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Decomposability Bundle of Measures and Differentiability of Lipschitz Functions

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Chapter 1

Introduction

In this thesis, we study a result on the differentiability of Lipschitz functions with respect to general measures, established by Alberti and Marchese in [1]. According to Rademacher's classical theorem, every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable almost everywhere, where the notion of "almost everywhere" is taken with respect to the Lebesgue measure $\mathscr{L}^n$. This result extends naturally to any measure that is absolutely continuous with respect to $\mathscr{L}^n$.

However, Rademacher's theorem does not immediately generalize to arbitrary Borel measures. Theorem 1.1 in [1] addresses this gap by identifying, for an arbitrary finite Borel measure μ, the largest set of directions at each point in $\mathbb{R}^n$ along which a Lipschitz function is differentiable at μ-almost every point. Specifically, for μ-almost every point x, there exists a linear subspace $V(\mu, x) \subseteq \mathbb{R}^n$ such that every Lipschitz function is differentiable at x in all directions of $V(\mu, x)$.

The aim of this thesis is to develop the mathematical framework necessary to understand and prove the first part of Theorem 1.1 in [1]. To this end, we build up the required background in measure theory, geometric measure theory, and the theory of rectifiable sets and measures, gradually leading to the final proof.

Structure of the Thesis

- **Chapter 2** contains a review of notation and preliminary concepts. Section 2.1 introduces standard notation used throughout the thesis. Section 2.2.1 covers basic notions and properties of measures, followed by Section 2.2.2, which introduces Hausdorff measures and their properties. These sections are primarily based on [6]. In Section 2.2.3, we introduce the Grassmannian manifold $\mathrm{Gr}(\mathbb{R}^n, k)$, along with essential geometric properties that will be used in later chapters.

- **Chapter 3** is dedicated to Lipschitz functions. In Section 3.1, we define Lipschitz maps and discuss a connection with Hausdorff measuresas well as an extension theorem. Section 3.2 presents Rademacher's theorem and an approximation result to Lipschitz functions by $\mathcal{C}^1$ functions outside a set of small measure. This approximation is instrumental in understanding rectifiability in Chapter 4. The main reference for these sections is [6]. Finally, Section 3.3 introduces the area formula and demonstrates its application in computing surface measures of graphs of Lipschitz functions, following [5].

- **Chapter 4** focuses on rectifiability and tangent measures. In Section 4.1, we define approximate tangent planes, k-rectifiable sets, and k-rectifiable measures. We characterize such sets as those that can be covered, up to $\mathscr{H}^k$-null sets, by countably many $\mathcal{C}^1$ graphs. This section concludes with a criterion for rectifiability, supported by a geometric lemma relating cones and Lipschitz functions. Section 4.2 introduces the concept of blow-ups of measures and defines tangent measures. We explore the relation between densities of Radon measures, tangent measures, and uniform measures. The we prove a characterization of rectifiable measures in terms of their tangent measures. In particular, we prove that for a rectifiable measure, tangent measures exist and are unique at almost every point, and conversely, that a Radon measure whose tangent measures are multiples of $\mathscr{H}^k$ restricted to linear subspaces is rectifiable. This chapter follows [5].

- **Chapter 5** is dedicated to the proof of Theorem 1.1 from [1]. In Section 5.1, we introduce the notion of the *decomposability bundle* of a finite measure μ, which at almost every point gives the minimal subspace that contains the directions along which μ can be decomposed into 1-rectifiable components. Section 5.2 introduces the *differentiability bundle* of a Lipschitz function, which assigns to each point the collection of subspaces in which the function is differentiable. Finally, in Section 5.3, we prove that the differentiability bundle of μ is contained in the set of largest subspaces in which Lipschitz functions are differentiable — completing the proof of the first part of Theorem 1.1.

This thesis is intended to serve as both an exposition of the tools needed to understand Alberti and Marchese's result and a detailed presentation of the proof itself. We hope it offers a clear path through the combination of analysis, geometry, and measure theory that in which the results on differentiability of Lipschitz functions with respect to general measures relies on.

Chapter 2

Preliminaries

2.1 Notation

This section defines the symbols and conventions used throughout the thesis, particularly those related to measures and function spaces. We denote by

$\mathscr{L}^n$ the Lebesgue measure in $\mathbb{R}^n$.

ω_k the Lebesgue measure of the unit ball in $\mathbb{R}^k$, i.e. $\mathscr{L}^k\big(B(\mathbf{0},1)\big) = \omega_k$, where $B(\mathbf{0},1)$ represents the open ball centered at $\mathbf{0}$ and radius 1 in $\mathbb{R}^n$.

$df_{\mathbf{x}}$ differential of a map f on $\mathbb{R}^n$ at $\mathbf{x} \in \mathbb{R}^n$.

JA the Jacobian determinant of a matrix $A \in M_{mxk}(\mathbb{R})$, i.e. $JA = \sqrt{\det\big(A^t A\big)}$.

$d(A,B)$ the distance between two sets $A, B \subseteq \mathbb{R}^n$, i.e.

$$d(A,B) = \inf\Big\{ \|\mathbf{x} - \mathbf{y}\| \,:\, \mathbf{x}, A,\, \mathbf{x} \in B\Big\},$$

where $\|\mathbf{x}\| = \|\mathbf{x}\|_2$ refers to the euclidean norm in $\mathbb{R}^n$.

2^X the class of all subsets of a set X.

$A \setminus B$ the diffference of two sets A, B, i.e.

$$A \setminus B = \{x \in A : x \notin B\}.$$

$\mathbb{R}^+$ the set of all positive real numbers.

A^c the complement of a set $A \subseteq X$, i.e.

$$A^c = X \setminus A.$$

$f\#\mu(A)$ the push-forward of the measure μ induced by f, where μ is a measure on X and $f : X \longrightarrow Y$ a map, i.e., for each $A \subseteq Y$,

$$f\#\mu(A) = \mu\big(f^{-1}(A)\big).$$

$L^1(\mathbb{R}^n, \mu)$ the class of functions f such that $\int |f| d\mu < \infty$, where μ is a Radon measure on $\mathbb{R}^n$.

$L^1_{loc}(\mathbb{R}^n, \mu)$ the class of functions f such that $\int_K |f| d\mu < \infty$, for each compact set $K \subseteq \mathbb{R}^n$, where μ is a Radon measure on $\mathbb{R}^n$.

$L^1(\mathbb{R}^n)$ the class of functions f such that $\int |f| d\mathscr{L}^n < \infty$.

$L^1_{loc}(\mathbb{R}^n, \mu)$ the class of functions f such that $\int_K |f| d\mathscr{L}^n < \infty$, for each compact set $K \subseteq \mathbb{R}^n$.

$A + B$ the sum of two sets $A, B \subseteq X$ in a vector space X, i.e.

$$A + B = \big\{x + y : x \in A,\, y \in B\big\}.$$

2.2 Preliminaries

In the present section we collect the core ideas from measure theory and geometry that support the later chapters.

2.2.1 Measure Theory

We introduce the notion of Outer Measure, Measurable set, Borel Measure and Borel Regular Measure.

Definition 2.2.1 (Outer Measure). Let X be a set, we say that a mapping $\mu : 2^X \longrightarrow [0, \infty]$ is an outer measure if

(1) $\mu(\emptyset) = 0$.

(2) (Subadditivity) If $A, A_i \in 2^X$, for $i \in \mathbb{N}$ such that $A \subseteq \cup_{i=1}^{\infty} A_i$, then

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

Definition 2.2.2. Let μ be an outer measure on X and $A \subseteq X$. We denote by $\mu \llcorner A$ to the outer measure defined, for each $C \subseteq X$, as

$$\mu \llcorner E(C) = \mu(C \cap E)$$

Definition 2.2.3 (Measurable Set). Let μ be an outer measure on a set X and $A \subseteq X$. We say that A is μ-measurable if, for any $C \subseteq X$, we have that

$$\mu(C) = \mu(C \setminus A) + \mu(C \cap A).$$

Remark 2.2.1. Let μ be an outer measure on a set X and A be a μ-measurable set. Notice that, by definition of outer measure if $A \subseteq C \subseteq X$, then $\mu(A) \leq \mu(C)$. Hence, the assumption that Definition 2.2.3 introduces is that, for all $C \subseteq X$

$$\mu(C) \geq \mu(C \setminus A) + \mu(C \cap A)$$

For further details on the properties of measurable sets, see 1.1. Theorem 1 [6].

In some texts as [6] or [9], the Definition 2.2.1 corresponds to the definition of measure. We follow a different approach, by calling measure to those outer measures restricted to the class of their measurable sets. The class of measurable sets fits into a particular type of structure, the σ-algebra, which we introduce next.

Definition 2.2.4. Let X be a set, we say that $\mathcal{A} \subseteq 2^X$ is a σ-algebra if the following three conditions hold:

(1) $\emptyset, X \in \mathcal{A}$.

(2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

(3) If $A_k \in \mathcal{A}$ for $k \in \mathbb{N}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Remark 2.2.2. Let μ be a measure on X, then the collection of all μ-measurable sets is a σ-algebra.

Definition 2.2.5 (Borel σ-algebra). Let (X, τ) be a topological space. The Borel σ-algebra of (X, τ) is the smallest σ-algebra of X containing all open sets of X, we denote it by $\beta(X)$. Let $A \in \beta(X)$, we say that A is a Borel set in X or just a Borel set.

We introduce now the the notion of Borel measure, Regular Outer Measure, Borel regular measure, Radon measure.

Definition 2.2.6 (Regular Measure). Let μ be an outer measure on a set X. We say that μ is a regular if for each $A \subseteq X$ there exists a μ-measurable set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

Definition 2.2.7 (Borel Measure). Let μ be an outer measure on X and (X, τ) a topological space. We say that μ is a Borel measure if every Borel set is μ-measurable.

Definition 2.2.8 (σ-finite Measure in $\mathbb{R}^n$). Let μ be a Borel measure. Then, we say that μ is σ-finite if, for any $K \subseteq \mathbb{R}^n$ a compact set, $\mu(K) < \infty$.

Definition 2.2.9 (Borel Regular Measure). Let (X, τ) be a topological space and μ a Borel measure on X. We say that μ is Borel regular if for each $A \subseteq X$, there exists some $\tilde{A}$ Borel set such that $A \subseteq \tilde{A}$ and $\mu(A) = \mu(\tilde{A})$.

Definition 2.2.10 (Radon Measure). Let (X, τ) be a topological space and μ a Borel regular measure on X. We say that μ is Radon if, for any $K \subseteq X$ compact, we have that $\mu(K) < \infty$.

Definition 2.2.11 (Support of a Measure). Let μ be a Borel regular measure on a topological space (X, τ). We define the support of μ by

$$\operatorname{supp}(\mu) := \big\{ \mathbf{x} \in X \,:\, \mu(U) > 0 \text{ for all } \mathbf{x} \in U \big\}.$$

From now on we focus on measures and outer measures defined on R^n.

Lemma 2.2.12 (1.1 Theorem 3 [6])**.** Let μ be a Borel regular measure on $\mathbb{R}^n$. Suppose that E is a μ-measurable set such that $\mu(E) < \infty$. Then $\mu \llcorner E$ is a Radon measure.

Proof. We call $\nu := \mu \llcorner E$, it is clear that ν is finite over compacts, since $\nu(\mathbb{R}^n) = \mu(E) < \infty$. We prove now that ν is a Borel regular measure. Notice that we can assume E to be Borel. Indeed, there exists $\tilde{E}$ Borel such that $E \subseteq \tilde{E}$ and $\mu(E) = \mu(\tilde{E})$. We have to prove that $\mu \llcorner E = \mu \llcorner \tilde{E}$. Notice that E is measurable, so we obtain that $\mu(\tilde{E} \setminus E) = \mu(\tilde{E}) - \mu(E) = 0$.

Hence, for any set $C \subseteq \mathbb{R}^n$,

$$\begin{aligned}\mu \llcorner \tilde{E}(C) &= \mu(C \cap \tilde{E}) \\ &= \mu\big((C \cap \tilde{E}) \setminus E\big) + \mu(C \cap \tilde{E} \cap E) \\ &\leq \mu(\tilde{E} \setminus E) + \mu(C \cap E) \\ &= \mu \llcorner E(C).\end{aligned}$$

Then, assuming E to be a Borel set, we have to prove that, for any $C \subseteq \mathbb{R}^n$, there exists some Borel set $\tilde{C}$ such that $C \subseteq \tilde{C}$ and $\nu(C) = \nu(\tilde{C})$. Since μ is Borel regular, there exists D Borel such that $C \cap E \subseteq D$ and $\mu(C \cap E) = \mu(D)$. We check on $\tilde{C} := D \cup E^c$. First, $C = (E \cap C) \cup (E^c \cap C) \subseteq D \cup E^c = \tilde{C}$. Finally, we have that

$$\nu(\tilde{C}) = \mu(\tilde{C} \cap E) = \mu(D \cap E) \leq \mu(D) = \mu(C \cap E) = \nu(C). \quad \square$$

We claim two important results for Borel regular and Radon measures.

Lemma 2.2.13 (1.1 Lemma 1 [6])**.** Let μ be a Borel measure on $\mathbb{R}^n$ and let $E \subseteq \mathbb{R}^n$ be a Borel set. Then

(1) If $\mu(E) < \infty$, for each $\epsilon > 0$ there exists C_ϵ a closed set such that $C_\epsilon \subseteq E$ and $\mu(E \setminus C_\epsilon) \leq \epsilon$.

(2) If μ is a Radon measure, for each $\epsilon > 0$ there exists an open set U_ϵ such that $E \subseteq U_\epsilon$ and $\mu(U_\epsilon \setminus E)$.

Lemma 2.2.14 (1.1. Theorem 4 [6])**.** Let μ be a Radon measure on $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$. Then

(1)
$$\mu(A) = \inf\big\{\mu(U) : A \subseteq U,\, U \text{ an open set}\big\}.$$

(2) If A is μ-measurable, then
$$\mu(A) = \sup\big\{\mu(U) : C \subseteq A,\, C \text{ a closed set}\big\}.$$

We introduce now the notion of measurable function.

Definition 2.2.15 (**Measurable Function**)**.** Let (X, τ_1) and (Y, τ_2) two topological spaces, let μ be an outer measure on X and $f : X \longrightarrow Y$ a map. We say that f is

μ-measurable if, for each $U \subseteq Y$ open set, $f^{-1}(U)$ is μ-measurable.

For further details on measurable functions and details on the integral we refer to the first chapter of [6].

Definition 2.2.16 (k-density of a Set at a Point). Let μ be a Radon measure on $\mathbb{R}^n$, $k \geq 0$ a real number and $\mathbf{x} \in \mathbb{R}^n$. Then we define

- The **upper k-density** of μ at $\mathbf{x} \in \mathbb{R}^n$ by
$$\Theta^{*k}(\mu, \mathbf{x}) \equiv \limsup_{r \searrow 0} \frac{\mu\big(B(\mathbf{x}, r)\big)}{\omega_k r^k}.$$
- The **lower k-density** of μ at $\mathbf{x}$ by
$$\Theta^{k}_{*}(\mu, \mathbf{x}) \equiv \liminf_{r \searrow 0} \frac{\mu\big(B(\mathbf{x}, r)\big)}{\omega_k r^k}.$$
- When both $\Theta^k_*(\mu, \mathbf{x}) = \Theta^{*k}(\mu, \mathbf{x})$, we call this quantity the **k-density** of μ at $\mathbf{x}$ and we denote it by
$$\Theta^k(\mu, \mathbf{x}).$$

Defintion 2.2.16 and the following one are closely related to the notion of derivative among Radon measures and the Lebesgue Decomposition Theorem, for further details, see 1.6. and 1.7. in [6].

Definition 2.2.17 (Lebesgue Density Points). Let $A \subseteq \mathbb{R}^n$ and $\mathbf{x} \in A$. We say that $\mathbf{x}$ is a point of $\mathscr{L}^n$ density 1 if

$$\lim_{r \searrow 0} \frac{\mathscr{L}^n\big(B(\mathbf{x}, r) \cap A\big)}{\mathscr{L}^n\big(B(\mathbf{x}, r)\big)} = 1.$$

We say that $\mathbf{x}$ is a point of $\mathscr{L}^n$ density 0 if

$$\lim_{r \searrow 0} \frac{\mathscr{L}^n\big(B(\mathbf{x}, r) \cap A\big)}{\mathscr{L}^n\big(B(\mathbf{x}, r)\big)} = 0.$$

Lemma 2.2.18 (1.7. Corollary 3 [6])**.** Let $E \subseteq \mathbb{R}^n$ be a $\mathscr{L}^n$-measurable set. Then

$$\lim_{r \searrow 0} \frac{\mathscr{L}^n\big(B(\mathbf{x}, r) \cap A\big)}{\mathscr{L}^n\big(B(\mathbf{x}, r)\big)} = 1 \qquad \text{for } \mathscr{L}^n \text{ almost every } \mathbf{x} \in E.$$

and
$$\lim_{r\searrow 0}\frac{\mathscr{L}^n\big(B(\mathbf{x},r)\cap A\big)}{\mathscr{L}^n\big(B(\mathbf{x},r)\big)}=0 \qquad \text{for } \mathscr{L}^n \text{ almost every } \mathbf{x}\notin E.$$

2.2.2 Hausdorff Measures

Hausdorff measures generalize Lebesgue measures to non-integer and arbitrary dimensions. This section introduces their formal construction, motivation, and key properties. These measure will be key in the characterization of rectifiability in chapter 6 and the study of the decomposability bundle in chapter 7.

The main source for the following chapter is [6], chapter 2.

Definition 2.2.19 (**Hausdorff Measure**)**.** Let $A\subseteq\mathbb{R}^n$ be any set and let $0\le k<\infty$ and $0\le\delta\le\infty$ two real numbers. Using the following notation:

$$\mathscr{H}^k_\delta\big(A\big)\coloneqq\left\{\sum_{j=1}^{\infty}\omega_k\left(\frac{\operatorname{diam}(C_j)}{2}\right)^k \;\middle|\; A\subseteq\bigcup_{j=1}^{\infty}C_j,\ \operatorname{diam}(C_j)\le\delta\right\},$$

where

$$\omega_k\coloneqq\frac{\pi^{\frac{k}{2}}}{\Gamma\big(\frac{k}{2}+1\big)},$$

and Γ is the Gamma function. We define k**-dimensional Hausdorff measure on** $\mathbb{R}^n$ by

$$\mathscr{H}^k(A)\coloneqq\lim_{\delta\to 0}\mathscr{H}^k_\delta\big(A\big)=\sup_{\delta>0}H^k_\delta\big(A\big).$$

Remark 2.2.3. In Definition 2.2.19, the constant ω_k plays a normalization role, indeed, for $k\in\mathbb{N}$, $\omega_k=\mathscr{L}^k\big(B(\mathbf{0},1)\big)$, which is involved in the result Theorem 2.2.28. On the other hand, tit ensures that, given a k-dimensional linear plane $V\subseteq\mathbb{R}^n$, $\mathscr{H}^k\llcorner V\in\mathcal{U}^k(\mathbb{R}^n)$ (see Definition 4.2.3).

We prove now that Definition 2.2.19 leads to a Borel regular measure.

Lemma 2.2.20 (2.1. Theorem 1 [6])**.** Let $k\in[0,\infty)$. Then H^k_δ is an outer measure on $\mathbb{R}^n$.

Proof. It is clear that $\mathscr{H}^k_\delta$ is valued in $[0,\infty)$ and that $\mathscr{H}^k_\delta(\emptyset)=0$. Let us prove that it is σ-additive:

Let $\big\{A_i\big\}_{i=1}^{\infty}$ be a family of sets in $\mathbb{R}^n$. Consider now coverings for each of the elements of

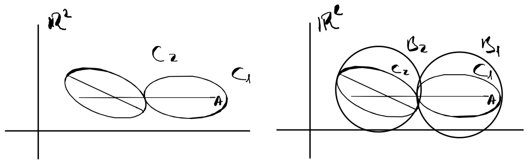

Figure 2.1: Example of Cover for a set A

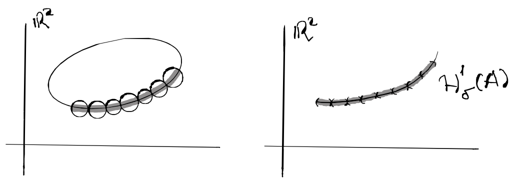

Figure 2.2: $\mathscr{H}^1_\delta$ of a segment in $\mathbb{R}^2$

the family so that $A_i \subseteq \cup_{j=1}^{\infty} C_{ij}$ such that $\operatorname{diam}(C_{ij}) \leq \delta$ for all $i, j \in \mathbb{N}$. Then the family $\{C_i\}_{i,\in\mathbb{N}}$ is a covering of the set $\cup_{i=1}^{\infty} A_i$ with diameter less or equal than δ. Hence

$$\mathscr{H}^k_\delta\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\omega_k}{2^k} \operatorname{diam}\big(C_{ij}\big)^k.$$

We take now the infima over all coverings of the elements of the family, concluding then that

$$\mathscr{H}^k_\delta\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathscr{H}^k_\delta\big(A_i\big). \qquad \square$$

Lemma 2.2.21 (2.1. Theorem 1 [6])**.** Let $k \in [0, \infty)$. Then $\mathscr{H}^k$ is an outer measure on $\mathbb{R}^n$.

Proof. Since for any $\delta > 0$, $\mathscr{H}_\delta^k$ is an measure, then it is clear that $\mathscr{H}^k$ verifies that $\mathscr{H}^k(\emptyset) = 0$ and $\mathscr{H}^k(A) \geq 0$ for any $A \subseteq \mathbb{R}^n$. Let us focus on the σ-additivity, shich also leverages on the σ-additivity of $\mathscr{H}_\delta^k$. Notice that for every $\gamma > \delta > 0$ the converings involved in the evaluation of a set A by $\mathscr{H}_\gamma^k$ are also among the available ones to evaluate $\mathscr{H}_\delta^k$ and hence $\mathscr{H}_\gamma^k(A) \geq \mathscr{H}_\delta^k(A)$ and $\mathscr{H}_\delta^k(A) \geq \mathscr{H}^k(A)$. Thus, for any family of sets in $\mathbb{R}^n$, $\{A_i\}_{i=1}^\infty$:

$$\mathscr{H}_\delta^k\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \mathscr{H}_\delta^k(A_i) \leq \sum_{i=1}^\infty \mathscr{H}^k(A_i).$$

We conclude now by letting $\delta \to 0$. $\square$

Proposition 2.2.22 (2.1. Theorem 1 [6])**.** let $k \in [0, \infty)$. Then $\mathscr{H}^k$ is a Borel regular measure on $\mathbb{R}^n$.

Proof.

Claim 1: *$\mathscr{H}^k$ is a Borel measure.*

In view of Charatheodory's criterion (theorem A.1.1), let us consider two sets $A, B \subseteq \mathbb{R}^n$ such that $d(A, B) > 0$. Let us use the outer measures $\mathscr{H}_\delta^k$, where $0 < \delta < \frac{1}{4}d(A, B)$. Fix a covering of $A \cup B \subseteq \cup_{i=1}^\infty C_i$ with $\operatorname{diam}(C_i) \leq \delta$, then we can consider two disjoint families of indexes I_A for those i such that $C_i \cap A \neq \emptyset$ and I_B for those i such that $C_i \cap B \neq \emptyset$, notice that for $I' = I_A \cup I_B$ the family $\{C_i\}_{i \in I'} \subseteq \{C_i\}_{i \in I}$ is a covering of $A \cup B$ and thus

$$\begin{aligned}\sum_{i=1}^\infty \frac{\omega_k}{2^k}\operatorname{diam}(C_i)^k &\geq \sum_{i \in I_A} \frac{\omega_k}{2^k}\operatorname{diam}(C_i)^k + \sum_{i \in I_B} \frac{\omega_k}{2^k}\operatorname{diam}(C_i)^k \\ &\geq \mathscr{H}_\delta^k(A) + \mathscr{H}_\delta^k(B).\end{aligned}$$

We can conclude by considering the infium over the coverings $\{C_i\}_{i=1}^\infty$ of $A \cup B$ and letting δ fo to 0.

Claim 2: *$\mathscr{H}^k$ is Borel regular.*

We want to prove that for any $A \subseteq \mathbb{R}^n$ there exists a Borel set $\tilde{A} \subseteq \mathbb{R}^n$ such that $A \subseteq \tilde{A}$ and $\mathscr{H}^k(A) = \mathscr{H}^k(\tilde{A})$.

We assume $\mathscr{H}^k(A) < \infty$, otherwise we can approximate the measure of A by $\mathbb{R}$. Notice that $\operatorname{diam}(A) = \operatorname{diam}(\bar{A})$, hence, we can consider only closed sets in the definition of $\mathscr{H}_\delta^k$.

Consider now coverings $A \subseteq \cup_{i=1}^{\infty} C_i^j$ such that $\operatorname{diam}(C_i^j) \leq \dfrac{1}{k}$ for al $i \in \mathbb{N}$ and

$$\sum_{i=1}^{\infty} \frac{\omega_k}{2^k} \operatorname{diam}\big(C_i^j\big)^k \leq \mathscr{H}^k_{\frac{1}{j}}(A) + \frac{1}{j}.$$

Now define $A_j := \cup_{i=1}^{\infty} C_i^j$ and $\tilde{A} = \cap_{i=1}^{\infty} A_k$. $\square$

Remark 2.2.4. Let $k \in \{1, 2, ..., n\}$, then $\mathscr{H}^k$ is not a Radon measure, but if $E \subseteq \mathbb{R}^n$ such that $\mathscr{H}^k(E) < \infty$, by Lemma 2.2.12, $\mathscr{H}^k \llcorner E$ is a Radon measure.

We introduce now some elementary properties of the Hausdorff Measures and the notion of Hausdorff dimension, which formalizes the concept of s-dimensional set in $\mathbb{R}^n$, with $s \in [0, n]$.

Proposition 2.2.23 (2.1 Theorem 2 [6])**.** In $\mathbb{R}^n$:

(1) $\mathscr{H}^0$ is the counting measure.

(2) $\mathscr{H}^1 = \mathscr{L}^1$ on $\mathbb{R}^n$.

(3) $\mathscr{H}^s \equiv 0$ on $\mathbb{R}^n$ for all $s > n$.

(4) $\mathscr{H}^s(\lambda A) = \lambda^s \mathscr{H}^s(A)$ for $\lambda > 0$ and $A \subseteq \mathbb{R}^n$.

(5) $\mathscr{H}^s\big(L(A)\big)$ for each affine isometry $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$.

Lemma 2.2.24 (2.2. Lemma 1 [6])**.** Let $k > 0$ and $A \subseteq \mathbb{R}^n$ and $\mathscr{H}^k_\delta(A) = 0$ for some $0 < \delta \leq \infty$. Then $\mathscr{H}^k(A) = 0$.

Remark 2.2.5. Notice that $\mathscr{H}^k(A) \geq \mathscr{H}^k_\delta(A)$, for all $\delta > 0$. Hence, if $\mathscr{H}^k(A) = 0$, we have that $\mathscr{H}^k_\delta(A) = 0$ for each $\delta > 0$.

Lemma 2.2.25 (2.2. Lemma 2 [6])**.** Let $A \subseteq \mathbb{R}^n$ and $0 \leq k < t < \infty$. Then

(1) If $\mathscr{H}^k(A) < \infty$, then $\mathscr{H}^t(A) > 0$, then $\mathscr{H}^k(A) = 0$.

(2) If $\mathscr{H}^t(A) < \infty$, then $\mathscr{H}^k = +\infty$.

Based on the previous two lemmas, we can define the Hausdorff dimension of a set in $\mathbb{R}^n$.

Definition 2.2.26 (Hausdorff Dimension of a Set in $\mathbb{R}^n$). Let $E \subseteq \mathbb{R}^n$, we define

$$\dim_{\mathscr{H}}(E) := \inf \big\{0 \leq k < \infty : \mathscr{H}^k(E) = 0\big\}.$$

Remark 2.2.6. Let $E \subseteq \mathbb{R}^n$, when we say that it is k-dimensional, we refer to its Hausdorff dimension in $\mathbb{R}^n$. Notice that for a set $E \subseteq \mathbb{R}^n$ such that $\dim_{\mathscr{H}}(E) = k$, we do not know any information about the measure of the set, it can be $\mathscr{H}^k(E) \in \mathbb{R}^+ \cup \{0, \infty\}$.

We introduce now Theorem 2.2.28, which relies on 2.2.27, which can be proved based on the notion of Steiner symmetrization. For further details see 2.2. in [6].

Proposition 2.2.27 (Isodiametric Inequality - 2.2. Theorem 1 [6]). Let $E \subseteq \mathbb{R}^n$. Then

$$\mathscr{L}^n(E) \leq \frac{\omega_k}{2^k} \operatorname{diam}(E)^k.$$

Theorem 2.2.28 (2.3. Theorem 2 [6])**.** $\mathscr{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$.

Related to the Hausdorff Measure and its properties as a generalization of the Lebesgue measure in $\mathbb{R}^n$ we introduce now the definition of density of a point in a given set.

Definition 2.2.29 (k-density of a Set at a Point). Let $k \in [0, \infty)$, $E \subseteq \mathbb{R}^n$ a Borel set and $\mathbf{x} \in \mathbb{R}^n$.

- The upper k-density of E at $\mathbf{x}$ is defined as

$$\Theta^{*k}(E, \mathbf{x}) := \Theta^{*k}\big(\mathscr{H}^k \llcorner E, \mathbf{x}\big).$$

- The lower k-density of E at $\mathbf{x}$ is defined as

$$\Theta^k_*(E, \mathbf{x}) := \Theta^k_*\big(\mathscr{H}^k \llcorner E, \mathbf{x}\big).$$

- When it exists, we define the k-density for E at $\mathbf{x}$ by

$$\Theta^k(E, \mathbf{x}) := \Theta^k\big(\mathscr{H}^k \llcorner E, \mathbf{x}\big).$$

The following two propositions follow from 2.56 [2]. Proposition 2.2.30 can be seen as an analogous result to Lemma 2.2.18 for Hausdorff densities (see Definition 2.2.16 and Definition 2.2.29).

Proposition 2.2.30 (2.15. [5])**.** Le $E \subseteq \mathbb{R}^n$ be a Borel set and $k > 0$ such that $\mathscr{H}^k(E) < \infty$. Then

(1) $\Theta^{*k}(E, \mathbf{x}) = 0$ for $\mathscr{H}^k$-almost every $\mathbf{x} \notin E$.

(2) $2^{-k} \leq \Theta^{*k}(E, \mathbf{x}) \leq 1$ for $\mathscr{H}^k$-almost every $\mathbf{x} \in E$.

Remark 2.2.7. The only information that we know about the lower density is that it is bounded by the upper density. There are sets such that their $\mathscr{H}^k$ measure is finite for which the lower density vanishes for $\mathscr{H}^k$-almost every point in the set.

Proposition 2.2.31 (2.16. [5])**.** Let μ be a Radon measure on $\mathbb{R}^n$ and k a non-negative real number such that, for μ-almost every $\mathbf{x}$,

$$0 < \Theta^{*k}(\mu, \mathbf{x}) < \infty.$$

Then there exists an k-dimensional set E and a Borel function f such that $\mu = f\mathscr{H}^k \llcorner B$.

2.2.3 The Grassmannian in $\mathbb{R}^n$

Definition 2.2.32 (**Grassmannian**)**.** We define de Grassmannian of dimension $k \leq n$ in $\mathbb{R}^n$ as

$$Gr(\mathbb{R}^n, k) \equiv \big\{C \subseteq \mathbb{R}^n : V \text{ linear subspace such that } \dim(V) = k\big\}.$$

We define de Grassmannian in $\mathbb{R}^n$ as

$$Gr(\mathbb{R}^n) = \bigcup_{k=1}^{n} Gr(\mathbb{R}^n, k).$$

In $(\mathbb{R}^n, \|\cdot\|)$, all norms induce the same topology. Consider a metric $d_{\|\cdot\|} = d$ on $\mathbb{R}^n$, and let $\mathcal{K}(\mathbb{R}^n)$ be the family of all non-empty compact subsets of $\mathbb{R}^n$. We consider the Hausdorff metric on $\mathcal{K}(\mathbb{R}^n)$,i.e. given $K, L \in \mathcal{K}(\mathbb{R}^n)$,

$$d_H(K, L) := \max\left\{\sup_{x\in K}\inf_{y\in L} d(x, y),\ \sup_{x\in L}\inf_{y\in K} d(x, y)\right\}.$$

This metric d_H induces the Vietoris topology on $\mathcal{K}(\mathbb{R}^n)$. We have that, since $(\mathbb{R}^n, d)$ is a Polish space (i.e., a completely metrizable, separable topological space), so $(\mathcal{K}(\mathbb{R}^n), d_H)$ is a

Polish space as well. Moreover, since $(\overline{B(0,1)}, d)$ is a compact metric space, $\big(\mathcal{K}(\overline{B(0,1)}), d_H\big)$ is a compact metric space (and hence Polish as well). For further details, we refer to chapter 2 section "Spaces of Compact Sets" in [10]. We may embed the Grassmannian in $\mathbb{R}^n$ into $\mathcal{K}(\mathbb{R}^n)$ by identifying each $V \in \mathrm{Gr}(\mathbb{R}^n)$ with the following map

$$\begin{aligned} \phi : \mathrm{Gr}(\mathbb{R}^n) &\longrightarrow \mathcal{K}(\overline{B(0,1)}) \\ V &\longmapsto V \cap \overline{B(0,1)}. \end{aligned}$$

We claim that ϕ is an isometric embedding, for the metric over $\mathrm{Gr}(\mathbb{R}^n)$ induced by the pullback given by ϕ, i.e.

$$d_{\mathrm{Gr}}(V, W) := d_H(\phi(V), \phi(W)).$$

- Non-negativity is immediate.
- If $d_{\mathrm{Gr}}(V, W) = 0$, then $V \cap \overline{B(0,1)} = W \cap \overline{B(0,1)}$, and hence $V = W$ (since linear subspaces are determined by their unit ball intersections).
- Triangle inequality:

$$\begin{aligned} d_{\mathrm{Gr}}(V, W) = d_H(\phi(V), \phi(W)) &\leq d_H(\phi(V), \phi(\tilde{W})) + d_H(\phi(\tilde{W}), \phi(W)) \\ &= d_{\mathrm{Gr}}(V, \tilde{W}) + d_{\mathrm{Gr}}(\tilde{W}, W). \end{aligned}$$

Chapter 3

Rademacher's Theorem and Approximation of Lipschitz Functions

3.1 Lipschitz Maps.

Definition 3.1.1 (Lipschtiz Maps). Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a map $f : X \longrightarrow Y$ is Lipschitz if there exists some $C > 0$ such that, for any $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2),$$

then we will say that $f \in \mathrm{Lip}(X, Y)$, when $X = \mathbb{R}^n$ and $Y = \mathbb{R}$, we will denote the space of Lipschitz functions by $\mathrm{Lip}(\mathbb{R}^n)$. Finally, we define

$$\mathrm{Lip}(f) \coloneqq \inf \Big\{ C > 0 \mid d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2) \text{ holds for any } x_1, x_2 \in X \Big\}.$$

We say that $\mathrm{Lip}(f)$ is the Lipschitz constant associated to f.

Lemma 3.1.2 (2.4. Theorem 1 [6])**.** Let $f : \mathbb{R}^k \longrightarrow \mathbb{R}^m$ be a Lipschitz function, $E \subseteq \mathbb{R}^n$ and $0 \leq k < \infty$. Then

$$\mathscr{H}^k\big(f(E)\big) \leq \mathrm{Lip}(f)\mathscr{H}^k(E).$$

Proof. Fix $\delta > 0$ and consider a covering of $E \subseteq \cup_{i=1}^{\infty} C_i$ such that $\mathrm{diam}(C_i) \leq \delta$. Then $f(E) \subseteq \cup_{i=1}^{\infty} f(C_i)$ and $\mathrm{diam}\big(f(C_i)\big) \leq \mathrm{Lip}(f)\mathrm{diam}(C_i) \leq \mathrm{Lip(f)}\delta$, so $\{f(C_i)\}_{i=1}^{\infty}$ is among the valid coverings of $f(E)$ for $\mathscr{H}^k_{\mathrm{Lip}(f)\delta}$. Hence,

$$\begin{aligned}\mathscr{H}^k_{\mathrm{Lip}(f)\delta}\big(f(E)\big) &\leq \sum_{i=1}^{\infty} \frac{\omega_k}{2^k} \mathrm{diam}\big(f(C_i)\big)^k \\ &\leq \big(\mathrm{Lip}(f)\big)^k \sum_{i=1}^{\infty} \frac{\omega_k}{2^k} \mathrm{diam}(C_i)^k\end{aligned}$$

Now consider the infimum over all covering of E with diameter lower or equal than δ, obtaining the result. $\square$

Lemma 3.1.3 (2.17 [5])**.** Let $A \subseteq \mathbb{R}^n$ be a set and $f : A \longrightarrow \mathbb{R}^m$ a Lipschitz function. Then, there exists a Lipschitz function $\tilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $\tilde{f}_{|A} = f$

Proof. It is enough to prove it for the case where f is real-valued, for bigger m we can take the function $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m)$ and $\mathrm{Lip}(\tilde{f}) \leq \max\{\mathrm{Lip}(\tilde{f}_1), \mathrm{Lip}(\tilde{f}_2), ..., \mathrm{Lip}(\tilde{f}_m)\}$, where $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function such that $\tilde{f}_{i|A} = f_i$. Now consider $\tilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\tilde{f}(\mathbf{x}) \coloneqq \inf \Big\{ f(\mathbf{y}) + \mathrm{Lip}(f) \left\| \mathbf{y} - \mathbf{x} \right\| \ : \ \mathbf{y} \in A \Big\}.$$

Notice that, for any $\mathbf{x} \in A$, we have that $\tilde{f}(\mathbf{x}) = f(\mathbf{x})$, so $\tilde{f}_{|A} = f$. Let us see that $\tilde{f}$ is Lipscthiz: for any $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$ and each $\epsilon > 0$, there exist $\mathbf{y}^{\mathbf{x}}_\epsilon, \mathbf{y}^{\mathbf{z}}_\epsilon \in A$ such that

$$\tilde{f}(\mathbf{x}) \geq f(\mathbf{y}^{\mathbf{x}}_\epsilon) + \mathrm{Lip}(f) \left\| \mathbf{x} - \mathbf{y}^{\mathbf{x}}_\epsilon \right\| - \epsilon \tag{3.1}$$

and, by definition of $\tilde{f}$, we have that

$$\tilde{f}(\mathbf{z}) \leq f(\mathbf{y}^{\mathbf{x}}_\epsilon) + Lip(f) \left\| \mathbf{z} - \mathbf{y}^{\mathbf{x}}_\epsilon \right\|. \tag{3.2}$$

Hence,

$$\begin{aligned}\tilde{f}(\mathbf{z}) - \tilde{f}(\mathbf{x}) &\leq f(\mathbf{y}^{\mathbf{x}}_\epsilon) + \mathrm{Lip}(f) \left\| \mathbf{x} - \mathbf{y}^{\mathbf{x}}_\epsilon \right\| - \Big(f(\mathbf{y}^{\mathbf{x}}_\epsilon) + \mathrm{Lip}(f) \left\| \mathbf{z} - \mathbf{y}^{\mathbf{x}}_\epsilon \right\| - \epsilon \Big) \\ &= \mathrm{Lip}(f) \Big(\left\| \mathbf{x} - \mathbf{y}^{\mathbf{x}}_\epsilon \right\| - \left\| \mathbf{z} - \mathbf{y}^{\mathbf{x}}_\epsilon \right\| \Big) + \epsilon \\ &\leq \mathrm{Lip}(f) \left\| \mathbf{x} - \mathbf{z} \right\| + \epsilon.\end{aligned}$$

Analogously to 3.1 and 3.2, we can prove that

$$\tilde{f}(\mathbf{x}) - \tilde{f}(\mathbf{z}) \leq \mathrm{Lip}(f) \left\| \mathbf{x} - \mathbf{z} \right\| + \epsilon.$$

Thus

$$|\tilde{f}(\mathbf{x}) - \tilde{f}(\mathbf{z})| \leq \mathrm{Lip}(f)\, \|\mathbf{x}-\mathbf{z}\| + \epsilon \leq \mathrm{Lip}(f)\, \|\mathbf{x}-\mathbf{z}\|, \quad \square$$

for any $\epsilon > 0$, hence

$$|\tilde{f}(\mathbf{x}) - \tilde{f}(\mathbf{z})| \leq \mathrm{Lip}(f)\, \|\mathbf{x}-\mathbf{z}\|. \quad \square$$

3.2 Approximation of Lipschitz Functions

Theorem 3.2.1 (**Rademacher Theorem -** 3.1. Theorem 2 [6])**.** Let $f \in \mathrm{Lip}\big(\mathbb{R}^n\big)$, then f is differentiable for $\mathscr{L}^n$-almost every $\mathbf{x} \in \mathbb{R}^n$.

Notation. Let $C \subseteq \mathbb{R}^n$ be a closed set, $K \subseteq \mathbb{R}^n$ a compact set such that $K \subseteq C$, $\delta > 0$ and $f : C \longrightarrow \mathbb{R}$ and $h : C \longrightarrow \mathbb{R}^n$ two given functions:

- $R(\mathbf{y},\mathbf{x}) := \dfrac{f(\mathbf{y}) - f(\mathbf{x}) - h(\mathbf{x})(\mathbf{y}-\mathbf{x})}{\|\mathbf{x}-\mathbf{y}\|}$, for any $\mathbf{x},\mathbf{y} \in C$, $\mathbf{x} \neq \mathbf{y}$.
- $\Phi_K(\delta) := \sup\Big\{\big|R(\mathbf{y},\mathbf{x})\big| \,:\, \mathbf{0} < \|\mathbf{x}-\mathbf{y}\| \leq \delta,\ \mathbf{x},\mathbf{y} \in K\Big\}$.

Theorem 3.2.2 (**Whitney's Extension Theorem -** 6.5. Theorem 1 [6])**.** Let f and h be two continuous real-valued functions defined over a closed set $C \subseteq \mathbb{R}^n$ such that, for any $K \subseteq \mathbb{R}^n$ compact set we have

$$\Phi_k(\delta) \xrightarrow[\delta\to 0]{} 0.$$

Then there exists a function $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

1. $\bar{f}$ is $\mathcal{C}^1(\mathbb{R}^n)$.
2. $\bar{f}_{|C} = f$ and $d\bar{f}_{|C} = h$.

The Theorem 3.2.3 can be obtained as a corollary of Theorem 3.2.2. It allows us to replace the Lipschitz functions in the definition of rectifiable set, 4.1.3, by functions of class $\mathcal{C}^1(\mathbb{R}^k;\mathbb{R}^{n-k})$, see Proposition 4.1.6.

In this text, we show the proof of the first statement of Theorem 3.2.3, the second follows from the construction of $\bar{f}$ during the proof of 3.2.2, for further details, see 6.6. Theorem 1 [6].

Theorem 3.2.3 (**Approximation of Lipschitz functions -** 6.6.1 Theorem 1 [6])**.** Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz function. For each $\epsilon > 0$, there exists a function $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ of class $\mathcal{C}^1$ such that

$$\mathscr{L}^n\left(\left\{\mathbf{x} \in \mathbb{R}^n : \bar{f}(\mathbf{x}) \neq f(\mathbf{x})\right\} \cup \left\{\mathbf{x} \in \mathbb{R}^n : d\bar{f}_{\mathbf{x}} \neq df_{\mathbf{x}}\right\}\right) \leq \epsilon.$$

Moreover,

$$\sup_{\mathbf{x}\in\mathbb{R}^n} \|\nabla f_{\mathbf{x}}\|_\infty \leq c_n \mathrm{Lip}(f),$$

for some constant $c_n > 0$ depending only on n.

Proof. By Theorem 3.2.1, there exists a set $R \subseteq \mathbb{R}^n$ such that $\mathscr{L}^n(R) = 0$ and $df_{\mathbf{x}}$ exists for every $\mathbf{x} \in R$.

First we approximate f in the ball $B_1 \coloneqq B(\mathbf{0}, 1) \cap R$.

Using Theorem A.1.2, there exists a compact set $K_1 \subseteq \mathbb{R}^n$ such that the function $df_{\mathbf{x}} : K_1 \longrightarrow \mathbb{R}$ is continuous and $\mathscr{L}^n(B_1 \setminus K_1) \leq \dfrac{\epsilon}{2^3}$. Using the same notation as in Theorem 3.2.2, we write

$$d(\mathbf{x}) \coloneqq df_x \qquad \text{for each } \mathbf{x} \in K_1$$

and

$$R(\mathbf{y}, \mathbf{x}) = \frac{f(\mathbf{y}) - f(\mathbf{x}) - df_{\mathbf{x}}(\mathbf{y} - \mathbf{x})}{|\mathbf{x} - \mathbf{y}|} \qquad \text{for each } \mathbf{x}, \mathbf{y} \in K_1.$$

We would like to show that

$$\Phi\left(\frac{1}{l}\right) = \sup\left\{|R(\mathbf{y}, \mathbf{x})| : \mathbf{0} < \|\mathbf{x} - \mathbf{y}\| < \frac{1}{l},\ \mathbf{x}, \mathbf{y} \in K_1\right\} \xrightarrow[l\to\infty]{} 0.$$

For this purpose, let us define

$$\phi_l(\mathbf{x}) \coloneqq \sup\left\{|R(\mathbf{y}, \mathbf{x})| : \mathbf{0} < \|\mathbf{x} - \mathbf{y}\| < \tfrac{1}{l},\ \mathbf{y} \in K_1\right\} \text{ for each } \mathbf{x} \in K_1.$$

By choice of d, it is clear that $\phi_l(\mathbf{x}) \xrightarrow[l\to\infty]{} 0$ for each $\mathbf{x} \in K_1$.

Now, by Theorem A.1.3, there exists a set $F_1 \subseteq K_1$ such that $\mathscr{L}^n(K_1 \setminus F_1) < \dfrac{\epsilon}{2^3}$ where $\left\|\phi_{l|K_1}\right\|_\infty \xrightarrow[l\to\infty]{} 0$. Hence,

$$\Phi(\tfrac{1}{l}) = \sup_{\mathbf{x}\in K_1} |\phi_l(\mathbf{x})| = \left\|\phi_{l|K_1}\right\|_\infty \xrightarrow[l\to\infty]{} 0.$$

Notice that $F_1 \subseteq K_1$ can be taken closed, see the proof of Theorem A.1.3. Finally, applying Theorem 3.2.2, there exists a function $\bar{f}_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\bar{f}_1 \in \mathcal{C}^1(\mathbb{R}^n)$ $f_{|F_1} = \bar{f}_{1|F_1}$ and

$d\bar{f}_{1|F_1} = df_{|F_1}$. Finally, we obtain that

$$\mathscr{L}^n\Big(\Big\{\mathbf{x} \in B_1 \,:\, \bar{f}_1(\mathbf{x}) \neq f_1(\mathbf{x})\Big\} \cup \Big\{\mathbf{x} \in B_1 \,:\, d\bar{f}_{1\mathbf{x}} \neq df_{1\mathbf{x}}\Big\}\Big) \leq \mathscr{L}^n(B_1 \setminus K_1) + \mathscr{L}^n(K_1 \setminus F_1) \leq \frac{\epsilon}{2^2}.$$

For B $j \geq 2$, and writing $B_j = B(\mathbf{0}, j) \cap R$, following the same procedure as above, we approximate f over the annulus $\tilde{B}_j \coloneqq B_j \setminus B_{j-1}$ by a function $\bar{f}_j$ of class $\mathcal{C}^1$ such that

$$\mathscr{L}^n\Big(\Big\{\mathbf{x} \in \tilde{B}_j \,:\, \bar{f}_j(\mathbf{x}) \neq f(\mathbf{x})\Big\} \cup \Big\{\mathbf{x} \in \tilde{B}_j \,:\, d\bar{f}_{j\mathbf{x}} \neq df_{\mathbf{x}}\Big\}\Big) \leq \mathscr{L}^n(\tilde{B}_j \setminus K_j) + \mathscr{L}^n(K_j \setminus F_j) \leq \frac{\epsilon}{2^{j+1}}.$$

We define $\bar{f}(\mathbf{x}) = \bar{f}_j(\mathbf{x})$ for $\mathbf{x} \in \tilde{B}_j$ which is a function of class $\mathcal{C}^1$ $\mathscr{L}^n$-almost everywhere (possibly excluding the intersection of the patches $S(\mathbf{0}, j) = \{\|\mathbf{x}\| = j\}$). Moreover, writing (for a more synthetic notation) $\tilde{B}_1 = B_1$, we have

$$\begin{aligned}\mathscr{L}^n\Big(\Big\{\mathbf{x} \in \mathbb{R}^n \,:\, \bar{f}(\mathbf{x}) \neq f(\mathbf{x})\Big\} \cup \Big\{\mathbf{x} \in \mathbb{R}^n \,:\, d\bar{f}_{\mathbf{x}} \neq df_{\mathbf{x}}\Big\}\Big) &\leq \sum_{j=1}^{\infty} \mathscr{L}^n(\tilde{B}_j \setminus K_j) + \mathscr{L}^n(K_j \setminus F_j) \\ &\leq \epsilon. \quad \square\end{aligned}$$

3.3 The Area Formula

The following result relates the definition of Hausdorff measure in $\mathbb{R}^n$ with the classical notion of the volume of a smooth manifold in differential geometry and is key to study the properties of non-smooth surfaces. For further details on its proof see sections 4.1. and 4.2. in [5], and section 3.3. in [6].

Theorem 3.3.1 (**Area Formula -** 4.3. [5])**.** Let $E \subseteq \mathbb{R}^k$ be a Borel set and $f : E \longrightarrow \mathbb{R}^n$ a Lipschitz map, then

$$\int_{f(E)} \mathscr{H}^0\big(f^{-1}(\{\mathbf{x}\})\big) d\mathscr{H}^k(\mathbf{x}) = \int_E J df_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}).$$

Remark 3.3.1. If Γ is a Borel subset of a k-dimensional Lipschitz graph, i.e.

$$\Gamma = \mathrm{Graph}(f) \coloneqq \Big\{(\mathbf{x}, f(\mathbf{x})) \;\Big|\; \mathbf{x} \in G\Big\},$$

where $f \in \mathrm{Lip}\big(\mathbb{R}^k; \mathbb{R}^{n-k}\big)$. Then G is Borel. Indeed, $\mathbb{R}^{n-k}$ is second-countable, the Borel σ-algebra of a numerable product of second-countable topological spaces coincides with the product of their Borel σ-algebras, so $\Pi_{n-k}(\Gamma)$ is Borel (where Π_{n-k} represents the projection from $\mathbb{R}^n$ onto the last $n-k$ coordinates). Furthermore, f is Borel-measureable, so $G = f^{-1}\big(\Pi_{n-k}(\Gamma)\big)$ is a Borel set.

Remark 3.3.2. We can use Theorem 3.3.1 to compute the surface measure of Lipschitz graphs. Consider a Borel subset Γ of a k-dimensional Lipschitz graph in $\mathbb{R}^n$. There exists a function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^{n-k}$ and a Borel set B (see Remark 3.3.1) such that

$$\Gamma = \big\{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in B\big\}.$$

Applying Theorem 3.2.1, we know that f is differentiable in a set $\tilde{B} \subseteq B$ such that $\mathscr{L}^k(B \setminus \tilde{B}) = 0$, we may apply Theorem 3.3.1 to the function

$$\begin{aligned} F : B &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto F(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x})), \end{aligned}$$

and noticing that F is injective in B we obtain

$$\mathscr{H}^k\big(\Gamma\big) = \mathscr{H}^k(F(B)) = \int_{\tilde{B}} JdF_{\mathbf{x}} d\mathscr{L}^k(\mathbf{x}).$$

Remark 3.3.3. The area formula leads to a changes of variables formula (see 3.3.3 Theorem 2 [6]) which can be applied to prove that given the graph Γ, of a map of class $\mathcal{C}^1$ from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$, $\mathscr{H}^k \llcorner \Gamma$ coincides with the volume measure on Γ, and we can extend the concept to Lipschitz maps (see 3.3.4. [6]).

Notation: Let ρ be a function on $\mathbb{R}^n$, we denote by

$$\rho^+(\mathbf{x}) = \begin{cases} \rho(\mathbf{x}) & \text{if } \rho(\mathbf{x}) \geq 0 \\ 0 & \text{if } \rho(\mathbf{x}) < 0 \end{cases}$$

and

$$\rho^-(\mathbf{x}) = \begin{cases} -\rho(\mathbf{x}) & \text{if } \rho(\mathbf{x}) \leq 0 \\ 0 & \text{if } \rho(\mathbf{x}) > 0 \end{cases}$$

Corollary 3.3.2. Let $B \subseteq \mathbb{R}^n$, $F : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ be an injective Lipschitz function and

$\rho : \mathbb{R}^n \longrightarrow \mathbb{R}$ a Borel function. Then

$$\int_{F(G)} \rho(\mathbf{x}) d\mathscr{H}^k(\mathbf{x}) = \int_G \rho(F(\mathbf{y})) JdF_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}).$$

Sketch of the Proof. We first prove this result for simple functions $\rho(\mathbf{x}) = \sum_{k=1}^{m} \alpha_k 1_{I_k}$, where $m, \alpha_k \geq 0$ and $I_k \subseteq \mathbb{R}^n$ is Borel measureable for all $k \in \mathbb{N}$. Let now G_k be the Borel sets (see Remark 3.3.1) such that $F(G) \cap I_k = F(G_k)$. By linearity of the integral and Theorem 3.3.1

$$\begin{aligned}
\int_{F(G)} \rho(\mathbf{x}) d\mathscr{H}^k(\mathbf{x}) &= \int_{F(G)} \sum_{k=1}^{m} \alpha_k 1_{I_k}(\mathbf{x}) d\mathscr{H}^k(\mathbf{x}) \\
&= \sum_{k=1}^{m} \alpha_k \int_{F(G_k)} d\mathscr{H}^k(\mathbf{x}) \\
&= \sum_{k=1}^{m} \alpha_k \int_{G_k} dJF_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}) \\
&= \int_G \sum_{k=1}^{m} \alpha_k 1_{G_k}(\mathbf{y}) dJF_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}) \\
&= \int_G \sum_{k=1}^{m} \alpha_k 1_{I_k}(F(\mathbf{y})) dJF_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}) \\
&= \int_G \rho(F(\mathbf{y})) dJF_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}).
\end{aligned}$$

Next we prove for non-negative functions. Let us take a non-negative function ρ, it can can be written as the limit of a monotocally increasing sequence simple functions $\{\alpha_m\}_{m=1}^{\infty}$ such that $\alpha_m(\mathbf{x}) := \sum_{k \leq m} \frac{1}{k} 1_{I_k}(\mathbf{x})$, where

$$\begin{aligned}
I_1 &:= \left\{ \mathbf{x} \in \mathbb{R}^n : \rho(\mathbf{x}) \geq 1 \right\} \\
I_k &:= \left\{ \mathbf{x} \in \mathbb{R}^n : \rho(\mathbf{x}) \geq \tfrac{1}{k} + \sum_{l \leq k-1} \tfrac{1}{l} 1_{I_l} \right\}.
\end{aligned}$$

Using the monotone convergence theorem we can conclude the result for non-negative ρ. Finally, we use the linearity of the integral and the decomposition of $\rho = \rho^+ - \rho^-$ to conclude the proof. $\square$

Chapter 4

Rectifiability and Tangent Measures

4.1 Approximate Tangent Planes and Rectifiability

4.1.1 Approximate Tangent Planes

The following concept is intrinsic to the definition of approximate tangent plane and will be key in the characterization of rectifiable sets.

Definition 4.1.1 (k -Cone). Let $V \in Gr(\mathbb{R}^n, k)$, $V^{\perp}$ its orthogonal complement on $\mathbb{R}^n$, then we denote by $P_{V^{\perp}} : \mathbb{R}^n \longrightarrow V$ and $P_{V^{\perp}} : \mathbb{R}^n \longrightarrow V^{\perp}$ to the orthogonal projections on V and $V^{\perp}$, respectively. Let $s > 0$ and $V \in Gr(\mathbb{R}^n, k)$. We define the **k-cone centered at x** by

$$C(\mathbf{x}, V, s) \coloneqq \mathbf{x} + C(V, s),$$

where

$$C(V, s) \coloneqq \Big\{ \mathbf{x} \in \mathbb{R}^n : \ \|P_{V^{\perp}}(\mathbf{x})\| \leq s \, \|P_V(\mathbf{x})\| \Big\}.$$

Definition 4.1.2 (Approximate tangent k-plane for a set at a point). Let $A \subseteq \mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$ and $V \in Gr(\mathbb{R}^n, k)$. We say that V is a k-dimensional approximate tangent plane for A at $\mathbf{x}$ if the following two conditions hold:

(1) $\Theta^{*k}(A, \mathbf{x}) > 0$.

(2) For all $s \in (0, 1)$,

$$\lim_{r \searrow 0} \frac{1}{r^k} \mathscr{H}^k \Big([A \cap B(\mathbf{x}, r)] \setminus C(\mathbf{x}, V, s) \Big) = 0.$$

We write $\text{apTan}_k(\mathbf{x}, A)$ for the set of all approximate tangent planes for A at $\mathbf{x}$. If the set is a singleton, $\text{apTan}(\mathbf{x}, A) = \{V\}$, shall we write $V = Tan_k(\mathbf{x}, A)$.

Remark 4.1.1. When $\mathbf{x}$ belongs to a smooth k-dimensional surface S and $V = \text{Tan}(\mathbf{x}, S)$ is the tangent plane for S at $\mathbf{x}$, then $\text{Tan}(\mathbf{x}, S) = \text{apTan}(\mathbf{x}, S)$. See proof of Theorem 4.2.9.

4.1.2 Rectifiability

Next we introduce the notion of k-rectifiable sets which are essentially those ones which can be covered by Lipschitz maps $\mathscr{H}^k$ almost everywhere.

Definition 4.1.3 (k-rectifiable set). Let $E \subseteq \mathbb{R}^n$ a Borel set and $k \in \{1, 2, ..., n\}$, we say that E is k-rectifiable if there exists a countable family $\{\Gamma_i\}_{i\in\mathbb{N}}$ of graphs of Lipschitz maps in $\text{Lip}(\mathbb{R}^k; \mathbb{R}^{n-k})$ such that

$$\mathscr{H}^k\Big(E \setminus \bigcup_{i\in\mathbb{N}} \Gamma_i\Big) = 0.$$

Definition 4.1.4 (Purely k -unrectifiable Set). Let $F \subseteq \mathbb{R}^n$ be a Borel set. We say that F is purely k-unrectifiable if, for every k-rectifiable set E, $\mathscr{H}^k(F \cap E) = 0$.

Definition 4.1.5 (k-Rectifiable Measure). Let μ be a measure on $\mathbb{R}^n$, we say that it is k-rectifiable if there exist a k-rectifiable set $E \subseteq \mathbb{R}^n$ and a Borel-measureable function f such that

$$\mu = f\mathscr{H}^k \llcorner E.$$

The following result helps us to understand rectifiable sets as a generalization of $\mathcal{C}^1$ surfaces.

Proposition 4.1.6. Let $E \subseteq \mathbb{R}^n$ be a Borel set. Then, it is k-rectifiable if, and only if there exists a countable family of $\mathcal{C}^1(\mathbb{R}^k; \mathbb{R}^{n-k})$ functions such that the family of their graphs $\{\Gamma_i\}_{i=1}^{\infty}$ are such that

$$\mathscr{H}^k\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

Proof. First, notice that if one of the directions is immediate, since a map of class $\mathcal{C}^1$ is Lipschitz. Let us then prove that given a k-rectifiable set E we can find a countable family of maps of class $\mathcal{C}^1$ Let us consider E, $\{f_i\}_{i=1}^{\infty}$, $\{\Gamma_i\}_{i=1}^{\infty}$ as in Definition 4.1.3. By intersecting

E with countably many balls $\overline{B(\mathbf{x},1)}$ where $\mathbf{x} \in E \cap \mathbb{Q}^n$, we can prove the result for each locality $\overline{B(\mathbf{x},1)} \cap E = F$. Notice that, since the family $\{f_i\}_{i=1}^{\infty}$ is countable, if there exists a family of functions $\{\rho_{i,j}\}_{i,j\in\mathbb{N}}$ of class $\mathcal{C}^1(\mathbb{R}^k;\mathbb{R}^{n-k})$, with graphs $\tilde{\Gamma}_{i,j} = \text{Graph}(\rho_{i,j})$, such that

$$\mathscr{H}^k\left(\Gamma_i \setminus \bigcup_{j=1}^{\infty} \tilde{\Gamma}_{i,j}\right) = 0,$$

then, since all sets are Borel, we obtain that

$$\mathscr{H}^k\left(E \setminus \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} \tilde{\Gamma}_{i,j}\right) \leq \mathscr{H}^k\left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) + \sum_{i=1}^{\infty} \mathscr{H}^k\left(\Gamma_i \setminus \bigcup_{j=1}^{\infty} \tilde{\Gamma}_{i,j}\right) = 0.$$

Hence, it is sufficient to focus the prove when we have a single Lipschtiz map $f : G \longrightarrow \mathbb{R}^{n-k}$, where $G \subseteq$ is a Borel set, and whose graph$(f) = \Gamma$, covers E for $\mathscr{H}^k$ for almost every $\mathbf{x}$.

By Theorem 3.2.3 there exists some map $\rho_1 \in \mathcal{C}^1(\mathbb{R}^k;\mathbb{R}^{n-k})$ such that

$$\mathscr{L}^k\big(C_1\big) \leq 1,$$

where

$$C_1 \coloneqq \Big\{\mathbf{x} \in G \,:\, f(\mathbf{x}) \neq \rho_1(\mathbf{x})\Big\} \cup \Big\{\mathbf{x} \in G \,:\, df(\mathbf{x}) \neq d\rho_1(\mathbf{x})\Big\}.$$

Let us iterate this process, constructing a set of maps ρ_k, $k \geq 2$, of class $\mathcal{C}^1$ verify that

$$\mathscr{L}^k\Big(C_k\Big) \leq \frac{1}{2^k},$$

where

$$C_k \coloneqq \Big\{\mathbf{x} \in C_{k-1} \,:\, f(\mathbf{x}) \neq \rho_k(\mathbf{x})\Big\} \cup \Big\{\mathbf{x} \in \mathbb{R}^k \,:\, df(\mathbf{x}) \neq d\rho_k(\mathbf{x})\Big\},$$

for $k \geq 2$, moreover, we can assume C_k to be Borel. Notice that $C_{k+1} \subseteq C_k \subseteq G$, for $k \geq 1$. We call $\tilde{\Gamma}_k = \text{Graph}(\rho_k)$, and $F(\mathbf{x}, f(\mathbf{x}))$. Then, all sets are Borel and $\mathscr{H}^k(\cup_{k=1}^{\infty}\tilde{\Gamma}_k) \leq \mathscr{H}^k <$ $(\Gamma) < \infty$, by Theorem 3.3.1, dominated converge and applying the definition of the sets C_k, we obtain that

$$\begin{aligned}
\mathscr{H}^k\bigg(\Gamma\setminus\bigcup_{j=1}^{\infty}\tilde{\Gamma}_k\bigg) &= \mathscr{H}^k(\Gamma)-\mathscr{H}^k\bigg(\bigcup_{k=1}^{\infty}\tilde{\Gamma}_k\bigg)\\
&\leq \int_G JdF_{\mathbf{x}}d\mathscr{L}^k(\mathbf{x})-\sum_{k=1}^{\infty}\int_{C_k^c} JdR_{i\mathbf{x}}d\mathscr{L}^k(\mathbf{x})\\
&= \int_{G\setminus\bigcup_{k=1}^{\infty}C_k^c} JdF_{\mathbf{x}}d\mathscr{L}^k(\mathbf{x})+\sum_{k=1}^{\infty}\int_{C_k^c} JdF_{\mathbf{x}}-JdR_{i\mathbf{x}}d\mathscr{L}^k(\mathbf{x})\\
&= \int_{G\setminus\bigcup_{k=1}^{\infty}C_k^c} JdF_{\mathbf{x}}d\mathscr{L}^k(\mathbf{x}).
\end{aligned}$$

Hence, we finish the proof with the following claim.

Claim: $\mathscr{L}^k\bigg(G\setminus\bigcup_{k=1}^{\infty}C_k^c\bigg)=0.$

Notice that for any $m\geq 1$ we have that

$$G\setminus\bigcup_{k=1}^{m+1}C_k^c = G\cap\bigcap_{k=1}^{m+1}C_k = G\cap C_{m+1}\subseteq G\cap C_m = G\setminus\bigcup_{k=1}^{m}C_k^c,$$

moreover, recalling that all sets are Borel, then $\mathscr{L}^k\Big(G\setminus C_1\Big)\leq\mathscr{L}^k(G)<\infty$, and hence

$$\mathscr{L}^k\bigg(G\setminus\bigcup_{k=1}^{\infty}C_k^c\bigg)=\lim_{k\to\infty}\mathscr{L}^k\Big(G\cap C_k\Big)=0.\quad\square$$

4.1.3 The Rectifiability and Approximate Tangent Planes

In this section we aim to proof a characterization of rectifiable sets and measures by approximate tangent planes, which essentially states that, given a set, the existence of an approximate tangent plane for almost every of its point guarantees its rectifiability and vice versa. The main reference for this section is [5].

The following lemma formalizes the intuitive relation between k-cones and Lipschitz maps, see figure 4.1.

Lemma 4.1.7 (Geometric Lemma - 4.7. [5]**).** Let $A\subseteq\mathbb{R}^n$ be a set such that there exist $V\in Gr(\mathbb{R}^n,k)$ and $s>0$ such that

$$A\subseteq C(\mathbf{x},V,s)\qquad\text{for every } \mathbf{x}\in A$$

Then there exists a Lipschitz map $f : V \longrightarrow V^{\perp}$ such that

$$A \subseteq \text{Graph}(f) = \Big\{ (\mathbf{x}, f(\mathbf{x})) \,:\, \mathbf{x} \in V \Big\}.$$

Proof. We will first prove the following claim, using the notation as in Definition 4.1.1.

Claim: *for each* $\mathbf{x} \in P_V(A)$*, there exists a unique* $\mathbf{y} \in V^{\perp}$ *such that* $(\mathbf{x}, \mathbf{y}) \in P_V(A)$

Let V be the hyperplane considered in the statement, via a rotation, we may assume V to be horizontal, and let $\{\mathbf{x}_1, ..., \mathbf{x}_k, \mathbf{y}_1, ..., \mathbf{y}_{n-k}\}$ be an orthogonal system of coordinates such that

$$V = \Big\{ (x_1, ..., x_k, y_1, , ..., y_{n-k}) \,:\, x_1, ..., x_k \in \mathbb{R}^k \text{ and } y_1 = ... = y_{n-k} = 0 \Big\}$$

The map $P_{V|A} : A \longrightarrow V$ is injective, since $A \subseteq C(\mathbf{x}, V, s)$ for all $\mathbf{x} \in A$. Indeed, given $\mathbf{x}, \mathbf{y} \in A$ such that $P_V(\mathbf{x}) = P_V(\mathbf{y})$ then, by Pythagoras, linearity of the projections and $\text{Lip}(P_V) = 1$ and Definition 4.1.1, we obtain that

$$\|\mathbf{x} - \mathbf{y}\| = \|P_V(\mathbf{x}) - P_V(\mathbf{y})\| + \|P_{V^{\perp}}(\mathbf{x}) - P_{V^{\perp}}(\mathbf{y})\| \leq (1 + s) \|P_V(\mathbf{x}) - P_V(\mathbf{y})\| = 0.$$

Hence, we can define a function $f : P_V(A) \longrightarrow V^{\perp}$ such that

$$A = \Big\{ (\mathbf{x}, f(\mathbf{x})) \,:\, \mathbf{x} \in P_{|V}(A) \Big\}.$$

To see that $f \in \text{Lip}(\mathbb{R}^k; \mathbb{R}^{n-k})$, notice that by definition of f, for any $\mathbf{x}, \mathbf{y} \in P_V(A)$,

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq s \|P_V(\mathbf{x} - \mathbf{y}, P_{V^{\perp}}(\mathbf{x}) - P_{V^{\perp}}(\mathbf{y}))\| = s \|\mathbf{x} - \mathbf{y}\|.$$

By Lemma 3.1.3 we can extend f to a Lipschitz function $\tilde{f} : V \longrightarrow V^{\perp}$ such that $\tilde{f}_{|A} = f$, concluding the proof. $\square$

Remark 4.1.2. Notice that given any k-dimensional linear plane, for any $\alpha > 0$ we can find a constant $c(\alpha)$, only dependent on α such that for any $\mathbf{y} \notin C(\mathbf{x}, V, 4\alpha)$ we have that $B\big(\mathbf{z}, c(\alpha) \|x - y\|\big) \cap C(\mathbf{x}, V, 2\alpha) = \emptyset$.

The minimum distance from $\mathbf{y}$ to the cones $C(\mathbf{x}, V, 4\alpha)$ and $C(\mathbf{x}, V, 2\alpha)$ is given by the orthogonal projections on their border, let us call them $\mathbf{z}$ and $\mathbf{z}'$, respectively. Via a rotation we can focus on the section given by the rectangle triangle defined by $\mathbf{y}$, $\mathbf{z}'$ and $\mathbf{x}$, so that we can work on the plane containing such a triangle, with the side $\mathbf{z}' - \mathbf{x}$ on the horizontal axes. It is then clear that $c(\alpha) := \sin(\alpha) = \dfrac{\|\mathbf{z}' - \mathbf{z}\|}{\|\mathbf{z} - \mathbf{x}\|} < 1$ verifies the property, since

$$r := \frac{\|\mathbf{z}' - \mathbf{z}\|}{\|\mathbf{z} - \mathbf{x}\|} \|\mathbf{y} - \mathbf{x}\| < \|\mathbf{z}' - \mathbf{z}\|,$$

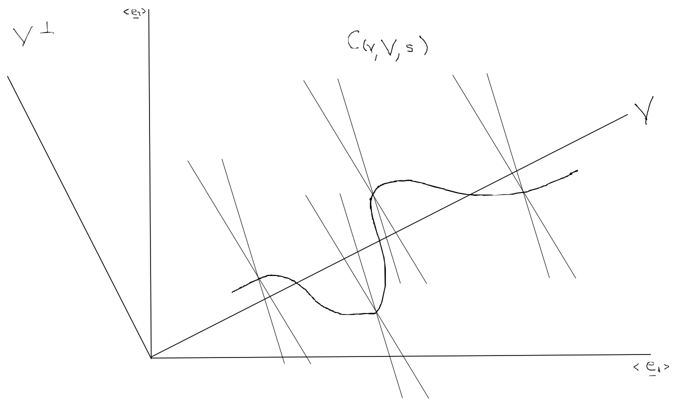

Figure 4.1: Lipschitz graph and cones

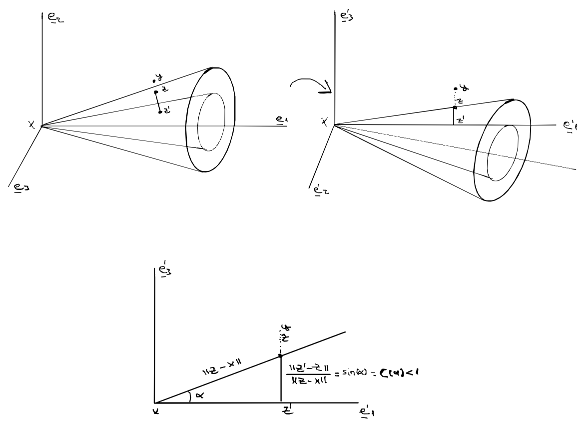

Figure 4.2: Idea of the proof of Remark 4.1.2 in $\mathbb{R}^2$

so the distance between $B(\mathbf{y}, r)$ and $C(\mathbf{x}, V, 2\alpha)$ is given by the point on the intersection of the border of the ball and the segment with end points $\mathbf{y}$ and $\mathbf{z}'$ and is always positive, see figure 4.2.

Theorem 4.1.8 (Rectifiability Criterion). Let $E \subseteq \mathbb{R}^n$ be a Borel set such that $0 < \mathscr{H}^k(E) < \infty$. Assume that, for $\mathscr{H}^k$-almost every $\mathbf{x} \in E$, the following two conditions hold:

(1) $\Theta_*^k(E, \mathbf{x}) > 0$.

(2) There exists an approximate tangent plane V for E at $\mathbf{x}$.

Then E is rectifiable.

Remark 4.1.3. Although we will not show that all the sets involved in the following proof are Borel, the main idea relies on Lemma A.2.1

Remark 4.1.4. The condition on the lower k-density in Theorem 4.1.8 will be essential during the proof of this result, in particular we will use that

$$\mathscr{H}^k\Big(E \setminus \bigcup_{i,j=1}^{\infty} F_{i,j}\Big) = \mathscr{H}^k\Big(E \setminus \bigcup_{i,j=1}^{\infty} \Big\{\mathbf{x} \in E : \mathscr{H}^k\big(E \cap B(\mathbf{x}, r)\big) \geq \tfrac{r^k}{j}, \text{ for all } r < \tfrac{1}{i}\Big\}\Big) = 0. \tag{4.1}$$

Indeed, since

$$\Theta_*^k(E, \mathbf{x}) = \lim_{s \to 0}\left(\inf_{r<s}\left\{\frac{\mathscr{H}^k\big(E \cap B(\mathbf{x}, r)\big)}{\omega_k r^k}\right\}\right) = \Theta > 0$$

for $\mathscr{H}^k$-almost every $\mathbf{x} \in E$, for any $\delta_{\mathbf{x}}$ such that $\Theta > \delta_{\mathbf{x}} > 0$ we have that there exists $s_0^{\mathbf{x}} > 0$ such that

$$\mathscr{H}^k\big(E \cap B(\mathbf{x}, r)\big) \geq \omega_k r^k(\Theta - \delta_{\mathbf{x}}) > 0$$

for any $r < s_0^{\mathbf{x}}$. Hence, considering $i_{\mathbf{x}} > \frac{1}{s_0^{\mathbf{x}}}$ and $j_x > \frac{1}{\omega_k(\Theta - \delta_{\mathbf{x}})}$ we have that

$$\mathscr{H}^k\big(E \cap B(\mathbf{x}, r)\big) > \frac{r^k}{j_{\mathbf{x}}},$$

for all $r < \frac{1}{i_{\mathbf{x}}}$. We can choose $i_{\mathbf{x}}, j_{\mathbf{x}}$ to be natural numbers, so 4.1 holds.

Proof of Theorem 4.1.8. For $i, j \in \mathbb{N}$, we consider the sets

$$F_{i,j} = \left\{\mathbf{x} \in E : \mathscr{H}^k\big(E \cap B(\mathbf{x}, r)\big) \geq \frac{r^k}{j} \text{ for all } r < \frac{1}{i}\right\}.$$

In view of Remark 4.1.4, we have that for $\mathscr{H}^k$-almost every $\mathbf{x} \in E$ belongs to $\cup_{i,j=1}^{\infty} F_{i,j}$. Since $\mathcal{F} := \{F_{i,j}\}_{i,j=1}^{\infty}$ is a countable family, it is enough to focus on an arbitrary $F_{i,j}$, we fix i, j and omit them in the notation of the sets.

Let us consider now a family of k-linear planes $\{V_1, V_2, ..., V_M\}$ such that for every k-dimensional linear plane V, there exists some $m \in \{1, 2, ..., M\}$ so that

$$C(\mathbf{0}, V, \alpha) \subseteq C(\mathbf{0}, V_m, 2\alpha). \tag{4.2}$$

For any fixed $\epsilon > 0$, any $m \in \{1, 2, ..., M\}$ and each $l \in \mathbb{N}$ define the set

$$G^{\epsilon}_{l,m} := \left\{ \mathbf{x} \in F : \frac{1}{r^k} \mathscr{H}^k\Big(E \cap B(\mathbf{x}, r) \setminus C(\mathbf{x}, V_m, 2\alpha)\Big) \leq \epsilon, \text{ for all } r < \frac{1}{l} \right\}$$

By the assumptions of the theorem and 4.2 we see that

$$F \subseteq \bigcup_{m=1}^{M} \bigcup_{l=1}^{\infty} G^{\epsilon}_{l,m}.$$

We need to check that there exists some fixed $\epsilon > 0$ such that $G^{\epsilon}_{l,m}$ is rectifiable for all l and m. We will see that there exists a constant $c(\alpha, j)$, such that this happens if $\epsilon < c(\alpha, j)$. In particular, we will show that there exists a Lipschitz function f^{ϵ} such that, calling $\delta^{\epsilon}_j = \delta := \frac{1}{3\max\{l,j\}}$,

$$G^{\epsilon}_{l,m} \cap B(\mathbf{y}, \delta) \subseteq \mathrm{Graph}(f^{\epsilon}).$$

Let us also call $G = G^{\epsilon}_{l,m} \cap B(\mathbf{y}, \delta)$.

Claim: *define* $c(\alpha, j) := \dfrac{\big[c(\alpha)\big]^k}{3^k j}$ *(see Remark 4.1.2). Then, for every* $0 < \epsilon < c(\alpha, j)$ *we have that*

$$G \subseteq C(\mathbf{x}, V_m, 4\alpha) \qquad \text{for every } \mathbf{x} \in G. \tag{4.3}$$

Notice that, if the claim holds, the proof of the theorem follows from Lemma 4.1.7. We argue by contradiction and assume that 4.3 does not hold. Then, there exist $\mathbf{x}, \mathbf{z} \in G$ such that $\mathbf{z} \notin C(\mathbf{x}, V, 4\alpha)$. Since $G = G^{\epsilon}_{l,m} \cap B(\mathbf{y}, \delta) \subseteq B(\mathbf{y}, \delta)$ then $\|z - x\| \leq \frac{2}{3}\delta$. By Remark 4.1.2 we know that

$$B(\mathbf{z}, c(\alpha)\, \|\mathbf{z} - \mathbf{x}\|) \subseteq \mathbb{R}^n \setminus C(\mathbf{x}, V_m, 4\alpha). \tag{4.4}$$

Now, by the definition of G in terms of δ and $G^{\epsilon}_{l,m}$, the fact that $\mathbf{x} \in G$, and that $\|\mathbf{z} - \mathbf{x}\| \leq \frac{2}{3}\delta$, we obtain

$$\frac{\mathscr{H}^k\Big(E \cap B(\mathbf{x}, \|\mathbf{z} - \mathbf{x}\|\, 3^{-1})\Big)}{3^{-k}\, \|\mathbf{z} - \mathbf{x}\|^k} \geq \frac{1}{j}. \tag{4.5}$$

Hence, by 4.4 and 4.5, we write

$$\frac{\big(c(\alpha)\big)^k \|\mathbf{z}-\mathbf{x}\|^k}{j3^k} \leq \mathscr{H}^k\Big(B\Big(\mathbf{z}, \tfrac{c(\alpha)\|\mathbf{z}-\mathbf{x}\|)}{3}\Big)\cap E\Big) \leq \mathscr{H}^k\Big(B\big(\mathbf{x}, \|\mathbf{z}-\mathbf{x}\|\,\big)\cap E\setminus C\big(\mathbf{x}, V_m, 2\alpha\big)\Big). \tag{4.6}$$

But by construction $G^\epsilon_{l,m} \subseteq G$, since $\mathbf{x}\in G$ and, $\|\mathbf{z}-\mathbf{x}\| < \delta$, we obtain

$$\mathscr{H}^k\Big(B\big(\mathbf{x}, \|\mathbf{z}-\mathbf{x}\|\,\big)\cap E\setminus C\big(\mathbf{x}, V_m, 2\alpha\big)\Big) \leq \epsilon\,\|\mathbf{z}-\mathbf{x}\|^k\,.$$

Gathering 4.6 and the choice of ϵ in the claim, we have that

$$\frac{\big(c(\alpha)\big)^k \|\mathbf{z}-\mathbf{x}\|^k}{j3^k} \leq \epsilon\,\|\mathbf{z}-\mathbf{x}\|^k\,,$$

so $\epsilon \geq c(\alpha, j)$ which yields to a contradiction. $\square$

4.2 Tangent Measures and Characterization of Rectifiability

Next we introduce the notion of rectifiability and tangent measures and prove that a set is k-rectifiable if and only if there exists an approximate k-tangent plane at almost all of its points. The main source for this section is [5].

Definition 4.2.1 (Blow-up of a Measure). Let μ be a measure on $\mathbb{R}^n$, $\mathbf{x}\in\mathbb{R}^n$ and $r > 0$ a positive real number. We define

$$\mu_{\mathbf{x},r}(B) \coloneqq \mu(\mathbf{x}+rB) \qquad \text{for every Borel set } B\subseteq\mathbb{R}^n.$$

The Definition 4.2.1 formalizes a notion of blow-up of μ at $\mathbf{x}$, the intuitive idea is that the smaller r becomes, the more we zoom in on μ at $\mathbf{x}$.

Definition 4.2.2 (k-Dimensional Tangent Measure to μ at $\mathbf{x}$). Let μ be a measure on $\mathbb{R}^n$, $\mathbf{x}\in\mathbb{R}^n$, and $\alpha > 0$. We say that a measure ν on $\mathbb{R}^n$ is a k-dimensional tangent measure to μ at $\mathbf{x}$ if there exists some sequence of real numbers $\{r_i\}_{i=1}^\infty$ such that $r_i \to 0$ and

$$\frac{\mu_{\mathbf{x},r_i}}{r_i^k} \overset{*}{\rightharpoonup} \nu.$$

We denote by $\operatorname{Tan}_k(\mu,\mathbf{x})$ the set of all k-dimensional tangent measures to μ at $\mathbf{x}$.

The following definition and results add context to the study of the tangent planes of measures with real density.

Definition 4.2.3 (Uniform Measure on $\mathbb{R}^n$). Let $k > 0$ and μ be a non-zero Radon measure on $\mathbb{R}^n$. We say that μ is k-uniform if

$$0 < \mu\big(B(\mathbf{x}, r)\big) = \omega_k r^k < \infty \qquad \text{for } \mathbf{x} \in \operatorname{supp}(\mu) \text{ and all } r > 0.$$

Remark 4.2.1. Id addition to Remark 2.2.3, it is important to point that there exist k-uniform measures which are not of the form $\mathscr{H}^k \llcorner V$, where $V \subseteq \mathbb{R}^n$ is linear plane, for further details, see section 6.1 in [5].

Proposition 4.2.4 (3.4 [9])**.** Let μ and ν be two Borel regular measures on $\mathbb{R}^n$ such that , for all $\mathbf{x} \in \mathbb{R}^n$ and all $r > 0$,

$$\begin{aligned} 0 < \mu\big(B(\mathbf{x}, r)\big) = g(r) < \infty, \\ 0 < \nu\big(B(\mathbf{x}, r)\big) = h(r) < \infty. \end{aligned}$$

Then there exists a positive number $c > 0$ such that $\mu = c\nu$.

Proof. Since both measures are Borel regular, it is suffiecient to prove the result for an arbitrary open set $U \subseteq \mathbb{R}^n$. We can assume U to be bounded. Indeed, if the result is proved for open and bounded sets, given an arbitrary open set $U' \subseteq \mathbb{R}^n$, we have that $\mu\big(U' \cap B(\mathbf{0}, n)\big) = c\mu\big(U' \cap B(\mathbf{0}, n)\big)$, for each $n \in \mathbb{N}$, since we have sequence of increasing sequence of ν and μ-measurable sets, we can take the limit when n goes to infinity, getting the result.

Since μ and ν are Borel measures, so h and g are Borel functions. Then, applying Fatou's

Lemma and Fubini we obtain

$$\begin{aligned}
\mu(U) &= \int_U \lim_{r\searrow 0} \frac{\nu\big(U\cap B(\mathbf{x},r)\big)}{h(r)} d\mu(\mathbf{x}) \\
&\le \lim_{r\searrow 0} \frac{1}{h(r)} \int_U \nu\big(U\cap B(\mathbf{x},r)\big) d\mu(\mathbf{x}) \\
&= \lim_{r\searrow 0} \frac{1}{h(r)} \int_U \int_{U\cap B(\mathbf{x},r)} d\nu(\mathbf{y}) d\mu(\mathbf{x}) \\
&= \lim_{r\searrow 0} \frac{1}{h(r)} \int_U \int_{B(\mathbf{y},r)} d\mu(\mathbf{x}) d\nu(\mathbf{y}) \\
&= \lim_{r\searrow 0} \int_U \frac{\mu\big(B(\mathbf{y},r)\big)}{h(r)} d\nu(\mathbf{y}) \\
&= \nu(U) \lim_{r\searrow 0} \frac{g(r)}{h(r)}.
\end{aligned}$$

Hence, we have that

$$\mu(U) \le \nu(U) \lim_{r\searrow 0} \frac{g(r)}{h(r)}. \tag{4.7}$$

Repeating the argument but inverting the roles of ν and μ we get that

$$\nu(U) \le \mu(U) \lim_{r\searrow 0} \frac{h(r)}{g(r)} = \mu(U) \left(\limsup_{r\searrow 0} \frac{g(r)}{h(r)} \right)^{-1}. \tag{4.8}$$

Combining 4.7 and 4.8, we obtain that

$$\limsup_{r\searrow 0} \frac{g(r)}{h(r)} \le \frac{\mu(U)}{\nu(U)} \le \liminf_{r\searrow 0} \frac{g(r)}{h(r)}.$$

Thus, there exists some

$$\lim_{r\searrow 0} \frac{g(r)}{h(r)} \in (0,\infty),$$

noticing that this number is independent on U, we have finished. □

Proposition 4.2.4 tells us that there are "not so many" uniform measures. Indeed, we present the following corollary (for further details on when $\mathrm{supp}(\mu) \neq \mathbb{R}^n$ we refer to [9]).

Corollary 4.2.5 (14.11 [9])**.** Let μ be a n-uniform measure on $\mathbb{R}^n$. Then there exists some constant $c > 0$ such that $\mu = c\mathscr{L}^n$.

The following two results are proved in [5].

Proposition 4.2.6 (3.4. [5])**.** Let μ be a Radon measure on $\mathbb{R}^n$, $k > 0$, and E a Borel set such that $\mu(E) > 0$: In addition, we assume that, for μ -almost every $\mathbf{x} \in E$,

$$0 < \Theta_*^k(\mu, \mathbf{x}) = \Theta^{*k}(\mu, \mathbf{x}) < \infty.$$

Then, for μ-almost every $\mathbf{x} \in E$ we have that

$$\emptyset \neq \operatorname{Tan}_k(\mu, \mathbf{x}) \subseteq \big\{\Theta(\mu, \mathbf{x})\nu : \nu \in \mathcal{U}^k(\mathbb{R}^n)\big\}.$$

Proposition 4.2.7 (3.5. [5])**.** If $\mathcal{U}^k(\mathbb{R}^n) \neq \emptyset$, then k is a non-negative integer lesser or equal than n.

We present the following result in a more particular setup, for rectifiable measures (we will see in the proofs, these measures have real density in almost every point of their support), it is the aim of the following two sections to prove Theorem 4.2.8.

Theorem 4.2.8 (**Characterization of rectifiable Measures**- 4.8. [5])**.** A measure μ on $\mathbb{R}^n$ is k-rectifiable if and only if, for μ almost every $\mathbf{x} \in \mathbb{R}^n$ there exist $c_{\mathbf{x}} \geq 0$ and a k-dimensional linear plane $V_{\mathbf{x}} \subseteq \mathbb{R}^n$ such that

$$\frac{\mu_{\mathbf{x},r}}{r^k} \stackrel{*}{\rightharpoonup} c_{\mathbf{x}} \mathscr{H}^k \llcorner V_{\mathbf{x}} \qquad \text{as } r \searrow 0.$$

Remark 4.2.2. This result characterizes rectifiable measures and hence rectifiable sets. In particular, the measures of the form $c_{\mathbf{x}} \mathscr{H}^k \llcorner V_{\mathbf{x}}$ are also called flat measures.

Hence, by Proposition 4.1.6 and Remark 4.2.3, a set is k-rectifiable if and only if it has k-dimensional flat measures at almost every point. This tells us that rectifiable sets are the measure theoretic analogous to differential manifolds.

4.2.1 From Rectifiability to Tangent Measures

The following theorem and proof can be found in 4.8 [5]

Remark 4.2.3. During the proof of the theorem 4.2.9, and assuming the function f to be in $\mathcal{C}^1(\mathbb{R}^k)$, the resulting k-dimensional vector space characterizing the tangent space of $\mu = \mathscr{H}^k \llcorner E$ coincides with the tangent space to the graph of f in the classical sense of differential geometry.

Proposition 4.2.9 (Tangent Space of a rectifiable Set). Let $E \subseteq \mathbb{R}^n$ be a k-rectifiable and $\mu = \mathscr{H}^k \llcorner E$. Then, for $\mathscr{H}^k$-almost every $\mathbf{x} \in E$ there exists a k-dimensional linear space $V_{\mathbf{x}} \subseteq \mathbb{R}^n$ such that

$$\mathrm{Tan}_k\big(\mu, \mathbf{x}\big) = \big\{\mathscr{H}^k \llcorner V_{\mathbf{x}}\big\}.$$

Proof. We start by assuming E to be a subset of a Lipschitz graph, Graph(f), where $f \in \mathrm{Lip}\big(\mathbb{R}^k; \mathbb{R}^{n-k}\big)$. Let $G \subseteq \mathbb{R}^k$ be the Borel set (see Remark 3.3.1) such that

$$E = \Big\{(\mathbf{x}, f(\mathbf{x})) \;\Big|\; \mathbf{x} \in G\Big\}. \tag{4.9}$$

Claim: *for $\mathscr{H}^k$ almost every $(\mathbf{z}, f(\mathbf{z})) \in E \subseteq \mathbb{R}^n$,*

$$\frac{1}{r^k}\big(\mathscr{H}^k \llcorner E\big)_{(\mathbf{z},f(\mathbf{z})),r} \overset{*}{\rightharpoonup} \mathscr{H}^k \llcorner V_{\mathbf{z}}, \tag{4.10}$$

where $V_{\mathbf{z}} := \big\{(\mathbf{y}, df_{\mathbf{z}}(\mathbf{y})) \mid \mathbf{y} \in \mathbb{R}^k\big\}$, analogously to the ususal definition of tangent space to a smooth graph.

Let $H \subseteq G$ be the set of points that verify the following properties: for any $\mathbf{z} \in H$

(1) $\mathbf{z}$ is a point of density 1 with respect to $\mathscr{L}^k$. Notice that G is a Borel set and thus it is $\mathscr{L}^k$-measurable, so, by Lemma 2.2.18, the density with respect to $\mathscr{L}^k$ is 1 for $\mathscr{L}^k$ almost every point in G.

(2) f is differentiable at $\mathbf{z}$. Using Theorem 3.2.1 this holds for $\mathscr{L}^k$ almost every $\mathbf{z} \in G$.

(3) df is approximately continuous at $\mathbf{z}$. Since f is continuous so it is Borel-measurable, thus its differential, when it exists, is Borel-measurable as well, and hence Lebesgue continuous (see 1.7. Theorem 3 [6]).

Let F be the map

$$\begin{aligned} F : \mathbb{R}^k &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto F(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x})) \end{aligned}$$

To simplify the notation, calling Graph$(F) = \Gamma$ and taking into account the invariant translation properties of the $\mathscr{L}^k$ and $\mathscr{H}^k$, we may suppose $(\mathbf{z}_0, f(\mathbf{z}_0)) \in \Gamma$ to be such that $\mathbf{z}_0 = \mathbf{0}$ and $f(\mathbf{z}_0) = \mathbf{0}$. We will prove that, for any test function $\rho \in \mathcal{C}_c(\mathbb{R}^n)$, with $\mathrm{supp}(f) \subseteq B(\mathbf{0}, r)$,

$$\left| \underbrace{\int_{\mathbb{R}^n} \rho(\mathbf{x}) d\big(\mathscr{H}^k \llcorner V_{\mathbf{0}}\big)(\mathbf{x})}_{1.} - \underbrace{\int_{\mathbb{R}^n} \frac{\rho(\mathbf{x})}{r^k} d\big(\mathscr{H}^k \llcorner E\big)_{\mathbf{0},r}(\mathbf{x})}_{2.} \right| \xrightarrow[r\to 0]{} 0. \tag{4.11}$$

By Corollary 3.3.2, we can rewrite term 2. in 4.11 as

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\rho(\mathbf{x})}{r^k} d\big(\mathscr{H}^k \llcorner E\big)_{\mathbf{0},r}(\mathbf{x}) &= \frac{1}{r^k} \int_E \rho\big(\tfrac{\mathbf{x}}{r}\big) d\mathscr{H}^k(\mathbf{x}) \\ &= \frac{1}{r^k} \int_H \rho\Big(\tfrac{F(\mathbf{y})}{r}\Big) JdF_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}) + \frac{1}{r^k} \int_{F(G\setminus H)} \rho\Big(\tfrac{F(\mathbf{z})}{r}\Big) d\mathscr{H}^k(\mathbf{x}). \end{aligned} \tag{4.12}$$

By Lemma 3.1.2 and Theorem 2.2.28 , we obtain that $\mathscr{H}^k(F(G \setminus H)) = 0$ and noticing that for $\mathbf{z} \in \operatorname{supp}\Big(\rho\big(\frac{1}{r}F(\cdot)\big)\Big)$ the following holds

$$\|\mathbf{z}\| = \left\| \frac{F(\mathbf{y})}{r} \right\| \leq \frac{Lip(F)\, \|\mathbf{y}\|}{r} \leq c, \tag{4.13}$$

then in addition to 4.12, term 2 in 4.11 becomes

$$\int_{\mathbb{R}^n} \frac{\rho(\mathbf{x})}{r^k} d\big(\mathscr{H}^k \llcorner E\big)_{\mathbf{0},r}(\mathbf{x}) = \frac{1}{r} \int_E \rho\Big(\frac{\mathbf{x}}{r}\Big) d\mathscr{H}^k(\mathbf{x}) = \frac{1}{r^k} \int_{H \cap B\left(\mathbf{0}, \frac{cr}{Lip(F)}\right)} \rho\Big(\tfrac{F(\mathbf{y})}{r}\Big) JdF_{\mathbf{y}} d\mathscr{L}^k(\mathbf{y}). \tag{4.14}$$

Regarding term 1. of 4.11, applying Corollary 3.3.2, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \rho(\mathbf{x}) d\big(\mathscr{H}^k \llcorner V_{\mathbf{0}}\big)(\mathbf{x}) &= \int_{\mathbb{R}^k} \rho(dF_{\mathbf{0}}(\mathbf{y})) JdF_{\mathbf{0}} d\mathscr{L}^k(\mathbf{y}) \\ &= \frac{1}{r^k} \int_{\mathbb{R}^k} \rho\Big(\tfrac{dF_{\mathbf{0}}(\mathbf{y})}{r}\Big) JdF_{\mathbf{0}} d\mathscr{L}^k(\mathbf{y}). \end{aligned} \tag{4.15}$$

However, since $\operatorname{supp}(\rho) \subseteq B(\mathbf{0}, c)$, combining 4.13 with $\|dF_0(\mathbf{z})\| \leq \operatorname{Lip}(F)$, we can rewrite 4.15 as

$$\underbrace{\frac{1}{r^k} \int_{H \cap B\left(\mathbf{0}, rc(\operatorname{Lip}(F))^{-1}\right)} \rho\Big(\tfrac{dF_{\mathbf{0}}(\mathbf{y})}{r}\Big) JdF_{\mathbf{0}} d\mathscr{L}^k(\mathbf{y})}_{1.1.} + \underbrace{\frac{1}{r^k} \int_{H^c \cap B\left(\mathbf{0}, rc(\operatorname{Lip}(F))^{-1}\right)} \rho\Big(\tfrac{dF_{\mathbf{0}}(\mathbf{y})}{r}\Big) JdF_{\mathbf{0}} d\mathscr{L}^k(\mathbf{y})}_{1.2.}.$$

The point $\mathbf{0}$ has $\mathscr{L}^k$-density 1 in G, then

$$\lim_{r\to 0} \frac{\mathscr{L}^k\Big(B(\mathbf{0},r)\cap G^c\Big)}{r^k} = 0.$$

Hence, for $r<1$:

$$|1.2.| \leq \sup_{\mathbf{y}\in B\left(\mathbf{0},\frac{c}{Lip(F)}\right)} \left|\rho\Big(\frac{dF_{\mathbf{0}}(\mathbf{y})}{r}\Big)\right| \cdot \Big|JdF_{\mathbf{0}}\Big| \frac{1}{r^k}\mathscr{L}^k\bigg(B\Big(\mathbf{0}, rc(\mathrm{Lip}(F))^{-1}\Big)\cap G^c\bigg) \xrightarrow[r\to 0]{} 0. \quad (4.16)$$

All in sum, we can write 4.11 as

$$\left|\frac{1}{r^k}\int_{H\cap B\left(\mathbf{0},\frac{cr}{Lip(F)}\right)} \rho\Big(\frac{dF_0(\mathbf{y})}{r}\Big)JF_0 - \rho\big(\frac{F(\mathbf{y})}{r}\big)JdF_{\mathbf{y}}d\mathscr{L}^k(\mathbf{y})\right| \leq \quad (4.17)$$

$$\underbrace{\frac{1}{r^k}\int_{H\cap B\left(\mathbf{0},cr(Lip(F))^{-1}\right)} \left|\rho\Big(\frac{dF_0(\mathbf{y})}{r}\Big)JdF_{\mathbf{y}} - \rho\Big(\frac{F(\mathbf{y})}{r}\Big)JdF_{\mathbf{y}}\right| d\mathscr{L}^k(\mathbf{y})}_{\mathrm{I}} + \quad (4.18)$$

$$\underbrace{\frac{1}{r^k}\int_{H\cap B\left(\mathbf{0},cr(Lip(F))^{-1}\right)} \left|\rho\Big(\frac{dF_0(\mathbf{y})}{r}\Big)JdF_{\mathbf{0}} - \rho\Big(\frac{dF_{\mathbf{0}}(\mathbf{y})}{r}\Big)JdF_{\mathbf{y}}\right| d\mathscr{L}^k(\mathbf{y})}_{\mathrm{II}} \quad (4.19)$$

Now, to estimate term I, observe that the function $\rho(\frac{\cdot}{r})$ is continuous and, since

$$\|dF_{\mathbf{0}}(\mathbf{y}) - F(\mathbf{y})\| = o(\|\mathbf{y}\|) \xrightarrow[r\to 0]{} 0,$$

then

$$\sup_{\mathbf{y}\in B\left(0,cr(Lip(F))^{-1}\right)} \left|\rho\Big(\frac{dF_{\mathbf{0}}(\mathbf{y})}{r}\Big) - \Big(\frac{F(\mathbf{y})}{r}\Big)\right| \xrightarrow[r\to 0]{} 0. \quad (4.20)$$

Moreover, $\mathscr{L}^k\Big(B(0,cr)\Big) = r^k c^k \omega_k$ and

$$\sup_{\mathbf{y}\in B\left(\mathbf{0},c(Lip(F))^{-1}\right)} \Big|JdF_{\mathbf{y}}\Big| < \infty,$$

so

$$|\mathrm{I}|\leq c^k\omega_k \sup_{\mathbf{y}\in B\left(\mathbf{0},c(Lip(F))^{-1}\right)} \left|JdF_{\mathbf{y}}\right| \sup_{\mathbf{y}\in B\left(0,cr(Lip(F))^{-1}\right)} \left|\rho\left(\frac{dF_{\mathbf{0}}(\mathbf{y})}{r}\right)-\left(\frac{F(\mathbf{y})}{r}\right)\right| \xrightarrow[r\to 0]{} 0.$$

Finally, for term II, since dF is Lebesgue continuous at $\mathbf{0}$:

$$|\mathrm{II}|\leq \sup_{\mathbf{y}\in B\left(\mathbf{0},\frac{c}{Lip(F)}\right)} \left|\rho\left(\frac{dF_{\mathbf{0}}(\mathbf{z})}{r}\right)\right| \frac{1}{r^k}\int_{B\left(\mathbf{0},\frac{cr}{Lip(F)}\right)} \left|JdF_{\mathbf{0}}-JdF_{\mathbf{y}}\right| d\mathscr{L}^k(\mathbf{y}) \xrightarrow[r\to 0]{} 0. \quad \square$$

Lemma 4.2.10 (Locality of $\mathbf{Tan_k(\boldsymbol{\mu},x)}$- 3.12. [5]). Let μ be a measure on $\mathbb{R}^n$ and $f \in L^1(\mu;\mathbb{R}^n)$ a non-negative Borel measurable function, and k a non-negative real number. Then

$$\mathrm{Tan}_k(f\mu,\mathbf{x}) = f(\mathbf{x})\mathrm{Tan}_k(\mu,\mathbf{x}) \quad \text{ for } \mu\text{-almost every} \mathbf{x}\in\mathbb{R}^n$$

Remark 4.2.4. Lemma 4.2.10 tells us that when we rescale a measure μ by a Borel function, then the tangent measures on a point only depend on the value of f at the point, regardless of the behavior of f elsewhere.

Remark 4.2.5. Let $E \subseteq \mathbb{R}^n$ be a Borel set, $\mathbf{x}\in\mathbb{R}^n$, and let $C_{\mathbf{x}}$ be another Borel set containing $\mathbf{x}$ such that $\mu(C_x) < \infty$. Then $1_{C_{\mathbf{x}}\cap E} \in L^1(\mu;\mathbb{R}^n)$, hence

$$\mathrm{Tan}_k(\mu\llcorner E,\mathbf{x}) = \mathrm{Tan}_k(1_E\mu,\mathbf{x}) = \mathrm{Tan}_k(1_{E\cap C_{\mathbf{x}}}\mu,\mathbf{x}) = 1_{E\cap C_{\mathbf{x}}}(\mathbf{x})\mathrm{Tan}_k(\mu,\mathbf{x}) = \mathrm{Tan}_k(\mu,\mathbf{x})$$

for μ almost every $\mathbf{x}\in E$.

Lemma 4.2.10 and Proposition 4.2.9, allow us to conclude the following result, which corresponds to the second implication of Theorem 4.2.8).

Proposition 4.2.11 (Tangent Space of rectifiable Measures). Let μ be a k-rectifiable measure on $\mathbb{R}^n$, then for μ almost every $\mathbf{x}\in\mathbb{R}^n$ there exists a constant $c_{\mathbf{x}} \geq 0$ and a k-dimensional linear subspace $V_{\mathbf{x}} \subseteq \mathbb{R}^n$ such that

$$\mathrm{Tan}_k(\mu,\mathbf{x}) = \left\{c_{\mathbf{x}}\mathscr{H}^k\llcorner V_{\mathbf{x}}\right\}.$$

4.2.2 From Tangent Measures to Rectifiability

We use now the notions of blow-up, tangent measures and Theorem 4.1.8 to prove the first direction of the main result of this chapter, Theorem 4.2.8, which characterizes rectifiable measures in term of their tangent measures.

Proposition 4.2.12. Let μ be a measure such that for μ almost every $\mathbf{x}$ there exist $c_{\mathbf{x}} > 0$ and a k-dimensional linear plane $V_{\mathbf{x}}$ such that

$$\frac{\mu_{\mathbf{x},r}}{r^k} \overset{*}{\rightharpoonup} c_{\mathbf{x}} \mathscr{H}^k \llcorner V_{\mathbf{x}} \text{ as } r \searrow 0.$$

Then μ is a k-rectifiable measure.

The next proof follows the following schema: reduction to a suitable scenario to apply Theorem 4.1.8 via Lemma 4.2.10 and Proposition A.1.4 to perform the calculations.

Proof. We assume μ to be a finite measure. The result is proved for finite measures, it follows as direct consequence for Radon measures, by considering the restriction of μ to a closed ball, $\mu_n \llcorner \overline{B(\mathbf{0}, n)}$, such that $\mathbf{x} \in B(\mathbf{0}, n)$.

We apply Proposition 2.2.31 to infer that there exist a Borel function f and a k-dimensional set E.

$$\mu = f \mathscr{H}^k \llcorner E. \tag{4.21}$$

Since $\mathscr{H}^k$ is a Borel regular measure, we may suppose E to be a Borel.

Claim 1:

$$\Theta^{*k}(\mu, \mathbf{x}) = \Theta^k_*(\mu, \mathbf{x}) = c_{\mathbf{x}} = \lim_{r \searrow 0} \frac{\mu\big(B(\mathbf{x}, r)\big)}{r^k}. \tag{4.22}$$

To prove 4.22 we may apply Proposition A.1.4 to an arbitrary subsequence $\{\mu_{\mathbf{x},r_i}\}_{i=1}^{\infty}$ and its limit $c_{\mathbf{x}} \mathscr{H}^k \llcorner V_{\mathbf{x}}$. We need to prove that $c_{\mathbf{x}} \mathscr{H}^k\big(\partial B(\mathbf{x}, r) \cap V_{\mathbf{x}}\big) = 0$, which follows by the properties of the Hausdorff measure, in particular noticing that $V_{\mathbf{x}} \cap \partial B(\mathbf{x}, r)$ is either the empty set, a point or a smooth s-dimensional curve , with $s < k$).

Hence,

$$\lim_{i \to \infty} \frac{\mu\big(B(\mathbf{x}, r_i)\big)}{r_i^k} = \lim_{i \to \infty} \frac{\mu_{\mathbf{x},r_i}\big(B(\mathbf{0}, 1)\big)}{r_i^k} = c_{\mathbf{x}} \mathscr{H}^k \llcorner V_{\mathbf{x}}\big(B(\mathbf{0}, 1)\big) = c_{\mathbf{x}} \omega_k = \omega_k \Theta^k_*(E, \mathbf{x}).$$

Claim 2: E is rectifiable.

To prove claim 2, we study the k-dimensional tangent measures of μ at $\mathbf{x}$ and use Theorem 4.1.8. To do this, we perform a countable decomposition of E for a more suitable setup.

Notice that $f(\mathbf{x}) > 0$ for $\mathscr{H}^k$ almost every $\mathbf{x} \in E$, otherwise the density of μ is zero in a set A such that $\mu(A) > 0$, by 4.22, this contradicts that $c_{\mathbf{x}} > 0$ at μ-almost every $\mathbf{x}$. Consider the sets $E_0 \coloneqq \big\{\mathbf{x} \in E : f(\mathbf{x}) = 0\big\}$, $E_j \coloneqq E \cap \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) > 0\}$, so that we have

$$E = \bigcup_{j=0}^{\infty} E_j.$$

Since $\mu(E_0) = 0$, it is sufficient to prove that E_j is rectifiable for some $j \in \mathbb{N}$.

Let us first check that $\Theta_*^k\big(E_c, \mathbf{x}\big) > 0$. To do so, we consider the measure $\nu \coloneqq \mathscr{H}^k \llcorner E_c$, and by Lemma 4.2.10,

$$\mathrm{Tan}_k(\mathscr{H}^k \llcorner E_c, \mathbf{x}) = \big\{\tfrac{1}{f(\mathbf{x})} c_{\mathbf{x}} \mathscr{H}^k \llcorner V_{\mathbf{x}}\big\} \qquad \text{for } \mathscr{H}^k \text{almost every } \mathbf{x} \in E_c.$$

Arguing as for 4.22, since $\frac{1}{f(\mathbf{x})} c_{\mathbf{x}} \mathscr{H}^k\big(V_{\mathbf{x}} \cap \partial B(\mathbf{x}, r)\big) = 0$, via Proposition A.1.4 and choosing any subsequence $\{r_i\}_{i=1}^{\infty}$ such that $r_i \to 0$,

$$\Theta_*^k(E_c, \mathbf{x}) = \liminf_{i\to\infty} \frac{\nu\big(B(\mathbf{x}, r_i)\big)}{r_i^k} = \liminf_{i\to\infty} \frac{\nu_{\mathbf{x}, r_i}\big(B(0,1)\big)}{r_i^k} = \tfrac{1}{f(\mathbf{x})} c_{\mathbf{x}} \mathscr{H}^k\big(V_{\mathbf{x}} \cap B(\mathbf{0}, 1)\big) = \tfrac{c_{\mathbf{x}}}{f(\mathbf{x})}\omega_k > 0$$

for $\mathscr{H}^k$-almost every $\mathbf{x} \in E_c$. To prove that $0 < \mathscr{H}^k(E_c) < \infty$, it remains to show that $\mathscr{H}^k(E_c) < \infty$. Indeed,

$$\mathscr{H}^k(E_c) \le \tfrac{1}{c} \int_{\{f>c\}} df \mathscr{H}^k \llcorner E(\mathbf{x}) \le f \mathscr{H}^k \llcorner E\big(\mathbb{R}^n\big) = \mu\big(\mathbb{R}^n\big) < \infty.$$

Finally, we check that, locally, the measure of E is concentrated in cones. In particular, we define these cones based on the affine plane $\mathbf{x} + V_{\mathbf{x}}$; fix $\alpha > 0$, an arbitrary sequence $\{r_i\}_{i=1}^{\infty}$ such that $r_i \searrow 0$, and note that $B(\mathbf{x}, 1) \setminus C(\mathbf{x}, V_x, \alpha)$ is an open set such that

$$\frac{c_{\mathbf{x}}}{f(\mathbf{x})} \mathscr{H}^k\big(V_{\mathbf{x}} \cap \partial(B(\mathbf{x}, 1) \setminus C(\mathbf{x}, V_{\mathbf{x}}, \alpha))\big) \le \frac{c_{\mathbf{x}}}{f(\mathbf{x})} \mathscr{H}^k\big(V_{\mathbf{x}} \cap \partial B(\mathbf{x}, 1)\big) = 0.$$

we can apply again Proposition A.1.4

$$\begin{aligned}
\limsup_{i\to\infty} \frac{1}{r_i^k} \mathscr{H}^k\Big(E_c \cap B(\mathbf{x}, r_i) \setminus C(\mathbf{x}, V_{\mathbf{x}}, \alpha)\Big) &= \limsup_{i\to\infty} \frac{1}{r_i^k} \nu_{\mathbf{x}, r_i}\Big(B(\mathbf{0}, 1) \setminus C(\mathbf{0}, V_{\mathbf{x}}, \alpha)\Big) \\
&= \frac{c_{\mathbf{x}}}{f(\mathbf{x})} \mathscr{H}^k\big(V_{\mathbf{x}} \cap (B(\mathbf{0}, 1) \setminus C(\mathbf{0}, V_{\mathbf{x}}, \alpha))\big) \\
&= 0. \quad \square
\end{aligned}$$

Chapter 5

Differentiability of Lipschitz Functions

5.1 The Decomposability Bundle of a Measure

This section introduces the decomposability bundle, and essential in understanding Alberti–Marchese's results on Lipschitz differentiability.

Along the following chapters we will assume the measures to be finite and Borel in $\mathbb{R}^n$, the results can be extended to σ-finite Borel measures in $\mathbb{R}^n$.

Remark 5.1.1. Given a measure space (I, dt) and $\{\mu_t\}_{t\in I}$ a family of measures, we aim to define a finite measure $\displaystyle\int_I \mu_t dt$, defined on $\mathbb{R}^n$ such that for any E Borel set:

$$\left[\int_I \mu_t dt\right](E) = \int_I \mu_t(E)dt$$

In (1) we define the the form of the elements of the family. Then we consider conditions (2) and (3) to make $\displaystyle\int_I \mu_t dt$ a measure, in particular, (2) allows us to properly state conditions (3) and (4).

Finally, we want this measure to be absolutely continuous with respect to μ, so we establish (4). This definition will help us to study the behavior of μ along 1-rectifiable sets in $\mathbb{R}^n$.

Definition 5.1.1 ($\boldsymbol{\mu}$ -integrable family). Let (I, dt) be a finite measure space. For each $t \in I$, let μ_t be a real measure on $\mathbb{R}^n$ such that

(1) (Rectifiability) $\mu_t = \mathscr{H}^1 \llcorner E_t$, where E_t is a 1- rectifiable set.

(2) (Measurability) For any Borel set E in $\mathbb{R}^n$, the map $t \longmapsto \mu_t(E)$ is dt-measurable.

(3) (Finiteness)

$$\int_I |\mu_t(\mathbb{R}^n)| dt < \infty.$$

(4) (Absolute continuity with respect to μ)

$$\int_I \mu_t \ll \mu.$$

We say that a pair $\big(\{\mu_t\}_{t\in I}(I, dt)\big)$ holding the four previous assumption is a μ-integrable family. We denote by $\mathscr{F}_\mu$ to the family of all μ-integrable families. Given an element $\big(\{\mu_t\}_{t\in I}(I, dt)\big) \in \mathscr{F}_\mu$, we denote it by $\{\mu_t : t \in I\}$.

Remark 5.1.2. Notice that for each $t \in I$ the E_t is 1-rectifiable, and $\operatorname{apTan}\big(E_t, \mathbf{x}\big) = \operatorname{Tan}\big(E_t, \mathbf{x}\big)$ for $\mathscr{H}^1$-almost every $\mathbf{x} \in E_t$ (see proof of Proposition 4.2.11). Hence we will directly speak of $\operatorname{Tan}\big(E_t, \mathbf{x}\big)$ from now on.

Remark 5.1.3. By definition of the measure space (I, dt), we can get a probability on I (assuming dt not to be degenerated) by normalizing it.

Definition 5.1.2 (Closed family of maps). Let $\mathcal{F}$ be a family of maps, we say that $\mathscr{F}$ is **closed under countable intersection** if for any countable subfamily $\big\{V_i\big\}_{i\in\mathbb{N}} \subseteq \mathscr{F}$, the map defined as $V(x) = \cap_{i=1}^{\infty} V_i(x)$ is a member of $\mathscr{F}$.

Definition 5.1.3 ($\boldsymbol{\mu}$-minimal element). Let μ be a measure on $\mathbb{R}^n$ and $\mathscr{G}$ a family of maps from $\mathbb{R}^n$ to $\operatorname{Gr}\big(\mathbb{R}^n\big)$. If there exists an element $\tilde{V} \in \mathscr{G}$ such that, for every $V \in \mathscr{G}$,

$$\tilde{V}(\mathbf{x}) \subseteq V(\mathbf{x}) \qquad \text{for } \mu\text{-almost every } \mathbf{x} \in \mathbb{R}^n,$$

we say that $\tilde{V}$ is **$\boldsymbol{\mu}$-minimal in $\mathscr{G}$**.

Remark 5.1.4. Notice that $\tilde{V}$ is unique module μ zero sets.

We introduce now the concept of essential span of a family vector fields. The idea is to consider a minimal (in the sense of Definition 5.1.3) Borel map from $\mathbb{R}^n$ to $\operatorname{Gr}(\mathbb{R}^n)$ such that contains all images of a given family of vector fields, up to module μ-zero measure sets. We use this notion to find a Borel map into the Grassmannian such that it is μ-minimal and contains all possible directions in which we can decompose μ (the directions will be the elements of the tangent space to the measure, when they exist).

Definition 5.1.4 (μ-essential span of a family of vector fields). Let μ be a measure on $\mathbb{R}^n$, $\mathscr{F}$ be a family of Borel vector fields on $\mathbb{R}^n$ and $\mathscr{G}_\mu$ a family of all Borel maps from $\mathbb{R}^n$ to $\mathrm{Gr}\big(\mathbb{R}^n\big)$ such that, $V \in \mathscr{G}_\mu$ if for every $V \in \mathscr{F}$

$$\tau(\mathbf{x}) \in V(\mathbf{x}) \qquad \text{for } \mu\text{-almost every } \mathbf{x} \in \mathbb{R}^n.$$

If $\mathscr{G}_\mu$ admits a μ-minimal element, we call it the **μ-essential span of $\mathscr{F}$**

Remark 5.1.5. Notice that the definition of essential span of $\mathscr{F}_\mu$ is not completely correct in the sense that is not unique, but the differences between the μ-minimal elements are restricted to μ-zero measure sets. We consider this detail as an abuse of language committed for the sake of clarity

Lemma 5.1.5 (Existence of μ-minimal element in $\mathscr{G}$ - 2.3. [1]). Le μ be a measure, and let $\mathscr{G}$ be a family of Borel maps from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$ which is closed under countable intersection. Then $\mathscr{G}$ admits a μ-minimal element.

Proof. We write

$$\Phi(V) := \int_{\mathbb{R}^n} \dim\big(V(\mathbf{x})\big) d\mu(\mathbf{x}).$$

Noticing that $\Phi \geq 0$, we can take a minimizing sequence $\big\{V_i\big\}_{i\in\mathbb{N}}$ such that

$$\Theta := \inf_{V \in \mathcal{G}_\mu} \Phi(V) = \liminf_{i\to\infty} \Phi(V_i).$$

Let us define $V := \cap_{i=1}^{\infty} V_i$.

Claim: *V is a μ-minimal element in $\mathscr{G}$.*

It is clear that $V \in \mathcal{G}_\mu$. Arguing by contradiction, let us assume that V is not minimal, then there exists a set $A \subseteq \mathbb{R}^n$ with $\mu(A) > 0$ such that for all $\mathbf{x} \in A$ then there is an element $V'_{\mathbf{x}} \in \mathscr{G}$ such that $V'_{\mathbf{x}}(\mathbf{x}) \nsubseteq V(\mathbf{x})$.
Now we consider the Borel map defined as $\tilde{V}(\mathbf{x}) = V'_{\mathbf{x}}(\mathbf{x}) \cap V(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^n$. Since $\tilde{V}(\mathbf{x}) \subset V(\mathbf{x})$ (strictly contained) for all $\mathbf{x} \in A$, hence

$$\begin{aligned}\Phi(\tilde{V}) &= \int_{\mathbb{R}^n\setminus A} \dim\big(\tilde{V}(\mathbf{x})\big) + \int_A \dim\big(\tilde{V}(\mathbf{x})\big)\\ &< \int_{\mathbb{R}^n\setminus A} \dim\big(V(\mathbf{x})\big) + \int_A \dim\big(V(\mathbf{x})\big) = \Phi(V) = \Theta,\end{aligned}$$

yielding a contradiction, since $\tilde{V} \in \mathscr{G}$ and Θ is the infimum of the values for Φ over $\mathscr{G}$. $\square$

Remark 5.1.6. Consider a finite measure μ and a μ-integrable family $\{\mu_t : t \in I\}$, then for each member of the family, $\mu_t = \mathscr{H}^1 \llcorner E_t$, the approximate tangent plane $\mathrm{Tan}(E_t, \mathbf{x})$ is defined and unique for $\mathscr{H}^1$-almost every $\mathbf{x} \in E_t$.

Definition 5.1.6 (**Family of Decompositions of a Measure**). Let μ be a finite measure on $\mathbb{R}^n$. We denote by $\mathscr{F}_\mu$ the class of all μ-integrable families such that $\mathrm{Tan}(E_t, \mathbf{x})$ is well defined across each family for dt-almost every $t \in I$. Moreover, we require the elements of $\mathscr{F}_\mu$ considered as vector fields to be Borel measurable and we will refer to it by the **family of decompositions of** μ.

Remark 5.1.7. Given an element in $\{\mu_t : t \in I\} \in \mathscr{F}_\mu$, and noticing that, when it exists, the approximate tangent plane of E_t is a 1-dimensional linear space in $\mathbb{R}^n$. We consider the map

$$\begin{aligned} F : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto F(\mathbf{x}) = \begin{cases} \mathrm{Tan}(E_t, \mathbf{x}) & \text{when it is well defined in } \{\mu_t : t \in I\} \\ \mathbf{0} & \text{otherwise} \end{cases} \end{aligned}$$

, where we are fixing a direction in $\mathrm{Tan}(E_t, \mathbf{x})$. Thus, we will refer to the elements of the family $\mathscr{F}_\mu$ described above as if they were Borel vector fields on $\mathbb{R}^n$.

Definition 5.1.7 (**Family of Spans of the Decomposition of a Measure**). Let μ be a finite measure on $\mathbb{R}^n$, and $\mathscr{F}_\mu$ its family of decompositions. We denote by $\mathscr{G}_\mu$ the family of all Borel maps from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$ such that if $V \in \mathscr{G}_\mu$, then, for each element $\{\mu_t = \mathscr{H}^1 \llcorner E_t : t \in I\} \in \mathscr{F}_\mu$,

$$\mathrm{Tan}(E_t, \mathbf{x}) \in (\subseteq) V(\mathbf{x}) \text{ for } \mathscr{H}^1\text{-almost every } \mathbf{x} \in E_t \text{ and } dt\text{-almost every } t \in I.$$

We refer to $\mathscr{G}_\mu$ as the **family of spans of the decomposition of $\boldsymbol{\mu}$** or simply as the family of spans of μ.

Remark 5.1.8. Notice that $\mathscr{G}_\mu$ is a family of Borel maps, closed under countable intersection.

We now introduce the notion of decomposability bundle of a measure μ. It describes the μ-minimal Borel map containing all the tangent directions given by the family of possible decompositions of μ, i.e. the μ-essential span of the family of decompositions of μ.

Definition 5.1.8 (**Decomposability Bundle of a Measure**). Let μ be a measure on $\mathbb{R}^n$, $\mathscr{F}_\mu$ the family of decompositions of μ and $\mathscr{G}_\mu$ the family of spans of μ. We denote by

$$\begin{aligned} V(\mu,\cdot) : \mathbb{R}^n &\longrightarrow Gr\big(\mathbb{R}^n\big) \\ \mathbf{x} &\longmapsto V(\mu,\mathbf{x}) \end{aligned}$$

any of the μ-minimal elements of $\mathscr{G}_\mu$, and call them the **decomposability bundle of** μ.

Remark 5.1.9. The decomposability bundle of a measure μ is well defined up to μ-zero measure sets.

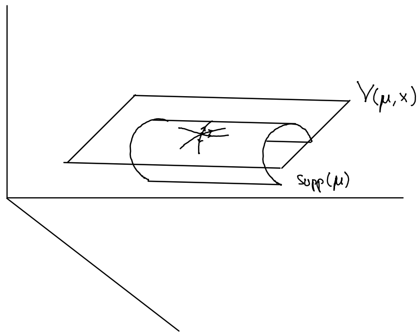

Figure 5.1: Decomposability Bundle

Definition 5.1.9. Let μ be a measure on $\mathbb{R}^n$ and fix a decomposition of μ, $F = \{\mathscr{H}^1 \llcorner E_t : t \in I\} \in \mathscr{F}_\mu$. We denote by $\mathscr{G}_\mu(F)$ to the family of all Borel maps V from $\mathbb{R}^n$ to $Gr\big(\mathbb{R}^n\big)$ such that $\mathrm{Tan}\big(E_t, \mathbf{x}\big) \subseteq V(\mathbf{x})$ for μ-almost every $\mathbf{x}$ and dt-almost every $t \in I$. We denote by $V(\mu, F, \cdot)$ to the μ-minimal element(s) of such a family.

Remark 5.1.10. Let $\mathcal{M}_+$ be the space of non-negative, finite measures on $\mathbb{R}^n$ endowed with the weak-* topology and the resulting Borel σ-algebra. We define:

$$\mathcal{R} := \left\{ \mathscr{H}^1 \llcorner E \in \cap \mathcal{M} \,\middle|\, E \text{ is 1-rectifiable and } \mathscr{H}^1(E) < +\infty \right\}.$$

For a fixed family $\{\mu_t\}_{t\in I} \subset \mathscr{F}_\mu$, we can assume $(I, \mathcal{I})$ to be a probability space and that $\mathscr{H}^1(E_t) < +\infty$ for all $t \in I$ (this does not affect our definition of $\mathcal{G}_\mu$, since it is true μ-almost everywhere). We call

$$\begin{aligned} F : I &\longrightarrow \mathcal{R} \\ t &\longmapsto \mu_t \end{aligned}$$

Then define $\psi : \mathcal{I} \to \mathcal{R}$ as a probability on $\mathcal{M}$ by

$$\psi(A) := \lambda(F^{-1}(A)).$$

Notice that ψ has support in $\mathcal{R}$ and a first moment:

$$\int_{\mathcal{M}} \lambda d\psi(\lambda) = \int_I \mathscr{H}^1 \llcorner E_t \, dt \ll \mu. \tag{5.1}$$

By 5.1, we obtain that

$$\sup_{t\in I} \mathscr{H}^1(E_t) < +\infty. \tag{5.2}$$

Hence, we may define $\mathscr{G}_\mu$ as the class of all Borel maps $\mathbf{x} \mapsto V(\mathbf{x})$ such that for all probabilities ψ supported on $\mathcal{R}$ such that their first moment (as a measure on $\mathbb{R}^n$) is absolutely continuous with respect to μ, and such that for ψ-almost every $t \in \mathcal{R}$,

$$\operatorname{Tan}(E_t, \mathbf{x}) \subseteq V(\mathbf{x}) \quad \text{for } \mathscr{H}^1\text{-almost every } x \in E.$$

We consider now $\mathcal{M}_+(\mathbb{R}^n)$ endowed with the weak-* topology, by Theorem see A.2.5, any tight and uniformly bounded set is relatively compact.Then, if we consider a family $\{\mu_t\}_{t\in I} \in \mathscr{F}_\mu$ then $\mu_t \in \mathcal{R}$ for all $t \in I$ and, by 5.2, then $\sup_{t\in I} \mathscr{H}^1(E_t) < +\infty$: Hence, applying Proposition A.2.4, there exists a k-parametrization ϕ of a probability ψ supported on $\{\mu_t\}_{t\in I}$ such that for any $A \subset \{\mu_t : t \in I\}$

$$\psi(A) = \mathcal{L}^1([0,1] \cap \phi^{-1}(A)).$$

So $\psi = \phi_\# \mathcal{L}^1 \llcorner [0,1]$.

All together, $\mathcal{G}_\mu$ remains the same if for any element $\{\mu_t\}_{t\in I} \subset \mathcal{F}_\mu$ we impose the condition $\mu_t(\mathbb{R}^n) < +\infty$ and

$$(I, \mathcal{I}) = ([0,1], \mathcal{L}^1 \llcorner [0,1]).$$

The following result will be used as a technical result during the proof of proposition 5.1.13.

Lemma 5.1.10 (Proposition 2.11. [1]). Let μ be a finite measure on $\mathbb{R}^n$, then

1. For any $F \in \mathcal{F}_\mu$,

$$V(\mu, \mathbf{x}) \subseteq V(\mu, F, \mathbf{x}) \text{ for } \mu\text{-almost every } \mathbf{x} \in \mathbb{R}^n.$$

2. There exists an element $G \in \mathcal{F}_\mu$ such that

$$V(\mu, G, \mathbf{x}) = V(\mu, \mathbf{x}) \text{ for } \mu\text{-almost every } \mathbf{x} \in \mathbb{R}^n.$$

Proof. We have that for $F = \{\mu_t : t \in I\} \in \mathscr{F}_\mu$, $\mathrm{Tan}\big(E_t, \mathbf{x}\big) \subseteq V(\mu, \mathbf{x})$ for μ_t-almost every $\mathbf{x}$ and dt-almost every $t \in I$, thus $V(\mu, \cdot) \in \mathscr{G}_\mu(F)$. $V(\mu, F, \cdot)$ is μ-minimal in $\mathscr{G}_\mu(F)$, so we have that

$$V(\mu, \mathbf{x}) \subseteq V(\mu, F, \mathbf{x}) \text{ for } \mu\text{-almost every}\mathbf{x},$$

getting 1.

For 2, we write the functional Φ such that for each $F \in \mathscr{F}_\mu$,

$$\Phi(F) := \int_{\mathbb{R}^n} \dim\big(V(\mu, F, \mathbf{x})\big) d\mu(\mathbf{x}).$$

Claim 1: *there exists an element $G \in F_\mu$ such that maximizes Φ over $\mathcal{F}_\mu$*

Consider a maximizing sequence $\big\{F_i\big\}_{i=1}^{\infty} \subseteq \mathscr{F}_\mu$ such that

$$\Theta := \sup_{F \in \mathscr{F}_\mu} \Phi(F) = \limsup_{i \to \infty} \Phi(F_i),$$

where $F_i = \big\{\mu_{t_i} : t \in I_i\big\}$ and $\mu_{t_i} = \mathscr{H}^1 \llcorner E_{t_i}$, where (I_i, dt_i) is a probability space, for each $i \in \mathbb{N}$. We can take $I_i \cap I_j = \emptyset$ for all $i \neq j$ (see Remark 5.1.3), moreover, we can assume that $\int_{I_i} \mu_{t_i} dt_i$ is a probability, for each $i \in \mathbb{N}$.

Let us consider now $I := \bigcup_{i=1}^{\infty} I_i$ and the probability space (I, dt), where $dt_{|I_i} = \frac{1}{2^i} dt_i$ and $\mu_t = \mu_{t_i}$ if $t \in I \cap I_i$. We define then $G := \{\mu_t : t \in I\}$.

Let us first check that G is a μ-integrable family:

(1) $\mu_t = \mathscr{H}^1 \llcorner E_t$ where $E_t = E_{t_i}$ is a 1-rectifiable set for $t \in I \cap I_i$.

(2) For every Borel set $E \subseteq \mathbb{R}^n$, $t \mapsto \mu_t(E) = \sum_{i=1}^{\infty} 1_{I_i}(t) \mu_{t_i}(E)$ is dt-measurable.

(3) Finiteness:

$$\int_I \mu_t(\mathbb{R}^n)dt \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \int_{I_i} \mu_{t_i}(\mathbb{R}^n)dt_i < \infty.$$

(4) $\int_I \mu_t dt \ll \mu$: if $A \subseteq \mathbb{R}^n$ is such that $\mu(A) = 0$, then

$$\int_I \mu_t(A)dt \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \int_{I_i} \mu_{t_i}(A)dt_i = 0.$$

Moreover, $G \in \mathscr{F}_\mu$: indeed, for each decomposition F_i, $\operatorname{Tan}\big(E_{t_i}, \mathbf{x}\big)$ is well defined for all $\mathbf{x} \notin C_i$ and all $t \notin J_i$ where $\mathscr{H}^1(C_i) = 0$ and $dt_i(J_i) = 0$, so it will be well defined for all $\mathbf{x} \notin C = \cup_i^\infty C_i$ and all $t \notin J = \cup_{i=1}^\infty J_i$.

Finally, let us prove that G is indeed a maximum for Φ on $\mathscr{F}_\mu$: noticing that $\mathscr{G}_\mu(G) \subseteq \mathscr{G}_\mu(F_i)$ for each $i \in I$, by definition of $V(\mu, F_i, \cdot)$ as the μ-minimal element in $\mathscr{G}_\mu(F_i)$, we have that $V(\mu, F_i, \mathbf{x}) \subseteq V(\mu, G, \mathbf{x})$ for μ-almost every $\mathbf{x}$. This implies that

$$\Theta = \limsup_{i \to \infty} \Phi(F_i) = \limsup_{i \to \infty} \int_{I_i} \dim\big(V(\mu, F_i, \mathbf{x})\big) d\mu(\mathbf{x}) \leq \int_I \dim\big(V(\mu, G, \mathbf{x})\big) d\mu(\mathbf{x}) = \Phi(G).$$

Claim 2:

$$V(\mu, G, \mathbf{x}) = V(\mu, \mathbf{x}) \textit{ for } \mu\textit{-almost every } \mathbf{x} \in \mathbb{R}^n.$$

For any $F_0 = \big\{\mu_{t_0} : t_0 \in I_0\big\} \in \mathcal{F}_\mu$:

$$V(\mu, F_0, \mathbf{x}) \subseteq V(\mu, G, \mathbf{x}) \text{ for } \mu\text{-almost every} \mathbf{x} \in \mathbb{R}^n.$$

Otherwise, there exists a set A of measure $\mu(A) > 0$, consider the element $F_0 \cup G = \big\{\mu_t : t \in I'\big\} \in \mathcal{F}_\mu$, where $I' = I_0 \cup I$ (where we can choose I_0 such that $I_0 \cap I = \emptyset$) and $dt' = \frac{1}{2}(dt + dt_0)$, $\mu_t = \mu_{t_0}$ when $t \in I_0$ and $\mu_t = \mu_t$ when $t \in I$. Then $\dim\big(V(\mu, G, \mathbf{x})\big) < \dim\big(V(\mu, F \cup G, \mathbf{x})\big)$ for all $\mathbf{x} \in A$, so

$$\Phi(G) < \Phi\big(G \cup F\big),$$

a contradiction. So we have proved that $V(\mu, G, \cdot)$ is a μ-minimal element if $\mathscr{G}_\mu$, so $V(\mu, G, \mathbf{x}) = V(\mu, \mathbf{x})$ for μ-almost every $\mathbf{x} \in \mathbb{R}^n$. □

Lemma 5.1.11. Let μ and μ' be two measures on $\mathbb{R}^n$ such that $\mu \ll \mu'$ and consider $F \in \mathscr{F}_\mu \cap \mathscr{F}_{\mu'}$. Then $\mathscr{G}_\mu(F) \subseteq \mathscr{G}_{\mu'}(F)$. In particular,

$$V(\mu', F, \mathbf{x}) \subseteq V(\mu, F, \mathbf{x}).$$

Proof. Let $F = \{\mathscr{H}^1 \llcorner E_t : t \in I\}$, take $V \in \mathscr{G}_\mu$ and consider the set $A := \{\mathbf{x} \in \mathbb{R}^n : \mathrm{Tan}(E_t, \mathbf{x}) \nsubseteq V(\mathbf{x})\}$, by definition of $\mathscr{G}_\mu$, $\mu(A) = 0$, but $\mu' \ll \mu$, so $\mu'(A) = 0$, hence $V \in \mathscr{G}_{\mu'}(F)$.

In particular, since $V(\mu, F, \cdot) \in \mathscr{G}_\mu \subseteq \mathscr{G}_{\mu'}$ and $V(\mu', F, \cdot)$ is μ'-minimal in $\mathscr{G}_{\mu'}$ we have that $V(\mu', F, \mathbf{x}) \subseteq V(\mu, F, \mathbf{x})$ for μ'-almost every $\mathbf{x}$. $\square$

The following result will be used during the proof of the last point of proposition 5.1.13.

Lemma 5.1.12 (Lemma 7.4., [1])**.** For every measure μ on $\mathbb{R}^n$ one of the following (mutually incompatible) alternatives holds:

(1) μ is supported on a purely unrectifiable set E.

(2) There exists a nontrivial measure of the form $\mu' = \int_I \mu_t \, dt$ where μ' is absolutely continuous with respect to μ and each μ_t is the restriction of $\mathcal{H}^1$ to some 1-rectifiable set E_t.

Proposition 5.1.13. **[Properties of de Decomposability Bundle -** 2.9. [1]] Let μ and μ' be two finite measures on $\mathbb{R}^n$. Then:

(1) Strong Locality Principle: If $\mu' \ll \mu$ then

$$V(\mu, \mathbf{x}) = V(\mu', \mathbf{x}) \text{ for } \mu'\text{-almost every } \mathbf{x}.$$

Moreover, let $E \subseteq \mathbb{R}^n$ be a set such that $1_E \mu' \ll \mu$, then

$$V(\mu', \mathbf{x}) = V(\mu, \mathbf{x}) \text{ for } \mu\text{-almost every} \mathbf{x} \in E.$$

(2) If S is a k-dimensional $\mathcal{C}^1$ surface and $\mathrm{supp}(\mu) \subseteq S$. Then

$$V(\mu, \mathbf{x}) \subseteq \mathrm{Tan}(S, \mathbf{x}) \text{ for } \mu\text{-almost every } \mathbf{x}.$$

(3) Let E be a k-rectifiable set and let μ be such that $\mu \ll \mathscr{H}^k \llcorner E$, then

$$V(\mu, \mathbf{x}) = \mathrm{Tan}(E, \mathbf{x}) \text{ for } \mu\text{-almost every } \mathbf{x} \in E.$$

In particular $\mu \ll \mathscr{L}^k$ then $V(\mu, \mathbf{x}) = \mathbb{R}^n$ for μ-almost every $\mathbf{x} \in \mathbb{R}^n$.

(4) $V(\mu, \mathbf{x}) = \{\mathbf{0}\}$ for μ-almost every if, and only if, $\operatorname{supp}(\mu) \subseteq E$ where E is purely unrectifiable.

Proof. Using lemma 5.1.10, let us choose $G' \in \mathscr{F}_{\mu'}$ and $G \in \mathscr{F}_\mu$ such that

$$V(\mu, \mathbf{x}) = V(\mu, G, \mathbf{x}) \qquad \text{for } \mu\text{-almost every } \mathbf{x},$$
$$V(\mu', \mathbf{x}) = V(\mu, G', \mathbf{x}) \qquad \text{for } \mu\text{-almost every } \mathbf{x}$$

With the usual notation $G = \{\mu_t = \mathscr{H}^1 \llcorner E_t : t \in I\}$ and $G' = \{\mu'_t = \mathscr{H}^1 \llcorner E'_t : t \in I'\}$.

(1) Proof of the first statement:

Claim 1.1: $V(\mu', \mathbf{x}) = V(\mu', G', \mathbf{x}) \subseteq V(\mu, \mathbf{x})$ *for* μ'*-almost every* $\mathbf{x}$.

If G' is in the family of decompositions of μ, i.e. $G' \in \mathscr{F}_\mu$, then the claim is proved by statement 1 in lemmas 5.1.10 and 5.1.11, because

$$V(\mu', \mathbf{x}) = V(\mu', G', \mathbf{x}) \subseteq V(\mu, G', \mathbf{x}) \subseteq V(\mu', \mathbf{x}) \quad \text{for } \mu'\text{-almost every } \mathbf{x}.$$

It remains to show that $\int_{I'} \mu'_t dt \ll \mu$, which holds since

$$\int_{I'} \tilde{\mu}'_t dt \ll \mu' \ll \mu.$$

Claim 1.2: $V(\mu, \mathbf{x}) \subseteq V(\mu', \mathbf{x}) = V(\mu', G', \mathbf{x})$ *for* μ' *-almost every* $\mathbf{x}$.

The idea is to reduce this case to the previous one by finding a suitable set C such that $1_C\mu \ll \mu'$. Notice that since μ' is Borel and finite, choosing $\epsilon > 0$ arbitrarily, we may consider closed sets $C_\epsilon \subseteq \operatorname{supp}(\mu')$ such that $\mu'\big(C_\epsilon^c\big) = \mu'\big(\operatorname{supp}(\mu') \setminus C_\epsilon\big) < \epsilon$. Then, for any $\tilde{\epsilon} > 0$ and the set $C := \cup_{\epsilon>0} C_\epsilon$ the following inequality holds

$$\mu'\Big(\operatorname{supp}(\mu') \setminus \bigcup_{\epsilon>0} C_\epsilon\Big) \leq \tilde{\epsilon}.$$

Hence $\mu'(C) = \mu'\big(\operatorname{supp}(\mu')\big)$. It is enough then to prove that

$$V(\mu, \mathbf{x}) \subseteq V(\mu', \mathbf{x}) \text{ for } \mu\text{-almost every } \mathbf{x} \in C_\epsilon.$$

Notice that $1_C\mu \ll \mu' \ll \mu$. Consider now the family $G'' := \{1_C\mu_t : t \in I\}$ arguing as for the previous claim and using the definition of measure it is easy to show that $G'' \in \mathscr{F}_{\mu'} \cap \mathscr{F}_\mu$

By lemmas 5.1.5 and 5.1.11 the following holds

$$V(1_C\mu, G'', \mathbf{x}) \subseteq V(\mu', G'', \mathbf{x}) \subseteq V(\mu', \mathbf{x}) \quad \text{for } \mu\text{-almost every} \mathbf{x} \in C$$

Moreover, $V(1_C\mu, G'', \mathbf{x}) = V(\mu, G'', \mathbf{x})$ for every μ-almost every $\mathbf{x} \in C$ (by the μ zero dependence in the choice of decomposability bundles). Finally, as a consequence of lemma 4.2.10, $V(\mu, G'', \mathbf{x}) = V(\mu, G, \mathbf{x}) = V(\mu, \mathbf{x})$, since C is a Borel set such that $\mu(C) < \infty$, for every μ almost every $\mathbf{x} \in C$ we have that

$$\operatorname{apTan}(\mu_t, \mathbf{x}) = \operatorname{apTan}(1_C\mu_t, \mathbf{x})$$

so G and G'' provide with the same tangent planes for μ-almost every $\mathbf{x} \in C$, finishing the proof of the first statement in (1).

Claim 1.3*: Let $E \subseteq \mathbb{R}^n$ such that $1_E\mu' \ll \mu$, then $V\big(\mu', \mathbf{x}\big) = V\big(\mu, \mathbf{x}\big)$ for μ'-almost every $\mathbf{x} \in E$.*

Applying the previous result to $1_E\mu' \ll \mu$ and $1_E\mu' \ll \mu'$ we get that

$$V\big(\mu', \mathbf{x}\big) = V\big(1_E\mu', \mathbf{x}\big) = V\big(\mu, \mathbf{x}\big) \text{ for } \mu'\text{-almost every } \mathbf{x} \in E.$$

(2) Proof of the second statement:

Notice that, since $\operatorname{supp}(\mu) \subseteq S$,

$$0 = \mu\big(\mathbb{R}^n \setminus S\big) = \int_I \mu_t\big(\mathbb{R}^n \setminus S\big)dt = \int_I \mu_t\big(E_t \setminus S\big)dt,$$

which implies that for dt-almost every $t \in I$, $E_t \subseteq S$ for $\mathscr{H}^1$-almost every $\mathbf{x} \in E_t$. Hence for dt-almost every $t \in I$, $\operatorname{Tan}\big(E_t, \mathbf{x}\big) \subseteq \operatorname{Tan}\big(S, \mathbf{x}\big)$ for $\mathscr{H}^1$-almost every $\mathbf{x} \in E_t$, thus

$$V\big(\mu, \mathbf{x}\big) = V\big(\mu, G, \mathbf{x}\big) \subseteq \operatorname{Tan}\big(S, \mathbf{x}\big) \text{ for } \mu\text{-almost every } \mathbf{x}.$$

(3) Proof of the third statement:

By the first statement of this proposition, we have that

$$V(\mu, \mathbf{x}) = V(\mathscr{H}^k \llcorner E, \mathbf{x}) \text{ for } \mu\text{-almost every } \mathbf{x}.$$

Hence, we will prove the result directly for $\mu = \mathscr{H}^k \llcorner E$. We can consider a more suitable setup by using proposition 4.1.6, which tells us that we can assume S to be a k-dimensional surface of class $\mathcal{C}^1$ such that $E \subseteq S$ for $\mathscr{H}^k$-almost every $\mathbf{x} \in E$. Let $\rho : A \longrightarrow S$ be a diffeomorphism parameterizing S. We assume that $A \subseteq \mathbb{R}^k$ is an open and bounded set (otherwise we cover A by counterimages of balls centerd at S).

Then $\mathrm{Tan}\big(E, \mathbf{x}\big) = \mathrm{Tan}\big(S, \mathbf{x}\big)$ for $\mathscr{H}^k$-almost every $\mathbf{x} \in E$ and by statement 2 of the present proposition, $V(\mu, \mathbf{x}) \subseteq \mathrm{Tan}\big(E, \mathbf{x}\big) = \mathrm{Tan}\big(S, \mathbf{x}\big)$ for μ-almost every $\mathbf{x}$.

Claim 3.1: $\mathrm{Tan}\big(S, \mathbf{x}\big) \subseteq V(\mu, \mathbf{x})$ *for μ-almost every* $\mathbf{x}$.

Let us set $E' := \rho^{-1}(E) \subseteq \mathbb{R}^k$ and $\mu' := \mathscr{L}^k \llcorner E'$. We will cover E' with segments (A is bounded). Let us choose a non-trivial vector $\mathbf{x}_0 \in \mathbb{R}^k$ and define, for each $\mathbf{y} \in \langle \mathbf{x}_0 \rangle^\perp$ the sets

$$E'_{\mathbf{y}} := E' \cap \big\{\mathbf{y} + t\mathbf{x}_0 \,:\, t \in \mathbb{R}\big\}.$$

Let us consider $I := P_{\langle \mathbf{x}_0\rangle^\perp}\Big(\bigcup_{y \in \langle \mathbf{x}_0\rangle^\perp} E'_{\mathbf{y}}\Big)$ and $d\mathbf{y} := \mathscr{H}^{k-1} \llcorner I$. For any ball $B \subseteq \mathbb{R}^k$, by Fubini we have

$$\begin{aligned}
\int_I \mu_{\mathbf{y}}(B) d\mathscr{H}^{k-1}(\mathbf{y}) &= \int_I \mathscr{H}^1(E'_{\mathbf{y}} \cap B) d\mathscr{H}^{k-1}(\mathbf{y}) \\
&= \int_I \int_{E'_{\mathbf{y}} \cap B} d\mathscr{H}^1(\mathbf{x}) d\mathscr{H}^{k-1}(\mathbf{y}) \\
&= \int_{P_{\langle \mathbf{x}_0\rangle^\perp}\left(\bigcup_{y \in \langle \mathbf{x}_0\rangle^\perp} E'_{\mathbf{y}}\right)} \int_{\{\mathbf{y} + t\mathbf{x}_0 \,:\, t \in \mathbb{R}\} \cap B \cap E'} d\mathscr{H}^1(\mathbf{x}) d\mathscr{H}^{k-1}(\mathbf{y}) \\
&= \int_{E' \cap B} d\mathscr{H}^k(\mathbf{z}) \\
&= \int_B d\mathscr{L}^k \llcorner E'(\mathbf{z}) \\
&= \mathscr{L}^k \llcorner E'(B) \\
&= \mu'(B).
\end{aligned}$$

Then

$$\big\{\mathscr{H}^1 \llcorner E'_{\mathbf{y}} \,:\, \mathbf{y} \in I\big\} \in \mathscr{F}_{\mu'}.$$

Now we go back to $\mathbb{R}^n$, set $E_{\mathbf{y}} := \rho(E'_{\mathbf{y}})$ and $\mu_{\mathbf{y}} := \mathscr{H}^1 \llcorner E_{\mathbf{y}}$. Taking into account that ρ is a diffeomorphism and using the formula of the area (proposition 3.3.1), we claim:

Claim 3.2:

$$\mu \ll \int_I \mu_{\mathbf{y}} d\mathbf{y} \text{ and } \int_I \mu_{\mathbf{y}} d\mathbf{y} \ll \mu.$$

Consider a ball $B \subseteq \mathbb{R}^n$, First, notice that the $G(\mathbf{x}) = (\mathbf{x}, \rho(\mathbf{x}))$ is an injective Lipschitz function. Now call $B' = \rho^{-1}(B)$ and write

$$\begin{aligned}\int_I \mu_{\mathbf{y}}(B)d\mathbf{y} &= \int_I \int_{B\cap E_{\mathbf{y}}} d\mathscr{H}^1(\mathbf{x})\mathscr{H}^{k-1}(\mathbf{y})\\ &= \int_I \mathscr{H}^1\big(G(E'_{\mathbf{y}} \cap B'))\big)d\mathscr{H}^{k-1}(\mathbf{y})\\ &= \int_I \int_{E'_{\mathbf{y}}\cap B'} JdF_{\mathbf{x}}d\mathscr{L}^1(\mathbf{x})\mathscr{H}^{k-1}(\mathbf{y})\\ &= 0.\end{aligned}$$

And this holds if and only if,

$$\int_I \int_{E'_{\mathbf{y}}\cap B'} d\mathscr{L}^1(\mathbf{x})\mathscr{H}^{k-1}(\mathbf{y}) = \int_I \int_{E'_{\mathbf{y}}\cap B'} d\mathscr{H}^1(\mathbf{x})\mathscr{H}^{k-1}(\mathbf{y}) = \mathscr{H}^k(E'\cap B') = \mathscr{L}^k(E'\cap B') = 0$$

which again holfd if and only if,

$$\mu(B) = \mathscr{H}^k \llcorner E(B) = \int_{E'\cap B'} JdF_{\mathbf{x}}d\mathscr{L}^k(\mathbf{x}) = 0.$$

This shows, together with Lemma A.2.1, that $\big\{\mu_{\mathbf{y}} : \mathbf{y} \in I\big\} \in \mathscr{F}_\mu$. Hence, $\mathrm{Tan}\big(E_{\mathbf{y}}, \mathbf{x}\big) \subseteq V\big(\mu, \mathbf{x}\big)$ for $\mathscr{H}^1$-a.e $\mathbf{x} \in E'_{\mathbf{y}}$ and $\mathscr{H}^{k-1}$-almost every $\mathbf{y} \in I$, i.e. for μ-almost every $\mathbf{x}$.

Now we can repeat this process k times taking $\mathbf{x}_0$ as an element in a basis of $\mathbb{R}^k$, concluding that $\mathrm{Tan}\big(S, \mathbf{x}\big) = \mathrm{Tan}\big(E, \mathbf{x}\big) \subseteq V\big(\mu, \mathbf{x}\big)$ for μ-almost every $\mathbf{x}$.

(3) Proof of the fourth statement:

We will proof that if $\mathrm{supp}(\mu) \subseteq E$ with E a purely unrectifiable set, then $V\big(\mu, \mathbf{x}\big) = \{\mathbf{0}\}$ for μ-almost every $\mathbf{x}$. The remaining implication follows from lemma 5.1.12

Since $\int_I \mu_t dt \ll \mu$, we have that

$$0 = \mu\Big(\mathbb{R}^n \setminus E\Big) = \int_I \mu_t\Big(\mathbb{R}^n \setminus E\Big)dt = \int_I \mathscr{H}^1\Big(E_t \setminus E\Big)dt.$$

which implies that $E_t \subseteq E$ for $\mathscr{H}^1$-almost every $\mathbf{x}$ and dt-almost every $t \in I$. But E is purely unrectifiable, so

$$\mathscr{H}^1(E \cap E_t) = 0.$$

Hence, $\mu_t\big(\mathrm{supp}(\mu)\big) = 0$ and $\mu_t = 0$ for dt-almost every $t \in I$, so $V\big(\mu, \mathbf{x}\big) = V\big(\mu, G, \mathbf{x}\big) = \{\mathbf{0}\}$ for μ-almost every $\mathbf{x}$ $\square$.

Corollary 5.1.14 (2.9. [1])**.** Let μ be a finite measure and consider $\{\nu_t : s \in J\}$ holding (2) and (3) in the definition of a μ-integrable family (5.1.1). Then

(1) If $\displaystyle\int_J \nu_s ds \ll \mu$, then

$$V(\nu_s, \mathbf{x}) \subseteq V(\mu, \mathbf{x}) \text{ for } \nu_s\text{-almost every } \mathbf{x} \text{ and } ds\text{-almost every } s \in J.$$

(2) If $\mu \ll \displaystyle\int_J \nu_s ds$ and $\nu_s = \mathscr{H}^k \llcorner E_s$ where E_s is a k-rectifiable set, then

$$\dim\big(V(\mu, \mathbf{x})\big) \geq k \text{ for } \mu\text{-almost every } \mathbf{x}.$$

Proof. To prove (1), we can restrict (J, ds) to be a probability space. Then $\{\nu_s : s \in J\} \in \mathscr{F}_\mu$. Take now $G_s = \{\tilde{\nu}_{s,t} : t \in I\} \in \mathscr{F}_{\nu_s}$ such that

$$V(\nu_s, \mathbf{x}) = V(\nu_s, G_s, \mathbf{x}) \text{ for } \nu_s\text{-almost every } \mathbf{x}$$

for each $s \in J$, where $I = [0,1]$ and $\mathscr{L}^1 \llcorner [0,1]$.

Claim 1: *$F = \{\bar{\nu}_{s,t} : (s,t) \in J \times I\} \in \mathcal{F}_\mu$, where $ds \times \mathscr{L}^1 \llcorner [0,1] = d(s,t)$ is the probability defined over $J \times I$ and $\bar{\nu}_{s,t} = \big(\int_I \nu(\mathbb{R}^n) dt\big)^{-1} \tilde{\nu}_{s,t}$* . We check the

(1) For any $E \subseteq \mathbb{R}^n$, $(s,t) \mapsto \bar{\nu}_{s,t}(E)$ is measurable. Indeed, we need to choose the families G_s, for each $s \in J$, in a measurable way, so we can apply a measurable selection theorem to ensure the measurability of this map in both variables.

(2) $\int_{J\times I} \bar{\nu}_{s,t} d(s,t) \leq 1$

(3) If $B \subseteq \mathbb{R}^n$ is such that $\mu(B) = 0$, then $\int_J \tilde{\nu}(B) ds = 0$, so $\nu_s(B) = 0$ for ds-almost every $s \in J$ and since $\int_I \bar{\nu}_{s,t} dt \ll \tilde{\nu}_s$

$$\int_{J\times I} \bar{\nu}_{s,t}(B) d(s,t) = 0.$$

Thus,

$$V\big(\mu, F, \mathbf{x}\big) \subseteq V\big(\mu, \mathbf{x}\big) \text{ for } \mu\text{-almost every } \mathbf{x}.$$

In particular, this holds for ν_s-almost every $\mathbf{x}$ and ds-almost every $s \in J$.

Finally, since $\tilde{\nu}_{s,t} = \mathscr{H}^1 \llcorner E_{s,t}$ verifies by construction that for a.e. $t \in [0,1]$,

$$\mathrm{Tan}\big(E_{s,t}, \mathbf{x}\big) \in V\big(\mu, F, \mathbf{x}\big) \text{ for } \mu\text{-almost every } \mathbf{x} \text{ and } ds\text{-almost every } s \in J.$$

To prove (2), note that if $\mu = \int_J \nu_s ds$ then in particular $\int_J \nu_s ds \ll \mu$, so by the first statement of the present corollary, we get that

$$V(\nu_s, \mathbf{x}) \subseteq V(\mu, \mathbf{x}) \text{ for } \nu_s \text{ and } ds\text{-almost every } s$$

But $\nu_s = \mathscr{H}^k \llcorner E_s$, so by statement (3) in Proposition 5.1.13, we have that $V(\nu_s, \mathbf{x}) = \operatorname{Tan}\big(E_s, \mathbf{x}\big)$ for ν_s-almost every $\mathbf{x}$ and $\dim(\operatorname{Tan}\big(E_s, \mathbf{x})) = \dim(V(\nu_s, \mathbf{x})) \leq \dim(V(\mu, \mathbf{x}))$ for ν_s-almost every $\mathbf{x}$ and ds-almost every $s \in J$ so for μ-almost every $\mathbf{x}$ ($\mu = \int_J \nu_s ds$). $\square$

5.2 Diferentiability Bundle

During this chapter all measures are Borel and functions Borel-measurable. Measures will be finite and positive.

Definition 5.2.1 (Differentiability with respect to a subspace). We say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable with respect to a linear subspace $V \subseteq \mathbb{R}^n$ if there exists a linear map $L : V \to \mathbb{R}^m$ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = L\mathbf{h} + o(\|\mathbf{h}\|)$$

for all $\mathbf{h} \in V$. We call L the derivative of f with respect to V and denote it by $d_{V(\mathbf{x})} f$.

Remark 5.2.1. L is unique: consider $L, \tilde{L}$ holding this property, then $\hat{L}(h) = L(h) - \tilde{L}(h) = o(\|h\|)$. Fixing a direction $\frac{\mathbf{h}}{\|\mathbf{h}\|} \in V$, we get, by linearity, that $\hat{L}(\mathbf{h}) = 0$, for every $\mathbf{h} \in V$.

The following definition is inspired by the notion of derivative assignment introduced in [8].

Definition 5.2.2 (Differentiability bundle - 3.1. [1]**).** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function and $x \in \mathbb{R}^n$. We will denote by $\mathcal{D}(f, \mathbf{x})$ the set of all linear subspaces of $\mathbb{R}^n$ with respect to which f is differentiable at $\mathbf{x}$:

$$\begin{aligned} \mathcal{D}(f, \cdot) : \mathbb{R}^n &\longrightarrow \operatorname{Gr.}(\mathbb{R}^n) \\ \mathbf{x} &\longmapsto \mathcal{D}(f, \mathbf{x}). \end{aligned}$$

We will call $\mathcal{D}(f, \cdot)$ the differentiability bundle of f.Finally, we will denote by

$$\begin{aligned} \mathcal{D}^*(f, \cdot) : \mathbb{R}^n &\longrightarrow \operatorname{Gr.}(\mathbb{R}^n) \\ \mathbf{x} &\longmapsto \mathcal{D}^*(f, \mathbf{x}), \end{aligned}$$

where $\mathcal{D}^*(f, \mathbf{x}) \subseteq \mathcal{D}(f, \mathbf{x})$ contains all linear spaces with maximum dimension.

Remark 5.2.2. As stated in section in the section Measurability of the Appendix, $\mathcal{D}^*(f, \cdot)$ and $\mathcal{D}(f, \cdot)$ are Borel-measurable maps.

Definition 5.2.3 (Deviation from Linearity). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function , $\mathbf{x} \in \mathbb{R}^n$, $V \subseteq \mathbb{R}^n$ a linear subspace, $L : V \to \mathbb{R}$ a linear function , and $\delta > 0$. We define

$$m(f, \mathbf{x}, V, L, \delta) := \sup_{\substack{\mathbf{h} \in V, \\ 0 < \|\mathbf{h}\| \leq \delta}} \frac{|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - L\mathbf{h}|}{\|\mathbf{h}\|}$$

and $m(f, x, V, L, \delta) := 0$ when $V = \{\mathbf{0}\}$.

Remark 5.2.3. Notice that f is differentiable at $\mathbf{x}$ with respect to V if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $m(f, \mathbf{x}, V, d_{V(\mathbf{x})} f, \delta) \leq \epsilon$, that is, $m(f, \mathbf{x}, V, d_{V(\mathbf{x})} f, \delta) \to 0$ as $\delta \to 0$.

Lemma 5.2.4 (3.4. [1])**.** Let be f, x and V as in Definition 5.2.3, W and W' linear subspaces of V. In addition, let L and L' be two linear functions on V. Then, setting $m := m(f, \mathbf{x}, W, L, \delta)$ and $m' := m(f, \mathbf{x}, W', L', \delta)$. Then,

$$m \leq m' + \|L' - L\| + (\mathrm{Lip}(f) + |L'|) d_{Gr}(W, W'),$$

where the norm for linear functionals is the operator norm.

Proposition 5.2.5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function and let E be a 1-rectifiable set in $\mathbb{R}^n$. Let V be a Borel map

$$\begin{aligned} V : E &\longrightarrow \mathrm{Gr}(\mathbb{R}^n) \\ \mathbf{x} &\longmapsto V(\mathbf{x}). \end{aligned}$$

Then, if $V(x) \in \mathcal{D}(f, x)$ for every $x \in E$, then

$$V(\mathbf{x}) + \mathrm{Tan}(E, \mathbf{x}) \in \mathcal{D}(f, \mathbf{x}) \quad \text{for } \mathscr{H}^1\text{-almost every } \mathbf{x} \in E.$$

Corollary 5.2.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function and let E be a 1-rectifiable set in $\mathbb{R}^n$. Let $V : E \to \mathrm{Gr}(\mathbb{R}^n)$ be a Borel map such that $V(\mathbf{x}) \in \mathcal{D}^*(f, \mathbf{x})$ for every $\mathbf{x} \in E$, then

$$\operatorname{Tan}(E, \mathbf{x}) \subseteq V(\mathbf{x}) \quad \text{for } \mathscr{H}^1\text{-almost every } \mathbf{x} \in E.$$

Proof of Corollary 5.2.6.

Provided that $V(\mathbf{x}) \in \mathcal{D}^*(f, \mathbf{x})$ for each $\mathbf{x} \in E$, we know that

$$V(\mathbf{x}) + \operatorname{Tan}(E, \mathbf{x}) \in \mathcal{D}(f, \mathbf{x}) \qquad \text{for } \mathscr{H}^1\text{-almost every } \mathbf{x},$$

, since $V(\mathbf{x})$ is of maximal dimension in $\mathcal{D}(f, \mathbf{x})$, we have that

$$V(\mathbf{x}) + \operatorname{Tan}(E, \mathbf{x}) = V(\mathbf{x}),$$

Hence, $\operatorname{Tan}(E, \mathbf{x}) \subseteq V(\mathbf{x})$ for $\mathscr{H}^1$-almost every $\mathbf{x} \in E$. $\square$

To prove Proposition 5.2.5, we will prove that f is differentiable at every

$$\mathbf{h} + \boldsymbol{\tau} \in V(\mathbf{x}) + \operatorname{Tan}(E, \mathbf{x})$$

for every $\mathbf{x} \in \widetilde{E} := \bigcup_{j=1}^{\infty} E_j$, with $\mathscr{H}^1(E \setminus \widetilde{E}) = 0$, where E_j are suitable subsets of graphs of curves of class $\mathcal{C}^1$, the ones from Proposition 4.1.6.

We will spend the first part of the proof preparing the setting $\widetilde{E}$.

Definition 5.2.7 (**Modulus of Continuity**). Let $F : X \longrightarrow \mathbb{R}$ be a continuous function on a metric space X and $g : [0, \infty) \to [0, \infty)$ a function. We say that g is a modulus of continuity if

(1) g is non-decreasing.

(2) $g(0) = 0$.

(3) $|f(x) - f(y)| \leq g(d(x, y))$, for all $x, y \in X$.

Proof of Proposition 5.2.5. Since $V(\mathbf{x}) \in \mathcal{D}(f, \mathbf{x})$ for each $\mathbf{x} \in E$, then f is differentiable with respect to $V(\mathbf{x})$ at $\mathbf{x}$. Calling $d_{V(\mathbf{x})} f$ the corresponding linear function such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = d_{V(\mathbf{x})} f(\mathbf{h}) + o_{\mathbf{x}}(\|\mathbf{h}\|)$$

for each $\mathbf{h} \in V(\mathbf{x})$. By Proposition 4.1.6, there exists a countable family of curves of class C^1 such that $\mathscr{H}^1\big(E \setminus \cup_{j=1}^{\infty} E_j'\big) = 0$. Applying Theorem A.1.2 a countable amount of times, we can cover each E'_j by sets $\{E_{j,i}\}_{i\in\mathbb{N}}$ such that

$$\mathscr{H}^1\left(E'_j \setminus \cup_{i=1}^{\infty} E_{j,i}\right) = 0,$$

and:

(1) The map $V : E_{j,i} \to \mathrm{Gr}(\mathbb{R}^n)$ such that $\mathbf{x} \mapsto V(\mathbf{x})$ is continuous.

(2) Provided that

$$\begin{aligned} d_V f : E_{j,i} &\longrightarrow (\mathbb{R}^n)^* \\ \mathbf{x} &\longmapsto d_{V(\mathbf{x})} f \end{aligned}$$

is Borel measurable (see lemma A.2.11), where $(\mathbb{R}^n)^*$ is the dual of $\mathbb{R}^n$, then $D_V f$ is continuous.

By Theorem A.1.3, applied on each $E_{j,i}$, there exists a countable family of Borel sets such that $\mathscr{H}^1\left(E_{j,i} \setminus \cup_{k=1}^{\infty} E_{j,i,k}\right) = 0$ where

(3) For each $h \in V(\mathbf{x})$, in

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = d_{V(\mathbf{x})} f\mathbf{h} + o_{\mathbf{x}}(\|\mathbf{h}\|),$$

the remainder $o_{\mathbf{x}}(\|\mathbf{h}\|)$ is uniform in each $E_{j,j}$, i.e., there exists a modulus of continuity g such that

$$m(f, \mathbf{x}, V(\mathbf{x}), d_{V(\mathbf{x})} f, \|\mathbf{h}\|) \le g(\|\mathbf{h}\|) \qquad \text{for each } \mathbf{h} \in V(\mathbf{x}) \text{ and each } \mathbf{x} \in E_{j,i,k}$$

Claim 1: *f is differentiable with respect to* $\mathrm{Tan}(E, \mathbf{x})$ *for $\mathscr{H}^1$-almost every $\mathbf{x} \in E$.*

By choice of $E_{j,i,k}$, these are subsets of $\mathcal{C}^1$ curves. Hence, for all $j, i, k \in \mathbb{N}$, we have that

$$\mathrm{Tan}(E_{j,i,k}, x) = \mathrm{Tan}(E'_j, \mathbf{x}) \qquad \text{for all } \mathbf{x} \in E_{j,i,k}.$$

Composing f with each curve and applying Theorem 3.2.1 to this 1-dimensional Lipschitz function, and via Proposition 3.3.1 (see Remark 3.3.3) we get the claim for $\mathscr{H}^1$-almost every $\mathbf{x}$. Together with the previous line, we can follow the same procedure to build the family of sets $E_{j,i,k}$ to adjust them $E_{j,i,k}$ up to $\mathscr{H}^1$-null sets so that the following properties hold (from now we restrict will call the new family of sets E_j),

(4) Denoting the differential of f at $\mathbf{x} \in E_j$ with respect to $\mathrm{Tan}(E_j, \mathbf{x})$ by $d_{\mathrm{Tan}(E_j,\mathbf{x})} f$, there exists a modulus of continuity g' as in [(3)], for $d_{V(\mathbf{x})} f$ and $V(\mathbf{x})$.

(5) Since $\operatorname{apTan}(E,\mathbf{x}) = \operatorname{Tan}(E,\mathbf{x}) = \operatorname{Tan}(E_j,\mathbf{x})$ for each $\mathbf{x} \in E_j$, consider now $\boldsymbol{\tau} \in \operatorname{Tan}(E,\mathbf{x})$, then there exists some $\mathbf{x}' \in E_j$ such that

$$\mathbf{x}' = \mathbf{x} + \boldsymbol{\tau} + o_(\|\boldsymbol{\tau}\|).$$

We finish the proof with the following claim:

Claim 2: *f is differentiable with respect to $V(\mathbf{x}) + \operatorname{Tan}(E,\mathbf{x})$ at every $\mathbf{x} \in E_j$.*

Fix $\mathbf{x} \in E_j$ and assume that $\operatorname{Tan}(E,\mathbf{x}) \nsubseteq V(\mathbf{x})$ (otherwise we are finished). We prove that, for each $\mathbf{h} \in V(\mathbf{x})$ and each $\boldsymbol{\tau} \in \operatorname{Tan}(E,\mathbf{x})$,

$$f(\mathbf{x}+\mathbf{h}+\boldsymbol{\tau}) - f(\mathbf{x}) = d_{V(\mathbf{x})}f(\mathbf{h}) + d_{\operatorname{Tan}(E_j,\mathbf{x})}f(\boldsymbol{\tau}) + o_(\|\mathbf{h}\| + \|\boldsymbol{\tau}\|).$$

For $\boldsymbol{\tau}$, there exists $\mathbf{x}' \in E_j$ such that $\mathbf{x}' = \mathbf{x} + \boldsymbol{\tau} + o_(\|\boldsymbol{\tau}\|)$ as in [(5)].

We write

$$\begin{aligned} f(\mathbf{x}+\mathbf{h}+\boldsymbol{\tau}) - f(\mathbf{x}) - d_{V(\mathbf{x})}f(\mathbf{h}) - d_{\operatorname{Tan}(E_j,\mathbf{x})}f(\tau) &= \underbrace{f(\mathbf{x}+\mathbf{h}+\boldsymbol{\tau}) - f(\mathbf{x}'+\mathbf{h}) + f(\mathbf{x}') - f(\mathbf{x}+\boldsymbol{\tau})}_{\mathrm{I}} \\ &+ \underbrace{f(\mathbf{x}'+\mathbf{h}) - f(\mathbf{x}') - d_{V(\mathbf{x})}f\mathbf{h}}_{\mathrm{II}} \\ &+ \underbrace{f(\mathbf{x}+\boldsymbol{\tau}) - f(\mathbf{x}) - d_{\operatorname{Tan}(E_j,\mathbf{x})}f\boldsymbol{\tau}}_{\mathrm{III}} \end{aligned}$$

Estimates of each term:

$|\mathrm{I}| \leq 2\operatorname{Lip}(f)\,\|\mathbf{x}+\boldsymbol{\tau}-\mathbf{x}'\| = o(\|\boldsymbol{\tau}\|)$.

$|\mathrm{II}| \leq m(f,\mathbf{x}',V(\mathbf{x}),d_{V(\mathbf{x})}f,\|\mathbf{h}\|)\,\|\mathbf{h}\|$, which , by Lemma 5.2.4,

$$\begin{aligned} m(f,\mathbf{x}',V(\mathbf{x}),d_{V(\mathbf{x})}f,\|\mathbf{h}\|)\,\|\mathbf{h}\| &= m(f,\mathbf{x}',V(\mathbf{x}'),d_{V(\mathbf{x}')}f,\|\mathbf{h}\|) + (1+2\operatorname{Lip}(f)g'(\|\boldsymbol{\tau}\|))\,\|\mathbf{h}\| \\ &\leq (g(\|\mathbf{h}\|) + (1+\operatorname{Lip}(f)g'(\|\boldsymbol{\tau}\|))\,\|\mathbf{h}\| = o(\|\boldsymbol{\tau}\| + \|\mathbf{h}\|). \end{aligned}$$

$|\mathrm{III}| \leq o(\|\boldsymbol{\tau}\|)$, by [(4)]. $\square$

5.3 Differentiability of Lipschitz Functions

Theorem 5.3.1 (**Differentiability of Lipschitz Functions -** 1.1 [1])**.** Let μ be a Borel regular finite measure on $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz function. Then:

(1) $V(\mu, \mathbf{x}) \in \mathcal{D}^*(f, \mathbf{x})$ for μ-almost every $\mathbf{x}$.

(2) There exists a Lipschitz function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that for any $\mathbf{h} \notin V(\mu, x)$, the derivative of g at the direction $\mathbf{h}$ does not exist for μ-almost every $\mathbf{x}$.

We prove the first statement of the theorem.

Proof. By lemma A.2.11, the function

$$\begin{aligned} \mathcal{D}^*(f, \cdot) : \mathbb{R}^n &\longrightarrow \mathrm{Gr}(\mathbb{R}^n) \\ \mathbf{x} &\longmapsto \mathcal{D}^*(f, \mathbf{x}) \end{aligned}$$

is a Borel measurable, closed-valued multifunction from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$. Applying Kuratowski and Ryll-Nardzewski's selection theorem (see A.2.9) (given the properties of the Grassmannian, see section 2.2.32), we can find a Borel-measurable selection

$$\begin{aligned} V : \mathbb{R}^n &\longrightarrow \mathrm{Gr}(\mathbb{R}^n) \\ \mathbf{x} &\longmapsto V(\mathbf{x}) \end{aligned}$$

with $V(\mathbf{x}) \in \mathcal{D}^*(f, \mathbf{x})$ for every $\mathbf{x}$. By Corollary (5.2.6), considering a decomposition of μ, $\{\mathcal{H}^1 \llcorner E_t : t \in I\} \in \mathscr{F}_\mu$, with sets E_t, $t \in I$, we have that

$$\mathrm{Tan}(E_t, \mathbf{x}) \subseteq V(\mathbf{x}) \quad \text{for } \mathcal{H}^1\text{-almost every } \mathbf{x} \in E_t,$$

for each $t \in I$. Thus, $V \in \mathscr{G}_\mu$ and hence, we have that

$$V(\mu, \mathbf{x}) \subseteq V(\mathbf{x})$$

for μ-almost every $\mathbf{x}$. Since f is differentiable with respect to $V(\mathbf{x})$ for every $\mathbf{x}$, so it is with respect to $V(\mu, \mathbf{x})$ for μ-almost every $\mathbf{x}$. $\square$

Appendix A

General Geometric Theory

This appendix presents foundational results used throughout the proofs but are not at the core of the main ideas of this text.

A.1 Geometric Measure Theory Results

The following result will be used to prove that Hausdorff measures are Borel, see in chapter 1, proposition 2.2.22.

Theorem A.1.1 (**Caratheodory's Criterion** - 1.1. Theorem 5 [6])**.** Let μ be an measure on $\mathbb{R}^n$. If for any tow sets $A, B \subseteq \mathbb{R}^n$ such that $d(A, B) > 0$ then $\mu(A \cup B) = \mu(A) + \mu(B)$, then μ is Borel.

Theorem A.1.2 (**Lusin's Theorem** - 1.2. Theorem 2 [6])**.** Let μ be a Borel regular measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a μ-measurable function. Assume $A \subseteq \mathbb{R}^n$ to be a μ-measurable set such that $\mu(A) < \infty$. For each $\epsilon > 0$ there exists a compact set $K \subseteq \mathbb{R}^n$ such that

1. $\mu(A \setminus K) < \epsilon$.

2. $f_{|K}$ is continuous.

The following result holds when μ is a measure on a space X and the functions have their images on a separable metric space.

We write the proof of the following theorem since it will illustrate the proof in theorem of approximation of Lipschitz Functions 3.2.3

Theorem A.1.3 (**Egoroff's Theorem** - 1.3. Theorem 3 [6])**.** Let μ be a measure on $\mathbb{R}^n$ and suppose $f_l : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is μ-measurable functions for $l \in \mathbb{N}$. Let $A \subseteq \mathbb{R}^n$ be a μ-measurable set with $\mu(A) < \infty$, and $f_l(\mathbf{x}) \to g(\mathbf{x})$ for μ-almost every $\mathbf{x} \in A$. Then, for each $\epsilon > 0$, there exists a μ-measurable set $C \subseteq A$ such that

1. $\mu(A \setminus C) < \epsilon$
2. $\left\| f_{l|C} - g_{|C} \right\|_\infty \xrightarrow[l\to\infty]{} 0$

Proof. Let us consider the following sets,

$$C_{i,j} \equiv \bigcup_{l=j}^{\infty} \Big\{ \mathbf{x} \in \mathbb{R}^n : |f_l(\mathbf{x}) - g(\mathbf{x})| > \frac{1}{2^i} \Big\}.$$

Now, notice that $C_{i,j+1} \subseteq C_{i,j}$, hence $A \cap C_{i,j}$ is a decreasing sequence with its first element such that $\mu(A \cap C_{i,0}) \leq \mu(A) < \infty$, and the sequence converge to g μ almost everywhere, hence

$$\lim_{j\to\infty} \mu(A \cap C_{i,j}) = \mu\left(A \cap \bigcap_{l=1}^{\infty} C_{i,j} \right) = 0.$$

So there exists $N_i \in \mathbb{N}$ such that

$$\mu(A \cap C_{i,N_i}) < \frac{\epsilon}{2^i}$$

We call $C := A \setminus \bigcup_{i=1}^{\infty} C_{i,N_i}$, so we can write

$$\mu(A \setminus C) \leq \sum_{i=1}^{\infty} \mu(A \cap C_{i,N_i}) \leq \epsilon.$$

So for each $i \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for every $\mathbf{x} \in C$ and every $l \geq N$, $|f_l(\mathbf{x}) - g(\mathbf{x})| \leq \frac{1}{2^i}$ holds, i.e. $\left\| f_{l|C} - g_{|C} \right\|_\infty \xrightarrow[l\to 0]{} 0$. □

Proposition A.1.4 (2.7. [5])**.** Le $\{\nu_i\}_{i=1}^{\infty}$ be a sequence such that converges weakly to a measure ν. Then

(1) For any open set B
$$\liminf_{i\to\infty} \nu(A) \geq \nu(A).$$

(2) For any compact set K
$$\limsup_{i\to\infty} \nu(K) \leq \nu(K).$$

Hence,

(3) $\nu_i(A) \to \nu(A)$ for every bounded open set A such that $\nu(\partial A) = 0$.

(4) For any point $\mathbf{x}$ there exists a set $S_{\mathbf{x}} \subseteq \mathbb{R}^+$ at most countable such that
$$\nu_i\big(B(\mathbf{x},\rho)\big) \xrightarrow[i\to\infty]{} \nu\big(B(\mathbf{x},\rho)\big)$$
for every $\rho \in \mathbb{R}^+ \setminus S_{\mathbf{x}}$.

A.2 Measurability

The following lemma is useful to proof the measurability of the sets in the proof of the Rectifiability Criterion 4.1.8, as well as in the proof of the third statement of proposition 5.1.13

Lemma A.2.1. Let $E \subseteq \mathbb{R}^n$ be a Borel set and $r > 0$, then the map $\mathbf{x} \mapsto \mathscr{H}^k \llcorner E(B(\mathbf{x}, r))$ is a Borel map on $\mathbb{R}^n$.

Proof. Let us consider the map
$$\begin{aligned}\Phi : E &\longrightarrow [0, +\infty]\\ \mathbf{x} &\longmapsto \Phi(\mathbf{x}) = \mathscr{H}^k\big(E \cap B(\mathbf{x}, r)\big).\end{aligned}$$

By definition of Hausdorff measure (see 2.2.19), and defining $\Phi_\delta(\mathbf{x}) := \mathscr{H}^k_\delta(E \cap B(\mathbf{x}, r))$ for each $\mathbf{x} \in \mathbb{R}^n$, we have that $\Phi(\mathbf{x}) = \lim_{\delta\to 0} \Phi_\delta(\mathbf{x})$. It is enough then to prove Φ_δ to be a Borel measurable function, for each $\delta > 0$.

Moreover, we can find a countable family of balls, of rational centers and radii, $U = \{B_i\}_i^\infty$, such that
$$\Phi_\delta(\mathbf{x}) = \inf\Big\{\sum_{i=1}^{\infty} \operatorname{diam}(B_i)^s \,:\, E \cap B(\mathbf{x}, r) \subseteq \cup_{i=1}^\infty B_i \text{ with } B_i \in U \text{ for every } i \in \mathbb{N}\Big\}.$$

Now, by considering the functions
$$\phi_N(\mathbf{x}) := \inf\Big\{\sum_{i=1}^{N} \operatorname{diam}(B_i)^s \,:\, E\cap B(\mathbf{x}, r) \subseteq \cup_{i=1}^N B_i \text{ with } B_i \in U \text{ for every } i \in \{1, 2, ..., N\}\Big\},$$

we may approximate now Φ_δ from above:

$$\Phi_\delta(\mathbf{x}) = \liminf_{N\to\infty} \phi_N(\mathbf{x}).$$

In order to conclude the proof, we prove the following claim.

Claim: ϕ_N *is upper continuous for each* $N \in \mathbb{N}$.

We aim to prove that

$$\limsup_{\mathbf{x}\to\mathbf{x}_0} \phi_N(\mathbf{x}) \le \phi(\mathbf{x}_0).$$

Fix $\epsilon > 0$ arbitrarily, by the definition of ϕ_N we can consider a covering $\{B_i\}_{i=1}^N$ such that $E \cap B(\mathbf{x}_0, r) \subseteq \cup_{i=1}^N B_i$ and

$$\sum_{i=1}^N \mathrm{diam}(B_i)^k \le \phi_N(\mathbf{x}_0) + \epsilon.$$

If $\mathbf{x}$ is close enough to $\mathbf{x}_0$ then $\{B_i\}_{i=1}^N$ is a valid covering for $E \cap B(\mathbf{x}_0, r)$ as well, hence

$$\phi_N(\mathbf{x}) \le \sum_{i=1}^N \mathrm{diam}(B_i)^k \le \phi_N(\mathbf{x}_0) + \epsilon.$$

We finish the proof by considering the upper limit and pointing to the arbitrarity of $\epsilon > 0$. □

Definition A.2.2 (Relatively Compact Metric Space). Let (K, d) be a metric space. We say that it is a relatively compact space if for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε.

Definition A.2.3 (K-Parametrization). Let (K, d) be a relatively compact metric space and μ be a probability measure on K. We call parametrization of μ a measurable application $G : [0,1] \to K$ such that for all $A \subseteq K$,

$$\mu(A) = \mathcal{L}^1([0,1] \cap G^{-1}(A)).$$

Proposition A.2.4 (Skorokhod-$A.3.$ [3]**).** Let (K, d) be a relatively compact metric space and μ be a probability measure on K. Then there exists a parametrization of μ.

Theorem A.2.5 (Prokhorov)-2.3.4. [4]**).** Let X be a complete separable metric space, $\mathcal{M}(X)$ the set of all Borel finite outer measures on X and let $\mathcal{A} \subseteq \mathcal{M}(X)$. Then $\mathcal{A}$ is relatively compact in the weak* topology of $\mathcal{M}(X)$ if and only if the following two properties hold:

(1) $\sup\{\mu(\mathbb{R}^n) : \mu \in \mathcal{A}\} < +\infty$,

(2) $\mathcal{A}$ is tight, i.e., for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R}^n$ such that

$$\mu(\mathbb{R}^n \setminus K_\varepsilon) < \varepsilon \quad \text{for all } \mu \in \mathcal{A}.$$

The following definitions are extracted from 5.1. in [10],

Definition A.2.6 (**Multifunction**)**.** We say that $F : X \to Y$ is a multifunction if $F(x) \subseteq Y$, with $F(x) \neq \emptyset$, for each $x \in X$.

Definition A.2.7 (**Closed-valued multifunction**)**.** We say that a multifunction $F : X \to Y$ is closed-valued if $F(x)$ is closed for every $x \in X$.

Definition A.2.8 (**Selection of a multifunction**)**.** Let $F : X \to Y$ be a multifunction. A selection of F is a map $f : X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$.

In [10], we can find the following theorem, stated for a domain X (non-empty) and an algebra $\mathcal{B}$ over X. A proof can be found in [7].

Theorem A.2.9 (**Kuratowski and Ryll-Nardzewski-**5.2.1. [10])**.** Let Y be a Polish space, $F : \mathbb{R}^n \to Y$ a Borel measurable, closed-valued multifunction. Then F admits a Borel selection.

The following result follows from remark 5.1.1 in [10]

Lemma A.2.10 (**Measurability of Multifunctions**)**.** Let X be a topological space and Y a compact metric space, and $F : X \to Y$ a closed-valued multifunction. If F is measurable as a map from X to the space of non-empty closed subsets of Y, then F is Borel measurable.

Based on the previous notions and results, as well as on the notion of deviation from linearity (3.3 in [1]), the following results are proved in [1]:

Lemma A.2.11 (3.5, [1])**.** Let f be a Lipschitz function on $\mathbb{R}^n$. Then $\mathcal{D}(f, \mathbf{x})$ and $\mathcal{D}^*(f, \mathbf{x})$ are closed, nonempty subsets of $\mathrm{Gr}(\mathbb{R}^n)$ for every $\mathbf{x}$.

Moreover, $\mathbf{x} \mapsto \mathcal{D}(f, \mathbf{x})$ and $\mathbf{x} \mapsto \mathcal{D}^*(f, \mathbf{x})$ are Borel-measurable, closed-valued multifunctions from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$.

Lemma A.2.12 (3.6, [1])**.** Let f be a Lipschitz function on $\mathbb{R}^n$, E a Borel set in $\mathbb{R}^n$, and $\mathbf{x} \mapsto V(\mathbf{x})$ a Borel map from E to $\mathrm{Gr}(\mathbb{R}^n)$ such that $V(\mathbf{x}) \in \mathcal{D}(f, \mathbf{x})$ for every $\mathbf{x}$.

For every $\mathbf{x} \in E$, we denote by $d_{V(\mathbf{x})} f$ the derivative of f at $\mathbf{x}$ with respect to $V(\mathbf{x})$, and we extend it to a linear function on $\mathbb{R}^n$ by setting $d_{V(\mathbf{x})} f(\mathbf{h}) = \mathbf{0}$ for every $\mathbf{h} \in V(\mathbf{x})^\perp$. Then $\mathbf{x} \mapsto d_{V(\mathbf{x})} f$ is a Borel map from E to the dual of $\mathbb{R}^n$.

Bibliography

[1] G. Alberti and A. Marchese. On the differentiability of lipschitz functions with respect to measures in the euclidean space. *Geometric and Functional Analysis*, 26(1):1–66, 2016.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford university press, 2000.

[3] M. Bernot, V. Caselles, and J.-M. Morel. *Optimal transportation networks: models and theory*, volume 1955. Springer Science & Business Media, 2009.

[4] V. I. Bogachev. *Weak convergence of measures*, volume 234. American Mathematical Society Providence, 2018.

[5] C. De Lellis. Lecture notes on rectifiable sets, densities, and tangent measures. *Preprint*, 23(1.2), 2006.

[6] L. Evans. *Measure theory and fine properties of functions*. Routledge, 2018.

[7] K. Kuratowski and C. Ryll-Nardzewski. A general theorem on selectors. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys*, 13(6):397–403, 1965.

[8] O. Maleva and D. Preiss. Directional upper derivatives and the chain rule formula for locally lipschitz functions on banach spaces. *Transactions of the American Mathematical Society*, 368(7):4685–4730, 2016.

[9] P. Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, volume 44. Cambridge university press, 1999.

[10] S. M. Srivastava. *A course on Borel sets*, volume 180. Springer Science & Business Media, 2008.