the intertemporal utility functions of the two household members.[7] This problem is solved by choosing the appropriate decision rule $(c_{1,A}, c_{2,A}, q_{1,A}, q_{2,A}, \epsilon_A)$ and is subject to the wealth process (6) and the initial condition (x_t, t). Let $(c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A)$ denote the decision rule that solves this problem. We can then write the value function of agent i in state A as

$$V_{i,A}(x_t, t) = J_{i,A}(x_t, t; c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A) , \tag{8}$$

and the value function of the household in state A as

$$V_A(x_t, t) = J_A(x_t, t; c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A) . \tag{9}$$

However, note that this decision rule will not necessarily be the one that maximizes (7), because we are restricting our set of admissible decision rules to those that are intertemporal equilibria, in the sense that they satisfy the definition of Ekeland and Lazrak (2010). This is due to the fact that the household may have inconsistent time preferences, which happens if $\rho_1 \neq \rho_2$.

In summary, the agents will make decisions on consumption, investment and premium while they are both still alive. When one agent has died, the other agent makes decisions on consumption and investment until she too dies. To place our model in its proper place in the literature, we have an infinite horizon model with no retirement and no terminal function. This is different from the commonly used intertemporal utility functions in Pliska and Ye (2007) and in Richard (1975), but similar to, for example, Bruhn and Steffensen (2011).

4. Solving the model

To solve the model, we apply the method that was used by Bruhn and Steffensen (2011) and Wei et al. (2020). First, we solve the problem for state B, where only one agent is alive. Then, having obtained the value functions for that problem, we insert them into the household's intertemporal utility function of state A, equation (7), where both agents are alive, and solve the problem for state A. Hence, the model is solved using a kind of two-stage backward induction.

4.1. State B: the household with one agent

We begin by transforming the intertemporal utility function (2) from one with random terminal time into one with no terminal time (infinite horizon). Various versions of this method are used in much of the literature on life insurance models, and it goes back at least to Yaari (1965). The purpose is to remove one source of randomness, the time of death, thus turning the problem into one of a more familiar kind, in which the planning horizon is either fixed or, as in our case, infinite. The set of steps necessary to transform the intertemporal utility function into the desired form differs somewhat between the various papers in literature depending on the particular assumptions of the models. For the present model, the transformed intertemporal utility function of the states when one agent is alive is given below in Proposition 1, which is proved in the appendix.

Proposition 1. *Suppose that the intertemporal utility function is given by (2) and that the time of death τ_i follows the exponential distribution (1). Then the intertemporal utility function can also be written as*

$$J_{i,B}(x_t, t; c_{i,B}, \epsilon_B) = \mathbb{E}\left[\int_t^\infty e^{-(\rho_i+\lambda_i)(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right]. \tag{10}$$

This intertemporal utility function can be interpreted as an infinite sum of utilities discounted at the rate of $\rho_i + \lambda_i$. We will say that $\rho_i + \lambda_i$ is the *effective discount rate* of the problem with one agent. Notice that the expectation in (10) is no longer conditional on $\tau_i > t$, since the randomness associated with τ_i has been removed.

The intertemporal utility function (10) has an effective discount rate that is constant. This implies that a commitment solution is an intertemporal equilibrium. Hence, this is a trivial problem to solve. The stationary Markov strategies are

$$c^*_{i,B}(x_t) = \left(\frac{\lambda_i + \rho_i - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu - r)^2}{(\sigma\gamma)^2} \right) (x_t + I_i/r)$$

and

$$\epsilon^*_B(x_t) = \frac{\mu - r}{\gamma\sigma^2} \frac{x_t + I_i/r}{x_t} ,$$

which implies that the value function for this problem is

[7] We may expand the model by adding weights to the individual intertemporal utility functions. For simplicity, we solve the model with both weights equal to one.

$$V_{i,B}(x_t) = \left(\frac{\lambda_i + \rho_i - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu-r)^2}{(\sigma\gamma)^2} \right)^{-\gamma} \frac{(x_t + I_i/r)^{1-\gamma}}{1-\gamma} . \tag{11}$$

We also note that the stochastic process for wealth becomes

$$dx_t = \left(\frac{1-\rho_i-\lambda_i}{\gamma} + \frac{(\mu-r)^2}{\gamma\sigma^2} \frac{1+\gamma}{2\gamma} \right) x_t dt + \left(\frac{1-\rho_i-\lambda_i}{\gamma r} + \frac{(\mu-r)^2}{\gamma\sigma^2} \frac{1-\gamma}{2\gamma r} \right) I_i dt + \frac{\mu-r}{\gamma\sigma} x_t dz_t . \tag{12}$$

Due to the presence of the second term on the right hand side of (12), there's no guarantee that wealth will stay positive. On the other hand, if we set $I_i = 0$ (so that there is no wage income), the term disappears and we obtain a geometric Brownian motion, which implies that wealth stays positive almost surely.

We note that the rate at which money is withdrawn for consumption, and the amount of money invested in the risky asset, are both linear in $x_t + I_i/r$. Moreover, since

$$\frac{1}{r} = \int_t^\infty e^{-r(s-t)} ds$$

for any t, we see that I_i/r is the present value of the future sum of wage incomes, as discounted by the rate of return of the safe asset. Richard (1975) refers to this term as the agent's "human capital". We can think of $x_t + I_i/r$ as the present value of wealth that is not contingent upon any investment decision.

4.2. State A: The household with two agents

We now proceed to the model of state A, in which both agents are alive. The intertemporal utility function is (7) and the wealth dynamics are (6). As before, we transform the intertemporal utility function from one with random terminal time into one with infinite horizon. The result is given in Proposition 2, which is proved in the appendix.

Proposition 2. *Suppose that the intertemporal utility function is given by (7) and that, for each i, the time of death τ_i follows the exponential distribution (1). Then the intertemporal utility function can also be written as*

$$\begin{aligned} &J_A(x_t, t; c_{1,A}, c_{2,A}, q_{1,A}, q_{2,A}, \epsilon_A) \\ &= \mathbb{E}\left[\int_t^\infty e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} \frac{c_{1,s}^{1-\gamma} + \lambda_2 \Theta_1^{-\gamma} (x_s + I_1/r + q_{2,s}/\eta_2)^{1-\gamma}}{1-\gamma} ds \right. \\ &\left. + \int_t^\infty e^{-(\rho_2+\lambda_1+\lambda_2)(s-t)} \frac{c_{2,s}^{1-\gamma} + \lambda_1 \Theta_2^{-\gamma} (x_s + I_2/r + q_{1,s}/\eta_1)^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right], \end{aligned} \tag{13}$$

where

$$\Theta_i = \frac{\lambda_i + \rho_i - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu-r)^2}{(\sigma\gamma)^2} .$$

Equation (13) gives the transformed intertemporal utility function for the household in state A. It follows that the transformed value function for agent i in state A, defined by (8), can be written as

$$V_{i,A}(x_t) = \mathbb{E}\left[\int_t^\infty e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} \frac{(c^*_{i,A,s})^{1-\gamma} + \lambda_{-i}\Theta_i^{-\gamma}(x_s + I_i/r + q^*_{-i,A,s}/\eta_{-i})^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right] . \tag{14}$$

We see that the intertemporal utility function can be interpreted as the sum of the two sums of discounted utilities, each with a rather complicated utility function which depends on consumption, premium and the wealth, and effective discount rates of $\rho_1 + \lambda_1 + \lambda_2$ and $\rho_2 + \lambda_1 + \lambda_2$. Provided that $\rho_1 \neq \rho_2$, the two sums have different discount rates, which implies that a commitment solution to the problem will not be an intertemporal equilibrium.

We want to find an intertemporal equilibrium solution for the problem with the intertemporal utility function given by (13) and wealth dynamics given by (6). We will now present a proposition that gives the dynamic programming equation for this problem. Since the intertemporal utility function given by equation (13) is the sum of two sums of discounted utilities, each with its own discount rate, the dynamic programming equation is non-standard. The proof is in the appendix.

Proposition 3. *Let the intertemporal utility function of the household be given by (13) and the wealth dynamics by (6). Moreover, let (14) be the value function of agent i. Then the solution to the dynamic programming equation*

$$\begin{aligned} &\max_{c_1,c_2,q_1,q_2,\epsilon} \left\{ \frac{c_1^{1-\gamma} + \lambda_2\Theta_1^{-\gamma}(x + I_1/r + q_2/\eta_2)^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma} + \lambda_1\Theta_2^{-\gamma}(x + I_2/r + q_1/\eta_1)^{1-\gamma}}{1-\gamma} \right. \\ &+ \left(\frac{dV_{1,A}}{dx}(x) + \frac{dV_{2,A}}{dx}(x) \right) (rx + \epsilon(\mu - r)x - c_1 - c_2 - q_1 - q_2 + I_1 + I_2) \end{aligned}$$

$$+\left(\frac{d^2V_{1,A}}{dx^2}(x)+\frac{d^2V_{2,A}}{dx^2}(x)\right)\frac{(\epsilon\sigma x)^2}{2}\Bigg\}$$
$$=\mathbb{E}\left[\frac{dV_{1,A}}{dt}(x)\,\middle|\,x\right]+(\rho_1+\lambda_1+\lambda_2)V_{1,A}(x)+\mathbb{E}\left[\frac{dV_{1,A}}{dt}(x)\,\middle|\,x\right]+(\rho_2+\lambda_1+\lambda_2)V_{2,A}(x)$$

is a stationary Markov intertemporal equilibrium decision rule.

The decision rule yielded by Proposition 3 is an intertemporal equilibrium decision rule in the sense that it satisfies the definition due to Ekeland and Pirvu (2008) and Ekeland and Lazrak (2010). The mathematical details of the definition are discussed in the appendix.

To proceed with finding the solution, we have to make a conjecture on the value functions. Given the structure of the dynamics and the utility functions, it is natural to make the conjecture[8]

$$V_{i,A}(x_t)=\frac{A_i^{-\gamma}(x_t+B_i)^{1-\gamma}}{1-\gamma}\ . \tag{15}$$

This implies that the value functions for the model when both agents are alive have the same general form as the value functions for the problems when one agent is alive. Indeed, the conjecture we make here is standard in the literature when the utility function for each agent is given by a power function. (See, for example, Bruhn and Steffensen (2011) and Wei et al. (2020).) Of course, in general, we should expect to obtain $A_1\neq A_2$ and $B_1\neq B_2$.

With the conjecture (15), the left hand side of the dynamic programming equation in Proposition 3 is

$$\max_{c_1,c_2,q_1,q_2,\epsilon}\left\{\frac{c_1^{1-\gamma}+\lambda_2\Theta_1^{-\gamma}(x+I_1/r+q_2/\eta_2)^{1-\gamma}}{1-\gamma}+\frac{c_2^{1-\gamma}+\lambda_1\Theta_2^{-\gamma}(x+I_2/r+q_1/\eta_1)^{1-\gamma}}{1-\gamma}\right.$$
$$+((A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma})(rx+\epsilon(\mu-r)x-c_1-c_2-q_1-q_2+I_1+I_2)$$
$$\left.-\gamma(A_1^{-\gamma}(x+B_1)^{-1-\gamma}+A_2^{-\gamma}(x+B_2)^{-1-\gamma})\frac{(\epsilon\sigma x)^2}{2}\right\}\ .$$

The maximization problem implicit in this equation yields the decision rules

$$c_{i,A}^*(x_t)=((A_1(x_t+B_1))^{-\gamma}+(A_2(x_t+B_2))^{-\gamma})^{-1/\gamma}\ ,$$
$$q_{i,A}^*(x_t)=\eta_i\left(\frac{\lambda_i}{\eta_i}\right)^{-1/\gamma}\left(\left(\frac{\Theta_{-i}}{A_1(x_t+B_1)}\right)^{\gamma}+\left(\frac{\Theta_{-i}}{A_2(x_t+B_2)}\right)^{\gamma}\right)^{-1/\gamma}-\eta_i(x_t+I_{-i}/r)$$

and

$$\epsilon_A^*(x_t)=\frac{\mu-r}{\gamma\sigma^2x_t}\frac{(A_1(x_t+B_1))^{-\gamma}+(A_2(x_t+B_2))^{-\gamma}}{A_1^{-\gamma}(x_t+B_1)^{-1-\gamma}+A_2^{-\gamma}(x_t+B_2)^{-1-\gamma}}\ .$$

Thus, we have obtained expressions for $\{c_{1,A}^*,c_{2,A}^*,q_{1,A}^*,q_{2,A}^*,\epsilon_A^*\}$ in terms of the constants $\{A_1,A_2,B_1,B_2\}$, which depend on the parameters of the model.

In order to find the equilibrium decision rules, what remains for us is to find the set of constants in the set $\{A_1,A_2,B_1,B_2\}$. Specifically, we would like to derive a system of equations that can be solved analytically or (if necessary) numerically. However, this is not straightforward, and we have not been able to solve this problem for the general case. Instead, in the remainder of this paper, we will focus on the particular case of when there is no wage income, so that $I_1=I_2=0$. For the interested reader, the appendix includes a discussion on the difficulties of finding the set of constants for the general case.

4.3. The case of no wage income

When there is no wage income, it can be inferred that the correct conjecture on the value functions is such that $B_1=B_2=0$. In other words, we obtain

$$V_{i,A}(x_t)=\frac{A_i^{-\gamma}x_t^{1-\gamma}}{1-\gamma}\ .$$

As a consequence, the decision rules $\{c_{1,A}^*,c_{2,A}^*,q_{1,A}^*,q_{2,A}^*,\epsilon_A^*\}$ become

$$c_{i,A}^*(x_t)=(A_1^{-\gamma}+A_2^{-\gamma})^{-1/\gamma}x_t\ ,$$
$$q_{i,A}^*(x_t)=\eta_i\Theta_{-i}^{-1}(\lambda_i/\eta_i)^{1/\gamma}(A_1^{-\gamma}+A_2^{-\gamma})x_t-\eta_ix_t$$

and

$$\epsilon_A^*(x_t)=\frac{\mu-r}{\gamma\sigma^2}\ .$$

[8] Chang (2004) has an extensive discussion on how to infer the general form of the value functions from the dynamics and the utility functions.

We now only have to find two constants, namely A_1 and A_2. For this, we need a system with two equations. To proceed, we take the total derivatives with respect to time of the value functions, which give us the two equations that we need.[9] We then insert our conjectures on the value functions and the strategies $\{c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A\}$ as defined above and divide by $x_t^{1-\gamma}/(1-\gamma)$. We obtain the system

$$\begin{aligned}0 = &\left(1+\lambda_{-i}\Theta_i^{-1}(\lambda_{-i}/\eta_{-i})^{\frac{1-\gamma}{\gamma}}\right)A_i^{-\gamma}(A_1^{-\gamma}+A_2^{-\gamma})^{\frac{1-\gamma}{\gamma}} \\ &-(1-\gamma)\left(2+\eta_1\Theta_2^{-1}(\lambda_1/\eta_1)^{1/\gamma}+\eta_2\Theta_1^{-1}(\lambda_2/\eta_2)^{1/\gamma}\right)(A_1^{-\gamma}+A_2^{-\gamma})^{-1/\gamma} \\ &+\left((1-\gamma)(r+\eta_1+\eta_2)-(\rho_i+\lambda_1+\lambda_2)+(1-\gamma)\frac{(\mu-r)^2}{2\gamma\sigma^2}\right),\end{aligned} \tag{16}$$

for $i \in \{i, 2\}$. In this system, the variable x_t has been removed, leaving us with only the parameters of the model, and with A_1 and A_2. Hence, although the system yields no analytic solution, we can use it to solve the model numerically.

Since we cannot find an explicit solution for A_1 and A_2, not much can be said analytically about the impact of the parameters of the model on the decision rules $\{c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A\}$. We need to defer such investigations to the numeric analysis in the next section. However, we can make the following observations (for the case of no wage income).

- The two agents have the exact same consumption, even though their discount rates differ. Hence, differences in discount rates do not result in differences in consumption. The same result is found in a similar but deterministic model in de Paz et al. (2014)).
- Consumption and premium are both linear in wealth, and the fraction of wealth invested in the risky asset is constant. These are standard results in the literature.
- The premiums are potentially negative. A negative premium implies that the household receives a flow of income in exchange for making a lump sum payment to the insurance company at the time of death of the first agent. This is an undesirable result. In the numeric solutions in Section 5, the parameter values are such that the premiums are always positive.

4.4. *The case of no wage income and logarithmic utility functions*

In closing this section, we will make some observations on the special case $\gamma = 1$, which corresponds to logarithmic utility functions. In this case, the system (16) simplifies to

$$0 = (1+\lambda_{-i}\Theta_i^{-1})A_i^{-1} - (\rho_i+\lambda_1+\lambda_2),$$

where $\Theta_i = \lambda_i + \rho_i$. This yields the analytic solution

$$A_i = \rho_i + \lambda_i .$$

Hence, the equilibrium decision rules for state A become

$$\begin{aligned}c^*_{i,A}(x_t) &= ((\rho_1+\lambda_1)^{-1}+(\rho_2+\lambda_2)^{-1})^{-1}x_t , \\ q^*_{i,A}(x_t) &= \frac{\lambda_i}{\rho_{-i}+\lambda_{-i}}((\rho_1+\lambda_1)^{-1}+(\rho_2+\lambda_2)^{-1})^{-1}x_t - \eta_i x_t\end{aligned}$$

and

$$\epsilon^*_A(x_t) = \frac{\mu-r}{\sigma^2} .$$

We see that in this case, consumption depends only on the discount rates and mortality rates, and is independent of all other parameters, for example the premium-insurance ratios. The premium q_i decreases in η_i, as should be expected, but is independent of η_{-i}.

5. Numeric solutions

In this section, we use numeric solutions to investigate the behavior of the household when both agents are alive (state A) and there is no wage income. That is, we infer properties of the decision rules $\{c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A\}$ when $I_1 = I_2 = 0$. The reason for imposing that there is no wage income is, as was mentioned in the previous section, that only under this condition are we able to derive a system of equations that can be used to find numeric expressions for the control variables.

In looking for a numeric solution, we can impose the conditions $A_1 > 0$ and $A_2 > 0$, which must hold because the value functions must have the appropriate sign. This is so because if $0 < \gamma < 1$, the utility function is always positive, which means that the value functions must be positive, while if $\gamma > 1$ the utility function is always negative, so the value functions must be negative. Hence, we can limit our search to solutions such that A_1 and A_2 are positive.

For our base specification, we have taken parameter values primarily from Koo and Lim (2021), which is a recent paper that also utilizes constant mortality rates and an infinite planning horizon. The parameter values are $\gamma = 1.2$, $\rho_1 = \rho_2 = 0.7$, $\lambda_1 = \lambda_2 = 0.1$, $\eta_1 = \eta_2 = 0.04$, $r = 0.3$, $\mu = 0.06$ and $\sigma = 0.02$. Also, we set $x = 1$ throughout the analysis.[10] Hence, in our base specification, the two agents are identical in that they have the same discount rate, mortality rate and premium-insurance ratio. In the following numeric solutions, we will create asymmetries by varying one or several

[9] If the agents have the same discount rate ($\rho_1 = \rho_2$), one can solve the problem with two agents by taking the derivative of a single value function for the whole household. This is the approach taken in, for example, Wei et al. (2020).

[10] Notice that since the decision rules are linear, the wealth dynamics will have the form of a geometric Brownian Motion, meaning that, in general, there will be no steady state to which wealth converges.

Table 3
Asymmetric discount rates.

ρ_1	c_1, c_2	q_1	q_2	ϵ	$\mathbb{E}(dx)$	$V_1(x)$	$V_2(x)$
0.4	0.370678	1.14154	1.83948	0.625	-3.67363	-9.81568	-6.63464
0.5	0.419134	1.296	1.73556	0.625	-3.82108	-7.94254	-6.25283
0.6	0.464632	1.44103	1.65025	0.625	-3.97179	-6.63464	-5.90945
0.7	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913

Table 4
Asymmetric mortality rates.

λ_1	c_1, c_2	q_1	q_2	ϵ	$\mathbb{E}(dx)$	$V_1(x)$	$V_2(x)$
0.05	0.487987	0.832979	1.61809	0.625	-3.3783	-6.07669	-5.75045
0.1	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913
0.15	0.53291	2.3415	1.55966	0.625	-4.91822	-5.18614	-5.45495

parameters at a time, keeping all other parameters fixed to their benchmark values, which allows us to investigate the effects of specific asymmetries on the behavior of the household.

5.1. Asymmetric discount rates

In order to investigate the effects of asymmetric discount rates on the behavior of the household, we vary ρ_1, the discount rate of agent 1. The results are given in Table 3. The columns, starting from the left, gives the discount rate of agent 1, followed by the consumptions (recall that $c_1 = c_2$), the premiums, the investment in the risky asset and the expected rate of change in wealth (i.e. the drift of the wealth process). We also provide the value functions of state A for each agent. First, we note that consumption increases in ρ_1, which is to be expected. The more impatient is the household, the more it will consume. As a result, wealth will decrease faster, as can be seen in the column for the expected rate of change in wealth.

Secondly, we note that when ρ_1 increases, the premium of agent 1 increases. The agent spends more on life insurance when he is less patient. This result was obtained already in the single agent model of Pliska and Ye (2007), and is thus confirmed here. For the intuition behind this result, see Remark 5.1 below. For agent 2, however, the premium decreases. Here we finally see how the discount rate of one agent impacts the purchase of life insurance by the other agent. *When one agent becomes less patient, the household shifts spending to the premium of that agent from the premium of the other agent.* One reason for this is that the premium q_2 is costly in the immediate present, while the benefits for agent 1 are enjoyed over an extended period of time in the future. Those future benefits are less valuable to agent 1 if he is impatient. This is a new result, not obtained in multiple agent models that assume symmetric discount rates, such as Bruhn and Steffensen (2011), Kwak et al. (2011), Bayraktar and Young (2013) and Wei et al. (2020).

The two premiums converge when $\rho_1 = 0.7$, because in this case the agents have the same discount rate. We also note that the total amount of spending on premiums, $q_1 + q_2$, increases in ρ_1, although the effect is not that large. Hence, we may say that when the household as a whole becomes less patient, it spends more money on both consumption and premiums, which leads to a faster depletion of its wealth.

Thirdly, we note that the change in the discount rate of agent 1 has no impact on the fraction of wealth that is invested in the risky asset. This we already knew from the discussion in Section 4.

Remark 5.1. The fact that an agent purchases less life insurance when she becomes more patient may seem counter-intuitive. Afterall, shouldn't more patience imply more willingness to invest into a future life insurance payment? The answer to this seeming paradox is that given how the wealth dynamics of our model are set up, the life insurance payment depends only on the premium at the time of death, and is independent of the premium at any previous time. Hence, what the agent is purchasing with the premium is the life insurance what would be paid out should she die *right now*. This is reflected by the fact that, in the transformed intertemporal utility function, the life insurance payment appears in what is effectively the utility function. The premium is therefore, strictly speaking, no different from consumption: the agent withdraws money from the budget to receive utility in the present.[11] This interesting aspect of the model is present already in Richard (1975) and in Pliska and Ye (2007), as well as in most papers based on those two.[12] □

5.2. Asymmetric mortality rates

In order to investigate the effects of asymmetric mortality rates on the behavior of the household, we vary λ_1, the mortality rate of agent 1. The results are given in Table 4. First, we see that consumption increases in the mortality rate. The sooner an agent expects to die, the more he consumes. This is an intuitive result. What is, perhaps, less intuitive is that, if one agent's mortality rate increases, the other agent also consumes more. (Again, recall that the agents always have the same consumption.)

Secondly, the premium of agent 1 increases quickly in λ_1, while the premium of agent 2 decreases, and much more slowly. To understand why this happens, we need to grasp that an increase in the mortality rate of agent 1 affects the premiums through three different channels.

- There is an increase in the probability that agent 1 dies before agent 2. This provides incentive to increase q_1 so that agent 2 will receive a higher life insurance payment.

[11] It is true that the utility obtained by the survivor from the life insurance payment is enjoyed in future, but it's discounted by the discount rate of the survivor, which in our model may differ from the discount rate of the deceased.

[12] As was mentioned earlier, a different approach can be found in Bayraktar and Young (2013), where the premium is a lump-sum payment.

Table 5
Asymmetric insurance-premium ratios.

η_1	c_1, c_2	q_1	q_2	ϵ	$\mathbb{E}(dx)$	$V_1(x)$	$V_2(x)$
0.04	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913
0.08	0.51351	0.838637	1.59683	0.625	-3.41373	-5.52617	-5.59913
0.12	0.516265	0.538758	1.60561	0.625	-3.12815	-5.45495	-5.59913

Table 6
The effect of which agent dies first.

λ_1	ρ_1	c_1, c_2	q_1	q_2	ϵ	$\mathbb{E}(dx)$	$V_1(x)$	$V_2(x)$
0.2	0.6	0.51351	2.87649	1.59683	0.625	-5.45158	-5.52617	-5.59913
0.1	0.7	0.51072	1.58793	1.58793	0.625	-4.14856	-5.59913	-5.59913
0.05	0.75	0.505108	0.863607	1.57004	0.625	-3.39512	-5.67386	-5.67386

- There is a decrease in $V_{1,B}(x)$, the intertemporal utility that agent 1 expects to receive when and if he becomes the sole survivor. This provides incentive to decrease q_2, since the life insurance payment received by agent 1 is less valuable in terms of utility. That is, when agent 1 expects to live a shorter amount of time, then he has less time to transform the insurance payment that he receives when agent 2 dies into utility. Mathematically, this effect is due to the presence of the constant Θ_1 in the intertemporal utility function (13).
- The effective discount rates increase, which provides incentive to increase both premiums.

The change in the premiums that we observe in Table 4 is the net effect of those three separate effects. The first contributes to the increase in q_1, and the second effect is the cause of the decrease in q_2. The third effect contributes to the fact that the total spending on premiums, $q_1 + q_2$, increases.

5.3. Asymmetric premium-insurance ratios

In order to investigate the effects of asymmetric premium-insurance ratios on the behavior of the household, we vary η_1, the premium-insurance ratio of agent 1. The results are given in Table 5. Again, we see that consumption increases, although only slightly. This is because when life insurance for agent 1 becomes more expensive, the household shifts spending from premiums to consumption. Note that this is not an intertemporal shift, i.e. the household is not trading future wealth for present consumption. As we discussed above, the premium is effectively the purchase of present utility, as is consumption. Hence, when the household reduces the premium of agent 1 in order to increase consumption, it is shifting spending from one source of present utility to another. Moreover, the household is not just shifting money to consumption, but also to life insurance for agent 2, although the effect is small.

The main effect of the increase in η_1 is, however, a decrease in overall spending on present utility, i.e. a decrease in $c_1 + c_2 + q_1 + q_2$, and consequently an increase in saving. Hence, the drift term of the wealth process becomes smaller (in absolute terms). The household is consuming its wealth at a slower rate.

5.4. Isolating the effect of which agent dies first

Suppose that we increase the parameter λ_1, keeping all other parameters constant. As we discussed above, this change affects the household through three different channels. However, if we increase λ_1 and simultaneously decrease ρ_1 by the same amount, two of these channels are mostly neutralized. This is because the effective discount rate of agent 1 is unchanged (including the one which in (13) appears inside the constant Θ_1, which is the effective discount of the state when only agent 1 is alive). For agent 2, the effective discount rate which appears inside the constant Θ_2 is also unchanged, while the one which appears explicitly in (13), namely $\rho_2 + \lambda_1 + \lambda_2$ is not. However, the channel that is not affected at all is the probability that agent 1 will die before agent 2. This implies that a change in the behavior of the household will primarily be the result of a change in that probability. Thus, we will have (almost) isolated the effect of one of the three channels.

In Table 6 we vary λ_1 and ρ_1 in such a way that the sum $\lambda_1 + \rho_1$ is constant. Hence, the effective discount rate of agent 1 is not changing. When the discount rate increases, agent 1 has incentive to increase consumption. But at the same time his life expectancy is increasing, which provides incentive to increase saving. The net effect is a very modest change in consumption, which is due to the fact that one of the effective discount rates of agent 2 has decreased slightly (since it contains λ_1). In contrast, we see a very large decrease in the premium of agent 1. The reason is that the probability that agent 1 will die first is decreasing, which increases the probability that the premium he spends will not result in the household receiving an insurance payment.

6. Conclusion

In this paper, we have solved a life insurance model for a household with two agents. The most important feature of our model is that we allow the agents to have different constant discount rates, and we derived a dynamic programing equation for finding a cooperative intertemporal equilibrium solution. Thus, our model combines the treatment of inconsistent time preferences and the presence of a household with more than one agent, something which seems to not have been done previously in life insurance models. In solving the model, we assumed that for each agent, the time of death is exponentially distributed, so that each agent has constant discount and mortality rates. This assumption proved to be useful for reasons of tractability. Through numeric analysis we were able to confirm results that have been established in previous papers, and also to find new results. In particular, we found that the purchase of life insurance of an agent decreases in the discount rate of the other agent.

Expansions of the model might go in several directions.

- More than two agents. A model with an arbitrary number of agents in the household was constructed in Bruhn and Steffensen (2011), but without taking the issue of inconsistent time preferences into consideration.

- Nonconstant mortality rates. For example, one might assume that the mortality rates follow Gompertz's Law, as has often been the case in the literature.
- Nonconstant discount rates. This would imply not only that the household as a whole has inconsistent time preferences, but also that each agent on his own has inconsistent time preferences.
- In the appendix of this paper, we derive a dynamic programming equation for finding the intertemporal cooperative equilibrium (in the sense of Ekeland and Lazrak (2010)) in a model with two agents that have different discount rates. This result can be applied to other economic research than life insurance.

CRediT authorship contribution statement

Joakim Alderborn: Formal analysis, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The author declares that he has no competing interests.

Data availability

No data was used for the research described in the article.

Appendix A

A.1. Proof of Proposition 1

First, we acknowledge that for $s > t$ and $i \in \{1, 2\}$, the expectation of $1_{\tau_i > s}$ conditional on $\tau_i > t$ can be rewritten as

$$\mathbb{E}\left[1_{\tau_i > s} \mid \tau_i > t\right] = \mathbb{P}(\tau_i > s \mid \tau_i > t) = \frac{\mathbb{P}(\tau_i > s\ ,\ \tau_i > t)}{\mathbb{P}(\tau_i > t)} = \frac{\mathbb{P}(\tau_i > s)}{\mathbb{P}(\tau_i > t)} = e^{-\lambda_i (s-t)}. \tag{17}$$

Then, to rewrite (2) into (10) we apply the following steps.

$$\begin{aligned} J_{i,B}(x_t, t; c_{i,B}, \epsilon_B) &= \mathbb{E}\left[\int_t^{\tau_i} e^{-\rho_i(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} ds \,\middle|\, \tau_i > t, x_t\right] \\ &= \mathbb{E}\left[\int_t^{\infty} e^{-\rho_i(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} 1_{\tau_i > s} ds \,\middle|\, \tau_i > t, x_t\right] \\ &= \mathbb{E}\left[\int_t^{\infty} e^{-\rho_i(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} \mathbb{E}\left[1_{\tau_i > s} \mid \tau_i > t\right] ds \,\middle|\, x_t\right] \\ &= \mathbb{E}\left[\int_t^{\infty} e^{-(\rho_i + \lambda_i)(s-t)} \frac{c_{i,s}^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t\right]. \end{aligned} \tag{18}$$

In the second equality, we change the limit of integration from τ_i to infinity and insert the random variable $1_{\tau_i > s}$. The third equality follows because τ_i is independent of the Brownian motion z_t. The fourth equality follows from (17).

A.2. Proof of Proposition 2

First, we acknowledge that for $s > t$ and $i \in \{1, 2\}$, the expectation of $1_{\bar{\tau} > s}$ conditional on $\bar{\tau} > t$ can be rewritten as

$$\begin{aligned} &\mathbb{E}\left[1_{\bar{\tau} > s} \mid \bar{\tau} > t\right] = \mathbb{P}(\bar{\tau} > s \mid \bar{\tau} > t) = \mathbb{P}(\tau_1 > s, \tau_2 > s \mid \tau_1 > t, \tau_2 > t) \\ &= \mathbb{P}(\tau_1 > s \mid \tau_1 > t)\mathbb{P}(\tau_2 > s \mid \tau_2 > t) = \frac{e^{-\lambda_1 s}}{e^{-\lambda_1 t}} \frac{e^{-\lambda_2 s}}{e^{-\lambda_2 t}} = e^{-(\lambda_1 + \lambda_2)(s-t)}\ . \end{aligned} \tag{19}$$

Then, to rewrite (7) into (13), we first set $i = 1$ and note that the expectation in (5) contains two terms which can be treated separately. For the first term, we have

$$\begin{aligned} &\mathbb{E}\left[\int_t^{\bar{\tau}} e^{-\rho_1(s-t)} \frac{c_{1,s}^{1-\gamma}}{1-\gamma} ds \,\middle|\, \bar{\tau} > t, x_t\right] \\ &= \mathbb{E}\left[\int_t^{\infty} e^{-\rho_1(s-t)} \frac{c_{1,s}^{1-\gamma}}{1-\gamma} 1_{\bar{\tau} > s} ds \,\middle|\, \bar{\tau} > t, x_t\right] \\ &= \mathbb{E}\left[\int_t^{\infty} e^{-\rho_1(s-t)} \frac{c_{1,s}^{1-\gamma}}{1-\gamma} \mathbb{E}\left[1_{\bar{\tau} > s} \mid \bar{\tau} > t\right] ds \,\middle|\, x_t\right] \end{aligned} \tag{20}$$

$$= \mathbb{E}\left[\int_t^\infty e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} \frac{c_{1,s}^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right],$$

where the third equality follows from (19). For the second term in (5) we have

$$\mathbb{E}\left[e^{-\rho_1(\bar{\tau}-t)} V_{1,B}(x_{\bar{\tau}} + q_{2,\bar{\tau}}/\eta_2) 1_{\tau_2=\bar{\tau}} \mid \bar{\tau} > t, x_t\right]$$
$$= \mathbb{E}\left[e^{-\rho_1(\tau_2-t)} V_{1,B}(x_{\tau_2} + q_{2,\tau_2}/\eta_2) 1_{\tau_2<\tau_1} \mid \tau_1 > t, \tau_2 > t, x_t\right]$$
$$= \mathbb{E}\left[\int_t^\infty \lambda_2 e^{-\lambda_2(s-t)} e^{-\rho_1(s-t)} V_{1,B}(x_s + q_{2,s}/\eta_2) 1_{s<\tau_1} ds \,\middle|\, \tau_1 > t, x_t \right]$$
$$= \mathbb{E}\left[\int_t^\infty \lambda_2 e^{-\lambda_2(s-t)} e^{-\rho_1(s-t)} V_{1,B}(x_s + q_{2,s}/\eta_2) \mathbb{E}\left[1_{s<\tau_1} \mid \tau_1 > t\right] ds \,\middle|\, x_t \right].$$

Recognizing that $\mathbb{E}\left[1_{s<\tau_1} \mid \tau_1 > t\right] = e^{-\lambda_1(s-t)}$ and, from (11), that

$$V_{1,B}(x_t + q_{2,t}/\eta_2)$$
$$= \left(\frac{\lambda_1 + \rho_1 - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2} \frac{(\mu - r)^2}{(\sigma\gamma)^2} \right)^{-\gamma} \frac{(x_t + q_{2,t}/\eta_2 + I_1/r)^{1-\gamma}}{1-\gamma}$$
$$= \Theta_1^{-\gamma} \frac{(x_t + q_{2,t}/\eta_2 + I_1/r)^{1-\gamma}}{1-\gamma},$$

the second term in (5) becomes

$$\mathbb{E}\left[\int_t^\infty \lambda_2 e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} \Theta_1^{-\gamma} \frac{(x_t + q_{2,t}/\eta_2 + I_1/r)^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right]. \tag{21}$$

Combining (20) and (21) we obtain

$$\mathbb{E}\left[\int_t^\infty e^{-(\rho_1+\lambda_1+\lambda_2)(s-t)} \frac{c_{1,s}^{1-\gamma} + \lambda_2 \Theta_1^{-\gamma} (x_s + I_1/r + q_{2,s}/\eta_2)^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right]. \tag{22}$$

In a similar way, by setting $i = 2$, (5) can be written as

$$\mathbb{E}\left[\int_t^\infty e^{-(\rho_2+\lambda_1+\lambda_2)(s-t)} \frac{c_{2,s}^{1-\gamma} + \lambda_1 \Theta_2^{-\gamma} (x_s + I_2/r + q_{1,s}/\eta_1)^{1-\gamma}}{1-\gamma} ds \,\middle|\, x_t \right]. \tag{23}$$

Combining (22) and (23) we obtain (13), the transformed intertemporal utility function.

A.3. Proof of Proposition 3

We want to derive the dynamic programming for an intertemporal equilibrium solution to a problem with stochastic dynamics, in which the intertemporal utility function is the sum of two sums of utilities, each with its own constant discount rate. In a deterministic setting and with an arbitrary number of agents, this was done in de Paz et al. (2014). Here, we present the corresponding proof for the case of two agents and with stochastic dynamics. As was mentioned above, our definition of an equilibrium solution is due to Ekeland and Pirvu (2008) and Ekeland and Lazrak (2010), and we will construct the dynamic programming equation in such a way that it satisfies this definition.

For notational simplicity, we will define

$$U_i(c_{i,s}, q_{-i,s}, x_s) = \frac{c_{i,s}^{1-\gamma} + \lambda_{-i} \Theta_i^{-\gamma} (x_s + I_i/r + q_{-i,s}/\eta_{-i})^{1-\gamma}}{1-\gamma},$$

for $i \in \{1, 2\}$. First, we recognize that for each $t \geq 0$, we have

$$J_A(x_t, t; c_{1,A}^*, c_{2,A}^*, q_{1,A}^*, q_{2,A}^*, \epsilon_A^*) = V_{1,A}(x_t, t) + V_{2,A}(x_t, t).$$

That is, the value of the intertemporal utility function when we apply the decision rule $\{c_{1,A}^*, c_{2,A}^*, q_{1,A}^*, q_{2,A}^*, \epsilon_A^*\}$ is equal to the sum of the value functions. Next, for a given t, we consider the decision rules that are given by the constants in the set $\{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}\}$ if $s \in [t, t + \Delta t]$ and by $\{c_{1,A}^*, c_{2,A}^*, q_{1,A}^*, q_{2,A}^*, \epsilon_A^*\}$ if $s > t + \Delta t$. We will call this decision rule Φ, and we let $\bar{x}_s$ be the trajectory of the wealth process when the household follows Φ. The intertemporal utility function when this decision rule is applied is

$$J_A(x_t, t; \Phi) =$$
$$= \mathbb{E}\left[\sum_{i=1}^{2} \int_t^{t+\Delta t} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)} U_i(\bar{c}_{i,s}, \bar{q}_{-i,s}, \bar{x}_s) ds \right.$$

$$+\sum_{i=1}^{2}\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}U_i(c^*_{i,s},q^*_{-i,s},x^*_s)ds \;\Big|\; x_t \Bigg].$$

The idea behind this decision rule is that $[t, t+\Delta t]$ is the period in which the household is able to commit its future selves to a certain behavior. We can refer to this as the "period of commitment". Hence, the household at t is in control of its behavior over a nonzero interval of time, as in the discrete time case, and can therefore influence the value of the intertemporal utility function. Next, suppose that for any Φ (that is, any set of constants $\{\bar{c}_1, \bar{c}_2, \bar{q}_1, \bar{q}_2, \bar{\epsilon}\}$), it is the case that

$$\lim_{\Delta t\to 0^+}\frac{J_A(x_t,t;c^*_{1,A},c^*_{2,A},q^*_{1,A},q^*_{2,A},\epsilon^*_A)-J_A(x_t,t;\Phi)}{\Delta t}\geq 0\,.$$

If this condition is satisfied, we say that $\{c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A\}$ is the equilibrium decision rule. This is the definition of an equilibrium solution due to Ekeland and Lazrak (2010).

By taking the limit of Δt, the period of commitment vanishes, and the numerator approaches zero. But the expression is scaled up by the denominator, so that the behavior within the period of commitment remains influential. The intuition is that if $\{c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A\}$ is indeed the equilibrium decision rule, the expression must be nonnegative because no other decision rule can result in a larger value of the intertemporal utility function. Of course, if we set $\Phi = \{c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A\}$, the expression evaluates to zero. Hence, we can also define the equilibrium decision rule to be any Φ that solves

$$0=\max_{\bar{c}_1,\bar{c}_2,\bar{q}_1,\bar{q}_2,\bar{\epsilon}}\left\{\lim_{\Delta t\to 0^+}\frac{J_A(x_t,t;\Phi)-J_A(x_t,t;c^*_{1,A},c^*_{2,A},q^*_{1,A},q^*_{2,A},\epsilon^*_A)}{\Delta t}\right\}, \tag{24}$$

which provides us with a dynamic programming equation for finding the equilibrium solution. To make the dynamic programming equation workable, we need to rewrite it into something more familiar. To this end, we notice that

$$\begin{aligned} &J_A(x_t,t;\Phi)-J_A(x_t,t;c^*_1,c^*_2,q^*_1,q^*_2,\epsilon^*)= \\ &=\mathbb{E}\Bigg[\sum_{i=1}^{2}\int_{t}^{t+\Delta t} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(\bar{c}_{i,s},\bar{q}_{-i,s},\bar{x}_s)-U_i(c^*_{i,s},q^*_{-i,s},x^*_s))ds+ \\ &+\sum_{i=1}^{2}\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(c^*_{i,s},q^*_{-i,s},\bar{x}_s)-U_i(c^*_{i,s},q^*_{-i,s},x^*_s))ds \;\Big|\; x_t\Bigg]. \end{aligned} \tag{25}$$

Moreover,

$$\begin{aligned} &\mathbb{E}\left[\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}U_i(c^*_{i,s},q^*_{-i,s},\bar{x}_s)ds \;\Big|\; x_t\right]= \\ &\mathbb{E}\left[V_{i,A}(\bar{x}_{t+\Delta t})+\int_{t+\Delta t}^{\infty}(e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}-e^{-(\rho_i+\lambda_1+\lambda_2)(s-t-\Delta t)})U_i(c^*_{i,s},q^*_{-i,s},\bar{x}_s)ds \;\Big|\; x_t\right] \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\mathbb{E}\left[\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}U_i(c^*_{i,s},q^*_{-i,s},x^*_s)ds \;\Big|\; x_t\right]= \\ &\mathbb{E}\left[V_{i,A}(x^*_{t+\Delta t})+\int_{t+\Delta t}^{\infty}(e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}-e^{-(\rho_i+\lambda_1+\lambda_2)(s-t-\Delta t)})U_i(c^*_{i,s},q^*_{-i,s},x^*_s)ds \mid x_t\right]. \end{aligned} \tag{27}$$

Combining (26) and (27) we have

$$\begin{aligned} &\mathbb{E}\left[\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(c^*_{i,s},q^*_{-i,s},\bar{x}_s)-U_i(c^*_{i,s},q^*_{-i,s},x^*_s))ds \mid x_t\right]= \\ &\mathbb{E}\Big[V_{i,A}(\bar{x}_{t+\Delta t})-V_{i,A}(x^*_{t+\Delta t})+ \\ &(o(\Delta)-(\rho_i+\lambda_1+\lambda_2)\Delta t)\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(c^*_{i,s},q^*_{-i,s},\bar{x}_s)-U_i(c^*_{i,s},q^*_{-i,s},x^*_s))ds \;\Big|\; x_t\Big], \end{aligned} \tag{28}$$

where we have used the Taylor expansion $e^{(\rho_i+\lambda_1+\lambda_2)\Delta t} = 1 + (\rho_i + \lambda_1 + \lambda_2)\Delta t + o(\Delta t)$, and $o(\Delta t)$ are functions of Δt that converge to zero at a faster rate than Δt. Combining this with (25) and dividing by Δt, we have

$$\frac{J_A(x_t,t;\Phi)-J_A(x_t,t;c^*_{1,A},c^*_{2,A},q^*_{1,A},q^*_{2,A},\epsilon^*_A)}{\Delta t}=$$

$$= \mathbb{E}\left[\sum_{i=1}^{2} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(\bar{c}_{i,s}, \bar{q}_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds + \right.$$
$$\sum_{i=1}^{2}\left(\frac{V_{i,A}(\bar{x}_{t+\Delta t}) - V_{i,A}(x_t)}{\Delta t} - \frac{V_{i,A}(x^*_{t+\Delta t}) - V_{i,A}(x_t)}{\Delta t}\right) +$$
$$\left.\sum_{i=1}^{2}(o(\Delta)/\Delta t - \rho_i - \lambda_1 - \lambda_2) \int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \,\middle|\, x_t\right].$$

When we take the limit $\Delta t \to 0^+$, we can treat each of the terms in the above expression separately. We have

$$\lim_{\Delta t \to 0^+} \mathbb{E}\left[\frac{1}{\Delta t} \int_{t}^{t+\Delta t} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(\bar{c}_{i,s}, \bar{q}_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \,\middle|\, x_t\right] =$$
$$U_i(\bar{c}_{i,t}, \bar{q}_{-i,t}, \bar{x}_t) - U_i(c^*_{i,t}, q^*_{-i,t}, x^*_t)\ ,$$

which follows from L'Hopital's Rule. We have

$$\lim_{\Delta t \to 0^+} \mathbb{E}\left[\frac{V_{i,A}(\bar{x}_{t+\Delta t}) - V_{i,A}(x_t)}{\Delta t} \,\middle|\, x_t\right] = -(\rho_i + \lambda_1 + \lambda_2)V_{i,A}(x_t) +$$
$$\frac{dV_{i,A}}{dx_t}(x_t)(rx_t + \bar{\epsilon}_t(\mu - r)x_t - \bar{c}_{1,t} - \bar{c}_{2,t} - \bar{q}_{1,t} - \bar{q}_{2,t} + I_1 + I_2) + \frac{d^2V_{i,A}}{dx^2}(x_t)\frac{(\bar{\epsilon}_t\sigma x_t)^2}{2}$$

and

$$\lim_{\Delta t \to 0^+} \mathbb{E}\left[\frac{V_{i,A}(x^*_{t+\Delta t}) - V_{i,A}(x_t)}{\Delta t} \,\middle|\, x_t\right] = -(\rho_i + \lambda_1 + \lambda_2)V_{i,A}(x_t) +$$
$$\frac{dV_{i,A}}{dx_t}(x_t)(rx_t + \epsilon^*_t(\mu - r)x_t - c^*_{1,t} - c^*_{2,t} - q^*_{1,t} - q^*_{2,t} + I_1 + I_2) + \frac{d^2V_{i,A}}{dx^2}(x_t)\frac{(\epsilon^*_t\sigma x_t)^2}{2}.$$

For the last terms, we have

$$0 = \lim_{\Delta t \to 0^+} \mathbb{E}\left[(o(\Delta)/\Delta t - \rho_i - \lambda_1 - \lambda_2)\right.$$
$$\left.\int_{t+\Delta t}^{\infty} e^{-(\rho_i+\lambda_1+\lambda_2)(s-t)}(U_i(c^*_{i,s}, q^*_{-i,s}, \bar{x}_s) - U_i(c^*_{i,s}, q^*_{-i,s}, x^*_s))ds \,\middle|\, x_t\right]$$

because the utility functions converge. Hence, putting everything together we obtain

$$\lim_{\Delta t \to 0^+} \frac{J_A(x_t, t; \Phi) - J_A(x_t, t; c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A)}{\Delta t} \tag{29}$$
$$= \sum_{i=1}^{2}(U_i(\bar{c}_{i,t}, \bar{q}_{-i,t}, \bar{x}_t) - U_i(c^*_{i,t}, q^*_{-i,t}, x^*_t))$$
$$+ \sum_{i=1}^{2}\left(\frac{dV_{i,A}}{dx_t}(x_t)(rx_t + \bar{\epsilon}_t(\mu - r)x_t - \bar{c}_{1,t} - \bar{c}_{2,t} - \bar{q}_{1,t} - \bar{q}_{2,t} + I_1 + I_2) + \frac{d^2V_{i,A}}{dx^2}(x_t)\frac{(\bar{\epsilon}_t\sigma x_t)^2}{2}\right)$$
$$- \sum_{i=1}^{2}\left(\frac{dV_{i,A}}{dx_t}(x_t)(rx_t + \epsilon^*_t(\mu - r)x_t - c^*_{1,t} - c^*_{2,t} - q^*_{1,t} - q^*_{2,t} + I_1 + I_2) + \frac{d^2V_{i,A}}{dx^2}(x_t)\frac{(\epsilon^*_t\sigma x_t)^2}{2}\right).$$

Finally, by recognizing that

$$\frac{dV_{i,A}}{dx_t}(x_t)(rx_t + \epsilon^*_t(\mu - r)x_t - c^*_{1,t} - c^*_{2,t} - q^*_{1,t} - q^*_{2,t} + I_1 + I_2) + \frac{d^2V_{i,A}}{dx^2}(x_t)\frac{(\epsilon^*_t\sigma x_t)^2}{2}$$
$$+ U_i(c^*_{i,t}, q^*_{-i,t}, x^*_t) = \mathbb{E}\left[\frac{dV_{i,A}}{dt}(x_t) \,\middle|\, x_t\right] + (\rho_i + \lambda_1 + \lambda_2)V_{i,A}(x_t)$$

we obtain Proposition 3.

A.4. In search of a solution when there is wage income

In order to find the constants $\{A_1, A_2, B_1, B_2\}$, we attempt the following method. We first create a system of two equations by taking the time derivative of (14) for both agents. With $\{c^*_{1,A}, c^*_{2,A}, q^*_{1,A}, q^*_{2,A}, \epsilon^*_A\}$ as we derived for the case when there is wage income, the system becomes

$$0 = (rx + I_1 + I_2 + (\eta_1 + \eta_2)x)$$

$$\frac{((A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma})^{\frac{1-\gamma}{\gamma}}}{1-\gamma}(A_i(x+B_i))^{-\gamma}-(\rho_i+\lambda_1+\lambda_2)\frac{x+B_i}{1-\gamma}$$
$$+\frac{(\mu-r)^2}{\gamma\sigma^2}\frac{(A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma}}{(A_1(x+B_1))^{-\gamma-1}+(A_2(x+B_2))^{-\gamma-1}}-2((A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma})^{-1/\gamma}$$
$$-(x+B_i)\frac{(\mu-r)^2}{2\gamma\sigma^2}\left(\frac{(A_1(x+B_1))^{-\gamma}+(A_2(x+B_2))^{-\gamma}}{(A_1(x+B_1))^{-\gamma-1}+(A_2(x+B_2))^{-\gamma-1}}\right)^2$$
$$+\eta_1\left(\frac{\lambda_1}{\eta_1}\right)^{1/\gamma}\left(\left(\frac{\Theta_1}{A_1(x+B_1)}\right)^{\gamma}+\left(\frac{\Theta_1}{A_2(x+B_2)}\right)^{\gamma}\right)^{-1/\gamma}$$
$$+\eta_2\left(\frac{\lambda_2}{\eta_2}\right)^{1/\gamma}\left(\left(\frac{\Theta_2}{A_1(x+B_1)}\right)^{\gamma}+\left(\frac{\Theta_2}{A_2(x+B_2)}\right)^{\gamma}\right)^{-1/\gamma}$$

for $i \in \{1,2\}$. At this point, what we would like to do is to separate each equation in the system into two equations, based on which terms are multiplied by x and which are not. We would then obtain the four equations that we need to solve for the four parameters in $\{A_1, A_2, B_1, B_2\}$. However, there is no way to separate the terms in the equations based on powers of x. The reason is the presence of the terms B_1 and B_2, which are part of our conjecture for the value functions. It is clear that the system is solvable if $B_1 = B_2$, because in that case one can divide by $x + B_i$ in order to remove the complicating factors. But attempting to prove conditions under which $B_1 = B_2$ is not straightforward.

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