

Quantum Teleportation of an Unknown State via a Partially Entangled Qutrit Pair

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Abstract: Quantum teleportation allows the perfect transmission of an unknown quantum state using a shared entangled resource and a classical channel alone. In this work, we study the teleportation of three-level quantum systems, known as qutrits, using entangled three-dimensional pairs. We demonstrate teleportation under maximal entanglement, and develop two complementary probabilistic strategies to herald successful state recovery when the resource is only partially entangled. Both approaches yield the same maximum success probability, governed entirely by the smallest component of the entangled resource. We further extend these results to quantum systems of arbitrary finite dimension, obtaining a universal closed-form expression for the maximum achievable success probability.

Keywords: Quantum teleportation, partial entanglement, unitary corrections, Bell states, projective measure

SDGs: 4.4, 9.5, 9.8

I. INTRODUCTION

We study the quantum teleportation of three-level systems—commonly referred to as qutrits in the quantum information literature—using qutrit entangled pairs. Teleportation via maximally or partially entangled level-two (qubit) pairs has been extensively treated in the literature [1]; by contrast, comparatively few works address the qutrit case. The present paper aims to fill this gap by extending the standard teleportation protocol and its probabilistic variants to qutrit resource states and later on into d -dimensional cases.

The canonical teleportation protocol proceeds as follows [1]. Two spatially separated parties, conventionally denoted Alice and Bob, share a bipartite entangled resource. Alice is additionally in possession of an arbitrary unknown quantum state $|\psi\rangle$ that she wishes to transmit to Bob. She performs a joint Bell-basis measurement on her subsystem—comprising her half of the shared entangled pair together with $|\psi\rangle$ —and communicates the outcome to Bob over a classic channel. Upon receiving this message, Bob applies the appropriate local unitary correction to his subsystem, thereby recovering $|\psi\rangle$ exactly. This protocol is state-independent: its validity is not contingent on the specific form of $|\psi\rangle$. We therefore parametrise the unknown input state in the computational basis as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle, \quad |\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1, \quad (1)$$

with the orthonormal basis vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The generalised Bell basis for two-qutrit systems, as

derived in [2, 3], reads

$$\begin{aligned} |\varphi_{k0}\rangle &= \frac{1}{\sqrt{3}}(|00\rangle + \omega^k|11\rangle + \omega^{2k}|22\rangle), \\ |\varphi_{k1}\rangle &= \frac{1}{\sqrt{3}}(|01\rangle + \omega^k|12\rangle + \omega^{2k}|20\rangle), \\ |\varphi_{k2}\rangle &= \frac{1}{\sqrt{3}}(|02\rangle + \omega^k|10\rangle + \omega^{2k}|21\rangle), \end{aligned} \quad (2)$$

where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity ($\omega^3 = 1$) and $k = 0, 1, 2$. It is important to note that $1 + \omega + \omega^2 = 0$. These nine maximally entangled states form a complete orthonormal basis for the two-qutrit Hilbert space. Throughout this work the subscripts A , B , and C label Alice's share of the entangled pair, Bob's share, and the unknown input qutrit, respectively.

Assuming Alice and Bob share the maximally entangled state $|\varphi_{00}\rangle_{AB}$, the full three particles state of the system can be decomposed in the Bell basis of the CA subsystem as

$$\begin{aligned} |\Phi\rangle_{CAB} &= |\psi\rangle_C |\varphi_{00}\rangle_{AB} \\ &= \frac{1}{\sqrt{3}} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_C (|00\rangle_{AB} + |11\rangle_{AB} + |22\rangle_{AB}) \\ &= \frac{1}{3} \sum_{k=0}^2 \left[|\varphi_{k0}\rangle_{CA} \begin{pmatrix} \alpha \\ \beta\omega^{2k} \\ \gamma\omega^k \end{pmatrix}_B + |\varphi_{k1}\rangle_{CA} \begin{pmatrix} \gamma\omega^k \\ \alpha \\ \beta\omega^{2k} \end{pmatrix}_B \right. \\ &\quad \left. + |\varphi_{k2}\rangle_{CA} \begin{pmatrix} \beta\omega^{2k} \\ \gamma\omega^k \\ \alpha \end{pmatrix}_B \right]. \end{aligned} \quad (3)$$

Upon Alice's Bell measurement, the CA system collapses onto one of the nine Bell states $|\varphi_{kn}\rangle_{CA}$ with equal probability $1/9$. Bob's conditional state is then related to $|\psi\rangle$ by a known unitary. For instance, if the outcome is $|\varphi_{12}\rangle_{CA}$, Bob holds $|\xi_{12}\rangle_B = \beta\omega^2|0\rangle_B + \gamma\omega|1\rangle_B + \alpha|2\rangle_B$, where is only a unitary correction R_{12} from the original

unknown particle. The full set of nine correction unitaries is

$$\begin{aligned} R_{k0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^k & 0 \\ 0 & 0 & \omega^{2k} \end{pmatrix}, & R_{k1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^k \\ \omega^{2k} & 0 & 0 \end{pmatrix}, \\ R_{k2} &= \begin{pmatrix} 0 & 0 & 1 \\ \omega^k & 0 & 0 \\ 0 & \omega^{2k} & 0 \end{pmatrix}, \end{aligned} \quad (4)$$

for $k = 0, 1, 2$. Equipped with Alice's measurement message (k, n) , Bob applies R_{kn} to his subsystem and recovers $|\psi\rangle$ exactly, completing the teleportation.

Deterministic teleportation as described above requires the shared resource to be maximally entangled. In practice, decoherence and channel noise typically degrade the entanglement, so it is essential to analyse the protocol under partially entangled resources. Without loss of generality a partially entangled two-qutrit state can be written as

$$|\chi_{00}\rangle_{AB} = a|00\rangle_{AB} + b|11\rangle_{AB} + c|22\rangle_{AB}, \quad (5)$$

(Schmidt decomposition) where a, b, c can be taken real and positive, $a^2 + b^2 + c^2 = 1$, and also $a \geq b \geq c > 0$. The maximally entangled case corresponds to $a = b = c = 1/\sqrt{3}$. Substituting this resource into the three particles state yields

$$\begin{aligned} |\Omega\rangle_{CAB} &= |\psi\rangle_C |\chi_{00}\rangle_{AB} \\ &= \frac{1}{\sqrt{3}} \sum_{k=0}^2 \left[|\varphi_{k0}\rangle_{CA} \begin{pmatrix} \alpha a \\ \beta b \omega^{2k} \\ \gamma c \omega^k \end{pmatrix}_B + |\varphi_{k1}\rangle_{CA} \begin{pmatrix} \gamma a \omega^k \\ \alpha b \\ \beta c \omega^{2k} \end{pmatrix}_B \right. \\ &\quad \left. + |\varphi_{k2}\rangle_{CA} \begin{pmatrix} \beta a \omega^{2k} \\ \gamma b \omega^k \\ \alpha c \end{pmatrix}_B \right]. \end{aligned} \quad (6)$$

A Bell measurement by Alice now projects Bob's subsystem onto a state untraceable by itself. Bob does not universally retrieve the original state, if $|\chi_{00}\rangle$ is not maximally entangled, upon acting the unitary R_{kn} .

II. TELEPORTATION DETECTION VIA ANCILLARY SYSTEMS

When the shared resource is only partially entangled, teleportation succeeds in a subset of runs. Identifying these successful instances requires coupling an ancillary system to the protocol via a suitable unitary interaction. Two complementary detection strategies are presented below. In the first, Bob appends a qubit ancilla and applies a carefully engineered interaction unitary, following the approach of Li *et al.* [4] extended here to the three-dimensional setting. In the second, Alice forgoes the standard Bell measurement in favor of a Positive Operator-Valued Measure (POVM), leaving Bob's subsystem undisturbed until the classical result is commu-

nicated. Throughout this section, $|S_i\rangle$ denotes the i -th basis state of the ancillary Hilbert space.

A. Unitary Interaction at Bob's Station

Following Alice's Bell measurement, Bob holds state of the form $|\theta_{kn}\rangle_B = a\kappa|0\rangle + b\rho|1\rangle + c\eta|2\rangle$, where κ, ρ and η are combinations of α, β, γ and powers of ω . Here it is demonstrated the procedure Bob uses for the $|\theta_{k0}\rangle$. Bob extends his Hilbert space by a qubit ($d = 2$) ancilla initialised in $|S_0\rangle$, forming the joint state

$$|S_0\rangle|\theta_{k0}\rangle_B = |S_0\rangle(a\alpha|0\rangle + b\beta\omega^{2k}|1\rangle + c\gamma\omega|2\rangle), \quad (7)$$

Bob then applies the unitary $\mathcal{U}$ acting on the joint ancilla-system Hilbert space:

$$\mathcal{U}[|S_0\rangle|\theta_{k0}\rangle] = |S_0\rangle(K_0|\theta_{k0}\rangle) + |S_1\rangle(K_1|\theta_{k0}\rangle), \quad (8)$$

where one can build the unitary $\mathcal{U}$ so that $K_0|\theta_{k0}\rangle \propto |\psi\rangle$ and build the rest of the transformation to make it unitary. In this particular case, it is convenient to write the unitary transformation in the $\{|S_0\rangle, |S_1\rangle\} \otimes \{|0\rangle, |1\rangle, |2\rangle\}$ basis as

$$\mathcal{U} = \left(\begin{array}{ccc|ccc} c/a & 0 & 0 & 0 & 0 & -U_a \\ 0 & c/b & 0 & 0 & -U_b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & U_b & 0 & 0 & c/b & 0 \\ U_a & 0 & 0 & 0 & 0 & c/a \end{array} \right), \quad (9)$$

where $U_a \equiv \sqrt{1 - (c/a)^2}$ and $U_b \equiv \sqrt{1 - (c/b)^2}$. One may identify K_0 as the top-left 3×3 matrix and K_1 as the bottom-left. Therefore the unitary transformation upon the joint ancilla-Bob's particle is,

$$\begin{aligned} \mathcal{U}[|S_0\rangle|\theta_{k0}\rangle] &= \left[c|S_0\rangle(\alpha|0\rangle + \beta\omega^{2k}|1\rangle + \gamma\omega^k|2\rangle) \right. \\ &\quad \left. + |S_1\rangle(\beta\omega^{2k}\sqrt{b^2 - c^2}|1\rangle + \alpha\sqrt{a^2 - c^2}|2\rangle) \right]. \end{aligned} \quad (10)$$

The $|S_0\rangle$ branch, is precisely $c \cdot R_{k0}|\psi\rangle_B$ —i.e., a successfully teleported state up to the known correction R_{k0} —whereas the $|S_1\rangle$ branch is orthogonal garbage. A projective measurement on the ancilla therefore heralds success ($|S_0\rangle$) and failure ($|S_1\rangle$).

Extending this analysis to the full nine-outcome Bell measurement on Alice's side and tracing over all post-measurement branches, the globally extended three par-

ticles evolution reads

$$\begin{aligned} \mathcal{U}[|S_0\rangle(|\psi\rangle_C|\chi_{00}\rangle_{AB})] = & \\ \frac{1}{\sqrt{3}} \sum_{k=0}^2 |\varphi_{k0}\rangle_{CA} & \left(c|S_0\rangle(R_{k0}|\psi\rangle_B) + |S_1\rangle \begin{pmatrix} 0 \\ \beta\omega^{2k}\sqrt{b^2-c^2} \\ \alpha\sqrt{a^2-c^2} \end{pmatrix}_B \right) \\ & + |\varphi_{k1}\rangle_{CA} \left(c|S_0\rangle(R_{k1}|\psi\rangle_B) + |S_1\rangle \begin{pmatrix} 0 \\ \alpha\sqrt{b^2-c^2} \\ \gamma\omega^k\sqrt{a^2-c^2} \end{pmatrix}_B \right) \\ & + |\varphi_{k2}\rangle_{CA} \left(c|S_0\rangle(R_{k2}|\psi\rangle_B) + |S_1\rangle \begin{pmatrix} 0 \\ \gamma\omega^k\sqrt{b^2-c^2} \\ \beta\omega^{2k}\sqrt{a^2-c^2} \end{pmatrix}_B \right) \end{aligned}$$

yielding an overall heralded success probability of $3c^2$.

One may identify $\mathcal{U}$ as a time-evolution operator $\mathcal{U}(t)$ with the parametrisation $c/a \equiv \cos \lambda t$ and $c/b \equiv \cos \mu t$, from which an effective interaction Hamiltonian $\mathcal{H}$ can be derived via $\mathcal{U}(t) = e^{-i\mathcal{H}t}$,

$$\mathcal{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i\lambda \\ 0 & 0 & 0 & 0 & -i\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\mu & 0 & 0 & 0 & 0 \\ i\lambda & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

B. POVM Detection at Alice's Station

An alternative strategy is to perform the detection at Alice's end, leaving Bob's subsystem undisturbed until the very end. Rather than projecting onto the Bell basis, Alice implements a Positive Operator-Valued Measure (POVM) on the CA system. To construct this POVM, it is used the method described in [5], Alice expands $|\Omega\rangle_{CAB}$ in a product basis rather than the Bell basis:

$$\begin{aligned} |\Omega\rangle_{CAB} = & |\psi\rangle_C |\chi_{00}\rangle_{AB} \\ = & \frac{1}{\sqrt{3}} \sum_{k=0}^2 \left[(a|00\rangle + b\omega^k|11\rangle + c\omega^{2k}|22\rangle)_{CA} \begin{pmatrix} \alpha \\ \beta\omega^{2k} \\ \gamma\omega^k \end{pmatrix}_B \right. \\ & + (a|01\rangle + b\omega^k|12\rangle + c\omega^{2k}|20\rangle)_{CA} \begin{pmatrix} \gamma\omega^k \\ \alpha \\ \beta\omega^{2k} \end{pmatrix}_B \\ & \left. + (a|02\rangle + b\omega^k|10\rangle + c\omega^{2k}|21\rangle)_{CA} \begin{pmatrix} \beta\omega^{2k} \\ \gamma\omega^k \\ \alpha \end{pmatrix}_B \right]. \end{aligned} \quad (12)$$

This expansion reveals three mutually orthogonal sub-

spaces of the CA Hilbert space,

$$\begin{aligned} \Pi_0 &= \{a|00\rangle + b\omega^k|11\rangle + c\omega^{2k}|22\rangle\}_{k=0}^2, \\ \Pi_1 &= \{a|01\rangle + b\omega^k|12\rangle + c\omega^{2k}|20\rangle\}_{k=0}^2, \\ \Pi_2 &= \{a|02\rangle + b\omega^k|10\rangle + c\omega^{2k}|21\rangle\}_{k=0}^2, \end{aligned} \quad (13)$$

whose mutual orthogonality $\langle v|u\rangle = \langle u|w\rangle = \langle w|v\rangle = 0$ (for $|v\rangle \in \Pi_0$, $|u\rangle \in \Pi_1$, $|w\rangle \in \Pi_2$) can be verified directly. Each subspace Π_n is spanned by three states that are *not* mutually orthogonal for general a, b, c ; consequently, the measurement cannot resolve them via a projective measurement. A POVM, however, is not subject to this restriction.

Alice first applies a subspace selector operator $\mathcal{P}$ (e.g., $\mathcal{P} = \sum_j |jj\rangle\langle jj|$ for Π_0) to identify the subspace Π_n in which the CA system lies—a step that can be performed perfectly since the subspaces are orthogonal. Conditioned on this outcome, Alice then applies a POVM within Π_n to discriminate among its three non-orthogonal elements. This is realised by appending a four-dimensional ancilla (one dimension more than the subspace size) in state $|S_0\rangle$ and applying a unitary $\mathcal{U}$ of the block form

$$\mathcal{U} = \left(\begin{array}{c|c} \cdots & \cdots \\ \hline K_i & \cdots \\ \hline \cdots & \cdots \end{array} \right), \quad (14)$$

so that the evolution of an arbitrary state $|\phi\rangle \in \Pi_n$ is

$$\mathcal{U}[|S_0\rangle|\phi\rangle] = \sum_{j=0}^3 |S_j\rangle(K_j|\phi\rangle). \quad (15)$$

Unitarity of $\mathcal{U}$ imposes the completeness relation $\sum_{j=0}^3 K_j^\dagger K_j = \mathbb{I}_3$. The POVM elements are then $M_j = K_j^\dagger K_j$, with $M_3 = \mathbb{I} - M_0 - M_1 - M_2$ absorbing the inconclusive outcomes. Denoting the three elements of an arbitrary Π_n by $\{|v_i\rangle\}_{i=0}^2$, one imposes

$$\mathcal{U}[|S_0\rangle|v_i\rangle] = |S_i\rangle(K_i|v_i\rangle) + |S_3\rangle(K_3|v_i\rangle), \quad i = 0, 1, 2, \quad (16)$$

with $K_l|v_i\rangle = 0$ for $l \neq i, 3$. One then seeks POVM elements [6],

$$M_i = \lambda|m_i\rangle\langle m_i|, \quad i = 0, 1, 2, \quad M_3 = \mathbb{I} - \sum_{i=0}^2 M_i, \quad (17)$$

where the dual vectors $|m_i\rangle$ satisfy the orthogonality condition $\langle v_i|m_j\rangle = \delta_{ij}$, and the scalar $\lambda > 0$ is maximized subject to $M_3 \geq 0$.

Each POVM outcome $|S_i\rangle$ ($i = 0, 1, 2$) heralds that the CA system collapsed onto $|v_i\rangle_{CA}$, which is unambiguously correlated with a specific state of Bob's subsystem; Bob then applies the corresponding correction R_{kn} to recover $|\psi\rangle$. The outcome $|S_3\rangle$ is inconclusive and the run is discarded. This means that a measure only in the

ancillary system measures not only if teleportation has occurred or not but also which unitary correction Bob has to apply if teleportation came through.

As a concrete realisation, consider the projector $\mathcal{P} = |00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22|$ selecting Π_0 , whose three spanning states are

$$\begin{aligned} |v_0\rangle &= a|00\rangle + b|11\rangle + c|22\rangle, \\ |v_1\rangle &= a|00\rangle + b\omega|11\rangle + c\omega^2|22\rangle, \\ |v_2\rangle &= a|00\rangle + b\omega^2|11\rangle + c\omega|22\rangle. \end{aligned} \quad (18)$$

Assembling these as columns of a 3×3 matrix V and noting that the dual vectors are the columns of $Q = (V^{-1})^\dagger$, one obtains

$$\begin{aligned} |m_0\rangle &= \frac{1}{3} \left(\frac{1}{a}|00\rangle + \frac{1}{b}|11\rangle + \frac{1}{c}|22\rangle \right), \\ |m_1\rangle &= \frac{1}{3} \left(\frac{1}{a}|00\rangle + \frac{\omega}{b}|11\rangle + \frac{\omega^2}{c}|22\rangle \right), \\ |m_2\rangle &= \frac{1}{3} \left(\frac{1}{a}|00\rangle + \frac{\omega^2}{b}|11\rangle + \frac{\omega}{c}|22\rangle \right). \end{aligned} \quad (19)$$

The positivity constraint $M_3 \geq 0$ is most stringent along the smallest Schmidt coefficient. Evaluating M_3 explicitly,

$$M_3 = \text{diag} \left(1 - \frac{\lambda}{3a^2}, 1 - \frac{\lambda}{3b^2}, 1 - \frac{\lambda}{3c^2} \right), \quad (20)$$

the constraint is $\lambda \leq 3c^2$. Saturating this bound gives the optimal $\lambda = 3c^2$, and the success-heralding POVM elements become

$$M_k = \frac{1}{3} \begin{pmatrix} \frac{c^2}{a^2} & \frac{c^2}{ab}\omega^{2k} & \frac{c}{a}\omega^k \\ \frac{c^2}{ab}\omega^k & \frac{c^2}{b^2} & \frac{c}{b}\omega^{2k} \\ \frac{c}{a}\omega^{2k} & \frac{c}{b}\omega^k & 1 \end{pmatrix}, \quad k = 0, 1, 2, \quad (21)$$

with $M_3 = \text{diag}(1 - c^2/a^2, 1 - c^2/b^2, 0)$. Each state $|v_i\rangle$ is detected with probability $p(v_i) = \langle v_i|M_i|v_i\rangle = 3c^2$, and the inconclusive probability is $p(\text{fail}) = 1 - 3c^2$. An identical analysis for Π_1 and Π_2 yields the same bound, confirming that the maximum achievable success probability under a POVM strategy is

$$p_{\max} = 3c^2, \quad (22)$$

not depending of the subspace. In the maximally entangled limit $a = b = c = 1/\sqrt{3}$, one recovers $p_{\max} = 1$ and deterministic teleportation is restored.

III. GENERALISATION TO ARBITRARY DIMENSIONAL SYSTEMS

Having established the qutrit ($d = 3$) case in detail, we now extend the foregoing analysis to arbitrary d -dimensional quantum systems also treated in [6]. The unknown input state is

$$|\psi\rangle = \sum_{j=0}^{d-1} \alpha_j |j\rangle, \quad \sum_{j=0}^{d-1} |\alpha_j|^2 = 1. \quad (23)$$

The generalised d -dimensional Bell basis, with d^2 elements labelled by (k, n) with $k, n = 0, \dots, d-1$, is [1]

$$|\varphi_{kn}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega^{kj} |j\rangle \otimes |(j+n) \bmod d\rangle, \quad (24)$$

where $\omega = e^{2\pi i/d}$. If Alice obtains outcome $|\varphi_{kn}\rangle_{CA}$, Bob applies the correction

$$R_{kn} = \sum_{j=0}^{d-1} \omega^{kj} |j\rangle \langle (j+n) \bmod d| \quad (25)$$

to recover $|\psi\rangle$ deterministically, provided the shared resource is maximally entangled.

For partial entanglement, the Schmidt decomposition of the shared resource state takes the form

$$|\varphi_{00}\rangle_{AB} = \sum_{j=0}^{d-1} a_j |jj\rangle_{AB}, \quad (26)$$

with $\sum_j a_j^2 = 1$ and $a_0 \geq a_1 \geq \dots \geq a_{d-1} > 0$. To measure the probability of teleportation, it is safe to use the POVM method as the robustness of this approach makes it easy to scale it to larger dimensions. As in the three-dimensional case, one identifies d orthogonal subspaces of the CA measurement space,

$$\Pi_n = \left\{ \sum_{j=0}^{d-1} a_j \omega^{kj} |j\rangle \otimes |(j+n) \bmod d\rangle \right\}_{k=0}^{d-1}, \quad (27)$$

each subspace is constructed by d non-orthogonal states $\{|v_k^{(n)}\rangle\}_{k=0}^{d-1}$. The POVM construction proceeds identically to the $d = 3$ case: one appends a $(d+1)$ -dimensional ancilla in $|S_0\rangle$, constructs the POVM elements $M_i = \lambda |m_i\rangle\langle m_i|$ with $\langle v_i|m_j\rangle = \delta_{ij}$, and maximizes λ subject to $M_d = \mathbb{I} - \sum_{i=0}^{d-1} M_i \geq 0$ that measures the inconclusive outcome.

Assembling the spanning states of Π_n as columns of the matrix V , one recognises the factorization $V = DO$, where

$$D = \text{diag}(a_0, a_1, \dots, a_{d-1}), \quad (28)$$

and O is the $d \times d$ a matrix with entries $O_{jk} = \omega^{jk}$:

$$O = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{d-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{d-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{d-1} & \omega^{d-2} & \dots & \omega \end{pmatrix}. \quad (29)$$

The inverse $V^{-1} = O^{-1}D^{-1} = O^\dagger D^{-1}/d$ is easily computed exploiting the unitarity of the O matrix. The resulting M_d operator is diagonal,

$$M_d = \text{diag} \left(1 - \frac{\lambda}{a_0^2 d}, 1 - \frac{\lambda}{a_1^2 d}, \dots, 1 - \frac{\lambda}{a_{d-1}^2 d} \right), \quad (30)$$

and the positivity constraint $M_d \geq 0$ yields $\lambda \leq a_{d-1}^2 d$. Setting $\lambda = a_{d-1}^2 d$ maximizes the success probability, giving

$$p_{\max}^{(d)} = a_{d-1}^2 d, \quad (31)$$

which is governed exclusively by the smallest Schmidt coefficient of the resource state. In the maximally entangled limit $a_j = 1/\sqrt{d}$ for all j , one recovers $p_{\max}^{(d)} = 1$ as required. The factorization $V = DO$ is what makes the construction tractable: D carries all the dependence on the Schmidt weights, while O is unitary up to the constant $\sqrt{d}$, so the dual vectors and the optimal λ are existing in any d -dimensional state.

This result establishes that, for any partially entangled resource in any finite dimension d , there exists a POVM-based protocol that heralds successful teleportation with probability $a_{d-1}^2 d$, after which Bob's subsystem is related to $|\psi\rangle$ by the known correction R_{kn} .

IV. CONCLUSIONS

We have presented a systematic treatment of quantum teleportation for three-level systems, extending the standard qubit formalism to the qutrit setting and beyond. Several conclusions can be drawn from this work.

First, deterministic teleportation of an arbitrary qutrit state is achievable whenever Alice and Bob share a maximally entangled three-dimensional pair. The protocol requires a Bell measurement over the nine-dimensional CA subspace, followed by one of nine local unitary corrections R_{kn} applied by Bob upon receiving Alice's classical message.

Second, when the shared resource is only partially entangled—parametrised by Schmidt coefficients $a \geq b \geq c > 0$ —exact recovery of the unknown state is no longer guaranteed in every run. We showed that this limitation can be overcome probabilistically through two complementary detection strategies: (i) a unitary interaction

at Bob's station, coupling a qubit ancilla and projecting onto a success subspace; and (ii) a positive operator-valued measure (POVM) at Alice's station that unambiguously discriminates among the non-orthogonal states spanning each Bell subspace. Both approaches converge on the same optimal heralded success probability

$$p_{\max} = 3c^2, \quad (32)$$

confirming the equivalence of the two strategies and showing that the bottleneck is entirely determined by the smallest coefficient of the resource.

Third, the POVM construction generalizes cleanly to arbitrary finite dimensions d . The key algebraic ingredient — the factorization $V = DO$ of the subspace — allows an analytic solution for the dual POVM vectors in all dimensions. The resulting maximum success probability,

$$p_{\max}^{(d)} = a_{d-1}^2 d, \quad (33)$$

is a dimension-dependent formula that reduces to the known qubit ($d = 2$) and qutrit ($d = 3$) results as special cases, and correctly recovers $p_{\max}^{(d)} = 1$ in the maximally entangled limit.

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Teleportació quàntica d'un estat desconegut via una parella entrellaçada de qutrits

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Resum: La teletransportació quàntica permet la transmissió perfecta d'un estat quàntic desconegut utilitzant un mitjà entrellaçat compartit i un canal clàssic. En aquest treball, estudiem la teletransportació de sistemes quàntics de tres nivells, coneguts com a qutrits, utilitzant parells tridimensionals entrellaçats. Demostrem la teleportació sota entrellaçament màxim i desenvolupem dues estratègies probabilístiques complementàries per anunciar una recuperació de l'estat desconegut quan el mitjà només està parcialment entrellaçat. Ambdós enfocaments donen la mateixa probabilitat màxima d'èxit, governada completament pel component més petit de la parella entrellaçada. Ampliem aquests resultats a sistemes quàntics de dimensió finita arbitrària, obtenint una expressió universal de forma tancada per a la màxima probabilitat d'èxit assolible.

Paraules clau: Teletransportació quàntica, entrellaçament parcial, correccions unitàries, estats de Bell, mesura projectiva

ODS: Aquest TFG està relacionat amb els Objectius de Desenvolupament Sostenible (SDGs)

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures	X		

El contingut d'aquest TFG, part d'un grau universitari de Física, es relaciona amb l'ODS 4, i en particular amb la fita 4.4, ja que contribueix a l'educació a nivell universitari. També es pot relacionar amb l'ODS 9, fita 9.5, perquè aporta a la investigació científica, i 9.8, perquè tracta sobre un protocol d'informació i comunicació.