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Riesz bases of exponentials for finite unions of intervals

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Introduction

The origins of modern harmonic analysis date back to 1822, when the French mathematician Joseph B. Fourier published *Théorie analytique de la chaleur*. In this work, Fourier studied the propagation of the heat using partial differential equations. However, his truly revolutionary idea was the assertion that any arbitrary function could be represented as an infinite linear combination of sines and cosines [12]. In modern terminology, this statement implies that any function in $L^2[-\pi, \pi]$ can be expressed as a Fourier series of complex exponentials of the form $\{e^{int}\}_{n \in \mathbb{Z}}$ with integer frequencies.

Fourier series have been proved to be of great utility in areas as diverse as signal processing, acoustics, circuit designs, crystallography or telecommunication systems [13].

Nevertheless, restricting the frequencies to the integers presents an important limitation when analyzing non-periodic functions or irregular domains. Efforts to relax this constraint led to the development of non-harmonic Fourier analysis in the early 30s. The foundational work in this field was due to Raymond Paley and Norbert Wiener, who investigated the possibility of representing functions using perturbations of the sequence $\{e^{int}\}_{n \in \mathbb{Z}}$ obtained by replacing integer frequencies with sequences of real numbers [14]. This concept of deforming an orthonormal basis was later generalized by Marcel Riesz, who introduced what is currently known as Riesz bases.

Over the following years, investigation in non-harmonic analysis focused on extending the notion of Riesz bases of exponentials to functions defined on different domains. Specifically, due to its importance in sampling band-limited signals [6], research turned to the existence of Riesz bases of exponentials over finite union of intervals. The main result in this direction was due to Gady Kozma and Shahaf Nitzan in their paper *Combining Riesz bases* [8], published in 2014. The objective of this thesis is to provide a comprehensive analysis of their work.

Abstract

This thesis presents a proof of the existence of exponential Riesz bases for the space of square-integrable functions defined on a finite union of intervals. For this purpose, we begin with the study of basis theory in Hilbert spaces. In this context, we define Riesz bases, which can be understood as controlled deformations of orthonormal bases.

Subsequently, we particularize the results obtained for Riesz bases to the space L^2 . Our analysis focuses on the strong connection between square-integrable functions defined on a bounded interval and entire functions of exponential type. This relation is established by the Fourier transform. In this part, we follow the work of Paley and Wiener.

Finally, we introduce the necessary framework to combine Riesz bases, which constitutes the essential tool to prove our main result. This thesis concludes with the existence proof originally developed by Gady Kozma and Nitzan Shahaf.

Chapter 1

Introduction to Riesz bases

In this first chapter we aim to introduce the notion of Riesz bases in an infinite-dimensional Hilbert space. To this end, the first two sections focus on Schauder and orthonormal bases, which provide the natural background for our study. The remaining sections explore the essential properties of Riesz bases and the concept of stability.

1.1 Schauder bases

Throughout this chapter, let H denote an infinite-dimensional Hilbert space over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers. It is well-known that a basis for H seeks to provide a unique representation for each vector of the space. Therefore, one might consider defining a basis for H as a set of linearly independent vectors whose finite linear combinations generate H . Such sets are known as Hamel bases. However, Hamel bases present a significant limitation: since their existence relies on Zorn's Lemma - which is equivalent to the Axiom of choice - they are, in general, non-constructive. (See [1]). For this reason, Schauder bases are of far greater applicability, as they incorporate a useful notion of convergence for infinite series.

Definition 1.1. (Schauder bases) *A sequence $\{\phi_k\}_{k=1}^{\infty} \subset H$ is a Schauder basis for H if for each vector $x \in H$ there exists a unique sequence of scalars $\{c_k\}_{k=1}^{\infty}$ such that*

$$\|x - \sum_{k=1}^n c_k \phi_k\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

In this case, we write

$$x = \sum_{k=1}^{\infty} c_k \phi_k$$

Henceforth, the term basis will always refer to Schauder basis.

It is important to note that not all sequences of H that generate the entire space are bases. This distinction lies in the uniqueness. Sequences that generate H but do not necessarily ensure a unique representation of each vector of the space are said to be complete sequences:

Definition 1.2. Let $\{\phi_k\}_{k=1}^{\infty}$ be a sequence of H and denote by $\text{span}\{\phi_k\}_{k=1}^{\infty}$ the set of all finite linear combinations of $\{\phi_k\}_{k=1}^{\infty}$ with coefficients in $\mathbb{K}$. A sequence of vectors $\{\phi_k\}_{k=1}^{\infty} \subset H$ is said to be complete if its span is dense in H , i.e. if

$$H = \overline{\text{span}\{\phi_k\}_{k=1}^{\infty}}$$

In Hilbert spaces, since the closure of the span of $\{\phi_k\}_{k=1}^{\infty}$ is a closed subspace of H , we can express H as the sum of $\overline{\text{span}\{\phi_k\}_{k=1}^{\infty}}$ and its orthogonal complement (cf. [2]). Therefore, $\{\phi_k\}_{k=1}^{\infty}$ is complete if and only if the zero vector is the only element in H orthogonal to every element of $\{\phi_k\}_{k=1}^{\infty}$.

Example 1.3. Let $\phi_n = \{\delta_{n,i}\}_{i=1}^{\infty}$ be the sequence whose n 'th term is 1 and all other terms are zero. The set $\{\phi_n\}_{n \in \mathbb{N}}$ is a Schauder basis for the Hilbert space of square-summable sequences, ℓ^2 . However, if we introduce an additional nonzero vector to $\{\phi_n\}_{n \in \mathbb{N}}$, the resulting set remains a complete sequence but is no longer a basis.

1.2 Orthonormal bases

One of the most useful notions for representing elements in a Hilbert space is that of orthogonality. A basis whose vectors are mutually orthogonal and, in addition, of unit norm, is said to be an orthonormal basis. Such bases will play a central role in the study of Riesz bases. Consequently, we shall recall several of their properties. See [3] for further details.

Proposition 1.4. Let H be a Hilbert space and denote by $\langle \cdot, \cdot \rangle$ its scalar product. The following statements are equivalent:

1. The sequence $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal basis for H .
2. The sequence $\{\phi_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence in H .
3. The sequence $\{\phi_k\}_{k=1}^{\infty}$ is orthonormal and, for every element $x \in H$, Parseval's identity holds, i.e.

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, \phi_k \rangle|^2 \quad \forall x \in H \quad (1.1)$$

Proof. Statements (1) and (2) are trivially equivalent. For the proof of statement (3), consult [3]. $\square$

Proposition 1.4 implies that an incomplete orthonormal sequence $\{\phi_k\}_{k=1}^{\infty}$ does not verify Parseval's identity for all $x \in H$. Nevertheless, Bessel's inequality (Equation 1.2) is still valid:

Proposition 1.5. *Let $\{\phi_k\}_{k=1}^{\infty} \subset H$ be an orthonormal sequence, not necessarily complete. Then Bessel's inequality holds:*

$$\sum_{i=1}^{\infty} |\langle x, \phi_k \rangle|^2 \leq \|x\|^2 \quad \forall x \in H \quad (1.2)$$

Proof. See [3]. $\square$

The fundamental consequence of defining bases with the property of orthogonality is the simplicity of representation of any element in H :

Let $x \in H$, if $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal basis for H , the vector x can be expressed directly as

$$x = \sum_{k=1}^{\infty} \langle x, \phi_k \rangle \phi_k \quad (1.3)$$

This expansion, known as the Fourier series of x , identifies its coefficients with the projections of x onto the orthogonal directions defined by the basis. Consequently, Parseval's identity (Equation 1.1) acts as a generalized form of the Pythagorean theorem, where the squared norm of x is decomposed into the sum of the squares of the norms of its orthogonal projections.

Furthermore, the Fourier expansion reveals a deep connection between the Hilbert space H and the space of square-summable sequences ℓ^2 . Specifically, Bessel's inequality (Equation 1.2) ensures that the Fourier coefficients of each element in H form a sequence in ℓ^2 , while the Riesz-Fischer Theorem (cf. [3]) guarantees that every sequence in ℓ^2 defines a unique element in H . Therefore, we conclude that all Hilbert spaces with an orthonormal basis are isomorphic to ℓ^2 via the mapping $h : H \rightarrow \ell^2$ given by $h(x) = \{\langle x, \phi_k \rangle\}_{k=1}^\infty$.

Example 1.6. For any interval $I \subset \mathbb{R}$ of length $L \in \mathbb{R}$, the family $\{\frac{1}{\sqrt{L}}e^{2\pi i n t/L}\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(I)$ with the inner product

$$\langle f, g \rangle = \int_I f(x) \overline{g(x)} dx$$

for any $f, g \in L^2(I)$. In particular, the case $L = 1$ provides the basis $\{e^{2\pi i n t}\}_{n \in \mathbb{Z}}$ for $L^2[0, 1]$.

1.3 Riesz bases

Let $S \subset \mathbb{R}$ be a finite union of intervals. We are interested in replicating the exponential bases of Example 1.6 for the space $L^2(S)$. To this end, for a given set $\Lambda \subset \mathbb{R}$, we define the system of exponential functions $E(\Lambda)$ as

$$E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda} \quad \text{where} \quad e_\lambda(t) = e^{2\pi i \lambda t}$$

Initially, one might seek an orthonormal basis of the form $E(\Lambda)$ for $L^2(S)$. However, requiring orthogonality is too restrictive. Indeed, the orthogonality of a set $E(\Lambda)$ requires that the Fourier transform of the indicator function χ_S satisfies

$$\widehat{\chi_S}(2\pi(\mu - \lambda)) = \int_{-\infty}^{\infty} \chi_S(t) e^{-2\pi i(\mu - \lambda)t} dt = \langle e_\lambda, e_\mu \rangle = 0$$

for any distinct $\lambda, \mu \in \Lambda$. Besides, a necessary condition for $E(\Lambda)$ to be an orthonormal basis is that Λ be periodic [4]. These two conditions together imply that even the union of two intervals may not admit an orthonormal basis of exponentials. Such is the case for the set $S = [0, 1] \cup [2, 2 + \sqrt{3}]$.

Consequently, it is necessary to weaken the restriction of orthogonality. This motivates the definition of Riesz bases, which can be understood as a controlled deformation of an orthonormal basis.

1.3.1 Definition and properties of Riesz bases

Definition 1.7. Let H be an infinite-dimensional Hilbert space. A sequence $\{f_k\}_{k=1}^{\infty} \subset H$ is a Riesz basis if it is the image of an orthonormal basis under a bounded invertible linear operator from H to H .

Although a Riesz basis is not necessarily orthogonal, it does not differ significantly from an orthonormal one. This is evident in Theorem 1.8, where the boundedness of the operator associated with the Riesz basis guarantees that the quantities in the Parseval's identity (Equation 1.1) are still comparable.

Theorem 1.8. Let H be an infinite-dimensional Hilbert space. The following statements are equivalent:

1. The sequence $\{f_k\}_{k=1}^{\infty}$ forms a Riesz basis for H
2. The sequence $\{f_k\}_{k=1}^{\infty}$ is complete in H and there exist positive constants A and B such that

$$A \sum_{i=1}^n |c_k|^2 \leq \left\| \sum_{k=1}^n c_k f_k \right\|^2 \leq B \sum_{k=1}^n |c_k|^2 \quad (1.4)$$

for an arbitrary positive integer n and arbitrary scalars $c_1, \dots, c_n \in \mathbb{K}$.

3. The sequence $\{f_k\}_{k=1}^{\infty}$ is minimal (i.e. no vector from the system lies in the closed span of the rest) and there exist positive constants c and C such that

$$c \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq C \|f\|^2 \quad (1.5)$$

for every $f \in H$.

Proof. See [5]

□

A sequence $\{f_k\}_{k=1}^{\infty} \subset H$ which satisfies Equation 1.4 but is not necessarily complete is said to be a Riesz sequence. Such sequences preserve the notion of linear independence, as can be easily seen from the left-hand side inequality of Equation 1.4, but they may not generate the entire space. Similarly, a sequence $\{f_k\}_{k=1}^{\infty} \subset H$ which satisfies Equation 1.5 but is not necessarily minimal is called a frame. As we will see in Lemma 1.9, a frame generates the entire space, although

each element of the space may not have a unique representation.

It is clear then that both types of sequences lack an important property to be a basis. However, by combining both sequences, we obtain the following useful Lemma:

Lemma 1.9. *A system of vectors $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space is a Riesz basis if and only if it is both a Riesz sequence and a frame.*

Proof. The necessary condition for being a Riesz basis is trivial from Theorem 1.8. Conversely, assume that $\{f_k\}_{k=1}^{\infty}$ is both a Riesz sequence and a frame. It is enough to show that the sequence is complete or, equivalently, that any element $f \in H$ satisfying $\langle f, f_k \rangle = 0$ for all $k \geq 1$ must be the zero vector.

Since $\{f_k\}_{k=1}^{\infty}$ is a frame, the left-hand side inequality in Equation 1.5 holds. Thus, if $\langle f, f_k \rangle = 0$ for all $k \geq 1$, then $\|f\| = 0$. It follows immediately that $f = 0$. $\square$

1.3.2 Series expansions in Riesz bases

Having established the structure of a Riesz basis, we now focus on the representation of elements of H with such basis. For this purpose, we define biorthogonal sequences as follows:

Definition 1.10. *Let δ_{kn} be the Kronecker's delta. Two sequences $\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty} \subset H$ are said to be biorthogonal if*

$$\langle f_k, g_n \rangle = \delta_{kn}$$

for all natural number $k, n \geq 1$.

Geometrically, obtaining the series expansion of an element $f \in H$ with respect to a Riesz basis $\{f_k\}_{k=1}^{\infty}$ consists of projecting the vector f onto the directions determined by the biorthogonal sequence. Theorem 1.11 gives the explicit expansion of an element of a Hilbert space using Riesz basis:

Theorem 1.11. *If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for H , there exists a unique sequence $\{g_k\}_{k=1}^{\infty}$ in H such that*

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k \quad \forall f \in H \quad (1.6)$$

Besides, $\{g_k\}_{k=1}^\infty$ is also a Riesz basis, $\{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty$ are biorthogonal, the series 1.6 converges for all $f \in H$ and, by duality,

$$f = \sum_{k=1}^{\infty} \langle f, f_k \rangle g_k \quad \forall f \in H$$

Proof. Cf. [6] □

It is worth noting that the proof of Theorem 1.11 provides an explicit expression for $\{g_k\}_{k=1}^\infty$. Specifically, let T be a bounded invertible linear operator and let $\{\phi_k\}_{k=1}^\infty$ be an orthonormal basis. If $\{f_k\}_{k=1}^\infty$ is a Riesz basis such that $f_k = T\phi_k$ for all $k \geq 1$, then $g_k = (T^{-1})^*\phi_k$. This relation ensures that the sequence $\{g_k\}_{k=1}^\infty$ is uniformly bounded. Indeed, for each $k \geq 1$, we have

$$\|g_k\| = \|(T^{-1})^*\phi_k\| \leq \|(T^{-1})^*\| \|\phi_k\|$$

Since $\{\phi_k\}_{k=1}^\infty$ is orthonormal and the adjoint of the inverse of a bounded operator is also bounded, the assertion follows.

1.3.3 Stability of bases

We end this chapter by analyzing quantitatively the stability of basis, i.e, how much an orthonormal basis can be perturbed while still remaining a Riesz basis. The fundamental criterion for this study is due to Paley and Wiener.

Theorem 1.12. (Paley-Wiener stability criterion) *Let $\{f_k\}_{k=1}^\infty$ be a Riesz basis for a Banach space X , and suppose that, for all $n \in \mathbb{N}$, the sequence $\{h_k\}_{k=1}^\infty \subset X$ satisfies*

$$\left\| \sum_{k=1}^n c_k (f_k - h_k) \right\| \leq \lambda \left\| \sum_{k=1}^n c_k f_k \right\| \quad (1.7)$$

for some constant λ such that $0 \leq \lambda < 1$ and all choices of the scalars $c_1, \dots, c_n$. Then $\{h_k\}_{k=1}^\infty$ is also a Riesz basis.

Proof. We follow the proof presented in [5], which relies on the fact that a bounded linear operator U on a Banach space is invertible provided that

$$\|I - U\| < 1$$

Let us define a linear operator $T : X \rightarrow X$ by setting

$$T\left(\sum_{k=1}^{\infty} c_k f_k\right) = \sum_{k=1}^{\infty} c_k (f_k - h_k)$$

Note that the operator T is well-defined, as the convergence of $\sum_{k=1}^{\infty} c_k f_k$ implies the convergence of $\sum_{k=1}^{\infty} c_k (f_k - h_k)$ by assumption. Furthermore, the continuity of the norm and the Inequality 1.7 guarantee that T is bounded and satisfies $\|T\| \leq \lambda < 1$. It follows that $I - T$ is invertible. Since the relation $(I - T)f_k = h_k$ holds for all $k \geq 1$, the assertion follows. □

In Hilbert spaces, we can rewrite Theorem 1.12 as follows:

Corollary 1.13. *If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis in a Hilbert space H , then there exists some constant K such that every sequence $\{h_k\}_{k=1}^{\infty} \subset H$ which satisfies*

$$\sum_{k=1}^{\infty} |\langle f, f_k - h_k \rangle|^2 \leq K \|f\|^2 \quad \forall f \in H \quad (1.8)$$

is also a Riesz basis in H .

Proof. We shall prove that the Inequality 1.8 implies the hypothesis of the Paley-Wiener criterion (Theorem 1.12).

To this end, let us define a mapping $T : H \rightarrow \ell^2$ by setting

$$T(f) = \{\langle f, f_k - h_k \rangle\}_{k=1}^{\infty}$$

Evidently, the linearity of the inner product ensures that T is linear. Moreover, Equation 1.8 can be rewritten as

$$\|T(f)\|^2 \leq K \|f\|^2,$$

which implies that T is a bounded operator satisfying

$$\|T\| \leq \sqrt{K}$$

The adjoint operator $T^* : \ell^2 \rightarrow H$ takes the form

$$T^*(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k (f_k - h_k)$$

To justify this expression, consider an arbitrary $\{c_k\}_{k=1}^\infty \in \ell^2$ and an element $g \in H$. Then

$$\begin{aligned} \langle T^*(\{c_k\}), g \rangle_H &= \left\langle \sum_{k=1}^\infty c_k(f_k - h_k), g \right\rangle_H = \sum_{k=1}^\infty c_k \langle f_k - h_k, g \rangle_H = \\ &= \sum_{k=1}^\infty c_k \overline{\langle g, f_k - h_k \rangle_H} = \langle \{c_k\}, \{\langle g, f_k - h_k \rangle_H\} \rangle_{\ell^2} = \langle \{c_k\}, T(g) \rangle_{\ell^2} \end{aligned}$$

Since $\|T\| = \|T^*\|$, the adjoint operator T^* is also bounded by $\sqrt{K}$. Therefore, the Inequality 1.7 follows directly from the boundedness of T^* combined with the fact that $\{f_k\}_{k=1}^\infty$ is a Riesz basis. Indeed, for any $n \in \mathbb{N}$ and all choices scalars, we have

$$\left\| \sum_{k=1}^n c_k(f_k - g_k) \right\| \leq \sqrt{K} \sum_{k=1}^n |c_k|^2 \leq \sqrt{\frac{K}{A}} \left\| \sum_{k=1}^n c_k f_k \right\|$$

where A is the bound on the left-hand side of Inequality 1.4

Finally, assuming $K \leq A$ guarantees that $0 \leq \lambda = \sqrt{\frac{K}{A}} < 1$, as desired. $\square$

The specific application of Corollary 1.13 to exponential basis will be explored in the next chapter. However, we need to introduce first the Paley-Wiener space.

Chapter 2

The Paley-Wiener space

Let A be a positive real number. This chapter begins by revisiting the Fourier transform and the concept of entire function of exponential type. Both concepts will be essential for exploring the profound relation between the Hilbert space $L^2([-A, A])$ and the Paley-Wiener space in the third section. This connection allows the study of functions in $L^2([-A, A])$ in terms of entire functions. Furthermore, it simplifies the analysis of the space $L^2(S)$ for any bounded set $S \subset \mathbb{R}$, which is the topic of the fourth section. We conclude this chapter by reformulating the Paley-Wiener criterion (Theorem 1.12) for exponential bases in $L^2(S)$.

2.1 The Fourier transform

Consider the space L^2 with the inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx.$$

Our characterization of the space $L^2(S)$ for a bounded subset $S \subset \mathbb{R}$ is based on the Fourier transform. As presented in Definition 2.1, this transformation is initially introduced for the space $L^1(\mathbb{R})$ in order to guarantee the convergence of the integral.

Definition 2.1. Let $f \in L^1(\mathbb{R})$. The Fourier transform of f , denoted by $\hat{f}$, is defined by the integral

$$\hat{f}(t) := \int_{-\infty}^{\infty} f(x) e^{-ixt} dx$$

Similarly, if both $f, \hat{f} \in L^1(\mathbb{R})$, then the inverse Fourier transform is defined as follows:

Theorem 2.2. (The Inversion Theorem) *If $f, \hat{f} \in L^1(\mathbb{R})$ and if*

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{ixt} dt$$

then g is continuous, tends to 0 when $|x| \rightarrow \infty$ and $f(x) = g(x)$ a.e.

Proof. See [2] □

However, the integral in Definition 2.1 does not converge for every function $f \in L^2(\mathbb{R})$. Therefore, given that $L^1(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the Fourier transform is extended to $L^2(\mathbb{R})$ by continuity. This result is known as the Plancherel theorem.

Theorem 2.3. (Plancherel Theorem) *One can associate to each $f \in L^2(\mathbb{R})$ a function $\hat{f} \in L^2(\mathbb{R})$ so that the following properties hold:*

1. *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $\hat{f}$ is the Fourier transform defined in Definition 2.1.*
2. *For every $f \in L^2(\mathbb{R})$, $\|\hat{f}\|_{L^2}^2 = 2\pi \|f\|_{L^2}^2$.*
3. *The mapping $f \rightarrow \hat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.*
4. *The following relation exists between f and $\hat{f}$: If*

$$g_A(t) = \int_{-A}^A f(x) e^{-ixt} dx \quad \text{and} \quad h_A(x) = \frac{1}{2\pi} \int_{-A}^A \hat{f}(t) e^{ixt} dt$$

then $\|g_A - \hat{f}\|_{L^2} \rightarrow 0$ and $\|h_A - f\|_{L^2} \rightarrow 0$ when $A \rightarrow \infty$.

Proof. Cf. [2] □

All concepts in this section has been covered in Anàlisi Harmònica i Teoria del Senyal.

2.2 Entire functions of exponential type

We will show in Section 3 that the inverse Fourier transform of a function $f \in L^2(S)$ can be associated with an entire function of exponential type. Thus, let us recall several properties of such functions.

Definition 2.4. *An entire function f is said to be of exponential type if there exist positive constants B and C such that*

$$|f(z)| \leq C e^{B|z|} \quad \forall z \in \mathbb{C}$$

In this case, we say that f is of exponential type τ if τ is the smallest nonnegative number such that for every $\epsilon > 0$, there exists a positive constant C_ϵ such that

$$|f(z)| \leq C_\epsilon e^{(\tau+\epsilon)|z|}$$

Notice from Definition 2.4 that the concept of exponential type ensures that f does not globally increase faster than some exponential, whereas the specific value τ captures the exact asymptotic growth of f .

If f is square integrable over the real line, it is also square integrable over any line parallel to the real axis. This is the Plancherel-Pólya Theorem.

Theorem 2.5. (Plancherel-Pólya) *If $f(z)$ is an entire function of exponential type τ and if*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

then

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 dx \leq e^{2\tau|y|} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Proof. Cf. [5] □

Finally, an important corollary to Theorem 2.5 is the following:

Corollary 2.6. *Let $f(z)$ be an entire function of exponential type τ and suppose that*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

If $\{\lambda_k\}_{k=1}^{\infty}$ is an increasing sequence of real numbers such that

$$\lambda_{k+1} - \lambda_k \geq \epsilon > 0 \quad \forall k > 0$$

then

$$\sum_{k=1}^{\infty} |f(\lambda_k)|^2 \leq B \int_{-\infty}^{\infty} |f(x)|^2 dx$$

where B is a constant that depends only on τ and ϵ .

Proof. Since f is an entire function, $|f|^2$ is subharmonic (Cf. [2]). Therefore, for any $a \in \mathbb{C}$ and $r \in \mathbb{R}^+$, $|f|^2$ satisfies

$$|f(a)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})|^2 dt$$

Consider a strictly increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that $\lambda_{k+1} - \lambda_k \geq \epsilon$ for all $k \geq 1$. For any radius $r \in \mathbb{R}^+$, applying the previous inequality to each element of $\{\lambda_k\}_{k=1}^{\infty}$ and summing over k yields

$$\sum_{k \geq 1} |f(\lambda_k)|^2 \leq \frac{1}{2\pi} \sum_{k \geq 1} \int_0^{2\pi} |f(\lambda_k + re^{it})|^2 dt$$

Notice that by setting $R = \epsilon/2$, the corresponding discs $B(\lambda_k, R)$ do not overlap. Therefore, multiplying both sides by r and integrating with respect to r from 0 to R , we obtain

$$\sum_{k \geq 1} |f(\lambda_k)|^2 \leq \frac{1}{\pi R^2} \iint_{\Omega} |f(x + iy)|^2 dx dy$$

where $\Omega = \bigcup_{k \geq 1} B(\lambda_k, R)$.

Since $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$, the set Ω is contained in the horizontal strip $\mathbb{R} \times [-R, R]$. It follows immediately that

$$\sum_{k \geq 1} |f(\lambda_k)|^2 \leq \frac{1}{\pi R^2} \int_{-R}^R \int_{-\infty}^{\infty} |f(x + iy)|^2 dx dy$$

Finally, by Theorem 2.5

$$\sum_{k \geq 1} |f(\lambda_k)|^2 \leq \frac{1}{\pi R^2} \int_{-R}^R e^{2\tau|y|} \int_{-\infty}^{\infty} |f(x)|^2 dx dy = B \int_{-\infty}^{\infty} |f(x)|^2 dx$$

where B is the constant defined by

$$B = \frac{1}{\pi R^2} \int_{-R}^R e^{2\tau|y|} dy$$

□

2.3 The Paley-Wiener theorem

For now, we restrict ourselves to $L^2([-A, A])$. Our approach to $L^2([-A, A])$ consists in characterizing the set of functions whose Fourier transform belongs to $L^2([-A, A])$. This will allow us to analyze $L^2([-A, A])$ in terms of the properties of this specific set of functions. To this end, we proceed by observing first the behavior of the inverse Fourier transform of an element of $L^2([-A, A])$.

Proposition 2.7. *Let $\phi \in L^2([-A, A])$ and consider the inverse Fourier transform of ϕ extended to the complex numbers, i.e.,*

$$f(z) = \frac{1}{2\pi} \int_{-A}^A \phi(t) e^{izt} dt \quad \forall z \in \mathbb{C}$$

Then f is an entire function of exponential type at most A and its restriction to the real line is square integrable.

Proof. According to Morera's Theorem, if we prove that f is continuous and that the integral of f around an arbitrary triangle in the complex plane is 0, then f is an entire function.

Let $g(z, t) = \phi(t) e^{izt}$. Since g is continuous and integrable in every compact subset of $\mathbb{C} \times [-A, A]$, by the Dominated Convergence Theorem, f is continuous. Furthermore, we can apply Fubini's Theorem to obtain

$$\int_{\partial\Delta} \int_{-A}^A \phi(t) e^{izt} dt dz = \int_{-A}^A \phi(t) \left(\int_{\partial\Delta} e^{izt} dz \right) dt = 0$$

where the last equality is due to the Cauchy-Goursat Theorem applied to the function e^{izt} . Therefore, f is an entire function.

Moreover, the inequality

$$|f(z)| \leq \int_{-A}^A |\phi(t)| e^{-yt} dt \leq e^{A|y|} \int_{-A}^A |\phi(t)| dt \leq M e^{A|y|}$$

demonstrates that the function f is an entire function of exponential type at most A .

Finally, it follows from the Plancherel's Theorem (Theorem 2.3) that f is square integrable over the real line. Specifically, extending ϕ by zeros outside the interval $[-A, A]$ yields

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(t)|^2 dt < \infty$$

which completes the proof. □

As established in the previous section, the Fourier transform is an isomorphism. Consequently, the space $L^2([-A, A])$ is isomorphic to certain subset of the space of entire functions of exponential type at most A that are square integral over the real line. The Paley-Wiener Theorem establishes that this subset is, in fact, the entire space.

Theorem 2.8. (Paley-Wiener theorem) *Let f be an entire function of exponential type at most A such that*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

Then there exists a function $\phi \in L^2[-A, A]$ such that

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{izt} dt.$$

In particular, ϕ is the Fourier transform of f .

Proof. See [5]. □

We define the Paley-Wiener space as follows:

Definition 2.9. *Let $A > 0$. The Paley-Wiener space, which is denoted by PW_A , is the space of entire functions of exponential type at most A that are square integrable over the real axis. That is,*

$$PW_A = \{f \in H(\mathbb{C}) : f \text{ is of exponential type at most } A, \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty\}$$

Equivalently,

$$PW_A = \{f \in L^2(\mathbb{R}) : \text{Supp}(\hat{f}) \subset [-A, A]\}$$

Therefore, it is clear that the growth of an entire function is related to the support of its Fourier transform. This connection is preserved under differentiation:

Proposition 2.10. *Let $A > 0$ and $f \in PW_A$. Then $f' \in PW_A$ and the norm of f' satisfies*

$$\|f'\| \leq A\|f\|$$

Proof. If $f \in PW_A$, by Theorem 2.8, there exists a function $\phi \in L^2([-A, A])$ such that

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{izt} dt.$$

Differentiating with respect to z gives

$$f'(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} it\phi(t) e^{izt} dt.$$

Since $\phi \in L^2([-A, A])$, it follows that $it\phi \in L^2([-A, A])$. Hence, $f' \in PW_A$. Furthermore, applying Plancherel's Theorem (2.3), we obtain

$$\|f'\| = \frac{1}{\sqrt{2\pi}} \|t\phi\| \leq \frac{A}{\sqrt{2\pi}} \|\phi\| = A\|f\|,$$

as needed. □

2.4 Stability of bases in L^2

The concept of stability for Riesz bases is now revisited by applying the Paley-Wiener Theorem (Theorem 2.8). Let $S \subset \mathbb{R}$ be a bounded set of positive measure. A necessary condition for $E(\Lambda)$ to be a Riesz basis is that the sequence Λ must be separated.

Definition 2.11. A sequence $\{\lambda_n\}$ of real numbers is said to be separated if for some positive number ϵ

$$|\lambda_n - \lambda_m| \geq \epsilon \quad \text{whenever} \quad n \neq m$$

Lemma 2.12. If $S \subset \mathbb{R}$ is a bounded set of positive measure and $E(\Lambda)$ is a Riesz basis for $L^2(S)$, then Λ must be separated.

Proof. We argue by contradiction. Suppose that for every $\epsilon > 0$ there exist distinct values $\lambda, \mu \in \Lambda$ such that $|\lambda - \mu| < \epsilon$. Since $E(\Lambda)$ is a Riesz basis for $L^2(S)$, Theorem 1.11 guarantees the existence of a uniformly bounded sequence $\{g_k\}_{k=1}^\infty \subset L^2(S)$ biorthogonal to $E(\Lambda)$.

Let g_λ be the unique element in $\{g_k\}_{k=1}^\infty$ satisfying $\langle g_\lambda, e_\lambda \rangle = 1$ and $\langle g_\lambda, e_\mu \rangle = 0$. If we denote the Fourier transform of g_λ by G , we observe that

$$G(2\pi\lambda) = \langle g_\lambda, e_\lambda \rangle = 1 \quad \text{and} \quad G(2\pi\mu) = \langle g_\lambda, e_\mu \rangle = 0.$$

On the one hand, the boundedness of S implies that there exists a constant $A > 0$ such that $S \subset [-A, A]$. Consequently, $g_\lambda \in L^2[-A, A]$. This ensures, by Proposition 2.7, that $G \in PW_A$ and, in particular, that the restriction of G to the real line is differentiable. Therefore, by Barrow's rule, we have

$$|G(2\pi\lambda) - G(2\pi\mu)| = \left| \int_{2\pi\mu}^{2\pi\lambda} G'(x) dx \right| \leq 2\pi|\lambda - \mu| \sup_{x \in \mathbb{R}} |G'(x)|,$$

which can be rearranged to obtain the following lower bound

$$\|G'\|_\infty := \sup_{x \in \mathbb{R}} |G'(x)| \geq \frac{|G(2\pi\lambda) - G(2\pi\mu)|}{2\pi|\lambda - \mu|} = \frac{1}{2\pi|\lambda - \mu|} \geq \frac{1}{2\pi\epsilon}$$

On the other hand, let M be the uniform bound of the sequence $\{g_k\}_{k=1}^\infty \subset L^2(S)$. We observe that

$$|G'(x)| \leq \int_S |itg_\lambda(t)e^{-ixt}| dt = \int_S |tg_\lambda(t)| dt \leq A \int_S |g_\lambda(t)| dt$$

By the Cauchy-Schwarz inequality

$$\int_S |g_\lambda(t)| dt \leq \sqrt{\int_S |1|^2 dt} \sqrt{\int_S |g_\lambda(t)|^2 dt} = \sqrt{|S|} \|g_\lambda\| \leq \sqrt{|S|} M$$

where $|S|$ denotes the measure of S

Combining both inequalities yields

$$\frac{1}{2\pi\epsilon} \leq \|G'\|_\infty \leq AM\sqrt{|S|}$$

which leads to a contradiction given that ϵ can be chosen to be arbitrary small. Thus, Λ is separated. $\square$

It is worth noting that a system of exponential functions with a separated sequence of frequencies may not form Riesz basis. However, such a system always verify the Bessel inequality for some positive constant.

Lemma 2.13. *If $S \subset \mathbb{R}$ is a bounded set and $\Lambda = \{\lambda_k\}_{k=1}^\infty$ is separated, then the system $E(\Lambda)$ satisfies the Bessel inequality with a positive constant C , i.e*

$$\sum_{k=1}^\infty |\langle f, e_{\lambda_k} \rangle|^2 \leq C \|f\|^2 \quad \forall f \in L^2(S)$$

Proof. Let F be the Fourier transform of an arbitrary function $f \in L^2(S)$, for which $F(2\pi\lambda_k) = \langle f, e_{\lambda_k} \rangle$ for every $\lambda_k \in \Lambda$. Given that S is bounded, there exists a constant $A > 0$ such that $S \subset [-A, A]$. Therefore, $L^2(S) \subset L^2[-A, A]$, which implies that $F \in PW_A$. Moreover, since the set Λ is separated, by Corollary 2.6, the function F evaluated at Λ satisfies

$$\sum_{k=1}^\infty |F(2\pi\lambda_k)|^2 \leq B \int_{-\infty}^\infty |F(x)|^2 dx$$

The proof is completed by applying Plancherel's theorem. $\square$

Altogether, these considerations lead to a reformulation of the Paley-Wiener Criterion (Theorem 1.12) for bases of exponential functions in $L^2(S)$.

Corollary 2.14. *Let $S \subset \mathbb{R}$ be a bounded set of positive measure and Λ be a sequence of real numbers such that $E(\Lambda)$ is a Riesz basis in $L^2(S)$. Then there exists a constant $\eta = \eta(S, \Lambda)$ such that if a sequence $\Gamma = \{\gamma_k\}_{k=1}^\infty$ satisfies*

$$|\lambda_k - \gamma_k| < \eta \quad \forall k > 0$$

then $E(\Gamma)$ is also a Riesz basis in $L^2(S)$.

Proof. It is enough to show that $E(\Gamma)$ satisfies the hypothesis of Corollary 1.13. To verify this condition, let $f \in L^2(S)$ be an arbitrary function and denote the Fourier transform of f by F . Since S is bounded, F belong to the Paley-Wiener space PW_A , where A is a positive constant such that $S \subset [-A, A]$. It follow from Proposition 2.10 that $F' \in PW_A$ and that $\|F'\| \leq A\|F\|$. Besides, the Paley-Wiener Theorem (Theorem 2.8) ensures that F' is the image under the Fourier transform of some function $h \in L^2[-A, A]$. Hence, by Plancherel's Theorem (Theorem 2.3), the inequality $\|h\| \leq A\|f\|$ holds. Indeed,

$$\|h\| = \frac{1}{\sqrt{2\pi}} \|F'\| \leq \frac{A}{\sqrt{2\pi}} \|F\| = A\|f\|$$

By Barrow's rule, for each $k \geq 1$, we can express the difference $F(2\pi\lambda_k) - F(2\pi\gamma_k)$ as an average of F' along the segment connecting $2\pi\gamma_k$ and $2\pi\lambda_k$:

$$F(2\pi\lambda_k) - F(2\pi\gamma_k) = \int_{2\pi\gamma_k}^{2\pi\lambda_k} F'(s) ds = 2\pi(\lambda_k - \gamma_k) \int_0^1 F'(2\pi\xi_k(u)) du$$

where $\xi_k(u) := (1-u)\gamma_k + u\lambda_k$.

Applying the Cauchy-Schwarz inequality to the integral in u , we obtain

$$|F(2\pi\lambda_k) - F(2\pi\gamma_k)|^2 \leq (2\pi)^2 |\lambda_k - \gamma_k|^2 \int_0^1 |F'(2\pi\xi_k(u))|^2 du.$$

By hypothesis, $|\lambda_k - \gamma_k| < \eta$ for all $k > 0$. Summing over k and interchanging the sum and the integral (since all terms are non-negative), we have

$$\sum_{k=1}^{\infty} |\langle f, e_{\lambda_k} - e_{\gamma_k} \rangle|^2 = \sum_{k=1}^{\infty} |F(2\pi\lambda_k) - F(2\pi\gamma_k)|^2 \leq (2\pi\eta)^2 \int_0^1 \sum_{k=1}^{\infty} |\langle h, e_{\xi_k(u)} \rangle|^2 du,$$

where we have used that $F'(2\pi\xi) = \langle h, e_{\xi} \rangle$.

Now, fix $u \in [0, 1]$. Since $\xi_k(u) = (1-u)\gamma_k + u\lambda_k$, it follows that $|\xi_k(u) - \lambda_k| = (1-u)|\gamma_k - \lambda_k| < \eta$, so that each $\xi_k(u)$ is at a distance of less than η from the corresponding λ_k . According to Lemma 2.12, Λ is separated, i.e., $|\lambda_n - \lambda_m| \geq \epsilon$ for all $n \neq m$. Therefore, if $2\eta < \epsilon$, the reverse triangle inequality implies that

$$|\xi_n(u) - \xi_m(u)| \geq |\lambda_n - \lambda_m| - |\xi_n(u) - \lambda_n| - |\xi_m(u) - \lambda_m| \geq \epsilon - 2\eta > 0 \quad (n \neq m),$$

which means the family $\{\xi_k(u)\}_k$ is separated with the same constant $\epsilon - 2\eta$ for all u .

By Bessel's inequality (Lemma 2.13) applied to h , there exists a constant C depending only on $\epsilon - 2\eta$ which, combined with the inequality $\|h\| \leq A \|f\|$, yields

$$\sum_{k=1}^{\infty} |\langle h, e_{\xi_k(u)} \rangle|^2 \leq C \|h\|^2 \leq CA^2 \|f\|^2, \quad \forall u \in [0, 1].$$

Integrating this uniform bound over $u \in [0, 1]$, we get

$$\sum_{k=1}^{\infty} |\langle f, e_{\lambda_k} - e_{\gamma_k} \rangle|^2 \leq (2\pi\eta)^2 CA^2 \|f\|^2.$$

Finally, since C and A do not depend on η , we can choose η small enough such that $(2\pi\eta)^2 CA^2 \leq K$. This implies that the previous inequality satisfies the hypothesis of Corollary 1.13, as needed.

□

Chapter 3

Union of Riesz bases

Let $S \subset \mathbb{R}$ be a bounded set of positive measure. In this chapter, we present a technical result concerning the union of Riesz bases. Specifically, this result guarantees the existence of a Riesz basis of exponential functions for $L^2(S)$ by combining existing Riesz bases.

We begin by stating a necessary condition to construct Riesz bases.

3.1 Landau's density

Definition 3.1. Let $\Lambda \subset \mathbb{R}$ a separated sequence of real numbers. We define the upper and lower densities of Λ , respectively by

$$D^-(\Lambda) = \liminf_{r \rightarrow \infty} \frac{\min_{|I|=r} |\Lambda \cap I|}{r} \quad D^+(\Lambda) = \limsup_{r \rightarrow \infty} \frac{\max_{|I|=r} |\Lambda \cap I|}{r}$$

where the maximum and minimum are taken over all intervals $I \subset \mathbb{R}$ of length r .

Furthermore, Λ is said to be uniformly distributed if $D^-(\Lambda) = D^+(\Lambda)$. In this case, we denote this value simply by $D(\Lambda)$.

The fundamental requirement for $E(\Lambda)$ to be a Riesz basis is due to H.J Landau [7], who established the following necessary condition on the density of the set Λ :

Theorem 3.2. Let $S \subset \mathbb{R}$ be a bounded measurable set and Λ a separated sequence of real numbers.

1. If $E(\Lambda)$ is a Riesz sequence in $L^2(S)$, then $D^+(\Lambda) \leq |S|$.
2. If $E(\Lambda)$ is a frame in $L^2(S)$, then $D^-(\Lambda) \geq |S|$

The first statement in Theorem 3.2 implies that if the set Λ is too dense, $E(\Lambda)$ loses the linear independence required to be a Riesz sequence. Meanwhile, the second part indicates that if Λ is not dense enough, the space $L^2(S)$ is not generated by $E(\Lambda)$. Therefore, we conclude that $E(\Lambda)$ is a Riesz basis if and only if the density $D(\Lambda)$ is exactly equal to $|S|$.

3.2 Combining Riesz bases

Without loss of generality, we will restrict our study to $S \subset [0, 1]$. This restriction is justified by the fact that the Riesz basis property of an exponential system is preserved under translation and rescaling of the set S . Indeed, this transformation induces an isometry between the corresponding L^2 spaces and therefore preserves the inequality 1.8 and the completeness of the system.

3.2.1 Constructing Riesz bases

Let $S \subset [0, 1]$ and let $N \in \mathbb{N}$. Our goal is to construct a Riesz basis of exponential functions for $L^2(S)$ with $\Lambda \subset \mathbb{N}$ combining existing Riesz bases $E(\Lambda_1), \dots, E(\Lambda_N)$ for certain L^2 spaces related to $L^2(S)$. In other words, we seek a Riesz basis $E(\Lambda)$ in $L^2(S)$ such that $\Lambda \subset \mathbb{N}$ is the union of the sets $\Lambda_1, \dots, \Lambda_N$. Hence, $\Lambda_1, \dots, \Lambda_N$ must be subsets of $\mathbb{N}$ and, by Theorem 3.2, their union must have density $D(\Lambda) = |S|$. Such restriction will be guaranteed by imposing that for each $n \in \{1, \dots, N\}$, the system $E(\Lambda_n)$ is a basis for a suitable space.

To this end, notice that S can be partitioned into the sets

$$B_n = \left\{ t \in S : t + \frac{l}{N} \in S \text{ for exactly } n \text{ integers } l \right\}$$

for $n \in \{1, \dots, N\}$. This is equivalent to divide S into intervals of length $1/N$ and taking the points of S whose translations using multiples of $1/N$ lie in S exactly n times.

If we assume that $\Lambda_1, \dots, \Lambda_N \subset N\mathbb{Z}$, then the exponential functions of each set $E(\Lambda_n)$ are periodic with period equal to $1/N$. Therefore, we can focus our study to the interval $[0, 1/N]$. It is then natural to define A_n as the folding of B_n to $[0, \frac{1}{N}]$, i.e. A_n is obtained by cutting B_n in N pieces, translating each piece to $[0, \frac{1}{N}]$ and taking their union. Specifically, for each $n \in \{1, \dots, N\}$, the set A_n is defined by

$$A_n = \{t \in [0, \frac{1}{N}] : t + \frac{l}{N} \in S \text{ for exactly } n \text{ values of } l \in \{0, 1, \dots, N-1\}\} \quad (3.1)$$

An explicit example of the construction of the sets A_n is illustrated in Figure 3.1.

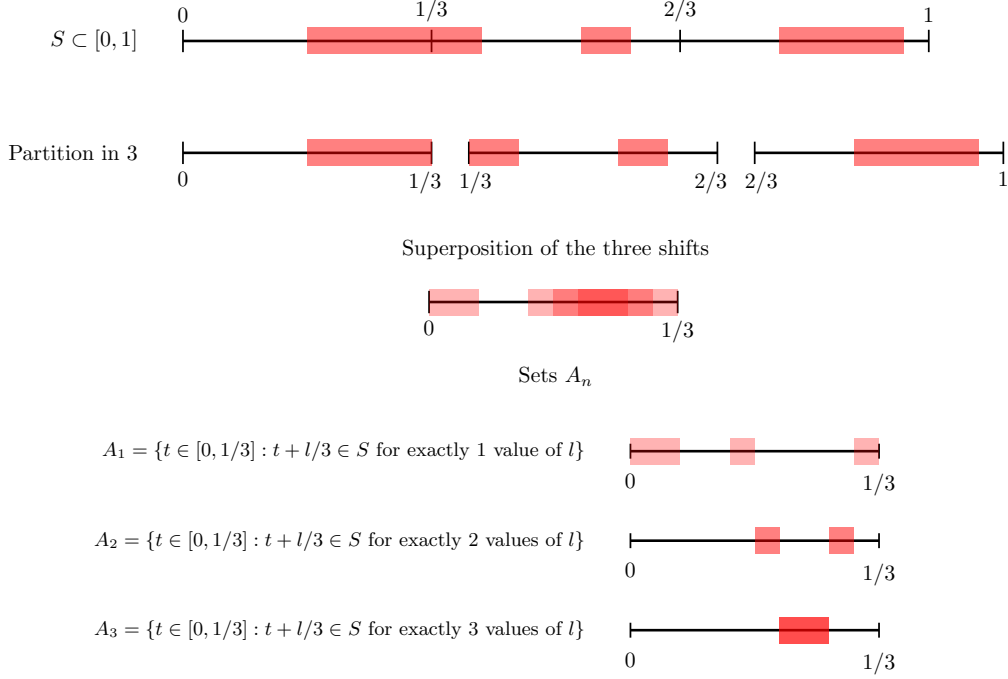


Figure 3.1: Construction of the sets A_n (Equation 3.1) for an arbitrary set $S \subset [0, 1]$ in the case $N = 3$

However, if for each $n \in \{1, \dots, N\}$ we obtain a set Λ_n for which $E(\Lambda_n)$ is Riesz basis for $L^2(A_n)$, we cannot construct a Riesz basis for $L^2(S)$ simply by combining these bases. Indeed, due to the folding, the union of all $\Lambda_1, \dots, \Lambda_N$ would not satisfy Landau's Theorem 3.2. In fact, the density of the union would be less than $|S|$.

This issue can be solved by introducing multiple copies of A_n in the sets over which the system $E(\Lambda_n)$ constitute a Riesz basis. Since the number of copies to be introduced is exactly the number of superpositions produced during the folding of B_n to $[0, 1/N]$, i.e n for A_n , we define

$$A_{\geq n} = \bigcup_{k=n}^N A_k$$

It is then clear that if each $E(\Lambda_n)$ is a Riesz basis for $L^2(A_{\geq n})$ and $\Lambda_1, \dots, \Lambda_N$ are disjoint, then the union of $\Lambda_1, \dots, \Lambda_N$ verify the necessary condition imposed by Theorem 3.2. If $\Lambda_1, \dots, \Lambda_N$ are not disjoint, we can instead define the union

$$\Lambda = \bigcup_{j=1}^N (\Lambda_j + j).$$

This leads to the following lemma:

Lemma 3.3. *Let S be a bounded measurable subset of $[0, 1]$. Suppose there exist sets $\Lambda_1, \dots, \Lambda_N \subset \mathbb{N}\mathbb{Z}$ such that, for every $n \in \{1, \dots, N\}$, the system $E(\Lambda_n)$ is a Riesz basis for $L^2(A_{\geq n})$. Then, the system $E(\Lambda)$, where Λ*

$$\Lambda = \bigcup_{j=1}^N (\Lambda_j + j),$$

is a Riesz basis for $L^2(S)$.

Proof. Due to the length of the proof, we divide it in different subsections:

3.2.2 Preliminary reduction

The property of being a Riesz basis is invariant under a unitary operator. Therefore, for each $n \in \{1, \dots, N\}$, the image of $E(\Lambda_n)$ under the operator

$$U_n(f) = e^{2\pi i n t} f$$

is still a Riesz basis for $L^2(A_{\geq n})$. This new basis, say it $E(\Lambda'_n)$, satisfies that $\Lambda'_n \subset \mathbb{N}\mathbb{Z} + n$ and

$$\Lambda = \bigcup_{j=1}^N \Lambda'_j$$

Thus, Lemma 3.3 is equivalent to verify the following statement: if $E(\Lambda'_n)$ is a Riesz basis for $L^2(A_{\geq n})$, then $E(\Lambda)$ is a Riesz basis for $L^2(S)$. For this purpose, we just need to prove that $E(\Lambda)$ is a frame and a Riesz sequence and then apply Lemma 1.9.

From now on, denote Λ'_n simply by Λ_n to avoid cumbersome notation.

The proof that $E(\Lambda)$ is both a frame and a Riesz sequence relies on the construction of a Vandermonde matrix associated to the bases $E(\Lambda_1), \dots, E(\Lambda_N)$. Hence, let us recall its definition:

Definition 3.4. Let $N \in \mathbb{N}$ and let $\alpha_1, \dots, \alpha_N \subset \mathbb{R}$. The Vandermonde matrix generated by the constants $\alpha_1, \dots, \alpha_N \subset \mathbb{R}$ is

$$V = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{N-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_N & \alpha_N^2 & \dots & \alpha_N^{N-1} \end{pmatrix}$$

It is well-known that the determinant of V is given by

$$\det(V) = \prod_{1 \leq i < j \leq N} (\alpha_j - \alpha_i)$$

Consequently, V is invertible if and only if $\alpha_1, \dots, \alpha_N \subset \mathbb{R}$ are all distinct.

3.2.3 Frame

We begin the proof by showing that $E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}$ is a frame. Therefore, we need to verify that for every $f \in L^2(S)$, the inequality

$$c \|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \leq C \|f\|^2 \quad (3.2)$$

holds for some positive constants c and C .

On the one hand, the right-hand side inequality in Equation 3.2 is the Bessel inequality, which is satisfied immediately given that $S \subset [0, 1]$, $\Lambda \subset \mathbb{Z}$ and $E(\Lambda)$ is an orthonormal sequence in $L^2[0, 1]$.

On the other hand, for any $f \in L^2(S)$ and $n \in \{1, \dots, N\}$, we define f_n as the restriction of f to B_n and extended by zeros to S if needed.

The left hand-side inequality in the definition of frame (Equation 3.2) is a direct consequence of the following inequality:

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \geq c \|f_n\|^2 - \sum_{k=1}^{n-1} \|f_k\|^2 \quad \forall n \in \{1, \dots, N\} \quad (3.3)$$

where c is a positive constant not depending on f . Indeed, if $\{\delta_n\}_{n=1}^N$ is an arbitrary sequence of positive numbers such that $\sum_{n=1}^N \delta_n = 1$, then using Equation 3.3 we have that

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 = \sum_{n=1}^N \delta_n \sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \geq \sum_{n=1}^N \delta_n (c \|f_n\|^2 - \sum_{k=1}^{n-1} \|f_k\|^2)$$

Rearranging and relabeling the indices of the second summation, we obtain

$$\sum_{n=1}^N \delta_n (c \|f_n\|^2 - \sum_{k=1}^{n-1} \|f_k\|^2) = \sum_{n=1}^N (c\delta_n - \sum_{k=n+1}^N \delta_k) \|f_n\|^2$$

Without loss of generality, we can take $\{\delta_n\}_{n=1}^N$ satisfying

$$\delta_n > \frac{2}{c} \sum_{k=n+1}^N \delta_k \quad \forall n \in \{1, \dots, N\} \quad (3.4)$$

Essentially, the construction of this sequence proceeds by backward recursion: we take $\delta_N = 1$ and, for each $1 \leq n < N$, we determine δ_n by imposing the condition 3.4, assuming that the values $\{\delta_{n+1}, \dots, \delta_N\}$ are already known. Once we end this process, we normalize the succession by dividing all terms by $\sum_{n=1}^N \delta_n$.

Let $c_1 = \frac{c}{2} \min \delta_n$. Using the relation imposed by Equation 3.4 together with the fact that the functions f_n have disjoint supports, we have

$$\sum_{n=1}^N (c\delta_n - \sum_{k=n+1}^N \delta_k) \|f_n\|^2 \geq \sum_{n=1}^N \frac{c\delta_n}{2} \|f_n\|^2 \geq c_1 \sum_{n=1}^N \|f_n\|^2 = c_1 \|f\|^2$$

Hence,

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \geq c_1 \|f\|^2$$

as needed.

It remains to prove 3.3. For this purpose, recall that for any $x, y \in \mathbb{C}$, the inequality $|x + y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$ holds.

Proof:

$$(\sqrt{\frac{1}{2}}|x|^2 - |y|^2)^2 \geq 0 \quad \implies \quad \frac{1}{2}|x|^2 + 2|y|^2 \geq 2|x||y|$$

This implies that

$$|x + y|^2 = |x|^2 + |y|^2 + 2\operatorname{Re}(x\bar{y}) \geq |x|^2 + |y|^2 - 2|x||y| \geq$$

$$|x|^2 + |y|^2 - \frac{1}{2}|x|^2 - 2|y|^2 = \frac{1}{2}|x|^2 - |y|^2$$

as required.

Therefore, for a fixed $n \in \{1, \dots, N\}$, the previous inequality implies that

$$|\langle f, e_\lambda \rangle|^2 = |\langle \sum_{k=1}^N f_k, e_\lambda \rangle|^2 \geq \frac{1}{2} |\langle \sum_{k=n}^N f_k, e_\lambda \rangle|^2 - |\langle \sum_{k=1}^{n-1} f_k, e_\lambda \rangle|^2$$

Define $f_{\geq n} = \sum_{k=n}^N f_k$ to avoid cumbersome notation. If we sum over $\lambda \in \Lambda$, then

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 &\geq \frac{1}{2} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \sum_{\lambda \in \Lambda} |\langle \sum_{k=1}^{n-1} f_k, e_\lambda \rangle|^2 \geq \\ &\geq \frac{1}{2} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \left\| \sum_{k=1}^{n-1} f_k \right\|^2 \end{aligned}$$

where the last inequality follows from Bessel's inequality with constant 1, which is satisfied since $\Lambda \subset \mathbb{Z}$ and $S \subset [0, 1]$.

Since f_k have disjoint supports, then

$$\left\| \sum_{k=1}^{n-1} f_k \right\|^2 = \sum_{k=1}^{n-1} \|f_k\|^2$$

Taken together, we obtain that

$$\sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \geq \frac{1}{2} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 - \sum_{k=1}^{n-1} \|f_k\|^2 \quad \forall n \in \{1, \dots, N\}$$

At this point, notice that to show Equation 3.3, we only need to verify that

$$\sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 \geq c \|f_n\|^2$$

For any $\lambda \in \Lambda_j$, the projection of $f_{\geq n}$ onto e_λ can be estimated by splitting the interval $[0, 1]$ into a partition of intervals of length $1/N$ and exploiting the translational properties of the exponential functions. Specifically, we begin by decomposing the inner product as follows:

$$\begin{aligned} \langle f_{\geq n}, e_\lambda \rangle &= \int_0^1 f_{\geq n}(t) \overline{e_\lambda}(t) dt = \sum_{l=0}^{N-1} \int_0^{\frac{1}{N}} f_{\geq n}(t + \frac{l}{N}) \overline{e_\lambda}(t + \frac{l}{N}) dt = \\ &= \int_0^{\frac{1}{N}} \sum_{l=0}^{N-1} f_{\geq n}(t + \frac{l}{N}) e_{-\lambda}(t + \frac{l}{N}) dt \end{aligned}$$

Let us define the auxiliary function h_j by

$$h_j(t) = \mathbb{1}_{A_{\geq n}}(t) \sum_{l=0}^{N-1} f_{\geq n}(t + \frac{l}{N}) e(-\frac{jl}{N}),$$

where we have included the indicator function $\mathbb{1}_{A_{\geq n}}$ because if $t \notin A_{\geq n}$, then $t + \frac{l}{N} \notin B_{\geq n}$ for any $l \in \{0, \dots, N-1\}$ and every term $f_{\geq n}(t + \frac{l}{N})$ vanishes.

Recalling that $\lambda \in N\mathbb{Z} + j$, the inner product simplifies directly to

$$\langle f_{\geq n}, e_{\lambda} \rangle = \int_0^{\frac{1}{N}} h_j(t) e_{-\lambda}(t) dt = \langle h_j, e_{\lambda} \rangle$$

Now, fix $j \leq n$. By hypothesis, $E(\Lambda_j)$ is a Riesz basis for $L^2(A_{\geq j})$. Thus, given that h_j is supported on $A_{\geq n} \subset A_{\geq j}$, the left-hand side inequality in the definition of frame (Equation 3.2) implies that

$$\sum_{\lambda \in \Lambda_j} |\langle f_{\geq n}, e_{\lambda} \rangle|^2 = \sum_{\lambda \in \Lambda_j} |\langle h_j, e_{\lambda} \rangle|^2 \geq c \|h_j\|^2 \quad (3.5)$$

where c is a positive constant.

Since Equation 3.5 holds for every $j \in \{1, \dots, N\}$, we can extend it to the union of frequencies $\Lambda = \bigcup_{j=1}^N \Lambda_j$ to obtain

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_{\lambda} \rangle|^2 &= \sum_{j=1}^N \sum_{\lambda \in \Lambda_j} |\langle f_{\geq n}, e_{\lambda} \rangle|^2 \geq \sum_{j=1}^n \sum_{\lambda \in \Lambda_j} |\langle f_{\geq n}, e_{\lambda} \rangle|^2 \geq^{(3.5)} \\ &\geq c \sum_{j=1}^n \|h_j\|^2 \geq c \sum_{j=1}^n \|h_j \cdot \mathbb{1}_{A_n}\|^2 \end{aligned}$$

where in the last inequality is a direct consequence of the inclusion $A_n \subset A_{\geq n}$.

To evaluate the right-hand side of the previous inequality, it is worth noting that, for every $t \in A_n$, the values of $\{h_j(t)\}_j$ are given by the product of the $n \times N$ matrix $L = \{e(-jl/N)\}_{j,l}$ and the vector $v = \{f_{\geq n}(t + l/N)\}_l$. However, $t \in A_n$ guarantees that only n different values of the vector v are nonzero. Restricting to these components, only an $n \times n$ submatrix of L contributes to the product. Let $L_{n \times n}$ be this submatrix. Observe that $L_{n \times n}$ is a Vandermonde matrix which is invertible because the numbers $e(-l/N)$ are different for every $l \in \{0, \dots, N-1\}$. The invertibility of $L_{n \times n}$ implies that its inverse, say it $L_{n \times n}^{-1}$, is bounded. Since this property holds for all $n \times n$ submatrices of L , there exists a constant C only

depending on N that bounds the norm of the inverses over the entire collection of these matrices. Hence, for every $t \in A_n$ there exist a submatrix $L_{n \times n}$ of L such that

$$\|\{f_{\geq n}(t + \frac{l}{N})\}_l\| = \|L_{n \times n}^{-1} \cdot \{h_j(t)\}_j\| \leq \|L_{n \times n}^{-1}\| \|\{h_j(t)\}_j\| \leq C \|\{h_j(t)\}_j\|$$

which implies that

$$\sum_{j=1}^n |h_j(t)|^2 \geq \frac{1}{C} \sum_{l=0}^{N-1} |f_{\geq n}(t + \frac{l}{N})|^2$$

Integrating this inequality over A_n we obtain that:

$$\sum_{j=1}^n \|h_j \cdot \mathbb{1}_{A_n}\|^2 = \int_{A_n} \sum_{j=1}^n |h_j(t)|^2 dt \geq \frac{1}{C} \int_{A_n} \sum_{l=0}^{N-1} |f_{\geq n}(t + \frac{l}{N})|^2 dt$$

Finally, since A_n is the folding of B_n and the support of each f_k is B_k , the restriction of $f_{\geq n}$ to the set B_n is precisely f_n , we can evaluate the integral in the right-hand side of the previous inequality as follows:

$$\begin{aligned} \int_{A_n} \sum_{l=0}^{N-1} |f_{\geq n}(t + \frac{l}{N})|^2 &= \sum_{l=0}^{N-1} \int_{A_n} |f_{\geq n}(t + \frac{l}{N})|^2 dt = \\ &= \int_{B_n} |f_{\geq n}(t)|^2 dt = \int_{B_n} |f_n(t)|^2 dt = \|f_n\|^2 \end{aligned}$$

Therefore,

$$\sum_{\lambda \in \Lambda} |\langle f_{\geq n}, e_\lambda \rangle|^2 \geq c \|f_n\|^2$$

and $E(\Lambda)$ is a frame.

3.2.4 Riesz sequence

$E(\Lambda)$ is a Riesz sequence if and only if there exist positive constants A and B such that, for every finite sequence of scalars $\{c_\lambda\}_{\lambda \in \Lambda}$, the following inequality holds:

$$A \sum_{\lambda \in \Lambda} |c_\lambda|^2 \leq \left\| \sum_{\lambda \in \Lambda} c_\lambda f_\lambda \right\|^2 \leq B \sum_{\lambda \in \Lambda} |c_\lambda|^2 \quad (3.6)$$

Let $l^2(\Lambda)$ be the set of square-summable sequences from Λ to $\mathbb{C}$ and let $\{c_\lambda\}_{\lambda \in \Lambda} \subset l^2(\Lambda)$ be a finite sequence. The right-hand side inequality in the definition of Riesz

sequence (Equation 3.6) is an immediate consequence of the fact that $\Lambda \subset \mathbb{Z}$ and $E(\Lambda)$ is orthonormal in $[0, 1]$. Indeed,

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda \right\|_{L^2(S)}^2 \leq \left\| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda \right\|_{L^2([0,1])}^2 = \sum_{\lambda \in \Lambda} |a_\lambda|^2$$

Hence, to prove that $E(\Lambda)$ is a Riesz sequence we must verify Equation 3.7:

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda \right\|^2 \geq c \sum_{\lambda \in \Lambda} |a_\lambda|^2 \quad (3.7)$$

Again, it suffices to show that for every $n \in \{1, \dots, N\}$ we have

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda \right\|^2 \geq c \sum_{\lambda \in \Lambda_n} |a_\lambda|^2 - \sum_{j=n+1}^N \sum_{\lambda \in \Lambda_j} |a_\lambda|^2 \quad (3.8)$$

The proof that the inequality in 3.8 implies the left-hand side inequality in the Riesz sequence condition (Equation 3.7) is analogous to the proof that 3.3 implies the left-hand side inequality in the definition of frame (Equation 3.2). The only difference is that the indices of the second sum in the right-hand side of Equation 3.8 ranges from $n+1$ to N instead of 1 to $n-1$. This difference changes the sequence $\{\delta_n\}_{n=1}^N$, which now satisfies

$$\delta_n \geq \frac{2}{c} \sum_{k=1}^{n-1} \delta_k \quad \forall n \geq 1 \quad (3.9)$$

and can be easily constructed by recursion: we first take $\delta_1 = 1$ and then, for each $1 < n \leq N$, we determine δ_n by imposing condition 3.9, assuming that the values $\{\delta_1, \dots, \delta_{n-1}\}$ are already known. The last step is normalize the succession by dividing all terms by $\sum_{n=1}^N \delta_n$.

Therefore, we shall verify Inequality 3.8. To this end, fix an $n \in \{1, \dots, N\}$. Applying $|x + y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$ to the sum over Λ , we obtain

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda \right\|^2 = \int_S \left| \sum_{\lambda \in \Lambda} a_\lambda e_\lambda \right|^2 dt \geq \frac{1}{2} \int_S \left| \sum_{j=1}^n \sum_{\lambda \in \Lambda_j} a_\lambda e_\lambda \right|^2 dt - \int_S \left| \sum_{j=n+1}^N \sum_{\lambda \in \Lambda_j} a_\lambda e_\lambda \right|^2 dt$$

To bound the second integral in the right-hand side of the previous inequality, recall that $S \subset [0, 1]$ and $\Lambda \subset \mathbb{Z}$. This implies that $E(\Lambda)$ is orthonormal in $L^2[0, 1]$. It follows that

$$\int_S \left| \sum_{j=n+1}^N \sum_{\lambda \in \Lambda_j} a_\lambda e_\lambda \right|^2 dt \leq \left\| \sum_{j=n+1}^N \sum_{\lambda \in \Lambda_j} a_\lambda e_\lambda \right\|_{L^2([0,1])}^2 = \sum_{j=n+1}^N \sum_{\lambda \in \Lambda_j} |a_\lambda|^2$$

Consequently, substituting this bound to the previous inequality, we obtain:

$$\left\| \sum_{\lambda \in \Lambda} a_{\lambda} e_{\lambda} \right\|^2 \geq \frac{1}{2} \int_S \left| \sum_{j=1}^n \sum_{\lambda \in \Lambda_j} a_{\lambda} e_{\lambda} \right|^2 dt - \sum_{j=n+1}^N \sum_{\lambda \in \Lambda_j} |a_{\lambda}|^2$$

For brevity, let us define the auxiliary function f by

$$f = \mathbb{1}_S \sum_{j=1}^n \sum_{\lambda \in \Lambda_j} a_{\lambda} e_{\lambda}$$

Using this notation, the proof is complete once we verify

$$\int_S |f(t)|^2 dt \geq c \sum_{\lambda \in \Lambda_n} |a_{\lambda}|^2$$

The structure of the proof consists of reducing the domain of integration to $A_{\geq n}$ in order to exploit the hypothesis that $E(\Lambda_n)$ is a Riesz basis for $L^2(A_{\geq n})$. For this purpose, fix a point $t \in A_{\geq n}$ and an index $l \in \{0, \dots, N-1\}$ such that $t + \frac{l}{N} \in S$.

Since $\lambda \in N\mathbb{Z} + j$, evaluating f at this point yields

$$\begin{aligned} f\left(t + \frac{l}{N}\right) &= \sum_{j=1}^n \sum_{\lambda \in \Lambda_j} a_{\lambda} e_{\lambda}\left(t + \frac{l}{N}\right) = \sum_{j=1}^n \sum_{\lambda \in \Lambda_j} a_{\lambda} e\left(\lambda t + \frac{\lambda l}{N}\right) = \\ &= \sum_{j=1}^n e\left(\frac{j l}{N}\right) \sum_{\lambda \in \Lambda_j} a_{\lambda} e_{\lambda}(t) \end{aligned}$$

where the last step is due to the periodicity of the exponential function.

Analogously to the proof that $E(\Lambda)$ is a frame, let us consider the values of $l \in \{0, \dots, N-1\}$ for which $t + \frac{l}{N} \in S$. Assuming that there are k such values, the vector $\{f(t + \frac{l}{N})\}_l$ can be represented as the product of a $k \times n$ matrix $L = \{e(\frac{j l}{N})\}_{l,j}$ and the vector $\{\sum_{\lambda \in \Lambda_j} a_{\lambda} e_{\lambda}(t)\}_j$.

Notice that any $n \times n$ minor of L is an invertible Vandermonde matrix. Therefore, if C denotes the uniform bound for the norms of the inverses over all such $n \times n$ submatrices and $\{l_1, \dots, l_n\}$ is an arbitrary sequence of n distinct possible values of l , it follows that

$$\sum_{l=0}^{N-1} |f(t + \frac{l}{N})|^2 \geq \sum_{j=1}^n |f(t + \frac{l_j}{N})|^2 \geq \frac{1}{C} \sum_{j=1}^n | \sum_{\lambda \in \Lambda_j} a_\lambda e_\lambda(t) |^2 \geq \frac{1}{C} | \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(t) |^2 \quad (3.10)$$

Now, we integrate f over its domain. First, since $B_{\geq n} \subset S$, we obtain

$$\int_S |f(t)|^2 dt \geq \int_{B_{\geq n}} |f(t)|^2 dt$$

Second, we use that $A_{\geq n}$ is the folding of $B_{\geq n}$ to guarantee that

$$\int_{B_{\geq n}} |f(t)|^2 dt = \int_{A_{\geq n}} \sum_{l=0}^{N-1} |f(t + \frac{l}{N})|^2 dt$$

Finally, it follows from Equation 3.10 that

$$\int_{A_{\geq n}} \sum_{l=0}^{N-1} |f(t + \frac{l}{N})|^2 dt \geq \frac{1}{C} \int_{A_{\geq n}} | \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(t) |^2 dt$$

Combining this result with the fact that $E(\Lambda_n)$ is a Riesz basis over $A_{\geq n}$, we conclude that

$$\int_S |f(t)|^2 dt \geq \frac{1}{C} \int_{A_{\geq n}} | \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(t) |^2 dt = \frac{1}{C} \| \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda \|_{L^2(A_{\geq n})}^2 \geq \frac{A}{C} \sum_{\lambda \in \Lambda_n} |a_\lambda|^2$$

where A is the bound from Equation 1.4.

This completes the proof. □

Chapter 4

Existence of Riesz bases of exponentials

We are now in a position to state and prove the existence of Riesz bases of exponential functions for L^2 spaces on a finite union of intervals.

Theorem 4.1. *Let $S \subset \mathbb{R}$ be a finite union of intervals. There exists a set $\Lambda \subset \mathbb{R}$ such that the functions $\{e^{2\pi i\lambda t}\}_{\lambda \in \Lambda}$ form a Riesz basis in $L^2(S)$. Moreover, if $S \subset [0, 1]$ then Λ may be chosen to satisfy $\Lambda \subset \mathbb{Z}$.*

Section 4.1 begins by analyzing the case of a single interval, which establishes the necessary framework to address finite unions of intervals in Section 4.2. We conclude with the proof of Theorem 4.1.

4.1 Riesz bases for a single interval

The existence of a Riesz basis of exponential functions on a single interval relies on the stability of the orthonormal basis $E(\mathbb{Z}) = \{e^{int}\}_{n \in \mathbb{Z}}$. An initial result in this direction is Claim 4.2:

Claim 4.2. *Let η be a constant satisfying the conditions of the Paley-Wiener criterion (Corollary 2.14) for $S = [0, 1]$ and the Riesz basis $E(\mathbb{Z})$. There exists a sequence $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2[0, \eta]$.*

Proof. It is well known that $E(\mathbb{Z})$ is an orthonormal basis for $L^2([0, 1])$. By a change of variables, it follows that $E((\frac{1}{\eta})\mathbb{Z})$ is an orthogonal basis for $L^2[0, \eta]$. Furthermore, Corollary 2.14 implies that if a set $\Lambda = \{\lambda_n : n \in \mathbb{Z}\}$ satisfies the condition

$|n/\eta - \lambda_n| \leq 1$ for all $n \in \mathbb{Z}$, then $E(\Lambda)$ is a Riesz basis for $L^2([0, \eta])$. Indeed, for each $n \in \mathbb{Z}$, the inequality $|n/\eta - \lambda_n| \leq 1$ is equivalent to $|n - \eta\lambda_n| \leq \eta$. Under this condition, Corollary 2.14 guarantees that the scaled set $\eta\Lambda = \{\eta\lambda_n : n \in \mathbb{Z}\}$ is a Riesz basis for $L^2([0, 1])$. Consequently, rescaling back by η ensures that Λ is a Riesz basis for $L^2([0, \eta])$.

Finally, if $\lfloor \cdot \rfloor$ denotes the floor function, we observe that $\lambda_n = \lfloor \frac{n}{\eta} \rfloor$ satisfies $|n/\eta - \lambda_n| \leq 1$ for each $n \in \mathbb{Z}$. Furthermore, $\Lambda = \{\lfloor \frac{n}{\eta} \rfloor : n \in \mathbb{Z}\}$ is a subset of $\mathbb{Z}$. Thus, we conclude that the collection $E(\Lambda)$ is a Riesz basis for $L^2([0, \eta])$ with frequencies in $\mathbb{Z}$, as needed. $\square$

Not every subinterval of $[0, 1]$ admits the construction of a Riesz basis simply through stability properties. However, appropriately combining the Riesz bases obtained perturbing $E(\mathbb{Z})$ is enough to construct a Riesz basis for any subinterval within $[0, 1]$.

Lemma 4.3. *Let $S \subset [0, 1]$ be an interval. Then there exists a sequence $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2(S)$.*

Proof. The property of being a Riesz basis is invariant under isometric isomorphisms and, in particular, it is invariant under translations. We may assume then, without loss of generality, that $S = [0, b]$. Fix η as in Claim 4.2 and N such that $\{Nb\} \leq \eta$, where $\{\cdot\}$ denotes the fractional part of a number. The existence of this N is trivial if b is rational and an immediate consequence of the Weyl's equidistribution Theorem (Appendix A.1) if b is irrational.

Let us now find for each $n \in \{1, \dots, N\}$ a Riesz basis for $L^2(A_{\geq n})$ for our set S and N . Recalling that $\lfloor \cdot \rfloor$ denotes the integer part of a number, we get that

$$b = \frac{\lfloor Nb \rfloor}{N} + \frac{\{Nb\}}{N}$$

Thus, there are $\lfloor Nb \rfloor$ complete intervals of length $1/N$ in S , while the $(\lfloor Nb \rfloor + 1)$ -th interval is of the form $[\frac{\lfloor Nb \rfloor}{N}, b]$. This last interval has measure $\frac{\{Nb\}}{N}$. It clearly follows that

$$\begin{aligned} A_{\geq n} &= [0, \frac{1}{N}] & 1 \leq n \leq \lfloor Nb \rfloor \\ A_{\geq \lfloor Nb \rfloor + 1} &= [0, \frac{\{Nb\}}{N}] \\ A_{\geq n} &= \emptyset & \lfloor Nb \rfloor + 2 \leq n \leq N \end{aligned}$$

Therefore, a Riesz basis for the set $[0, 1/N]$ is $N\mathbb{Z}$, which is only the orthonormal basis $E(\mathbb{Z})$ in $[0, 1]$ rescaled by N . Similarly, for $[0, \{Nb\}/N]$ we can scale this interval by N and then apply Claim 4.2, which holds since $\{Nb\} < \eta$.

Lemma 3.3 completes the proof. □

4.2 Riesz bases for finite unions of intervals

Let us now consider a finite union of intervals S within $[0, 1]$. If the endpoints of these intervals are linearly independent over the rationals, Lemma 4.3 can be extended to construct a Riesz basis for $L^2(S)$.

Claim 4.4. *Let $S \subset [0, 1]$ be a finite union of intervals $S = \bigcup_{j=1}^L [a_j, b_j]$. If the numbers $\{a_1, \dots, a_L, b_1, \dots, b_L, 1\}$ are linearly independent over the rationals then there exists a sequence $\Lambda \subset \mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis over S .*

Proof. Since the elements of the set $\{a_1, \dots, b_L, 1\}$ are linearly independent over the rationals, by the multidimensional Weyl's Equidistribution Theorem (Theorem A.2), the vectors

$$(\{Na_1\}, \dots, \{Na_L\}, \{Nb_1\}, \dots, \{Nb_L\}),$$

taken over all $N \in \mathbb{N}$, are equidistributed in $[0, 1]^L$. In particular, there exists a specific N such that

$$\{Na_1\}, \dots, \{Na_L\} < \frac{1}{2} < \{Nb_1\}, \dots, \{Nb_L\} \quad (4.1)$$

Motivated by the construction of $A_{\geq n}$ (Equation 3.1), we define the function ϕ in such a way that $A_{\geq n} = \{t \in [0, 1/N] : \phi(t) \geq n\}$ for all $n \in \{1, \dots, N\}$, i.e.,

$$\phi(t) = \#\{l \in \mathbb{N} : 0 \leq l \leq N-1, t + l/N \in S\}$$

Observe that ϕ increases at every $\{Na_i\}/N$ and decreases at every $\{Nb_i\}/N$. This behavior follows from the fact that as t crosses a specific $\{Na_i\}/N$ (respectively, $\{Nb_i\}/N$), some translated point $t + l/N$ enters (leaves) the interval $[a_i, b_i]$. Moreover, the distribution of $\{Na_1\}, \dots, \{Na_L\}, \{Nb_1\}, \dots, \{Nb_L\}$ imposed in condition 4.1 ensures that ϕ is increasing on $[0, 1/2N]$ and decreasing on $[1/2N, 1/N]$. Therefore, for every $n \in \{1, \dots, N\}$, the set $A_{\geq n}$ is an interval.

Since each $A_{\geq n}$ is an interval in $[0, 1/N]$, it follows that $NA_{\geq n} \subset [0, 1]$ for all n . Consequently, Lemma 4.3 guarantees the existence of a Riesz basis with frequencies in $\mathbb{Z}$ for each $NA_{\geq n}$. The proof concludes by rescaling these bases back to $A_{\geq n}$, which now have frequencies in $N\mathbb{Z}$, and then applying Lemma 3.3 to obtain a Riesz basis for S . $\square$

4.2.1 Interlacement

Let $S \subset [0, 1]$ be a finite union of intervals $S = \bigcup_{j=1}^L [a_j, b_j]$. If the endpoints $\{a_1, \dots, a_L, b_1, \dots, b_L, 1\}$ are not linearly independent over the rationals, the sets $A_{\geq n}$ constructed in Claim 4.4 may not be intervals. Therefore, Lemma 4.3 is not enough to guarantee the existence of a Riesz basis for $L^2(S)$. This motivates the definition of auxiliary sets related to $A_{\geq n}$ whose structure can be precisely determined. The construction of these sets is introduced in Claim 4.5.

Claim 4.5. *Let L be a positive integer and $a_1 \leq \dots \leq a_L$ and $b_1 \leq \dots \leq b_L$ be $2L$ numbers in $[0, 1]$. Let*

$$\psi(t) = \sum_{i=1}^L \mathbb{1}_{[0, b_i]}(t) + \sum_{i=1}^L \mathbb{1}_{[a_i, 1]}(t)$$

and

$$C_{\geq n} = \psi^{-1}[n, 2L] \quad 1 \leq n \leq 2L$$

Then the sets $C_{\geq n}$ are all unions of at most L intervals considered cyclically; that is, by imposing an equivalence relation between 0 and 1. Moreover, if there exists some $n \in \{1, \dots, 2L\}$ such that $C_{\geq n}$ is a union of exactly L intervals, then the sequences $\{a_i\}_{1 \leq i \leq L}$ and $\{b_i\}_{1 \leq i \leq L}$ interlace, i.e. either $a_1 \leq b_1 \leq a_2 \dots$ or $b_1 \leq a_1 \leq b_2 \dots$

Proof. By construction, ψ behaves as step function whose value increments exclusively at the points $\{a_i\}_{1 \leq i \leq L}$ and decrements only at $\{b_i\}_{1 \leq i \leq L}$. Therefore, each set $C_{\geq n}$ decomposes into a union of cyclic intervals of the form $[a_i, b_j]$ for some choice of indices $1 \leq i, j \leq L$. Furthermore, given that there are only $2L$ possible endpoints paired without repetition, each set $C_{\geq n}$ consists of at most L intervals.

We proceed by contradiction for the second part of the proof. Suppose that the sequences $\{a_i\}_{1 \leq i \leq L}$ and $\{b_i\}_{1 \leq i \leq L}$ do not interlace. By definition, this implies that, for some $1 \leq l \leq L$, there is no b_i in the interval $[a_l, a_{l+1}]$. Since every interval is of the form $[a_i, b_j]$ for some choice of $1 \leq i, j \leq L$, it follows that any

set $C_{\geq n}$ containing an interval with end-point at a_l , does not include an interval with endpoint at a_{l+1} . Consequently, the maximum number of disjoint intervals forming each $C_{\geq n}$ is $L - 1$. $\square$

Notice that Claim 4.5 implies that if the sequences $\{a_i\}_{1 \leq i \leq L}$ and $\{b_i\}_{1 \leq i \leq L}$ do not interlace, then each $C_{\geq n}$ is the union of at most $L - 1$ intervals. This property will be essential to prove Theorem 4.1 by induction over the number of intervals L . Hence, let us study the property of interlacement in the interval $[0, 1]$ considered cyclically.

Definition 4.6. Let $a_1, \dots, a_L, b_1, \dots, b_L \in [0, 1]$ and let $\{\cdot\}$ denote the fractional part of a number. The sequences $\{a_k\}_{1 \leq k \leq L}$ and $\{b_k\}_{1 \leq k \leq L}$ do interlace cyclically if for some permutations τ and σ of the set $\{1, \dots, L\}$ and for some $x \in [0, 1]$, we obtain

$$\{a_{\sigma(1)} + x\} \leq \{b_{\tau(1)} + x\} \leq \{a_{\sigma(2)} + x\} \leq \dots \leq \{b_{\tau(L)} + x\}$$

We introduce the displacement x to unify all cyclic configurations into a single representation, forcing the sequence to always begin with an element a .

The following lemma provides an auxiliary result regarding interlacement that we will use in the proof of Theorem 4.1. The proof of Lemma 4.7 is due to Fedor Nazarov.

Lemma 4.7. Let $L \in \mathbb{N}$ and let $a_1, \dots, a_L, b_1, \dots, b_L \in \mathbb{R}$ be all different. Then there exist infinitely many $N \in \mathbb{N}$ such that $\{Na_i\}_{1 \leq i \leq L}$ and $\{Nb_i\}_{1 \leq i \leq L}$ do not interlace.

Proof. We argue by contradiction. Suppose that $\{Na_j\}_{1 \leq j \leq L}$ and $\{Nb_j\}_{1 \leq j \leq L}$ interlace for all but finitely many $N \in \mathbb{N}$. We will reach a contradiction by estimating the following quantities:

$$s_N = \left| \sum_{j=1}^L e(Na_j) - \sum_{j=1}^L e(Nb_j) \right|^2$$

$$S_K = \sum_{N=1}^K s_N$$

We shall first estimate the size of S_K . In order to avoid cumbersome notation, let us define ξ_j and ε_j by

$$\xi_j = \begin{cases} e(a_j) & j \in \{1, \dots, L\} \\ e(b_{j-L}) & j \in \{L+1, \dots, 2L\} \end{cases}$$

$$\varepsilon_j = \begin{cases} 1 & j \in \{1, \dots, L\} \\ -1 & j \in \{L+1, \dots, 2L\} \end{cases}$$

Expanding the squared norm in S_K , we obtain

$$S_K = \sum_{j=1}^{2L} \sum_{k=1}^{2L} \varepsilon_j \varepsilon_k \sum_{N=1}^K \zeta_j^N \overline{\zeta_k^N}$$

where the terms with $\varepsilon_j \varepsilon_k = 1$ correspond to $|\sum_{j=1}^L e(Na_j)|^2$ and $|\sum_{j=1}^L e(Nb_j)|^2$ while those with $\varepsilon_j \varepsilon_k = -1$ are the cross terms.

When $j = k$, the product $\zeta_j^N \overline{\zeta_k^N}$ is the squared norm of a complex exponential and therefore equal to 1. In contrast, the sum over $\zeta_j^N \overline{\zeta_k^N}$ when $j \neq k$ can be treated as a geometric progression. It follows that

$$S_K = 2LK + \sum_{j \neq k} \varepsilon_j \varepsilon_k \frac{\zeta_j \overline{\zeta_k} - (\zeta_j \overline{\zeta_k})^{K+1}}{1 - \zeta_j \overline{\zeta_k}}$$

Let I denote the sum over the indices $j \neq k$ in the previous expression for S_K . Since $|\zeta_j \overline{\zeta_k}| \neq 1$, there exists for each $j \neq k$ a positive number δ_{jk} such that $|1 - \zeta_j \overline{\zeta_k}| > \delta_{jk}$. Taking $\delta = \min_{j \neq k} \delta_{jk}$, then

$$|I| \leq \frac{|\zeta_j \overline{\zeta_k}| + |(\zeta_j \overline{\zeta_k})^{K+1}|}{\delta} \leq \frac{2}{\delta}$$

Therefore,

$$S_K = 2LK + O(1)$$

where $O(1)$ depends on L and on ζ_j , but is independent of K .

Turning now to s_N , we first express s_N as $|\alpha|^2$, where $\alpha := \sum_{j=1}^L e(Na_j) - \sum_{j=1}^L e(Nb_j)$. We assume that $\alpha \neq 0$ since otherwise the result is trivial. Define P as a function from $\mathbb{C}$ to $\mathbb{R}$ such that $Px = \langle x, \alpha/|\alpha| \rangle_{\mathbb{R}^2}$. Geometrically, P is the projection on the line $\alpha\mathbb{R}$ rotated to $\mathbb{R}$. Since P is linear and $P\alpha = |\alpha|$, we can express s_N as

$$s_N = |P(\alpha)|^2 = \left| \sum_{j=1}^L P(e(Na_j)) - \sum_{j=1}^L P(e(Nb_j)) \right|^2$$

Now, order the collection of points $\{e(Na_i)\} \cup \{e(Nb_j)\}$ for $1 \leq i, j \leq L$ counterclockwise, starting from $\alpha/|\alpha|$, and denote this sequence as $\{d_j\}_{1 \leq j \leq 2L}$. Notice

that $e(Na_j) = e(\{Na_j\})$ and $e(Nb_j) = e(\{Nb_j\})$ due to the periodicity of the complex exponential. Hence, the assumption of interlacement guarantees that either $d_{2j} \in \{e(Na_k)\}_k$ and $d_{2j+1} \in \{e(Nb_k)\}_k$ or vice-versa. In both cases, it follows that

$$s_N = \left| \sum_{j=1}^L P(d_{2j-1}) - P(d_{2j}) \right|^2$$

By construction, the sequence $\{d_j\}_j$ rotates counterclockwise in the unit circle. Thus, the differences $P(d_{2j-1}) - P(d_{2j})$ are positive up to a certain index j_0 . This follows from the fact that, as one moves counterclockwise along the unit circle starting from $\alpha/|\alpha|$, the projection decreases until it reaches its minimum at the point $-\alpha/|\alpha|$ and then begins to increase as the rotation continues. Hence

$$\left| \sum_{j=1}^L P(d_{2j-1}) - P(d_{2j}) \right| \leq \max \left\{ \sum_{j=1}^{j_0} P(d_{2j-1}) - P(d_{2j}), \sum_{j=j_0+1}^L P(d_{2j}) - P(d_{2j-1}) \right\}$$

Moreover, the intervals $[P(d_{2j-1}), P(d_{2j})]$ and $[P(d_{2j}), P(d_{2j-1})]$ are disjoint for $j \leq j_0$ and for $j_0 < j$ respectively. Therefore, the sum of the lengths of these intervals in each regime ($j \leq j_0$ or $j_0 < j$) is bounded by the diameter of the unit circle, which is equal to 2. Then, if the sequences $\{\{Na_i\}\}_i$ and $\{\{Nb_i\}\}_i$ interlace, the quantity s_N is bounded by 4.

Altogether, it follows that for all $K \in \mathbb{N}$

$$2LK + O(1) = S_K = \sum_{N=1}^K s_N \leq 4K + O(1)$$

where the $O(1)$ term on the right-hand side is due to the finite set of indices N for which the sequences $\{\{Na_i\}\}_i$ and $\{\{Nb_i\}\}_i$ do not interlace.

This leads to a contradiction for $L \geq 3$. Indeed, if the sequences $\{\{Na_i\}\}_i$ and $\{\{Nb_i\}\}_i$ interlace for all but finitely many $N \in \mathbb{N}$, the previous inequality is no longer satisfied as $K \rightarrow \infty$.

Consider now the case $L = 2$. Under this assumption, we need to strengthen the bound for s_N . Fix an arbitrary small $\epsilon > 0$. The original proof of this case uses the Dirichlet approximation theorem to guarantee that for any integer N there exists an integer n in the range $N \leq n \leq N + \lceil \epsilon^{-4} \rceil$ such that the values na_1, na_2, nb_1 and nb_2 are all within distance less or equal than ϵ from the nearest integer. (See [8] and [9])

Our approach relaxes this requirement on the frequency of such integers n by replacing $\lceil \epsilon^{-4} \rceil$ with an integer Q . In other words, we only demand that for some Q and for any integer N there exists a suitable n in the range $N \leq n \leq N + Q$. This modification does not alter the rest of the proof, which is adapted from the proof by Nazarov.

Let $||x||$ denote the distance of x to the nearest integer and let

$$U = \{x \in [0, 1]^4 : \forall i \in \{1, \dots, 4\}, ||x_i|| < \epsilon\}$$

Consider also the translational map T defined by $T(x) = \{x + \alpha\}$, where $\alpha = (a_1, a_2, b_1, b_2)$.

On the one hand, if $a_1, a_2, b_1, b_2, 1$ are linearly independent over the rationals, by the multidimensional Weyl's equidistribution Theorem (Theorem A.2), the orbit of T is dense in $[0, 1]^4$. Consequently, for any $x \in [0, 1]^4$, there exists an integer $k \geq 1$ such that $T^k(x) \in U$. This implies that $[0, 1]^4 = \bigcup_{k=1}^{\infty} T^{-k}(U)$. Since $[0, 1]^4$ is compact, this cover admits a finite subcover for $[0, 1]^4$, i.e, there exists an integer Q such that $[0, 1]^4 = \bigcup_{k=1}^Q T^{-k}(U)$. Therefore, for any integer N , the point $\{N\alpha\}$ must belongs to $T^{-k}(U)$ for some $1 \leq k \leq Q$, which implies that $T^k(N\alpha) = \{(N+k)\alpha\} \in U$. Since k depends on N , we denote it as k_N . It is then clear that for any N , the integer $n = N + k_N$ satisfies $N \leq n \leq N + Q$ and guarantees that the values na_1, na_2, nb_1 and nb_2 are all within distance less or equal than ϵ from the nearest integer.

On the other hand, the case where $a_1, a_2, b_1, b_2, 1$ are linearly dependent over the rationals is analogous but restricting the argument to the proper submanifold of $[0, 1]^4$ where the sequence $(\{ka_1\}, \{ka_2\}, \{kb_1\}, \{kb_2\})_{k \in \mathbb{N}}$ is dense. More precisely, if there exist $c_1, c_2, c_3, c_4, c_5 \in \mathbb{Q}$ such that $c_1a_1 + c_2a_2 + c_3b_1 + c_4b_2 = c_5$, we can apply the same procedure used in the linearly independent case to the subspace defined by the relation $mc_1x_1 + mc_2x_2 + mc_3x_3 + mc_4x_4 = 0 \pmod{1}$, where m is the minimum common multiple of c_1, c_2, c_3, c_4, c_5 and $(x_1, x_2, x_3, x_4) \in [0, 1]^4$. Since this subspace contains a neighborhood of the origin and it is compact, we can similarly deduce the existence of integers n that satisfy the desired property.

Let us now consider these n 's. Since $|e^{2\pi i s} - 1| = 2|\sin \pi s| \leq 2\pi \text{dist}(s, \mathbb{Z})$, it follows that the quantities s_n verifies:

$$\begin{aligned} s_n^{1/2} &= |e(na_1) + e(na_2) - e(nb_1) - e(nb_2)| \leq \\ &\leq |e(na_1) - 1| + |e(na_2) - 1| + |1 - e(nb_1)| + |1 - e(nb_2)| \leq C\epsilon \end{aligned}$$

Choosing ϵ small enough such that $C\epsilon \leq 1$, we ensure that for all $K \geq 1$ there is a proportion of indices $N \in \{1, \dots, K\}$, say it $c_1 K$, for which $s_N \leq 1$. Furthermore, if we have interlacement for all but finitely many $N \leq K$, the quantities s_N with interlacement are bounded by 4, as previously seen. Taken together, we obtain

$$S_K = \sum_{N=1}^K s_N \leq 4 \cdot (1 - c_1)K + 1 \cdot c_1 K + O(1) \leq (4 - c_2)K + O(1) \quad (4.2)$$

where $O(1)$ takes into account the finite set of indices for which there are no interlacement.

Finally, as $K \rightarrow \infty$, Equation 4.2 contradicts the fact that $S_K = 2LK + O(1)$, as desired. □

4.2.2 Existence of Riesz bases for finite union of intervals

We end this work by proving Theorem 4.1, which guarantees the existence of a Riesz basis of exponential functions for a finite union of intervals in the general case.

Proof. It suffices to show that for any $S \subset [0, 1]$, there exists a Riesz basis for $L^2(S)$ of the form $\{e^{2\pi i n t}\}_{n \in \Lambda}$ with $\Lambda \subset \mathbb{Z}$. Indeed, any set $S \subset \mathbb{R}$ formed by a finite union of intervals can be mapped to a subset $S' \subset [0, 1]$ through a suitable translation and rescaling. Since the property of being a Riesz basis is invariant under such transformations, the existence of a Riesz basis of the form $\{e^{2\pi i n t}\}_{n \in \Lambda'}$ with $\Lambda' \subset \mathbb{Z}$ for $L^2(S')$ guarantees the existence of a Riesz basis of the form $\{e^{2\pi i n t}\}_{n \in \Lambda}$ with $\Lambda \subset \mathbb{R}$ for $L^2(S)$, as required.

Therefore, we may assume without loss of generality that $S \subset [0, 1]$. We proceed by induction on L , where L is the number of intervals in the union S . The case $L = 1$ is given by Lemma 4.3. Suppose now that the induction hypothesis is true for any union of fewer than L intervals, i.e, assume that all unions of less than L intervals have a Riesz basis of the form $E(\Lambda)$ with $\Lambda \subset \mathbb{Z}$. We shall prove that the assumption still holds for a union of L intervals.

Denote $S = \bigcup_{i=1}^L [a_i, b_i]$. By Lemma 4.7, there are infinitely many $N \in \mathbb{N}$ such that the sequences $\{Na_i\}_{1 \leq i \leq L}$ and $\{Nb_i\}_{1 \leq i \leq L}$ do not interlace. Take N large enough that every interval in S contains at least one element of the form $\frac{k}{N}$ for

some $k \in \{0, \dots, N\}$. It follows that each interval $[a_i, b_i]$ can be partitioned into translations of $[\{Na_i\}/N, 1/N]$, $[0, \{Nb_i\}/N]$ and $[0, 1/N]$. This decomposition becomes clear by expressing $[a_i, b_i]$ as

$$\left[\frac{\lfloor Na_i \rfloor}{N} + \frac{\{Na_i\}}{N}, \frac{\lfloor Nb_i \rfloor}{N} + \frac{\{Nb_i\}}{N} \right]$$

To apply the induction hypothesis, we require a set with fewer than L intervals. Since the sequences $\{a'_i\}_{1 \leq i \leq L}$ and $\{b'_i\}_{1 \leq i \leq L}$, where

$$a'_i = \{Na_i\} \quad b'_i = \{Nb_i\},$$

do not interlace, Claim 4.5 applied to these values ensures that the sets $C_{\geq m} = \psi^{-1}[m, 2L]$ with $m \in \{0, \dots, 2L\}$ and

$$\psi(t) = \sum_{i=1}^L \mathbb{1}_{[0, b'_i]}(t) + \sum_{i=1}^L \mathbb{1}_{[a'_i, 1]}(t)$$

have at most $L - 1$ intervals.

Examine now the relation between $C_{\geq m}$ and the sets

$$A_{\geq n} = \{t \in [0, 1/N] : \phi(t) \geq n\}$$

for $n \in \{1, \dots, N\}$ and

$$\phi(t) = \#\{l \in \mathbb{N} : 0 \leq l \leq N - 1, t + l/N \in S\}$$

As noted in Claim 4.4, ϕ increases at every $\{Na_i\}/N$ and decreases at every $\{Nb_i\}/N$. Therefore, each $A_{\geq n}$ is either a finite union of intervals of the form $[\{Na_i\}/N, \{Nb_j\}/N]$, equal to $[0, 1/N]$ or empty. Similarly, as deduced in Claim 4.5, each $C_{\geq n}$ is either $[0, 1]$ or a union of cyclic intervals of the form $[a'_i, b'_j] = [\{Na_i\}, \{Nb_j\}]$, for some $1 \leq i, j \leq L$.

Furthermore, for a specific ordering of the sequences $\{a'_i\}_i$ and $\{b'_i\}_i$, the jumps of ϕ and ψ occur in the same order and with the same location up to a factor $1/N$, i.e. $\{Na_i\}/N$ for ϕ while $\{Na_i\}$ for ψ and $\{Nb_i\}/N$ for ϕ while $\{Nb_i\}$ for ψ . Hence, both functions exhibit the same growth and decay patterns and, as a consequence, each $A_{\geq n}$ is equal to $C_{\geq m}/N$ for some $m \in \{1, \dots, 2L\}$ not necessarily equal to n because of the pieces $[0, 1/N]$.

Since each $C_{\geq m}$ consists of at most $L - 1$ intervals, the inductive hypothesis ensures the existence of an exponential Riesz basis $E(\Lambda')$ with $\Lambda' \subset \mathbb{Z}$ for each $L^2(C_{\geq m})$. Moreover, the fact that each $A_{\geq n}$ is equal to some $C_{\geq m}$ rescaled by $1/N$ implies that each $L^2(A_{\geq n})$ possesses a Riesz basis of exponential functions with frequencies in $N\mathbb{Z}$. Finally, Lemma 3.3 completes the induction. □

Appendix A

Uniform distribution of real sequences

Let $\alpha \in \mathbb{R}$ be an irrational number. Weyl's equidistribution Theorem ensures that the sequence of fractional parts $\{k\alpha\}_{k \in \mathbb{N}}$ is uniformly distributed in $[0, 1]$:

Theorem A.1. (Weyl's equidistribution theorem) *If α is irrational, then the sequence of fractional parts $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \dots$ satisfies*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \# \{1 \leq k \leq n : \{k\alpha\} \in [a, b]\} = b - a$$

for each pair a, b such that $[a, b] \subset [0, 1]$

Proof. Cf. [10]

□

Weyl's equidistribution Theorem generalizes to N dimensions as follows:

Theorem A.2. *Let $N \in \mathbb{N}$. If the real numbers $\alpha_1, \alpha_2, \dots, \alpha_N, 1$ are linearly independent over the rationals, then the sequence $(\{k\alpha_1\}, \{k\alpha_2\}, \dots, \{k\alpha_N\})_{k \in \mathbb{N}}$ is uniformly distributed in $[0, 1]^N$, i.e.,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \# \{1 \leq k \leq n : (\{k\alpha_1\}, \{k\alpha_2\}, \dots, \{k\alpha_N\}) \in \prod_{j=1}^N [a_j, b_j]\} = \prod_{j=1}^N (b_j - a_j)$$

for all intervals $\prod_{j=1}^N [a_j, b_j] \subset \mathbb{R}^N$

Proof. See [11]

□

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