STRICT REARRANGEMENT INEQUALITIES: NONEXPANSIVITY AND PERIODIC GAGLIARDO SEMINORMS

GYULA CSATÓ AND ALBERT MAS

Abstract. This paper deals with the behavior of the periodic Gagliardo seminorm under two types of rearrangements, namely under a periodic, and respectively a cylindrical, symmetric decreasing rearrangement. Our two main results are Pólya-Szegő type inequalities for these rearrangements. We also deal with the cases of equality.

Our method uses, among others, some classical nonexpansivity results for rearrangements for which we provide some slight improvements. Our proof is based on the ideas of [Frank and Seiringer, *Non-linear ground state representations and sharp Hardy inequalities*, J. Funct. Anal., 2008], where a new proof to deal with the cases of equality in the nonexpansivity theorem was given, albeit in a special case involving the rearrangement of only one function.

Contents

1. Introduction

1.1. **The periodic Gagliardo seminorm and rearrangements.** The goal of this paper is to establish Pólya-Szegő type inequalities for Gagliardo seminorms in a periodic setting. Let us start by defining, for $0 < s < 1$, $1 \leq p < +\infty$, and a function $u : \mathbb{R}^n \to \mathbb{R}$ which is 2π-periodic in the variable x_1, the periodic Gagliardo seminorm

$$[u]^{\mathrm{per}}_{W^{s,p}} := \left(\int_{\{x \in \mathbb{R}^n :\, -\pi < x_1 < \pi\}} dx \int_{\mathbb{R}^n} dy \, \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \right)^{1/p}; \tag{1.1}$$

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throughout this work we will use the notation $\mathbb{R}^n = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}\}$. This seminorm has been used in the literature in some specific situations. For instance, if $E \subset \mathbb{R}^n$ is 2π-periodic in the variable x_1 then, taking u equal to the characteristic function χ_E and $p = 1$, we have that $\mathcal{P}_s(E) := [\chi_E]^{\mathrm{per}}_{W^{s,1}}$ is the periodic fractional perimeter introduced in [13]. $\mathcal{P}_s(E)$ was successfully used in [10] to construct periodic surfaces with constant nonlocal mean curvature. Whereas if $p = 2$ and $n = 1$ then $[\,\cdot\,]^{\mathrm{per}}_{W^{s,2}}$ coincides with the kinetic part of the Lagrangian (or energy) studied in [11] to address the variational formulation, symmetry, and regularity properties of periodic solutions to semilinear equations involving the fractional Laplacian.

The celebrated Pólya-Szegő inequality states that, for sufficiently smooth open sets $\Omega \subset \mathbb{R}^n$,

$$\|\nabla(u^{*n})\|_{L^p(\Omega^{*n})} \leq \|\nabla u\|_{L^p(\Omega)} \quad \text{for all } u \in W^{1,p}_0(\Omega), \tag{1.2}$$

where u^{*n} denotes the radially symmetric decreasing (Schwarz) rearrangement of the function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and Ω^{*n} denotes the ball centered at the origin and with the same volume as Ω. The vanishing boundary condition is important here. Without it the inequality does not hold true in general; see the work of Kawohl [18, Section II.4, Example 2.2]. We are interested in analogues of this inequality for the periodic Gagliardo seminorm. We will establish two theorems dealing with its behavior under two different types of rearrangements. Let us briefly introduce these rearrangements; we refer to Section 2 for the precise definition. The first one is the *periodic rearrangement* $u^{*\,\mathrm{per}}$, which is defined as follows: first perform a Steiner symmetrization with respect to the hyperplane $\{x_1 = 0\}$ of the function u resticted to $(-\pi, \pi) \times \mathbb{R}^{n-1}$, and then extend this function to $\mathbb{R}^n$ in a 2π-periodic way with respect to the variable x_1. The second rearrangement is the *cylindrical rearrangement* with respect to the x_1-axis. It is denoted by $u^{*n,1}$ and it is obtained by performing the Schwarz rearrangement in $\mathbb{R}^{n-1}$ of $u(x_1, \cdot)$ for each frozen value of x_1.

The behavior of the periodic Gagliardo seminorm under these two rearrangements has been addressed for the first time by Dávila, del Pino, Dipierro, and Valdinoci [13], but only in the special case $p = 1$ and if u is a characteristic function. Their only result in this direction is [13, Proposition 13] which establishes that $\mathcal{P}_s(E^{*n,1}) \leq \mathcal{P}_s(E)$, where $E^{*n,1} := \{(\chi_E)^{*n,1} > 0\}$. We note that their proof also works for any function u (and not only for $u = \chi_E$) as long as $p = 1$ or $p = 2$, with slight modifications, however not for other values of p. The same argument has also been used in [22, Lemma 4.2]. Regarding the periodic rearrangement, the authors of [13] only conjectured that $\mathcal{P}_s(E^{*\,\mathrm{per}}) \leq \mathcal{P}_s(E)$, where $E^{*\,\mathrm{per}} := \{(\chi_E)^{*\,\mathrm{per}} > 0\}$. We proved this conjecture in [10], as well as the analogous inequality for the Gagliardo seminorm in the case $p = 2$ and $n = 1$ in [11]. Indeed, more general seminorms than (1.1) are considered in this last work; see Remark 5.3.

Let us briefly comment why these special cases are simpler, for both cylindrical and periodic rearrangements. First of all, for u and v general, the factorization

$$|u(x) - v(y)|^p = u^p(x) + v^p(y) - 2u(x)v(y)$$

only holds for $p = 2$. We mention that it also holds if $p = 1$, $u = \chi_E$, and $v = \chi_F$ for any E, $F \subset \mathbb{R}^n$. Hence in these cases one can directly apply Riesz rearrangement inequalities to obtain Pólya-Szegő inequalities. Secondly, we will have to deal with sections of the kernel $|x - y|^{-(n+sp)}$ appearing in the seminorm (1.1), namely, the function

$$t \in (0, +\infty) \mapsto g(t) := (t^2 + a^2)^{-(n+ps)/2},$$

where $t = |x_1 - y_1|$ and $a = |x' - y'|$. If $n = 1$ then $a = 0$ and the function g is convex. However, if $n > 1$ then g is concave for t near the origin whenever $a \neq 0$, and this causes considerable difficulties when dealing with the periodic rearrangement.

Our first main result is a full characterization of the behavior of the seminorm $[\,\cdot\,]^{\mathrm{per}}_{W^{s,1}}$ under periodic and cylindrical rearrangements. In particular, it includes the above mentioned partial results. In order to simplify the exposition, we only give here a brief summary of our main Theorems 5.1 and 5.2, where we also deal with the cases of equality in the inequalities (1.3).

Theorem 1.1. *The periodic Gagliardo seminorm does not increase under periodic and cylindrical rearrangements. That is to say, if* $0 < s < 1$ *and* $1 \leq p < +\infty$*, then*

$$[u^{*\,\mathrm{per}}]^{\mathrm{per}}_{W^{s,p}} \leq [u]^{\mathrm{per}}_{W^{s,p}} \quad \text{and} \quad [u^{*\,n,1}]^{\mathrm{per}}_{W^{s,p}} \leq [u]^{\mathrm{per}}_{W^{s,p}} \tag{1.3}$$

for all $u : \mathbb{R}^n \to \mathbb{R}$ *which is measurable and* 2π*-periodic in the variable* x_1*.*

We emphasize that dealing with the cases of equality is considerably harder than establishing these inequalities; the reader will find the full description of the cases of equality in Section 5.

Let us now make some comments on Theorem 1.1 in comparison with the known results in the nonperiodic local, nonperiodic nonlocal, and periodic local cases. We first concentrate on the nonperiodic setting. The nonlocal counterpart of the classical Pólya-Szegő inequality (1.2) was first proven by Almgren and Lieb [1] and states that

$$[u^{*\,n}]_{W^{s,p}(\mathbb{R}^n)} \leq [u]_{W^{s,p}(\mathbb{R}^n)}, \qquad \text{where } [u]^p_{W^{s,p}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}. \tag{1.4}$$

When studying the cases of equality, there is a major difference between the local and nonlocal case if $p > 1$. It is easy to see that equality in the classical Pólya-Szegő inequality (1.2) does not force u to be equal to $u^{*\,n}$ modulo translations. Let us illustrate this with a simple example for $n = 1$. Take a symmetric decreasing function v such that $\mathrm{supp}(v) \subset (-2, 2)$ and $v(x) = 1$ for all $x \in [-1, 1]$. Then, take $0 < \epsilon < 1$ and another nontrivial symmetric decreasing function φ with $\mathrm{supp}(\varphi) \in (-\epsilon, \epsilon)$, and define $\varphi_t(x) := \varphi(x - t)$. Clearly, for every $1 \leq p < +\infty$ and every $t \in (\epsilon - 1, 1 - \epsilon)$, it holds that

$$\int_{-2}^{2} |(v + \varphi_t)'|^p = \int_{-2}^{2} |(v + \varphi_0)'|^p = \int_{-2}^{2} |((v + \varphi_t)^{*1})'|^p, \tag{1.5}$$

but any translation of $v + \varphi_t$ differs from $(v + \varphi_t)^{*\,1}$ if $t \neq 0$. However, if one excludes *flat* (i.e., horizontal) parts of the graph of u, then equality in the Pólya-Szegő inequality (1.2) does indeed force u to be equal to its rearrangement modulo translations. A first general theorem for a fairly big class of functions excluding *flat* parts (for instance analytic functions) was given by Kawohl [17]. This was generalized by Brothers and Ziemer [7], of which a simpler proof was later given by Ferone and Volpicelli [14].

In contrast to the local case, equality in the nonlocal Pólya-Szegő inequality (1.4) is sufficient to conclude that a translation of u agrees with its rearrangement, if $p > 1$. Whereas for $p = 1$ the conclusion in case of equality in (1.4) remains the same as in the local case: all superlevel sets of u must be balls, but not necessarily centered at the same point (as in example (1.5)); see Frank and Seiringer [15, Appendix A]. Our characterization of the cases of equality in Theorem 1.1 is analogous, that is, if $p > 1$ the function must be a translate of the corresponding rearranged function, whereas if $p = 1$ the superlevel sets must be translates of the corresponding rearranged superlevel sets, but the translation might depend on the level.

When it comes to the rearrangement $u^{*\,\mathrm{per}}$ of a function u periodic in the variable x_1, there is no interest in studying the periodic version of the local Pólya-Szegő inequality (1.2) for $n = 1$. Indeed, the inequality remains trivially true in each period. To see this, assume that u is 2π-periodic and with $u \in W^{1,p}(0, 2\pi)$. Then simply consider the periods $\Omega_a := (a + 2k\pi, a + 2(k + 1)\pi)$, $k \in \mathbb{Z}$, for some a with $|u(a)| = \min |u|$. As $|u| - \min |u| \in W^{1,p}_0(\Omega_a)$, one can simply work separately in each period. A similar argument works for $n = 2$, *if* we require constant periodic boundary conditions $u(-\pi, x_2) = u(\pi, x_2) = c$ for all $x_2 \in \mathbb{R}$, since the Pólya-Szegő inequality (1.2) remains true also for Steiner symmetrization with vanishing boundary conditions; see Kawohl [18, Section II.7, Corollary 2.32].

In this regard, we were not able to find in the literature a result that states or yields the following inequality: if $\Omega = (-\pi, \pi) \times \mathbb{R}$, then

$$\int_\Omega |\nabla(u^*)|^p \leq \int_\Omega |\nabla u|^p \quad \text{for all } u \in W^{1,p}(\Omega) \text{ with } u(-\pi, x_2) = u(\pi, x_2) \text{ for all } x_2 \in \mathbb{R},$$

where u^* is the Steiner rearrangement with respect to $\{x_1 = 0\}$. This inequality can be proven using a periodic polarization as introduced in Friedberg and Luttinger [16] and proceeding as in Brock and Solynin [6, Sections 5, 6, and 8]. The only explicit result of a periodic local Pólya-Szegő type inequality that we have found is the following result by Cox-Kawohl [12, Proposition 2.1]: Let $\Omega = (-\pi, \pi) \times (0, 1)$ and let u satisfy $u(-\pi, x_2) = u(\pi, x_2)$ for all $x_2 \in (0, 1)$. Then,

$$\int_\Omega dx_1 dx_2\, x_2 \left(\frac{1}{x_2^2} (u^*)^2_{x_1} + (u^*)^2_{x_2} \right) \leq \int_\Omega dx_1 dx_2\, x_2 \left(\frac{1}{x_2^2} u^2_{x_1} + u^2_{x_2} \right),$$

where u_{x_1} and u_{x_2} denote the partial derivatives of u.

The study of periodic versions of the nonlocal Pólya-Szegő inequality (1.4), in contrast to the local case, becomes highly nontrivial even in dimension one. Firstly, the Gagliardo seminorm (or other nonlocal energies) feels the changes in all other periods and one cannot simply localize the problem to one period. Secondly, even if one could study only one period, there is no useful inequality of the type (1.4) for bounded domains Ω, i.e., when the double integral in the seminorm (1.4) is replaced by $\Omega \times \Omega$. Indeed, symmetrization can increase the Gagliardo seminorm on domains; see for instance Li and Wang [19]. For these two reasons one has to deal with the presence of competing terms when treating nonlocal periodic energy functionals. We have overcome these difficulties (in the case of the periodic rearrangement) by combining the following two ingredients via the Laplace transform: the nonexpansivity Theorem 1.3 below and the monotonicity of the fundamental solution of the heat equation with periodic boundary conditions.

1.2. **A general nonexpansivity result for rearrangements.** Our proof of the Pólya-Szegő type inequalities in Theorem 1.1 strongly relies on the two very general nonexpansivity inequalities described in Theorems 1.2 and 1.3 below. We present them here since they are of independent interest and contain some additions to earlier known versions. For instance, in there we also treat the case of equality for the convolution kernel equal to the absolute value $t \mapsto |t|$, which is not strictly convex. Moreover, our method to prove them is a generalization of the one used in Frank and Seiringer [15, Lemma A.2], who gave en elegant and quite elementary new proof of the case of equality in such a nonexpansivity inequality.

The classical nonexpansivity property of the L^p distance of two functions under rearrangement can be seen as a generalization of the Riesz rearrangement inequality, which states that

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, u(x)g(x-y)v(y) \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, u^{*n}(x)g^{*n}(x-y)v^{*n}(y). \tag{1.6}$$

Under the assumption that $g = g^{*n}$ is radially decreasing in the whole space, the cases of equality were first characterized by Lieb [20]. Let us now state in the following theorem the generalization of this inequality, and we shall comment on its history and other versions thereafter. As stated here, it is the generalization of [15, Lemma A.2] to two functions. In the theorem, $|\Omega|$ denotes the measure of a set $\Omega \subset \mathbb{R}^n$.

Theorem 1.2. *Let J be a nonnegative, convex function in $\mathbb{R}$ with $J(0) = 0$, and let $g \in L^1(\mathbb{R}^n)$ be a nonnegative function. Then,*

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J(u^{*n}(x) - v^{*n}(y))g^{*n}(x-y) \leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J(u(x) - v(y))g(x-y) \tag{1.7}$$

for all pairs of nonnegative measurable functions $u, v : \mathbb{R}^n \to [0, +\infty)$ such that the right hand side of (1.7) is finite, and $|\{u > \tau\}|$ and $|\{v > \tau\}|$ are finite for all $\tau > 0$.

If, in addition, J is strictly convex and g is a radially symmetric decreasing[1] function, then equality in (1.7) holds if and only if one of the following two cases occur:

- *(i) One of the two functions u or v is zero almost everywhere in $\mathbb{R}^n$ (and the other one can be anything).*
- *(ii) There exists $a \in \mathbb{R}^n$ such that $u(x) = u^{*n}(x-a)$ and $v(y) = v^{*n}(y-a)$ for almost every $x \in \mathbb{R}^n$ and almost every $y \in \mathbb{R}^n$.*

If $J(t) = |t|$ and $g = g^{*n}$ is decreasing, then equality in (1.7) holds if and only if for every $\tau \in (0, \min\{\operatorname{ess\,sup} u, \operatorname{ess\,sup} v\})$ there exists $z_\tau \in \mathbb{R}^n$ such that

$$\{u > \tau\} = \{u^{*n} > \tau\} - z_\tau, \quad \{v > \tau\} = \{v^{*n} > \tau\} - z_\tau \quad \textit{up to sets of measure zero.}$$

The description of the cases of equality for strictly convex J is a special case of Burchard and Hajaiej [9, Theorem 2], which dealt with more general convolution kernels than a convex function J and multiple integrals[2]. However, [9] does not cover the cases of equality in the case $J(t) = |t|$. Note also that [9] states the inequality only in the case that $g = g^{*n}$. To the best of our knowledge, the method of polarization used in [9] can only work in that special case when $g = g^{*n}$.

Note that we characterize the cases of equality only under an additional assumption on g. For a general J there is probably little hope of a simple characterization of the cases of equality if we drop the assumption $g = g^{*n}$. As far as we know, the only case where this has been done is Burchard [8] for $J(t) = |t|^2$ —in which case Theorem 1.2 reduces to the Riesz rearrangement inequality by expanding the product $(u(x) - v(y))^2 = u(x)^2 + v(y)^2 - 2u(x)v(y)$. In the next paragraph, when we talk about the cases of equality, we will always assume that $g = g^{*n}$ is radially decreasing.

Observe that for $g = \delta$ (the Dirac delta function understood as a measure) and $J(t) = |t|^p$ Theorem 1.2 is the well known nonexpansivity inequality for the L^p-norm, whereas for $J(t) = |t|^2$ it is the Riesz rearrangement inequality, as mentioned above. The inequality involving general

[1]Here, and throughout the whole paper, we call *decreasing* what some other authors call *stricltly decreasing.*

[2]To compare with [9, Theorem 2], we must mention that if J is convex then $F(y_1, y_2) := -J(y_1 - y_2)$ is supermodular. See [5] for the case when J is a C^2 function.

g and J is due to Almgren and Lieb [1, Corollary 2.3] under the additional (but unnecessary) assumptions $J(-t) = J(t)$ and $J(u)$, $J(v) \in L^1(\mathbb{R}^n)$. A proof of the inequality without these two assumptions and for more general (k,n)-Steiner rearrangements, but requiring $g = g^{*n}$, can be found in Brock and Solynin [6, Lemma 8.2]. This last proof uses polarization techniques and, therefore, only works if $g = g^{*n}$. In the special case $u = v$ and $g = g^{*n}$ the two unnecessary assumptions were also removed by Frank and Seiringer [15, Lemma A.2], and indeed their proof also works for general g. A characterization of the cases of equality, again in the special case $u = v$, for strictly convex J, can be found in [6, Theorem 8.1]. In addition, [15] characterizes the cases of equality under the assumption $u = v$ (not being aware of the work of [6]), for strictly convex J, but also for $J(t) = |t|$. We however need the inequality, and in particular the characterization of the equality cases, in the most general form when u is not necessarily equal to v. This generalization is necessary for our application to Gagliardo seminorms, namely, to the second inequality in Theorem 1.1. In there we deal with cylindrical rearrangement and we will apply Theorem 1.2 in $\mathbb{R}^{n-1}$ to the functions $x' \mapsto u(x_1, x')$ and $v(y') := u(y_1, y')$ for two different frozen values x_1, $y_1 \in \mathbb{R}$.

For the proof of the first inequality in Theorem 1.1, we will also need a version of the nonexpansivity Theorem 1.2 on the circle. This is the content of Theorem 1.3. There are some subtle differences with the nonperiodic result in the hypothesis, proof, and conclusion. In its statement, u^* denotes the Steiner rearrangement of u with respect to the origin in the interval $(-\pi, \pi)$, which is the same as the Schwarz rearrangement in dimension one. Moreover, for a function g which is 2π-periodic in $\mathbb{R}$, recall that $g^{*\,\mathrm{per}}$ denotes the 2π-periodic extension of the function $(g\chi_{(-\pi,\pi)})^*$ (i.e., the Schwarz rearrangement of g once restricted to one period); see Section 2 for the detailed definition.

Theorem 1.3. *Let J be a nonnegative, convex function in $\mathbb{R}$, and let g be a nonnegative measurable 2π-periodic function with $g \in L^1(-\pi, \pi)$. Then,*

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J(u^*(x) - v^*(y)) g^{*\,\mathrm{per}}(x-y) \le \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J(u(x) - v(y)) g(x-y) \qquad (1.8)$$

for every pair of nonnegative measurable functions $u, v : (-\pi, \pi) \to \mathbb{R}$ such that the right hand side of (1.8) *is finite.*

If, in addition, J is strictly convex, $g = g^{*\,\mathrm{per}}$, and g is decreasing in $(0, \pi)$, then equality holds in (1.8) *if and only if one of the two cases occur:*

(*i*) *One of the two functions u or v is constant almost everywhere in $(-\pi, \pi)$ (and the other one can be anything).*

(*ii*) *If u and v are extended to $\mathbb{R}$ in a 2π-periodic way, then there exists $z \in \mathbb{R}$ such that $u(x) = u^{*\,\mathrm{per}}(x+z)$ and $v(y) = v^{*\,\mathrm{per}}(y+z)$ for almost every x, $y \in \mathbb{R}$.*

If $J(t) = |t|$, $g = g^{*\,\mathrm{per}}$, g is decreasing in $(0, \pi)$, and u and v are extended to $\mathbb{R}$ in a 2π-periodic way, then equality in (1.8) *holds if and only if for every $\tau \in (\operatorname{ess\,inf} u, \operatorname{ess\,sup} u) \cap (\operatorname{ess\,inf} v, \operatorname{ess\,sup} v)$ there exists $z_\tau \in \mathbb{R}$ such that*

$$\{u > \tau\} = \{u^{*\,\mathrm{per}} > \tau\} - z_\tau, \quad \{v > \tau\} = \{v^{*\,\mathrm{per}} > \tau\} - z_\tau \quad \text{up to sets of measure zero.} \qquad (1.9)$$

The theorem can also be stated requiring u, v, and g to be defined in $\mathbb{R}$ and to be 2π-periodic, or alternatively as a result on the circle. Note that, as a consequence of Theorem 1.3, if $J(t) = |t|$ and the supremum of one of the functions u and v is less than or equal to the infimum of the other one, then equality in (1.8) holds, whereas if the infima and suprema of both functions coincide, then (1.9) yields that all the superlevel sets of u and v are of the form $\bigcup_{k\in\mathbb{Z}}(I+2k\pi)$ for some interval $I \in \mathbb{R}$ depending on the level.

It must be mentioned that most of the statements of Theorem 1.3 are special cases of Burchard and Hajaiej [9, Theorem 2]. However, the same comparing comments apply as those already mentioned right after Theorem 1.2. In addition, we have the improvement that $J(0) = 0$ is not assumed in Theorem 1.3 (as it was in Theorem 1.2), in contrary to [9] who do assume that $J(0) = 0$. Removing this assumption, in particular in the case that $\inf J$ is not attained, is not straightforward.

As for the nonexpansivity Theorem 1.2 in $\mathbb{R}^n$, Theorem 1.3 is also a generalization of a Riesz rearrangement inequality, but now of a version on the circle. This Riesz rearrangement inequality is due to Baernstein [2] and is not as well known as the classical one in $\mathbb{R}^n$. There exists a generalization to the n-dimensional sphere, where in (1.6) the space $\mathbb{R}^n$ is replaced by $\mathbb{S}^{n-1}$, the rearrangement is by geodesic balls around a chosen pole, and $g(x-y)$ is replaced by $K(x\cdot y)$, where $K : [-1,1] \to [0,+\infty)$ is a nondecreasing function. In this form (i.e., with $g = g^{*\,\mathrm{per}}$) the Riesz rearrangement inequality on the circle $\mathbb{S}^1$ was first proven independently by Baernstein and Taylor [4] and by Friedberg and Luttinger [16]. They also generalized it to the sphere $\mathbb{S}^{n-1}$ with $n \geq 2$ in [4], and on $\mathbb{S}^1$ but to a product of an arbitrary finite number of functions in [16]. We point out that these results cannot be deduced from the classical Riesz rearrangement inequality (1.6) of the euclidean case. It is still an open problem when $n \geq 2$ how a more general form of the Riesz rearrangement inequality on the sphere would look like —i.e., not assuming K to be symmetric and also rearranging it in some way; see [3, Notes and Comments 8.11]. A study of the cases of equality for the Riesz rearrangement inequality on the sphere can be found in Burchard and Hajaiej [9], respectively Baernstein [3]. We will gather a summary of these results on the circle in Theorem 3.1.

Our proofs of the two nonexpansivity Theorems 1.2 and 1.3 are very much inspired by [15, Lemma A2], which we generalize in the case of the Schwarz rearrangement to two functions u and v. Moreover, we give a slight simplification, since we do not need to consider the second derivative of J in our proof, as is done in [15]. In short, the proof consists of generalizing the known facts about the case $J(t) = t^2$, i.e., the Riesz rearrangement inequality, to any convex function J. We will prove Theorem 1.3 in full detail in Section 3, since it is slightly harder and less known among the nonexpansivity results. Then, we will outline the proof of Theorem 1.2 in Section 4, giving the details only where the proof differs from the periodic case.

2. Some preliminaries on rearrangements

Let us start by recalling some basic facts on symmetrization and settle some notation. All sets and functions appearing in this section are assumed to be measurable with respect to the n-dimensional Lebesgue measure, where n will vary by the context. We will frequently use the layer cake representation of a nonnegative function u,

$$u(x) = \int_0^{+\infty} dt\, \chi_{\{u>t\}}(x) = \sup\{t : x \in \{u > t\}\}.$$

In this section we will deal with different types of rearrangements. We recall the definitions and most important properties that all these rearrangements have in common. To every measurable set $\Omega \subset \mathbb{R}^n$, $n \geq 1$, we associate $\Omega^\star$, a set of same n-dimensional Lebesgue measure as Ω, which we shall denote by $|\Omega|$, and with the following property: if $\Omega_1 \subset \Omega_2$ then $\Omega_1^\star \subset \Omega_2^\star$. For a characteristic function one defines $(\chi_\Omega)^\star := \chi_{\Omega^\star}$. More generally, for every measurable function

u defined in $\mathbb{R}^n$ or in a subset of it, we associate a rearranged function $u^\star$ by rearranging the superlevel sets of $|u|$, that is,

$$u^\star(x) := \int_0^{+\infty} dt\, \chi_{\{|u|>t\}^\star}(x) = \sup\{t :\, x \in \{|u| > t\}^\star\}. \tag{2.1}$$

This is well defined if $|\{|u| > t\}| < +\infty$ for all $t > 0$. Hence $u^\star$ is nonnegative and the definition (2.1) yields that, for every $\epsilon > 0$,

$$\{|u| > t + \epsilon\}^\star \subset \{u^\star > t\} \subset \{|u| > t\}^\star \text{ and, thus, } |\{|u| > t + \epsilon\}| \leq |\{u^\star > t\}| \leq |\{|u| > t\}|.$$

This in turn easily gives that $u^\star$ is equimeasurable with $|u|$, i.e., $|\{u^\star > t\}| = |\{|u| > t\}|$ for all $t > 0$. Hence, up to a set of n-dimensional Lebesgue measure zero, it holds that

$$\{u^\star > t\} = \{|u| > t\}^\star. \tag{2.2}$$

Moreover, if for some measurable subset $\Omega \subset \mathbb{R}^n$ with $|\Omega| < +\infty$ it holds that

$$|\{u^\star > t\} \cap \Omega^\star| = |\{|u| > t\} \cap \Omega| \quad \text{for all } t > 0,$$

and $F : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function, then

$$\int_\Omega F(|u|) = \int_{\Omega^\star} F(u^\star) \tag{2.3}$$

—actually, this identity holds true for slightly more general functions F. This follows immediately from Fubini's theorem, writing

$$\begin{aligned}\int_\Omega F(|u|) &= \int_\Omega dx\, \Big(F(0) + \int_0^{|u(x)|} dt\, F'(t)\Big)\\ &= F(0)|\Omega| + \int_\Omega dx \int_0^{+\infty} dt\, \chi_{(0,|u(x)|)}(t) F'(t)\\ &= F(0)|\Omega| + \int_0^{+\infty} dt\, F'(t)|\{|u| > t\} \cap \Omega|.\end{aligned}$$

This argument also shows that (2.3) holds in the case that $|\Omega| = +\infty$ if one further assumes, for example, that $F(0) = 0$.

In the sequel * will be one of the four rearrangements (or alternatively called symmetrizations) we will use in the different sections of this work, where $A \subset \mathbb{R}^n$ is a measurable set:

A^* : Steiner symmetrization in $\mathbb{R}^n$ with respect to the hyperplane $\{x_1 = 0\}$.

$A^{*\,n}$: Schwarz symmetrization in $\mathbb{R}^n$.

$A^{*\,\mathrm{per}}$: periodic symmetrization in $\mathbb{R}^n$ with respect to the variable x_1.

$A^{*\,n,1}$: cylindrical symmetrization in $\mathbb{R}^n$ with respect to the variable x_1.

Clearly $A^* = A^{*\,1}$. Be aware that we do not follow the notation of (k, n)-Steiner symmetrizations, where $1 \leq k \leq n$ and $(1, n)$, $(n-1, n)$, and (n, n) correspond to Steiner, cylindrical, and Schwarz symmetrization, respectively; see for instance [6, Section 4] where that notation was used.

We start with A^*, respectively its periodic version. For a measurable set $A \subset \mathbb{R}$ with finite measure, its rearrangement is defined as the open interval $A^* := \frac{1}{2}(-|A|, |A|)$, and the rearrangement of a function $u : \Omega \to \mathbb{R}$, defined in some subset $\Omega \subset \mathbb{R}$, is $u^* : \Omega^* \to \mathbb{R}$ defined by the layer cake formula (2.1).

For $A \subset \mathbb{R}^n$ we denote its section $A_{x'} := \{x_1 \in \mathbb{R} : (x_1, x') \in A\} \subset \mathbb{R}$ for every $x' \in \mathbb{R}^{n-1}$. If $A \subset \mathbb{R}^n$, then its Steiner symmetrization with respect to the hyperplane $\{x_1 = 0\}$ is defined by

$$A^* := \bigcup_{x' \in \mathbb{R}^{n-1}} (A_{x'})^* \times \{x'\},$$

with the agreement that if $|(A_{x'})^*| = 0$ then $(A_{x'})^* \times \{x'\} = \emptyset$. For a set $B \subset \mathbb{R}^n$ which is 2π-periodic in the variable x_1, we define $B^{*\,\mathrm{per}}$ by

$$B^{*\,\mathrm{per}} := \bigcup_{k \in \mathbb{Z}} \Big(\big(B \cap ((-\pi, \pi) \times \mathbb{R}^{n-1})\big)^* + 2k\pi e_1 \Big), \quad e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n.$$

It follows from the definitions that, for every $A \subset \mathbb{R}^n$ and every 2π-periodic set $B \subset \mathbb{R}^n$,

$$(A_{x'})^* = (A^*)_{x'} \quad \text{and} \quad (B_{x'})^{*\,\mathrm{per}} = (B^{*\,\mathrm{per}})_{x'} \quad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{2.4}$$

The periodic rearrangement of a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ which is 2π-periodic in the variable x_1 is defined by

$$u^{*\,\mathrm{per}}(x) := \int_0^{+\infty} dt\, \chi_{\{|u|>t\}^{*\,\mathrm{per}}}(x) = \sup\{t : x \in \{|u| > t\}^{*\,\mathrm{per}}\}.$$

It easily follows from (2.4) and the identity $\{|u(\cdot, x')| > t\} = \{|u| > t\}_{x'}$, that

$$(u(\cdot, x'))^{*\,\mathrm{per}}(x_1) = u^{*\,\mathrm{per}}(x_1, x') \quad \text{for all } x_1 \in \mathbb{R} \text{ and all } x' \in \mathbb{R}^{n-1}, \tag{2.5}$$

where the left hand side is simply the periodic rearrangement in $\mathbb{R}$ for every frozen value x'.

We now recall also the definition of the Schwarz symmetrization of a set. If $\Omega \subset \mathbb{R}^n$ is a measurable set of finite measure, then

$$\Omega^{*\,n} := \text{the open ball of } \mathbb{R}^n \text{ centered at the origin and of same measure as } \Omega.$$

We will use the following property of rearrangements in a crucial way and, therefore, we state it in a lemma and provide its proof for the sake of completeness.

Lemma 2.1. *Let $n \geq 1$, $\Omega \subset \mathbb{R}^n$ be measurable, $u : \Omega \to \mathbb{R}$ be a measurable function, and $G : [0, +\infty) \to [0, +\infty)$ be a nondecreasing and lower semicontinuous function. Then,*

$$(G \circ |u|)^{*\,n} = G \circ (u^{*\,n}) \quad \text{and} \quad (G \circ |u|)^{*\,\mathrm{per}} = G \circ (u^{*\,\mathrm{per}}), \tag{2.6}$$

where for the second identity we assume that $n = 1$, $\Omega = \mathbb{R}$, and u is 2π-periodic.

This lemma is sometimes stated with and sometimes without the assumption "lower semincontinuous" in the literature. Let us clarify this point. If G is nondecreasing, but not necessarily lower semicontinuous, then the identities (2.6) still hold true for a.e. $x \in \Omega^*$, but not necessarily everywhere. Note that the rearrangement of a function is always lower semicontinuous because rearranged sets are *open* balls. Consider, for instance, the following example: $\Omega = (-1, 1)$, $u(x) = u^*(x) = 1 - |x|$, and $G(t) = 1$ if $|t| \geq 1/2$ and 0 elsewhere. Then $(G \circ |u|)^*$ is the characteristic function of $(-1/2, 1/2)$, but $G \circ (u^*)$ is the characteristic function of the closed interval $[-1/2, 1/2]$.

Proof of Lemma 2.1. In view of the layer cake representations, we have that

$$(G \circ |u|)^{*\,n}(x) = \inf G + \int_{\inf G}^{\sup G} dt\, \chi_{\{G \circ |u| > t\}^{*\,n}}(x),$$

$$G \circ (u^{*\,n})(x) = \inf G + \int_{\inf G}^{\sup G} dt\, \chi_{\{G \circ (u^{*\,n}) > t\}}(x).$$

Therefore, it is enough to show that $\{G\circ|u|>t\}^{*n}=\{G\circ(u^{*n})>t\}$ for all $\inf G<t<\sup G$. Since G is nondecreasing, lower semincontinuous, and $\inf G<t<\sup G$, the set $G^{-1}(t,+\infty)$ is equal to an open interval $(t_G,+\infty)$ for some $t_G>0$. Hence, it is enough to show that $\{|u|>t_G\}^{*n}=\{u^{*n}>t_G\}$, which is precisely (2.2). Note that, for $\star=*n$, the identity (2.2) holds as an equality of sets (not only in measure) because both sides of the identity are open balls. This proves the first identity in (2.6).

The second identity in (2.6), which refers to the periodic rearrangement, follows from the first one applied to $\Omega=(-\pi,\pi)$. More precisely, since both functions $(G\circ|u|)^{*\,\mathrm{per}}$ and $G\circ(u^{*\,\mathrm{per}})$ are 2π-periodic, it is enough to prove the identity in $(-\pi,\pi)$, and the latter follows from the fact that $f^*=f^{*\,\mathrm{per}}$ in $(-\pi,\pi)$ for every f. □

We will also use the following rather elementary lemma to deal with sets of measure zero.

Lemma 2.2. *Let $m\in\mathbb{N}$ and $f,\,g:\mathbb{R}^n\to[0,+\infty)$ be two nonnegative measurable functions.*

(i) *Suppose that there exist a countable dense set $\{\tau_k\}_{k\in\mathbb{N}}\subset(0,+\infty)$ and a sequence of sets of zero m-dimensional Lebesgue measure $\{N_k\}_{k\in\mathbb{N}}$ such that, for every $k\in\mathbb{N}$,*

$$\chi_{\{f>\tau_k\}}(y)=\chi_{\{g>\tau_k\}}(y)\quad \textit{for all } y\in\mathbb{R}^n\setminus N_k.$$

Then $f=g$ almost everywhere in $\mathbb{R}^n$. In particular, if $\{f>\tau\}=\{g>\tau\}$ in measure for a.e. $\tau>0$, then $f=g$ almost everywhere in $\mathbb{R}^n$.

(ii) *Suppose that $\{f>\tau\}$ is a ball (open or closed or containing some subsets of its boundary) for almost every $\tau>0$. Then, $\{f>\tau\}$ is a ball for every $\tau>0$ such that $|\{f>\tau\}|<+\infty$, in the sense that $\{f>\tau\}$ contains an open ball of the same measure.*

Let us stress that, in Lemma 2.2 (i), we are assuming that the sets $\{f>\tau\}$ and $\{g>\tau\}$ only differ on a set of measure zero, but this set may depend on τ. In this regard, if one actually knows that for a.e. $\tau>0$ the sets $\{f>\tau\}$ and $\{g>\tau\}$ only differ on a set of measure zero independent of τ, the fact that $f=g$ almost everywhere in $\mathbb{R}^n$ would follow directly from the layer cake decomposition of f and g. However, the dependence on τ of the zero measure set makes the conclusion of Lemma 2.2 (i) not completely trivial.

Observe also that Lemma 2.2 (ii) may not hold if $|\{f>\tau\}|<+\infty$ is not assumed. For example, think on the balls $B_n(ne_1)$, whose boundary passes through the origin for every n. Then, the union of the balls over all n gives a half space.

Proof of Lemma 2.2. Let us first address the proof of (i). We will see that $|\{|f-g|>0\}|=0$. Note that

$$\{|f-g|>0\}=\bigcup_{j=1}^{+\infty}\{|f-g|>1/j\}=\bigcup_{j=1}^{+\infty}\Big(\{f-g>1/j\}\cup\{g-f>1/j\}\Big).$$

Hence, it is enough to check that $|\{f-g>1/j\}|=|\{g-f>1/j\}|=0$ for all $j\geq1$. We will prove that $|\{f-g>1/j\}|=0$; the case of $|\{g-f>1/j\}|$ follows analogously.

Assume that $x\in\{f-g>1/j\}$, which means that $f(x)-g(x)>1/j$. Since $\{\tau_k\}_{k\in\mathbb{N}}$ is dense in $(0,+\infty)$, there exists $k_0\in\mathbb{N}$ such that $g(x)\leq\tau_{k_0}<f(x)$. Therefore, $x\in\{f>\tau_{k_0}\}\cap\{g\leq\tau_{k_0}\}=\{f>\tau_{k_0}\}\setminus\{g>\tau_{k_0}\}$. From this we deduce that

$$\{f-g>1/j\}\subset\bigcup_{k\in\mathbb{N}}\Big(\{f>\tau_k\}\setminus\{g>\tau_k\}\Big).$$

Now, the fact that $\chi_{\{f>\tau_k\}}(y)=\chi_{\{g>\tau_k\}}(y)$ for all $y\in\mathbb{R}^n\setminus N_k$ and that $|N_k|=0$ yields $|\{f>\tau_k\}\setminus\{g>\tau_k\}|=0$ for all $k\in\mathbb{N}$. Thus, $|\{f-g>1/j\}|=0$ for all $j\geq1$, as desired.

We now prove (ii). Let $\tau_i \downarrow \tau$ be a monotone sequence such that $\{f > \tau_i\}$ are balls with radii R_i. From the inclusions $\{f > \tau_i\} \subset \{f > \tau_{i+1}\}$, we get that the sequence $\{R_i\}_i$ is monotone nondecreasing and has a limit R_∞, which is finite because $|\{f > \tau\}| < +\infty$ by assumption. The centers of the balls $\{f > \tau_i\}$ must have a converging subsequence, and it can be easily seen that the limit of such a sequence must be unique. One then concludes the proof of (ii) simply using that $\{f > \tau\} = \bigcup_{i=1}^{+\infty}\{f > \tau_i\}$. □

3. Nonexpansivity under periodic rearrangement

The aim of this section is to prove Theorem 1.3. The proof is based on the following version of the Riesz rearrangement inequality on the circle.

Theorem 3.1 ([2, 3, 4, 9, 16])**.** *Let $f, h : (-\pi, \pi) \to \mathbb{R}$ be two nonnegative measurable functions and $g : (-2\pi, 2\pi) \to [0, +\infty)$ be a 2π-periodic measurable function. Then,*

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, f(x)g(x-y)h(y) \leq \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, f^*(x)g^{*\,\mathrm{per}}(x-y)h^*(y). \tag{3.1}$$

If in addition $g = g^{*\,\mathrm{per}}$, g is decreasing in $(0, \pi)$, and the left-hand side of (3.1) is finite, then equality holds in (3.1) if and only if at least one of the following conditions holds:

- (i) *One of the two functions f or h is constant almost everywhere (and the other one can be anything).*
- (ii) *If f and h are extended to $\mathbb{R}$ in a 2π-periodic way, then there exists $z \in \mathbb{R}$ such that $f(x) = f^{*\,\mathrm{per}}(x + z)$ and $h(x) = h^{*\,\mathrm{per}}(x + z)$ for almost every $x \in \mathbb{R}$.*

The inequality (3.1) in the case $g = g^{*\,\mathrm{per}}$ was first discovered, independently, in [4] and [16]. Both references contain more general inequalities: [4] deals with the sphere $\mathbb{S}^n$, whereas [16] deals with a product of more than three functions defined in $\mathbb{R}$. The result [3, Theorem 7.3] is also more general than Theorem 3.1, as it deals with a Riesz rearrangement inequality on the sphere $\mathbb{S}^n$ and not only on the circle. Moreover, [3] treats more general functions of $f(x)$ and $h(y)$ than simply the product $f(x)h(y)$. Without the assumption $g = g^{*\,\mathrm{per}}$, the inequality (3.1) can be found in [2]. We mention that we stated the inequality in this general form, but we will actually only use the case when $g = g^{*\,\mathrm{per}}$.

Instead, the statement in Theorem 3.1 concerning equality in (3.1) follows from Burchard and Hajaiej [9, Theorem 2], who treated the case of equality in $\mathbb{S}^n$ for the first time. For this result, we also cite [3, Theorem 7.3] since, being less general than [9], fits precisely with our setting.

Let us finally mention that we find [3] to be the simplest reference for looking up all the statements of Theorem 3.1 that we need in the present work.

For the proof of Theorem 1.3, it will be convenient to introduce the following abbreviation

$$E[u, v, g] := \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J(u(x) - v(y))g(x - y),$$

where $J : \mathbb{R} \to [0, +\infty)$ is some convex function. Moreover, whenever we will assume that $g = g^{*\,\mathrm{per}}$, as is the setting when studying cases of equality, as well as in all our applications, we will omit that argument and will simply use the notation

$$E[u, v] := E[u, v, g] \quad \text{ if } g = g^{*\,\mathrm{per}}.$$

The main idea of the proof is based on that of the nonexpansivity of the L^p norm under rearrangement without the convolution kernel g (see [21, Theorem 3.5]), but one uses in the proof the periodic Riesz rearrangement inequality of Theorem 3.1 instead of the Hardy-Littlewood inequality $\int fh \leq \int f^*h^*$. This same strategy of the proof was also used by Frank and Seiringer [15, Appendix A, Lemma A.2].

Before starting the proof let us make some comments on some of the hypothesis' of the theorem. Note that, contrary to the case $u = v$, strict convexity of J in $[0, +\infty)$ or $(-\infty, 0]$ is not enough to characterize cases of equality, as in [15, first paragraph in the proof of Lemma A.2]. Consider, for instance, a function J which is identically zero in the negative real line and strictly convex in $[0, +\infty)$. Then, for any pair of nonnegative functions u and v such that $\inf v \geq \sup u$ it holds that $E[u, v] = 0 = E[u^*, v^*]$.

Remark 3.2. If J is an even, nonnegative, and convex function with its minimum at 0, then the inequality in Theorem 1.3 remains true even if we drop the assumption on the sign of u and v, since $|u - v| \geq ||u| - |v||$, but one would have to ensure by some other assumption than $|\{u > \tau\}| < +\infty$ that the rearrangement is well defined. For instance, if $J(t) = t^2$ then the inequality still holds, as can be easily seen by the factorization $(u - v)^2 = u^2 + v^2 - 2uv$, the fact that $u \leq |u|$, and Theorem 3.1. That (ii) implies the equality in (1.8) obviously remains true in this special case, but that (i) implies the equality in (1.8) does not hold if we do not assume u and v to be nonnegative, even for $J(t) = t^2$. Consider, for instance, that $u = c > 0$ is some positive constant, in which case

$$\begin{aligned} E[u,v] &= \int_{-\pi}^{\pi} dx\, g(x) \int_{-\pi}^{\pi} dy\, (c - v(y))^2 = \int_{-\pi}^{\pi} dx\, g(x) \int_{-\pi}^{\pi} dy\, (c^2 - 2cv(y) + v^2(y)) \\ &= \int_{-\pi}^{\pi} dx\, g(x) \int_{-\pi}^{\pi} dy\, (c^2 - 2cv(y) + (v^*)^2(y)). \end{aligned}$$

Now, for any $v \geq 0$ not necessarily symmetric, we have $\int v = \int |v| = \int v^*$. However, if v takes on negative values on a set of positive measure, then $\int v < \int |v| = \int v^*$, and therefore $E[u^*, v^*]$ is strictly smaller than $E[u, v]$. Thus, the condition of (i) is satisfied, since u is constant, but there is no equality in (1.8).

Proof of Theorem 1.3. The proof of the theorem will be divided in five steps. We will first assume that $\min J$ exists to carry out the proof of the theorem for this case in Steps 1 to 4. The proof of the case that $\inf J$ is not attained is slightly different and is given in Step 5. Recall that we write $E[u, v]$ instead of $E[u, v, g]$ whenever $g = g^{*\,\mathrm{per}}$.

Step 1 (preliminaries):

The purpose of this first step is to show that we can assume that $J(0) = 0$. To see this, let $J(t_0) = \min J$ and consider the new function $\widetilde{J}(t) = J(t + t_0) - \min J$, which vanishes at $t = 0$, is nonnegative, and convex. Denote the corresponding functional for $\widetilde{J}$ by $\widetilde{E}[u, v, g]$. The term in $\widetilde{E}[u, v, g]$ involving $\min J$ is equal to

$$-\min J \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, g(x - y) = -2\pi \min J\, \|g\|_{L^1(-\pi,\pi)} = -2\pi \min J\, \|g^{*\,\mathrm{per}}\|_{L^1(-\pi,\pi)}, \qquad (3.2)$$

which does not depend on u and v. Therefore, this term will have no effect when showing the inequality or studying the case of equality.

Let us first show that it is sufficient to prove the inequality for $\widetilde{E}$ to get it also for E. Assume that the inequality holds for $\widetilde{J}$ and any two nonnegative functions u and v. On the one hand, if $t_0 \geq 0$, by (3.2) we have

$$\begin{aligned}
E[u,v,g] &= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, \widetilde{J}(u(x) - (v(y) + t_0))g(x-y) + 2\pi \min J\, \|g\|_{L^1(-\pi,\pi)} \\
&= \widetilde{E}[u, v + t_0, g] + 2\pi \min J\, \|g\|_{L^1(-\pi,\pi)} \\
&\geq \widetilde{E}[u^*, (v+t_0)^*, g^{*\,\mathrm{per}}] + 2\pi \min J\, \|g^{*\,\mathrm{per}}\|_{L^1(-\pi,\pi)} \\
&= \widetilde{E}[u^*, v^* + t_0, g^{*\,\mathrm{per}}] + 2\pi \min J\, \|g^{*\,\mathrm{per}}\|_{L^1(-\pi,\pi)} \\
&= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, \widetilde{J}(u^*(x) - (v^*(y) + t_0))g^{*\,\mathrm{per}}(x-y) + 2\pi \min J\, \|g^{*\,\mathrm{per}}\|_{L^1(-\pi,\pi)} \\
&= E[u^*, v^*, g^{*\,\mathrm{per}}],
\end{aligned}$$

where we have used that $v, t_0 \geq 0$ and, hence, $v + t_0$ is nonnegative and $(v+t_0)^* = v^* + t_0$. On the other hand, if $t_0 < 0$ one proceeds in a similar way, but using now the inequality for $\widetilde{J}$ and the nonnegative functions $u - t_0$ and v.

Let us now assume that $g = g^{*\,\mathrm{per}}$, that $E[u,v] = E[u^*,v^*]$, and that we can characterize the cases of equality for $\widetilde{E}$. If $t_0 \geq 0$ then, using that

$$\begin{aligned}
\widetilde{E}[u, v+t_0] &= E[u,v] - 2\pi \min J\, \|g\|_{L^1(-\pi,\pi)} \\
&= E[u^*, v^*] - 2\pi \min J\, \|g^{*\,\mathrm{per}}\|_{L^1(-\pi,\pi)} = \widetilde{E}[u^*, (v+t_0)^*]
\end{aligned}$$

and that $(v+t_0)^* = v^* + t_0$, the cases of equality for E are also characterized. As before, if $t_0 < 0$ simply consider $u - t_0$, for which $(u - t_0)^* = u^* - t_0$.

All these considerations show that it is enough to prove the theorem for $\widetilde{J}$ or, equivalently, that we can assume that $J(0) = 0$ without loss of generality.

Step 2 (proof of the inequality):

In this section we will prove (1.8). In view of Step 1, to do it we can assume that $\min J = J(0) = 0$. Let us decompose $J = J_+ + J_-$, where $J_+(t) := J(t)$ for $t \geq 0$ and $J_+(t) := 0$ for $t < 0$, and define the corresponding functionals

$$E_\pm[u,v,g] := \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J_\pm(u(x) - v(y))g(x-y). \tag{3.3}$$

Note that, by the assumptions on J, $J_\pm$ are nonnegative convex functions with $J_\pm(0) = 0$. In particular, J_+ is nondecreasing in $\mathbb{R}$.

Using this decomposition of J, it is clear that it is enough to prove the inequality for $E_\pm$ to get it also for E. Actually, we only need to show that E_+ is nonincreasing under rearrangement. Once this is proven, it easily follows that E_- is nonincreasing under rearrangement too by applying the proof for done for E_+ to $t \mapsto J_-(-t)$ (which is also nonnegative, convex, and vanishes in $(-\infty, 0]$) and interchanging the roles of u and v. Therefore, from now on we will only focus on the proof of (1.8) for J_+.

Note that J_+' is nondecreasing, nonnegative, and continuous almost everywhere (see, for instance, [23, Theorem 25.3]) and hence, redefining J' on a set of measure zero, we can assume that J_+' is lower semicontinuous in $\mathbb{R}$. We chose this representative of J_+' because, then, any translation of J_+' satisfies the assumptions of the function G in Lemma 2.1. Also, recalling that $J_+ = 0$ in $(-\infty, 0]$, we have $J_+'(0) = 0$.

Since J_+ is locally Lipschitz continuous, it is also absolutely continuous on any bounded set, and therefore

$$J_+(u(x)-v(y)) = \int_{u(x)}^{v(y)} d\tau\, \frac{d}{d\tau}\Big(J_+(u(x)-\tau)\Big) = -\int_{u(x)}^{v(y)} d\tau\, J_+'(u(x)-\tau).$$

Recall that $J_+' = 0$ in $(-\infty, 0]$, thus

$$\begin{aligned} J_+(u(x)-v(y)) &= \int_{v(y)}^{+\infty} d\tau\, J_+'(u(x)-\tau) = \int_0^{+\infty} d\tau\, J_+'(u(x)-\tau)\chi_{\{v\le\tau\}}(y) \\ &= \int_0^{+\infty} d\tau\, J_+'(u(x)-\tau)(1-\chi_{\{v>\tau\}}(y)). \end{aligned}$$

Since $E_+[u,v,g]$ is finite, an application of Fubini's theorem gives

$$E_+[u,v,g] = \int_0^{+\infty} d\tau \left(\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J_+'(u(x)-\tau)g(x-y)(1-\chi_{\{v>\tau\}}(y))\right). \tag{3.4}$$

We will now distinguish first a special case and then we will treat the general case.

Step 2.1 (proof of the inequality for u or $|J'|$ bounded):

Note first that if u is bounded, that is, if $\operatorname{ess\,sup} u \le M < +\infty$ for some $M > 0$, then we can estimate, for any $\tau \ge 0$,

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J_+'(u(x)-\tau)g(x-y) \le 2\pi \|g\|_{L^1(-\pi,\pi)} J_+'(M-\tau) < +\infty.$$

Similarly, if $|J'|$ is bounded (which occurs, for instance, if $J(t) = |t|$), the double integral on the left hand side is finite too. Therefore, in any case we can split $E_+[u,v,g]$ given in (3.4) as

$$E_+[u,v,g] = \int_0^{+\infty} d\tau\, \big(A(u,g,\tau) - B(u,v,g,\tau)\big), \tag{3.5}$$

where A and B are both finite and nonnegative for every τ, and are given by

$$\begin{aligned} A(u,g,\tau) &:= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J_+'(u(x)-\tau)g(x-y), \\ B(u,v,g,\tau) &:= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J_+'(u(x)-\tau)g(x-y)\chi_{\{v>\tau\}}(y). \end{aligned} \tag{3.6}$$

Observe that $A(u,g,\tau) \ge B(u,v,g,\tau)$, hence the integrand on the right hand side of (3.5) is nonnegative.

We will show that, for every $\tau \ge 0$, the functional A is unaltered under the rearrangement of u, v, and g, and that B does not decrease. Once we have shown this, the inequality of the theorem is shown as follows, using the abbreviations $A(\tau) \equiv A(u,g,\tau)$ and $A^*(\tau) \equiv A(u^*, g^{*\,\mathrm{per}}, \tau)$, and analogously for B:

$$\begin{aligned} E_+[u,v,g] &= \int_0^{+\infty} d\tau\, (A(\tau) - B(\tau)) = \int_0^{+\infty} d\tau\, (A^*(\tau) - B(\tau)) \\ &= \int_0^{+\infty} d\tau\, (A^*(\tau) - B^*(\tau) + B^*(\tau) - B(\tau)) \\ &= E_+[u^*, v^*, g^{*\,\mathrm{per}}] + \int_0^{+\infty} d\tau\, (B^*(\tau) - B(\tau)) \ge E_+[u^*, v^*, g^{*\,\mathrm{per}}]. \end{aligned} \tag{3.7}$$

Observe that one cannot assume in general that $\int d\tau\, A(\tau)$ and $\int d\tau\, B(\tau)$ are finite.[3] In principle, only the difference $A(\tau) - B(\tau)$ is known to be integrable in $\tau \in (0, +\infty)$.

It only remains to show the claimed behaviors of A and B under rearrangement. We first deal with A. Since g is 2π-periodic and nonnegative, we have $\int_{-\pi}^{\pi} dy\, g(x-y) = \|g\|_{L^1(-\pi,\pi)} = \|g^{*\,\mathrm{per}}\|_{L^1(-\pi,\pi)}$ for all x. Hence, it is sufficient to show that

$$\int_{-\pi}^{\pi} dx\, J_+'(u(x)-\tau) = \int_{-\pi}^{\pi} dx\, J_+'(u^*(x)-\tau). \tag{3.8}$$

Using that $\int_{-\pi}^{\pi} f = \int_{-\pi}^{\pi} f^*$ for the nonnegative function $f := J_+' \circ (u-\tau)$ we can first replace the integrand on the left hand side of (3.8) by $(J_+' \circ (u-\tau))^*$, and then apply Lemma 2.1 with $G(s) := J_+'(s-\tau)$ for all $s \geq 0$, since $u = |u|$ by assumption. We therefore get

$$(J_+'(u-\tau))^*(x) = J_+'(u^*(x)-\tau), \tag{3.9}$$

and (3.8) follows.

The claim that B does not decrease under rearrangement follows from Theorem 3.1 using (3.9) and (2.2).

Step 2.2 (proof of the inequality for general u):

We will now remove the assumption that u is bounded. To do it, for $M > 0$ define

$$u_M := \min\{u, M\} \quad \text{and} \quad (u^*)_M := \min\{u^*, M\}. \tag{3.10}$$

Observe that if we replace in the integrand of (3.3), in the case J_+, the function u by u_M, then this integrand is nondecreasing for every (x, y) as M increases. Hence the monotone convergence theorem gives that

$$\lim_{M\uparrow+\infty} E_+[u_M, v, g] = E_+[u, v, g] \quad \text{and} \quad \lim_{M\uparrow+\infty} E_+[(u^*)_M, v^*, g^{*\,\mathrm{per}}] = E_+[u^*, v^*, g^{*\,\mathrm{per}}].$$

The proof of the inequality for general u follows from Step 2.1 and the fact that $(u^*)_M = (u_M)^*$.

Step 3 (case of equality for J strictly convex):

Throughout this step we will assume that J is strictly convex, that $g = g^{*\,\mathrm{per}}$ is decreasing in $(0, \pi)$, and shall write $E[u, v]$ instead of $E[u, v, g]$. The purpose of this step is to show that there is equality in (1.8) if and only if (i) and/or (ii) in Theorem 1.3 holds.

Step 3.1 (case of equality for u or v constant, or u and v as in Theorem 1.3 (ii)):

Note that (ii) trivially yields equality in (1.8).

We will show here that if one of the functions u or v is constant (that is, if Theorem 1.3 (i) holds), then there is equality in (1.8). Let us assume that $u(x) = c \geq 0$ for all x; the case of v constant is analogous. Then, by the 2π-periodicity of g,

$$E[u, v] = \int_{-\pi}^{\pi} dy\, J(c - v(y)) \left(\int_{-\pi}^{\pi} dx\, g(x-y) \right) dy = \|g\|_{L^1(-\pi,\pi)} \int_{-\pi}^{\pi} dy\, J(c - v(y)).$$

Now, by the equimeasurability indentity (2.3) applied to $F(t) := J(c-t)$ we get that

$$\int_{-\pi}^{\pi} dy\, J(c - v(y)) = \int_{-\pi}^{\pi} dy\, J(c - v^*(y)).$$

This shows that $E[u, v] = E[u^*, v^*]$.

[3]This would be essentially equivalent to assuming, for instance if $J(t) = |t|$ and $g \equiv 1$, that $u \in L^1(-\pi, \pi)$, since $\int d\tau A(\tau) = 2\pi\|u\|_{L^1}$.

Step 3.2 (case of equality for u and v bounded):

In this step we will show that if both u and v are bounded, neither u nor v are constant, and there is equality in (1.8), then Theorem 1.3 (*ii*) holds.

Set $S := \operatorname{ess\,sup} v$ and

$$I_S(t) := \begin{cases} J(t-S) - tJ'(-S) - J(-S) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

If J is not differentiable at $-S$, then take instead of $J'(-S)$ in the above formula any value of a subgradient (i.e., the slope of a tangent touching the graph of J at $-S$). Note that $I_S(0) = 0$. Therefore, I_S is a convex and nonnegative function, and strictly convex in $[0, +\infty)$. Also, observe that $u + S - v \geq 0$, thus

$$J(u(x) - v(y)) = I_S(u(x) + S - v(y)) + (u(x) + S - v(y))J'(-S) + J(-S) \tag{3.11}$$

for all x, y. Regarding the term $(u(x) + S - v(y))J'(-S)$, by the 2π-periodicity of g we have that (recall also that we are assuming u and v bounded)[4]

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, (u(x) + S - v(y))g(x-y) = \|g\|_{L^1(-\pi,\pi)} \Big(\int_{-\pi}^{\pi} (u+S) - \int_{-\pi}^{\pi} v \Big).$$

Hence, this expression coming from $tJ'(-S)$ with $t = u(x) + S - v(y)$ in the definition of $I_S(t)$ is unaltered by the rearrangement of u and v because $(u+S)^* = u^* + S$. The same conclusion holds for the expression coming from $J(-S)$ in the definition of $I_S(t)$. Therefore, defining $u_S := u + S$, the assumption of equality in (1.8) and (3.11) yield

$$\begin{aligned} E_S[u, v] &:= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, I_S(u_S(x) - v(y))g(x-y) \\ &= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, I_S((u_S)^*(x) - v^*(y))g(x-y) = E_S[u^*, v^*], \end{aligned} \tag{3.12}$$

where we have used that $(u_S)^* = (u^*)_S$. Since $I_S(t) = 0$ for $t \leq 0$, we can proceed exactly as we did in Step 2 to obtain (3.4) and write

$$E_S[u, v] = \int_0^{+\infty} d\tau \left(\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, I_S'(u_S(x) - \tau)g(x-y)(1 - \chi_{\{v>\tau\}}(y)) \right). \tag{3.13}$$

Since we assumed that u is bounded, u_S too. Thus, all conclusions of Step 2 hold with E_+ replaced by E_S and A, B replaced by A_S, B_S where

$$\begin{aligned} A_S(u_S, \tau) &:= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, I_S'(u_S(x) - \tau)g(x-y), \\ B_S(u_S, v, \tau) &:= \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, I_S'(u_S(x) - \tau)g(x-y)\chi_{\{v>\tau\}}(y). \end{aligned}$$

In particular, A_S is unaltered under rearrangement and the analog of (3.7) for I_S' and u_S holds. Therefore, using that the integrand of B_S is nonnegative, that B_S does not decrease under

[4]Note that this step will not work in $\mathbb{R}$ instead of $(-\pi, \pi)$, since the integrals may be infinite —compare with the proof of Theorem 1.2 in Section 4; see in particular the paragraph after (4.4).

rearrangement, (3.12), and (3.13), we must have that $B_S(u_S, v, \tau) = B_S((u_S)^*, v^*, \tau)$, i.e.,

$$\begin{aligned}\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, I_S'(u_S(x)-\tau) g(x-y) \chi_{\{v>\tau\}}(y) \\ = \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, (I_S'(u_S-\tau))^*(x) g(x-y) \chi_{\{v^*>\tau\}}(y)\end{aligned} \tag{3.14}$$

for almost every $\tau > 0$ (here we have also used (3.9) with I_S' instead of J_+'). We will therefore be in position to apply the conclusions of Theorem 3.1, as done below.

We now use the assumption that u and v are not constant almost everywhere. Firstly, this means that

$$\chi_{\{v>\tau\}}(y) \text{ is not constant in } y \text{ for any } \tau \in (\operatorname{ess\,inf} v, \operatorname{ess\,sup} v) = (\operatorname{ess\,inf} v, S) \neq \emptyset.$$

For this range of τ we also obtain that

$$u_S(x) - \tau \geq S - \tau > 0. \tag{3.15}$$

Since I_S is strictly convex in $[0, +\infty)$, I_S' is increasing and, therefore, one-to-one in the positive real line. Thus,

$$I_S'(u(x)-\tau) \text{ is not constant in } x \text{ for any } \tau \in (\operatorname{ess\,inf} v, S).$$

Assuming that u and v are extended to $\mathbb{R}$ in a 2π-periodic way, from (3.14) and the case of equality in Theorem 3.1 we deduce that for almost every $\tau \in (\operatorname{ess\,inf} v, S)$ there exists $z_\tau \in \mathbb{R}$ such that

$$\begin{aligned} I_S'(u_S(x)-\tau) &= (I_S'(u_S-\tau))^{*\,\mathrm{per}}(x+z_\tau) = I_S'((u_S)^{*\,\mathrm{per}}(x+z_\tau)-\tau), \\ \chi_{\{v>\tau\}}(y) &= \chi_{\{v^{*\,\mathrm{per}}>\tau\}}(y+z_\tau) \end{aligned} \tag{3.16}$$

for almost every $x, y \in \mathbb{R}$, where we have also used Lemma 2.1 as in (3.9) in the first line in (3.16). Since I_S' is one-to-one in the positive real line, the first identity is equivalent to

$$u_S(x) = (u_S)^{*\,\mathrm{per}}(x+z_\tau) \text{ for a.e. } x \in \mathbb{R}, \text{ for a.e. } \tau \in (\operatorname{ess\,inf} v, S). \tag{3.17}$$

From this it follows that z_τ is indeed independent of τ because u is not constant. In more detail, that u is not constant yields that $(u_S)^{*\,\mathrm{per}}$ is a nontrivial symmetric decreasing function in $(-\pi, \pi)$, which in particular yields that $(u_S)^{*\,\mathrm{per}}(\cdot + z_{\tau_1}) = (u_S)^{*\,\mathrm{per}}(\cdot + z_{\tau_2})$ almost everywhere if and only if $z_{\tau_1} = z_{\tau_2}$ modulo 2π, since $(u_S)^{*\,\mathrm{per}}$ has minimal period 2π.

Finally, using that $(u_S)^{*\,\mathrm{per}} = u^{*\,\mathrm{per}} + S$, (3.17), and the fact that z_τ is indeed independent of τ, we conclude that $u(x) = u^{*\,\mathrm{per}}(x+z)$ for some $z \in \mathbb{R}$ and almost every $x \in \mathbb{R}$. In addition, from (3.16) we now obtain that

$$\chi_{\{v>\tau\}}(y) = \chi_{\{v^{*\,\mathrm{per}}>\tau\}}(y+z) \text{ for a.e. } y \in \mathbb{R}, \text{ for a.e. } \tau \in (\operatorname{ess\,inf} v, \operatorname{ess\,sup} v).$$

This clearly holds also for all $\tau \in (0, +\infty) \setminus (\operatorname{ess\,inf} v, \operatorname{ess\,sup} v)$ by definition of the rearrangement. Thus, using Lemma 2.2 (*i*) we get that $v(y) = v^{*\,\mathrm{per}}(y+z)$ for almost every $y \in \mathbb{R}$.

Step 3.3 (case of equality for general u and v):

To complete the proof of Step 3, it only remains to remove the boundedness assumption in Step 3.2. This is precisely what we will do here. More precisely, we will show that if neither u nor v are constant and there is equality in (1.8), then Theorem 1.3 (*ii*) holds.

Observe first that the equality $E[u, v] = E[u^*, v^*]$ forces the validity of both equalities

$$E_+[u, v] = E_+[u^*, v^*] \quad \text{and} \quad E_-[u, v] = E_-[u^*, v^*], \tag{3.18}$$

where $E_\pm$ have been introduced in (3.3). This is because $E = E_+ + E_-$, E_+ and E_- are nonnegative, and both are nonincreasing under rearrangement as shown in Step 2.

Let u_M and v_M be defined as in (3.10). Observe that u and v are assumed to be nonconstant, hence u_M and v_M are also nonconstant for all M big enough (in order to use Step 3.2). Setting $\phi_u^M := \max\{0, u - M\}$, and analogously for v, we have

$$u = u_M + \phi_u^M, \qquad v = v_M + \phi_v^M.$$

We will prove below that $E_+[u_M, v_M] = E_+[(u_M)^*, (v_M)^*]$ using the first identity of (3.18). Considering $J_-(-t)$ and interchanging the roles of u and v, it is easily seen that the same argument will prove the corresponding statement for J_-, that is to say, that the second identity of (3.18) yields $E_-[u_M, v_M] = E_-[(u_M)^*, (v_M)^*]$. Then, assuming for the moment that the identities $E_\pm[u_M, v_M] = E_\pm[(u_M)^*, (v_M)^*]$ have been proven, they yield that $E[u_M, v_M] = E[(u_M)^*, (v_M)^*]$. At this point we are in position to apply Step 3.2, which proves that, for every $M > 0$, $u_M(x) = (u_M)^{*\,\mathrm{per}}(x + z)$ and $v_M(x) = (v_M)^{*\,\mathrm{per}}(x + z)$ for almost every $x \in \mathbb{R}$, for some $z \in \mathbb{R}$ depending a priori on M. The fact that z is indeed independent of M follows by an argument as the one below (3.17). By letting $M \uparrow +\infty$ and using that $(u^*)_M = (u_M)^*$ in $(-\pi, \pi)$ (and analogously for v), we finally get that $u(x) = u^{*\,\mathrm{per}}(x+z)$ and $v(x) = v^{*\,\mathrm{per}}(x+z)$ for some $z \in \mathbb{R}$ and almost every $x \in \mathbb{R}$.

Therefore, it only remains to prove that $E_+[u_M, v_M] = E_+[(u_M)^*, (v_M)^*]$. We first claim that for every $x, y \in (-\pi, \pi)$ the following identity holds:

$$J_+(u(x) - v(y)) = J_+(u_M(x) - v_M(y)) + J_+(\phi_u^M(x) - \phi_v^M(y)) + F_M(\phi_u^M(x), v_M(y)), \tag{3.19}$$

where

$$F_M(\phi, v) := J_+(\phi + M - v) - J_+(\phi) - J_+(M - v).$$

We will show (3.19) by considering four different cases, depending on where (x, y) lies.

- *Case* $(x, y) \in \{u > M\} \times \{v > M\}$:
  In this case

  $$u_M(x) = M, \quad v_M(y) = M, \quad u(x) = M + \phi_u^M(x), \quad v(y) = M + \phi_v^M(y),$$

  and

  $$F_M(\phi_u^M(x), v_M(y)) = J_+(\phi_u^M(x) + M - M) - J_+(\phi_u^M(x)) - J_+(M - M) = 0,$$

  since, as showed in Step 1, we can assume without loss of generality that $J(0) = 0$. From this we get (3.19).
- *Case* $(x, y) \in \{u \le M\} \times \{v > M\}$:
  In this case $u(x) - v(y) \le 0$ and hence $J_+(u(x) - v(y)) = 0$. Similarly, as $v_M(y) = M$, we have $J_+(u_M(x) - v_M(y)) = J_+(u(x) - M) = 0$, and since $\phi_u^M(x) = 0 \le \phi_v^M(y)$ we also get that $J_+(\phi_u^M(x) - \phi_v^M(y)) = 0$. Finally, we obtain

  $$F_M(\phi_u^M(x), v_M(y)) = J_+(M - v_M(y)) - J_+(0) - J_+(M - v_M(y)) = 0.$$
- *Case* $(x, y) \in \{u > M\} \times \{v \le M\}$:
  In this case

  $$u(x) = M + \phi_u^M(x), \quad v(y) = v_M(y), \quad u_M(x) = M, \quad \phi_v^M(y) = 0.$$

Therefore,

$$\begin{aligned} J_+(u(x)-v(y)) &= J_+(M+\phi_u^M(x)-v_M(y)),\\ J_+(u_M(x)-v_M(y)) &= J_+(M-v_M(y)),\\ J_+(\phi_u^M(x)-\phi_v^M(y)) &= J_+(\phi_u^M(x)),\\ F_M(\phi_u^M(x),v_M(y)) &= J_+(\phi_u^M(x)+M-v_M(y))-J_+(\phi_u^M(x))-J_+(M-v_M(y)), \end{aligned}$$

which proves (3.19) in the present case.

- *Case* $(x,y)\in\{u\le M\}\times\{v\le M\}$:
  In this case

$$u(x)=u_M(x),\quad v(y)=v_M(y),\quad \phi_u^M(x)=0,\quad \phi_v^M(y)=0,$$

and

$$F_M(\phi_u^M(x),v_M(y))=J_+(M-v_M(y))-J_+(M-v_M(y))=0.$$

In conclusion, (3.19) holds for all $x,y\in(-\pi,\pi)$.

From (3.19) we obtain the following representation of $E_+[u,v]$:

$$E_+[u,v]=E_+[u_M,v_M]+E_+[\phi_u^M,\phi_v^M]+\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}dy\,F_M(\phi_u^M(x),v_M(y))g(x-y). \tag{3.20}$$

Note that the third term on the right hand side of (3.20) involving F_M is nonnegative. This follows from the definition of F_M and the fact that any convex function f with $f(0)=0$ (in our case $f:=J_+$) satisfies, for every $a,b\ge 0$ (in our case $a:=\phi_u^M(x)$ and $b:=M-v_M(y)$), that

$$f(a+b)-f(a)-f(b)=\int_a^{a+b}dt\,f'(t)-f(b)\ge\int_a^{a+b}dt\,f'(t-a)-f(b)=0.$$

Hence all three terms on the right side of (3.20) are nonnegative and finite.

We will show that the double integral involving F_M on the right hand side of (3.20) does not increase under the rearrangement of u and v. Once we have shown this, we can combine the inequalities $E_+[u_M,v_M]\ge E_+[(u^*)_M,(v^*)_M]$ and $E_+[\phi_u^M,\phi_v^M]\ge E_+[\phi_{u^*}^M,\phi_{v^*}^M]$ from Step 2.2 (recall that $(u_M)^*=(u^*)_M$ and observe that $(\phi_u^M)^*=\phi_{u^*}^M$) with (3.20) also for u^* and v^* to obtain, in view of the assumption $E_+[u,v]=E_+[u^*,v^*]$, that all inequalities must be equalities. In particular, we deduce that $E_+[u_M,v_M]=E_+[(u_M)^*,(v_M)^*]$, as desired.

Thus, it only remains to show that for any pair of functions $\phi\ge 0$ and $0\le v\le M$, it holds that

$$\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}dy\,F_M(\phi(x),v(y))g(x-y)\ge\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}dy\,F_M(\phi^*(x),v^*(y))g(x-y) \tag{3.21}$$

whenever the double integral on the left hand side is finite. To prove this inequality, let us first observe that the integral of the third term in the definition of F_M is finite, since J_+ is nondecreasing in $\mathbb{R}$ and, thus,

$$\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}dy\,J_+(M-v(y))g(x-y)\le 2\pi\|g\|_{L^1(-\pi,\pi)}J(M)<+\infty.$$

Therefore, we can split the left hand side of (3.21) as

$$\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}dy\,F_M(\phi(x),v(y))g(x-y)=\Sigma_M[\phi,v]-\int_{-\pi}^{\pi}dx\int_{-\pi}^{\pi}dy\,J_+(M-v(y))g(x-y),$$

where

$$\Sigma_M[\phi, v] := \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy \Big(J_+(\phi(x) + M - v(y)) - J_+(\phi(x)) \Big) g(x-y).$$

To prove (3.21), it is sufficient to show that Σ_M does not increase under rearrangement because, using (2.3) with $F(t) := J_+(M-t)$, we have

$$\int_{-\pi}^{\pi} dy\, J_+(M - v(y)) = \int_{-\pi}^{\pi} dy\, J_+(M - |v(y)|) = \int_{-\pi}^{\pi} dy\, J_+(M - v^*(y)).$$

At this point, we now proceed in a similar way as how we obtained (3.4). More precisely, we use the identity

$$\begin{aligned} J_+(\phi(x) + M - v(y)) - J_+(\phi(x)) &= \int_{\phi(x)}^{\phi(x)+M-v(y)} d\tau\, J_+'(\tau) = \int_0^{M-v(y)} d\tau\, J_+'(\phi(x)+\tau) \\ &= \int_0^M d\tau\, J_+'(\phi(x)+\tau)(1 - \chi_{\{v > M-\tau\}}(y)) \end{aligned}$$

to write

$$\Sigma_M[\phi, v] = \int_0^M d\tau \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J_+'(\phi(x)+\tau) g(x-y)(1 - \chi_{\{v>M-\tau\}}(y)).$$

Finally, that Σ_M does not increase under rearrangement follows from Theorem 3.1 and the monotone convergence theorem—exactly as in Step 2, where we have shown the same result for E_+ instead of Σ_M. In conclusion, (3.21) holds and Step 3.3 is now complete.

Step 4 (case of equality for $J(t) = |t|$):

Since $J_+'(u(x) - \tau) = 1$ if $u(x) > \tau$ and zero elsewhere, using (3.6) and (3.7) we see that $E[u, v] = E[u^*, v^*]$ if and only if, for almost every $\tau > 0$,

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, \chi_{\{u>\tau\}}(x) g(x-y) \chi_{\{v>\tau\}}(y) = \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, \chi_{\{u^*>\tau\}}(x) g(x-y) \chi_{\{v^*>\tau\}}(y).$$

From this it follows immediately that (1.9) yields $E[u, v] = E[u^*, v^*]$. So it remains to show the converse.

Note that neither of the two functions $\chi_{\{u>\tau\}}$ and $\chi_{\{v>\tau\}}$ is constant almost everywhere if and only if

$$\tau \in (\operatorname{ess\,inf} u, \operatorname{ess\,sup} u) \cap (\operatorname{ess\,inf} v, \operatorname{ess\,sup} v). \tag{3.22}$$

Thus, from the case of equality in Theorem 3.1 we obtain that, for almost every τ in the range given by (3.22) there exists $z_\tau \in \mathbb{R}$ such that

$$\chi_{\{u>\tau\}}(x) = \chi_{\{u^{*\,\mathrm{per}}>\tau\}}(x + z_\tau), \qquad \chi_{\{v>\tau\}}(x) = \chi_{\{v^{*\,\mathrm{per}}>\tau\}}(x + z_\tau) \quad \text{for a.e. } x \in \mathbb{R}. \tag{3.23}$$

In particular, for almost every τ as in (3.22), $\{u > \tau\}$ and $\{v > \tau\}$ are "2π-periodic unions of intervals". Then, thanks to Lemma 2.2 (*ii*) applied to each of these intervals, (3.23) actually holds for every τ as in (3.22).

Step 5 (case that $\inf J$ is not attained):

Throughout Step 1 to 4 we were assuming that $\inf J$ was attained, that is, that there exists $t_0 \in \mathbb{R}$ such that $J(t_0) \le J(t)$ for all $t \in \mathbb{R}$, and we proved Theorem 1.3 in this particular case. It remains to prove the theorem in the case when $\inf J$ is not attained. Since J is nonnegative by assumption, we have that $0 \le \inf J < +\infty$ and thus, without loss of generality, we can assume that $\inf J = 0$, arguing as in (3.2) in Step 1. Note also that a convex function in the real line which does not attain its infimum must be either increasing or decreasing (this can be seen arguing by contradiction). Therefore, we can also assume without loss of generality that J is increasing and, based on the previous observation, that $J(-\infty) := \lim_{t\downarrow -\infty} J(t) = 0$; in the case that J is decreasing simply consider $J(-t)$. Now the splitting $J = J_+ + J_-$ is not necessary (and not possible), but since $J(-\infty) = 0$ we have as in Step 2 that

$$\begin{aligned} J(u(x) - v(y)) &= \int_{-\infty}^{u(x)-v(y)} d\tau\, J'(\tau) = \int_{v(y)}^{+\infty} d\tau\, J'(u(x) - \tau) \\ &= \int_0^{+\infty} d\tau\, J'(u(x) - \tau) \left(1 - \chi_{\{v>\tau\}}(y)\right). \end{aligned}$$

Thus, we obtain the same expression for $E[u, v, g]$ as in (3.4), with the only difference that J_+ is replaced by J, but recall that now J' is nondecreasing (by the convexity of J) and nonnegative (since J is increasing) in $\mathbb{R}$, as it was J'_+ in Step 2. With this in hand, the inequality (1.8) in Theorem 1.3 can be proven exactly as in Step 2.

Next, we study the case of equality. First observe that in Step 3.2 we did not use the assumption that $\inf J$ is attained, nor the splitting $J = J_+ + J_-$. Hence we only have to deal with the case of equality in the case that u or v (or both) are unbounded. Let us first assume that only v might be unbounded, but that u is bounded. We will use the same abbreviations and argument as in Step 3.3. Let us write

$$J(u(x) - v(y)) = J(u(x) - v_M(y)) + J(u(x) - M - \phi_v^M(y)) - J(u(x) - M).$$

It is immediate to check this identity by distinguishing the two cases $v(y) \le M$ and $v(y) > M$. If we define this time

$$\Sigma_M[u, \phi] := \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J(u(x) - M - \phi(y)) g(x - y),$$

then we obtain

$$E[u, v] = E[u, v_M] + \Sigma_M[u, \phi_v^M] - \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J(u(x) - M) g(x - y).$$

The third integral is finite, because J is increasing and therefore $J(u(x)-M) \le J(\operatorname{ess\,sup} u - M)$ for almost every x. By assumption $E[u, v]$ is finite, and thus all three terms on the right hand side of the previous expression are finite too (because E and Σ_M are nonnegative). As in Step 3.3, using (2.3) we see that the third term involving $J(u(x) - M)$ on the right hand side is unaltered under the rearrangement of u. Moreover, $E[u, v_M]$ and $\Sigma_M[u, \phi_v^M]$ do not increase under the rearrangements of u, v_M, and ϕ_v^M, in view of the inequality of the theorem. Hence the equality $E[u, v] = E[u^*, v^*]$ forces the equality $E[u, v_M] = E[u^*, (v_M)^*]$ to hold. Now, u and v_M are both bounded and we can conclude as in Step 3.2, which, we recall, does not use the assumption that $\inf J = \min J$.

Let us finally assume that both u and v are unbounded, which is the last remaining case. We will repeat the previous argument, using that Theorem 1.3 holds true for general v, but u bounded. This time use the identity

$$J(u(x) - v(y)) = J(u_M(x) - v(y)) + J(\phi_u^M(x) + M - v(y)) - J(M - v(y)),$$

and set

$$\Sigma_M[\phi, v] := \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, J(\phi(x) + M - v(y)) g(x - y).$$

Note that $J(M - v(y)) \leq J(M) < +\infty$ because J is increasing. Thus, we can proceed exactly as in the previous case and show that the equality $E[u,v] = E[u^*,v^*]$ forces the equality $E[u_M,v] = E[(u_M)^*,v^*]$ to hold. Since we have already dealt with the case that one of the two functions is bounded, u_M in the present case, we are done letting $M \uparrow +\infty$ as in the beginig of Step 3.3.

□

4. Nonexpansivity under Schwarz rearrangement

We will now prove the nonexpansivity result under standard Schwarz symmetrization, namely Theorem 1.2. As already mentioned in the introduction, we will only detail those steps which differ from the proof of the periodic case given in Section 3. Let us use the abbreviations

$$\begin{aligned} \overline{E}[u,v,g] &:= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J(u(x) - v(y)) g(x-y), \\ \overline{E}[u,v] &:= \overline{E}[u,v,g] \quad \text{ if } g = g^{*n}, \end{aligned}$$

for two nonnegative measurable functions $u, v : \mathbb{R}^n \to \mathbb{R}$. Before giving the proof, we shall mention some elementary observations on the hypothesis of the theorem for the sake of completeness.

Note that the assumption $|\{u > \tau\}| < +\infty$ is necessary to give sense to the theorem (in contrary to the periodic case), otherwise the rearrangement is in general not defined.

Concerning the hypothesis of nonnegativeness of u and v, a similar comment as Remark 3.2 for the periodic version of the theorem is valid here. For instance if $J(t) = |t|^p$, $p \geq 1$, then the inequality remains true even for sign-changing functions u and v. But the hypothesis of nonnegativeness cannot be relaxed for the statement (i) of the theorem to hold true. To see this we give a different example than the one of Remark 3.2, which does not work in the present setting, since infinite integrals would appear if we take u or v to be a nonzero constant function. Instead, consider the strictly convex function J given by

$$J(t) = \begin{cases} t^2 & \text{if } t \leq 0, \\ \frac{t^2}{2} & \text{if } t > 0. \end{cases}$$

Let $v \equiv 0$ and u be some nonpositive function with compact support, not identical to zero. Then,

$$\int_{\mathbb{R}^n} J(u^{*n}) = \int_{\mathbb{R}^n} \frac{(u^{*n})^2}{2} = \int_{\mathbb{R}^n} \frac{|u|^2}{2} < \int_{\mathbb{R}^n} |u|^2 = \int_{\mathbb{R}^n} J(u),$$

which yields $\overline{E}[u^{*n}, v^{*n}] < \overline{E}[u,v]$. Thus, $\overline{E}$ decreases under the rearrangement although v is identically zero.

The proof of Theorem 1.2 will actually show that the conclusion of (ii) holds true if one only requires that $J_+ := \chi_{[0,+\infty)} J$ is strictly convex and the supremum of u is greater than or equal to the supremum of v (or, similarly, that $J_- := J - J_+$ is strictly convex and $\operatorname{ess\,sup} v \geq \operatorname{ess\,sup} u$).

Proof of Theorem 1.2. The proof follows the same outline as that of Theorem 1.3, hence we will only carry out the steps which are different. The decomposition $J = J_+ + J_-$ is identical as in the proof of Theorem 1.3, and also expressing $\overline{E}_+$ in terms of the derivative of J_+. Since $\overline{E}_+[u,v,g]$ is finite, Fubini's theorem gives

$$\overline{E}_+[u,v,g] = \int_0^{+\infty} d\tau\, (A(u,g,\tau) - B(u,v,g,\tau)),$$

where A and B are defined as in (3.6) with the only difference that the integrals are over $\mathbb{R}^n \times \mathbb{R}^n$ instead of $(-\pi,\pi)\times(-\pi,\pi)$, that is,

$$A(u,g,\tau) := \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J_+'(u(x)-\tau)g(x-y),$$
$$B(u,v,g,\tau) := \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J_+'(u(x)-\tau)g(x-y)\chi_{\{v>\tau\}}(y).$$

Step 1 (proof of the inequality):

Let us first assume that either u is bounded or that $|J'|$ is bounded —the latter occurring for instance if $J(t) = |t|$. Note that if $\operatorname{ess\,sup} u = M < +\infty$ then $A(u,g,\tau)$ is finite because $g \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} dx\, J_+'(u(x)-\tau) = \int_{\{u>\tau\}} dx\, J_+'(u(x)-\tau) \le |\{u>\tau\}| J_+'(M-\tau) < +\infty.$$

Similarly, if $|J'|$ is bounded one obtains as well that $A(u,g,\tau)$ is finite. As in the proof of Theorem 1.3, using Lemma 2.1, we get $A(u,g,\tau) = A(u^{*n}, g^{*n}, \tau)$. Since $A(u,g,\tau)$ is finite we get that $B(u,v,g,\tau)$ is finite too for almost every τ, and therefore

$$\overline{E}_+[u,v,g] = \overline{E}_+[u^{*n}, v^{*n}, g^{*n}] + \int_0^{+\infty} d\tau\, (B(u^{*n}, v^{*n}, g^{*n}, \tau) - B(u,v,g,\tau)),$$

where $B(u^{*n}, v^{*n}, g^{*n}, \tau)$ and $B(u,v,g,\tau)$ are finite for almost every $\tau > 0$. Hence it is sufficient to show that $B(u,v,g,\tau) \le B(u^{*n}, v^{*n}, g^{*n}, \tau)$. This last inequality follows from the classical Riesz rearrangement inequality (1.6); see for instance [21, Theorem 3.7]. This proves (1.7) in the case that u or $|J'|$ is bounded.

If neither u nor $|J'|$ is bounded, then one can proceed exactly as in Step 2.2 of the proof of Theorem 1.3 by defining u_M as in (3.10) and using the monotone convergence theorem.

Step 2 (case of equality for strictly convex J under a boundedness assumption):

From now on, we will omit in all the functionals $\overline{E}, A, B, \dots$ the argument $g = g^{*n}$, as it is unaltered under rearrangement.

If u is constant, then by the hypothesis $|\{u > \tau\}| < +\infty$ for every $\tau > 0$ we must have that $u \equiv 0$. In this case there is equality in (1.7) for any v (whether J is strictly convex or not). Since the same argument applies if v is constant, from now on we can assume that neither u nor v is constant.

In this step we will study the case of equality under the assumption that either u and v are both bounded or that $|J'|$ is bounded. By Step 1 it holds that

$$\overline{E}_+[u,v] = \overline{E}_+[u^{*n}, v^{*n}] + \int_0^{+\infty} d\tau\, (B(u^{*n}, v^{*n}, \tau) - B(u,v,\tau)) \tag{4.1}$$

and, therefore, that $B(u,v,\tau) = B(u^{*n}, v^{*n}, \tau)$ for almost every $\tau > 0$. Note that, from the assumption that $|\{u > \tau\}|$ and $|\{v > \tau\}|$ are finite for all $\tau > 0$, we see that $\operatorname{ess\,inf} u =$

$\operatorname{ess\,inf} v = 0$. Let us now assume $\operatorname{ess\,sup} u \geq \operatorname{ess\,sup} v$ (for the other case, in all of the following argument interchange the roles of $J_+(t)$ and $J_-(-t)$). Hence we have

$$\mathcal{R}(v) := (\operatorname{ess\,inf} v, \operatorname{ess\,sup} v) \subset (\operatorname{ess\,inf} u, \operatorname{ess\,sup} u) =: \mathcal{R}(u).$$

We now apply the strict Riesz rearrangement inequality to the second term on the right hand side of (4.1); see Baernstein [3, Theorem 2.15 (b)] or Lieb and Loss [21, Theorem 3.9].[5] Observe first that if $\tau \in \mathcal{R}(v)$ then neither $J'_+(u-\tau)$ nor $\chi_{\{v>\tau\}}$ is constant (in particular, they are not identical to zero), and we therefore obtain from $B(u, v, \tau) = B(u^{*n}, v^{*n}, \tau)$ that there exists $a_\tau \in \mathbb{R}^n$ depending a priori on τ such that

$$J'_+(u(x)-\tau) = J'_+(u^{*n}(x-a_\tau)-\tau) \;\text{ and }\; \chi_{\{v>\tau\}}(y) = \chi_{\{v^{*n}>\tau\}}(y-a_\tau) \tag{4.2}$$

for a.e. $x \in \mathbb{R}^n$ and for a.e. $y \in \mathbb{R}^n$. Since J_+ is strictly convex in $[0, +\infty)$, we have that J'_+ is increasing and, therefore, one-to-one in $[0, +\infty)$. Using this together with the fact that J'_+ is identical to zero in $(-\infty, 0)$, we deduce that, for almost every $\tau \in \mathcal{R}(v)$,

$$\{u > \tau\} = \{u^{*n} > \tau\} + a_\tau \quad \text{in measure} \tag{4.3}$$

and

$$u(x) = u^{*n}(x - a_\tau) \quad \text{for a.e. } x \in \{u^{*n} > \tau\} + a_\tau. \tag{4.4}$$

We will now show that a_τ is indeed independent of τ and thus, taking $\tau \downarrow 0$ in (4.4), that u is a translate of a symmetric decreasing function.[6]

Suppose that $s, \tau \in \mathcal{R}(v)$ are such that $s < \tau$. We claim that (4.3) and (4.4) applied to s and τ gives

$$\{u > \tau\} = \{u^{*n} > \tau\} + a_s \quad \text{in measure.} \tag{4.5}$$

Let us first show that, up to a set of measure zero, the set on the left hand side is included in the set on the right hand side. If $x \in \{u > \tau\} \subset \{u > s\}$ then by (4.3) $x \in \{u^{*n} > s\} + a_s$. We now use (4.4) with s and obtain that $\tau < u(x) = u^{*n}(x - a_s)$, and thus $x \in \{u^{*n} > \tau\} + a_s$. This proves that

$$\{u > \tau\} \subset \{u^{*n} > \tau\} + a_s\,.$$

Now the two sets have the same measure, and thus they have to be equal in measure, which proves (4.5). Since the sets $\{u^{*n} > \tau\}$ are nonempty bounded balls centered at the origin for every $\tau \in \mathcal{R}(v) \subset \mathcal{R}(u)$, the identities (4.3) and (4.5) force $a_\tau = a_s$. This shows that $a_\tau \equiv a$ is independent of τ, which gives, by letting $\tau \downarrow 0 = \operatorname{ess\,inf} u$ and using Lemma 2.2 (i), that

$$u(x) = u^{*n}(x - a) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Finally, we now use the second identity in (4.2) to obtain that

$$\chi_{\{v>\tau\}}(y) = \chi_{\{v^{*n}>\tau\}}(y - a) \quad \text{for a.e. } y \in \mathbb{R}^n.$$

Since this holds for almost every $\tau \in \mathcal{R}(v)$, we obtain from Lemma 2.2 (i) that $v(y) = v^{*n}(y-a)$ for a.e. $y \in \mathbb{R}^n$.

[5][21, Theorem 3.9] should include two hypotheses which are needed for its validity —and which are explicitly stated in most of the results of [21, Chapter 3]. Namely, that the functions f and h of its statement must vanish at infinity and that neither f nor h is identically zero. Indeed, there is always equality in the Riesz rearrangement inequality if one of these two functions is constant.

[6]Let us point out the difference with Step 3.2 of the proof of the corresponding theorem in the periodic case. There we had in (3.15) that $u_S(s) - \tau > 0$ for all τ in the given range and (4.4) followed for all x. This trick of "shifting" u by defining I_S does not work in the case where $(-\pi, \pi)$ is replaced by $\mathbb{R}^n$, because some integrals might not be finite now. However, this was not the only reason why we introduced I_S: it was also due to the fact that in the periodic setting we may not have $\operatorname{ess\,inf} u = \operatorname{ess\,inf} v = 0$, which causes some technical difficulties.

Step 3 (case of equality for $J(t) = |t|$):

The proof of this case is exactly the same as that of the corresponding result in Theorem 1.3. Observe that since $|\{u > \tau\}|$ and $|\{v > \tau\}|$ are finite for all $\tau > 0$ by assumption, we have $\operatorname{ess\,inf} u = \operatorname{ess\,inf} v = 0$.

Step 4 (case of equality for unbounded u or v, and unbounded $|J'|$)[7]

In Step 2 we studied the case of equality under the assumption that either u and v are both bounded or that $|J'|$ is bounded. The purpose of this step is study the case of equality without this assumption.

Let t_i be a sequence of positive real numbers such that $\lim_{i\uparrow+\infty} t_i = +\infty$ and J_+ is differentiable at t_i (the use of points of differentiability is not really necessary to carry out this step but it makes the proof less heavy). Define $J_{+,i}$ as the convex function in $\mathbb{R}$ which coincides with J_+ in $(-\infty, t_i]$ and is a linear function with slope $J_+'(t_i)$ in $(t_i, +\infty)$, namely

$$J_{+,i}(t) := J_+(t) \quad \text{if } t \le t_i\,, \qquad J_{+,i}(t) := J_+(t_i) + (t - t_i)J_+'(t_i) \quad \text{if } t > t_i\,.$$

This gives an increasing sequence $J_{+,i}(t) \le J_{+,i+1}(t) \le J_+(t)$ wich converges to $J_+(t)$ for every $t \in \mathbb{R}$ as $i \uparrow +\infty$. A analogous cutt-off $J_{-,i}$ can be done for J_-, choosing some sequence of negative real numbers $s_i \downarrow -\infty$ and replacing J_- in $(-\infty, s_i)$ by a suitable linear function. Let us define

$$\overline{E}^i_\pm[u,v] := \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J_{\pm,i}(u(x) - v(y))g(x-y).$$

We claim that if $\overline{E}$ is unaltered under the rearrangement of u and v, then for every i we have

$$\overline{E}^i_+[u,v] = \overline{E}^i_+[u^{*n}, v^{*n}] \quad \text{ and } \quad \overline{E}^i_-[u,v] = \overline{E}^i_-[u^{*n}, v^{*n}]. \tag{4.6}$$

First observe that the equality $\overline{E}[u,v] = \overline{E}[u^{*n}, v^{*n}]$ yields the validity if (4.6) without the superscript i because $\overline{E}_\pm$ do not increase under Schwarz rearrangement. Hence, we only need to show that from the equality $\overline{E}_+[u,v] = \overline{E}_+[u^{*n}, v^{*n}]$ it follows the first equality of (4.6), since the corresponding second identity of (4.6) would then follow by considering $J(-t)$.

Let us decompose $\overline{E}_+ = \overline{E}^i_+ + \overline{\Sigma}_i$, where

$$J_+ = J_{+,i} + \Phi_i \quad \text{ and } \quad \overline{\Sigma}_i[u,v] := \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, \Phi_i(u(x) - v(y))g(x-y).$$

Observe that Φ is a convex function, because it is identically zero in $(-\infty, t_i]$ and in $[t_i, +\infty)$ it is the (nonnegative) difference of a convex function and a linear function. Hence, by Step 1, $\overline{\Sigma}_i$ does not increase under rearrangement. Now, by assumption we have that

$$\overline{E}^i_+[u,v] + \overline{\Sigma}_i[u,v] = \overline{E}^i_+[u^{*n}, v^{*n}] + \overline{\Sigma}_i[u^{*n}, v^{*n}],$$

where all terms are nonnegative. Since both $\overline{E}^i_+$ and $\overline{\Sigma}_i$ do not increase under rearrangement, we obtain (4.6) for $\overline{E}^i_+$, as desired.

Since $|J'_{\pm,i}|$ are bounded we can now apply the argument of Step 2 to the identities of (4.6). The first identity of (4.6) gives for instance that, for a.e. $\tau \ge 0$,

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J'_{+,i}(u(x)-\tau)g(x-y)\chi_{\{v>\tau\}}(y) = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} dy\, J'_{+,i}(u^{*n}(x)-\tau)g(x-y)\chi_{\{v^{*n}>\tau\}}(y).$$

[7]We believe this proof is a simplification of the proof given in [15, Lemma A.2] in the special case $u = v$, where the second derivative of J was used to carry out this step.

Let us assume as in Step 2 that the supremum of u is greater than or equal to the supremum of v, thus $\operatorname{ess\,sup} u = +\infty$ because u or v is unbounded by assumption. Observe that $J_{+,i}$ is strictly convex only in $(0, t_i)$. Therefore the arguments of Step 2 have to be actually slightly modified (this is the only difference to Step 2). If we define $\mathcal{R}(v) := (0, \operatorname{ess\,sup} v)$, then the analogue of (4.2) holds for almost every $\tau \in \mathcal{R}(v)$ —since $\tau < \operatorname{ess\,sup} v$ we have that $\chi_{\{v>\tau\}}$ is not indentical to zero, and since $\operatorname{ess\,inf} u = 0$, $\operatorname{ess\,sup} u = +\infty$, and $J'_{+,i}(t) = J'_+(t_i) > 0$ for all $t > t_i$ we have that $J'_{+,i}(u - \tau)$ is not indentical to zero— and rewrites as

$$J'_{+,i}(u(x) - \tau) = J'_{+,i}(u^{*n}(x - a_{\tau,i}) - \tau) \text{ and } \chi_{\{v>\tau\}}(y) = \chi_{\{v^{*n}>\tau\}}(y - a_{\tau,i}).$$

From the first identity one deduces (since $J'_{+,i}$ is one-to-one in $(0, t_i)$ and if $u < t_i$ then $u - \tau < t_i$) that, as in (4.3),

$$\{\tau < u < t_i\} = \{\tau < u^{*n} < t_i\} + a_{\tau,i} \quad \text{in measure,}$$

and

$$u(x) = u^{*n}(x - a_{\tau,i}) \quad \text{for a.e. } x \in \{\tau < u^{*n} < t_i\} + a_{\tau,i}, \tag{4.7}$$

for a.e. $\tau \in \mathcal{R}(v)$. Arguing as in (4.5) one obtains that if $0 < s \le \tau$ then

$$\{\tau < u^{*n} < t_i\} + a_{\tau,i} = \{\tau < u^{*n} < t_i\} + a_{s,i}.$$

Since $\{\tau < u^{*n} < t_i\}$ is either a bounded annulus, or the empty set (if $t_i \le \tau$ we have an empty set, and if $t_i > \tau$ we have an annulus since $\operatorname{ess\,sup} u = +\infty$ and $\{u > \tau\}$ has finite measure for all $\tau > 0$) we obtain that $a_{\tau,i} \equiv a_i$ is independent of τ. Letting $\tau \downarrow 0$ in (4.7) gives

$$u(x) = u^{*n}(x - a_i) \quad \text{for a.e. } x \in \{u < t_i\}.$$

One easily sees that $a_i \equiv a$ cannot depend on i, and by letting $t_i \uparrow +\infty$ we obtain that $u = u^{*n}(x - a)$.

Exactly as in Step 2, we also obtain that $v(y) = v^{*n}(y - a)$. □

5. Rearrangements and periodic Gagliardo seminorms

Let us define, for $1 \le p < +\infty$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ which is 2π-periodic in the variable x_1, the following periodic Gagliardo seminorm

$$[u]^{\text{per}}_{W^{s,p}} := \left(\int_{\{x \in \mathbb{R}^n :\, -\pi < x_1 < \pi\}} dx \int_{\mathbb{R}^n} dy \, \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \right)^{1/p}.$$

Indeed, if a set $E \subset \mathbb{R}^n$ is 2π-periodic in the variable x_1, then taking u equal to the characteristic function χ_E gives that $[\chi_E]^{\text{per}}_{W^{s,1}} = 2\mathcal{P}_s(E)$, where

$$\mathcal{P}_s[E] := \int_{E \cap \{-\pi < x_1 < \pi\}} dx \int_{\mathbb{R}^n \setminus E} dy \, \frac{1}{|x - y|^{n+s}}$$

is the periodic fractional perimeter functional introduced in [13]; here we have also used that $\mathcal{P}_s(\mathbb{R}^n \setminus E) = \mathcal{P}_s(E)$ by the periodicity of E. More generally, one can show that for every function u which is 2π-periodic in the variable x_1 it holds that

$$[u]^{\text{per}}_{W^{s,1}} = 2 \int_{-\infty}^{+\infty} dt \, \mathcal{P}_s(\{u > t\}). \tag{5.1}$$

We call this identity the periodic fractional coarea formula, which follows from the identity

$$|u(x) - u(y)| = \int_{-\infty}^{+\infty} dt \, \big(\chi_{\{u>t\}}(x) \chi_{\{u \le t\}}(y) + \chi_{\{u \le t\}}(x) \chi_{\{u>t\}}(y) \big)$$

and Fubini's theorem.

Our first result of this section deals with the behavior of the periodic Gagliardo seminorm under periodic rearrangement. Recall that $u^{*\,\mathrm{per}}(x_1,x') = (u(\cdot,x'))^{*\,\mathrm{per}}(x_1)$ where, for a periodic function f in $\mathbb{R}$, $f^{*\,\mathrm{per}}$ is simply the standard Schwarz (or Steiner) symmetrization of f resticted to one period centered at 0 and then extended in a periodic way.

Theorem 5.1. *Let $n \geq 1$ be an integer, $0 < s < 1$, $1 \leq p < +\infty$, and $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function that is 2π-periodic in the variable x_1 and with finite seminorm $[u]^{\mathrm{per}}_{W^{s,p}}$. Then,*

$$[u^{*\,\mathrm{per}}]^{\mathrm{per}}_{W^{s,p}} \leq [u]^{\mathrm{per}}_{W^{s,p}}\,. \tag{5.2}$$

Moreover, the following holds:

(*i*) *If $p > 1$, or $p = 1$ and u is the characteristic function of some measurable set E, then there is equality in* (5.2) *if and only if $u = \pm|u|$ and, modulo translations, $|u|$ is nonincreasing in $(0,\pi)$ in the variable x_1 and even with respect to $\{x_1 = 0\}$. More precisely, there is equality in* (5.2) *if and only if*

$$u(x_1,x') = \pm u^{*\,\mathrm{per}}(x_1 + a, x') \quad \text{for a.e. } (x_1,x') \in \mathbb{R}\times\mathbb{R}^{n-1},$$

for some $a \in \mathbb{R}$.

(*ii*) *If $p = 1$ and u is continuous then there is equality in* (5.2) *if and only if $u = \pm|u|$ and for almost every $\tau \in \mathbb{R}$ there exists $a_\tau \in \mathbb{R}$ such that the superlevel sets satisfy*

$$\{|u| > \tau\} = \{|u| > \tau\}^{*\,\mathrm{per}} + a_\tau e_1,$$

up to redefining u on a set of vanishing n-dimensional Lebesgue measure. Here $e_1 := (1,0,\ldots,0)$ denotes the first coordinate direction of $\mathbb{R}^n$.

One of the statements of (*ii*), namely the "if", does not require that u is continuous. Any function which is of the form described in (*ii*), whether continuous or not, preserves the periodic Gagliardo seminorm $[\,\cdot\,]^{\mathrm{per}}_{W^{s,1}}$ under periodic rearrangement. This follows from (*i*) and the periodic fractional coarea formula (5.1). Note that if u is as in (*ii*), this does not force u to be (modulo translations) symmetric and nonincreasing in the variable x_1 in $(-\pi,\pi)\times\mathbb{R}^{n-1}$, even if all its superlevel sets are. Consider for example the function $u : (-2,2) \to \mathbb{R}$, given by $u(x) = \chi_{(-1,2)}(x) + \chi_{(-1,0)}(x)$, and extended in a 4-periodic way to $\mathbb{R}$. Then $u^{*\,\mathrm{per}}$ is, up to a translation, the 4-periodic extension of $\chi_{(-1,2)} + \chi_{(0,1)}$. It is easy to check that for $p = 1$ the periodic Gagliardo seminorms of u and $u^{*\,\mathrm{per}}$ are equal. On the other hand, we were not able to establish whether the continuity hypothesis on u is necessary or not in the "only if" statement of (*ii*).

Our second result of this section concerns the behavior of the periodic Gagliardo seminorm under a cylindrical rearrangement, more precisely under the spherical rearrangement in $\{x_1\}\times\mathbb{R}^{n-1}$ for every frozen $x_1 \in \mathbb{R}$. We recall here its definition (given in Section 2), which is

$$u^{*\,n,1}(x_1,x') := u(x_1,\cdot)^{*\,n-1}(x'),$$

where $v^{*\,n-1}$ denotes the usual Schwarz symmetrization (spherical decreasing rearrangement) of $v : \mathbb{R}^{n-1} \to \mathbb{R}$. In other words, $u^{*\,n,1}$ is cylindrically symmetric around the x_1-axis and nonincreasing in $|x'|$. As stated in the following theorem, this rearrangement also decreases the periodic Gagliardo seminorm, although this has nothing to do with periodicity.

Theorem 5.2. *Let $n \geq 2$ be an integer, $0 < s < 1$, $1 \leq p < +\infty$, and $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function that is 2π-periodic in the variable x_1 and with finite seminorm $[u]^{\mathrm{per}}_{W^{s,p}}$. Then,*

$$\big[u^{*\,n,1}\big]^{\mathrm{per}}_{W^{s,p}} \leq [u]^{\mathrm{per}}_{W^{s,p}}. \tag{5.3}$$

Moreover, the following holds:

(*i*) *If $p > 1$ then the equality in (5.3) holds if and only if $u = \pm|u|$ and there exists $a \in \mathbb{R}^{n-1}$ such that $u(x_1, x') = \pm u^{*\,n,1}(x_1, x' + a)$ for a.e. $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$.*

(*ii*) *If $p = 1$ and u is continuous then the equality in (5.3) holds if and only if $u = \pm|u|$ and for almost every $\tau > 0$ there exists $a_\tau \in \mathbb{R}^{n-1}$ such that the superlevel sets satisfy*

$$\{|u| > \tau\} = \{|u| > \tau\}^{*\,n,1} + (0, a_\tau),$$

up to redefining u on a set of vanishing n-dimensional Lebesgue measure.

Let us start with the proof of Theorem 5.1, stating that the periodic Gagliardo seminorm decreases under periodic rearrangement. For this purpose it is convenient to write the periodic Gagliardo seminorm in a different form using the Laplace transform. We shall denote the Gamma function by $\Gamma(\lambda) := \int_0^{+\infty} dt\, t^{\lambda-1} e^{-t}$ for $\lambda > 0$. By the change of variables $t \mapsto zt$ we obtain the Laplace transform of the function $t \mapsto t^{\lambda-1}/\Gamma(\lambda)$, which is

$$z^{-\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} dt\, t^{\lambda-1} e^{-zt} \quad \text{ for } z > 0.$$

Using this identity with

$$\lambda = \frac{n + sp}{2} \quad \text{ and } \quad z = |x' - y'|^2 + (x_1 - y_1 + 2k\pi)^2,$$

combined with the identity

$$\big([u]^{\mathrm{per}}_{W^{s,p}}\big)^p = \int_{\mathbb{R}^{n-1}} dx' \int_{-\pi}^{\pi} dx_1 \int_{\mathbb{R}^{n-1}} dy' \int_{-\pi}^{\pi} dy_1 \sum_{k\in\mathbb{Z}} \frac{|u(x) - u(y)|^p}{|x - y + 2k\pi e_1|^{n+sp}}$$

for functions u which are 2π-periodic in the variable x_1, gives the following representation of the periodic Gagliardo seminorm:

$$\big([u]^{\mathrm{per}}_{W^{s,p}}\big)^p = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} dt\, t^{\lambda-1} \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^{n-1}} dy'\, \Phi(u, x', y', t) e^{-|x'-y'|^2 t}, \tag{5.4}$$

where

$$\Phi(u, x', y', t) := \int_{-\pi}^{\pi} dx_1 \int_{-\pi}^{\pi} dy_1\, |u(x_1, x') - u(y_1, y')|^p \sum_{k\in\mathbb{Z}} e^{-(x_1 - y_1 + 2k\pi)^2 t}.$$

In view of representation (5.4) it will be sufficient to show that the functional Φ does not increase if we replace u by $u^{*\,\mathrm{per}}$, for every $x', y' \in \mathbb{R}^{n-1}$ and every $t > 0$. Although this rearrangement inequality for Φ is now a 1-dimensional problem, its proof relies on two nontrivial ingredients. The first one is Theorem 1.3 which we will apply, for every given $t > 0$, to the kernel

$$z \mapsto g(z, t) := \sum_{k\in\mathbb{Z}} e^{-(z+2k\pi)^2 t}. \tag{5.5}$$

This will show that Φ does not increase under periodic rearrangement. Regarding the assumptions in Theorem 1.3, it is obvious that g is 2π-periodic and even with respect to the variable z. The second nontrivial ingredient is the other hypothesis that g must satisfy, to be decreasing in $(0, \pi)$ with respect to the variable z. This property follows from the monotonicity of the heat kernel on the circle, which in turns follows from a maximum principle on its spatial derivative based on the periodicity of the circle; see for instance [10, Appendix B].

Remark 5.3. In [11, Theorem 1.4] we considered, for $n = 1$ and $p = 2$, a more general periodic Gagliardo seminorm where $|x-y|^{-(1+2s)}$ in (1.1) is replaced by a more general kernel $K(x-y)$. However, the assumptions on K are essentially such that the Riesz rearrangement inequality on the circle, i.e., Theorem 3.1, can be directly applied. For instance, a convexity assumption on K gives that $g(t) = \sum_{k\in\mathbb{Z}} K(|t + 2k\pi|)$ satisfies the assumption in Theorem 3.1 on g being symmetric and decreasing in $(0, \pi)$, and hence $g = g^{*\,\mathrm{per}}$. The second possible assumption on K is such that after a Laplace transform one obtains a g as in (5.5) but with an additional nonnegative factor that only depends on t. Obviously such kind of generalizations are also possible in the n-dimensional case.

Proof of Theorem 5.1. The proof will be divided into several steps.

<u>*Step 1* (proof of the inequality):</u>

For the proof of (5.2), we can assume that $u \geq 0$ due to the inequality $|u(x) - u(y)| \geq ||u(x)| - |u(y)||$ and the fact that $u^{*\,\mathrm{per}} = |u|^{*\,\mathrm{per}}$, since $u^{*\,\mathrm{per}}$ is defined by rearranging the superlevel sets of $|u|$.

In view of (5.4), it is sufficient to show that Φ does not increase under periodic rearrangement, for every x', $y' \in \mathbb{R}^{n-1}$ and every $t > 0$. If g is defined as in (5.5), then we obtain from Theorem 1.3 —recall that $g(\cdot, t) = g(\cdot, t)^{*\,\mathrm{per}}$ is decreasing in $(0, \pi)$; see [10, Appendix B]— that

$$\begin{aligned}\Phi(u, x', y', t) &= \int_{-\pi}^{\pi} dx_1 \int_{-\pi}^{\pi} dy_1\, |u(x_1, x') - u(y_1, y')|^p g(x_1 - y_1, t)\\ &\geq \int_{-\pi}^{\pi} dx_1 \int_{-\pi}^{\pi} dy_1\, |u(\cdot, x')^*(x_1) - u(\cdot, y')^*(y_1)|^p g(x_1 - y_1, t),\end{aligned}$$

where here $u(\cdot, x')^*$ denotes the Steiner symmetrization of $u(\cdot, x')\chi_{(-\pi,\pi)}$. Now, using that for every $x' \in \mathbb{R}^{n-1}$ we have $u(\cdot, x')^* = u(\cdot, x')^{*\,\mathrm{per}}$ in $(-\pi, \pi)$, and the identity (2.5), we obtain

$$\Phi(u, x', y', t) \geq \Phi(u^{*\,\mathrm{per}}, x', y', t), \tag{5.6}$$

which concludes the proof of the inequality.

<u>*Step 2* (case of equality for $p > 1$):</u>

Firstly, observe that in case of equality u cannot change sign, due to the following two facts: $|u|^{*\,\mathrm{per}} = u^{*\,\mathrm{per}}$ and $|u(x) - u(y)| > ||u(x)| - |u(y)||$ on a set of positive measure in $\mathbb{R}^n \times \mathbb{R}^n$ if u changes sign. Hence $u = \pm|u|$. We assume from now on that $u \geq 0$.

In case of equality we must have that at least for one $t > 0$ there is equality in (5.6) —actually there must be equality for almost every $t > 0$, but one t is enough to conclude. Let us say for simplicity that $t = 1$ and abbreviate $g(z) \equiv g(z, 1)$. Thus, we must have that

$$\begin{aligned}&\int_{-\pi}^{\pi} dx_1 \int_{-\pi}^{\pi} dy_1\, |u(x_1, x') - u(y_1, y')|^p g(x_1 - y_1)\\ &\qquad = \int_{-\pi}^{\pi} dx_1 \int_{-\pi}^{\pi} dy_1\, |u^{*\,\mathrm{per}}(x_1, x') - u^{*\,\mathrm{per}}(y_1, y')|^p g(x_1 - y_1)\end{aligned} \tag{5.7}$$

for all $x' \in \mathbb{R}^{n-1} \setminus N$ and all $y' \in \mathbb{R}^{n-1} \setminus N(x')$, where $\mathcal{L}^{n-1}(N) = \mathcal{L}^{n-1}(N(x')) = 0$, and where g satisfies the hypothesis of Theorem 1.3 and, in addition, g is decreasing in $(0, \pi)$. Let us denote

$$\Lambda(u) := \left\{x' \in \mathbb{R}^{n-1} : u(\cdot, x') = c(x') \in \mathbb{R} \text{ is constant in the } x_1 \text{ variable}\right\}.$$

Then, by Theorem 1.3 and (2.5), if (x',y') is a pair satisfying the equality (5.7) and $x',y'\notin\Lambda(u)$, there exists $a(x',y')\in\mathbb{R}$ such that

$$\begin{aligned} u(x_1,x') &= u^{*\,\mathrm{per}}(x_1+a(x',y'),x') \quad \text{ for a.e. } x_1\in\mathbb{R},\\ u(y_1,y') &= u^{*\,\mathrm{per}}(y_1+a(x',y'),y') \quad \text{ for a.e. } y_1\in\mathbb{R}. \end{aligned}$$

From this we will now deduce, since $x'\notin\Lambda(u)$, that $a(x',y')$ is independent of x' and y'. To see this, one can argue as follows. First of all, if $x'\notin\Lambda(u)$ then $u^{*\,\mathrm{per}}(\cdot,x')$ is also nonconstant in x_1 and, therefore, $u^{*\,\mathrm{per}}(x_1+b,x')=u^{*\,\mathrm{per}}(x_1+d,x')$ forces that $b=d$ modulo 2π; see the lines immediately after (3.17) for a similar argument. Next, take another $z'\in\mathbb{R}^{n-1}\setminus\Lambda(u)$ different from y' for which it also holds that $u(x_1,x')=u^{*\,\mathrm{per}}(x_1+a(x',z'),x')$ for almost every $x_1\in\mathbb{R}$. Then, $a(x',y')=a(x',z')$ modulo 2π and, therefore, a is independent of the second variable. By symmetry, a is also independent of the first variable. Thus, we have shown that $u(x_1,x')=u^{*\,\mathrm{per}}(x_1+a,x')$ for almost every $x'\in\mathbb{R}^{n-1}\setminus\Lambda(u)$. But this also holds trivially for $x'\in\Lambda(u)$, and we obtain

$$u(x_1,x')=u^{*\,\mathrm{per}}(x_1+a,x') \quad \text{ for a.e. } x'\in\mathbb{R}^{n-1} \text{ and a.e. } x_1\in\mathbb{R}.$$

Step 3 (case of equality for $p=1$):

Let us first address the case (ii); the statement (i) of Theorem 5.1 which deals with $p=1$ will be commented afterwards.

In case of equality, at the beginning of Step 2 we have seen that (5.7) holds for all $x'\in\mathbb{R}^{n-1}\setminus N$ and all $y'\in\mathbb{R}^{n-1}\setminus N(x')$, where $\mathcal{L}^{n-1}(N)=\mathcal{L}^{n-1}(N(x'))=0$. However, since u is continuous, it is uniformly continuous on compact sets. The same holds true for $u^{*\,\mathrm{per}}$, since Steiner symmetrization does not increase the modulus of continuity of a function; see [6, Theorem 3.3, Section 4, and Remark 6.1]. From this and a limiting argument it follows that if u is continuous then (5.7) with $p=1$ actually holds for *all* $x',y'\in\mathbb{R}^{n-1}$, and not only up to sets of measure zero.

As for $p>1$, we can assume that $u\geq 0$. Then, we obtain from (5.7) and Theorem 1.3 that for every

$$\tau\in(\operatorname{ess\,inf} u(\cdot,x'),\operatorname{ess\,sup} u(\cdot,x'))\cap(\operatorname{ess\,inf} u(\cdot,y'),\operatorname{ess\,sup} u(\cdot,y'))=:\mathcal{R}(x',y') \tag{5.8}$$

there exists $a_\tau(x',y')\in\mathbb{R}$ such that

$$\begin{aligned} \{u(\cdot,x')>\tau\} &= \{u^{*\,\mathrm{per}}(\cdot,x')>\tau\}+a_\tau(x',y'),\\ \{u(\cdot,y')>\tau\} &= \{u^{*\,\mathrm{per}}(\cdot,y')>\tau\}+a_\tau(x',y') \end{aligned} \tag{5.9}$$

in measure. Consider for a moment only the first equation in (5.9). Note that a_τ cannot depend on y' because if $\tau\in(\operatorname{ess\,inf} u(\cdot,x'),\operatorname{ess\,sup} u(\cdot,x'))$ then $\{u^{*\,\mathrm{per}}(\cdot,x')>\tau\}$ is a nontrivial "periodic interval" —and hence all its translations which are not multiples of 2π are different. Thus, we obtain

$$\{u(\cdot,x')>\tau\}=\{u^{*\,\mathrm{per}}(\cdot,x')>\tau\}+a_\tau(x') \quad \text{ for all } \tau\in\mathcal{R}(x',y')$$

(this equality holds only in measure). In particular, and since (5.7) with $p=1$ holds for all $x',y'\in\mathbb{R}^{n-1}$, this last identity holds taking $x'=y'$, that is, for all $\tau\in\mathcal{R}(x',x')$ —this is the key point where the continuity assumption on u is used. Thus,

$$\{u(\cdot,x')>\tau\}=\{u^{*\,\mathrm{per}}(\cdot,x')>\tau\}+a_\tau(x') \quad \text{ for all } \tau>0. \tag{5.10}$$

Next, we return to (5.9), which gives that

$$\{u(\cdot, y') > \tau\} = \{u^{*\,\mathrm{per}}(\cdot, y') > \tau\} + a_\tau(x') \quad \text{for all } \tau \in \mathcal{R}(x', y').$$

Thus a_τ does not depend on x'. Now the claim follows immediately from the fact that $\{u > \tau\}$ is the union over all $x' \in \mathbb{R}^{n-1}$ of the sets $\{u(\cdot, x') > \tau\} \times \{x'\}$ (and from Fubini's theorem to deal with measure zero sets). In more detail, let $N(x') \subset \mathbb{R}$ be a set of zero $\mathcal{L}^1$-measure modulo which (5.10) holds. Then, the set

$$M = \bigcup_{x' \in \mathbb{R}^{n-1}} N(x') \times \{x'\}$$

has $\mathcal{L}^n$-measure zero by Fubini's theorem. Now by (5.10)

$$\chi_{\{u(\cdot,x')>\tau\}}(x_1) = \chi_{\{u^{*\,\mathrm{per}}(\cdot,x')>\tau\}}(x_1 - a_\tau) \quad \text{for all } x_1 \in \mathbb{R} \setminus N(x').$$

Hence this last identity holds for all $x \in \mathbb{R}^n \setminus M$, which yields

$$\chi_{\{u>\tau\}}(x) = \chi_{\{u^{*\,\mathrm{per}}>\tau\}}(x_1 - a_\tau, x')$$

for all $x \in \mathbb{R}^n \setminus M$.

Finally, let us address the statement (i) of Theorem 5.1 which deals with $p = 1$. Recall that in case of equality, at the beginning of Step 2 we have seen that (5.7) holds for all $x' \in \mathbb{R}^{n-1} \setminus N$ and all $y' \in \mathbb{R}^{n-1} \setminus N(x')$, where $\mathcal{L}^{n-1}(N) = \mathcal{L}^{n-1}(N(x')) = 0$. Now, if u is the characteristic function of a set E, we see that either $\mathcal{R}(x', y') = (0, 1)$ or $\mathcal{R}(x', y') = \emptyset$ in (5.8). Indeed, if E has positive measure, for every $x' \in \mathbb{R}^{n-1} \setminus N$ we can always choose $y' \in \mathbb{R}^{n-1} \setminus N(x')$ such that $\mathcal{R}(x', y') = \mathcal{R}(x', x')$. Therefore, the rest of the proof follows as in (5.10) and thereafter, showing that u is of the form described in Theorem 5.1 (ii). However, since u is the characteristic function of a set, there is only one nontrivial superlevel set and, thus, u is indeed of the form described in Theorem 5.1 (i). □

We now prove the second main theorem of this section, which concerns the periodic Gagliardo seminorm and the cylindrical rearrangement.

Proof of Theorem 5.2. We proceed as in the proof of Theorem 5.1 and obtain using the Laplace transform as in (5.4) the following representation of the periodic Gagliardo seminorm

$$([u]^{\mathrm{per}}_{W^{s,p}})^p = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} dt\, t^{\lambda-1} \int_{-\pi}^{\pi} dx_1 \int_{-\pi}^{\pi} dy_1\, \Psi(u, x_1, y_1, t) \sum_{k\in\mathbb{Z}} e^{-(x_1-y_1+2k\pi)^2 t},$$

where

$$\Psi(u, x_1, y_1, t) := \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^{n-1}} dy'\, |u(x_1, x') - u(y_1, y')|^p e^{-|x'-y'|^2 t}.$$

By Theorem 1.2 it holds that $\Psi(u^{*\,n,1}, x_1, y_1, t) \le \Psi(u, x_1, y_1, t)$ for almost every $x_1, y_1 \in (-\pi, \pi)$ and every $t > 0$ (not necessarily for every x_1 and y_1, since $f \in L^p((-\pi, \pi) \times \mathbb{R}^{n-1})$ only yields that $\int_{\mathbb{R}^{n-1}} dx'\, |f(x_1, x')|^p < +\infty$ for almost every x_1). From this the inequality in the theorem follows.

The study of the case of equality is almost identical to that of Theorem 5.1 and we will only carry out some details which are different. If there is equality in the inequality then

$$\Psi(u, x_1, y_1, t) = \Psi(u^{*\,n,1}, x_1, y_1, t)$$

for all $x_1 \in (-\pi, \pi) \setminus N$ and all $y_1 \in (-\pi, \pi) \setminus N(x_1)$, where $\mathcal{L}^1(N) = \mathcal{L}^1(N(x_1)) = 0$), for some $t \in (0, +\infty)$ (we will need only one $t \in (0, +\infty)$ to conclude).

Let us first address the case $p > 1$. In this case we can argue identically as in the proof of Theorem 5.1, with the only difference that we replace $\Lambda(u)$ by

$$\Lambda(u) := \{x_1 \in (-\pi, \pi) : \, u(x_1, x') = 0 \text{ for a.e. } x' \in \mathbb{R}^{n-1}\}.$$

We now address the case $p = 1$. Using the continuity of u —as we did at the beginning of Step 3 in the proof of Theorem 5.1— together with Theorem 1.2, we obtain that for every $x_1, y_1 \in (-\pi, \pi)$ and for all

$$\tau \in \mathcal{R}(x_1, y_1) := (0, \min\{\operatorname{ess\,sup} u(x_1, \cdot), \operatorname{ess\,sup} u(y_1, \cdot)\})$$

there exists $a_\tau(x_1, y_1) \in \mathbb{R}^{n-1}$ such that

$$\{u(x_1, \cdot) > \tau\} = \{u^{*\,n,1}(x_1, \cdot) > \tau\} - a_\tau(x_1, y_1),$$
$$\{u(y_1, \cdot) > \tau\} = \{u^{*\,n,1}(y_1, \cdot) > \tau\} - a_\tau(x_1, y_1).$$

This two identities hold up to sets of $\mathcal{L}^{n-1}$ measure zero (which might depend on x_1 respectively y_1). We can now argue similarly as in the proof of the case of equality for $p = 1$ of Theorem 5.1 to conclude that $a_\tau(x_1, y_1)$ is independent of x_1 and y_1. Finally, use that

$$\{u > \tau\} = \bigcup_{x_1 \in \mathbb{R}} \{x_1\} \times \{u(x_1, \cdot) > \tau\}$$

and Fubini theorem (to deal with sets of measure zero) to obtain the claim of the theorem. □

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G. Csató [1,2]

[1] Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

[2] Centre de Recerca Matemàtica, Edifici C, Campus Bellaterra, 08193 Bellaterra, Spain.

Email address: gyula.csato@ub.edu

A. Mas [1,2]

[1] Departament de Matemàtiques, Universitat Politècnica de Catalunya, Campus Diagonal Besòs, Edifici A (EEBE), Av. Eduard Maristany 16, 08019 Barcelona, Spain

[2] Centre de Recerca Matemàtica, Edifici C, Campus Bellaterra, 08193 Bellaterra, Spain.

Email address: albert.mas.blesa@upc.edu