DOCUMENTS DE TREBALL
DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E09/232

The induced 2-tuple linguistic generalized OWA operator and its application in linguistic decision making

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Abstract: We present the induced 2-tuple linguistic generalized ordered weighted averaging (2-TILGOWA) operator. This new aggregation operator extends previous approaches by using generalized means, order-inducing variables in the reordering of the arguments and linguistic information represented with the 2-tuple linguistic approach. Its main advantage is that it includes a wide range of linguistic aggregation operators. Thus, its analyses can be seen from different perspectives and we obtain a much more complete picture of the situation considered and are able to select the alternative that best fits with with our interests or beliefs. We further generalize the operator by using quasi-arithmetic means, and obtain the Quasi-2-TILOWA operator. We conclude this paper by analysing the applicability of this new approach in a decision-making problem concerning product management.

Keywords: 2-tuple linguistic aggregation operator; 2-tuple linguistic OWA operator; Linguistic generalized mean; Linguistic decision making.
JEL Classification: C44, C49, D81, D89.

Resumen: Se presenta el operador de media ponderada ordenada generalizada lingüística de 2 tuplas inducida (2-TILGOWA). Es un nuevo operador de agregación que extiende los anteriores modelos a través de utilizar medias generalizadas, variables de ordenación inducidas e información lingüística representada mediante el modelo de las 2 tuplas lingüísticas. Su principal ventaja se encuentra en la posibilidad de incluir a un gran número de operadores de agregación lingüísticos como casos particulares. Por eso, el análisis puede ser visto desde diferentes perspectivas de forma que se obtiene una visión más completa del problema considerado y seleccionar la alternativa que parece estar en mayor concordancia con nuestros intereses o creencias. A continuación se desarrolla una generalización mayor a través de utilizar medias cuasi-aritméticas, obteniéndose el operador Quasi-2-TILOWA. El trabajo finaliza analizando la aplicabilidad del nuevo modelo en un problema de toma de decisiones sobre gestión de la producción.

1. Introduction

The ordered weighted averaging (OWA) operator (Yager 1988) is a well-known aggregation operator for fusing numerical information and for decision making problems (Ahn and Park 2008; Alonso et al. 2008; Beliakov 2005; Beliakov et al. 2007; Canós y Liern 2008; Chiclana et al. 2007; Emrouznejad 2008; Liu 2008, 2009; Merigó and Gil-Lafuente 2008a, 2008b, 2008c; Wang 2008; Xu 2005, 2008a, 2008b; Yager 1993, 1996, 2007a, 2008; Yager and Kacprzyk 1997; and Zarghami et al. 2008). However, situations might arise in which the information available is vague or imprecise and we are unable to analyze it using numerical values. In such instances, we require an alternative method such as a qualitative approach based on linguistic assessments (Zadeh 1975). The literature describes various types of OWA operators using linguistic information (Bonissone 1982; Bustince et al. 2008; Herrera and Herrera-Viedma, 1997; Herrera et al. 1995, 2008; Herrera and Martínez 2000a, 2000b, 2001; Xu 2004, 2007, 2008; Yager 2007b; and Zadeh 1975). In this paper, we adopt the 2-tuple linguistic OWA (2-TLOWA) operator and its extensions (Herrera and Martínez 2000a; Wang and Hao 2006), and OWA operator based on the 2-tuple linguistic representation model introduced by Herrera and Martínez (2000a).

An interesting extension of this is the induced OWA operator (Yager and Filev 1999). This uses a more general formulation in the reordering process of the arguments by using order-inducing variables. Its application means we are able to deal with more complex situations that are not dependent on the values of the arguments, that is, on the degree of optimism. Since its introduction the IOWA operator has received considerable attention, see, for example, Chiclana et al. (2004, 2007); Merigó and Gil-Lafuente (2008b); and Yager (2003).

Further interesting extensions of the OWA operator are the generalizations that use generalized means (Dyckhoff and Pedrycz 1984) and quasi-arithmetic means, known respectively as the generalized OWA (GOWA) operator

(Karayiannis 2000; Yager 2004) and the Quasi-OWA operator (Beliakov 2005; Beliakov et al. 2007; Calvo et al. 2002; and Fodor et al. 1995). They generalize a wide range of aggregation operators such as the average, the OWA and the ordered weighted geometric (OWG) operator (Herrera et al. 2003).

Recently, Merigó and Gil-Lafuente (2008b) have suggested a generalization of the IOWA operator by using generalized means. This operator, known as the induced generalized OWA (IGOWA) operator, generalizes a wide range of aggregation operators such as the OWA and the IOWA operator. Note that a further generalization is possible by using quasi-arithmetic means (Quasi-IOWA operator).

Taking this generalization one step further, in this paper we present the induced 2-tuple linguistic generalized OWA (ILGOWA) operator. This represents an extension of the IGOWA operator for those cases in which the information available is assessed with linguistic variables in the form of the 2-tuple linguistic approach. In this way, we can generalize a wide range of 2-tuple linguistic aggregation operators including the 2-tuple induced linguistic OWA (2-TILOWA), the 2-TLOWA, the 2-tuple linguistic weighted average (2-TLWA), the 2-tuple linguistic generalized mean (2-TLGM), the 2-tuple linguistic weighted generalized mean (2-TLWGM) and the 2-tuple linguistic GOWA (2-TLGOWA), among others. The main advantage of this operator is that it includes a wide range of specific cases which enables us to consider many different situations and select the one that best fits with our interests.

We also present a further generalization of the 2-TILGOWA operator - the Quasi-2-TILOWA operator - by using quasi-arithmetic means. Note that while various approaches have been developed for dealing with linguistic information (Bonissone 1982; Herrera and Herrera-Viedma, 1997; Herrera et al. 1995, 2008; Herrera and Martínez 2000a, 2000b, 2001; Wang and Hao 2006; Xu 2004, 2007, 2008; and Zadeh 1975), in this paper we focus on the ideas of Herrera and Martínez (2000a, 2000b, 2001) and compute with words (CWW) directly. It should be noted, therefore, that this generalization can be seen as an initial step in the process of generalizing the 2-TLOWA operator with generalized means and quasi-arithmetic means, since further generalizations using other linguistic models are possible.

In our discussion of the applicability of the 2-TILGOWA operator, we are able to show that it is applicable in a wide range of situations. And we present a specific application of this new approach in a linguistic decision making problem concerning product management. We focus, in particular, on the selection of production strategies. The main conclusion we draw when using the 2-TILGOWA operator is that decisions can vary depending on the specific case used. Therefore, with the 2-TILGOWA operator, the decision maker obtains a more complete view of the problem and will select the decision that is in closest accordance with his interests.

The rest of this paper is organized as follows. Section 2 presents some basic concepts about the 2-tuple linguistic representation model, the 2-TLOWA operator and the IGOWA operator. In Section 3, we present the 2-TILGOWA operator and study some of its main properties. In Section 4 we analyze a wide range of families of 2-TILGOWA operators distinguishing between the weighting vector W and the parameter λ. Section 5 introduces the Quasi-ILOWA operator and Section 6 presents an application of the new approach in a linguistic decision making problem. Section 7 brings the paper to a close by summarizing its main conclusions.

2. Preliminaries

In this section, we briefly review the 2-tuple linguistic approach, the 2-tuple linguistic OWA and the induced generalized OWA operator.

2.1. The 2-tuple linguistic representation model

We are used to working in a quantitative setting, where the information is expressed by means of numerical values. However, many aspects of the real world cannot be assessed in a quantitative manner and we must work in a qualitative form, i.e., with vague or imprecise knowledge. In such instances, a better approach might be provided by the use of linguistic assessments rather than numerical values. The linguistic approach represents qualitative aspects as linguistic values by means of linguistic variables (Zadeh 1975).

In adopting this approach, we need to select the appropriate linguistic descriptors for the term set and its semantics. One way to generate the linguistic term set involves directly supplying the term set by considering all the terms distributed on a scale along which a total order is defined (Herrera and Herrera-Viedma 1997). For example, a set of seven terms S could be given as follows:

$$S = \{s_1 = N, s_2 = VL, s_3 = L, s_4 = M, s_5 = H, s_6 = VH, s_7 = P\}$$

where N = *None*, VL = *Very low*, L = *Low*, M = *Medium*, H = *High*, VH = *Very high*, P = *Perfect*. Typically, in such cases, the linguistic term set should have the following characteristics:

- A negation operator: $\text{neg}(s_i) = s_j$ such that $j = g+1-i$.
- Be ordered: $s_i \le s_j$ if and only if $i \le j$.
- Max operator: $\max(s_i, s_j) = s_i$ if $s_i \ge s_j$.
- Min operator: $\min(s_i, s_j) = s_i$ if $s_i \le s_j$.

Various approaches have been forwarded for dealing with linguistic information such as Bonissone (1982); Herrera and Herrera-Viedma (1997); Herrera et al. (1995, 2008); Herrera and Martínez (2000a, 2000b, 2001); Wang and Hao (2006); Xu (2004, 2007, 2008); Yager (2007b) and Zadeh (1975). In this paper, we adopt the approach suggested by Herrera and Martínez (2000a, 2000b, 2001). They developed a fuzzy linguistic representation model, which represents linguistic information by using a pair of values that they refer to as 2-tuple, (s, α), where s is a linguistic label and α is a numerical value representing the value of the symbolic translation. With this model, it is possible to undertake CWW processes without any loss of information, thereby overcoming one of the main limitations of earlier linguistic computational models (Bonissone 1982; Herrera et al. 1995 and Zadeh 1975).

Definition 1. Let β be the result of an aggregation of the indexes of a set of labels assessed in the linguistic label set $S = \{s_0, s_1, \ldots, s_g\}$, i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, where $g + 1$ is the cardinality of S. Let $i = \text{round}(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is known as the symbolic translation.

Note that the 2-tuple (s_i, α) that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5),$$

$$\Delta(\beta) = \begin{cases} s_i & i = round(\beta), \\ \alpha = \beta - i & \alpha \in [-0.5, 0.5). \end{cases} \quad (1)$$

where round is the usual round operation, s_i has the closest index label to β and α is the value of the symbolic translation. For further information on the 2-tuple linguistic representation model, see (Herrera and Martínez 2000a, 2000b, 2001).

2.2. The 2-tuple linguistic OWA operator

The 2-tuple linguistic OWA (2-TLOWA) operator is a linguistic aggregation operator that uses the 2-tuple linguistic representation model in the OWA operator. It can be defined as follows:

Definition 2. Let $\hat{S}$ be the set of the 2-tuples. A 2-TLOWA operator of dimension n is a mapping f: $\hat{S}^n \rightarrow \hat{S}$, which has an associated weighting vector W such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then:

$$f((s_1, \alpha_1), (s_2, \alpha_2), \ldots, (s_n, \alpha_n)) = \Delta(\sum_{j=1}^{n} w_j \beta_j^*) \tag{2}$$

where β_j^* *is the jth largest of the 2-tuples* (s_i, α_i).

Note that it is possible to distinguish between descending (2-TDLOWA) and ascending (2-TALOWA) orders. Note also that the weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the 2-TDLOWA (or 2-TLOWA) operator and w^*_{n+1-j} the jth weight of the 2-TALOWA operator. Following Herrera and Herrera-Viedma (1997), we can refer to the ascending order as the inverse 2-TLOWA operator.

By using a different weighting vector W, it is possible to study a wide range of families of 2-TLOWA operators including the olympic-2-TLOWA, the S-2-TOLWA, centered-2-TOLWA, etc. For further information, see, for example, Merigó and Gil-Lafuente 2008b; Xu 2005; or Yager 1993.

2.3. The induced generalized OWA operator

The induced generalized OWA (IGOWA) operator is an extension of the GOWA operator, with the difference that the reordering step of the IGOWA operator is not defined by the values of the arguments a_i, but rather by order inducing variables u_i, where the ordered position of the arguments a_i depends upon the values of the u_i. It can be defined as follows:

Definition 3. An IGOWA operator of dimension n is a mapping $IGOWA$: $R^n \rightarrow R$ defined by an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, a set of order-inducing variables u_i, and a parameter $\lambda \in (-\infty, \infty)$, according to the following formula:

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \quad (3)$$

where $(b_{1, \ldots,}\ b_n)$ is $(a_1,\ a_2, \ldots,\ a_n)$ reordered in decreasing order of the values of the u_i, the u_i are the order-inducing variables, and a_i are the argument variables.

Note that it is possible to generalize the IGOWA operator further by using quasi-arithmetic means. Then, we obtain the Quasi-IOWA operator, which can be defined as follows:

Definition 4. A Quasi-IOWA operator of dimension n is a mapping $QIOWA$: $R^n \rightarrow R$ defined by an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, and by a strictly monotonic continuous function $g(b)$, as follows:

$$QIOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(b_{(j)}\right) \right) \quad (4)$$

where the b_j are the argument values a_i of the Quasi-IOWA pairs $\langle u_i, a_i \rangle$ ordered in decreasing order of their u_i values.

As we can see, the difference between the IGOWA and the Quasi-IOWA, is that we replace b^λ with a general continuous strictly monotonic function $g(b)$.

3. The induced linguistic generalized OWA operator

The 2-TILGOWA operator is an extension of the OWA operator that uses linguistic assessments, generalized means and order-inducing variables in the reordering of arguments. By using linguistic information assessed by the 2-tuple linguistic representation model, we are able to represent uncertainty more completely without losing any information in the computing process. By using generalized means, we can generalize a wide range of mean operators including the arithmetic mean, the geometric mean and the quadratic mean. And by using order-inducing variables, we obtain a more general formulation of the reordering process that can deal with more complex situations that are not only dependent on the values of the arguments. The 2-TILGOWA operator provides a parameterized family of 2-tuple linguistic aggregation operators that includes the 2-TILOWA operator, the 2-TLOWA, the 2-tuple linguistic maximum, the 2-tuple linguistic minimum and the 2-tuple linguistic average (2-TLA), among others. It can be defined as follows:

Definition 5. Let $\hat{S}$ be the set of the 2-tuples. A 2-TILGOWA operator of dimension n is a mapping f: $\hat{S}^n \to \hat{S}$, which has an associated weighting vector W such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then:

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^{\lambda}\right)^{1/\lambda} \tag{5}$$

where β_j are the argument values (s_i, α_i) of the 2-TILGOWA triplets (u_i, s_i, α_i) ordered in decreasing order of their u_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Remark 1: Note that if $\lambda \leq 0$, we can only use positive numbers R^+, in order to obtain consistent results.

Remark 2: From a generalized perspective of the reordering step, it is possible to distinguish between the descending 2-TILGOWA (2-TDILGOWA) operator and the ascending 2-TILGOWA (2-TAILGOWA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the 2-TDILGOWA and w^*_{n-j+1} the jth weight of the 2-TAILGOWA operator.

Remark 3: If B is a vector corresponding to the ordered arguments $s_{\beta_j}{}^{\lambda}$, we shall call this the ordered argument vector and W^T is the transpose of the weighting vector, then, the 2-TILGOWA operator can be expressed as:

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \left(W^T B\right)^{1/\lambda} \tag{6}$$

Remark 4: Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the 2-TILGOWA operator can be expressed as:

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \left(\frac{1}{W} \sum_{j=1}^{n} w_j s_{\beta_j}^{\lambda} \right)^{1/\lambda} \tag{7}$$

The 2-TILGOWA operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. These properties can be demonstrated with the following theorems.

Theorem 1 (Commutativity). Assume f is the 2-TILGOWA operator, then

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = f((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n)) \tag{8}$$

where $((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n))$ is any permutation of the arguments $((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n))$.

Proof. Let

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^{\lambda} \right)^{1/\lambda} \tag{9}$$

$$f((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n)) = \Delta\left(\sum_{j=1}^{n} w_j X_j^{\lambda} \right)^{1/\lambda} \tag{10}$$

Since $((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n))$ is a permutation of $((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n))$, we have $\beta_j = X_j$, for all j, and then

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = f((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n)) \quad \square$$

Theorem 2 (Monotonicity). Assume f is the 2-TILGOWA operator, if $\beta_i \geq X_i$, for all i, then

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) \geq f((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n)) \qquad (11)$$

Proof. Let

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^{\lambda}\right)^{1/\lambda} \qquad (12)$$

$$f((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n)) = \left(\sum_{j=1}^{n} w_j s_{\chi_j}^{\lambda}\right)^{1/\lambda} \qquad (13)$$

Since $\beta_i \geq X_i$, for all i, it follows that, $\beta_j \geq X_j$, and then

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) \geq f((u_1, s'_1, \alpha'_1), \ldots, (u_n, s'_n, \alpha'_n)) \qquad \square$$

Theorem 3 (Bounded). Assume f is the 2-TILGOWA operator, then

$$\min\{(s_i, \alpha_i)\} \leq f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) \leq \max\{(s_i, \alpha_i)\} \qquad (14)$$

Proof. Let $\max\{(s_i, \alpha_i)\} = c$, and $\min\{(s_i, \alpha_i)\} = d$, then

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^{\lambda}\right)^{1/\lambda} \leq \left(\sum_{j=1}^{n} w_j c^{\lambda}\right)^{1/\lambda} =$$

$$\left(c^{\lambda} \sum_{j=1}^{n} w_j\right)^{1/\lambda} \qquad (15)$$

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^{\lambda}\right)^{1/\lambda} \geq \left(\sum_{j=1}^{n} w_j d^{\lambda}\right)^{1/\lambda} =$$

$$\left(d^{\lambda} \sum_{j=1}^{n} w_j\right)^{1/\lambda} \quad (16)$$

Since $\sum_{j=1}^{n} w_j = 1$, we obtain

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) \leq c \quad (17)$$

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) \geq d \quad (18)$$

Therefore,

$$\min\{(s_i, \alpha_i)\} \leq f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) \leq \max\{(s_i, \alpha_i)\} \quad \square$$

Theorem 4 (Idempotency). Assume f is the 2-TILGOWA operator, if $(s_i, \alpha_i) = (s_k, \alpha_k)$, for all i, then

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = (s_k, \alpha_k) \quad (19)$$

Proof. Since $s_{\alpha_i} = s_{\alpha}$, for all i, we have

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^{\lambda}\right)^{1/\lambda} = \Delta\left(\sum_{j=1}^{n} w_j \beta^{\lambda}\right)^{1/\lambda} =$$

$$\Delta\left(\beta^{\lambda} \sum_{j=1}^{n} w_j\right)^{1/\lambda} \quad (20)$$

Since $\sum_{j=1}^{n} w_j = 1$, we obtain

$$f((u_1, s_1, \alpha_1), \ldots, (u_n, s_n, \alpha_n)) = (s_k, \alpha_k) \qquad \square$$

Remark 5: Another interesting point to consider is the different measures available for characterizing the weighting vector. For example, we could consider the entropy of dispersion (Yager 1988), the divergence of W (Yager 2002) or the balance operator (Yager 1996). The entropy of dispersion is defined as follows:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \tag{21}$$

For the balance operator, we have:

$$BAL\,(W) = \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1}\right) w_j \tag{22}$$

And for the divergence of W:

$$DIV(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2 \tag{23}$$

Note that in this case, it is also possible to distinguish between descending and ascending orders.

Remark 6: An interesting point when analyzing induced linguistic aggregation operators is the problem of ties in the reordering step. To solve this problem, we recommend following the method developed by Yager and Filev (1999) whereby they replace each argument of the tied IOWA pair by its average. For the 2-TILGOWA operator, instead of using the arithmetic mean, we replace each argument of the tied 2-TILGOWA pair by its 2-TLGM depending on the parameter of λ.

Remark 7: As explained in Yager and Filev (1999) for the IOWA operator, we should note that the values used for the order-inducing variables of the IGOWA operator, can be drawn from any space that has a linear ordering. Thus, it is possible to use different kinds of attributes for the order-inducing variables; specifically, we can mix numbers with words in the aggregations (Yager and Filev 1999).

4. Families of 2-TILGOWA operators

In this section, we present a wide range of particular cases of 2-TILGOWA operators. We distinguish between the parameter λ and the weighting vector W.

4.1. Analysing the parameter λ

If we analyze different values of the parameter λ, we obtain another group of particular cases including the usual 2-TILOWA, the 2-TILOWG, the 2-TILOWHA and the 2-TILOWQA operator. Note that we can distinguish between descending and ascending orders in each of these cases.

Remark 8: When $\lambda = 1$, we obtain the 2-TILOWA operator.

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \sum_{j=1}^{n} w_j \beta_j \tag{24}$$

Note that if $w_j = 1/n$, for all i, we obtain the 2-TLA. If the ordered position of $u_i = i$, for all i, the 2-TLWA. And if $u_i = j$, for all i, then, we obtain the 2-TLOWA.

Remark 9: When $\lambda = 2$, we obtain the 2-TILOWQA operator.

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \left(\sum_{j=1}^{n} w_j \beta_j^2 \right)^{1/2} \tag{25}$$

If $w_j = 1/n$, for all i, we obtain the 2-tuple linguistic quadratic average (2-TLQA). If the ordered position of $u_i = i$, for all i, the 2-tuple linguistic weighted quadratic average (2-TLWQA). And if $u_i = j$, for all i, then, we obtain the 2-tuple linguistic ordered weighted quadratic averaging (2-TLOWQA) operator.

Remark 10: When $\lambda = 0$, we obtain the 2-TILOWG operator.

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \prod_{j=1}^{n} \beta_j^{w_j} \tag{26}$$

If $w_j = 1/n$, for all i, we obtain the 2-tuple linguistic geometric average (2-TLGA) and if the ordered position of $u_i = i$, for all i, the 2-tuple linguistic weighted geometric average (2-TLWGA). If $u_i = j$, for all i, then, we obtain the 2-tuple linguistic ordered weighted geometric averaging (2-TLOWGA) operator.

Remark 11: When $\lambda = -1$, we obtain the 2-TILOWHA operator.

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\beta_j}} \tag{27}$$

Note that if $w_j = 1/n$, for all i, we obtain the 2-tuple linguistic harmonic mean (2-TLHM) and if the ordered position of $u_i = i$, for all i, the 2-tuple linguistic weighted harmonic mean (2-TLWHM). If $u_i = j$, for all i, then, we obtain the 2-tuple linguistic ordered weighted harmonic averaging (2-TLOWHA) operator.

Note that we could analyze other families by using different values in the parameter λ. Note also that it is possible to study these families individually in a similar way to that reported in Sections 3 and 4.2.

4.2. Analysing the weighting vector W

By choosing a different manifestation of the weighting vector in the 2-TILGOWA operator, we are able to obtain different types of aggregation operators. For example, we can obtain the 2-tuple linguistic maximum, the 2-tuple linguistic minimum, the 2-tuple linguistic generalized mean (2-TLGM), the 2-TLWGM and the 2-TLGOWA operator.

Remark 12: The 2-tuple linguistic maximum is obtained if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. The 2-tuple linguistic minimum is obtained if $w_n = 1$ and $w_j = 0$, for all $j \neq n$. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we obtain the step-2-TILGOWA. The 2-TLGM is found when $w_j = 1/n$, for all i. The 2-TLWGM is obtained when the ordered position of i is the same as j. Finally, the 2-TLGOWA is found if the ordered position of u_i is the same as the ordered position of the values of the a_i.

Remark 13: The 2-tuple linguistic median can also be used as 2-TILGOWA operators. If n is odd we assign $w_{(n+1)/2} = 1$ and $w_{j^*} = 0$ for all others. If n is even then we assign, for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_{j^*} = 0$ for all others.

Remark 14: The olympic-2-TILGOWA is found when $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$. Note that if $n = 3$ or $n = 4$, the olympic-2-TILGOWA is transformed in the median-2-TILGOWA and if $m = n - 2$ and $k = 2$, the window-2-TILGOWA is transformed in the olympic-2-TILGOWA.

Remark 15: Following (Liu 2009), it is possible to develop a general form of the olympic-2-TILGOWA operator considering that $w_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$. Note that if $k = 1$, then, this general form becomes the usual olympic-2-TILGOWA. If $k = (n - 1)/2$, then, this general form becomes the median-2-TILGOWA aggregation.

Remark 16: Note that it is also possible to develop the contrary case of the general olympic-2-TILGOWA operator. In this case, $w_j = (1/2k)$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and $w_j = 0$, for all others, where $k < n/2$. Note that if $k = 1$, then, we obtain the contrary case of the median-2-TILGOWA.

Remark 17: A further type of aggregation that could be used is the E-Z 2-TILGOWA weights based on the E-Z IGOWA weights (Merigó and Gil-Lafuente, 2008b). In this case, we should distinguish between two classes. In the first class, we assign $w_{j*} = (1/q)$ for $j* = 1$ to q and $w_{j*} = 0$ for $j* > q$, and in the second class, we assign $w_{j*} = 0$ for $j* = 1$ to $n - q$ and $w_{j*} = (1/q)$ for $j* = n - q + 1$ to n.

Remark 18: The window-2-TILGOWA is found when $w_{j*} = 1/m$ for $k \leq j* \leq k + m - 1$ and $w_{j*} = 0$ for $j* > k + m$ and $j* < k$. Note that k and m must be positive integers such that $k + m - 1 \leq n$.

Remark 19: A further family of linguistic aggregation operator that could be used is the centered-2-TILGOWA operator, based on the OWA version (Yager, 2007a). We can define a 2-TILGOWA operator as a centered aggregation operator if it is symmetric, strongly decaying and inclusive.

- It is symmetric if $w_j = w_{j+n-1}$.
- It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$.
- It is inclusive if $w_j > 0$.

Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. (softly decaying centered-2-TILGOWA operator). Another particular situation of the centered-LGOWA operator appears if we remove the third condition (non-inclusive centered-2-TILGOWA operator).

Remark 20: Another interesting family is the S-2-TILGOWA operator based on the S-OWA operator (Yager 1993). It can be subdivided in three classes, the "orlike", the "andlike" and the generalized S-2-TILGOWA operator. The generalized S-2-TILGOWA operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-2-TILGOWA operator becomes the "andlike" S-2-TILGOWA operator and if $\beta = 0$, it becomes the "orlike" S-2-TILGOWA operator. Also note that if $\alpha + \beta = 1$, we obtain the 2-tuple induced linguistic generalized Hurwicz criteria.

Remark 21: Another interesting family is the nonmonotonic-2-TILGOWA operator that follows the ideas of (Yager, 1999). It is found when at least one of the weights w_j is lower than 0 and $\sum_{j=1}^{n} w_j = 1$. Note that a key aspect of this operator is that it does not always accomplish the monotonicity property. Therefore, this operator is not strictly a particular case of the 2-TILGOWA and can be seen as a different type of aggregation operator.

Remark 22: Using a similar methodology, many other families of 2-TILGOWA weights could be similarly developed as have been reported in many studies for the OWA operator, including Ahn and Park (2008); Beliakov (2005); Beliakov et al. (2007); Chiclana et al. (2007); Emrouznejad (2008); Liu (2008, 2009); Merigó and Gil-Lafuente (2008a, 2008b, 2008c); Xu (2005); and Yager (1993, 1996, 2007a).

Remark 23: Note that it is relatively straightforward to apply these methods to the 2-TILGOWA operator as the weights are not affected by the linguistic information. Obviously, more complex analyses might be undertaken in which the weights are also linguistic variables, but in this paper we do not tackle this problem.

5. Quasi-2-TILOWA operators

As explained in Beliakov (2005), a further generalization of the GOWA operator is possible using quasi-arithmetic means. Adopting the same methodology, we can suggest a similar generalization of the 2-TILGOWA operator by using these quasi-arithmetic means. We call this generalization the Quasi-2-TILOWA operator. Then, we obtain a more general formulation of the reordering process by using order inducing variables and this is able to deal with more complex situations. The Quasi-2-TILOWA operator can be defined as follows:

Definition 6. Let $\hat{S}$ be the set of the 2-tuples. A Quasi-2-TILOWA operator of dimension n is a mapping f: $\hat{S}^n \rightarrow \hat{S}$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(s_{\beta_j}\right)\right) \tag{28}$$

where s_{β_j} is the jth largest of the s_{α_i}.

As we can see, we replace $s_\beta{}^\lambda$ with a general continuous strictly monotone function $g(s_\beta)$. In this case, the weights of the ascending and descending versions are also related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the Quasi-2-TDILOWA and w^*_{n-j+1} the jth weight of the Quasi-2-TAILOWA operator.

Remark 24: As explained in the case of the 2-TILGOWA, if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the Quasi-2-TILOWA operator can be expressed as:

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = g^{-1}\left(\frac{1}{W}\sum_{j=1}^{n} w_j g\left(s_{\beta_j}\right)\right) \tag{29}$$

Remark 25: Note that all the properties and particular cases commented in the 2-TILGOWA operator are also included in this generalization. For example, we could study different families of Quasi-2-TILOWA operators such as the Quasi-2-TLA, the Quasi-2-TLWA, the Quasi-S-2-TILOWA, the Quasi-olympic-2-TILOWA, the Quasi-centered-2-TILOWA, etc.

Remark 26: Note also that the Quasi-2-TILOWA operator includes many other cases that are not included in the 2-TILGOWA such as the trigonometric 2-

TILOWA, the radical 2-TILOWA, the exponential 2-TILOWA, etc. These aggregations follow the same methodology as the OWA version (Beliakov et al. 2007) with the difference that now we are using linguistic information in the problem.

For the radical 2-TILOWA operator, we obtain:

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \left(\log_\gamma \left(\sum_{j=1}^{n} w_j \gamma^{1/s_{\beta_j}} \right) \right)^{-1} \quad (30)$$

For the trigonometric 2-TILOWA operator we form the following equations:

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \frac{2}{\pi} \arcsin \left(\sum_{j=1}^{n} w_j \sin \left(\frac{\pi}{2} s_{\beta_j} \right) \right) \quad (31)$$

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \frac{2}{\pi} \arccos \left(\sum_{j=1}^{n} w_j \cos \left(\frac{\pi}{2} s_{\beta_j} \right) \right) \quad (32)$$

$$f((u_1, s_1, \alpha_1), (u_2, s_2, \alpha_2), \ldots, (u_n, s_n, \alpha_n)) = \frac{2}{\pi} \arctan \left(\sum_{j=1}^{n} w_j \tan \left(\frac{\pi}{2} s_{\beta_j} \right) \right) \quad (33)$$

And for the exponential 2-TILOWA, we obtain: $\log_\gamma \left(\sum_{j=1}^{n} w_j \gamma^{s_{\beta_j}} \right)$, if $\gamma \neq 1$; and the 2-TILOWA if $\gamma = 1$.

6. Application in linguistic decision making

Many different applications are possible using the 2-TILGOWA operator. In principle, it can be applied to similar situations to those described when considering the OWA operator. Moreover, it has a range of other applications that can be developed in many different fields, including:

- Decision theory
- Statistics
- Economics
- Business decision making
- Mathematics
- Physics

Clearly, in each field, many different applications are also possible. For example, in business decision making we might consider financial problems (Merigó and Gil-Lafuente, 2007), human resource management, strategic management or product management, among others.

Below, we focus on an application of the 2-TILGOWA operator to a business decision-making problem. Specifically, we analyze a product management problem in which a company seeks to plan its production strategy for the forthcoming year. Let us assume they consider five alternatives:

- A_1 = Create a new product for high-income customers.
- A_2 = Create a new product for medium-income customers.
- A_3 = Create a new product for low-income customers.
- A_4 = Create a new product suitable for all customers.
- A_5 = Do not create a new product.

As the environment is highly uncertain, the enterprise's experts are unable to draw on numerical information in conducting their analysis. Rather, they have to rely on linguistic information assessed using the 2-tuple linguistic representation model. The results of these linguistic values are as follows. Note that in this example the experts use a set of seven terms *S* as follows:

$$S = \{s_1 = N, s_2 = VL, s_3 = L, s_4 = M, s_5 = H, s_6 = VH, s_7 = P\}$$

where *N* = *None*, *VL* = *Very low*, *L* = *Low*, *M* = *Medium*, *H* = *High*, *VH* = *Very high*, *P* = *Perfect*.

We next analyze the results obtained by using different types of 2-TILGOWA operators in order to see the range of different results in line with the attitude adopted by the company in the face of uncertainty. In this example, we consider the 2-tuple linguistic maximum, the 2-tuple linguistic minimum, the 2-TLA, the 2-TLGA, the 2-TLQA, the 2-TLWA, the 2-TLOWA operator, the 2-TILOWA, the 2-TILOWG, the 2-TILOWQA operator, the median-2-TILOWA and the olympic-2-TILOWA.

In evaluating these strategies, the experts consider the key factor as being the firm's economic situation over the forthcoming year. Following careful analysis, they consider five potential scenarios: S_1 = Very bad, S_2 = Bad, S_3 = Regular, S_4 = Good, S_5 = Very good. The expected linguistic results depending on situation N_i and alternative A_k are shown in Table 1. Note that the results are linguistic values represented with the 2-tuple linguistic approach.

Table 1: Linguistic payoff matrix

	N_1	N_2	N_3	N_4	N_5
A_1	$(s_4, 0.5)$	$(s_3, 0.1)$	$(s_3, 0)$	$(s_4, -0.4)$	$(s_6, -0.2)$
A_2	$(s_4, 0)$	$(s_5, 0.2)$	$(s_2, -0.3)$	$(s_4, 0.2)$	$(s_5, -0.1)$
A_3	$(s_2, -0.2)$	$(s_3, -0.3)$	$(s_3, 0.5)$	$(s_5, 0)$	$(s_6, -0.1)$
A_4	$(s_3, 0)$	$(s_5, -0.3)$	$(s_5, 0)$	$(s_4, -0.4)$	$(s_3, 0.2)$
A_5	$(s_5, 0)$	$(s_4, 0)$	$(s_4, -0.2)$	$(s_3, 0.5)$	$(s_2, 0.4)$

In this example, we assume that the experts assume the following weighting vector for all the cases: $W = (0.1, 0.2, 0.3, 0.4, 0.5)$. Note that this weighting vector is used as a weighted average in the 2-TLWA, while for the rest it is used to represent the attitudinal character of the enterprise. Note that the attitudinal character of the company is particularly complex since it involves the opinions of the different members sitting on the board of directors. Thus, the company's experts use order-inducing variables to represent the attitudinal character. The results are shown in Table 2.

Table 2: Order-inducing variables

	N_1	N_2	N_3	N_4	N_5
A_1	5	8	12	10	6
A_2	9	6	4	3	15
A_3	16	12	9	7	5
A_4	14	12	10	2	5
A_5	2	6	8	11	15

This information can then be aggregated in order to take a decision. The results are shown in Tables 3 and 4.

Table 3: Aggregate linguistic results 1

	Max	Min	LA	LGA	LQA	LWA
A_1	$(s_6, -0.2)$	$(s_3, 0)$	$(s_4, 0)$	$(s_4, -0.13)$	$(s_4, 0.13)$	$(s_4, 0.13)$
A_2	$(s_5, 0.2)$	$(s_2, -0.3)$	$(s_4, 0)$	$(s_4, -0.17)$	$(s_4, 0.18)$	$(s_4, 0.09)$
A_3	$(s_6, -0.1)$	$(s_2, -0.2)$	$(s_4, -0.22)$	$(s_3, 0.46)$	$(s_4, 0.06)$	$(s_4, 0.19)$
A_4	$(s_5, 0)$	$(s_3, 0)$	$(s_4, -0.1)$	$(s_4, -0.19)$	$(s_4, -0.02)$	$(s_4, -0.08)$
A_5	$(s_5, 0)$	$(s_2, 0.4)$	$(s_4, -0.26)$	$(s_4, -0.37)$	$(s_4, -0.17)$	$(s_3, 0.48)$

Table 4: Aggregate linguistic results 2

	LOWA	ILOWA	ILOWG	ILOWQA	Median	Olympic
A_1	$(s_4, -0.28)$	$(s_4, 0.15)$	$(s_4, 0.03)$	$(s_4, 0.26)$	$(s_3, 0.1)$	$(s_4, 0.16)$
A_2	$(s_4, -0.35)$	$(s_4, -0.07)$	$(s_4, -0.33)$	$(s_4, 0.11)$	$(s_5, 0.2)$	$(s_4, -0.37)$
A_3	$(s_4, -0.63)$	$(s_4, 0.19)$	$(s_4, -0.1)$	$(s_4, 0.43)$	$(s_3, 0.5)$	$(s_4, -0.27)$
A_4	$(s_4, -0.3)$	$(s_4, -0.04)$	$(s_4, -0.16)$	$(s_4, -0.01)$	$(s_5, 0)$	$(s_4, 0.3)$
A_5	$(s_3, 0.48)$	$(s_4, 0)$	$(s_4, -0.09)$	$(s_4, 0.07)$	$(s_4, -0.2)$	$(s_4, -0.24)$

As we see, different results are obtained depending on the linguistic aggregation operator used and, consequently, the decision maker can take different decisions. Note that more specific instances of the LGOWA operator could be considered in the analysis such as those described above in the previous sections.

A further interesting issue involves establishing an ordering for the production strategies. Note that this is particularly useful when we wish to consider more than one production strategy in the analysis. The results are shown in Table 5.

Table 5: Ordering of the production strategies

	Ordering		Ordering
Max	$A_3 \succ A_1 \succ A_2 \succ A_4 = A_5$	2-TLOWA	$A_1 \succ A_4 \succ A_2 \succ A_5 \succ A_3$
Min	$A_1 = A_4 \succ A_5 \succ A_3 \succ A_2$	2-TILOWA	$A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_2$
2-TLA	$A_1 = A_2 \succ A_4 \succ A_3 \succ A_5$	2-TILOWG	$A_1 \succ A_5 \succ A_3 \succ A_4 \succ A_2$
2-TLGA	$A_1 \succ A_4 \succ A_2 \succ A_5 \succ A_3$	2-TILOWQA	$A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$
2-TLQA	$A_2 \succ A_1 \succ A_3 \succ A_4 \succ A_5$	Median	$A_2 \succ A_4 \succ A_5 \succ A_3 \succ A_1$
2-TLWA	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$	Olympic	$A_4 \succ A_1 \succ A_5 \succ A_3 \succ A_2$

As we can see, depending on the linguistic aggregation operator used, the ordering of the production strategies differs. Thus, the decision maker can then consider a wide range of scenarios and select the specific case that best fits with his interests.

7. Conclusions

In this article we have presented the induced 2-tuple linguistic generalized OWA operator, an aggregation operator that uses generalized means, order-inducing variables in its reordering of arguments and uncertain information assessed with the 2-tuple linguistic representation model. We have demonstrated that this operator can be of great use because it generalizes a wide range of linguistic aggregation operators including the 2-TLA, the 2-TLWA, the 2-TLOWA, the 2-

TILOWA, the 2-TLWGM and the 2-TLGOWA, among others. The main advantage of this operator is that it makes it possible to consider a wide range of results depending on the particular type of 2-TILGOWA operator being used. In this way, the same problem can be viewed from a range of perspectives and the solution that best fits our interests can be selected.

In an additional step, we have further generalized the 2-TILGOWA operator using quasi-arithmetic means. This we have called the Quasi-2-TILOWA operator. In this case, the main advantage is that the more general formulation provided allows us to consider many situations that are not included in the 2-TILGOWA.

We have also discussed here the applicability of this new approach. We have demonstrated that it is applicable in a wide range of fields including decision theory, statistics, economics, etc. We have presented an application in a decision making problem concerning product management. The main advantage of using the 2-TILGOWA operator is that the decision maker obtains a more complete view of the problem because he is able to consider a wide range of situations and select the one that best fits with his interests.

In our future research, we wish to extend this approach to other situations that can be assessed by applying linguistic approaches. Our primary motivation is that we believe the computing with words process of the 2-tuple linguistic approach needs to be improved so as to make it more efficient. Moreover, we also wish to examine other decision-making applications in fields such as financial management and human resource selection.

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