UNIVERSITAT DE BARCELONA

The Multisymplectic Geometry of Classical Field Theories on Finite-Dimensional Covariant

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Tesi doctoral

The Multisymplectic Geometry of Classical Field Theories on Finite-Dimensional Covariant Phase Spaces

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I dedicate this thesis to the people I cherish most — my loving family:

To my parents, whose unconditional love, guidance, and sacrifices have been my foundation and greatest support,

and to my brother, whose quiet strength, kindness, and steady presence have always inspired me.

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"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

— Albert Einstein, *Geometry and Experience (1921).*

Abstract

This thesis presents new developments in the De Donder–Weyl formulation of classical field theories which treats space and time on the same footing. The formal geometric Lagrangian construction of field theories takes place on jet bundles which yields the standard Euler–Lagrange equations, while the equivalent De Donder–Weyl Hamiltonian formulation takes place on the affine dual jet bundles, yielding the Hamilton–De Donder–Weyl equations. The (dual) jet bundles act as finite-dimensional phase spaces and their multisymplectic geometry serves as a generalization of symplectic geometry in classical mechanics, thereby underlying the variational calculus and providing a powerful set of tools for the investigation of Noether symmetries and constraints. Moreover, the multisymplectic construction of relativistic field theories preserves covariance throughout the analysis of such theories and by working on sections of the (dual) jet bundles over spacetime, it is in fact possible to derive the covariant phase space formalism of Zuckerman [152], Crnković and Witten [34], and Lee and Wald [108] which is more well known in the physics literature and is, in general, infinite-dimensional.

Novel contributions of this thesis include new properties of De Donder–Weyl constraints and natural symmetries, which we provide as newly proven mathematical propositions which can be found in the following of our publications: [63, 75, 76]. This work also provides a multisymplectic construction of the Poisson brackets first introduced by Marsden *et al* [116] to obtain the Hamilton–De Donder–Weyl equations using the action functional in direct analogy with the analysis in classical mechanics; we show how our geometric interpretation of this Poisson bracket also yields the field equations and we also discuss the relation of this Poisson bracket to the aforementioned infinite-dimensional covariant phase space formalism. Finally, we provide a first step toward understanding the multisymplectic geometry associated with the BRST–BV analysis of gauge theories in the Lagrangian setting. After constructing the BRST symmetry, we find the corresponding multimomentum map and show that it yields the standard BRST charge up to a total derivative. The novelties presented in this thesis offer a new selection of geometric techniques that allow us to pass from the infinite-dimensional language of local functionals, to the finite-dimensional language of vector fields and differential forms while treating space and time on the same footing.

Key words: Physics, theoretical physics, differential geometry, gravitational fields.

La Geometria Multisimplèctica de les Teories de Camps Clàssiques en Espais de Fases Covariants de Dimensió Finita

Resum

Aquesta tesi presenta nous desenvolupaments en la formulació de De Donder–Weyl de les teories de camps clàssiques, que tracta l'espai i el temps en peu d'igualtat. La construcció geomètrica formal lagrangiana de les teories de camps té lloc sobre les varietats de jets, donant lloc a les equacions estàndard d'Euler–Lagrange, mentre que la formulació hamiltoniana equivalent de De Donder–Weyl té lloc sobre els jets duals afins, donant lloc a les equacions de Hamilton–De Donder–Weyl. Les varietats de jets (i els seus duals) actuen com espais de fases de dimensió finita, i la seva geometria multisimplèctica actua com una generalització de la geometria simplèctica en la mecànica clàssica, tot fonamentant el càlcul variacional i proporcionant un conjunt potent d'eines per a la investigació de les simetries de Noether i els lligams. A més, la construcció multisimplèctica de les teories de camps relativistes preserva la covariància al llarg de l'anàlisi d'aquestes teories i, treballant sobre seccions de les varietats de jets (i els seus duals) sobre l'espai-temps, és possible derivar el formalism de l'espai de fases covariant de Zuckerman [152], Crnković i Witten [34], i Lee i Wald [108], més conegut en la literatura física i, en general, de dimensió infinita.

Les contribucions originals d'aquesta tesi inclouen noves propietats dels lligams de De Donder–Weyl i simetries naturals, que presentem com a proposicions matemàtiques demostrades recentment i que es poden trobar a les següents publicacions: [63, 75, 76]. Aquest treball també proporciona una construcció multisimplèctica dels parèntesis de Poisson introduïts per Marsden *et al.* [116] per obtenir les equacions de Hamilton–De Donder–Weyl a partir del funcional d'acció, en anàloga directa amb l'anàlisi de la mecànica clàssica. A més, mostrem com la nostra interpretació geomètrica d'aquests parèntesis de Poisson també condueix a les equacions del camp i discutim la seva relació amb el formalisme covariant de dimensió infinita esmentat anteriorment. Finalment, oferim un primer pas cap a la comprensió de la geometria multisimplèctica associada a l'anàlisi BRST–BV de les teories de gauge en el marc lagrangià. Després de construir la simetria BRST, trobem la aplicació multimoment corresponent i mostrem que proporciona la càrrega BRST estàndard llevat d'una derivada total. Les novetats presentades en aquesta tesi ofereixen una nova selecció de tècniques geomètriques que ens permeten passar del llenguatge infinit-dimensional dels funcionals locals al llenguatge finit-dimensional dels camps de vectors i les formes diferencials, tot tractant l'espai i el temps en peu d'igualtat.

Paraules clau: Física, física teórica, geometria diferencial, camps gravitatoris.

Contents

Part I

Introduction

The concept of covariant phase space has a long-standing presence in theoretical physics at the foundational level of classical field theory, with origins which can be traced back to the development of the variational principle. The foundation of modern variational calculus was first introduced by Leonhard Euler in 1744[1] in the context of classical mechanics. Since then, variational techniques have played a crucial role across numerous areas of mathematics and physics. For conservative systems within classical physics, it is well established that the governing equations of motion—whether dynamical equations in mechanics or field equations in field theory—can be derived from a variational principle; this involves an *action functional* whose stationary configurations satisfy the system's equations of motion.

The notion of covariant phase space then arises from the desire to carry out the variational calculus and analysis of symmetries of relativistic field theories in a manner which respects the covariance of such field theories throughout the analysis in both the Lagrangian and Hamiltonian setting. In the case of Lorentz covariant field theories—that is, those which are specified by Lorentz invariant Lagrangians on fixed background spacetimes—the objective is to maintain Lorentz covariance throughout the analysis in which we identify and solve the equations of motion, identify the Noether symmetries and find the corresponding conserved currents, and identify all constraints if any exist. Similarly, in the case of generally covariant theories—i.e., theories with a dynamical background spacetime (or worldvolume) which under the group action of diffeomorphisms leaves the Lagrangian invariant—the objective is to maintain the diffeomorphism invariance of such theories throughout the full analysis. This, of course, is a challenge when constructing the canonical Hamiltonian from the canonical momenta which requires a splitting of space and time, thereby breaking the covariance of the field theory under investigation. Constructing the phase space of the theory in a covariant manner then requires a deeper understanding of the underlying differential geometry from which the variational calculus arises. However, the formal geometric construction of the variational calculus for field theories continues to be an evolving field of research, with much of the progress being made nowadays within the differential geometry and mathematical physics communities.

Yet, this field of research is also of great importance to the physics community, especially to those of us who are interested in gravitational theories. In most cases, including in the cases which are physically viable given the observations of our universe, gravity continues to evade a promising quantum description. On the theoretical side, this is partially due to the fact that space and time are treated on a separate footing in the Hamiltonian setting. Modern gravitational theories also fail to provide us with a full understanding of black holes; the microstate construction of the laws of thermodynamics for physical black holes is still an open problem and the information paradox presents an even more challenging problem which is not understood even in gravitational theories for which we have a consistent quantum theory, including those for which we have a microstate description of black holes. Much of the modern literature which addresses such problems in gravitational theories involves the use of covariant phase spaces. The objective of this thesis is to contribute to the ongoing development of the differential geometry underlying the variational calculus in classical field theory, with the intention of expanding our understanding of covariant phase spaces. This, in turn, could provide us with a more powerful set of tools for addressing some of the gaps in our understanding of quantum gravity and black holes.

Before discussing the underlying differential geometry of the variational calculus in classical field

[1] L. Euler: "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti". 1744.

theory, it is instructive to revisit the analogous geometry in classical mechanics which has been well understood for several decades now (see, for instance, [1, 9, 42, 61, 85, 110, 117, 122, 138]). In the context of mechanical systems, the variational calculus—often referred to as the *simple variational calculus*—involves single-variable functions, namely the generalized coordinates $q^i(t)$. In particular, one begins by constructing a *configuration manifold* Q on which the dynamical variables q^i are parameterized curves. More specifically, the Lagrangian setting takes place on the target space of the tangent bundle $TQ \to Q$ while the Hamiltonian setting takes place either on the target space of the symplectic cotangent bundle $T^*Q \to Q$ in the case of *regular* mechanical systems or, in the case of *singular* mechanical systems, on the primary constraint submaniofold $P_\circ \subset T^*Q$ which is presymplectic and is assumed to also have the structure of a fiber bundle over Q. Then, the action functional is formally identified as a spacetime integral mapping from the space of sections of such fiber bundles to $\mathbb{R}$ and the variational problem can be posed in terms of the (pre)symplectic structures on the aforementioned fibered manifolds. Moreover, the constraint analysis of singular mechanical systems can be carried out using the *Dirac–Bergmann theory of constraints* [7, 44] in which one extends the presymplectic structure on $P_\circ$ to all of T^*Q which is equipped with a canonical symplectic structure used to construct the Dirac Poisson bracket. The Dirac–Bergmann constraint analysis was later extended to the Lagrangian framework in [15, 16], where equivalence with the Hamiltonian setting is specified. Equivalently, one can perform the constraint analysis of singular systems in geometrical terms by means of the Gotay–Nester–Hinds approach [66, 71] and similar approaches such as [115] which make use of the presymplectic structure on $P_\circ$ to algorithmically identify the final constraint submanifold (if one exists) of the constrained mechanical system. The equivalent Lagrangian geometric constraint algorithms are presented in [70, 121] (see also [72]). Furthermore, *Noether symmetries* correspond to Lie group actions on Q whose natural (canonical) lifts to TQ and T^*Q preserve the corresponding (pre)symplectic structures.

Now, classical field theories describe the physics of some fields $\phi^A \in C^\infty(M)$ which are functions of several variables. We then use the *multiple variational calculus* to find the relevant field equations. The *configuration manifold* is identified as the target space of the *configuration bundle* $\pi : E \to M$, where the base space is taken to be m-dimensional spacetime (or worldvolume) and the *space of field configurations* is given by the space of smooth sections $\Gamma(\pi) = \{\phi : M \to E\}$. The covariant phase space formalism that is typically used in the physics literature to study relativistic field theories was introduced by Zuckerman [152], by Witten and Crnković in [34, 148], and was further developed by Lee and Wald in [108]. This covariant phase space formalism takes place on the space of field configurations which, under suitable conditions, is a Banach manifold that is typically infinite-dimensional and is equipped with a presymplectic form, obtained by integrating the *presymplectic current* over a Cauchy surface [108]. Similarly, the space of field configurations which satisfy the equations of motion is an infinite-dimensional manifold equipped with a symplectic form [34]. Moreover, Lee and Wald showed us in [108] how local symmetries are associated with first-class constraints. The first triumph of this formalism is that covariance is maintained in the canonical Hamiltonian formulation of relativistic field theories, and the symplectic structure within the space of solutions is independent of the choice of the Cauchy surface. This covariant phase space formalism was used successfully in the early years to quantize Jackiw–Teitelboim (JT) gravity [123, 124]. In addition, the rigorous differential geometry of the covariant phase space formalism provides us with a robust tool for analyzing boundary terms in relativistic field theories (see e.g. [82] for the state of the art), leading to Wald's geometric derivation of the first law of thermodynamics for stationary black holes [144] in terms of conserved quantities. The covariant phase space formalism was further developed through the study of conserved quantities in [92, 93, 145]; for more details on the covariant phase space formalism, see [97]. In more recent years, the covariant phase space formalism has continued to be useful in the study of black holes, including in Stephen Hawking's final publication [78] written with collaborators in an attempt to define near-horizon symmetries of black holes. The list of publications referenced in this work on the uses of the covariant phase space formalism is far from exhaustive.[2]

In the differential geometry literature, the Lagrangian formulation of classical field theories is typically carried out on *jet bundles* rather than on the space of field configurations as is typical in the physics literature. In order to formally carry out the variational calculus which leads to the *Euler–Lagrange equations* and investigate Noether symmetries of a first-order field theory, it is sufficient to work on the *first-order jet manifold* $J^1\pi$ which is defined as the manifold given by the set of equivalence classes of local sections of the configuration bundle $\pi : E \to M$, where two sections are equivalent at a point $x \in M$ if their Taylor expansions agree point-wise up to first order. Furthermore, the jet manifold $J^1\pi$ is an *affine bundle* over E. The use of jet bundles in the analysis of field theories comes with several geometric advantages. Firstly, $J^1\pi$ is a finite-dimensional manifold as opposed to the space of field configurations which is given as the space of local sections of the configuration bundle $\pi : E \to M$ and is thereby typically infinite-dimensional. Moreover, the bundle structure provides $J^1\pi$ with a variety of geometrically canonical structures, including the vertical endomorphism from which we obtain the *Poincaré–Cartan m-form* and the *(pre)multisymplectic form* called the *Poincaré–Cartan $(m+1)$-form*[3] (for more details on the multisymplectic Lagrangian formulation of first-order field theories see, e.g., [2, 46, 47, 54, 62, 131, 136]). The premultisymplectic form is then used in direct analogy with the presymplectic analysis of time-dependent Lagrangian mechanics [28, 38] to solve the variational problem, investigate Noether symmetries, and carry out the full constraint analysis of *singular* field theories . Furthermore, the (pre)multisymplectic structure of $J^1\pi$ can be viewed as the geometrically canonical origin of the (pre)symplectic structure of the infinite-dimensional phase space formalism. This allows us to pass from the use of functionals and other integrated quantities on infinite-dimensional functional spaces, to the use of tensors on $J^1\pi$. Moreover, the covariance of relativistic field theories is maintained on all of $J^1\pi$ throughout the (pre)symplectic analysis in which one obtains the field equations, investigates the Noether symmetries, and identifies any Lagrangian constraints that might exist. This is in contrast with the aforementioned infinite-dimensional covariant phase space formalism which, although maintains covariance throughout the analysis of a relativistic field theory, is only independent of the choice of a Cauchy surface on-shell.

Performing a covariant Hamiltonian analysis that is equivalent and geometrically analogous to the analysis conducted on jet bundles is a much more subtle task. A covariant Hamiltonian formalism was first explored in the 1930's through the pioneering work of Théophile De Donder and Hermann Weyl [36, 143] which today is often referred to in the literature as the *De Donder–Weyl Hamiltonian formalism*. Their Hamiltonian analysis begins on the space of *holonomic sections* of $J^1\pi$ and proceeds by replacing the spacetime derivatives of the fields with the *canonical multimomenta*. In particular, the multimomenta are constructed from the De Donder–Weyl Legendre transform in which we take the partial derivative of the Lagrangian function evaluated on holonomic sections of $J^1\pi$ with respect to the spacetime derivatives of the fields. The corresponding variational principle leads to the so-called *(Hamilton)–De Donder–Weyl equations* which are the field equations that are equivalent to the Euler–Lagrange equations. However, the formal geometric formulation of the De Donder–Weyl formalism was developed much later than the original works [25, 62, 67] and can be viewed as a direct generalization of the simple variational calculus in Hamiltonian mechanics to the multiple variational calculus. Instead of working with local sections, the De Donder–Weyl framework can be constructed in formal geometric terms on the *affine dual* of the first-jet bundle, denoted $J^1\pi^* \to E$. This bundle is given as the quotient bundle $J^1\pi^* \equiv \mathcal{M}\pi/\Lambda^m_1(\mathrm{T}^*\mathrm{E})$, where $\Lambda^m_1(\mathrm{T}^*\mathrm{E})$ denotes the bundle of π-semibasic m-forms on E and $\mathcal{M}\pi \equiv \Lambda^m_2\mathrm{T}^*E$ is the sub-bundle of $\Lambda^m\mathrm{T}^*E$ made up of the m-forms on E which vanish by the action of two π-vertical vector fields; the bundles $\mathcal{M}\pi$ and $J^1\pi^*$ are often called the *extended* and *restricted multimomentum bundles*.

[2] A few more references will be provided, primarily in Section 6 when we revisit the subject.

[3] The Poincaré–Cartan form itself is not canonical in formal geometric terms, but it is unique once the Lagrangian for some field theory is specified.

Now, given a Lagrangian on the jet bundle $J^1\pi$, we have the *extended Legendre transform* which maps from $J^1\pi$ to $\mathcal{M}\pi$ and the *restricted Legendre transform* which maps from $J^1\pi$ to $J^1\pi^*$.

The extended multimomentum bundle $\mathcal{M}\pi$ is equipped with the canonical multisymplectic $(m+1)$-form called the *multimomentum Liouville $(m+1)$-form.* This differential form pulls back by the extended Legendre transform to the Poincaré–Cartan $(m+1)$-form on $J^1\pi$. Additionally, $\mathcal{M}\pi$ is a fiber bundle over $J^1\pi^*$ with local sections called *Hamiltonian sections.* It follows that, through the pullback by Hamiltonian sections, the canonical Liouville $(m+1)$-form on $\mathcal{M}\pi$ endows $J^1\pi^*$ with a (pre)multisymplectic $(m+1)$-form called the *Hamitlon–Cartan $(m+1)$-form* which, in turn, pulls back to the Poincaré–Cartan $(m+1)$-form on $J^1\pi$ via the restricted Legendre transform. Moreover, the splitting of space and time in $J^1\pi^*$ follows from the foliation of the base space M into leaves consisting of Cauchy surfaces and the Hamilton–Cartan $(m+1)$-form then induces the presymplectic form of the infinite-dimensional covariant phase space formalism [50, 51, 68]. In this manner, we also recover the standard canonical Hamiltonian which corresponds to *total energy* and generates time evolution through the *canonical Poisson bracket* (which is also recovered from the space and time splitting of $J^1\pi^*$); furthermore, the Hamilton–De Donder–Weyl equations also reduce to the standard Hamilton equations of the canonical Hamiltonian formalism. Moreover, the splitting of space and time naturally emerges when studying boundaries in the multisymplectic framework; see [10, 90, 91] for the original work on boundaries within the multisymplectic framework.

However, constraints were not understood in the De Donder–Weyl formalism at the time of its development. A constraint analysis is required to fully determine the dynamics of a field theory when its De Donder–Weyl Legendre map is singular, which occurs when the multi-Hessian—defined as the matrix of second derivatives of the Lagrangian with respect to the spacetime derivatives of the fields—is singular. A geometric algorithm for performing the desired covariant constraint analysis was developed in [37, 39] for sufficiently well behaved constrained systems; namely, we require that the Legendre transform is *almost-regular* which ensures that it is a submersion onto its image, yielding a closed fibered submanifold $P_\circ \hookrightarrow J^1\pi^*$, the *primary constraint submanifold.* It follows that the premultisymplectic Poincaré–Cartan form on $J^1\pi$ is related by the Legendre map to the induced premultisymplectic Hamilton–Cartan form on $P_\circ$ and the geometric constraint algorithm then uses the premultisymplectic geometry of $J^1\pi$ and $P_\circ$ to obtain the final constraint submanifolds.[4] This analysis is geometrically analogous to how the presymplectic geometry of the phase spaces in classical mechanics is used to perform the geometric constraint analysis (developed in [28, 38, 70, 71, 115, 121]) of mechanical systems which is equivalent to the Dirac–Bergmann constraint procedure for mechanics.

Furthermore, *Noether symmetries* are given as Lie group actions on $J^1\pi$ and $J^1\pi^*$ (or on $P_\circ$ in the singular case) which preserve the corresponding (pre)multisymplectic $(m+1)$-forms. The resulting conserved quantities are called *multimomentum maps* and are the generalization of the *momentum maps* in (pre)symplectic mechanics. The multimomentum maps, represented as $(m-1)$-forms on the (dual) jet bundles, yield the standard Noether currents (which are conserved by Noether's first theorem) when pulled back to the base space M by local sections [40, 48, 54, 55, 58, 67, 132]. Noether's second theorem gives us the *Noether identities* which are relations among the Euler–Lagrange equations that arise due to local (gauge) symmetries in a Lagrangian field theory; more specifically, the variational derivatives are not all independent, and the relations among them are given explicitly by the Noether identities, which hold both on-shell and off-shell.

Jet bundles are also important because they are the natural setting on which the cohomological construction of the global variational principle in field theories takes place. The *k-order jet bundle* $\pi^k : J^k\pi \to M$ is given by the set of equivalence classes of sections $\phi : M \to E$ which agree point-wise to order k in their Taylor expansion while the *infinite-jet bundle* $\pi^\infty : J^\infty\pi \to M$ is given by those which agree in their full Taylor expansion. Furthermore, the infinite jet manifold

[4] It is worth noting that final constraint submanifolds only exist for well behaved theories which fortunately include the relevant physical theories such as General Relativity and Yang–Mills.

$J^\infty\pi$ has a bundle structure over jet manifolds of arbitrary order and so it follows that forms on $J^k\pi$ can be naturally lifted to $J^\infty\pi$. The variational principle in the Lagrangian setting is posed in cohomological terms on $J^k\pi$ by constructing the Euler–Lagrange complex with coboundary differential operator δ which yields the Euler–Lagrange equations [139, 140, 141]. The operator δ is of significant importance, producing the variation of fields in the variational principle through its natural application to sections ϕ. Additionally, δ acts on the Lagrangian form as the sum of the *Euler–Lagrange operator* plus a total derivative; in fact, the total derivative term gives rise precisely to the presymplectic structure on the space of field configurations in the infinite-dimensional covariant phase space formalism. Furthermore, the Euler–Lagrange complex is incorporated as an augmentation to the vertical part of the *variational bicomplex* [6, 8] which is naturally constructed on $J^\infty\pi$ due to the natural splitting of the exterior derivative on $J^\infty\pi$ into horizontal and vertical parts with respect to the projection $\pi^\infty : J^\infty\pi \to M$. Local symmetries leave the Lagrangian invariant up to a total derivative and the Noether currents which are given as $(m-1)$-forms belong to the *horizontal complex*. Consequently, as Noether currents are conserved on-shell, they are classified by the *characteristic cohomology* of degree $m-1$. Moreover, the Poincaré–Cartan forms on $J^k\pi$ also lie in the variational bicomplex [23]. It is also worth noting that the variational bicomplex is relevant in the infinite-dimensional covariant phase space formalism [113, 130].

The *BRST–BV (antifield)* formalism provides the most general and systematic framework for analyzing gauge-invariant field theories, particularly those with gauge algebras of more intricate structure, such as reducible or open algebras. The BRST antifield analysis is carried out by constructing the *BRST differential* on a jet manifold $J^k\pi$ which has been augmented by graded *ghost* and *antighost* degrees of freedom (see e.g., [12, 65, 84]). The differential operator δ is incorporated in the BRST antifield formalism as the *Koszul–Tate differential* which has *antighost number* -1 and implements the equations of motion (and any Noether identities) through its *homological resolution*. Gauge degrees of freedom are accounted for by the ghost fields and the *longitudinal cohomology* is constructed out of the *longitudinal derivative* which is given by the exterior derivative on $J^k\pi$ along the gauge orbits of the gauge group action. The nilpotent BRST differential is then constructed as the sum of the Koszul–Tate differential and the longitudinal derivative[5] and has *total ghost number* $+1$, thus acting as the coboundary operator of the BRST complex. Then, the natural application of the BRST differential to sections $j^k\phi : M \to J^k\pi$ is used to identify the physical observables which are classified by the zeroth BRST cohomology: the set of gauge invariant functionals that are constructed from sections which are solutions to the equations of motion. Furthermore, the BRST transformations of functionals can be regarded as canonical transformations in terms of the BRST antibracket $(\ , \ )$ which is a graded Poisson bracket on the augmented configuration manifold (see e.g., [65, 84]). Then, the generator of the BRST transformations of functionals is the appropriately augmented action functional S called the *proper solution* to the *master equation* $(S, S) = 0$ which ensures the nilpotence of the BRST transformations.

Despite the many insights and powerful tools developed through the study of the differential geometry which underlies variational calculus, a wide range of questions about field theories can still be posed in formal geometric terms. The primary advantage offered by the De Donder–Weyl approach is that it treats all spacetime coordinates on an equal footing, making it appealing for the analysis of relativistic field theories. The formal geometric construction takes place on $J^1\pi^*$, which serves as a finite-dimensional covariant phase space. In addition, the canonical geometric structures in the jet bundles $J^1\pi$ and $J^1\pi^*$ do not depend on the introduction of Cauchy surfaces, and they provide us with the formal geometric procedure for the formulation of the variational principle, analysis of symmetries, and identification of constraints, in direct analogy with geometric analysis in classical mechanics. Nevertheless, the differential geometry of jet bundles which underlies the analysis of field theories is still a developing field of research.

Firstly, we do not currently have a canonical Poisson bracket analysis of De Donder–Weyl

[5] In this work we focus our attention to the BRST formalism applied to irreducible gauge theories.

constraints *a la* Dirac; instead we use the geometric constraint analysis which follows from the premultisymplectic $(m+1)$-forms on $J^1\pi$ and $P_\circ \hookrightarrow J^1\pi^*$. In Section 3.3 we present some new lemmas and propositions regarding Lagrangian and De Donder–Weyl constraints. A Dirac constraint formalism on $J^1\pi^*$ does not yet exist partially due to the fact that we do not have a full understanding of Poisson structures on $J^1\pi^*$. However, Marsden *et al* showed in [116] how the variational principle leading to the De Donder–Weyl equations is posed from a Poisson bracket of functionals defined on the fibers of $J^1\pi^* \to M$. This Poisson structure was not understood at the time in terms of canonical structures on $J^1\pi^*$. In Section 6.3 we introduce a geometric origin of the Poisson bracket presented by Marsden *et al* in [116] coming from the Hamilton–Cartan form on $J^1\pi^*$; we also discuss the relation of this bracket with the (pre)symplectic structures of the infinite-dimensional covariant phase space formalism. A related open question involving De Donder–Weyl constraints asks if there exists a direct geometric relation between the corresponding constraint submanifolds of $J^1\pi^*$ and the canonical Dirac constraints on the instantaneous foliation of sections of $J^1\pi^*$ on which the infinite-dimensional covariant phase space formalism takes place. In Section 6.4 we find that the conserved Noether charge of Yang–Mills which leads to the first-class constraints in the infinite-dimensional covariant phase space formalism [108] is related to the corresponding multimomentum maps on $J^1\pi$ and $J^1\pi^*$ up to a total derivative.

Furthermore, the notions of canonical lifts and natural symmetries on the multimomentum bundles of (pre)multisymplectic De Donder–Weyl Hamiltonian field theories have not been rigorously studied in previous literature to our knowledge (although these notions are peripherally addressed in [67]). However, the notion of canonical lifts of Lie group actions and their generating vector fields on jet manifolds has been well understood at a rigorous level for many years (see e.g., [136]). Moreover, the *natural symmetries* on jet bundles which arise from such canonical lifts have been well understood in terms of (pre)multisymplectic geometry [2, 3, 35, 40, 46, 48, 52, 54, 62, 67, 105, 132, 136]; for an overview of symmetries in the (pre)multisymplectic Lagrangian setting, see [55, 58, 75]. In this work, we present a formal mathematical definition of the canonical lift of a configuration bundle diffeomorphism and of a vector field on the configuration bundle of a field theory to the extended and restricted multimomentum bundles on which the multisymplectic De Donder–Weyl Hamiltonian formulation of field theories takes place. The notion of natural symmetries in the multisymplectic De Donder–Weyl framework thereby follows, along with a variety of new propositions. We also give a preliminary definition of canonical lifts for field theories that are singular (or, more specifically, *almost-regular* [37, 134]) in the De Donder–Weyl sense. The De Donder–Weyl Hamiltonian formulation of almost-regular field theories takes place on an embedded primary constraint submanifold of the restricted multimomentum bundle. However, the definition we provide for the canonical lift of a group action to embedded submanifolds of $J^1\pi^*$ requires the invariance of the primary constraint submanifold under the lifted group action. This requirement is quite restrictive, and a more general definition would be desirable. Prior to this work, the usual way to obtain the appropriate symmetries in the De Donder–Weyl setting resulting from canonical lifts was to first work out the full analysis in the Lagrangian setting and then project the constraint and symmetry structures onto the Hamiltonian setting via the Legendre transform [75]. The results presented here allow us to go directly to the De Donder–Weyl formalism.

The final topic we address in this thesis is the Poincaré–Cartan geometry of $J^1\pi$ when performing the BRST–BV analysis. In Section 15.5 we show how the augmentation of the configuration manifold by ghost fields and antifields alters the vertical endomorphism on $J^1\pi$ for first-order field theories. We then construct the Poincaré–Cartan m-form from the appropriately extended Lagrangian on $J^1\pi$ given by the proper solution to the master equation. The Poincaré–Cartan m-form is also invariant under the BRST transformations since they are a symmetry of the extended Lagrangian. We compute the multimomentum map $(m-1)$-form on $J^1\pi$ associated with the BRST symmetry and show that it coincides with the standard BRST charge up to a total derivative when pulled back by holonomic sections and integrated over a Cauchy surface $\Sigma \subset M$; we work out the case of Yang–Mills theory explicitly.

Some of the novelties presented in this thesis can be found in our published works [63, 75, 76] while more recent novelties will appear in future publications.

Part II

Variational Calculus and Noether Symmetries

1 Preliminary Differential Geometry

This chapter gives an introductory presentation of the differential geometry of the fiber bundles on which we construct field theories. For a comprehensive textbook treatment of fiber bundles, the reader is referred to [107] for example. Some standard textbooks on jet bundles relevant for Sections 1.2 to 1.4 include [105, 133, 136]. See [25, 49, 81] for more details on multimomentum bundles, discussed in Section 1.6.

1.1 The Configuration Bundle

Let spacetime (or worldvolume) be an oriented m-dimensional Lorentzian manifold M with local coordinates x^μ and volume form with local expression $\mathrm{d}^m x \equiv \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{m-1}$ and metric with components $g_{\mu\nu}$ and signature $(- + \cdots +)$. Furthermore, let $\pi : E \to M : (x^\mu, y^A) \mapsto x^\mu$ be a fiber bundle (with $\dim E = m + n$) which serves as the ***configuration bundle*** of some classical field theory on M. We denote smooth local sections as $\phi : U_M \subset M \to E : x^\mu \mapsto (x^\mu, \phi^A(x))$, where $U_M \subset M$ is an open set and we recall that the $\phi^A \in C^\infty(M)$ are the fields of the theory under investigation. This means that the ***space of field configurations*** is given by the space of sections $\Gamma(\pi)$ which is typically infinite dimensional as the space of smooth functions on M usually is. Furthermore, we have the ***tangent bundle of*** E

$$\tau_E : TE \to E \ , \tag{1.1}$$

in which each fiber $T_x E$ over some point $x \in M$ is the tangent space of M at x. Sections of the tangent bundle are called ***vector fields*** on E, and the set of all such vector fields is denoted by $\mathfrak{X}(E)$. Next, we have the set of π***-vertical vectors*** $V\pi \equiv \ker(\pi_*) \subset TE$. The tangent bundle TE comes with the distinguished sub-bundle called the π***-vertical tangent bundle***:

$$\tau_E|_{V\pi} : V\pi \to E \ , \tag{1.2}$$

and sections of this bundle are called π***-vertical vector fields on*** E and are denoted $\mathfrak{X}^{V\pi}(E)$. In fact, we have the following short exact sequence of vector bundle morphisms:

$$0 \longrightarrow V\pi \longrightarrow TE \longrightarrow \pi^*(TM) \longrightarrow 0$$

Furthermore, the ***cotangent bundle of*** E

$$\widetilde{\tau}_E : T^*E \to E \ , \tag{1.3}$$

comes with the distinguished sub-bundle called the π***-horizontal cotangent bundle of*** E:

$$\widetilde{\tau}_E|_{\pi^*(T^*M)} : \pi^*(T^*M) \to E \ . \tag{1.4}$$

The target space $\pi^*(T^*M) \subset TE$ is the set of π-horizontal covectors on E which are those which annihilate π-vertical vectors in TE.

It is well known that the sub-bundles $\pi^*(T^*M)$ and $V\pi$ do not have distinguished complements [136]. Instead, one can obtain complements to these sub-bundles in a manner that is not canonical by introducing an ***Ehresmann connection on*** E which is defined as a type $(1,1)$-tensor field

∇ on E which satisfies $\nabla^*\alpha = \alpha$ for all π-semibasic forms α on E. The local expression for the connection is

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^A_\mu \frac{\partial}{\partial y^A}\right) \in \Gamma\left(\widetilde{\tau}_E|_{\pi^*(T^*M)}\right) \otimes \Gamma\left(\tau_E\right) \ , \tag{1.5}$$

given some smooth functions $\Gamma^A_\mu \in C^\infty(E)$. Upon making a choice for the connection ∇ by specifying the Γ^A_μ, the tangent bundle is split into the direct sum of the following two sub-bundles:

$$TE = H(\nabla) \otimes V\pi \ . \tag{1.6}$$

Similarly, the cotangent bundle of E can also be split as

$$T^*E = H^*(\nabla) \otimes V^*\pi \ . \tag{1.7}$$

However, these splittings are not canonical since they depend on the choice for the connection ∇. Instead, distinguished complements to $V\pi$ and π^*T^*M exist upon introducing *jet bundles* which carry the structure of *holonomic vectors* and *contact forms* as follows.

1.2 The First-order Jet Bundle $J^1\pi$

Given a point $x \in M$, two sections $\phi, \bar{\phi}$ of π are ***1st-order equivalent*** at x if $\phi(x) = \bar{\phi}(x)$ and $\partial_\mu\phi|_x = \partial_\mu\bar{\phi}|_x$ (where $\partial_\mu\phi \equiv \partial\phi/\partial x^\mu$), and hence $T_x\phi = T_x\bar{\phi}$. The corresponding equivalence classes, denoted $j^1_x\phi$, are called the ***1-jets*** of ϕ at x. Then, the ***first-order jet bundle*** $J^1\pi$ is defined as

$$J^1\pi \equiv \{j^1_x\phi : x \in M, \phi \in \Gamma(\pi)\} \ , \tag{1.8}$$

with $\dim J^1\pi = m+n+mn$. Furthermore, $J^1\pi$ has the structure of an affine bundle over E and M as $\pi^1_0\colon J^1\pi \to E$ and $\pi^1\colon J^1\pi \to M$. A section of π^1 is said to be a ***holonomic section*** if it is the ***first jet prolongation*** to $J^1\pi$ of a section $\phi \in \Gamma(\pi)$ denoted $j^1\phi : M \to J^1\pi : x^\mu \mapsto (x^\mu, \phi^A, \partial_\mu\phi^A)$; the space of holonomic sections will be denoted $\mathcal{J}^1\pi$. The ***holonomic lift to*** $J^1\pi$ of a tangent vector $\xi \in T_xM$ at the point $x \in M$ is defined as

$$\left(\phi_*(\xi), j^1_x\phi\right) \in (\pi^1_0)^*(TE) \ . \tag{1.9}$$

Sections of $\Lambda^m(\mathrm{T}J^1\pi) \equiv \overbrace{\mathrm{T}J^1\pi \wedge \ldots \wedge \mathrm{T}J^1\pi}^{(m)}$ are called ***m-multivector fields*** [6] on $J^1\pi$ and they are the contravariant skew-symmetric tensor fields of order m on $J^1\pi$. The set of m-multivector fields on $J^1\pi$ is denoted $\mathfrak{X}^m(J^1\pi)$. We will later show how a certain class of *Hamiltonian* m-multivector fields on $J^1\pi$ characterize the solutions to the variational problem posed on $J^1\pi$. A multivector field is said to be a ***holonomic multivector field*** if it is π^1-transverse, integrable, and its *integral sections* are holonomic sections of π^1_0. The space of holonomic vector on $J^1\pi$ is given as

$$\mathrm{Hol}(J^1\pi) = \mathrm{Span}\left\{\partial_\mu = \frac{\partial}{\partial x^\mu} + y^A_\mu \frac{\partial}{\partial y^A}\right\} \ , \tag{1.10}$$

which is the span of the ***total derivatives*** of smooth functions on E. It is also worth noting that a ***first-order differential equation*** on $\pi : E \to M$ is an embedded submanifold $\mathcal{S} \hookrightarrow J^1\pi$ and a ***solution*** is a local section $\phi : U_M \subset M \to E$ which satisfies $j^1_p\phi \in \mathcal{S}$ for every $p \in U_M$.

Now, the distinguished sub-bundle of TE given by the bundle $V\pi$ has a complement when pulled back by π^1_0; namely, the complement of $(\pi^1_0)^*V\pi$ is the sub-bundle of holonomic vectors on $J^1\pi$. It follows that,

$$(\pi^1_0)^*TE = (\pi^1_0)^*V\pi \oplus \mathrm{Hol}(J^1\pi) \ . \tag{1.11}$$

[6] We refer the reader to Appendix A for further details on multivector fields.

Moreover, $(\pi_0^1)^*TE$ can be canonically embedded as a sub-bundle of $TJ^1\pi$. Furthermore, $J^1\pi$ comes equipped with the ***contact*** **1*-forms***,

$$\theta^A = \mathrm{d}y^A - y^A_\mu \mathrm{d}x^\mu \ . \tag{1.12}$$

Note that the pullback of the contact form by a holonomic section vanishes:

$$j^1\phi^*\theta^A = 0 \ . \tag{1.13}$$

The sub-bundle of $(\pi_0^1)^*T^*E$ composed of the contact 1-forms is denoted $\mathrm{Cont}(J^1\pi)$ and is complement to the sub-bundle given by the pullback by π_0^1 of π-horizontal covectors, π^*T^*M, hence

$$(\pi_0^1)^*T^*E = (\pi_0^1)^*(\pi^*T^*M) \oplus \mathrm{Cont}(J^1\pi) \ . \tag{1.14}$$

As before, $(\pi_0^1)^*T^*E$ can be canonically embedded into $T^*J^1\pi$. Similarly to TE as seen in (1.11), the canonical splitting of T^*E into two distinguished sub-bundles occurs only upon pulling back by π_0^1 to $J^1\pi$.

Remark 1.1 *The first-jet bundle $\pi_0^1 : J^1\pi \to E$ is an affine bundle modeled on the vector bundle*

$$\tau_E|_{V\pi} \otimes \widetilde{\tau}_E|_{\pi^*(T^*M)} : V\pi \otimes \pi^*T^*M \to E \ . \tag{1.15}$$

Now, the ***tangent bundle of*** $J^1\pi$,

$$\tau_{J^1\pi} : TJ^1\pi \to J^1\pi \ , \tag{1.16}$$

comes with the distinguished sub-bundle called the π^1***-vertical tangent bundle***:

$$\tau_{J^1\pi}|_{V\pi^1} : V\pi^1 \to J^1\pi \ . \tag{1.17}$$

Furthermore, the ***cotangent bundle of*** $J^1\pi$,

$$\widetilde{\tau}_{J^1\pi} : T^*J^1\pi \to J^1\pi \ , \tag{1.18}$$

comes with the distinguished sub-bundle called the π^1***-horizontal cotangent bundle of*** $J^1\pi$,

$$\widetilde{\tau}_{J^1\pi}|_{(\pi^1)^*(T^*M)} : (\pi^1)^*(T^*M) \to J^1\pi \ . \tag{1.19}$$

Now we may ask if the distinguished sub-bundles $V\pi^1 \subset TJ^1\pi$ and $(\pi^1)^*(T^*M) \subset T^*J^1\pi$ have distinguished complements. Similarly to before, distinguished complements are obtained by lifting to the *second-order jet bundle* $J^2\pi$.

Instead, one can again obtain complements to these sub-bundles in a manner that is not canonical by introducing an ***Ehresmann connection on*** $J^1\pi$, now defined as a type $(1,1)$-tensor field on $J^1\pi$ which satisfies $\nabla^*\alpha = \alpha$ for all π^1-semibasic forms on $J^1\pi$ and has the following local expression:

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^A_\mu \frac{\partial}{\partial y^A} + \Gamma^A_{\mu\nu} \frac{\partial}{\partial y^A_\nu} \right) \in \Gamma\left(\widetilde{\tau}_{J^1\pi}|_{(\pi^1)^*(T^*M)} \right) \otimes \Gamma\left(\tau_{J^1\pi}\right) \ . \tag{1.20}$$

Then,

$$TJ^1\pi = H(\nabla) \oplus V\pi^1 \quad , \quad T^*J^1\pi = H^*(\nabla) \oplus V^*\pi^1 \quad . \tag{1.21}$$

The first-jet manifold $J^1\pi$ also comes equipped with the canonical type $(1,2)$ tensor known as the ***vertical endomorphism*** which is given as,

$$\mathcal{S} = \theta^A \otimes \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial y^A_\mu} \ . \tag{1.22}$$

The volume form $\mathrm{d}^m x$ on M can be lifted to both E and $J^1\pi$ by pullback of the corresponding projection maps π and π^1. It is typical to see the vertical endomorphism given in terms of its contraction with the volume form as a type $(m, 1)$ tensor:

$$\mathcal{S} = \theta^A \wedge \mathrm{d}^{m-1}x_\mu \otimes \frac{\partial}{\partial y^A_\mu} \ . \tag{1.23}$$

The vertical endomorphism vanishes on holonomic sections, and maps π^1_0-horizontal vector fields (that is, those which satisfy $(\pi^1_0)_* X = X_E$ for some $X_E \in E$) to π^1_0-vertical vector fields (those which satisfy $(\pi^1_0)_* X = 0$). A first-order field theory is given by a ***Lagrangian form*** $\mathbf{L} = \mathscr{L}\mathrm{d}^m x$ with ***Lagrangian function*** $\mathscr{L} \in C^\infty(J^1\pi)$. As we will discuss in the following section, the contact form above is the *vertical exterior derivative* of y^A as $\theta^A = \mathrm{d_v} y^A$. We then have the ***Poincaré–Cartan*** m***-form***,

$$\Theta_{\mathscr{L}} \equiv \mathcal{S}(\mathrm{d}\mathscr{L}) + \mathscr{L}\mathrm{d}^m x = \frac{\partial \mathscr{L}}{\partial y^A_\mu}\theta^A \wedge \mathrm{d}^{m-1}x_\mu + \mathscr{L}\mathrm{d}^m x \ , \tag{1.24}$$

where

$$\begin{aligned} \mathrm{d}^{m-1}x_\mu &\equiv i(\partial_\mu)\mathrm{d}^m x = \frac{1}{(m-1)!}\epsilon_{\mu\mu_2\cdots\mu_m}\mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_m}, \\ \mathrm{d}^{m-2}x_{\mu\nu} &\equiv i(\partial_\nu)i(\partial_\mu)\mathrm{d}^m x = \frac{1}{(m-2)!}\epsilon_{\mu\nu\mu_3\cdots\mu_m}\mathrm{d}x^{\mu_3} \wedge \cdots \wedge \mathrm{d}x^{\mu_m}, \\ &\vdots \\ \mathrm{d}^0 x_{\mu_1\cdots\mu_m} &\equiv i(\partial_{\mu_m})\cdots(\partial_{\mu_1})\mathrm{d}^m x = \epsilon_{\mu_1\cdots\mu_m}. \end{aligned} \tag{1.25}$$

The Poincaré–Cartan m-form $\Theta_{\mathscr{L}}$ is the *principle Lepage equivalent* of $\mathbf{L}$ (see e.g. [105, 136]). Furthermore, the ***Poincaré–Cartan*** $(m+1)$***-form*** is defined as,

$$\Omega_{\mathscr{L}} \equiv -\mathrm{d}\Theta_{\mathscr{L}} \ . \tag{1.26}$$

In the following chapters we will discuss how the ***Poincaré–Cartan*** structure shown above is used to carry out the *variational principle* and study *Noether symmetries* in first-order field theories.

1.3 The k-Jet Bundle $J^k\pi$

The k***-jet bundle*** $J^k\pi$ is defined as

$$J^k\pi \equiv \{j^k_x\phi : x \in M, \phi \in \Gamma(\pi)\} \ , \tag{1.27}$$

where now have the k***-jets*** denoted $j^k_x\phi$ are the equivalence class of sections that are k***-order equivalent*** at $x \in M$. The jet manifold $J^k\pi$ comes with projection map $\pi^k : J^k\pi \to M : \left(x^\mu, y^A, y^A_\mu, \ldots, y^A_{\mu_1\ldots\mu_k}\right) \mapsto x^\mu$ and the k***-jet prolongations*** are given as $j^k\phi : M \to J^k\pi : x^\mu \mapsto (x^\mu, \phi^A, \phi^A_\mu, \ldots, \phi^A_{\mu_1\ldots\mu_k})$. Note that the configuration bundle $\pi : E \to M$ is in fact the zeroth-jet bundle. Furthermore, given two jet bundles of different order, $J^k\pi$ and $J^l\pi$ with $0 \leq l<k$, it follows that $\pi^k_l : J^k\pi \to J^l\pi$ is a fiber bundle and $\pi^{k+1}_k : J^{k+1}\pi \to J^k\pi$ is an affine bundle.

Now, the ***holonomic lift to*** $J^{k+1}\pi$ of a tangent vector $\xi \in T_x M$ at the point $x \in M$ is defined as

$$\left(j^k\phi_*(\xi), j^{k+1}_x\phi\right) \in (\pi^{k+1}_k)^*(TJ^k\pi) \ . \tag{1.28}$$

In general, $J^k\pi$ comes equipped with the set space of holonomic vectors which is spanned by the total derivatives as

$$\mathrm{Hol}(J^k\pi) = \mathrm{Span}\left\{\partial_\mu = \frac{\partial}{\partial x^\mu} + \sum_{i=0}^{k-1} y^A_{\mu\nu_1\ldots\nu_i}\frac{\partial}{\partial y^A_{\nu_1\ldots\nu_i}}\right\} \ , \tag{1.29}$$

Now, the total derivatives of functions on $J^k\pi$ exist as holonomic vectors on $J^{k+1}\pi$. Furthermore, the contact 1-forms

$$\theta^A_{\mu_1\ldots\mu_i} = \mathrm{d}y^A_{\mu_1\ldots\mu_i} - y^A_{\nu\mu_1\ldots\mu_i}\mathrm{d}x^\nu \ , \tag{1.30}$$

with $i \in \{0, 1, \ldots, k\}$. Furthermore,

$$j^k\phi^*\theta^A_{\mu_1\ldots\mu_i} = 0 \ . \tag{1.31}$$

The set of 1-forms on $J^k\pi$ will be denoted $\mathrm{Cont}(J^k\pi)$. Now it follows that the distinguished subbundles $V\pi^k \subset TJ^k\pi$ and $(\pi^k)^*(T^*M) \subset T^*J^k\pi$ have distinguished complements when pulled back by π^{k+1}_k as,

$$(\pi^{k+1}_k)^*(TJ^k\pi) = (\pi^{k+1}_k)^*V\pi^k \oplus \mathrm{Hol}(J^{k+1}\pi) \ , \tag{1.32}$$

and

$$(\pi^{k+1}_k)^*(T^*J^k\pi) = (\pi^{k+1}_k)^*\Big((\pi^k)^*(T^*M)\Big) \oplus \mathrm{Cont}(J^{k+1}\pi) \ . \tag{1.33}$$

Moreover, it is possible to choose a connection on $J^k\pi$,

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \sum_{i=0}^{k-1}\Gamma^A_{\mu\nu_1\ldots\nu_i}\frac{\partial}{\partial y^A_{\nu_1\ldots\nu_i}}\right) \ , \tag{1.34}$$

by specifying the functions $\Gamma^A_{\mu\nu_1\ldots\nu_i} \in C^\infty(J^k\pi)$.

A ***Lagrangian system of order*** k consists of the k-jet manifold $J^k\pi$ and a ***Lagrangian density*** denoted $\mathbf{L} = \mathscr{L}\,\mathrm{d}^m x \in \Omega^m(J^k\pi)$, where $\mathscr{L} \in \mathrm{C}^\infty(J^k\pi)$ is the associated *Lagrangian function*. Note that the Lagrangian density is π^k-semibasic by definition. The Poincaré–Cartan m-form on $J^k\pi$ is given as:

$$\Theta_{\mathscr{L}} = \mathbf{L} + \sum_{r=0}^{k-1}\left(\sum_{l=0}^{k-1-r}(-1)^l\partial_{\mu_1\ldots\mu_l}\frac{\partial\mathscr{L}}{\partial y^A_{\nu_1\ldots\nu_r\mu_1\ldots\mu_l\rho}}\theta^A_{\nu_1\ldots\nu_r}\right)\wedge \mathrm{d}^{m-1}x_\rho \ . \tag{1.35}$$

The Poincaré–Cartan $\Theta_{\mathscr{L}}$ is the *principal Lepage equivalent* of the Lagrangian density $\mathbf{L}$ (see e.g. [105, 136]); the resulting variational problem is equivalent since $j^k\phi^*\mathbf{L} = j^k\phi^*\Theta_{\mathscr{L}}$, as a simple calculation in coordinates shows. We should note that the principal Lepage equivalent $\Theta_{\mathscr{L}}$ to $\mathbf{L}$ is determined uniquely only up to $k = 2$, relevant for second-order field theories.

1.4 The Infinite-Jet Bundle $J^\infty\pi$

Finally, we define the ***infinite-jet bundle*** $J^\infty\pi$ as the set of infinite-jets

$$J^\infty\pi \equiv \{j^\infty_x\phi : x \in M, \phi \in \Gamma(\pi)\} \ . \tag{1.36}$$

The infinite jet bundle $J^\infty\pi$ has induced local coordinates $(x^\mu, y^A, y^A_\mu, y^A_{\mu_1\mu_2}, \cdots)$ which can be written in terms of the multi-index $\mathbf{J} = \{\varnothing, \mu, \mu_1\mu_2, \cdots\}$ as $(x^\mu, y^A_{\mathbf{J}})$. Jet bundles of finite order lack some geometric features possessed by $J^\infty\pi$.

The exterior derivative on $J^\infty\pi$ splits canonically into horizontal and vertical parts $\mathrm{d} = \mathrm{d_h} + \mathrm{d_v}$ as

$$\mathrm{d_h} = \mathrm{d}x^\mu \otimes \partial_\mu = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + y^A_{\mu\mathbf{J}}\frac{\partial}{\partial y^A_{\mathbf{J}}}\right) \ , \tag{1.37}$$

$$\mathrm{d_v} = \left(\mathrm{d}y^A_{\mathbf{J}} - y^A_{\mu\mathbf{J}}\mathrm{d}x^\mu\right) \otimes \frac{\partial}{\partial y^A_{\mathbf{J}}} = \theta^A_{\mathbf{J}} \otimes \frac{\partial}{\partial y^A_{\mathbf{J}}} \ . \tag{1.38}$$

Now ∂_μ is the total derivative on $J^\infty\pi$ and the forms $\theta^A_{\mathbf{J}} \equiv \mathrm{d}y^A_{\mathbf{J}} - y^A_{\mu\mathbf{J}}\mathrm{d}x^\mu$ are the contact 1-forms on $J^\infty\pi$ and yield a basis for π^∞-vertical differential forms on $J^\infty\pi$ which is called the ***contact basis***. It follows that [6]:

$$j^\infty\phi^*(\mathrm{d_v}f) = 0 \quad , \quad j^\infty\phi^*\left(\mathrm{d_h}f\right) = \mathrm{d}\left(j^\infty\phi^* f\right) \ , \tag{1.39}$$

for some $f \in C^\infty(J^\infty\pi)$.

Furthermore, the exterior derivative on $J^\infty\pi$ is given as,

$$\mathrm{d} = \mathrm{d}x^\mu \otimes \frac{\partial}{\partial x^\mu} + \sum_{k=0}^{\infty} \mathrm{d}y^A_{\mu_1\ldots\mu_k} \otimes \frac{\partial}{\partial y^A_{\mu_1\ldots\mu_k}} = \mathrm{d}x^\mu \otimes \frac{\partial}{\partial x^\mu} + \mathrm{d}y^A_{\mathbf{J}} \otimes \frac{\partial}{\partial y^A_{\mathbf{J}}} \ . \tag{1.40}$$

Taking the exterior derivative on $J^\infty\pi$ to act on an arbitrary function $f \in C^\infty(J^\infty\pi)$ and composing with the jet prolongation $j^\infty\phi$ allows us to write,

$$\mathrm{d}f \circ j^\infty\phi = \left(\mathrm{d}x^\mu \otimes \frac{\partial f}{\partial x^\mu} + \mathrm{d}y^A \otimes \frac{\partial f}{\partial y^A} + \mathrm{d}y^A_\mu \otimes \frac{\partial f}{\partial y^A_\mu} + \cdots\right) \circ j^\infty\phi \ , \tag{1.41}$$

which gives the exterior derivative on the space of holonomic sections $\mathcal{J}^\infty\pi$. Formally, $\mathrm{d}f \circ j^\infty\phi$ is a *differential form along the map* $j^\infty\phi$. In the physics literature, this derivative is often denoted as:

$$\mathrm{d} = \mathrm{d}x^\mu \otimes \partial_\mu = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \phi^A_\mu \frac{\partial}{\partial \phi^A} + \phi^A_{\mu\nu}\frac{\partial}{\partial \phi^A_\nu} + \cdots\right) = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \phi^A_{\mu\mathbf{J}}\frac{\partial}{\partial \phi^A_{\mathbf{J}}}\right) \ . \tag{1.42}$$

This notation makes sense when the exterior derivative on $\mathcal{J}^\infty\pi$ shown above acts on functions given as $\mathcal{F}(\phi) = f \circ j^\infty\phi \in C^\infty(\mathcal{J}^\infty\pi)$.

Finally, we have the ***Ehresmann connection on*** $J^\infty\pi$ which is defined as a type $(1,1)$-tensor field ∇ on $J^\infty\pi$ which satisfies $\nabla^*\alpha = \alpha$ for all π^∞-semibasic forms α on $J^\infty\pi$. Now, the local expression for ∇ is given as

$$\nabla \equiv \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^A_\mu \frac{\partial}{\partial y^A} + \Gamma^A_{\mu\nu}\frac{\partial}{\partial y^A_\nu} + \cdots\right) = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^A_{\mu\mathbf{J}}\frac{\partial}{\partial y^A_{\mathbf{J}}}\right) \ . \tag{1.43}$$

The $\Gamma^A_{\mu\mathbf{J}}$ are now arbitrary functions on $J^\infty\pi$ that can be chosen to be $\Gamma^A_{\mu\mathbf{J}} = y^A_{\mu\mathbf{J}}$, yielding the natural flat connection admitted by $\pi^\infty : J^\infty\pi \to M$ which is locally spanned by the total derivative [8] as $\nabla = \mathrm{d}x^\mu \otimes \partial_\mu$.

We point out to the reader that we will denote both exterior derivatives on $J^\infty\pi$ and M as d and we will specify which of the exterior derivatives is being used; we will also share the notation for ∂_μ on both spaces.

1.5 The Euler–Lagrange Operator

We now introduce the *Euler–Lagrange operator*, relevant in the study of Lagrangian field theories. Other discussions on the Euler–Lagrange operator can be found in [6, 12, 23, 84, 139, 140, 141] for instance.

The ***Euler–Lagrange derivative on*** $J^\infty\pi$ acting on some arbitrary function $f \in C^\infty(J^\infty\pi)$ as,

$$\frac{\delta f}{\delta y^A} \equiv \frac{\partial f}{\partial y^A} - \partial_\mu\left(\frac{\partial f}{\partial y^A_\mu}\right) + \partial_\mu\partial_\nu\left(\frac{\partial f}{\partial y^A_{\mu\nu}}\right) + \cdots = (-1)^{|\mathbf{J}|}\partial_{\mathbf{J}}\left(\frac{\partial f}{\partial y^A_{\mathbf{J}}}\right) \ . \tag{1.44}$$

Furthermore, the ***Euler–Lagrange derivative on*** $\mathcal{J}^\infty\pi$ is defined by composing with $j^\infty\phi$ as

$$\frac{\delta f}{\delta y^A} \circ j^\infty\phi = (-1)^{|\mathbf{J}|}\partial_{\mathbf{J}} \left(\frac{\partial f}{\partial y^A_{\mathbf{J}}} \right) \circ j^\infty\phi \ . \tag{1.45}$$

Then the Euler–Lagrange derivative acts on some arbitrary function $\mathcal{F}(\phi) = f \circ j^\infty\phi \in C^\infty(\mathcal{J}^\infty\pi)$ as

$$\frac{\delta \mathcal{F}}{\delta \phi^A} \equiv (-1)^{|\mathbf{J}|}\partial_{\mathbf{J}} \left(\frac{\partial \mathcal{F}}{\partial \phi^A_{\mathbf{J}}} \right) = \frac{\delta f}{\delta y^A} \circ j^\infty\phi \ . \tag{1.46}$$

Finally, we define the ***Euler–Lagrange operator*** $\hat{E}$ on $J^\infty\pi$ as

$$\hat{E} \equiv \theta^A \otimes \frac{\delta}{\delta y^A} \ . \tag{1.47}$$

Now, since the Euler–Lagrange operator and derivatives are defined on $J^\infty\pi$ (and $\mathcal{J}^\infty\pi$) and, in general, we have the natural projection $\pi^\infty_k : J^\infty\pi \to J^k\pi$, it follows that the Euler–Lagrange operator and derivatives can also be constructed on $J^k\pi$ (and $\mathcal{J}^k\pi$) in general.

Notice that the Euler–Lagrange derivatives involve total derivatives. This means that for a Lagrangian function of order k, its Euler–Lagrange derivative is given on $J^{k+1}\pi$. Later, we will discuss in geometric detail how the *Euler–Lagrange equations*, defined by the vanishing of the Euler–Lagrange derivative of a given Lagrangian function, provide the solutions to the variational problem associated with that Lagrangian.

The submanifold $\Sigma \subset J^{k+1}\pi$ given by

$$\frac{\delta \mathscr{L}}{\delta y^A} \underset{\Sigma}{=} 0 \ , \tag{1.48}$$

for some Lagrangian $\mathscr{L} \in C^\infty(J^k\pi)$ will be called the ***Euler–Lagrange submanifold*** (or ***stationary surface***) on $J^{k+1}\pi$. Furthermore, the corresponding Euler–Lagrange derivative along $\mathcal{J}^{k+1}\pi$ is given as

$$\frac{\delta \mathscr{L}}{\delta \phi^A} = j^{k+1}\phi^* \left(\frac{\delta \mathscr{L}}{\delta y^A} \right) \ , \tag{1.49}$$

where we use the slight abuse of notation $j^{k+1}\phi^*\mathscr{L} = \mathscr{L}$. Moreover, we also have the submanifold $\Sigma_\phi \subset \mathcal{J}^{k+1}\pi$ given by

$$\frac{\delta \mathscr{L}}{\delta \phi^A} \underset{\Sigma_\phi}{=} 0 \ , \tag{1.50}$$

which we call the ***Euler–Lagrange submanifold*** (or ***stationary surface***) on $\mathcal{J}^{k+1}\pi$.

In this thesis, we work exclusively with first-order Lagrangians $\mathscr{L} \in C^\infty(J^1\pi)$ and it thereby follows that the Euler–Lagrange equations are ***second-order differential equations***, yielding the stationary surfaces $\Sigma \subset J^2\pi$ and $\Sigma_\phi \subset \mathcal{J}^2\pi$.

1.6 The Extended and Restricted Multimomentum Bundles

The geometric framework for the *De Donder–Weyl Hamiltonian formulation* of first-order field theories begins by first introducing the ***extended multimomentum bundle*** $\mathcal{M}\pi \equiv \Lambda^m_2\mathrm{T}^*E$ which is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields. This manifold has natural coordinates $(x^\mu,\, y^A, p^\mu_A, p)$, and hence $\dim \mathcal{M}\pi = m + n + mn + 1$. As $\mathcal{M}\pi \equiv \Lambda^m_2\mathrm{T}^*E$ is a sub-bundle of $\Lambda^m\mathrm{T}^*E$, the *multicotangent bundle* of E of order m, it follows that $\mathcal{M}\pi$ is endowed with a canonical form, the "tautological m-form" $\Theta \in \Omega^m(\mathcal{M}\pi)$, which is defined as follows: if $\mathrm{p} \equiv (y, \zeta) \in \mathcal{M}\pi$ with $y \in E$ and $\zeta \in \Lambda^m_2\mathrm{T}^*_yE$, then, for every $X_0, \ldots, X_{m-1} \in \mathrm{T}_{(y,\zeta)}(\mathcal{M}\pi)$,

$$\Theta_{(y,\zeta)}(X_0, \ldots, X_{m-1}) \equiv \zeta(\mathrm{T}_{(y,\zeta)}\widetilde{\sigma}(X_0), \ldots, \mathrm{T}_{(y,\zeta)}\widetilde{\sigma}(X_{m-1})) \ , \tag{1.51}$$

where $\widetilde{\sigma}\colon \mathcal{M}\pi \to E$ is the natural projection. Consequently, $\Omega = -\mathrm{d}\Theta \in \Omega^{m+1}(\mathcal{M}\pi)$ is a multisymplectic form. The local expressions of these forms are,

$$\Theta = p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + p\,\mathrm{d}^m x \quad , \quad \Omega = -\mathrm{d}p^\mu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \mathrm{d}p \wedge \mathrm{d}^m x \ , \tag{1.52}$$

and are known as the ***multimomentum Liouville*** m ***and*** $(m+1)$***-forms*** of $\mathcal{M}\pi$ respectively.

The quotient bundle $J^1\pi^* \equiv \mathcal{M}\pi/\Lambda^m_1(\mathrm{T}^*\mathrm{E})$, where $\Lambda^m_1(\mathrm{T}^*\mathrm{E})$ denotes the bundle of π-semibasic m-forms on E, is called the ***(restricted) multimomentum bundle*** of E and has natural coordinates (x^μ, y^A, p^μ_A).

Remark 1.2 *The multimomentum bundle $J^1\pi^* \to E$ is the affine dual bundle to $J^1\pi \to E$ and is modeled on the vector bundle $\pi^*TM \otimes V^*\pi \to E$ [25].*

Sections $\mathrm{h}\colon J^1\pi^* \to \mathcal{M}\pi$ of the bundle $\sigma : \mathcal{M}\pi \to J^1\pi^*$ are called ***Hamiltonian sections***. Hamiltonian sections can be used to define the forms,

$$\Theta_{\mathscr{H}} \equiv \mathrm{h}^*\Theta \in \Omega^m(J^1\pi^*) \ , \qquad \Omega_{\mathscr{H}} \equiv -\mathrm{d}\Theta_{\mathscr{H}} = \mathrm{h}^*\Omega \in \Omega^{m+1}(J^1\pi^*) \ , \tag{1.53}$$

which are called the ***Hamilton–Cartan*** m ***and*** $(m+1)$ ***forms*** of $J^1\pi^*$ associated with the Hamiltonian section h. A Hamiltonian section is locally specified by

$$\mathrm{h}(x^\mu, y^A, p^\mu_A) = \big(x^\mu, y^A, p^\mu_A, p = -\mathscr{H}(x^\nu, y^B, p^\nu_B)\big) \ , \tag{1.54}$$

where $\mathscr{H}$ is the ***De Donder–Weyl Hamiltonian function***. Then, the local expressions for the Hamilton–Cartan forms are:

$$\Theta_{\mathscr{H}} = p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \mathscr{H}\mathrm{d}^m x \quad , \quad \Omega_{\mathscr{H}} = -\mathrm{d}p^\mu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}\mathscr{H} \wedge \mathrm{d}^m x \ . \tag{1.55}$$

The form $\Omega_{\mathscr{H}}$ is multisymplectic and the pair $(J^1\pi^*, \Omega_{\mathscr{H}})$ is called a ***regular Hamiltonian system***. The natural projections (submersions) are listed as follows,

$$\tau : J^1\pi^* \to E : (x^\mu, y^A, p^\mu_A) \mapsto (x^\mu, y^A) \ , \tag{1.56}$$
$$\overline{\tau} : J^1\pi^* \to M : (x^\mu, y^A, p^\mu_A) \mapsto (x^\mu) \ , \tag{1.57}$$
$$\sigma : \mathcal{M}\pi \to J^1\pi^* : (x^\mu, y^A, p^\mu_A, p) \mapsto (x^\mu, y^A, p^\mu_A) \ , \tag{1.58}$$
$$\widetilde{\sigma} = \tau \circ \sigma \colon \mathcal{M}\pi \to E : (x^\mu, y^A, p^\mu_A, p) \mapsto (x^\mu, y^A) \ , \tag{1.59}$$
$$\overline{\sigma} = \pi \circ \widetilde{\sigma} \colon \mathcal{M}\pi \to M : (x^\mu, y^A, p^\mu_A, p) \mapsto (x^\mu) \ , \tag{1.60}$$

and are depicted in the following diagram:

$$\begin{tikzcd}
\mathcal{M}\pi \arrow[rr, "\sigma"] \arrow[dr, "\widetilde{\sigma}"] \arrow[ddr, "\overline{\sigma}"'] & & J^1\pi^* \arrow[dl, "\tau"'] \arrow[ddl, "\overline{\tau}"] \\
& E \arrow[d, "\pi"] & \\
& M &
\end{tikzcd} \tag{1.61}$$

Furthermore, sections of $\overline{\tau}$ will be denoted as $\psi : M \to J^1\pi^* : x^\mu \mapsto (x^\mu, \phi^A, \pi^\mu_A)$.

1.7 Locality

This section is devoted to discussing the notion of *locality* in precise mathematical terms. The terminology found in this section differs from what is often used in the mathematics literature and instead follows the terminology found in the physics literature, such as in [12]. Locality is a fundamental concept in field theory which originates from the treatment of fields and their derivatives as independent variables in the calculus of variations; namely, the equations of motion, field transformations, constraints, and gauge fixing conditions are expected to depend on the fields and their derivatives at each spacetime (or worldvolume) point. The jet bundle formulation of field theories provides us with a geometric origin for discussing locality.

A function $f \in C^\infty(U)$, $U \subset J^1\pi$, is called a ***local function***. Local functions satisfy the following lemma [12]:

Lemma 1.1 *Given a local function $f \in C^\infty(J^\infty\pi)$, it follows that*

$$\frac{\delta f}{\delta y^A} = 0 \quad \Leftrightarrow \quad f = \partial_\mu j^\mu \ , \tag{1.62}$$

for some $j^\mu \in C^\infty(J^\infty\pi)$.

A ***local form*** is a semi-basic on $J^\infty\pi$; therefore, a local p-form is written in local coordinates as,

$$\omega^p = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \ , \tag{1.63}$$

where the $\omega_{\mu_1\cdots\mu_p}$ are local functions. Such differential forms are also known as *horizontal forms* as discussed later in Section 14.

Furthermore, a ***local functional*** $F[\phi]$ is defined as the base space integral of a local function, given by the following map:

$$F[\phi] : \mathcal{J}^\infty\pi \to \mathbb{R} : j^\infty\phi \mapsto \int_M \mathrm{d}^m x \ j^\infty\phi^* f \ . \tag{1.64}$$

Note the following lemma [12]:

Lemma 1.2 *Given two local functionals $F[\phi]$ and $G[\phi]$, it follows that*

$$F[f,\phi] = G[g,\phi] \quad \Leftrightarrow \quad f - g = \partial_\mu j^\mu \quad \textit{i.e.} \quad \frac{\delta}{\delta y^A}(f-g) = 0 \ . \tag{1.65}$$

Locality requires that the equations of motion, field transformations, constraints, and gauge fixing conditions of some field theory are given in terms of local functions composed with jet prolongations $j^\infty\phi$. We then usually integrate over spacetime, or some hypersurface in order to obtain conserved charges or energy, for example. This means that the functions on the space of sections $\Gamma(\pi^\infty)$ which we use are those which are given by the pullback space of $j^\infty\phi$; that is, functions on $\mathcal{J}^\infty\pi$.

2 Multisymplectic Field Theory

We now turn our attention to the variational calculus of *regular* first-order field theories.

The original work on the multisymplectic formulation of classical field theories are the following: [4, 98, 99]. See [2, 46, 47, 54, 62, 131] for previous work on the multisymplectic Lagrangian formalism and [25, 62, 67] for more on the multisymplectic De Donder–Weyl Hamiltonian formalism for first-order field theories.

Regular field theories are naturally formulated on $J^1\pi$ and $J^1\pi^*$, both of which are *multisymplectic manifolds* when the field theory under investigation is regular. Given a differentiable manifold F, a differential form $\Omega \in \Omega^k(F)$ $(k \geq 2)$ is 1-***nondegenerate*** if, for every $\mathrm{p} \in F$ and $Y \in \mathfrak{X}(F)$, it follows that $i(Y)\Omega|_{\mathrm{p}} = 0 \iff Y|_{\mathrm{p}} = 0$. A differential form $\Omega \in \Omega^k(F)$ is called a ***multisymplectic form*** if it is closed and 1-nondegenerate and (F, Ω) is then called a ***multisymplectic manifold***. Furthermore, when Ω is closed and 1-degenerate, it is called a ***premultisymplectic form*** and (F, Ω) is called a ***premultisymplectic manifold***.

We show in detail how the jet bundle $J^1\pi$ is used to carry out the variational calculus in the Lagrangian formalism while the De Donder–Weyl Hamiltonian formalism takes place on the affine dual $J^1\pi^*$. It is also worth noting that in the case where M is taken to be $(0+1)$-dimensional, the entire geometric structure reduces to that of time-dependent mechanics, where the configuration bundle is (locally) given as $\pi : \mathbb{R} \times Q \to \mathbb{R}$ and the presymplectic geometry of $J^1\pi = \mathbb{R} \times TQ$ and $J^1\pi^* = \mathbb{R} \times T^*Q$ is used to carry out the variational calculus [28, 38].

Now, consider the general geometrical setting that consists of the fiber bundle $\kappa\colon F \to M$ with sections $\psi_\kappa : M \to F$, where $F = J^1\pi$ in the Lagrangian formulation and $F = J^1\pi^*$ in the De Donder–Weyl Hamiltonian formulation. The corresponding multisymplectic form $\Omega \in \Omega^{m+1}(F)$ must satisfy the so-called ***variational condition*** given as

$$i(Y_1)\, i(Y_2)\, i(Y_3)\Omega = 0 \ , \tag{2.1}$$

for all κ-vertical vector fields $Y_1, Y_2, Y_3 \in \mathfrak{X}^{\mathrm{V}(\kappa)}(F)$, in order to ensure that the geometric formulation of the standard variational calculus is well posed.

The characterization of solutions to the variational problem will be given in terms of multivector fields. We remind the reader to see Appendix A for required definitions and properties involving multivector fields.

2.1 Lagrangian Formalism

Given a Lagrangian $\mathbf{L} = \mathscr{L}(x^\mu, y^A, y^A_\mu)\mathrm{d}^m x \in \Omega^m(J^1\pi)$, the ***De Donder–Weyl Lagrangian energy function*** $E_{\mathscr{L}} \in C^\infty(J^1\pi)$ is defined as

$$E_{\mathscr{L}} \equiv \frac{\partial \mathscr{L}}{\partial y^A_\mu} y^A_\mu - \mathscr{L} \ . \tag{2.2}$$

The Lagrangian is said to be ***regular*** when the generalized Hessian matrix

$$H^{\mu\nu}_{AB} = \frac{\partial^2 \mathscr{L}}{\partial y^A_\mu \partial y^B_\nu} \tag{2.3}$$

is non-singular everywhere. We now rewrite the Poincaré–Cartan m-form $\Theta_{\mathscr{L}} \in \mathfrak{X}^m(J^1\pi)$ as

$$\Theta_{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial y^A_\mu} \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - E_{\mathscr{L}} \mathrm{d}^m x \ , \tag{2.4}$$

and the Poincaré–Cartan $(m+1)$-form $\Omega_{\mathscr{L}} \in \mathfrak{X}^{m+1}(J^1\pi)$ as

$$\Omega_{\mathscr{L}} = -\mathrm{d}\Theta_{\mathscr{L}} = -\mathrm{d}\left(\frac{\partial \mathscr{L}}{\partial y^A_\mu}\right) \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}E_{\mathscr{L}} \wedge \mathrm{d}^m x \ . \tag{2.5}$$

The field equations for this system are obtained from a variational action principle posed on $J^1\pi$. The *action* is a functional on the set of sections $\Gamma(\pi^1)$ given by $\Gamma(M, J^1\pi) \to \mathbb{R} : j^1\phi \mapsto \int_M (j^1\phi)^*\Theta_{\mathscr{L}}$ and is denoted simply as,

$$S[\phi] = \int_M (j^k\phi)^*\mathbf{L} = \int_M (j^1\phi)^*\Theta_{\mathscr{L}} \ . \tag{2.6}$$

The variational problem on $J^1\pi$ requires that we find the critical (stationary) sections of the action:

$$\delta S = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_M (j^1\phi)_s^*\mathbf{L} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_M (j^1\phi)_s^*\Theta_{\mathscr{L}} = 0 \ , \tag{2.7}$$

for variations $j^1\phi_s = \eta_s \circ j^1\phi$ of $j^1\phi$, where η_s is the flow of a π^1-vertical vector field on E compactly supported on M. Critical sections are characterized as follows:

$j^1\phi$ is a holonomic integral section of a class of integrable, locally decomposable, π^1-transverse multivector fields $\{\mathbf{X}_{\mathscr{L}}\} \subset \mathfrak{X}^m(J^1\pi)$ (that is, a *holonomic multivector field*) which satisfy

$$i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = 0 \quad , \quad \text{for every } \mathbf{X}_{\mathscr{L}} \in \{\mathbf{X}_{\mathscr{L}}\} \ . \tag{2.8}$$

Furthermore, we choose a representative of the class $\{\mathbf{X}_{\mathscr{L}}\}$ which is a normalized and π^1-transverse multivector field so that

$$i(\mathbf{X}_{\mathscr{L}})\mathrm{d}^m x = 1 \ . \tag{2.9}$$

The local expression for $\mathbf{X}_{\mathscr{L}}$ is given as,

$$\mathbf{X}_{\mathscr{L}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + H_\mu^A \frac{\partial}{\partial y^A} + H_{\mu\nu}^A \frac{\partial}{\partial y_\nu^A} \right) \ , \tag{2.10}$$

and the fact that this multivector field is holonomic ensures the SOPDE *condition* $H_\mu^A = y_\mu^A$ is satisfied. Then, it follows from equation (2.8) that the holonomic integral sections of $\mathbf{X}_{\mathscr{L}}$ is

$$i\left((j^1\phi)^{(m)}\right)(\Omega_{\mathscr{L}} \circ j^1\phi) = 0 \ , \tag{2.11}$$

which are the field equations. In the Lagrangian setting, these are in fact the ***Euler–Lagrange equations*** expressed as:

$$\frac{\partial \mathscr{L}}{\partial y^A} \circ j^1\phi - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial y_\mu^A} \circ j^1\phi \right) = 0 \ . \tag{2.12}$$

The field equations can be characterized equivalently as $(j^1\phi)^*\, i(X)\Omega_{\mathscr{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$; see also [46, 54, 62] for other equivalent statements.

Remark 2.1 *The variation of the action* (2.7) *for a first–order field theory is given explicitly as,*

$$\begin{aligned}
\delta S &= \int_M \mathrm{d}^m x \otimes \left\{ \left[\frac{\partial \mathscr{L}}{\partial \phi^A} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \phi_\mu^A} \right) \right] \delta\phi^A + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \phi_\mu^A} \delta\phi^A \right) \right\} \\
&= \int_M \left[\hat{E}(\boldsymbol{L} \circ j^1\phi) + \mathrm{d}^m x \otimes \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \phi_\mu^A} \delta\phi^A \right) \right] \\
&= \int_M j^1\phi * \left(\hat{E}(\boldsymbol{L}) - \mathrm{d_h}\theta_{\mathscr{L}} \right) \ ,
\end{aligned} \tag{2.13}$$

where,

$$\theta_{\mathscr{L}} \equiv \frac{\partial \mathscr{L}}{\partial y_\mu^A} \theta^A \wedge \mathrm{d}^{m-1}x_\mu = \frac{\partial \mathscr{L}}{\partial y_\mu^A} \mathrm{d_v} y^A \wedge \mathrm{d}^{m-1}x_\mu \ . \tag{2.14}$$

It therefore follows that (see [152])

$$\mathrm{d_v}\boldsymbol{L} = \hat{E}(\boldsymbol{L}) - \mathrm{d_h}\theta \quad , \quad \mathrm{d}\boldsymbol{L} = \hat{E}(\boldsymbol{L}) + \mathrm{d_h}\Theta_{\mathscr{L}} \quad . \tag{2.15}$$

2.2 De Donder–Weyl Hamiltonian Formalism

See, for instance, [25, 37, 48, 67, 81, 131] for more details on the multisymplectic analysis of the De Donder–Weyl Hamiltonian formalism.

The field equations are obtained from a variational principle on $J^1\pi^*$ in which the *action* is a functional on the set of sections $\Gamma(\overline{\tau})$ given by $\Gamma(M, J^1\pi^*) \to \mathbb{R} : \psi \mapsto \displaystyle\int_M \psi^*\Theta_{\mathscr{H}}$. The objective is to find sections that are critical (stationary),

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \int_M \psi_s^*\Theta_{\mathscr{H}} = 0 \ , \tag{2.16}$$

for variations $\psi_s = \eta_s \circ \psi$ of ψ, where η_s is the flow of a $\overline{\tau}$-vertical vector field on $J^1\pi^*$ compactly supported on M. Critical sections are characterized as follows:

ψ is an integral section of a class of integrable, locally decomposable, $\overline{\tau}$-transverse multivector fields $\{\mathbf{X}_{\mathscr{H}}\} \subset \mathfrak{X}^m(J^1\pi^*)$ which satisfy

$$i(\mathbf{X}_{\mathscr{H}})\Omega_{\mathscr{H}} = 0 \quad , \quad \text{for every } \mathbf{X}_{\mathscr{H}} \in \{\mathbf{X}_{\mathscr{H}}\} \ ; \tag{2.17}$$

then, if ψ is an integral section of $\mathbf{X}_{\mathscr{H}}$, the relation $\psi^{(m)} = \mathbf{X}_{\mathscr{H}} \circ \psi$ holds (see Appendix A). Furthermore, it is possible to choose a representative of the class $\{\mathbf{X}_{\mathscr{H}}\}$ which is a normalized, $\overline{\tau}$-transverse, multivector field such that

$$i(\mathbf{X}_{\mathscr{H}})\mathrm{d}^m x = 1 \ . \tag{2.18}$$

The local expression for $\mathbf{X}_{\mathscr{H}}$ is given as,

$$\mathbf{X}_{\mathscr{H}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + H_\mu^A \frac{\partial}{\partial y^A} + H_{A\mu}^\nu \frac{\partial}{\partial p_A^\nu} \right) \ . \tag{2.19}$$

Them, the field equations which follow from equation (2.17) for the integral sections of $\mathbf{X}_{\mathscr{H}}$ are

$$i(\psi^{(m)})(\Omega_{\mathscr{H}} \circ \psi) = 0 \ . \tag{2.20}$$

Now, these are the ***Hamilton–De Donder–Weyl equations*** expressed as:

$$\partial_\mu(y^A \circ \psi) = \frac{\partial \mathscr{H}}{\partial p_A^\mu} \circ \psi \quad , \quad \partial_\mu(p_A^\mu \circ \psi) = -\frac{\partial \mathscr{H}}{\partial y^A} \circ \psi \ . \tag{2.21}$$

Once again, the field equations can equivalently be characterized as $\psi^* i(X)\Omega_{\mathscr{H}} = 0$, for every $X \in \mathfrak{X}(J^1\pi^*)$. See also [49, 131] for other equivalent statements.

The ***extended Legendre map***, $\widetilde{\mathscr{F}\mathscr{L}}\colon J^1\pi \to \mathcal{M}\pi$, and the ***(restricted) Legendre map***, $\mathscr{F}\mathscr{L} \equiv \sigma \circ \widetilde{\mathscr{F}\mathscr{L}}\colon J^1\pi \to J^1\pi^*$, associated with a Lagrangian $\mathscr{L} \in C^\infty(J^1\pi)$ are given locally as

$$\begin{aligned}
&\widetilde{\mathscr{F}\mathscr{L}}^* x^\mu = x^\mu \quad , \quad \widetilde{\mathscr{F}\mathscr{L}}^* y^A = y^A \quad , \quad \widetilde{\mathscr{F}\mathscr{L}}^* p_A^\mu = \frac{\partial \mathscr{L}}{\partial y_\mu^A} \quad , \quad \widetilde{\mathscr{F}\mathscr{L}}^* p = \mathscr{L} - \frac{\partial \mathscr{L}}{\partial y_\mu^A} y_\mu^A \\
&\mathscr{F}\mathscr{L}^* x^\mu = x^\mu \quad , \quad \mathscr{F}\mathscr{L}^* y^A = y^A \quad , \quad \mathscr{F}\mathscr{L}^* p_A^\mu = \frac{\partial \mathscr{L}}{\partial y_\mu^A} \ .
\end{aligned}$$

The Lagrangian function is regular if, and only if, $\mathscr{F}\mathscr{L}$ is a local diffeomorphism. As a particular case, $\mathscr{L}$ is ***hyper-regular*** if $\mathscr{F}\mathscr{L}$ is a global diffeomorphism. It is worth noting that the regularity of a field theory is a quite demanding condition that is not met by many of the important physical field theories, e.g. Yang-Mills and General Relativity among others.

In the (hyper-)regular case, the map $\mathrm{h} \equiv \widetilde{\mathscr{F}\mathscr{L}} \circ \mathscr{F}\mathscr{L}^{-1}$ is a Hamiltonian section which is associated with $\mathscr{L}$ and, as $E_{\mathscr{L}} = \dfrac{\partial \mathscr{L}}{\partial y^A_\mu} y^A_\mu - \mathscr{L}$, the De Donder–Weyl Hamiltonian function is given as,

$$\mathscr{H} = (\mathscr{F}\mathscr{L}^{-1})^* E_{\mathscr{L}} = p^\mu_A (\mathscr{F}\mathscr{L}^{-1})^* y^A_\mu - (\mathscr{F}\mathscr{L}^{-1})^* \mathscr{L} \in C^\infty(J^1\pi^*) \ . \tag{2.22}$$

The following diagram illustrates the relations between the multisymplectic spaces discussed thus far:

$$\begin{tikzcd} & \mathcal{M}\pi \arrow[d, "\sigma"'] \\ J^1\pi \arrow[ur, "\widetilde{\mathscr{F}\mathscr{L}}"] \arrow[r, "\mathscr{F}\mathscr{L}"'] & J^1\pi^* \arrow[u, bend right, "\mathrm{h}"'] \end{tikzcd} \tag{2.23}$$

It follows that $\mathscr{F}\mathscr{L}^* \Theta_{\mathscr{H}} = \Theta_{\mathscr{L}}$ and $\mathscr{F}\mathscr{L}^* \Omega_{\mathscr{H}} = \Omega_{\mathscr{L}}$.

The pair $(J^1\pi^*, \Omega_{\mathscr{H}})$ is the *Hamiltonian system associated with the regular Lagrangian system* $(J^1\pi, \Omega_{\mathscr{L}})$.

3 Premultisymplectic Field Theory

We now turn our attention to field theories in which the Legendre transform $\mathscr{F}\mathscr{L} : J^1\pi \to J^1\pi^*$ is not invertible; such field theories are said to be ***singular***. Equivalently, field theories are singular when the corresponding multi-Hessian (2.3) is singular. The first formal geometric treatment of singular field theories in terms of premultisymplectic geometry was in [37]. Furthermore, see [39] for the development of the geometric constraint algorithm. Here we present new lemmas and propositions which further describe the premultisymplectic constraint structure; the new results we present in the section were published in [63].

Understanding singular theories is crucial, as they include most field theories of physical interest such as Yang–Mills and General Relativity. In order to ensure the existence of an equivalent De Donder–Weyl Hamiltonian formalism and to have control of the constraint structure arising from the variational problem, the singular theory under investigation must be sufficiently well behaved [37]. In particular, we require that the Legendre transform falls under the following definition:

Definition 3.1 *A field theory is said to be* ***almost-regular*** *if the following properties are satisfied:*

1. $P_\circ \equiv \mathrm{Im}\mathscr{F}\mathscr{L}$ *is an embedded submanifold of* $\jmath_\circ \colon P_\circ \hookrightarrow J^1\pi^*$*,*
2. $\mathscr{F}\mathscr{L}$ *is a submersion onto its image, and*
3. *for every* $\mathrm{p} \in J^1\pi$*, the fibers* $\mathscr{F}\mathscr{L}^{-1}(\mathscr{F}\mathscr{L}(\mathrm{p}))$ *are connected submanifolds of* $J^1\pi$*.*

A more restrictive definition of almost-regularity which implies the one given above is considered in [37] by demanding that: (i) $\widetilde{P}_\circ \equiv \mathrm{Im}\widetilde{\mathscr{F}\mathscr{L}}$ is an embedded submanifold $\widetilde{\jmath}_\circ \colon \widetilde{P}_\circ \hookrightarrow \mathcal{M}\pi$, (ii) $\widetilde{\mathscr{F}\mathscr{L}}$ is a submersion onto its image, and (iii) for every $\mathrm{p} \in J^1\pi$, the fibers $\mathscr{F}\mathscr{L}^{-1}(\mathscr{F}\mathscr{L}(\mathrm{p}))$ and $\widetilde{\mathscr{F}\mathscr{L}}^{-1}(\widetilde{\mathscr{F}\mathscr{L}}(\mathrm{p}))$ are connected submanifolds of $J^1\pi$. Both definitions are locally equivalent. Other alternative definitions are also considered in [59, 134].

An algorithmic method to find all constraints of an almost-regular field theory is needed in order to understand its full dynamics and ensure the existence of a stable solution to the variational problem. One of the most important geometric characteristics of almost-regular field theories is that the corresponding manifolds $J^1\pi$, $P_\circ \subset J^1\pi^*$, and $\widetilde{P}_\circ \subset \mathcal{M}\pi$ are all premultisymplectic. The geometric algorithm which gives the intrinsic global description of the constraint structure in premultisymplectic systems is developed in detail in [39] using the multivector field characterization of the variational principle and is given as a generalization of the algorithms in time-dependent singular mechanics arising from the presymplectic geometry of $J^1\pi = \mathbb{R}\times TQ$ and $J^1\pi^* = \mathbb{R}\times T^*Q$ [28, 38]. Here we give the corresponding geometric constraint algorithm in terms of local coordinates as opposed to the intrinsic global description given in [39].

Now, consider the general geometrical setting that consists of the fiber bundle $\kappa\colon F \to M$ with sections $\psi_\kappa : M \to F$, where $F = J^1\pi$ in the Lagrangian formulation and $F = P_\circ \subset J^1\pi^*$ in the De Donder–Weyl Hamiltonian formulation. The procedure for obtaining the constraint submanifolds of F terminates at some final constraint submanifold $\mathcal{C}_f \hookrightarrow F$ when the theory under investigation is a ***soluble constrained system***. The premultisymplectic form Ω must satisfy the variational condition (2.1) as in the regular case. Furthermore, the constraint analysis arises from the variational problem whose objective is now (in general terms) to find a final constraint submanifold $\mathcal{C}_f \hookrightarrow F$ and an integrable, locally decomposable m-vector field $\mathbf{X} \in \mathfrak{X}^m(F)$ on $\mathcal{C}_f$ such that

$$i(\mathbf{X})\Omega|_{\mathcal{C}_f} = 0 \ , \tag{3.1}$$

where $\mathbf{X}$ satisfies the normalized κ-transverse condition $i(\mathbf{X})\omega|_{\mathcal{C}_f} = 1$, where ω is the volume form lifted from M to F.

The algorithmic procedure consists of the following steps:

1. ***Compatibility conditions***: Compute

$$i(\mathbf{X})\Omega = 0 \ , \tag{3.2}$$

   and identify any equations,

$$\zeta_1 \underset{\mathcal{C}_1}{=} 0 \ , \tag{3.3}$$

   called ***compatibility constraints*** which define the set $\mathcal{C}_1 \subset F$ assumed to be an embedded submanifold submanifold $\mathcal{C}_1 \hookrightarrow F$.

2. ***Tangency conditions***: Now, given a compatibility constraint submanifold $\mathcal{C}_1$, ensure the stability of solutions to equation (3.2) by imposing the ***stability*** or ***tangency condition*** by requiring that all vector field components of $\mathbf{X}$ are tangent to $\mathcal{C}_1$; that is, given a multivector field $\mathbf{X} = \bigwedge_{\mu=0}^{m-1} X_\mu$ which solve (3.2), ensure that

$$\mathrm{L}(X_\mu)\zeta_1|_{\mathcal{C}_1} = 0 \ . \tag{3.4}$$

   This may produce a new constraint submanifold $\mathcal{C}_2 \hookrightarrow \mathcal{C}_1 \hookrightarrow F$; if this is the case, then it is necessary to impose the tangency of $\mathbf{X}$ to $\mathcal{C}_2$ as well. This procedure is repeated until it is finalized when no new constraints are produced, which occurs only for soluble constrained systems. Then, the *final constraint submanifold* $\mathcal{C}_f \hookrightarrow \ldots \mathcal{C}_1 \hookrightarrow F$ is found, for which $\mathbf{X}$ is everywhere tangent to $\mathcal{C}_f$.

3.1 Lagrangian Formalism

The Lagrangian constraint analysis arises from the variational problem whose objective is now to find a final constraint submanifold $\mathcal{S}_f \hookrightarrow J^1\pi$ and an integrable, locally decomposable m-vector field $\mathbf{X}_{\mathscr{L}} \in \mathfrak{X}^m(J^1\pi)$ on $\mathcal{S}_f$ such that

$$i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}}|_{\mathcal{S}_f} = 0 \ , \tag{3.5}$$

where $\mathbf{X}_{\mathscr{L}}$ satisfies the normalized π^1-transverse condition $i(\mathbf{X}_{\mathscr{L}})\omega|_{\mathcal{S}_f} = 1$ and the SOPDE condition $H^A_\mu = y^A_\mu$ (that is, $\mathbf{X}_{\mathscr{L}}$ is a *holonomic multivector field*), recalling the local expression for $\mathbf{X}_{\mathscr{L}}$ given by equation (2.10) as,

$$\mathbf{X}_{\mathscr{L}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + H^A_\mu \frac{\partial}{\partial y^A} + H^A_{\mu\nu} \frac{\partial}{\partial y^A_\nu} \right) \ . \tag{3.6}$$

However, such multivector fields are not, in general, holonomic and it is therefore sometimes necessary to impose this condition *ad hoc*. Imposing the *holonomy (*SOPDE*) condition* may give rise to additional constraints called SOPDE ***constraints***.

The steps of the constraint algorithm in the Lagrangian setting are as follows:

1. Identify from the field equations any compatibility constraints which specify the compatibility constraint submanifold $\mathcal{S}_1 \hookrightarrow J^1\pi$. In the Lagrangian setting for first-order field theories, the compatibility constraints yield first-order partial differential equations for the fields when pulled back by holonomic sections $j^1\phi$.

2. Impose the SOPDE condition $H^A_\mu = y^A_\mu$ if necessary. This may produce a SOPDE ***constraint submanifold*** $\mathcal{S}_2 \hookrightarrow \mathcal{S}_1 \hookrightarrow J^1\pi$.

3. Impose stability (tangency) of solutions $\mathbf{X}_{\mathscr{L}} = \bigwedge_{\mu=0}^{m-1} X_\mu$ to the variational problem by requiring
$$\mathrm{L}(X_\mu)\zeta_i|_{\mathcal{S}_i} = 0 \tag{3.7}$$
for all constraints ζ_i which arise, until no new constraints are found and the algorithm terminates (for soluble systems) at the final constraint submanifold $\mathcal{S}_f \hookrightarrow \cdots \hookrightarrow \mathcal{S}_2 \hookrightarrow \mathcal{S}_1 \hookrightarrow J^1\pi$.

3.2 De Donder–Weyl Hamiltonian Formalism

The constraint analysis in the De Donder–Weyl Hamiltonian setting is a bit for subtle as we first need to identify the premultisymplectic $(m+1)$-form on $P_\circ \subset J^1\pi^*$ where singular De Donder–Weyl Hamiltonian systems are defined. For almost-regular systems, $\widetilde{\jmath}_\circ \colon \widetilde{P}_\circ \hookrightarrow \mathcal{M}\pi$ is a $\widetilde{\sigma}$-transverse embedded submanifold (recall diagram (1.61)) and $\jmath_\circ \colon P_\circ \hookrightarrow J^1\pi^*$ is a τ-transverse embedded submanifold. Furthermore, $P_\circ$ is a fibered submanifold over E and M with natural projections

$$\tau_\circ \equiv \tau \circ \jmath_\circ \colon P_\circ \to E \quad , \quad \overline{\tau}_\circ \equiv \pi \circ \tau_\circ \colon P_\circ \to M \ . \tag{3.8}$$

We now have $\mathscr{F}\mathscr{L}_\circ$ which is the restriction of $\mathscr{F}\mathscr{L}$ onto its image $P_\circ$, given as $\mathscr{F}\mathscr{L} = \jmath_\circ \circ \mathscr{F}\mathscr{L}_\circ$. The full geometric structure of almost-regular field theories is illustrated in the following diagram:

$$\begin{tikzcd}
 & \widetilde{P}_\circ \arrow[rr, "\widetilde{\jmath}_\circ"] \arrow[d, "\widetilde{\sigma}_\circ"] & & \mathcal{M}\pi \arrow[d, "\sigma"'] \\
J^1\pi \arrow[ur, "\widetilde{\mathscr{F}\mathscr{L}}_\circ"] \arrow[r, "\mathscr{F}\mathscr{L}_\circ"'] \arrow[ddrr, "\pi^1"'] & P_\circ \arrow[u, bend left, "\widetilde{\mathrm{h}}"] \arrow[urr, "\mathrm{h}_\circ"] \arrow[rr, "\jmath_\circ"'] \arrow[dr, "\tau_\circ"] \arrow[ddr, "\overline{\tau}_\circ"'] & & J^1\pi^* \arrow[u, bend right, "\mathrm{h}"'] \arrow[dl, "\tau"'] \arrow[ddl, "\overline{\tau}"] \\
 & & E \arrow[d, "\pi"] & \\
 & & M &
\end{tikzcd}$$

The restriction of the projection σ to $\widetilde{P}_\circ$ is the (local) diffeomorphism $\widetilde{\sigma}_\circ \colon \widetilde{P}_\circ \to P_\circ$. Now let $\widetilde{\mathrm{h}} \equiv \widetilde{\sigma}_\circ^{-1}$ gives a Hamiltonian section

$$\mathrm{h}_\circ = \widetilde{\jmath}_\circ \circ \widetilde{\mathrm{h}} = \mathrm{h} \circ \jmath_\circ : P_\circ \to \mathcal{M}\pi \ . \tag{3.9}$$

The corresponding Hamilton–Cartan forms on $P_\circ$ are defined as

$$\Theta^\circ_{\mathscr{H}} = \mathrm{h}_\circ^*\Theta \in \varOmega^m(P_\circ) \quad , \quad \Omega^\circ_{\mathscr{H}} = -\mathrm{d}\Theta^\circ_{\mathscr{H}} = \mathrm{h}_\circ^*\Omega \in \varOmega^{m+1}(P_\circ) \ . \tag{3.10}$$

It follows that $\mathscr{F}\mathscr{L}_\circ^*\Theta^\circ_{\mathscr{H}} = \Theta_{\mathscr{L}}$ and $\mathscr{F}\mathscr{L}_\circ^*\Omega^\circ_{\mathscr{H}} = \Omega_{\mathscr{L}}$. Furthermore, there exists a De Donder–Weyl Hamiltonian function $\mathscr{H}_\circ \in C^\infty(P_\circ)$ such that $E_{\mathscr{L}} = \mathscr{F}\mathscr{L}_\circ^*\mathscr{H}_\circ$. It is necessary to use a connection on the bundle $\pi \colon E \to M$ in order to define the De Donder–Weyl Hamiltonian function globally [25, 46, 48]. However, the De Donder–Weyl Hamiltonian function can, in general, be defined locally without using a fiber bundle connection by using the trivial connection induced on the local chart.

The form $\Omega^\circ_{\mathscr{H}}$ is, in general, a premultisymplectic form and thus the pair $(P_\circ, \Omega^\circ_{\mathscr{H}})$ is the ***singular Hamiltonian system*** which is associated with the singular Lagrangian system $(\Omega_{\mathscr{L}}, J^1\pi)$. The De Domnder–Weyl Hamiltonian field equations are obtained from the variational principle for sections $\Gamma(\overline{\tau}_\circ)$ similarly to the regular case. The analog to the field equations (2.17) read

$$i(\mathbf{X}^\circ_{\mathscr{H}})\Omega^\circ_{\mathscr{H}} = 0 \quad , \quad \text{for every } \mathbf{X}^\circ_{\mathscr{H}} \in \{\mathbf{X}^\circ_{\mathscr{H}}\} \subset \mathfrak{X}^m(P_\circ) \ ; \tag{3.11}$$

and the corresponding field equation for the integral sections $\psi_\circ \colon M \to P_\circ$ of $\mathbf{X}^\circ_{\mathscr{H}}$ is

$$i\left(\psi_\circ^{(m)}\right)(\Omega^\circ_{\mathscr{H}} \circ \psi_\circ) = 0 \ . \tag{3.12}$$

Now, soluble Hamiltonian systems admit a final constraint submanifold $P_f \hookrightarrow P_\circ$ on which we have consistent solutions to the variational problem posed on $P_\circ$ (see [39, 63]). Now the constraint algorithm is carried out as follows:

1. The field equations may yield some *compatibility constraints* which specify the *compatibility constraint submanifold* $P_1 \hookrightarrow P_\circ$.
2. Impose stability (tangency) of solutions $\mathbf{X}^\circ_{\mathscr{H}} = \bigwedge_{\mu=0}^{m-1} X_\mu$ to the variational problem by requiring
$$\mathrm{L}(X_\mu)\zeta_i|_{P_i} = 0 \tag{3.13}$$
for all constraints ζ_i which arise, until no new constraints are found and the algorithm terminates (for soluble systems) at the final constraint submanifold $P_f \hookrightarrow \cdots \hookrightarrow P_1 \hookrightarrow P_\circ$.

3.3 Some Properties of Constraints

Now that the geometric constraint algorithm has been described in detail, it is possible to establish various properties of constraints by means of a local analysis. The first of such properties presented here concerns compatibility constraints. As a consequence of the variational condition (2.1), the premultisymplectic form can be expressed as the following splitting:

$$\Omega = \widetilde{\Omega} + \alpha \wedge \omega \ , \tag{3.14}$$

where $\widetilde{\Omega} \in \varOmega^{m+1}(F)$ and $\alpha \in \varOmega^1(F)$ are closed forms and ω is the volume form naturally lifted from M. In order to perform such a splitting globally, it is necessary to use an Ehresmann connection on the bundle $F \to M$ (although all the results are independent of this choice) [39, 38, 46]. However, since the field theories in this work are studied only locally, we will always use the connection on each local chart $U_F \subset F$ (with $\dim F = N+m$) induced by the natural connection on $\mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m$. It thereby follows that, in the Lagrangian and the Hamiltonian formulations of classical field theories, the (pre)multisymplectic forms are split in the following manner:

$$\Omega_{\mathscr{L}} = \widetilde{\Omega}_{\mathscr{L}} + \alpha_{\mathscr{L}} \wedge \omega \in \Omega^{m+1}(J^1\pi) \Rightarrow \widetilde{\Omega}_{\mathscr{L}} = -\mathrm{d}\left(\frac{\partial \mathscr{L}}{\partial y^A_\mu}\right) \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu \ , \tag{3.15}$$

$$\Omega^\circ_{\mathscr{H}} = \widetilde{\Omega}_{\mathscr{H}} + \alpha_{\mathscr{H}} \wedge \omega \in \Omega^{m+1}(P_\circ) \Rightarrow \widetilde{\Omega}_{\mathscr{H}} = -\mathrm{dp}^\mu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu \ , \tag{3.16}$$

where $\alpha_{\mathscr{L}} \wedge \omega = \mathrm{d}E_{\mathscr{L}} \wedge \mathrm{d}^m x$ and $\alpha_{\mathscr{H}} \wedge \omega = \mathrm{d}\mathscr{H}_\circ \wedge \mathrm{d}^m x$. As a consequence of splitting the premultisymplectic forms as shown above, the following lemma and subsequent propositions hold:

Lemma 3.1 *If $\mathbf{X} \in \mathfrak{X}^m(U_F)$ is a locally decomposable, κ-transverse multivector field on $U_F \subset F$, then*

$$i(\mathbf{X})\Omega = i(\mathbf{X})\widetilde{\Omega} + \widetilde{\alpha}_{\mathbf{X}} + (-1)^m f\,\alpha \ , \tag{3.17}$$

where $f = i(\mathbf{X})\omega \in \mathrm{C}^\infty(U_F)$ is a nonvanishing function and $\widetilde{\alpha}_{\mathbf{X}} \in \Omega^1(U_F)$ is a κ-semibasic 1-form (that is, $i(Y)\widetilde{\alpha}_{\mathbf{X}} = 0$, for every $Y \in \mathfrak{X}^{\mathrm{V}(\kappa)}(F)$).

(*Proof*) As $\mathbf{X} = X_0 \wedge \cdots \wedge X_{m-1} \in U_F$, it follows from (3.14) that

$$\begin{aligned} i(\mathbf{X})\Omega &= i(\mathbf{X})\widetilde{\Omega} + i(\mathbf{X})(\alpha \wedge \omega) \\ &= i(\mathbf{X})\widetilde{\Omega} + \sum_{\mu=0}^{m-1}(-1)^{m-1}\, i(X_\mu)\alpha\ i(X_0 \wedge \ldots \wedge X_{\mu-1} \wedge X_{\mu+1} \wedge \ldots \wedge X_{m-1})\omega + (-1)^m \alpha\ i(\mathbf{X})\omega \\ &= i(\mathbf{X})\widetilde{\Omega} + \sum_{\mu=0}^{m-1}(-1)^{m-1} g_\mu\ i(X_\mu)\alpha\,\mathrm{d}x^\mu + (-1)^m\alpha\ i(\mathbf{X})\omega \ . \end{aligned} \tag{3.18}$$

Then, denoting $f \equiv i(\mathbf{X})\omega$ and

$$\widetilde{\alpha}_{\mathbf{X}} = \sum_{\mu=0}^{m-1}(-1)^{m-1}\, i(X_\mu)\alpha\ i(X_0 \wedge \ldots \wedge X_{\mu-1} \wedge X_{\mu+1} \wedge \ldots \wedge X_{m-1})\omega \equiv \sum_{\mu=0}^{m-1}(-1)^{m-1}\, i(X_\mu)\alpha\, g_\mu\,\mathrm{d}x^\mu \ , \tag{3.19}$$

where $f, g_\mu \in \mathrm{C}^\infty(U)$ are non-vanishing functions and the result follows. ■

In particular, taking $i(\mathbf{X})\omega = 1$, it follows that $f = 1 = g_\mu$ and $i(\mathbf{X})\Omega = i(\mathbf{X})\widetilde{\Omega} + \widetilde{\alpha}_{\mathbf{X}} + (-1)^m\alpha$, with $\widetilde{\alpha}_{\mathbf{X}} = (-1)^{m-1}\, i(X_\mu)\alpha\,\mathrm{d}x^\mu$.

Proposition 3.1 *Let (F, Ω) be a premultisymplectic bundle $\kappa : F \to M$ where Ω satisfies the variational condition (2.1) and is written as the splitting (3.14). If $\mathcal{C}_1 \hookrightarrow F$ is the maximal submanifold on which solutions to the field equations (3.5) exist (the compatibility constraint submanifold), then the following condition holds at $\mathrm{p} \in U \subset \mathcal{C}_1$:*

$$[i(Z)\alpha](\mathrm{p}) = 0, \ \text{for every } Z \in \ker\widetilde{\Omega} \cap \ker\omega \ . \tag{3.20}$$

(*Proof*) As $\mathcal{C}_1$ is the maximal submanifold where the field equations are compatible, there exists a locally decomposable multivector $\mathbf{X} \in \mathfrak{X}^m(U_F)$ such that $i(\mathbf{X})\Omega = 0$ and $i(\mathbf{X})\omega \neq 0$. Then (3.17) holds and, as $\ker\omega = \mathfrak{X}^{\mathrm{V}(\kappa)}(F)$, if $Z \in \ker\widetilde{\Omega} \cap \ker\omega$, then $i(Z)\widetilde{\alpha}_{\mathbf{X}} = 0$, because $\widetilde{\alpha}_{\mathbf{X}}$ is κ-semibasic. Therefore, for every $Z \in \ker\widetilde{\Omega} \cap \ker\omega$ and for $\mathrm{p} \in U_F$ it follows that

$$0 = i(Z_\mathrm{p})\, i(\mathbf{X}_\mathrm{p})\Omega_\mathrm{p} = i(Z_\mathrm{p})\left[i(\mathbf{X}_\mathrm{p})\widetilde{\Omega}_\mathrm{p} + \widetilde{\alpha}_{\mathbf{X}_\mathrm{p}} + (-1)^m f\,\alpha_\mathrm{p}\right] = (-1)^m f\ i(Z_\mathrm{p})\alpha_\mathrm{p} \iff i(Z_\mathrm{p})\alpha_\mathrm{p} = 0 \ . \tag{3.21}$$

■

Thus, condition (3.20) is necessary for the compatibility of the field equations (3.5). In the particular case where $m = 1$ (i.e., time-dependent mechanics), (3.20) is a necessary and sufficient and therefore provides a geometric characterization of all compatibility constraints. However, when $m>1$, proving the converse of Proposition 3.1, which would assert that (3.20) fully determines the compatibility constraint submanifold $\mathcal{C}_1$, remains an open problem.

Finally, we turn our attention to the relationship between the vector fields in $\ker \mathscr{F}\mathscr{L}_*$ and the primary De Donder–Weyl Hamiltonian constraints arising from the Legendre map, in analogy with the corresponding structure observed in singular mechanical systems [15, 16].

Let $(J^1\pi,\ \Omega_{\mathscr{L}})$ be an almost-regular Lagrangian system. First observe that in the natural coordinates of $J^1\pi$, the map $\mathscr{F}\mathscr{L}_*$ is represented by the matrix

$$\mathrm{T}\mathscr{F}\mathscr{L} \equiv \begin{pmatrix} (\mathrm{Id})_{m\times m} & (0)_{m\times n} & (0)_{m\times nm} \\ (0)_{n\times m} & (\mathrm{Id})_{n\times n} & (0)_{n\times nm} \\ (0)_{nm\times m} & \left(\dfrac{\partial^2\mathscr{L}}{\partial y^B \partial y^A_\mu}\right) & \left(\dfrac{\partial^2\mathscr{L}}{\partial y^B_\nu \partial y^A_\mu}\right) \end{pmatrix} . \tag{3.22}$$

It follows that $Y_i = (\gamma_i)^A_\mu \dfrac{\partial}{\partial y^A_\mu} \in \ker \mathscr{F}\mathscr{L}_*$ with (local) component functions $(\gamma_i)^A_\mu \in \mathrm{C}^\infty(J^1\pi)$ which annihilate the Hessian, $\left(\dfrac{\partial^2\mathscr{L}}{\partial y^B_\nu \partial y^A_\mu}\right)\left((\gamma_i)^A_\mu\right) = (0)$, and hence $\ker \mathscr{F}\mathscr{L}_* \subset \mathfrak{X}^{\mathrm{V}(\pi^1_0)}(J^1\pi)$. Furthermore, is worth noting that

$$\ker \Omega_{\mathscr{L}} \cap \mathfrak{X}^{\mathrm{V}(\pi^1_0)}(J^1\pi) = \ker \mathscr{F}\mathscr{L}_* = \ker \widetilde{\Omega}_{\mathscr{L}} \cap \mathfrak{X}^{\mathrm{V}(\pi^1_0)}(J^1\pi) \subset \ker \widetilde{\Omega}_{\mathscr{L}} \cap \mathfrak{X}^{\mathrm{V}(\pi^1)}(J^1\pi) = \ker \widetilde{\Omega}_{\mathscr{L}} \cap \ker \omega . \tag{3.23}$$

Finally:

Proposition 3.2 *For every constraint $\zeta_i \in \mathrm{C}^\infty(J^1\pi^*)$ which locally defines the (primary Hamiltonian constraint) submanifold $P_\circ = \mathscr{F}\mathscr{L}(J^1\pi) \hookrightarrow J^1\pi^*$, there exists a vector field $Y_i = (\gamma_i)^A_\mu \dfrac{\partial}{\partial y^A_\mu} \in \ker \mathscr{F}\mathscr{L}_*$ such that*

$$(\gamma_i)^A_\mu = \mathscr{F}\mathscr{L}^*\left(\frac{\partial \zeta_i}{\partial p^\mu_A}\right) . \tag{3.24}$$

(*Proof*) As ζ_i is a primary Hamiltonian constraint it follows that $\mathscr{F}\mathscr{L}^*\zeta_i = 0$. Then,

$$\begin{aligned} 0 &= \mathrm{d}\mathscr{F}\mathscr{L}^*\zeta_i = \mathrm{d}(\zeta_i \circ \mathscr{F}\mathscr{L}) \\ &= \left(\frac{\partial \zeta_i}{\partial p^\mu_A} \circ \mathscr{F}\mathscr{L}\right)\frac{\partial p^\mu_A}{\partial y^B_\nu}\,\mathrm{d}y^B_\nu + \left(\frac{\partial \zeta_i}{\partial p^\mu_A} \circ \mathscr{F}\mathscr{L}\right)\frac{\partial p^\mu_A}{\partial y^B}\,\mathrm{d}y^B + \left(\frac{\partial \zeta_i}{\partial y^B} \circ \mathscr{F}\mathscr{L}\right)\mathrm{d}y^B \\ &= \mathscr{F}\mathscr{L}^*\left(\frac{\partial \zeta_i}{\partial p^\mu_A}\right)\left(\frac{\partial^2\mathscr{L}}{\partial y^B_\nu \partial y^A_\mu}\right)\mathrm{d}y^B_\nu + \left[\mathscr{F}\mathscr{L}^*\left(\frac{\partial \zeta_i}{\partial p^\mu_A}\right)\left(\frac{\partial^2\mathscr{L}}{\partial y^B \partial y^A_\mu}\right) + \mathscr{F}\mathscr{L}^*\left(\frac{\partial \zeta_i}{\partial y^B}\right)\right]\mathrm{d}y^B \\ \Longleftrightarrow\ & \mathscr{F}\mathscr{L}^*\left(\frac{\partial \zeta_i}{\partial p^\mu_A}\right)\left(\frac{\partial^2\mathscr{L}}{\partial y^B_\nu \partial y^A_\mu}\right) = 0 \ ,\ \mathscr{F}\mathscr{L}^*\left(\frac{\partial \zeta_i}{\partial p^\mu_A}\right)\left(\frac{\partial^2\mathscr{L}}{\partial y^B \partial y^A_\mu}\right) + \mathscr{F}\mathscr{L}^*\left(\frac{\partial \zeta_i}{\partial y^B}\right) = 0 \ , \end{aligned}$$

and equation (3.24) follows. ■

4 Canonical Lifts of Group Actions and Vector Fields

In preparation for our discussion on Noether symmetries in the following chapter, it is necessary first to discuss how Lie group actions are canonically lifted to the (pre)multi-symplectic manifolds on which we construct field theories. In Section 4.1 we give an exposition on how to canonically lift base space diffeomorphisms $M \to M$ to E when possible; furthermore, we discuss how the transformation of fields under diffeomorphisms of M can be understood as Lie derivatives of sections of the bundle $\pi : E \to M$, taken with respect to the infinitesimal vector fields generating the diffeomorphisms on the base manifold. Then, we will give a brief review on the canonical lift to $J^1\pi$ of group actions on the configuration manifold E in Section 4.2; such lifts are well known and can be found in Saunders' book [136]. Now, the definition of canonical lifts to $\mathcal{M}\pi$ and $J^1\pi^*$ is briefly discussed only in [67] to our knowledge; in Sections 4.3 and 4.4 we go into greater depth on canonical lifts to $\mathcal{M}\pi$ and $J^1\pi^*$, presenting some new results on the subject. Finally, in Section 4.5 we propose a definition of canonical lifts to $P_\circ \hookrightarrow J^1\pi^*$, relevant for the study of singular field theories.

We begin by revisiting how Lie groups act on manifolds. In particular, we focus on one-parameter subgroups of some Lie group G, which are given as Lie group homomorphisms from $\mathbb{R}$ to G. Consider a manifold P and a Lie group G with group action Φ on P written as

$$\Phi : G \times P \to \mathrm{Diff}(P) : g \mapsto \Phi_g \quad ; \quad g \in G \ , \tag{4.1}$$

and denote $\Phi_g : P \to P$ the corresponding diffeomorphisms induced by this action. If $\mathfrak{g}$ is the Lie algebra of G, every $\xi \in \mathfrak{g}$ induces a vector field on P

$$X_\xi = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} \Phi_{exp(\lambda\xi)} \ , \tag{4.2}$$

with the use of some parameter λ which is used to write $g \in G$ in terms of the exponential map

$$exp : \mathfrak{g} \to G : \xi \mapsto g = exp(\lambda\xi) \ . \tag{4.3}$$

4.1 Canonical Lifts to E

Classical field theories often exhibit symmetries that are associated with diffeomorphisms on the base manifold M (e.g., spacetime diffeomorphisms). However, it is not always the case that canonical lifts exist from M to E; for such lifts to exist, it is necessary for E to be a bundle with some additional natural structure. For instance, as in Yang–Mills and in several other relevant bosonic field theories in physics, E is a tensor bundle over M for which it is well-known that the canonical lifts from M to E exist. Other examples such as in Einstein–Cartan theory, E is the bundle of frames and spin connections which is also natural.

Given a group action $\Phi_M : M \to M$, the group action lifted to E, denoted $\Phi_E : E \to E$ satisfies

$$\Phi_M \circ \pi = \pi \circ \Phi_E \ , \tag{4.4}$$

hence,

$$\begin{array}{ccc} E & \xrightarrow{\Phi_E} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\Phi_M]{} & M \end{array}$$

Now, consider a vector field $\xi_M = -\xi^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ which generates infinitesimal diffeomorphisms $\Phi_M : M \to M$ associated with the infinitesimal coordinate transformations $x'^\mu = x^\mu + \xi^\mu(x)$.

Such diffeomorphisms are given by the one-parameter subgroup of local diffeomorphisms infinitesimally generated by ξ_M and are the only kind that we will be interested in. A generic vector field on E which projects to ξ_M has the coordinate expression

$$\xi_E = -\xi^\mu(x)\frac{\partial}{\partial x^\mu} - \xi^A(x,y)\frac{\partial}{\partial y^A} \in \mathfrak{X}(E) \ , \tag{4.5}$$

and generates the infinitesimal transformations $(x^\mu, y^A) \to (x^\mu + \xi^\mu(x), y^A + \xi^A(x,y))$ on E, with $\xi^A(x,y) \in C^\infty(E)$. The fields $\phi^A(x)$ are components of the sections $\phi = (x^\mu, \phi^A(x))$ of $\pi \colon E \to M$ and transform as the Lie derivatives with respect to ξ_M as

$$\delta_\xi \phi^A(x) \equiv \mathrm{L}(\xi_M)\phi^A(x) = \phi'^A(x) - \phi^A(x) = -\xi^\mu \partial_\mu \phi^A + \widetilde{\xi}^A \ , \tag{4.6}$$

for some $\widetilde{\xi}^A(x) \in C^\infty(M)$. These field transformations are referred to as the ***local variation*** of the fields. The term $-\xi^\mu\partial_\mu\phi^A$ in (4.6) is known as the ***transport term*** and $\widetilde{\xi}^A(x)$ are called the ***global variation*** of the fields, given by

$$\widetilde{\xi}^A(x) = \phi'^A(x') - \phi^A(x) \ . \tag{4.7}$$

Note that in this notation, $\phi'^A(x)$ is a different point on along the fiber over the spacetime point $x \in M$. Furthermore, it follows that $\phi'^A(x') \to \phi'^A(x)$ as $x'^\mu \to x^\mu$.

One can interpret the local field variations as follows [67, 73, 104]:

Definition 4.1 *Consider a section* $\phi \colon M \to E : x^\mu \to (x^\mu, \phi^A(x))$ *and let* $\xi_E \in \mathfrak{X}(E)$ *be a* π*-projectable vector field whose local expression is* (4.5) *and therefore projects to a vector field* $\xi_M = -\xi^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ *which generates infinitesimal diffeomorphisms* $\Phi_M \colon M \to M$*. The* ***generalized Lie derivative of the section*** ϕ ***by*** ξ_M *is the map* $\mathbb{L}(\xi_M)\phi \colon M \to \mathrm{T}E$ *defined as*

$$\mathbb{L}(\xi_M)\phi = \mathrm{T}\phi \circ \xi_M - \xi_E \circ \phi \ , \tag{4.8}$$

which is a vector field along ϕ*. This generalized Lie derivative has the form*

$$\mathbb{L}(\xi_M)\phi = (\phi, \mathrm{L}(\xi_M)\phi) = \Big(x^\mu, \xi^\nu\,\partial_\nu\phi^A - \xi^A \circ \phi\Big) \ , \tag{4.9}$$

and the section $\mathrm{L}(\xi_M)\phi \colon M \to V(\mathrm{T}\pi)$ *is called the* ***Lie derivative of the section*** ϕ ***by*** ξ_M *with its local expression in accordance with* (4.5) *for* ξ_E*.*

The above definition of the Lie derivative (4.8) is in agreement with the Lie derivative of $\phi^A(x)$ defined by the field transformations $\delta_\xi\phi^A(x)$ in (4.6). Then, the functions $\widetilde{\xi}^A(x)$ in (4.6) are given in terms of the component functions $\xi^A(x,y)$ of ξ_E and some local section ϕ as

$$\widetilde{\xi}^A(x) = \xi^A(x,y) \circ \phi \ . \tag{4.10}$$

For theories in which $\phi^A(x)$ are scalar fields, diffeomorphisms on the base space M produce field variations $\delta_\xi\phi^A(x)$ for which $\widetilde{\xi}^A(x) = 0$ and $\xi^A(x,y) = 0$.

Now, consider when the fields $\phi^A(x)$ are tensor fields of type $(k,l) \neq (0,0)$ on M, denoted $T \equiv (T^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s}(x)) \in \mathfrak{T}^{(k,l)}(\mathrm{T}M)$, so that E is a tensor bundle over M. Then, the functions $\widetilde{\xi}^A(x)$ are obtained from the induced tensor transformation laws in which the Jacobian matrices associated with the coordinate transformations $x^\mu \to x^\mu + \xi^\mu(x)$ are used to transform the indices of the tensor field as usual (see e.g. [111]):

$$(T')^{\mu_1,\dots,\mu_r}_{\nu_1,\dots,\nu_s}(x') = \left(\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\right)\cdots\left(\frac{\partial x'^{\mu_r}}{\partial x^{\alpha_r}}\right)\left(\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\right)\cdots\left(\frac{\partial x^{\beta_s}}{\partial x'^{\nu_s}}\right) T^{\alpha_1,\dots,\alpha_r}_{\beta_1,\dots,\beta_s}(x) \ . \tag{4.11}$$

The Taylor expansion, to first order, of the left-hand side of the equation above around x gives the transport term as

$$(T')^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x') = (T')^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x+\xi) = (T')^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x) + \xi^\lambda \, \frac{\partial (T')^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x)}{\partial x^\lambda} \ , \tag{4.12}$$

while the Taylor expansion of the Jacobian matrices on the right-hand side of equation (4.12) gives the global variation of the fields $\Delta T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x)$ as

$$\Delta T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s} = \sum_{\mu=\mu_1}^{\mu_r} \left(\frac{\partial \xi^\mu}{\partial x^\lambda}\right) T^{\mu_1,\ldots,(\mu\to\lambda),\ldots,\mu_r}_{\nu_1,\ldots,\nu_s} - \sum_{\nu=\nu_1}^{\nu_s} \left(\frac{\partial \xi^\lambda}{\partial x^\nu}\right) T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,(\nu\to\lambda),\ldots,\nu_s} \ . \tag{4.13}$$

Then, the local field variation given by the Lie derivative with respect to $\xi_M \in \mathfrak{X}(M)$ is

$$\delta T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x) = (T')^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x) - T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x) = \mathrm{L}(Z) T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s} = -\xi^\lambda \, \frac{\partial (T')^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}}{\partial x^\lambda} + \Delta T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s} \ . \tag{4.14}$$

It follows from (4.10) that,

$$\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} - \Delta^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x,T) \, \frac{\partial}{\partial T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}} \ , \tag{4.15}$$

where,

$$\Delta T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x) = \Delta^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x,T) \circ \phi \ . \tag{4.16}$$

Then, when E is a tensor bundle over M with coordinates $(x^\mu, T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s})$, it follows that (4.15) is the canonical lift of $\xi_M \in \mathfrak{X}(M)$ to E which is defined as follows.

Definition 4.2

1. *Let $\Phi_M \colon M \to M$ be a diffeomorphism. The* ***canonical lift of*** Φ_M ***to*** E *is the diffeomorphism $\Phi_E \colon E \to E$ such that, for every $(x, T_x) \in E$, with $T_x \in \mathfrak{T}^{(k,l)}(\mathrm{T}_x M)$,*

   $$\Phi_E(x, T_x) \equiv (\Phi_M(x), \mathcal{T}\Phi_M(T_x)) \ , \tag{4.17}$$

   where $\mathcal{T}\Phi_M$ denotes the canonical transformation of tensors on M induced by Φ_M. Thus,

   $$\pi \circ \Phi_E = \Phi_M \circ \pi \ . \tag{4.18}$$

2. *Let $\xi_M \in \mathfrak{X}(M)$ be a vector field whose flux is made of the local one-parameter groups of diffeomorphisms of M, denoted ϕ_t. The* ***canonical lift of*** ξ_M ***to*** E *is the vector field $\xi_E \in \mathfrak{X}(E)$ whose flux is made of the local one-parameter groups of diffeomorphisms $(\phi_E)_t$ which are the canonical lifts of ϕ_t to E.*

4.2 Canonical Lifts to $J^1\pi$

We now consider the canonical lifts to $J^1\pi$ of diffeomorphisms $\Phi_E \colon E \to E$ which arise from diffeomorphisms on the base space M as $\Phi_M \colon M \to M$ so $\pi \circ \Phi_E = \Phi_M \circ \pi$. Then, the ***canonical lift of*** Φ_E ***to*** $J^1\pi$ is the diffeomorphism $\Phi_{J^1\pi} = j^1\Phi_E \colon J^1\pi \longrightarrow J^1\pi$ defined point-wise $\bar{y} \in J^1\pi$ as

$$\Phi_{J^1\pi}(\bar{y}) \equiv (j^1\Phi_E)(\bar{y}) = j^1(\Phi \circ \phi \circ \Phi_M^{-1})(\Phi_M(x)) \ . \tag{4.19}$$

Furthermore, the canonical lift of $\xi_E \in \mathfrak{X}(E)$ to $J^1\pi$ is given by the following expression [67, 136]:

$$X_\xi = j^1\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} - \xi^A \frac{\partial}{\partial y^A} - \left(\frac{\partial \xi^A}{\partial x^\mu} + y^B_\mu \frac{\partial \xi^A}{\partial y^B} - y^A_\nu \frac{\partial \xi^\nu}{\partial x^\mu}\right) \frac{\partial}{\partial y^A_\mu} \in \mathfrak{X}(J^1\pi) \ . \tag{4.20}$$

These canonical lifts are characterized by the property that they leave the canonical structures of the jet bundle $J^1\pi$ invariant. In particular, the *contact module* and consequently the *canonical endomorphism* are invariant under such canonical lifts.

Now, the variation of the spacetime derivatives of the fields, $\delta_\xi \phi^A_\mu = \mathrm{L}(\xi_M)\partial_\mu \phi^A$, can also be characterized by the generalized Lie derivative $\mathrm{L}(\xi_M)j^1\phi$ of the first–jet prolongations $j^1\phi : M \to J^1\pi : x^\mu \mapsto (x^\mu, \phi^A, \phi^A_\mu)$ with respect to the vector field $\xi_M = -\xi^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$, where now

$$\mathbb{L}(\xi_M)j^1\phi = Tj^1\phi \circ \xi_M - j^1 Z \circ j^1\phi \ , \tag{4.21}$$

and $\mathbb{L}(\xi_M)j^1\phi = (\phi, \mathrm{L}(\xi_M)\phi)$. It thereby follows that

$$\mathrm{L}(\xi_M)(j^1\phi)^A_\mu = \delta\phi^A_\mu(x) = -\xi^\nu \frac{\partial \phi^A_\mu}{\partial x^\nu} - \frac{\partial \xi^\nu}{\partial x^\mu}\frac{\partial y^A}{\partial x^\nu} + \frac{\partial \widetilde{\xi}^A}{\partial x^\mu} + \frac{\partial \phi^B}{\partial x^\mu}\frac{\partial \widetilde{\xi}^A}{\partial \phi^B} \ , \tag{4.22}$$

where now $\widetilde{\xi}^A(x) = \xi^A(x,y) \circ j^1\phi$ as before. Furthermore, $\xi^A = 0$ for scalar fields transformed under spacetime diffeomorphisms and the equations above simplify accordingly.

In the case where the fields are tensor fields $T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x)$, the canonical lift $j^1\xi_E$ in (4.36) of ξ_E in (4.15) to $J^1\pi$ is

$$j^1\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} - \Delta^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x,T)\frac{\partial}{\partial T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}} - \Gamma^{\mu_1,\ldots,\mu_r}_{\alpha\nu_1,\ldots,\nu_s}\frac{\partial}{\partial T^{\mu_1,\ldots,\mu_r}_{\alpha\nu_1,\ldots,\nu_s}} \ , \tag{4.23}$$

where the $\Delta^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}(x,T) \in C^\infty(E)$ are again given by (4.13) and (4.16) while

$$\begin{aligned} \Gamma^{\mu_1,\ldots,\mu_r}_{\alpha\nu_1,\ldots,\nu_s} = &\sum_{\mu=\mu_1}^{\mu_r} \frac{\partial \xi^\mu}{\partial x^\lambda}\frac{\partial T^{\mu_1,\ldots,(\mu\to\lambda),\ldots,\mu_r}_{\nu_1,\ldots,\nu_s}}{\partial x^\alpha} - \sum_{\nu=\nu_1}^{\nu_s}\frac{\partial \xi^\lambda}{\partial x^\nu}\frac{\partial T^{\mu_1,\ldots,\mu_r}_{\nu_1,\ldots,(\nu\to\lambda),\ldots,\nu_s}}{\partial x^\alpha} \\ &+ \sum_{\mu=\mu_1}^{\mu_r}\frac{\partial \xi^\mu}{\partial x^\lambda} T^{\nu_1,\ldots,\nu_s} - \sum_{\nu=\nu_1}^{\nu_s}\frac{\partial \xi^\lambda}{\partial x^\nu} T^{\mu_1,\ldots,\mu_r}_{\alpha\nu_1,\ldots,(\nu\to\lambda),\ldots,\nu_s} - T_{\beta\nu_1,\ldots,\nu_s}{}^{\mu_1,\ldots,\mu_r}\frac{\partial \xi^\beta}{\partial x^\alpha} \ . \end{aligned} \tag{4.24}$$

Moreover, the canonical lift (4.20) can be generalized for vector fields on E that are not π-projectable. The local expression for such vector fields is given as

$$\xi_E = -\xi^\mu(x,y)\frac{\partial}{\partial x^\mu} - \xi^A(x,y)\frac{\partial}{\partial y^A} \in \mathfrak{X}(E) \ . \tag{4.25}$$

It follows that if ξ_E is not π projectable, then its canonical lift $j^1 Y \in \mathfrak{X}(J^1\pi)$ is the only vector field that is π^1_0 projectable to ξ_E and leaves the canonical structures of $J^1\pi$ invariant (see [46, 136] for details). Furthermore, the local expression for the canonical lift to $J^1\pi$ is then given as

$$X_\xi = j^1\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} - \xi^A \frac{\partial}{\partial y^A} - \left(\frac{\partial \xi^A}{\partial x^\mu} + y^B_\mu \frac{\partial \xi^A}{\partial y^B} - y^A_\nu\left(\frac{\partial \xi^\nu}{\partial x^\mu} + y^B_\mu\frac{\partial \xi^\nu}{\partial y^B}\right)\right)\frac{\partial}{\partial y^A_\mu} \ . \tag{4.26}$$

However, we will not be dealing with such vector fields in this work; all canonical lifts to $J^1\pi$ will be given by equation (4.20).

4.3 Canonical Lifts to $\mathcal{M}\pi$

We now turn our attention to defining canonical lifts from the configuration manifold E to the extended multimomentum bundle $\mathcal{M}\pi \equiv \Lambda^m_2 \mathrm{T}^* E$. Since $\mathcal{M}\pi$ is a bundle of forms over E, the canonical lift of diffeomorphisms from E to $\mathcal{M}\pi$ arises naturally, in a manner analogous to the construction in Definition 4.2 for the case where $\pi : E \to M$ is a tensor bundle. Also recall $\widetilde{\sigma} : \mathcal{M}\pi \to E$.

Definition 4.3 *Let $\Phi_E\colon E \to E$ be a diffeomorphism on E which induces a diffeomorphism $\Phi_M\colon M \to M$, on M; that is, $\Phi_M \circ \pi = \pi \circ \Phi_E$. The* ***canonical lift of*** Φ_E ***to*** $\mathcal{M}\pi$ *is the diffeomorphism $\Phi_{\mathcal{M}\pi}\colon \mathcal{M}\pi \to \mathcal{M}\pi$ such that, for every $\mathrm{p} = (y, \zeta) \in \mathcal{M}\pi$, with $\zeta \in \Lambda_2^m \mathrm{T}_y^* E$,*

$$\Phi_{\mathcal{M}\pi}(y, \zeta) \equiv (\Phi_E(y), (\Phi_E^{-1})_y^* \zeta) \ . \tag{4.27}$$

This definition is equivalent to the one given in [67].

Such $\Phi_{\mathcal{M}\pi}$ are $\widetilde{\sigma}$-fiber preserving diffeomorphisms: $\Phi_E \circ \widetilde{\sigma} = \widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi}$ (see Diagram (4.38)). Note that we will only be interested in lifting the one-parameter subgroup of local diffeomorphisms of E which are generated infinitesimally by some vector field $\xi_E \in \mathfrak{X}(E)$, those which in turn are the lifts of from M to E as discussed in Section 4.1.

The canonical lifts $\Phi_{\mathcal{M}\pi}$ preserve the canonical structures of $\mathcal{M}\pi$ which consist of the tautological form $\Theta \in \Omega^m(\mathcal{M}\pi)$ and the multisymplectic form $\Omega \in \Omega^{m+1}(\mathcal{M}\pi)$ as shown in the following proposition; this result is mentioned but not proven in [67].

Proposition 4.1 *If $\Phi_E\colon E \to E$ is a diffeomorphism, inducing a diffeomorphism $\Phi_M\colon M \to M$, and $\Phi_{\mathcal{M}\pi}$ is its canonical lift to $\mathcal{M}\pi$, then $\Phi_{\mathcal{M}\pi}^* \Theta = \Theta$ and $\Phi_{\mathcal{M}\pi}^* \Omega = \Omega$.*

(*Proof*) For every $y \in E$, $\zeta \in \Lambda_2^m \mathrm{T}_y^* E$, and $X_0, \ldots, X_{m-1} \in \mathrm{T}_{(y,\zeta)}(\mathcal{M}\pi)$, using the definition of Θ given in (1.51), and the property $\Phi_E^{-1} \circ \widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi} = \widetilde{\sigma}$, we obtain

$$\begin{aligned}
(\Phi_{\mathcal{M}\pi}^* \Theta)_{(y,\zeta)}(X_0, \ldots, X_{m-1}) &= \Theta_{(\Phi_E(y),(\Phi_E^{-1})^*\zeta)}\Big(\mathrm{T}_{(y,\zeta)}\Phi_{\mathcal{M}\pi}(X_0), \ldots, \mathrm{T}_{(y,\zeta)}\Phi_{\mathcal{M}\pi}(X_{m-1})\Big) = \\
((\Phi_E^{-1})_{\Phi_E(y)}^* \zeta)\Big(\mathrm{T}_{(\Phi_E(y),(\Phi_E^{-1})^*\zeta)}\widetilde{\sigma}\big(\mathrm{T}_{(y,\zeta)}\Phi_{\mathcal{M}\pi}(X_0)\big), &\ldots, \mathrm{T}_{(\Phi_E(y),(\Phi_E^{-1})^*\zeta)}\widetilde{\sigma}\big(\mathrm{T}_{(y,\zeta)}\Phi_{\mathcal{M}\pi}(X_{m-1})\big)\Big) = \\
((\Phi_E^{-1})_{\Phi_E(y)}^* \zeta)\Big(\mathrm{T}_{(y,\zeta)}(\widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi})(X_0), &\ldots, \mathrm{T}_{(y,\zeta)}(\widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi})(X_{m-1})\Big) = \\
\zeta\Big(\mathrm{T}_{\Phi_E(y)}\Phi_E^{-1}\big(\mathrm{T}_{(y,\zeta)}(\widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi})(X_0)\big), &\ldots, \mathrm{T}_y\Phi_E^{-1}\big(\mathrm{T}_{(y,\zeta)}(\widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi})(X_{m-1})\big)\Big) = \\
\zeta\Big(\mathrm{T}_{(y,\zeta)}(\Phi_E^{-1} \circ \widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi})(X_0), &\ldots, \mathrm{T}_{(y,\zeta)}(\Phi_E^{-1} \circ \widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi})(X_{m-1})\Big) = \\
\zeta\Big(\mathrm{T}_{(y,\zeta)}\widetilde{\sigma}(X_0), \ldots, \mathrm{T}_{(y,\zeta)}\widetilde{\sigma}(X_{m-1})\Big) &= \Theta_{(y,\zeta)}(X_0, \ldots, X_{m-1}) \ ,
\end{aligned} \tag{4.28}$$

hence $\Phi_{\mathcal{M}\pi}^* \Theta = \Theta$. The result for Ω is immediate. ■

Now, if $\xi_E \in \mathfrak{X}(E)$ is a π-projectable vector field (i.e., there exists some $\xi_M \in \mathfrak{X}(M)$ such that the local flows of ξ_M and ξ_E are π-related), then it is possible to define the canonical lift of ξ_E to $\mathcal{M}\pi$ as:

Definition 4.4 *Let $\xi_E \in \mathfrak{X}(E)$ be a π-projectable vector field whose flux is made of local one-parameter groups of diffeomorphisms of E, denoted ϕ_t. The* ***canonical lift of*** ξ_E ***to*** $\mathcal{M}\pi$ *is the vector field $Z_\xi \in \mathfrak{X}(\mathcal{M}\pi)$ whose flux is made of the local one-parameter groups of diffeomorphisms $\widetilde{\phi}_t$ which are the canonical lifts of ϕ_t to $\mathcal{M}\pi$.*

Proposition 4.2 *The canonical lift, $Z_\xi \in \mathcal{M}\pi$, of a π-projectable vector field $\xi_E \in \mathfrak{X}(E)$ has the following properties:*

1. *Z_ξ is $\widetilde{\sigma}$-projectable and $\widetilde{\sigma}_* Z_\xi = \xi_E$.*
2. *$\mathrm{L}(Z_\xi)\Theta = 0$ and $\mathrm{L}(Z_\xi)\Omega = 0$.*

This is a straightforward consequence of the previous definition and Proposition 4.1.

Such vector field lifts, $Z_\xi \in \mathfrak{X}(\mathcal{M}\pi)$, can be characterized using the canonical forms of $\mathcal{M}\pi$. First we define:

Definition 4.5 *Every vector field $\xi_E \in \mathfrak{X}(E)$ induces a form $\Gamma_\xi \in \Omega^{m-1}(\mathcal{M}\pi)$ defined as follows: for every point $(y,\zeta) \in \mathcal{M}\pi$,*

$$(\Gamma_\xi)_{(y,\zeta)} \equiv i(\xi_E|_y)\zeta \ . \tag{4.29}$$

As an immediate consequence of this definition we obtain the following result:

Proposition 4.3 *If $\xi_E \in \mathfrak{X}(E)$ is a π-projectable vector field and $Z_\xi \in \mathfrak{X}(\mathcal{M}\pi)$ is the canonical lift of ξ_E to $\mathcal{M}\pi$, then*

$$\Gamma_\xi = i(Z_\xi)\Theta \ . \tag{4.30}$$

(*Proof*) The $\widetilde{\sigma}$-projectivity property of the vector field Z_ξ means that, for every $(y,\zeta) \in \mathcal{M}\pi$, it follows that $\mathrm{T}_{(y,\zeta)}\widetilde{\sigma}(Z_\xi|_{(y,\zeta)}) = \xi_E|_y$. Then, taking into account the definition of the canonical m-form Θ, it follows that

$$(\Gamma_\xi)_{(y,\zeta)} = \zeta(\xi_E|_y) = \zeta\big(\mathrm{T}_{(y,\zeta)}\widetilde{\sigma}(Z_\xi|_{(y,\zeta)})\big) = \Theta_{(y,\zeta)}(Z_\xi|_{(y,\zeta)}) \ , \tag{4.31}$$

and the statement holds. ∎

From this proposition, we obtain:

Proposition 4.4 *If $\xi_E \in \mathfrak{X}(E)$ is a π-projectable vector field, then the canonical lift of ξ_E to $\mathcal{M}\pi$ is the unique vector field $Z_\xi \in \mathfrak{X}(\mathcal{M}\pi)$ such that,*

$$i(Z_\xi)\Omega = \mathrm{d}\Gamma_\xi \ . \tag{4.32}$$

That is, Z_ξ is the (global) Hamiltonian vector field associated with the $(m-1)$-form Γ_ξ.

(*Proof*) Taking into account that $\mathrm{L}(Z_\xi)\Theta = 0$, we obtain

$$i(Z_\xi)\Omega = -\,i(Z_\xi)\mathrm{d}\Theta = \mathrm{d}\,i(Z_\xi)\Theta - \mathrm{L}(Z_\xi)\Theta = \mathrm{d}(\Theta(Z_\xi)) = \mathrm{d}\Gamma_\xi \ . \tag{4.33}$$

Furthermore, given the local coordinate expression (4.5) of a π-projectable vector field $\xi_E \in \mathfrak{X}(E)$, it follows that

$$\Gamma_\xi = \xi^\nu p_A^\mu \,\mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\mu\nu} - \left(\xi^A p_A^\mu + \xi^\mu p\right) \mathrm{d}^{m-1}x_\mu \ , \tag{4.34}$$

$$\begin{aligned} \mathrm{d}\Gamma_\xi = {} & \xi^\nu \,\mathrm{d}p_A^\mu \,\mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\mu\nu} + \left(\frac{\partial \xi^A}{\partial x^\nu}p_A^\mu - \frac{\partial \xi^\nu}{\partial x^\nu}p_A^\mu - \frac{\partial \xi^B}{\partial y^A}p_B^\mu\right) \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu \\ & - \xi^A \,\mathrm{d}p_A^\mu \wedge \mathrm{d}^{m-1}x_\mu - \xi^\mu \,\mathrm{d}p \wedge \mathrm{d}^{m-1}x_\mu - \left(\frac{\partial \xi^A}{\partial x^\mu}p_A^\mu + \frac{\partial \xi^\mu}{\partial x^\mu}p\right) \mathrm{d}^m x \ . \end{aligned} \tag{4.35}$$

Using Proposition 4.4 and equation (1.52), it follows that the canonical lift of ξ_E to $\mathcal{M}\pi$ is uniquely given as

$$Z_\xi = -\xi^\mu \frac{\partial}{\partial x^\mu} - \xi^A \frac{\partial}{\partial y^A} - \left(\frac{\partial \xi^A}{\partial x^\nu}p_A^\mu - \frac{\partial \xi^\nu}{\partial x^\nu}p_A^\mu - \frac{\partial \xi^B}{\partial y^A}p_B^\mu\right) \frac{\partial}{\partial p_A^\mu} + \left(\frac{\partial \xi^A}{\partial x^\mu}p_A^\mu + \frac{\partial \xi^\mu}{\partial x^\mu}p\right) \frac{\partial}{\partial p} \ , \tag{4.36}$$

which is in agreement with the expression found in [67]. ∎

Remark 4.1 It is important to point out that the existence of a unique Hamiltonian vector field associated with the $(m-1)$-form Γ_ξ is assured as a consequence of the assumption that ξ_E is a π-projectable vector field; this is also shown in the above calculations in local coordinates.

4.4 Canonical Lifts to $J^1\pi^*$

We now consider a regular De Donder–Weyl Hamiltonian system $(J^1\pi^*, \Omega_{\mathscr{H}})$. If $\Phi_E \colon E \to E$ is a diffeomorphism on E which arises from a diffeomorphism $\Phi_M \colon M \to M$, then the induced map $\Phi_E^* \colon \Omega^m(E) \to \Omega^m(E)$ transforms π-semibasic forms into themselves. Hence, bearing in mind the definition of $J^1\pi^* \equiv \mathcal{M}\pi/\Lambda_1^m(\mathrm{T}^*\mathrm{E})$ and Definition 4.3, this implies that the canonical lifts $\Phi_{\mathcal{M}\pi} : \mathcal{M}\pi \to \mathcal{M}\pi$ induce diffeomorphisms on $J^1\pi^*$ as follows.

Definition 4.6 *Let $\Phi_E \colon E \to E$ be a diffeomorphism on E which induces a diffeomorphism $\Phi_M \colon M \to M$ on M and $\Phi_{\mathcal{M}\pi} \colon \mathcal{M}\pi \to \mathcal{M}\pi$ is the canonical lift of Φ_E to $\mathcal{M}\pi$. The **canonical lift of** Φ_E **to** $J^1\pi^*$ is the fiber preserving diffeomorphism $\Phi_{J^1\pi^*} \colon J^1\pi^* \to J^1\pi^*$ induced on $J^1\pi^*$ by $\Phi_{\mathcal{M}\pi}$ as*

$$\Phi_{J^1\pi^*} \circ \sigma = \sigma \circ \Phi_{\mathcal{M}\pi} \; . \tag{4.37}$$

This definition is equivalent to the one given in [67].

Since $\Phi_E \circ \widetilde{\sigma} = \widetilde{\sigma} \circ \Phi_{\mathcal{M}\pi}$, it follows that $\Phi_E \circ \widetilde{\sigma} = \widetilde{\sigma} \circ \Phi_{J^1\pi^*}$ and we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}\pi \arrow[r, "\Phi_{\mathcal{M}\pi}"] \arrow[d, "\sigma"] \arrow[dd, bend right, "\widetilde{\sigma}"'] & \mathcal{M}\pi \arrow[d, "\sigma"'] \arrow[dd, bend left, "\widetilde{\sigma}"] \\
J^1\pi^* \arrow[r, "\Phi_{J^1\pi^*}"] \arrow[d, "\tau"] & J^1\pi^* \arrow[d, "\tau"'] \\
E \arrow[r, "\Phi_E"] & E
\end{tikzcd} \tag{4.38}$$

Remark 4.2 Note that $J^1\pi^*$ is canonically diffeomorphic to $\pi^*\mathrm{T}M \otimes_E V^*E \otimes_E \pi^*\Lambda^m\mathrm{T}^*M$ (see, for instance, [49]). Therefore, for a diffeomorphism $\Phi_E \colon E \to E$ which restricts to a diffeomorphism $\Phi_M \colon M \to M$, the definition of the canonical lift of Φ_E to $J^1\pi^*$ can be stated straightforwardly in a natural way, using the same pattern as in Definition 4.3.

Then, as on $\mathcal{M}\pi$, we can define:

Definition 4.7 *Let $\xi_E \in \mathfrak{X}(E)$ be a π-projectable vector field whose flux is made of local one-parameter groups of diffeomorphisms of E, denoted ϕ_t. The **canonical lift of** ξ_E **to** $J^1\pi^*$ is the vector field $Y_\xi \in \mathfrak{X}(J^1\pi^*)$ whose flux is made of the local one-parameter groups of diffeomorphisms $\widehat{\phi}_t$ which are the canonical lifts of ϕ_t to $J^1\pi^*$.*

It follows from the local expression (4.36) that canonical lifts $Z_\xi \in \mathcal{M}\pi$ are σ-projectable vector fields. Therefore, from the above definitions we arrive at the following proposition.

Proposition 4.5 $\sigma_* Z_\xi = Y_\xi$.

Additionally, by Proposition 4.2 it follows that $\widetilde{\sigma}_* Z_\xi = \xi_E$, and therefore,

$$\xi_E = \widetilde{\sigma}_* Z_\xi = (\tau \circ \sigma)_* Z_\xi = \tau_*\big(\sigma_* Z_\xi\big) = \tau_* Y_\xi \; . \tag{4.39}$$

Then, the coordinate expression of the canonical lift Y_ξ is,

$$Y_\xi = -\xi^\mu \frac{\partial}{\partial x^\mu} - \xi^A \frac{\partial}{\partial y^A} - \left(\frac{\partial \xi^\mu}{\partial x^\nu} p_A^\mu - \frac{\partial \xi^\nu}{\partial x^\nu} p_A^\mu - \frac{\partial \xi^B}{\partial y^A} p_B^\mu \right) \frac{\partial}{\partial p_A^\mu} \ . \tag{4.40}$$

Now, let $(J^1\pi^*, \Omega_{\mathscr{H}})$ be the Hamiltonian system associated with a (hyper-)regular Lagrangian system $(J^1\pi, \Omega_{\mathscr{L}})$. If $X_\xi = j^1\xi_E \in \mathfrak{X}(J\pi)$ is the canonical lift to $J^1\pi$ (with local expression (4.20)) of a π-projectable vector field $\xi_E \in \mathfrak{X}(E)$ (with local expression (4.5)), then Y_ξ is related to X_ξ as follows.

Proposition 4.6 *If the Lagrangian $\mathscr{L}$ is (hyper-)regular and* $\mathrm{L}(X_\xi)\mathbf{L} = 0$*, then* $\mathscr{F}\mathscr{L}_* X_\xi = Y_\xi$.

(*Proof*) As the Lagrangian $\mathscr{L}$ is (hyper-)regular, $\mathscr{F}\mathscr{L}$ is a (global) diffeomorphism, and the matrix of the tangent map $\mathscr{F}\mathscr{L}_*$ in a natural coordinate system on $J^1\pi$ is given as

$$\mathscr{F}\mathscr{L}_* = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ \dfrac{\partial^2 \mathscr{L}}{\partial x^\mu \partial y_\nu^B} & \dfrac{\partial^2 \mathscr{L}}{\partial y^A \partial y_\nu^B} & \dfrac{\partial^2 \mathscr{L}}{\partial y_\mu^A \partial y_\nu^B} \end{pmatrix} . \tag{4.41}$$

Then, if

$$\mathscr{F}\mathscr{L}_* X_\xi = f^\mu \frac{\partial}{\partial x^\mu} + f^A \frac{\partial}{\partial y^A} + f_B^\nu \frac{\partial}{\partial p_\nu^B} \ , \tag{4.42}$$

while bearing in mind (4.20), then $f^\mu = -\xi^\mu(x^\nu)$, $f^A = -\xi^A(x^\nu, y^B)$, and

$$\begin{aligned} \mathscr{F}\mathscr{L}^* f_B^\nu =& - \xi^\mu \frac{\partial^2 \mathscr{L}}{\partial x^\mu \partial y_\nu^B} - \xi^A \frac{\partial^2 \mathscr{L}}{\partial y^A \partial y_\nu^B} - \left(\frac{\partial \xi^A}{\partial x^\mu} - y_\rho^A \frac{\partial \xi^\rho}{\partial x^\mu} + y_\mu^C \frac{\partial \xi^A}{\partial y^C} \right) \frac{\partial^2 \mathscr{L}}{\partial y_\mu^A \partial y_\nu^B} \ , \\ =& \frac{\partial}{\partial y_\nu^B} \left[\xi^\mu \frac{\partial \mathscr{L}}{\partial x^\mu} + \xi^A \frac{\partial \mathscr{L}}{\partial y^A} + \left(\frac{\partial \xi^A}{\partial x^\mu} - y_\rho^A \frac{\partial \xi^\rho}{\partial x^\mu} + y_\mu^C \frac{\partial \xi^A}{\partial y^C} \right) \frac{\partial \mathscr{L}}{\partial y_\mu^A} \right] \\ & - \frac{\partial}{\partial y_\nu^B} \left(\frac{\partial \xi^A}{\partial x^\mu} - y_\rho^A \frac{\partial \xi^\rho}{\partial x^\mu} + y_\mu^C \frac{\partial \xi^A}{\partial y^k} \right) \frac{\partial \mathscr{L}}{\partial y_\mu^A} \ , \\ =& \frac{\partial}{\partial y_\nu^B} \left[\xi^\mu \frac{\partial \mathscr{L}}{\partial x^\mu} + \xi^A \frac{\partial \mathscr{L}}{\partial y^A} + \left(\frac{\partial \xi^A}{\partial x^\mu} - y_\rho^A \frac{\partial \xi^\rho}{\partial x^\mu} + y_\mu^C \frac{\partial \xi^A}{\partial y^C} \right) \frac{\partial \mathscr{L}}{\partial y_\mu^A} \right] \\ & - \left(\frac{\partial \xi^A}{\partial y^B} \frac{\partial \mathscr{L}}{\partial y_\nu^A} - \frac{\partial \xi^\nu}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial y_\mu^B} \right) \ . \end{aligned} \tag{4.43}$$

Now, from using (4.20) it is straightforward to calculate the Lie derivative of the Lagrangian density with respect to X_ξ and obtain:

$$\begin{aligned} \mathrm{L}(X_\xi)\mathbf{L} &= \mathrm{L}(X_\xi)(\mathscr{L}\, \mathrm{d}^m x) \\ &= - \left[\xi^\mu \frac{\partial \mathscr{L}}{\partial x^\mu} + \xi^A \frac{\partial \mathscr{L}}{\partial y^A} + \left(\frac{\partial \xi^A}{\partial x^\mu} - y_\rho^A \frac{\partial \xi^\rho}{\partial x^\mu} + y_\mu^C \frac{\partial \xi^A}{\partial y^C} \right) \frac{\partial \mathscr{L}}{\partial y_\mu^A} + \mathscr{L} \frac{\partial \xi^\mu}{\partial x^\mu} \right] \mathrm{d}^m x \ . \end{aligned} \tag{4.44}$$

Then, if $\mathrm{L}(X_\xi)\mathbf{L} = 0$, it follows that

$$\mathscr{F}\mathscr{L}^* f_B^\nu = - \frac{\partial}{\partial y_\nu^B} \left(\mathscr{L} \frac{\partial \xi^\mu}{\partial x^\mu} \right) - \frac{\partial \xi^A}{\partial y^B} \frac{\partial \mathscr{L}}{\partial y_\nu^A} + \frac{\partial \xi^\nu}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial y_\mu^B} = - \frac{\partial \xi^\mu}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial y_\nu^B} - \frac{\partial \xi^A}{\partial y^B} \frac{\partial \mathscr{L}}{\partial y_\nu^A} + \frac{\partial \xi^\nu}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial y_\mu^B} \ , \tag{4.45}$$

hence,

$$f_B^\nu = - \frac{\partial \xi^\mu}{\partial x^\mu} p_B^\nu - \frac{\partial \xi^A}{\partial y^B} p_A^\mu + \frac{\partial \xi^\nu}{\partial x^\mu} p_B^\mu \ . \tag{4.46}$$

Then with $f^\mu = -\xi^\mu$ and $f^A = -\xi^A$, the expression above is in agreement with the expression (4.40) for the canonical lift $Y_\xi \in \mathfrak{X}(J^1\pi^*)$, and hence, $\mathscr{F}\mathscr{L}_* X_\xi = Y_\xi$. ■

Observe that, if $\mathrm{L}(X_\xi)\mathrm{d}^m x = 0$, then $\mathrm{L}(X_\xi)\mathbf{L} = 0 \iff \mathrm{L}(X_\xi)\mathscr{L} = 0$.

4.5 Canonical Lifts to Submanifolds of $J^1\pi^*$

Finally, we investigate the notion of canonical lifts to primary constraint submanifolds $P_\circ$ of $J^1\pi^*$ for singular De Donder–Weyl Hamiltonian field theories. In general, it is not possible to canonically lift group actions and vector fields from E to any submanifold of $J^1\pi^*$ unless some additional condition is imposed. Here, we present a preliminary definition for such lifts to $P_\circ$ under the rather restrictive condition that the lifted diffeomorphisms to $J^1\pi^*$ preserve the primary constraint submanifold.

As before, let $\jmath_\circ \colon P_\circ \hookrightarrow J^1\pi^*$ be a τ-transverse embedded submanifold of $J^1\pi^*$ so that $P_\circ$ is a fibered submanifold over E and M, with natural projections (3.8), arising from an almost-regular Lagrangian system as $P_\circ = \mathrm{Im}\mathscr{F}\mathscr{L}$.

Definition 4.8 *Let $\Phi_E \colon E \to E$ be a diffeomorphism which induces a diffeomorphism $\Phi_M \colon M \to M$, and let $\Phi_{J^1\pi^*} \colon J^1\pi^* \to J^1\pi^*$ be the canonical lift of Φ_E to $J^1\pi^*$. If $\Phi_{J^1\pi^*}$ leaves $P_\circ$ invariant, the* ***canonical lift of*** *Φ_E* ***to*** *$P_\circ \subset J^1\pi^*$ is the restriction of $\Phi_{J^1\pi^*}$ to $P_\circ$,*

$$\Phi_{P_\circ} \equiv \Phi_{J^1\pi^*}|_{P_\circ} \colon P_\circ \to P_\circ \ . \tag{4.47}$$

The canonically lifted infinitesimal vector field generators of such diffeomorphisms is then defined as follows:

Definition 4.9 *Let $\xi_E \in \mathfrak{X}(E)$ be a π-projectable vector field whose flux consists of one-parameter groups of local diffeomorphisms of E, denoted ϕ_t. Furthermore, suppose that the canonical lift $Y_\xi \in \mathfrak{X}(J^1\pi^*)$ of ξ_E to $J^1\pi^*$ is tangent to $P_\circ$. Then, the* ***canonical lift of*** *ξ_E* ***to*** *$P_\circ \subset J^1\pi^*$ is the vector field $Y^\circ_\xi \in \mathfrak{X}(P_\circ)$ whose flux consists of the one-parameter groups of local diffeomorphisms ϕ°_t which are the canonical lifts of ϕ_t to $P_\circ$, and hence*

$$(\jmath_\circ)_* Y^\circ_\xi = Y_\xi|_{P_\circ} \ . \tag{4.48}$$

The geometrical setting is depicted in the following diagram:

$$\begin{tikzcd}
\mathcal{M}\pi \arrow[r, "\sigma"'] \arrow[ddr, "\widetilde{\sigma}"'] & J^1\pi^* \arrow[r, "\Phi_{J^1\pi^*}"] & J^1\pi^* & \mathcal{M}\pi \arrow[l, "\sigma"] \arrow[ddl, "\widetilde{\sigma}"] \\
 & P_\circ \arrow[u, "\jmath_\circ"'] \arrow[r, "\Phi_{P_\circ}"] \arrow[d, "\tau_\circ"'] & P_\circ \arrow[u, "\jmath_\circ"'] \arrow[d, "\tau_\circ"] & \\
 & E \arrow[r, "\Phi_E"'] & E &
\end{tikzcd}$$

Now, consider the particular case in which $(P_\circ, \Omega^\circ_{\mathscr{H}})$ is the Hamiltonian system associated to an almost-Lagrangian system $(J^1\pi, \Omega_{\mathscr{L}})$ with final constraint submanifold $P_f \subseteq P_\circ$. Then, if $\mathcal{S}_f \subset J^1\pi$ is the final Lagrangian constraint submanifold, then $\mathscr{F}\mathscr{L}_\circ(\mathcal{S}_f) = P_f$ ([39, 63, 131]).

A natural question to ask is if there exists a result analogous to Proposition 4.6 for these kinds of singular field theories. However, in general, it is not the case that $\mathscr{F}\mathscr{L}_{\circ *} X_\xi = Y^\circ_\xi$, even when $\mathrm{L}(X_\xi)\mathbf{L} = 0$. This is due to the fact that the canonical lift $X_\xi \in \mathfrak{X}(J^1\pi)$ is not always $\mathscr{F}\mathscr{L}_\circ$-projectable on all of $J^1\pi$. Moreover, if $\mathrm{L}(X_\xi)\mathbf{L} = 0$ and X_ξ is $\mathscr{F}\mathscr{L}_\circ$-projectable on all of $J^1\pi$, then we can ask if $\mathscr{F}\mathscr{L}_{\circ *} X_\xi = Y^\circ_\xi$ always holds; this is difficult to prove in general, but the result holds at least in the singular field theories we study in this paper.

Fortunately, in some cases when X_ξ is not $\mathscr{F}\mathscr{L}_\circ$-projectable on all of $J^1\pi$, X_ξ is at least $\mathscr{F}\mathscr{L}_\circ$-projectable from the final Lagrangian constraint submanifold $\mathcal{S}_f \subset J^1\pi$ which projects onto the final

Hamiltonian constraint submanifold $P_f \subset J^1\pi^*$. Moreover, as $\mathscr{FL}_\circ$ is a surjective submersion, if $X_\xi \in \mathfrak{X}(J^1\pi)$ is $\mathscr{FL}_\circ$-projectable and tangent to $\mathcal{S}_f$, then $(\mathscr{FL}_\circ)_* X_\xi \in \mathfrak{X}(P_\circ)$ is tangent to P_f. Then, in such cases, assuming that $\mathrm{L}(X_\xi)\mathbf{L} = 0$ and that X_ξ is only $\mathscr{FL}_\circ$-projectable from points on $\mathcal{S}_f \subset J^1\pi$, we ask if $\mathscr{FL}_{\circ *} X_\xi|_{\mathcal{S}_f} = Y^\circ_\xi|_{P_f}$ always holds; this is also difficult to prove in general, but is straightforward to check when working with a specific field theory.

In closing, if $\xi_E \in \mathfrak{X}(E)$ is a canonical lift of a vector field $\xi_M \in \mathfrak{X}(M)$, then the canonical lift of ξ_E to $\mathcal{M}\pi$, $J^1\pi^*$, and $P_\circ$ are also called the ***canonical lifts*** of ξ_M to $\mathcal{M}\pi$, $J^1\pi^*$, and $P_\circ$, which are denoted $Z_\xi \in \mathfrak{X}(\mathcal{M}\pi)$, $Y_\xi \in \mathfrak{X}(J^1\pi^*)$, and $Y^\circ_\xi \in \mathfrak{X}(P_\circ)$ respectively. The same goes for canonical lifts of diffeomorphisms from M to $\mathcal{M}\pi$, $J^1\pi^*$, and $P_\circ$.

5 Noether Symmetries

We are now prepared to discuss *Noether symmetries* in formal geometric terms, providing a powerful framework for their analysis. The terminology used to describe symmetries can vary substantially throughout the literature; one of our aims in this chapter is to clarify some of the common terminology. For references on the treatment of Noether symmetries in the multisymplectic framework, see e.g. [67] and references therein.

Consider in general terms a (pre)multisymplectic phase space (P, Ω) and a Lie group G which acts as a diffeomorphism of P with group action denoted $\Phi : P \to P$. Furthermore, let $\mathfrak{g}$ be the Lie algebra of G and let $X_\xi \in \mathfrak{X}(P)$ be an infinitesimal generating vector field induced by some $\xi \in \mathfrak{g}$. Now, if

$$\Phi^*\Omega = \Omega \ , \tag{5.1}$$

then Φ is called a ***Noether symmetry***. Consequently,

$$\mathrm{L}(X_\xi)\Omega = 0 \ , \tag{5.2}$$

and X_ξ is usually called and an ***infinitesimal Noether symmetry***. Furthermore, if

$$\Phi^*\Theta = \Theta \ , \tag{5.3}$$

then

$$\mathrm{L}(X_\xi)\Theta = 0 \ , \tag{5.4}$$

and Φ is called an ***exact Noether symmetry*** while X_ξ is usually called an ***exact infinitesimal Noether symmetry***. We will use the terminology loosely here, referring to both Φ and X_ξ as (exact) Noether symmetries.

Furthermore, if (P, Ω) is the premultisymplectic phase space of some almost-regular field theory, then Noether symmetries are additionally diffeomorphisms which preserve the fibers of $P \to E$ and leave the final constraint submanifold $\mathcal{C}_f \hookrightarrow P$ is invariant as $\Phi|_{\mathcal{C}_f} : \mathcal{C}_f \to \mathcal{C}_f$. This means that the infinitesimal vector field which generates the group action must be everywhere tangent to P. We then introduce the following notation. The set of vector fields on P which are tangent to $\mathcal{C}$ will be denoted here as

$$\underline{\mathfrak{X}(\mathcal{C})} \equiv \{Y \in \mathfrak{X}(P) \mid \exists Y_\mathcal{C} \in \mathfrak{X}(\mathcal{C}) \ \mid \ \jmath_{\mathcal{C}*} Y_\mathcal{C} = Y|_\mathcal{C}\} \ . \tag{5.5}$$

The elements of $\underline{\mathfrak{X}(\mathcal{C})}$ are the vector fields $Y \in \mathfrak{X}(P)$ which, for some $Y_\mathcal{C} \in \mathfrak{X}(\mathcal{C})$, make the following diagram commutative:

$$\begin{tikzcd} P \arrow[r, "Y"] & \mathrm{T}P \\ \mathcal{C} \arrow[u, "\jmath_\mathcal{C}"] \arrow[r, "Y_\mathcal{C}"'] & \mathrm{T}\mathcal{C} \arrow[u, "\mathrm{T}\jmath_\mathcal{C}"'] \end{tikzcd}$$

Now, a vector field which produces a Noether symmetry on $J^1\pi$ may be $\mathscr{FL}$-projectable only from some of the Lagrangian constraint submanifold $\mathcal{S}_i \hookrightarrow J^1\pi$. If this is the case, then the vector field produced by the push-forward of the Legendre map is a vector field on the corresponding Hamiltonian constraint submanifold P_i. Furthermore, a Noether symmetry may exist only on one of the Lagrangian constraint submanifolds $\mathcal{S}_i$. If this is the case, then it follows that, on the Hamiltonian side, the Noether symmetry exists only on the corresponding constraint submanifold P_i. The precise description of these possible scenarios of how symmetries behave in the presence of premultisymplectic constraints are as follows:

1. There exists some $X \in \mathfrak{X}(J^1\pi)$ such that $L(X)\Omega_{\mathscr{L}} = 0$ and is $\mathscr{FL}$-projectable only from the constraint submanifold $\mathcal{S}_f \hookrightarrow J^1\pi$. Then, on the corresponding Hamiltonian constraint submanifold $P_f \subset P_\circ$, there exists the vector field $Y = \mathscr{FL}_* X|_{\mathcal{S}_f} \in \mathfrak{X}(P_f)$ such that $L(Y)\Omega_h^\circ|_{P_f} = 0$. Furthermore, the vector field $X \in \mathfrak{X}(J^1\pi)$ may or may not be the local extension of some $Z|_{\mathcal{S}_f}$ to $J^1\pi$ for some $Z \in \mathfrak{X}(J^1\pi)$.

2. There exists some $X \in \underline{\mathfrak{X}(\mathcal{S}_f)}$ such that $L(X)\Omega_{\mathscr{L}}|_{\mathcal{S}_f} = 0$ and X is $\mathscr{FL}$-projectable only from $\mathcal{S}_f$. Then, on the corresponding Hamiltonian constraint submanifold $P_f \subset P_\circ$, there exists the vector field $Y|_{P_f} = \mathscr{FL}_* X|_{\mathcal{S}_f} \in \underline{\mathfrak{X}(P_f)}|_{P_f}$ such that $L(Y)\Omega_{\mathrm{h}}^\circ|_{P_f} = 0$. Furthermore, it is possible to construct a local extension of Y to $P_\circ$, denoted as $\widetilde{Y} \in \mathfrak{X}(P_\circ)$; but $X \in \mathfrak{X}(J^1\pi)$ may or may not be the local extension of some $Z|_{\mathcal{S}_f}$ to $J^1\pi$ for some $Z \in \mathfrak{X}(J^1\pi)$.

Now, if a Noether symmetry on P arises as a canonical lift from E, then we say that the symmetry is ***natural***; the vast majority of Noether symmetries in physics are in fact natural symmetries. Furthermore, in the Lagrangian setting, if $j^1\phi\colon M \to J^1\pi$ is a holonomic solution to the Euler–Lagrange equations and $\Phi = j^1\varphi : J^1\pi \to J^1\pi$ is a natural Noether symmetry, then

$$\begin{aligned} (j^1(\varphi\circ\phi))^*\, i(X)\Omega_{\mathscr{L}} &= (j^k\phi)^*(j^k\varphi)^*\, i(X)\Omega_{\mathscr{L}} \\ &= (j^1\phi)^*\, i((j^1\varphi)_*^{-1}X)((j^1\varphi)^*\Omega_{\mathscr{L}}) = (j^1\phi)^*\, i(X')\Omega_{\mathscr{L}} = 0\ , \end{aligned} \tag{5.6}$$

since $X' = \Phi_*^{-1}X \in \mathfrak{X}(J^1\pi)$, $\Phi^*\Omega_{\mathscr{L}} = \Omega_{\mathscr{L}}$, and $\Phi^*\Omega_{\mathscr{L}} = 0$. Therefore $j^1(\varphi\circ\phi)$ is also a holonomic solution to the Euler–Lagrange equations, and it follows that every natural Noether symmetry transforms holonomic solutions to the Lagrangian field equations into equivalent holonomic solutions. Additionally, natural Noether symmetries which are exact satisfy the property that they preserve the Lagrangian density if, and only if, they preserve the Poincaré–Cartan form [46, 136],

$$\mathrm{L}(X_\xi)\mathbf{L} = 0 \ \Leftrightarrow \mathrm{L}(X_\xi)\Theta_{\mathscr{L}} = 0\ . \tag{5.7}$$

Proposition 5.1 *Every (infinitesimal) natural Noether symmetry transforms solutions of the Hamiltonian field equations into equivalent solutions.*

(*Proof*) Let $\psi\colon M \to J^1\pi^*$ be a section solution to the field equations; that is, $\psi^* i(X)\Omega_{\mathscr{H}} = 0$, for every $X \in \mathfrak{X}(J^1\pi^*)$. If $\Phi \in \mathrm{Diff}(J^1\pi^*)$ is a natural Noether symmetry, then it is the canonical lift of a diffeomorphism $\Phi_E\colon E \to E$ which restricts to a diffeomorphism $\Phi_M\colon M \to M$. Therefore, Φ preserves the fibration $\overline{\tau}\colon J^1\pi^* \to M$ and, hence, $\Phi\circ\psi\circ\Phi_M^{-1}$ is a section of $\overline{\tau}$. Then,

$$\begin{aligned} (\Phi\circ\psi\circ\Phi_M^{-1})^*\, i(X)\Omega_{\mathscr{H}} &= (\Phi_M^{-1})^*[\psi^*(\Phi^*\, i(X)\Omega_{\mathscr{H}})] = (\Phi_M^{-1})^*[\psi^*\, i(\Phi_*^{-1}X)(\Phi^*\Omega_{\mathscr{H}})] \\ &= (\Phi_M^{-1})^*[\psi^*\, i(X')\Omega_{\mathscr{H}}] = 0\ , \end{aligned}$$

since Φ_M^{-1} is a diffeomorphism and $\Phi^*\Omega_{\mathscr{H}} = \Omega_{\mathscr{H}}$, where $X' = \Phi_*^{-1}X \in \mathfrak{X}(J^1\pi^*)$ is an arbitrary vector field. Therefore, $\Phi\circ\psi\circ\Phi_M^{-1}$ is also a section solution to the field equations.

The proof in the infinitesimal case follows directly from this result and the definition. Moreover, in the almost-regular case, the argument carries over straightforwardly for natural symmetries which preserve the fibers of $P_\circ \to E$. ■

Furthermore, if Noether symmetry that is infinitesimally generated by a vector field that is vertical over M, then such a symmetry is usually called a ***gauge symmetry*** in the differential geometry literature. However, in the physics literature, a Noether symmetry is usually called a *gauge symmetry* if its generators are local functions on M; such symmetries are instead called *local symmetries* sometimes, including in the presentation of the infinite-dimensional covariant phase space formalism by Lee and Wald [108]. Moreover, these symmetries include spacetime (or world-volume) diffeomorphisms which would not be called *gauge symmetries* in the differential geometry literature, since such symmetries are infinitesimally generated on the configuration manifold E by vector fields which are expressed locally in equation (4.5) as

$$\xi_E = -\xi^\mu(x)\frac{\partial}{\partial x^\mu} - \xi^A(x,y)\frac{\partial}{\partial y^A} \in \mathfrak{X}(E) \ , \tag{5.8}$$

which is clearly not vertical over M. Nevertheless, such diffeomorphisms of the base space M can equivalently be generated by

$$\xi_E^{\mathrm{v}} = -Q^A\frac{\partial}{\partial y^A} = -\left(\xi^A - \xi^\nu y^A_\nu\right)\frac{\partial}{\partial y^A} \ , \tag{5.9}$$

known as the ***vertical*** (or ***characteristic***) ***representative*** of the transformation $\mathrm{L}(X_\xi)y^A$. This vector field is in fact a *generalized vector field* $J^1\pi \to TE$ and is therefore a vector along the projection $\pi^1_0 : J^1\pi \to E$. More specifically, $\xi^{\mathrm{v}}_E \in \mathfrak{X}^{V\pi}(E)$ as $\xi^{\mathrm{v}}_E : J^1\pi \to V\pi \subset TE$. Furthermore, the holonomic (canonical) lift of ξ^{v}_E to $J^k\pi$ is given as

$$X^{\mathrm{v}}_\xi = j^k\xi^{\mathrm{v}}_E = -\partial_{\mathbf{J}}Q^A\frac{\partial}{\partial y^A_{\mathbf{J}}} \in \mathfrak{X}^{V(\pi^k)}(J^k\pi) \ . \tag{5.10}$$

Such vector fields are called ***evolutionary vector fields*** [136].

5.1 Gauge Forms

We continue this section by discussing *gauge forms* as presented in [25]. First, we recall from [25] that as a result of the affine structure of $J^1\pi$, it is possible to make the following unique (one-to-one) identification between two differential m-forms of $\Omega^m(J^1\pi)$ as,

$$\overline{\omega} = (\rho + \rho^\mu_A y^A_\mu)\mathrm{d}^m x \quad \Leftrightarrow \quad \omega = \rho\,\mathrm{d}^m x + \rho^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu \ , \tag{5.11}$$

Now, if there exist some functions $f^\mu \in C^\infty(E)$ such that,

$$\rho = \frac{\partial f^\mu}{\partial x^\mu} \quad \text{and} \quad \rho^\mu_A = \frac{\partial f^\mu}{\partial y^A} \ , \tag{5.12}$$

then,

$$\overline{\omega} = \left(\frac{\partial f^\mu}{\partial x^\mu} + \frac{\partial f^\mu}{\partial y^A}y^A_\mu\right)\mathrm{d}^m x = \partial_\mu f^\mu \mathrm{d}^m x \quad \text{and} \quad \omega = \frac{\partial f^\mu}{\partial x^\mu}\mathrm{d}^m x + \frac{\partial f^\mu}{\partial y^A}\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu \ . \tag{5.13}$$

In such a case, ω is an exact form and we call $\overline{\omega}$ a ***gauge form***. Notice, that

$$\overline{\omega} = \mathrm{d_h}\left(f^\mu \mathrm{d}^{m-1}x_\mu\right) \in \Omega^{m,0}(J^1\pi) \quad \text{and} \quad \omega = \mathrm{d}\left(f^\mu \mathrm{d}^{m-1}x_\mu\right) \in \Omega^m(J^1\pi) \ . \tag{5.14}$$

Furthermore,

$$j^1\phi^*\omega = j^1\phi^*\overline{\omega} = \mathrm{d}\left[\left(f^\mu \mathrm{d}^{m-1}x_\mu\right)\circ j^1\phi\right] = \partial_\mu\left(f^\mu \circ j^1\phi\right)\mathrm{d}^m x \ . \tag{5.15}$$

Now, a gauge symmetry group action $\Phi : J^1\pi \to J^1\pi$ with infinitesimal generator $X_\xi \in \mathfrak{X}(J^1\pi)$ changes the Lagrangian density on $J^1\pi$ by some gauge form:

$$\Phi^*\mathbf{L} = \mathbf{L} + \overline{\alpha}_\xi \quad \Leftrightarrow \quad \mathrm{L}(X_\xi)\mathbf{L} = \overline{\alpha}_\xi \quad . \tag{5.16}$$

This is generally the definition of a Noether symmetry found in the physics literature. Furthermore, it follows that [25]

$$\Phi^*\Theta_{\mathscr{L}} = \Theta_{\mathscr{L}} + \alpha_\xi \quad \Leftrightarrow \quad \mathrm{L}(X_\xi)\Theta_{\mathscr{L}} = \alpha_\xi \quad . \tag{5.17}$$

where $\alpha_\xi \in \Omega^m(J^1\pi)$ is the exact form in correspondence with the gauge form $\overline{\alpha}_\xi \in \Omega^m(J^1\pi)$.

5.2 Hamiltonian Vector Fields and Hamiltonian Forms

Now, as an aside, we briefly discuss the notions of *Hamiltonian vector fields* and *Hamiltonian forms*nar in (pre)multisymplectic geometry. The following discussion closely follows [25, 26].

Given a (pre)multisymplectic manifold (P, Ω), a vector field $X \in \mathfrak{X}(P)$ is said to be ***locally Hamiltonian*** for this (pre)multisymplectic structure if,

$$\mathrm{L}(X)\Omega = 0 \ . \tag{5.18}$$

Furthermore, $X_\alpha \in \mathfrak{X}(P)$ is ***(globally) Hamiltonian*** if there exists some $\alpha \in \Omega^{m-1}(P)$ such that

$$i(X_\alpha)\Omega = -\mathrm{d}\alpha \ , \tag{5.19}$$

called a ***Hamiltonian*** $(m-1)$***-form***. The set of Hamiltonian vector fields, which we will denote as $\mathfrak{X}_{\mathscr{H}}(P)$, is a subset of the set of locally Hamiltonian vector fields $\mathfrak{X}_{\mathscr{LH}}(P)$. The module of Hamiltonian $(m-1)$-forms on P will be denoted as $\mathscr{H}^{m-1}(P)$.

The Lie bracket of two locally Hamiltonian vector fields X_α and X_β is also a Hamiltonian vector field, as

$$i\left([X_\alpha, X_\beta]\right)\Omega = \mathrm{d}\left(i(X_\alpha)i(X_\beta)\Omega\right) \ . \tag{5.20}$$

It thereby follows that $\mathscr{H}^{m-1}(P)$ is an ideal in the Lie subalgebra of locally Hamiltonian vector fields.

An algebra structure is defined on $\mathscr{H}^{m-1}(P)$ by

$$\{\alpha, \beta\} \equiv i(X_\alpha)i(X_\beta)\Omega \ , \tag{5.21}$$

where

$$[X_\alpha, X_\beta] = X_{\{\alpha,\beta\}} \ . \tag{5.22}$$

It follows that,

$$\{\{\alpha, \beta\}, \gamma\} + \{\{\gamma, \alpha\}, \beta\} + \{\{\beta, \alpha\}, \gamma\} = \mathrm{d}\left(i(X_\alpha)i(X_\beta)i(X_\gamma)\Omega\right) \ . \tag{5.23}$$

Hamiltonian forms are determined, in general, up to closed forms. The equivalence class of some Hamiltonian $(m-1)$-form modulo closed $(m-1)$-forms is written as

$$\widetilde{\alpha} = \left\{\alpha + \lambda | \forall\lambda \in \Omega^{m-1}(P),\ \mathrm{d}\lambda = 0\right\} \ . \tag{5.24}$$

Such equivalence classes belong to the following quotient space:

$$\widetilde{\mathscr{H}}^{m-1}(P) = \mathscr{H}^{m-1}(P)/Z^{m-1}(P) \ , \tag{5.25}$$

where $Z^{m-1}(P)$ is the space of closed $(m-1)$-forms on P. The bracket of $(m-1)$-forms is defined as

$$\left\{\widetilde{\alpha}, \widetilde{\beta}\right\} \equiv \widetilde{\{\alpha, \beta\}} \ . \tag{5.26}$$

Then, the Jacobi identity,

$$\left\{\left\{\widetilde{\alpha}, \widetilde{\beta}\right\}, \widetilde{\gamma}\right\} + \left\{\{\widetilde{\gamma}, \widetilde{\alpha}\}, \widetilde{\beta}\right\} + \left\{\left\{\widetilde{\beta}, \widetilde{\gamma}\right\}, \widetilde{\alpha}\right\} = 0 \ , \tag{5.27}$$

is satisfied as a consequence of (5.23). It is also important to note that Hamiltonian vector fields, as defined in (5.19), are only well-defined up to vector fields which lie in the kernel of Ω. In particular, if two Hamiltonian vector fields X and Y both correspond to the same Hamiltonian $(m-1)$-form, $i(X)\Omega = i(Y)\Omega = \mathrm{d}\alpha$, then, the vector field given by $X - Y$ lies in the kernel of Ω as $i(X-Y)\Omega = 0$; such vector fields are called ***characteristic vector fields***. The space of Hamiltonian vector fields modulo characteristic vector fields will be denoted $\widetilde{\mathfrak{X}}_{\mathcal{H}}(P)$. It follows that $\left(\widetilde{\mathcal{H}}^{m-1}(P), \{\ ,\ \}\right)$ is a Lie algebra which is isomorphic to $\left(\widetilde{\mathfrak{X}}_{\mathcal{H}}(P), [\ ,\ ]\right)$ [25].

5.3 Multimomentum Maps

We now turn our attention to the concept of conserved quantities and multimomentum maps.

Working with the infinitesimal generators of every $\xi \in \mathfrak{g}$ amounts to J_ξ to be linear in ξ as will be the case in the theories that are investigated in this paper. This makes it possible to define a linear map, $J|_{\mathrm{p}}$, by considering a point $\mathrm{p} \in P$ so that $J_{\mathrm{p}}(\xi) \equiv J_\xi|_{\mathrm{p}} \in \Lambda^{m-1}(\mathrm{T_p}P)$ gives $J|_{\mathrm{p}} : \mathfrak{g} \to \Lambda^{m-1}(\mathrm{T_p}P) : \xi \mapsto J_{\mathrm{p}}(\xi)$, for every $\xi \in \mathfrak{g}$. The linear map $J|_{\mathrm{p}}$ is an element of $\mathfrak{g}^* \otimes \Lambda^{m-1}(\mathrm{T_p}P)$. This structure makes it possible to construct another map, J, using $J(\mathrm{p}) \equiv J|_{\mathrm{p}}$ so that $J : P \to \mathfrak{g}^* \otimes \Lambda^{m-1}T^*P : \mathrm{p} \mapsto J(\mathrm{p})$, for every $\mathrm{p} \in P$. The map J is called the ***multimomentum map***, although this terminology is used to refer to both J and $J|_\xi$. This setting can be concisely specified through the natural pairing $\langle \cdot, \cdot \rangle$ between $\mathfrak{g}^*$ and $\mathfrak{g}$ written as

$$J_\xi|_z = \langle J(z), \xi \rangle \ . \tag{5.28}$$

The multimomentum map provides us with an $(m-1)$-form associated with the symmetry group G on P which we will denote simply as $J_\xi \in \Omega^{m-1}(P)$ and satisfies

$$i(X_\xi)\Omega = -\mathrm{d}J_\xi \ . \tag{5.29}$$

Now, since each Noether symmetry yields $\mathrm{L}(X_\xi)\Theta = \alpha_\xi = \mathrm{d}\beta_\xi$, it follows that

$$\mathrm{L}(X_\xi)\Omega = -\,\mathrm{L}(X_\xi)\mathrm{d}\Theta = -\mathrm{d}\,\mathrm{L}(X_\xi)\Theta = -\mathrm{d}\alpha_\xi = 0 \ . \tag{5.30}$$

Then, there exists a multimomentum map $J_\xi \in \Omega^{m-1}(P)$ which is a conserved quantity as,

$$\mathrm{d}J_\xi = -\,i(X_\xi)\Omega = i(X_\xi)\mathrm{d}\Theta = \mathrm{L}(X_\xi)\Theta - \mathrm{d}\,i(X_\xi)\Theta = \mathrm{d}\beta_\xi - \mathrm{d}\,i(X_\xi)\Theta \ . \tag{5.31}$$

It thereby follows that J_ξ given as,

$$J_\xi = -i(X_\xi)\Theta + \beta_\xi \ , \tag{5.32}$$

and is determined only up to exact $(m-1)$-forms $\mathrm{d}Q_\xi$. Furthermore, the sections $\psi : M \to P$ can be used to pull back the multimomentum map, yielding the ***Noether current***:

$$\mathcal{J}_\xi \equiv \psi^* J_\xi = j^\mu_\xi \mathrm{d}^{m-1}x_\mu \in \Omega^{m-1}(M) \ . \tag{5.33}$$

Now, a multimomentum map is ***Ad*-equivariant*** if the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "J_\xi"] \arrow[d, "\Phi_g"'] & \mathfrak{g}^* \otimes \Lambda^{m-1}T^*P \arrow[d, "\mathrm{Ad}^* \otimes \Lambda^{m-1}T^*\Phi_{g^{-1}}"] \\ P \arrow[r, "J_\xi"'] & \mathfrak{g}^* \otimes \Lambda^{m-1}T^*P \end{tikzcd} \tag{5.34}$$

Multimomentum maps that are not Ad*-equivariant are sometimes called ***weak multimomentum maps***.

Clearly from equation (5.31), multimomentum maps are Hamiltonian forms. In fact, the infinitesimal variation of the Poincarè–Cartarn form $\mathrm{L}(X_\xi)\Theta = \alpha(\xi) \in \Omega^m(P)$ is a 1-cocycle on the Lie algebra $\mathfrak{g}$.

$$\begin{tikzcd} & & & \mathfrak{g} \arrow[dl, "J_\xi"'] \arrow[d, "\Delta_g"] & \\ 0 \arrow[r] & \mathscr{H}^{m-1}(P) \arrow[r, "i"'] & \widetilde{\mathscr{H}}^{m-1}(P) \arrow[r, "h"'] & \widetilde{\mathfrak{X}}_{\mathscr{H}}(P) \arrow[r] & 0 \end{tikzcd} \tag{5.35}$$

Now define the ***multisymplectic bracket*** as,

$$\{J_\xi, J_\zeta\} \equiv i(X_\xi) i(X_\zeta)\Omega \ . \tag{5.36}$$

It follows that, for any $\xi, \zeta \in \mathfrak{g}$,

$$\{J_\xi, J_\zeta\} = J_{[\xi,\zeta]} + \alpha(\xi,\zeta) \ , \tag{5.37}$$

where $\alpha(\xi,\zeta) \in \Omega^{m-1}(P)$ is a 2-cocycle on $\mathfrak{g}$ (i.e. $\mathrm{d}_{\mathfrak{g}}\alpha(\xi,\zeta) = 0$). Furthermore, $\alpha(\xi,\zeta)$ vanishes (i.e. $H^2(\mathfrak{g})$=0) if, and only if, the multimomentum maps are Ad*-equivariant.

If the multimomentum map is not Ad*-equivariant, then $J_\xi : \mathfrak{g} \to \widetilde{\mathscr{H}}^{m-1}(P)$ is not a Lie algebra homomorphism. However, there exists a Lie algebra homomorphism from the central extension of $\mathfrak{g}$ (see Appendix D), namely, $\bar{J}_\xi : \mathfrak{g} \oplus \mathscr{H}^{m-1}(P) \to \widetilde{\mathscr{H}}^{m-1}(P)$. We then have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{H}^{m-1}(P) \arrow[r] \arrow[d] & \mathfrak{g} \oplus \mathscr{H}^{m-1}(P) \arrow[r] \arrow[d, "\bar{J}_\xi"] & \mathfrak{g} \arrow[r] \arrow[d, "\Delta_{\mathfrak{g}}"] & 0 \\ 0 \arrow[r] & \mathscr{H}^{m-1}(P) \arrow[r, "i"'] & \widetilde{\mathscr{H}}^{m-1}(P) \arrow[r, "h"'] & \widetilde{\mathfrak{X}}_{\mathscr{H}}(P) \arrow[r] & 0 \end{tikzcd} \tag{5.38}$$

The bracket on the centrally extended Lie algebra $\mathfrak{g} \oplus \mathscr{H}^{m-1}(P)$ is defined as,

$$[(\xi,\beta),(\zeta,\gamma)]_{\mathfrak{g}\oplus\mathscr{H}^{m-1}(P)} \equiv ([\xi,\zeta], \alpha(\xi,\zeta)) \quad , \quad \forall \xi,\zeta \in \mathfrak{g} \text{ and } \forall \beta,\gamma \in \mathscr{H}^{m-1}(P) \ . \tag{5.39}$$

Finally, we note that if $X_\xi \equiv j^1\xi_E \in \mathfrak{X}(J^1\pi)$ is the Noether symmetry arising from diffeomorphisms of M, then the Lagrangian multimomentum map is given as

$$J_{\mathscr{L}}(X_\xi) = -\, i(X_\xi)\Theta_{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial y^A_\mu} \left(\xi^A \mathrm{d}^{m-1}x_\mu - \xi^\nu y^A_\mu \mathrm{d}^{m-1}x_\nu - \xi^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\mu\nu} \right) + \mathscr{L}\, \xi^\nu \mathrm{d}^{m-1}x_\nu \ . \tag{5.40}$$

The corresponding Noether current is given by

$$j^\mu_\xi \mathrm{d}^{m-1}x_\mu = (j^1\phi)^* \left[-i\left(X_\xi\right)\Theta_{\mathscr{L}} \right] = \left[\left(\frac{\partial \mathscr{L}}{\partial y^A_\mu} \circ j^1\phi \right) \left(\widetilde{\xi}^A - \xi^\nu \partial_\nu \phi^A \right) + \xi^\mu \left(\mathscr{L} \circ j^1\phi \right) \eta^\mu{}_\nu \right] \mathrm{d}^{m-1}x_\mu \ . \tag{5.41}$$

The part of the Noether current shown above which corresponds to the infinitesimal spacetime transformations is linear in ξ and the ***canonical energy–momentum tensor***, denoted as usual as $T^\mu{}_\nu$, is defined from the terms contracted with ξ^ν above:

$$-\xi^\nu \left[\left(\frac{\partial L}{\partial y^A_\mu} \circ j^1\phi \right) \partial_\nu \phi^A - \left(\mathscr{L} \circ j^1\phi \right) \eta^\mu{}_\nu \right] \mathrm{d}^{m-1}x_\mu \equiv -\xi^\nu T^\mu{}_\nu \mathrm{d}^{m-1}x_\mu \ . \tag{5.42}$$

6 Covariant Phase Space Formalism and Poisson Structures

In this chapter we investigate the structure of the space of sections over spacetime (or worldvolume) M of the (pre)multisymplectic bundles studied thus far. These spaces are infinite-dimensional and are where one constructs the covariant phase space formalism [34, 148, 152] typically used in the physics literature (for the state of the art, see e.g. [29]-[32], [82, 113, 114]). Our exposition of the infinite-dimensional covariant phase space formalism summarizes the classic work of Lee and Wald [108], where they show how local symmetries yield constraints under a symplectic reduction procedure of a presymplectic form defined by integration over a Cauchy surface $\Sigma \subset M$. We also review how the presymplectic structure of the infinite-dimensional covariant phase space is derived from the multisymplectic Hamilton–Cartan form $\Omega_{\mathscr{H}} \in J^1\pi^*$ (this was originally done in [68] and in [51] along similar lines). However, the spaces of sections of the (pre)multisymplectic bundles are also relevant outside of the context of the infinite-dimensional covariant phase space formalism which involves the use of Cauchy surfaces.

As we have already discussed, the spaces of sections of the (pre)multisymplectic bundles are where functionals are defined. One of our new results shows how the Poisson bracket of functionals introduced by Marsden *et al* in [116] can be constructed from a bivector (denoted Ω^{-1}) which is inverse to the Hamilton–Cartan form $\Omega_{\mathscr{H}} \in J^1\pi^*$ in the sense that the contraction of the bivector with $\Omega_{\mathscr{H}}$ is equal to the volume form of some arbitrary spacetime hypersurface (i.e., we have the tensor contraction $\Omega^{-1} \cdot \Omega_{\mathscr{H}} = \mathrm{d}^{m-1}x$). By taking the tensor contraction of the inverse bivector Ω^{-1} with the exterior derivative of an arbitrary local function $f \in C^\infty(J^1\pi^*)$, and subsequently contracting with $\Omega_{\mathscr{H}}$, one recovers the Hamilton–De Donder–Weyl field equations. We will show how this is analogous to how the field equations are obtained in [116] by taking the Poisson bracket of some arbitrary local functional with the action functional of the theory under investigation. We also demonstrate how the covariant Poisson bracket of functionals in [116] is related to the symplectic structure defined in the infinite-dimensional covariant phase space formalism as presented by Lee and Wald [108]. This is achieved by restricting the inverse bivector so that $\Omega^{-1} \cdot \Omega_{\mathscr{H}}$ gives the volume form of an arbitrary Cauchy surface. The Hamilton–De Donder–Weyl equations are again obtained by calculating the Poisson bracket of some arbitrary functional with the action functional as in [116].

Finally, we present the relation between the multimomentum maps associated with Noether symmetries and the canonical first-class constraints associated with local symmetries as discussed by Lee and Wald [108]. It follows that the Poisson bracket of constraints in [108] is directly related to the (pre)multisymplectic bracket (5.21) of multimomentum maps on $P_\circ \subset J^1\pi^*$. Note that the relation between multimomentum maps and canonical constraints was first presented in the GIMMSY works [68] and [69], although their construction carries out the explicit splitting of space and time, while our construction maintains manifest Lorentz covariance.

6.1 Cauchy Surfaces and Foliations

We begin by briefly reviewing foliations in the context of fiber bundles. Consider the foliation of M with leaves consisting of Cauchy surfaces, denoted generically as $\Sigma \subset M$ (not to be confused with the stationary surface, also denoted as Σ, in other chapters of this thesis). The space of embeddings $\mathrm{Emb}(\Sigma, M)$ consists of maps $\tau \in \mathrm{Emb}(\Sigma, M)$ which give $\tau : \Sigma \hookrightarrow M : \tau(\Sigma) \mapsto \Sigma_\tau$ so that $\Sigma_\tau \subset M$ is a Cauchy surface. Moreover, take $\mathrm{d}^{m-1}x$ to be the induced volume on Σ_τ and n^μ as the components of the normal vector on Σ_τ.

Furthermore, let $\kappa : P \to M$ be a (pre)multisymplectic bundle on which we geometrically construct field theories and let $P_\tau \equiv \kappa^{-1}(\Sigma_\tau)$. It follows that $j_\tau : P_\tau \hookrightarrow P$ is an embedded submanifold since the pre-image of an embedded submanifold under a submersion is also an embedded submanifold [107]. Moreover, we can restrict sections $\psi\colon M \to P$ to the foliation given by $\mathrm{Emb}(\Sigma, M)$ so that $\psi_\tau : \Sigma_\tau \to P_\tau$.

We then have the following commutative diagram:

$$\begin{tikzcd} P_\tau \arrow[r, "j_\tau"] & P \\ \Sigma_\tau \arrow[u, "\psi_\tau"] \arrow[r, "\tau"'] & M \arrow[u, "\psi"'] \end{tikzcd}$$

6.2 The Symplectic and Presymplectic Forms on $\mathcal{J}^1\pi$

The content of this section closely reviews the presentation found in [108], in preparation for making contact with the multisymplectic formulation in the following sections.

Consider a first–order Lagrangian function on the space of holonomic sections: $\mathscr{L} \in C^\infty(\mathcal{J}^1\pi)$; the corresponding Lagrangian density is given as $\mathbf{L} = \mathscr{L}\mathrm{d}^m x$. The field variation $\delta\phi^A$ is a differential form on $\mathcal{J}^1\pi$ as given by the exterior derivative δ on $\mathcal{J}^1\pi$:

$$\delta = \delta\phi^A \otimes \frac{\partial}{\partial\phi^A} + \delta\phi^A_\mu \otimes \frac{\partial}{\partial\phi^A_\mu} \ . \tag{6.1}$$

Then, the variation of the Lagrangian (as in Remark 2.13) is given as

$$\begin{aligned} \delta\mathbf{L} = \delta\mathscr{L} \otimes \mathrm{d}^m x &= \left(\frac{\partial\mathscr{L}}{\partial\phi^A}\delta\phi^A + \frac{\partial\mathscr{L}}{\partial\phi^A_\mu}\delta\phi^A_\mu\right) \otimes \mathrm{d}^m x \\ &= \left[\frac{\partial\mathscr{L}}{\partial\phi^A} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\phi^A_\mu}\right)\right]\delta\phi^A \otimes \mathrm{d}^m x + \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\phi^A_\mu}\delta\phi^A\right) \otimes \mathrm{d}^m x \\ &\equiv \left(\frac{\delta\mathscr{L}}{\delta\phi^A}\delta\phi^A + \partial_\mu\theta^\mu\right) \otimes \mathrm{d}^m x \\ &= \frac{\delta\mathscr{L}}{\delta\phi^A}\delta\phi^A \otimes \mathrm{d}^m x + \mathrm{d}\left(\theta^\mu \otimes \mathrm{d}^{m-1}x_\mu\right) \ , \end{aligned} \tag{6.2}$$

recalling that $\dfrac{\delta\mathscr{L}}{\delta\phi^A}$ is the Euler–Lagrange derivative (1.46). We then pull back the boundary term $\theta^\mu \otimes \mathrm{d}^{m-1}x_\mu$ onto some Cauchy surface Σ and integrate to obtain

$$\theta[\phi] \equiv \int_\Sigma \mathrm{d}^{m-1}x_\mu \otimes \theta^\mu = \int_\Sigma \mathrm{d}^{m-1}x_\mu \otimes \left(\frac{\partial\mathscr{L}}{\partial\phi^A_\mu}\delta\phi^A\right) \ . \tag{6.3}$$

This differential form is often called the *presymplectic potential.* We then obtain the following presymplectic form:

$$\begin{aligned} \omega[\phi] \equiv \delta\theta[\phi] = \int_\Sigma \mathrm{d}^{m-1}x_\mu \otimes \delta\theta^\mu &= \int_\Sigma \mathrm{d}^{m-1}x_\mu \otimes \delta\left(\frac{\partial\mathscr{L}}{\partial\phi^A_\mu}\right) \wedge \delta\phi^A \\ &= \int_\Sigma \mathrm{d}^{m-1}x_\mu \otimes \left(\frac{\partial^2\mathscr{L}}{\partial\phi^A\phi^B_\mu}\delta\phi^A \wedge \delta\phi^B + \frac{\partial^2\mathscr{L}}{\partial\phi^A_\nu\phi^B_\mu}\delta\phi^A_\nu \wedge \delta\phi^B\right) \ . \end{aligned} \tag{6.4}$$

The manifest Lorentz covariance of all expressions is preserved, with the understanding that the vector n_μ is timelike since it is the normal vector to some Cauchy surface Σ.

Moreover, since partial derivatives commute with δ, working on the space of holonomic sections $\mathcal{J}^1\pi$ is equivalent to working on the space of field configurations $\mathscr{F}$, which is defined as the space of sections $\mathscr{F}(\phi) \equiv \Gamma(\pi) = \{\phi : M \to E : x^\mu \to (x^\mu, \phi^A)$. Furthermore, since the δ operator is an exterior derivative on $\mathscr{F}$, we can therefore think θ and ω as differential forms on $\mathscr{F}$ as

$$\theta_A(\delta\phi)^A = \theta[\phi] \quad , \quad \omega_{AB}(\delta\phi)^A(\delta\phi)^B = \omega[\phi] \quad . \tag{6.5}$$

The differential form ω_{AB} on $\mathscr{F}$ is presymplectic and is degenerate in field variations $\delta\phi^A$ with support away from the Cauchy surface Σ produce the degeneracy directions $(\delta\phi)^A$.

Now consider the degeneracy subspaces given as the collection of subspaces of the tangent space of $\mathscr{F}$ made up of the degeneracy vectors of ω_{AB}. The step of reduction which follows requires that the distribution degeneracy subspaces is a sub-bundle of the tangent bundle of $\mathscr{F}$. It is shown in [108] that the Lie bracket of any two degeneracy vector fields is also a degeneracy vector field. Then, by Frobenius' theorem, it follows that the distribution of degeneracy subspaces admits integral submanifolds and we define the equivalence relation between two field configurations $\phi_1 \sim \phi_2$ if and only if ϕ_1 and ϕ_2 lie in the same integral submanifold. Now, denote the set of equivalence classes of $\mathscr{F}$ as Γ which we assume to be a manifold and will serve as the phase space of the field theory under investigation. Furthermore, let $\pi : \mathscr{F} \to \Gamma$ be the natural projection, which assigns elements of $\mathscr{F}$ to its equivalence class; we shall assume that this map is continuous and surjective so that $\mathscr{F}$ is a fiber bundle over Γ.

We now wish to construct the symplectic form Ω_{ab} on phase space Γ. To do so, consider two tangent vectors v^A and w^A at some arbitrary point $\phi \in \mathscr{F}$ whose fiber is over the point $\gamma \in \Gamma$. Furthermore, consider the two tangent vectors at the point γ which are given by the pushforward of π as $\pi_* v^A = V^a \in T_\gamma\Gamma$ and $\pi_* w^A = W^a \in T_\gamma\Gamma$. We define the symplectic form Ω_{AB} by the following relation:

$$\Omega_{ab}V^aW^b = \omega_{AB}v^Aw^B \ . \tag{6.6}$$

It follows that,

$$\omega_{AB} = \pi^*\Omega_{ab} \ . \tag{6.7}$$

If Ω_{ab} is invertible (which is not guaranteed by nondegeneracy of Ω_{ab} on infinite dimensional spaces), then Ω^{ab} provides us with a Poisson structure on Γ. Furthermore, the solution submanifold of $\mathscr{F}$ is denoted $\bar{\mathscr{F}}$ and the image of $\pi : \mathscr{F} \to \Gamma$ is the constraint submanifold $\bar{\Gamma}$ so that $\pi : \bar{\mathscr{F}} \to \bar{\Gamma}$. The restriction of π onto the image of $\bar{\mathscr{F}}$ is denoted $\bar{\pi}$. The pullback of ω_{AB} onto $\bar{\mathscr{F}}$ is denoted $\bar{\omega}_{AB}$ and the pullback of Ω_{ab} onto $\bar{\Gamma}$ is denoted $\bar{\Omega}_{ab}$. Then,

$$\bar{\omega}_{AB} = \bar{\pi}^*\bar{\Omega}_{ab} \ . \tag{6.8}$$

Local symmetries on $\mathscr{F}$ are defined as field variations $\delta_\varepsilon\phi^A$ which leave the Lagrangian invariant up to total derivatives so that

$$\delta_\varepsilon\mathscr{L} = \partial_\mu\alpha^\mu \ , \tag{6.9}$$

for some $\alpha^\mu \in C^\infty(\mathcal{J}^1\pi)$. The ***Noether current*** is then defined as,

$$\mathcal{J}^\mu \equiv \theta^\mu_\varepsilon - \alpha^\mu = \frac{\partial\mathscr{L}}{\partial\phi^A}\delta_\varepsilon\phi^A - \alpha^\mu \ , \tag{6.10}$$

from which it follows that

$$\partial_\mu\mathcal{J}^\mu = \left(\delta_\varepsilon\mathscr{L} - \frac{\delta\mathscr{L}}{\delta\phi^A}\delta_\varepsilon\phi^A\right) - \delta_\varepsilon\mathscr{L} = -\frac{\delta\mathscr{L}}{\delta\phi^A}\delta_\varepsilon\phi^A \ , \tag{6.11}$$

It follows that the Noether current is conserved on-shell. Furthermore, the ***Noether charge*** is defined as

$$Q \equiv \int_\Sigma n_\mu\mathcal{J}^\mu \ . \tag{6.12}$$

It is proven in the original work [108] that Q is independent of the choice of Σ and in fact vanishes on-shell for compact Σ. Moreover, when Σ is noncompact, we require appropriate boundary conditions so that the conservation of the Noether current holds on-shell. Moreover, Wald showed us in the noncompact case how to obtain the first law of thermodynamics for stationary black holes by calculating the Noether charge associated with the timelike Killing vector [144] (see also [92, 93, 145]).

Now, another important result which is given as a theorem in [144] states that each local symmetry $(\delta_\varepsilon\phi)^A$ at each solution $\hat{\phi} \in \bar{\mathscr{F}}$ satisfies,

$$(\delta Q)_A = \omega_{AB}(\delta_\varepsilon\hat{\phi})^B \ . \tag{6.13}$$

Yang–Mills theory, in particular, exhibits the additional property that the equation above holds on all of $\mathscr{F}$.

Local symmetries on Γ are defined as the projection by π of local symmetries on $\mathscr{F}$ as

$$X^b = (\delta_\varepsilon\phi)^b = \pi_*(\delta_\varepsilon\phi^A) \ . \tag{6.14}$$

The constraint $\mathcal{C}$ defines $\bar{\Gamma} \subset \Gamma$ as,

$$\mathcal{C}|_{\bar{\Gamma}} = 0 \ . \tag{6.15}$$

Then,

$$\Omega_{ab}X^b = (\delta\mathcal{C})_a|_{\bar{\Gamma}} \ . \tag{6.16}$$

In the case of Yang–Mills theory, the equation above is satisfied everywhere on Γ and the constraint $\mathcal{C}$ is given as the projection of the Noether charge Q [108].

Now, by setting

$$X^a = \Omega^{ab}(\delta\mathcal{C})_b \ , \tag{6.17}$$

we have $\mathrm{L}(X^a)\Omega_{ab} = 0$. Furthermore, we have Ω^{ab} gives us a Poisson structure on Γ as,

$$\{\mathcal{C}_1, \mathcal{C}_2\} \equiv -\Omega^{ab}(\delta\mathcal{C}_1)_a(\delta\mathcal{C}_2)_b \ . \tag{6.18}$$

In fact,

$$\Omega_{ab}[X_1, X_2]^b = (\delta\{\mathcal{C}_1, \mathcal{C}_2\})_a \ , \tag{6.19}$$

as shown in [108].

Furthermore, Lemma 3 in [108] tells us that, at some solution $\hat{\phi} \in \bar{\mathscr{F}}$, the local symmetry $(\delta_\varepsilon\hat{\phi})^A$ is tangent to $\bar{\mathscr{F}}$. The projection by π of local symmetries $(\delta_\varepsilon\hat{\phi})^A$ on $\bar{\mathscr{F}}$ are called ***local symmetries on*** $\bar{\Gamma}$:

$$\widehat{X}^a \equiv \pi_*(\delta_\varepsilon\hat{\phi})^A \ . \tag{6.20}$$

It follows that

$$\bar{\Omega}_{ab}\widehat{X}^b = 0 \ . \tag{6.21}$$

Now, we wish to pass over to the De Donder–Weyl Hamiltonian setting and make contact with the Hamilton–Cartan forms $\Theta_{\mathscr{H}} \in \mathit{\Omega}^m(J^1\pi^*)$ and $\Omega_{\mathscr{H}} \in \mathit{\Omega}^{m+1}(J^1\pi^*)$ (or $\Theta^\circ_{\mathscr{H}} \in \mathit{\Omega}^m(P_\circ)$ and $\Omega^\circ_{\mathscr{H}} \in \mathit{\Omega}^{m+1}(P_\circ)$ for theories that are singular in the De Donder–Weyl sense). First, rewrite the presymplectic potential θ and presymplectic form ω from equations (6.3) and (6.4) as

$$\theta = \int_\Sigma \pi^\mu_A \delta\phi^A \otimes \mathrm{d}^{m-1}x_\mu \quad , \quad \omega = \int_\Sigma \delta\phi^A \wedge \delta\pi^\mu_A \otimes \mathrm{d}^{m-1}x_\mu \ . \tag{6.22}$$

These are related to the Hamilton–Cartan forms on $J^1\pi^*$ as follows (see e.g., [68]). Let $X, Y \in \mathfrak{X}^{V(\pi^1)}(J^1\pi)$:

$$X = \delta_x y^A \frac{\partial}{\partial y^A} + \delta_x p^\mu_A \frac{\partial}{\partial p^\mu_A} \quad , \quad Y = \delta_y y^A \frac{\partial}{\partial y^A} + \delta_y p^\mu_A \frac{\partial}{\partial p^\mu_A} \quad . \tag{6.23}$$

Then,

$$\theta(X) = \int_\Sigma j^1\phi^* \left[i(X)\Theta_{\mathscr{H}}\right] = \int_\Sigma j^1\phi^* \left(p^\mu_A \delta_x y^A\right) \mathrm{d}^{m-1}x_\mu \ , \tag{6.24}$$

and

$$\omega(X,Y) = \int_\Sigma j^1\phi^* \left[i(Y)i(X)\Omega_{\mathscr{H}}\right] = \int_\Sigma j^1\phi^* \left(\delta_x y^A \delta_y p^\mu_A - \delta_y y^A \delta_x p^\mu_A\right) \mathrm{d}^{m-1}x_\mu \ . \tag{6.25}$$

The product of the δ operators in the expression above act as the wedge product exterior derivatives, thus yielding the forms in equation (6.22).

Finally, as an aside, consider the pre-image of the projection map $\kappa : P \to M$ where $P = J^1\pi^*$ or $P = P_\circ$ and we denote both corresponding multimomenta generically as p^μ_A); furthermore, we have the embedded Cauchy surface $\Sigma \hookrightarrow M$. Since the pre-image of an embedded submanifold under a submersion is also an embedded submanifold [107], we have

$$P_\Sigma \equiv \kappa^{-1}(\Sigma) \subset P \ , \tag{6.26}$$

with embedding map denoted as $j_\Sigma : P_\Sigma \hookrightarrow P$. Moreover, the restriction of κ given as $\kappa_\Sigma : P_\Sigma \to \Sigma$ provides us with a sub-bundle of $\kappa : P \to M$.

As before, let n^μ be the components of the normal vector on Σ and let $\mathrm{d}^{m-1}x$ be the volume form on Σ so that the pullback of $\mathrm{d}^{m-1}x_\mu$ onto Σ is given as $n_\mu \mathrm{d}^{m-1}x$. Then, the Poincaré–Cartan forms on P_Σ are

$$\Theta_\Sigma = j^*_\Sigma \Theta_{\mathscr{L}} = n_\mu p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x \in \Omega^m(P_\Sigma) \ , \tag{6.27}$$

$$\Omega_\Sigma = j^*_\Sigma \Omega_{\mathscr{L}} = n_\mu \mathrm{d}y^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}^{m-1}x \in \Omega^{m+1}(P_\Sigma) \ . \tag{6.28}$$

These expressions are associated with the expressions for θ and ω in equation (6.22). However, the relevant variational correspondence between the exterior derivative operator δ on sections of $\kappa : P \to M$ and differential forms on P is discussed later, in Chapter 14, where instead $P = J^1\pi$, which is where the variational bicomplex is naturally defined.

6.3 Poisson Structure on the Fibers of $J^1\pi^*$

Recalling the expression for the multisymplectic Hamilton–Cartán form $\Omega_{\mathscr{H}}$ on $J^1\pi^*$,

$$\Omega_{\mathscr{H}} = \mathrm{d}y^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}\mathscr{H} \wedge \mathrm{d}^m x \ , \tag{6.29}$$

where now $\mathrm{d}^{m-1}x_\mu$ can be written as $n_\mu \mathrm{d}^{m-1}x$, corresponding to some arbitrary embedded hypersurface of M which is not necessarily a Cauchy surface. The hypersurface is then lifted to $J^1\pi^*$ as the pre-image of the projection map $J^1\pi^* \to M$ as before. We then define the bivector $\Omega^{-1} \in \mathfrak{X}^2(J^1\pi^*)$ along the normal direction of such a hypersurface as

$$\Omega^{-1} \equiv n^\mu \frac{\partial}{\partial p^\mu_A} \wedge \frac{\partial}{\partial y^A} \ . \tag{6.30}$$

This bivector is *inverse to* $\Omega_{\mathscr{H}}$ in the sense that their contraction yields the hypersurface volume form:

$$\Omega^{-1} \cdot \Omega_{\mathscr{H}} = n^\mu \mathrm{d}^{m-1}x_\mu = \mathrm{d}^{m-1}x \ . \tag{6.31}$$

Furthermore, we define the corresponding bivector along local sections $\psi : M \to J^1\pi^*$ as

$$\Omega^{-1} \equiv n^\mu \frac{\partial}{\partial \pi^\mu_A} \wedge \frac{\partial}{\partial \phi^A} = n^\mu \frac{\partial}{\partial p^\mu_A} \wedge \frac{\partial}{\partial y^A} \circ \psi \ . \tag{6.32}$$

Upon choosing this bivector along the normal direction to some Cauchy surface Σ, this bivector is *inverse* to the presymplectic form (6.22) of the infinite-dimensional covariant phase formalism from the previous section,

$$\omega = \int_\Sigma \mathrm{d}^{m-1}x_\mu \otimes \delta\phi^A \wedge \delta\pi^\mu_A \ , \tag{6.33}$$

in the sense that

$$\Omega^{-1} \cdot \omega = \int_\Sigma n^\mu \mathrm{d}^{m-1}x_\mu = \int_\Sigma \mathrm{d}^{m-1}x \ . \tag{6.34}$$

Now consider some functional given as

$$F[\phi] = \int_M \mathrm{d}^m x \mathcal{F}(\phi^A, \pi^\mu_A) = \int_M \psi^* \alpha_F \ , \tag{6.35}$$

in terms of some semi-basic form $\alpha_F = f\mathrm{d}^m x$. Furthermore, let

$$\Omega^{-1} \cdot \alpha_F = \mathrm{d}^m x \otimes \Omega^{-1}(\mathrm{d}f) = \mathrm{d}^m x \otimes X_f \ , \tag{6.36}$$

where,

$$X_f \equiv \Omega^{-1}(\mathrm{d}f) = n^\mu \left(\frac{\partial f}{\partial y^A} \frac{\partial}{\partial p^\mu_A} - \frac{\partial f}{\partial p^\mu_A} \frac{\partial}{\partial y^A} \right) \ . \tag{6.37}$$

Then, we pass the bivector through the integral as the bivector along sections, as in equation (6.32). It follows that

$$\Omega^{-1}(F[\phi]) = \int_M \Omega^{-1}(\psi^* \alpha_F) = \int_M \mathrm{d}^m x \otimes \Omega^{-1}(\delta\mathcal{F}) = \int_M \mathrm{d}^m x \otimes X_{\mathcal{F}} \tag{6.38}$$

where,

$$X_{\mathcal{F}} \equiv \Omega^{-1}(\delta\mathcal{F}) = n^\mu \left(\frac{\partial \mathcal{F}}{\partial \phi^A} \frac{\partial}{\partial \pi^\mu_A} - \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \frac{\partial}{\partial \phi^A} \right) \ . \tag{6.39}$$

Furthermore, upon passing the bivector through the pullback by local sections we obtain

$$\Omega^{-1}(F[\phi]) = \int_M \psi^* \Omega^{-1} \cdot \alpha_F = \int_M \psi^* \mathrm{d}^m x \otimes \Omega^{-1}(\mathrm{d}f) = \int_M \psi^* \mathrm{d}^m x \otimes X_f \ . \tag{6.40}$$

Now observe that,

$$i(X_f)\Omega_{\mathscr{H}} = -n^\mu \left[\frac{\partial f}{\partial y^A} \left(\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \frac{\partial \mathscr{H}}{\partial p^\mu_A} \mathrm{d}^m x \right) + \frac{\partial f}{\partial p^\mu_A} \left(\mathrm{d}p^\nu_A \wedge \mathrm{d}^{m-1}x_\nu + \frac{\partial \mathscr{H}}{\partial y^A} \mathrm{d}^m x \right) \right] \ . \tag{6.41}$$

It directly follows that

$$\psi^* i(X_f)\Omega_{\mathscr{H}} = -n^\mu \left[\frac{\partial \mathcal{F}}{\partial \phi^A} \left(\partial_\mu \phi^A - \frac{\partial \mathscr{H}}{\partial \pi^\mu_A} \right) + \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \left(\partial_\nu \pi^\nu_A + \frac{\partial \mathscr{H}}{\partial \phi^A} \right) \right] \mathrm{d}^m x \ , \tag{6.42}$$

which vanishes on-shell

$$\psi^* i(X_f)\Omega_{\mathscr{H}} \approx 0 \ . \tag{6.43}$$

Now, consider a second functional $G[\phi]$ with,

$$G[\phi] = \int_M \mathrm{d}^m x \mathcal{G}(\phi^A, \pi^\mu_A) = \int_M \psi^* \alpha_G = \int_M \psi^*(g\mathrm{d}^m x) \ , \tag{6.44}$$

and corresponding vectors,

$$X_g \equiv \Omega^{-1}(\mathrm{d}g) = \bar{n}^\mu \left(\frac{\partial g}{\partial y^A} \frac{\partial}{\partial p^\mu_A} - \frac{\partial g}{\partial p^\mu_A} \frac{\partial}{\partial y^A} \right) \ , \tag{6.45}$$

$$X_{\mathcal{G}} \equiv \Omega^{-1}(\delta \mathcal{G}) = \bar{n}^\mu \left(\frac{\partial \mathcal{G}}{\partial \phi^A} \frac{\partial}{\partial \pi^\mu_A} - \frac{\partial \mathcal{G}}{\partial \pi^\mu_A} \frac{\partial}{\partial \phi^A} \right) \ . \tag{6.46}$$

Contracting X_f and X_g into $\Omega_{\mathscr{H}}$ yields,

$$i(X_g) i(X_f) \Omega_{\mathscr{H}} = n^\mu \bar{n}^\nu \left(\frac{\partial f}{\partial y^A} \frac{\partial g}{\partial p^\nu_A} \mathrm{d}^{m-1} x_\mu - \frac{\partial f}{\partial p^\mu_A} \frac{\partial g}{\partial y^A} \mathrm{d}^{m-1} x_\nu \right) \ . \tag{6.47}$$

Setting $n^\mu = \bar{n}^\mu$ yields,

$$i(X_g) i(X_f) \Omega_{\mathscr{H}} = n^\mu n^\nu \left(\frac{\partial f}{\partial y^A} \frac{\partial g}{\partial p^\mu_A} - \frac{\partial f}{\partial p^\mu_A} \frac{\partial g}{\partial y^A} \right) \mathrm{d}^{m-1} x_\nu \ . \tag{6.48}$$

Then, using $n^\mu n_\nu = \delta^\mu_\nu$, we obtain

$$n_\rho \mathrm{d}x^\rho \wedge [i(X_g) i(X_f) \Omega_{\mathscr{H}}] = n^\mu \left(\frac{\partial f}{\partial y^A} \frac{\partial g}{\partial p^\mu_A} - \frac{\partial f}{\partial p^\mu_A} \frac{\partial g}{\partial y^A} \right) \mathrm{d}^m x \ . \tag{6.49}$$

Moreover, upon pulling back by sections ψ and integrating over M, we obtain the bracket of the functionals F and G given by the contraction with the bivector Ω^{-1} as,

$$\begin{aligned} \int_M \psi^* \left(n_\rho \mathrm{d}x^\rho \wedge [i(X_g) i(X_f) \Omega_{\mathscr{H}}] \right) &= \int_M \mathrm{d}^m x n^\mu \left(\frac{\partial f}{\partial y^A} \frac{\partial g}{\partial p^\mu_A} - \frac{\partial f}{\partial p^\mu_A} \frac{\partial g}{\partial y^A} \right) \circ \psi \\ &= \int_M \mathrm{d}^m x n^\mu \left(\frac{\partial \mathcal{F}}{\partial \phi^A} \frac{\partial \mathcal{G}}{\partial \pi^\mu_A} - \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \frac{\partial \mathcal{G}}{\partial \phi^A} \right) = \Omega^{-1}(F, G)[\phi] \end{aligned} \tag{6.50}$$

The expression above coincides with the Poisson bracket of functionals defined by Marsden *et al* in [116] which they use to obtain the De Donder–Weyl equations by taking the Poisson bracket of the action functional

$$S[\phi] = \int_M \psi^* \Theta_{\mathscr{H}} = \int_M \mathrm{d}^3 x \left(\pi^\mu_A \phi^A_\mu - \mathscr{H} \right) \ , \tag{6.51}$$

with some arbitrary functional F yields,

$$\{F, S\}_V = \int_M \mathrm{d}^m x n^\mu \left[\frac{\partial \mathcal{F}}{\partial \phi^A} \left(\partial_\mu \phi^A - \frac{\partial \mathscr{H}}{\partial \pi^\mu_A} \right) + \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \left(\partial_\nu \pi^\nu_A + \frac{\partial \mathscr{H}}{\partial \phi^A} \right) \right] \tag{6.52}$$

(where the total derivative of $\pi^\mu_A \phi^A$ is dropped as usual). It follows that setting $\{F, S\}_V = 0$ yields the Hamilton–De Donder–Weyl equations. Furthermore, the expression for $\{F, S\}_V$ above is in agreement with the multisymplectic calculation in equations (6.41) and (6.42) as

$$\begin{aligned} \{F, S\}_V &= - \int_M \psi^* \left(i(X_{\mathcal{F}}) \Omega_{\mathscr{H}} \right) \\ &= \int_M \psi^* \left[\left(n^\mu \frac{\partial f}{\partial y^A} \mathrm{d}y^A + n^\nu \frac{\partial f}{\partial p^\nu_A} \mathrm{d}p^\mu_A \right) \wedge \mathrm{d}^{m-1} x_\mu - n^\mu \left(\frac{\partial f}{\partial y^A} \frac{\partial \mathscr{H}}{\partial p^\mu_A} - \frac{\partial f}{\partial p^\mu_A} \frac{\partial \mathscr{H}}{\partial y^A} \right) \mathrm{d}^m x \right] \\ &= \int_M \mathrm{d}^m x n^\mu \left(\frac{\partial \mathcal{F}}{\partial \phi^A} \partial_\mu \phi^A + \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \partial_\nu \pi^\nu_A - \{ \mathcal{F}, \mathscr{H} \}_\mu \right) \ . \end{aligned} \tag{6.53}$$

We can similarly carry out the same analysis using $X_{\mathcal{F}}$ and the presymplectic form ω. First, obtain

$$i(X_{\mathcal{F}})(\omega^\mu \mathrm{d}^{m-1} x_\mu) = -n^\mu \left(\frac{\partial \mathcal{F}}{\partial \phi^A} \delta \phi^A \otimes \mathrm{d}^{m-1} x_\mu + \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \delta \pi^\nu_A \otimes \mathrm{d}^{m-1} x_\nu \right) \ . \tag{6.54}$$

Notice that this does not directly reproduce the Hamilton–De Donder equations as in equations (6.41) and (6.42). Instead they are obtained by contracting once again with the vector field associated with the local function $\mathscr{L} = \pi^\mu_A \phi^A_\mu - \mathscr{H}$ as follows:

$$X_{\mathscr{L}} \equiv \Omega^{-1}(\mathscr{L}\ ) = -\bar{n}^\mu \left[\left(\phi^A_\mu - \frac{\partial \mathscr{H}}{\partial \pi^\mu_A} \right) \frac{\partial}{\partial \phi^A} + \left(\partial_\nu \pi^\nu_A + \frac{\partial \mathscr{H}}{\partial \phi^A} \right) \frac{\partial}{\partial \pi^\mu_A} \right] , \tag{6.55}$$

and hence,

$$i\,(X_{\mathscr{L}})\, i\big(X_{\mathcal{F}}\big)(\omega^\mu \mathrm{d}^{m-1}x_\mu) = n^\mu \bar{n}^\nu \left[\frac{\partial \mathcal{F}}{\partial \phi^A} \left(\phi^A_\nu - \frac{\partial \mathscr{H}}{\partial \pi^\nu_A} \right) \mathrm{d}^{m-1}x_\mu + \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \left(\partial_\rho \pi^\rho_A + \frac{\partial \mathscr{H}}{\partial \phi^A} \right) \mathrm{d}^{m-1}x_\nu \right] . \tag{6.56}$$

Upon setting $n^\mu = \bar{n}^\mu$ and acting from the left by wedge product with $n_\rho \mathrm{d}x^\rho$, obtain

$$n_\rho \mathrm{d}x^\rho \wedge \left[i\,(X_{\mathscr{L}})\, i\big(X_{\mathcal{F}}\big)(\omega^\mu \mathrm{d}^{m-1}x_\mu) \right] = n^\mu \left[\frac{\partial \mathcal{F}}{\partial \phi^A} \left(\partial_\mu \phi^A - \frac{\partial \mathscr{H}}{\partial \pi^\mu_A} \right) + \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \left(\partial_\nu \pi^\nu_A + \frac{\partial \mathscr{H}}{\partial \phi^A} \right) \right] \mathrm{d}^m x , \tag{6.57}$$

which is in agreement (up to a negative sign) with the multisymplectic expression (6.42), and hence with the Poisson bracket expression (6.52) also, as:

$$-\psi^* i(X_f)\Omega_{\mathscr{H}} = \{F, S\}_n = n_\rho \mathrm{d}x^\rho \wedge \left[i\,(X_{\mathscr{L}})\, i\big(X_{\mathcal{F}}\big)(\omega^\mu \mathrm{d}^{m-1}x_\mu) \right] . \tag{6.58}$$

Furthermore, given two functionals $F = \int_M \mathrm{d}^m x \mathcal{F}$ and $G = \int_M \mathrm{d}^m x \mathcal{G}$, it follows that

$$i(X_{\mathcal{G}}) i(X_{\mathcal{F}})(\omega^\mu \mathrm{d}^{m-1}x_\mu) = n^\mu \left(\frac{\partial \mathcal{F}}{\partial \phi^A} \frac{\partial \mathcal{G}}{\partial \pi^\mu_A} - \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \frac{\partial \mathcal{G}}{\partial \phi^A} \right) . \tag{6.59}$$

Once again, upon setting $n^\mu = \bar{n}^\mu$ and acting from the left by wedge product with $n_\rho \mathrm{d}x^\rho$, we obtain

$$n_\rho \mathrm{d}x^\rho \wedge \left[i(X_{\mathcal{G}}) i(X_{\mathcal{F}})(\omega^\mu \mathrm{d}^{m-1}x_\mu) \right] = n^\mu \left(\frac{\partial \mathcal{F}}{\partial \phi^A} \frac{\partial \mathcal{G}}{\partial \pi^\mu_A} - \frac{\partial \mathcal{F}}{\partial \pi^\mu_A} \frac{\partial \mathcal{G}}{\partial \phi^A} \right) \mathrm{d}^m x , \tag{6.60}$$

and so it follows that

$$\{F, G\}_n = -\int_M \psi^* \left[n_\rho \mathrm{d}x^\rho \wedge (i(X_g) i(X_f) \Omega_{\mathscr{H}}) \right] = \int_M n_\rho \mathrm{d}x^\rho \wedge \left[i\,(X_{\mathcal{G}})\, i\big(X_{\mathcal{F}}\big)(\omega^\mu \mathrm{d}^{m-1}x_\mu) \right] . \tag{6.61}$$

6.4 Yang–Mills Theory

The Yang–Mills Lagrangian is given as

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a , \tag{6.62}$$

and we have the covariant derivative,

$$D^a_{\mu b} = \delta^a{}_b \partial_\mu - f^a_{cb} A^c_\mu \quad , \quad D_{\mu b}{}^a = \delta_b{}^a \partial_\mu + f^b_{ca} A^c_\mu \quad . \tag{6.63}$$

It is straightforward to obtain the expressions for the presymplectic potential and presymplectic form on $\mathscr{F}$ (see [108]):

$$\theta = \int_\Sigma \mathrm{d}^3 x n_\mu \pi^{\mu\nu}_a \delta A^a_\nu \quad , \quad \omega = \int_\Sigma \mathrm{d}^3 x n_\mu \delta \pi^{\mu\nu}_a \wedge \delta A^a_\nu \quad . \tag{6.64}$$

The expressions for the corresponding symplectic form Ω on phase space Γ is identical to the expression for ω on $\mathscr{F}$.

The local symmetry is generated by,

$$\delta_\Lambda A^a_\mu = D^a_{\mu b}\Lambda^b \quad , \quad \delta_\Lambda \pi^{\mu\nu}_a = f^c{}_{ab}\pi^{\mu\nu}_c\Lambda^c \quad , \tag{6.65}$$

and it follows that

$$\begin{aligned}(\Omega)_{ab}(\delta_\Lambda\phi)^b &= \int_\Sigma \mathrm{d}^3x n_\mu \left(f^c{}_{ab}\Lambda^b\pi^{\mu\nu}_c\delta A^a_\nu - D^a_{\nu b}\Lambda^b\delta\pi^{\mu\nu}_a\right)\\ &= -\int_\Sigma \mathrm{d}^3x n_\mu \delta(\pi^{\mu\nu}_a D^a_{\nu b}\Lambda^b)\\ &= \int_\Sigma \mathrm{d}^3x n_\mu \Lambda^b\delta(D^a_{\nu b}\pi^{\mu\nu}_a)\ ,\end{aligned} \tag{6.66}$$

and thereby follows that the constraint obtained is given as,

$$\mathcal{C}_\Lambda = \int_\Sigma \mathrm{d}^3x n_\mu \Lambda^b D^a_{\nu b}\pi^{\mu\nu}_a\ . \tag{6.67}$$

This constraint is in fact the one obtained from the premultisymplectic analysis and is in agreement with the corresponding multimomentum map (10.16) up to a total derivative, as

$$\mathcal{C}_\Lambda = \int_\Sigma j^*_\tau \left[J^\circ_{\mathscr{H}}(Y_\Lambda) + D^a_{\nu b}\left(\Lambda^b\pi^{[\mu\nu]}_a\right)\right]\ , \tag{6.68}$$

and hence,

$$J^\circ_{\mathscr{H}}(Y_\Lambda) = -i(Y_\Lambda)\Theta^\circ_{\mathscr{H}} = \pi^{[\mu\nu]}_a D^a_{\nu b}\Lambda^b\mathrm{d}^3x_\mu\ . \tag{6.69}$$

Now, recall that the relation between the (pre)multisymplectic form of the De Donder–Weyl Hamiltonian formalism is related to the (pre)symplectic forms of the infinite-dimensional covariant phase space formalism. This allows us to make a direct connection between the Poisson bracket of constraints (6.18) in the infinite-dimensional covariant phase space formalism and the premultisymplectic bracket (5.21) of multimomentum maps on $P_\circ \subset J^1\pi^*$.

For the state of the art on the formal definition of Poisson brackets on infinite-dimensional covariant phase space, we encourage the reader to see [29]-[32].

Part III

Examples

We now work out the (pre)multisymplectic geometry of a few specific field theories as explicit examples which are well known in the physics literature. The following examples have been presented in our published works [63, 75, 76].

7 Scalar Field Theories

In this chapter, we present a variety of scalar field theories. We begin in Section 7.1 with the standard Klein–Gordon field. Next, in Section 7.2 we present a version of Klein–Gordon which we call *canonical Klein–Gordon*, in which we introduce two separate fields, the standard Klein–Gordon field, and a field corresponding to the canonical momentum. By treating both fields on the same footing on the configuration manifold, we are able to obtain the standard results from instantaneous formalism. Furthermore, the canonical Klein–Gordon theory of Section 7.2 allows us to then construct the Electric and Magnetic Carrollian scalar field theories in Sections 7.3 and 7.4.

7.1 The Klein–Gordon Field

Klein–Gordon theory can be found in any standard quantum field theory textbook such as [129].

Consider a m-dimensional spacetime manifold M, equipped with the Minkowski metric $\eta_{\mu\nu}$ with signature $(- + ...+)$. The configuration bundle for Klein–Gordon theory is given as $\pi : E \to M : (x^\mu, \varphi) \to x^\mu$.

7.1.1 Lagrangian Formalism

The Lagrangian formalism takes place on the first-order jet bundle $J^1\pi$ which has natural coordinates $(x^\mu, \varphi, \varphi_\mu)$. The Lagrangian density $\mathbf{L}$ and Lagrangian function $\mathscr{L}(x^\mu, \varphi, \varphi_\mu)$ on $J^1\pi$ for the scalar field theory are

$$\mathbf{L} = \mathscr{L}(x^\mu, \varphi, \varphi_\mu)\mathrm{d}^m x = -\frac{1}{2}\left(\eta^{\mu\nu}\varphi_\mu\varphi_\nu + m^2\varphi^2\right)\mathrm{d}^m x \ , \tag{7.1}$$

from which the Lagrangian energy $E_{\mathscr{L}}$ is given by

$$E_{\mathscr{L}} \equiv \frac{\partial \mathscr{L}}{\partial \varphi_\mu}\varphi_\mu - \mathscr{L} = -\frac{1}{2}\eta^{\mu\nu}\varphi_\mu\varphi_\nu + \frac{1}{2}m^2\varphi^2 \ . \tag{7.2}$$

This is a regular Lagrangian system in the sense that the Hessian

$$\frac{\partial^2 \mathscr{L}}{\partial \varphi_\mu \partial \varphi_\nu} = -\eta^{\mu\nu} \ , \tag{7.3}$$

is regular. The jet bundle $J^1\pi$ comes equipped with the Poincaré-Cartan m-form $\Theta_{\mathscr{L}}$ and the multisymplectic $(m+1)$-form $\Omega_{\mathscr{L}}$ which are given by,

$$\Theta_{\mathscr{L}} = -\eta^{\mu\nu}\varphi_\nu \mathrm{d}\varphi \wedge \mathrm{d}^{m-1}x_\mu + \frac{1}{2}\left(\eta^{\mu\nu}\varphi_\mu\varphi_\nu - m^2\varphi^2\right)\mathrm{d}^m x \ , \tag{7.4}$$

$$\Omega_{\mathscr{L}} = \eta^{\mu\nu}\mathrm{d}\varphi_\nu \wedge \mathrm{d}\varphi \wedge \mathrm{d}^{m-1}x_\mu - \left(\eta^{\mu\nu}\varphi_\mu \mathrm{d}\varphi_\nu - m^2 \mathrm{d}\varphi\right) \wedge \mathrm{d}^m x \ . \tag{7.5}$$

The multisymplectic form $\Omega_{\mathscr{L}}$ is 1-non-degenerate as a result of the regularity of the Hessian (7.3).

The field equations are produced by a class of multivector fields $\{\mathbf{X}_{\mathscr{L}}\} \subset \mathfrak{X}^m(J^1\pi)$ that are locally decomposable, integrable, and π^1-transverse $(i(\mathbf{X}_{\mathscr{L}})\mathrm{d}^m x \neq 0)$. We use a representative of $\{\mathbf{X}_{\mathscr{L}}\}$ for which $i(\mathbf{X}_{\mathscr{L}})\mathrm{d}^m x = 1$, expressed locally as

$$\mathbf{X}_{\mathscr{L}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + D_\mu \frac{\partial}{\partial \varphi} + H_{\mu\nu} \frac{\partial}{\partial \varphi_\nu} \right) . \tag{7.6}$$

The field equations are then given as $i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = 0$, from which it follows that

$$\eta^{\mu\nu} H_{\mu\nu} - m^2 \varphi = 0 , \tag{7.7}$$
$$\eta^{\mu\nu} \left(D_\mu - \varphi_\mu\right) = 0 . \tag{7.8}$$

Furthermore, the solution to the variational problem on $J^1\pi$ requires that $X_{\mathscr{L}}$ must be *holonomic* multivector fields (the SOPDE condition). By definition, this means that they have holonomic integral sections $j^1\phi : M \to M \times \mathbb{R} : x^\mu \mapsto (x^\mu, \varphi, \partial_\mu \varphi)$, so $D_\mu = \partial_\mu \varphi$. Therefore, in the expression (7.6) must be $D_\mu = \varphi_\mu$, and this condition shows up in the field equations (7.8); this is expected as the SOPDE condition always shows up in the field equations when the Lagrangian is regular. The final field equations are obtained from (7.7) on holonomic integral sections, giving:

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \varphi - m^2 \varphi = 0 . \tag{7.9}$$

7.1.2 De Donder–Weyl Hamiltonian Formalism

The De Donder–Weyl Hamiltonian formulation of Klein–Gordon field theory is constructed on the extended multimomentum bundle $\mathcal{M}\pi$ which has local coordinates $(x^\mu, \varphi, p_\mu, p)$ and the restricted multimomentum bundle $J^1\pi^*$ which has natural coordinates (x^μ, φ, p_μ). The Klein–Gordon field theory is specified by a Hamiltonian section $\mathrm{h} = \left(x^\mu, \varphi, p_\mu, p = \frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu - \frac{1}{2} m^2 \varphi^2\right)$ so that the De Donder–Weyl Hamiltonian is given as

$$\mathscr{H} = -\frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu + \frac{1}{2} m^2 \varphi^2 \in C^\infty(J^1\pi^*) . \tag{7.10}$$

Furthermore, this Hamiltonian field theory is related to its Lagrangian version by the following Legendre map:

$$\mathscr{F}\mathscr{L}^* p^\mu = \frac{\partial \mathscr{L}}{\partial \varphi_\mu} = -\eta^{\mu\nu} \varphi_\nu , \tag{7.11}$$

which is invertible since the Hessian (7.3) is regular ($\mathscr{F}\mathscr{L}$ is a local diffeomorphism); it follows that

$$\left(\mathscr{F}\mathscr{L}^{-1}\right)^* \varphi_\mu = -\eta_{\mu\nu} p^\nu . \tag{7.12}$$

Furthermore, the De Donder–Weyl Hamiltonian can be produced by the Legendre map as,

$$\mathscr{H} = p^\mu \left(\mathscr{F}\mathscr{L}^{-1}\right)^* \varphi_\mu - \left(\mathscr{F}\mathscr{L}^{-1}\right)^* \mathscr{L} = -\frac{1}{2}\eta_{\mu\nu} p^\mu p^\nu + \frac{1}{2} m^2 \varphi^2 . \tag{7.13}$$

The bundle $J^1\pi^*$ is equipped with the Hamilton–Cartan m-form $\Theta_{\mathscr{H}}$ and multisymplectic $(m+1)$-form $\Omega_{\mathscr{H}}$ given by

$$\Theta_{\mathscr{H}} = p^\mu \mathrm{d}\varphi \wedge \mathrm{d}^{m-1} x_\mu + \frac{1}{2}\left(\eta_{\mu\nu} p^\mu p^\nu - m^2 \varphi^2\right) \mathrm{d}^m x , \tag{7.14}$$
$$\Omega_{\mathscr{H}} = -\mathrm{d}p^\mu \wedge \mathrm{d}\varphi \wedge \mathrm{d}^{m-1} x_\mu - \left(\eta_{\mu\nu} p^\nu \mathrm{d}p^\mu - m^2 \varphi \mathrm{d}\varphi\right) \wedge \mathrm{d}^m x . \tag{7.15}$$

The field equations are again produced by a class of multivector fields $\{\mathbf{X}_{\mathscr{H}}\} \subset \mathfrak{X}^m(J^1\pi^*)$ that are locally decomposable, integrable, and $\bar{\tau}$-transverse which satisfy $i(\mathbf{X}_{\mathscr{H}})\Omega_{\mathscr{H}} = 0$ for every $\mathbf{X}_{\mathscr{H}} \in \{\mathbf{X}_{\mathscr{H}}\}$. As above, we use a representative of $\{\mathbf{X}_{\mathscr{H}}\}$, expressed as,

$$\mathbf{X}_{\mathscr{H}} = \bigwedge_{\mu=0}^{m-1} X_\mu = \bigwedge_{\mu=0}^{m-1} \left(\frac{\partial}{\partial x^\mu} + D_\mu \frac{\partial}{\partial \varphi} + H^\rho_\mu \frac{\partial}{\partial p^\rho} \right) , \tag{7.16}$$

and the field equations $i(\mathbf{X}_{\mathscr{H}})\Omega_{\mathscr{H}} = 0$ give

$$H^\mu_\mu + m^2\varphi = 0 , \tag{7.17}$$

$$D_\mu + \eta_{\mu\nu}p^\nu = 0 . \tag{7.18}$$

Recall that the solution to the variational problem in the De Donder–Weyl Hamiltonian formalism requires that ψ are integral sections of $\mathbf{X}_{\mathscr{H}}$, hence $D_\mu = \partial_\mu\varphi$ and $H^\rho_\mu = \partial_\mu p^\rho$, giving the Hamilton–De Donder–Weyl equations:

$$\partial_\mu p^\mu + m^2\varphi = 0 , \tag{7.19}$$

$$\partial_\mu \varphi + \eta_{\mu\nu}p^\nu = 0 . \tag{7.20}$$

7.1.3 Symmetries

Lorentz transformations:

The symmetry group of the Klein–Gordon field theory on Minkowski spacetime consists of Lorentz transformations, written as

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu = (\delta^\mu{}_\nu - \omega^\mu{}_\nu)\, x^\nu \quad , \quad \varphi_\mu(x) \to \Lambda^\nu{}_\mu \varphi_\nu(x) = (\delta^\nu{}_\mu + \omega^\nu{}_\mu)\varphi_\nu , \tag{7.21}$$

where $\omega^\mu{}_\nu$ are the infinitesimal Lorentz matrices which are antisymmetric. It follows that the vector fields which generate the Lorentz transformations on spacetime M and the configuration bundle E are,

$$\xi_M = -\omega^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} \quad , \quad \xi_E = -\omega^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} , \tag{7.22}$$

respectively. The canonical lifts $X_\xi \in \mathfrak{X}(J^1\pi)$, $Z_\xi \in \mathfrak{X}(\mathcal{M}\pi)$, and $Y_\xi \in \mathfrak{X}(J^1\pi^*)$ are given by (4.20), (4.36), and (4.40), respectively, and they are

$$X_\xi = -\omega^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} + \omega^\nu{}_\mu \varphi_\nu \frac{\partial}{\partial \varphi_\mu} , \tag{7.23}$$

$$Z_\xi = -\omega^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} - \omega^\mu{}_\nu p^\nu \frac{\partial}{\partial p^\mu} , \tag{7.24}$$

$$Y_\xi = -\omega^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} - \omega^\mu{}_\nu p^\nu \frac{\partial}{\partial p^\mu} . \tag{7.25}$$

Furthermore, $\mathscr{F}\mathscr{L}_* X_\xi = Y_\xi$ trivially (see also Proposition 4.6).

It is straightforward to check that $\mathrm{L}(X_\xi)\mathbf{L} = 0 \Leftrightarrow \mathrm{L}(X_\xi)\Theta_{\mathscr{L}} = 0$. Furthermore, $\mathrm{L}(Y_\xi)\Theta_{\mathscr{H}} = 0$ and it follows that both X_ξ and Y_ξ are exact natural Noether symmetries. The corresponding momentum maps are:

$$J_{\mathscr{L}}(X_\xi) = \omega^\rho{}_\sigma x^\sigma \left[\eta^{\mu\nu}\varphi_\nu \mathrm{d}\varphi \wedge \mathrm{d}^{m-2}x_{\mu\rho} + \frac{1}{2}\left(\eta^{\mu\nu}\varphi_\mu\varphi_\nu - m^2\varphi^2\right)\mathrm{d}^{m-1}x_\rho \right] \in \Omega^{m-1}(J^1\pi) , \tag{7.26}$$

$$J_{\mathscr{H}}(Y_\xi) = \omega^\rho{}_\sigma x^\sigma \left[p^\mu \mathrm{d}\varphi \wedge \mathrm{d}^{m-2}x_{\mu\rho} + \frac{1}{2}\left(\eta_{\mu\nu}p^\mu p^\nu - m^2\varphi^2\right)\mathrm{d}^{m-1}x_\rho \right] \in \Omega^{m-1}(J^1\pi^*) . \tag{7.27}$$

7.2 Canonical Klein–Gordon From Geometric Constraints

Consider m-dimensional spacetime M with the Minkowski metric $\eta_{\mu\nu}$ with signature $(- + \ldots +)$ and two scalar fields, $\phi(x)$ and $\pi(x)$, given by the sections $\phi : M \to E : x^\mu \mapsto (x^\mu, \phi(x), \pi(x))$ of the configuration bundle $\pi : E \to M : (\phi, \pi) \mapsto x^\mu$.

7.2.1 Lagrangian Formalism

The scalar field theory investigated in this section is given by the canonical Klein–Gordon Lagrangian which originates from the canonical construction of the Klein–Gordon field on the canonical momentum phase space where the canonical momentum, $\pi(x)$, is given by the Legendre map $\pi(x) = \partial \mathscr{L}(x)/\partial\dot{\phi}(x)$. Conversely, in the construction of the canonical Klein–Gordon Lagrangian given here, $\pi(x)$ is proposed simply as a second variational field while the notion of the canonical Legendre map is abandoned and the relation that is typically obtained from the canonical Legendre map instead shows up as a SOPDE constraint. The jet bundle has coordinates $(x^\mu, \phi, \pi, \phi_\mu, \pi_\mu)$ and it thereby follows that the first-order jet prolongations of sections $\phi : M \to E$ have the expression $j^1\phi : M \to J^1\pi : x^\mu \mapsto (x^\mu, \phi(x), \pi(x), \partial_\mu\phi(x), \partial_\mu\pi(x))$.

Now, instead of writing the Lagrangian in terms of the fields themselves, the Lagrangian is written as a function on $J^1\pi$,

$$\mathscr{L} = \pi\phi_0 - \frac{1}{2}\pi^2 - \frac{1}{2}\phi_i\phi^i \in C^\infty(J^1\pi) \ . \tag{7.28}$$

This Lagrangian is singular as it is directly evident from the multi Hessian matrix, whose components are

$$\frac{\partial^2\mathscr{L}}{\partial\phi_0\partial\phi_0} = 0 \ , \ \frac{\partial^2\mathscr{L}}{\partial\phi_0\partial\phi_i} = 0 \ , \ \frac{\partial^2\mathscr{L}}{\partial\phi_i\partial\phi_j} = -\delta^{ij} \ , \ \frac{\partial^2\mathscr{L}}{\partial\pi_\mu\partial\pi_\nu} = 0 \ , \ \frac{\partial^2\mathscr{L}}{\partial\pi_\mu\partial\phi_\nu} = 0 \ . \tag{7.29}$$

The null vectors associated with $\dfrac{\partial^2\mathscr{L}}{\partial\phi_\mu\partial\phi_\nu}$ and $\dfrac{\partial^2\mathscr{L}}{\partial\pi_\mu\partial\pi_\nu}$ are

$$\gamma_\mu^{(\phi)} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad \gamma_\mu^{(\pi)} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad , \tag{7.30}$$

respectively. The Lagrangian energy is

$$E_\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi_\mu}\phi_\mu + \frac{\partial\mathscr{L}}{\partial\pi_\mu}\pi_\mu - \mathscr{L} = \frac{1}{2}(\pi^2 - \phi_i\phi^i) \in C^\infty(J^1\pi) \ , \tag{7.31}$$

and the Poincaré–Cartan forms are

$$\Theta_\mathscr{L} = \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_0 - \phi^i \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i - \frac{1}{2}(\pi^2 - \phi_i\phi^i)\mathrm{d}^m x \in \Omega^m(J^1\pi) \ , \tag{7.32}$$

$$\begin{aligned} \Omega_\mathscr{L} &= -\mathrm{d}\Theta_\mathscr{L} = \mathrm{d}\phi \wedge \mathrm{d}\pi \wedge \mathrm{d}^{m-1}x_0 - \mathrm{d}\phi^i \wedge \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i + (\pi\mathrm{d}\pi - \phi^i\mathrm{d}\phi_i) \wedge \mathrm{d}^m x \\ &\equiv \widetilde{\Omega}_\mathscr{L} + \mathrm{d}E_\mathscr{L} \wedge \mathrm{d}^m x \in \Omega^{m+1}(J^1\pi) \ . \end{aligned} \tag{7.33}$$

As $\mathscr{L}$ is a singular Lagrangian, $\Omega_\mathscr{L}$ is a premultisymplectic form. Now, take a locally decomposable multivector field $\mathbf{X}_\mathscr{L} = X_0 \wedge X_1 \wedge \ldots \wedge X_{m-1} \in \mathfrak{X}^m(J^1\pi)$, with components

$$X_\mu = \frac{\partial}{\partial x^\mu} + A_\mu\frac{\partial}{\partial\phi} + B_\mu\frac{\partial}{\partial\pi} + C_{\mu\nu}\frac{\partial}{\partial\phi_\nu} + D_{\mu\nu}\frac{\partial}{\partial\pi_\nu} \ , \tag{7.34}$$

which satisfies $i(\mathbf{X}_{\mathscr{L}})\mathrm{d}^m x = 1$. Then, the field equations are given as

$$
\begin{aligned}
0 = i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = &- B_0 \mathrm{d}\phi + A_0 \mathrm{d}\pi + C_i^i \mathrm{d}\phi - A^i \mathrm{d}\phi_i - \pi \mathrm{d}\pi + \phi^i \mathrm{d}\phi_i \\
&+ \left(B_0 A_\lambda - B_\lambda A_0 + C_{\lambda i} A_i - C_{ii} A_\lambda + \pi B_\lambda - \phi_i C_\lambda^i\right) \mathrm{d}x^\lambda \ ,
\end{aligned} \tag{7.35}
$$

which, by setting differential forms separately equal to zero, produces the following equations

$$
C_i^i - B_0 = 0 \ , \tag{7.36}
$$

$$
A_0 - \pi = 0 \ , \tag{7.37}
$$

$$
\phi_i - A_i = 0 \ , \tag{7.38}
$$

$$
B_0 A_\lambda - B_\lambda A_0 + C_{\lambda i} A_i - C_{ii} A_\lambda + \pi B_\lambda - \phi_i C_\lambda^i = 0 \ . \tag{7.39}
$$

Equation (7.39) is a combination of the first three equations and is thereby and identity. Equation (7.37) determines the coefficient A_0, equation (7.36) is a relation among the coefficients B_0 and C_i^i, and equation (7.38) is the spatial part of the holonomic or SOPDE condition for the $\partial/\partial\phi$ piece of the multivector field. As these equations show, no compatibility constraints appear. As it has been proven in Proposition 3.1, this can also be analyzed geometrically by enforcing

$$
i(Z)\mathrm{d}E_{\mathscr{L}} = 0 \quad , \quad \text{for every } Z \in \ker \widetilde{\Omega}_{\mathscr{L}} \cap \ker \mathrm{d}^m x \ , \tag{7.40}
$$

where, $\widetilde{\Omega}_{\mathscr{L}} = \mathrm{d}\phi \wedge \mathrm{d}\pi \wedge \mathrm{d}^{m-1}x_0 - \mathrm{d}\phi^i \wedge \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i$, as it is shown in (7.33). In this case

$$
\ker, \widetilde{\Omega}_{\mathscr{L}} \cap \ker \mathrm{d}^m x = \left\langle \frac{\partial}{\partial \phi_0}, \frac{\partial}{\partial \pi_\mu} \right\rangle \ , \tag{7.41}
$$

and it follows that

$$
i\left(\frac{\partial}{\partial \phi_0}\right) \mathrm{d}E_{\mathscr{L}} = 0 = i\left(\frac{\partial}{\partial \pi_\mu}\right) \mathrm{d}E_{\mathscr{L}} \ , \tag{7.42}
$$

which agrees with the fact that no compatibility constraints are produced in this system.

Now, imposing the SOPDE condition $A_\mu = \phi_\mu$, $B_\mu = \pi_\mu$ on $\mathbf{X}_{\mathscr{L}}$ (i.e. demanding that the multivector fields $\mathbf{X}_{\mathscr{L}}$ are holonomic), it follows that the relevant field equations left over are

$$
C_i^i - \pi_0 = 0 \ , \tag{7.43a}
$$

$$
\phi_0 - \pi = 0 \ . \tag{7.43b}
$$

At this stage, the first equation (7.43a) gives another relation for the coefficients C_i^i. Equation (7.43b) is a SOPDE constraint which defines the constraint submanifold $S_1 \hookrightarrow J^1\pi$; this is a constraint giving a relation between the fields and their partial derivatives. Notice that the Lagrangian function (7.28) on the submanifold S_1 takes the form of the standard Klein–Gordon Lagrangian:

$$
\mathscr{L}|_{S_1} = -\frac{1}{2}\phi_\mu \phi^\mu \ . \tag{7.44}
$$

As a final step, it is necessary to impose the tangency condition of the multivector fields $\mathbf{X}_{\mathscr{L}}$ to the submanifold S_1:

$$
\mathrm{L}(X_\mu)(\phi_0 - \pi) = C_{\mu 0} - B_\mu = 0 \quad (\text{on } S_1) \ . \tag{7.45}
$$

These are new relations among the coefficients of $\mathbf{X}_{\mathscr{L}}$ but are not constraints, therefore S_1 is the final constraint submanifold on which the constraint algorithm terminates.

Now, upon taking $(x^\mu, \phi(x), \pi(x), \phi_\nu(x), \pi_\nu(x))$ to be integral sections of the multivector field $\mathbf{X}_{\mathscr{L}}$ by setting

$$
A_\mu = \partial_\mu \phi \quad , \quad B_\mu = \partial_\mu \pi \quad , \quad C_{\mu\nu} = \partial_\mu \phi_\nu \quad , \quad D_{\mu\nu} = \partial_\mu \pi_\nu \ , \tag{7.46}
$$

the field equations on S_1 become

$$\partial_i \partial^i \phi - \dot{\pi} = 0 \ , \tag{7.47a}$$

$$\dot{\phi} - \pi = 0 \ . \tag{7.47b}$$

The combination of equations (7.47a) and (7.47b) gives the Klein–Gordon equation:

$$\partial_\mu \partial^\mu \phi = 0 \ . \tag{7.48}$$

Furthermore, the integral sections (7.46) which satisfy the field equations (7.47a) and (7.47b) satisfy the tangency condition on S_1 given by (7.45) particularly as a consequence of (7.47b).

7.2.2 De Donder–Weyl Hamiltonian Formalism

The De Donder–Weyl Hamiltonian formulation takes place on the bundle $J^1\pi^*$ which serves as the multimomentum phase space and has coordinates $(x^\mu, \phi, \pi, p^\mu_\phi, p^\mu_\pi)$, where the multimomenta p^μ_ϕ, p^μ_π are obtained from the Legendre map $\mathscr{F}\mathscr{L} : J^1\pi \to J^1\pi^*$,

$$\mathscr{F}\mathscr{L}^* p^i_\phi = \frac{\partial \mathscr{L}}{\partial \phi_i} = -\phi^i \quad , \quad \mathscr{F}\mathscr{L}^* p^0_\phi = \frac{\partial \mathscr{L}}{\partial \phi_0} = \pi \quad , \quad \mathscr{F}\mathscr{L}^* p^\mu_\pi = \frac{\partial \mathscr{L}}{\partial \pi_\mu} = 0 \ . \tag{7.49}$$

The first equation above is invertible and is therefore not a constraint. The second and third equations give the primary constraints $p^0_\phi - \pi = 0$, $p^\mu_\pi = 0$, which define the submanifold $\jmath_\circ : P_\circ \hookrightarrow J^1\pi^*$ with local coordinates $(x^\mu, \phi, \pi, p^i_\phi)$. Note that, by Proposition 3.2, the null vectors (7.30) are given as

$$\gamma^{(\phi)}_\mu = \mathscr{F}\mathscr{L}^* \frac{\partial}{\partial p^\mu_\phi} (p^0_\phi - \pi) \quad , \quad \gamma^{(\pi)}_\mu = \mathscr{F}\mathscr{L}^* \frac{\partial}{\partial p^\mu_\phi} (p^\mu_\pi) \ . \tag{7.50}$$

Furthermore, $\mathscr{F}\mathscr{L}$ maps onto the primary constraint submanifold $P_\circ \subset J^1\pi^*$ as a consequence of the fact that the Lagrangian function $\mathscr{L}$ is almost-regular and the restricted Legendre map $\mathscr{F}\mathscr{L}_\circ$ is given by $\mathscr{F}\mathscr{L} = \jmath_\circ \circ \mathscr{F}\mathscr{L}_\circ$.

The De Donder–Weyl Hamiltonian $\mathscr{H}_\circ \in \mathrm{C}^\infty(P_\circ)$ obtained from $E_\mathscr{L} = \mathscr{F}\mathscr{L}^*_\circ \mathscr{H}_\circ$ is given as,

$$\mathscr{H}_\circ = \frac{1}{2}(\pi^2 - p^i_\phi p_{\phi i}) \ . \tag{7.51}$$

The Hamilton–Cartan forms obtained from (1.53) are

$$\Theta^\circ_\mathscr{H} = \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_0 + p^i_\pi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i - \frac{1}{2}(\pi^2 - p^i_\phi p_{\phi i})\mathrm{d}^m x \in \Omega^m(P_\circ) \ , \tag{7.52}$$

$$\Omega^\circ_\mathscr{H} = \mathrm{d}\phi \wedge \mathrm{d}\pi \wedge \mathrm{d}^{m-1}x_0 + \mathrm{d}\phi \wedge \mathrm{d}p^i_\phi \wedge \mathrm{d}^{m-1}x_i + (\pi \mathrm{d}\pi - p_{\phi i}\mathrm{d}p^i_\phi) \wedge \mathrm{d}^m x \in \Omega^{m+1}(P_\circ) \ , \tag{7.53}$$

and we identify $\widetilde{\Omega}^\circ_\mathscr{H}$ as usual. Now, in order to produce the Hamiltonian field equations, take a locally decomposable multivector field $\mathbf{X}^\circ_\mathscr{H} = X_0 \wedge X_1 \wedge \cdots \wedge X_{m-1} \in \mathfrak{X}^m(P_\circ)$ satisfying the normalized transverse condition $i(\mathbf{X}^\circ_\mathscr{H})\mathrm{d}^m x = 1$, whose components are written as

$$X_\mu = \frac{\partial}{\partial x^\mu} + A_\mu \frac{\partial}{\partial \phi} + B_\mu \frac{\partial}{\partial \pi} + C^i_\mu \frac{\partial}{\partial p^i_\phi} \ . \tag{7.54}$$

Then, the field equation $i(\mathbf{X}^\circ_\mathscr{H})\Omega^\circ_\mathscr{H} = 0$ yields

$$B_0 + C^i_i = 0 \ , \tag{7.55}$$

$$A_0 - \pi = 0 \ , \tag{7.56}$$

$$A_i + p_{\phi i} = 0 \ , \tag{7.57}$$

$$\pi B_\lambda - p_{\phi i} C^i_\lambda + A_\lambda B_0 - A_0 B_\lambda + A_\lambda C^i_i - A_i C^i_\lambda = 0 \ . \tag{7.58}$$

Similarly to the Lagrangian formalism, equation (7.58) is a combination of the first three equations and is thereby an identity. Equations (7.56) and (7.57) determine the coefficients A_0 and A_i respectively and equation (7.55) is a relation among the coefficients B_0 and C^i_i. As before, no compatibility constraints appear; this can also be shown geometrically as in the Lagrangian formulation by searching for compatibility constraints via the application of Proposition 3.1: set $i(Z)\mathrm{d}\mathscr{H}_\circ = 0$ for every $Z \in \ker \widetilde{\Omega}^\circ_{\mathscr{H}} \cap \ker \mathrm{d}^m x$. However, as

$$\ker \widetilde{\Omega}^\circ_{\mathscr{H}} \cap \ker \mathrm{d}^m x = \{0\} \ , \tag{7.59}$$

it follows that condition (3.20) holds in agreement with the fact that there are no compatibility constraints (as one would expect from the Lagrangian formulation).

Now, taking $(x^\mu, \phi(x), \pi(x), p^i_\phi(x))$ to be integral sections of $\mathbf{X}^\circ_{\mathscr{H}}$ given by, and hence

$$A_\mu = \frac{\partial \phi}{\partial x^\mu} \quad , \quad B_\mu = \partial_\mu \pi \quad , \quad C^i_\mu = \partial_\mu p^i_\phi \ , \tag{7.60}$$

the field equations above become

$$\dot{\pi} + \partial_i p^i_\phi = 0 \ , \tag{7.61}$$
$$\dot{\phi} - \pi = 0 \ , \tag{7.62}$$
$$\partial_i \phi + p^i_\phi = 0 \ . \tag{7.63}$$

Plugging (7.62) and (7.63) into (7.61) yields

$$\ddot{\phi} - \partial_i \partial^i \phi = 0 \ , \tag{7.64}$$

which is again the Klein–Gordon equation as expected and thereby displaying the equivalence between the Lagrangian and the Hamiltonian formalisms.

7.2.3 Symmetries

The Lagrangian density associated with the Lagrangian function (7.28),

$$\mathbf{L} = \mathscr{L} \mathrm{d}^m x = \left(\pi \phi_0 - \frac{1}{2}\pi^2 - \frac{1}{2}\phi_i \phi^i \right) \mathrm{d}^m x \ , \tag{7.65}$$

is invariant under Lorentz transformations $\Lambda^\mu{}_\nu$:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \ , \ \phi'(x') = \phi(x) \ , \ \pi'(x') = \pi(x) \ , \ \phi'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\phi_\nu(x) \ , \ \pi'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\pi_\nu(x) \ . \tag{7.66}$$

Let the Lorentz transformations on M be infinitesimal so that the Lorentz matrices take the form $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$, where $\epsilon_{\mu\nu}$ is the infinitesimal antisymmetric Lorentz matrix. It follows that the spacetime coordinates transform as $x'^\mu = x^\mu + \xi^\mu(x)$ with $\xi^\mu(x) = \epsilon^\mu{}_\nu x^\nu$ and the resulting infinitesimal Lorentz transformations of the fields and their spacetime derivatives are given by the Lie derivatives of the first–jet prolongations $j^1\phi : M \to J^1\pi$ with respect to the vector field $\xi = -\xi^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ as defined in (4.22):

$$\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \phi \ , \\
\delta\pi(x) &\equiv \pi'(x) - \pi(x) = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \pi - \epsilon^\alpha{}_0 \phi_\alpha \ , \\
\delta\phi_\mu(x) &\equiv \phi'_\mu(x) - \phi_\mu(x) = -\epsilon^\alpha{}_\beta x^\beta \partial_\alpha \phi_\mu - \epsilon^\alpha{}_\mu \phi_\alpha \ , \\
\delta\pi_\mu(x) &\equiv \pi'_\mu(x) - \pi_\mu(x) = -\epsilon^\alpha{}_\beta x^\beta \partial_\alpha \pi_\mu - \epsilon^\alpha{}_\mu \pi_\alpha \ ,
\end{aligned} \tag{7.67}$$

which give $\delta\mathscr{L}(x) = -\dfrac{\partial}{\partial x^\alpha}\left(\epsilon^\alpha{}_\beta x^\beta \mathscr{L}(x)\right)$. The vector field which generates the Lorentz transformations above on $J^1\pi$ is

$$\begin{aligned} X_\Lambda &= -\epsilon^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} + \epsilon^\nu{}_0 \phi_\nu \frac{\partial}{\partial \pi} + \epsilon^\nu{}_\mu \phi_\nu \frac{\partial}{\partial \phi_\mu} + \epsilon^\nu{}_\mu \pi_\nu \frac{\partial}{\partial \pi_\mu} \\ &= -\epsilon^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} + \epsilon^i{}_0 \phi_i \frac{\partial}{\partial \pi} + \epsilon^i{}_0 \phi_i \frac{\partial}{\partial \phi_0} + \left(\epsilon^0{}_i \phi_0 + \epsilon^j{}_i \phi_j\right) \frac{\partial}{\partial \phi_i} + \epsilon^\nu{}_\mu \pi_\nu \frac{\partial}{\partial \pi_\mu} \in \mathfrak{X}(J^1\pi) \ . \end{aligned} \tag{7.68}$$

It follows that $\mathrm{L}(X_\Lambda)\mathbf{L} = 0$ holds everywhere on $J^1\pi$ while

$$\mathrm{L}(X_\Lambda)\Theta_{\mathscr{L}} = \epsilon^{0i}(\pi - \phi_0)(\mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i - \phi_i \mathrm{d}^m x) \ . \tag{7.69}$$

Also, observe that X_Λ is tangent to the constraint submanifold $S_1 \subset J^1\pi$ given by the constraint $\phi_0 - \pi = 0$ as desired since $\mathrm{L}(X_\Lambda)(\phi_0 - \pi) = 0$. However, X_Λ is not projectable onto E as a result of the component attached to $\dfrac{\partial}{\partial \pi}$; it is for this reason that the transformations (7.67) can be interpreted as the Lie derivatives of $j^1\phi$ only (as the sections ϕ do not provide the term associated with the second term of X_Λ that is not projectable onto E). Now $\mathrm{L}(X_\Lambda)\Theta_{\mathscr{L}}|_{S_1} = 0$, so the exact Noether symmetry exists only on the constraint submanifold S_1. Then, the corresponding momentum map $J_{\mathscr{L}}(X_\Lambda) = -i(X_\Lambda)\Theta_{\mathscr{L}}$ also exists only on the points of S_1 and is given by

$$J_{\mathscr{L}}(X_\Lambda)|_{S_1} = -\epsilon^i{}_\beta x^\beta \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-2}x_{0i} + \epsilon^j{}_\beta x^\beta \phi^i \mathrm{d}\phi \wedge \mathrm{d}^{m-2}x_{ij} - \frac{1}{2}\epsilon^i{}_\beta x^\beta (\pi^2 - \phi_j \phi^j)\mathrm{d}^{m-1}x_i \ . \tag{7.70}$$

In the Hamiltonian formalism on $P_\circ \subset J^1\pi^*$, the symmetry transformations are generated by $Y_\Lambda = \mathscr{FL}_{\circ *}\, X_\Lambda|_{S_1} \in \mathfrak{X}(P_\circ)$ which is given as

$$Y_\Lambda = -\epsilon^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} - \epsilon_{0i} p^i_\phi \frac{\partial}{\partial \pi} - \left(\epsilon^i{}_0 \pi + \epsilon^i{}_j p^j_\phi\right) \frac{\partial}{\partial p^i_\phi} \ . \tag{7.71}$$

The momentum map $J^\circ_{\mathscr{H}}(Y_\Lambda) = -i(Y_\Lambda)\Theta^\circ_{\mathscr{H}} \in \Omega^{m-1}(P_\circ)$ is given by

$$J^\circ_{\mathscr{H}}(Y_\Lambda) = -\epsilon^i{}_\beta x^\beta \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-2}x_{0i} - \epsilon^j{}_\beta x^\beta p^i_\phi \mathrm{d}\phi \wedge \mathrm{d}^{m-2}x_{ij} - \frac{1}{2}\epsilon^i{}_\beta x^\beta (\pi^2 - p_{\phi j} p^j_\phi)\mathrm{d}^{m-1}x_i \ . \tag{7.72}$$

7.3 Carrollian Electric Scalar Field Theory

We now turn our attention to theories which arise from taking the limit $c \to 0$ for the speed of light, first introduced in [77, 109]. Today, such theories are known as Carrollian field theories; for more details on such theories, see, for instance, [18, 19, 64, 80].

Again, consider m-dimensional spacetime M and two scalar fields, $\phi(x)$ and $\pi(x)$, given by the sections $\phi : M \to E : x^\mu \mapsto (x^\mu, \phi(x), \pi(x))$ of the configuration bundle $\pi : E \to M : (\phi, \pi) \mapsto x^\mu$. Now, the *electric Carrollian contraction* [80] of the canonical Klein–Gordon Lagrangian is performed by making the field redefinition given by $\phi(x) \to c\phi(x)$, $\pi(x) \to \frac{1}{c}\pi(x)$, and taking the limit $c \to 0$ for the speed of light. It follows that the Minkowski metric becomes degenerate as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -c^2 \mathrm{d}x^0 \otimes \mathrm{d}x^0 + \delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j \;\longrightarrow\; \delta_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j \ . \tag{7.73}$$

7.3.1 Lagrangian Formalism

The Lagrangian function $\mathscr{L} \in C^\infty(J^1\pi)$ obtained from the electric Carrollian contraction of the canonical Klein–Gordon Lagrangian is

$$\mathscr{L} = \pi\phi_0 - \frac{1}{2}\pi^2 \ , \tag{7.74}$$

which is clearly singular since

$$\frac{\partial^2 \mathscr{L}}{\partial\phi_\mu\partial\phi_\nu} = 0 \quad , \quad \frac{\partial^2 \mathscr{L}}{\partial\pi_\mu\partial\pi_\nu} = 0 \quad , \quad \frac{\partial^2 \mathscr{L}}{\partial\pi_\mu\partial\phi_\nu} = 0 \ . \tag{7.75}$$

The null vectors of the Hessians $\dfrac{\partial^2 \mathscr{L}}{\partial\phi_\mu\partial\phi_\nu}$ and $\dfrac{\partial^2 \mathscr{L}}{\partial\pi_\mu\partial\pi_\nu}$ are

$$\gamma_\mu^{(\phi)} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad , \quad \gamma_\mu^{(\pi)} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad , \tag{7.76}$$

respectively. The Lagrangian energy function and the Poincaré–Cartan forms are

$$E_{\mathscr{L}} = \frac{1}{2}\pi^2 \in C^\infty(J^1\pi) \ , \tag{7.77}$$

$$\Theta_{\mathscr{L}} = \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_0 - \frac{1}{2}\pi^2 \mathrm{d}^m x \in \Omega^m(J^1\pi) \ , \tag{7.78}$$

$$\Omega_{\mathscr{L}} = \mathrm{d}\phi \wedge \mathrm{d}\pi \wedge \mathrm{d}^{m-1}x_0 + \pi \mathrm{d}\pi \wedge \mathrm{d}^m x \in \Omega^{m+1}(J^1\pi) \ . \tag{7.79}$$

Since the Lagrangian function (7.74) is singular, $\Omega_{\mathscr{L}}$ is a premultisymplectic form. The field equations are obtained by using multivector fields $\mathbf{X}_{\mathscr{L}} = X_0 \wedge X_1 \wedge \cdots \wedge X_{m-1} \in \mathfrak{X}^m(J^1\pi)$ such that $i(\mathbf{X}_{\mathscr{L}})\mathrm{d}^m x = 1$, so it follows that their components are given by

$$X_\mu = \frac{\partial}{\partial x^\mu} + A_\mu \frac{\partial}{\partial \phi} + B_\mu \frac{\partial}{\partial \pi} + C_{\mu 0}\frac{\partial}{\partial \phi_0} + C_{\mu i}\frac{\partial}{\partial \phi_i} + D_{\mu\nu}\frac{\partial}{\partial \pi_\nu} \ . \tag{7.80}$$

Then the field equation is

$$0 = i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = A_0 \mathrm{d}\pi - B_0 \mathrm{d}\phi - \pi \mathrm{d}\pi + (A_\lambda B_0 - B_\lambda A_0 + \pi B_\lambda)\mathrm{d}x^\lambda \ . \tag{7.81}$$

which leads to

$$0 = - B_0 \ , \tag{7.82}$$

$$0 = A_0 - \pi \ , \tag{7.83}$$

$$0 = A_\lambda B_0 - B_\lambda A_0 + \pi B_\lambda \ . \tag{7.84}$$

Equations (7.82) and (7.83) determine the coefficients B_0 and A_0; equation (7.84) is an identity as a result of the previous equations as usual. There are no compatibility constraints, which is corroborated by applying Proposition 3.1:

$$\ker \widetilde{\Omega}_{\mathscr{L}} \cap \ker \mathrm{d}^m x = \left\langle \frac{\partial}{\partial \phi_\mu}, \frac{\partial}{\partial \pi_\mu} \right\rangle , \tag{7.85}$$

and $i\left(\dfrac{\partial}{\partial \phi_\mu}\right)\mathrm{d}E_{\mathscr{L}} = 0 = i\left(\dfrac{\partial}{\partial \pi_\mu}\right)\mathrm{d}E_{\mathscr{L}}$ is satisfied *a priori* without producing any constraints.

Imposing the SOPDE condition, $A_\mu = \phi_\mu$, $B_\mu = \pi_\mu$, the field equations (7.82) and (7.83) produce two SOPDE constraints

$$\pi_0 = 0 \quad , \quad \phi_0 - \pi = 0 \ , \tag{7.86}$$

which define the constraint submanifold $S_1 \hookrightarrow J^1\pi$. The tangency condition for $\mathbf{X}_{\mathscr{L}}$ on S_1 is imposec by

$$\mathrm{L}(X_\mu)(\pi_0) = D_{\mu 0} = 0 \quad , \quad \mathrm{L}(X_\mu)(\phi_0 - \pi) = C_{\mu 0} - B_\mu = 0 \quad (\text{on } S_1) \ . \tag{7.87}$$

It is evident above that no new constraints arise, hence, S_1 is the final constraint submanifold in the Lagrangian formalism.

Consider now $(x^\mu, \phi(x), \pi(x), \phi_\nu(x), \pi_\nu(x))$ which are integral sections of the multivector fields $\mathbf{X}_{\mathscr{L}}$ and therefore satisfy (7.46). Then, the SOPDE constraints (7.86) yield

$$\dot{\pi} = 0 \quad , \quad \dot{\phi} - \pi = 0 \ , \tag{7.88}$$

which can be combined giving

$$\ddot{\phi} = 0 \ , \tag{7.89}$$

as in [19]. The tangency conditions (7.87) lead to

$$\partial_\mu \pi_0 \equiv \frac{\partial \dot{\pi}}{\partial x^\mu} = 0 \quad , \quad \partial_\mu \pi = \partial_\mu \phi_0 \equiv \partial_\mu \dot{\phi} \quad ; \quad (\text{on } S_1) \ . \tag{7.90}$$

These equations are fulfilled by the solutions to the field equations, in particular, as a consequence of the second equation in (7.88).

7.3.2 De Donder–Weyl Hamiltonian Formalism

The multimomenta obtained from the Legendre map are

$$\mathscr{F}\mathscr{L}^* p^0_\phi \equiv \frac{\partial \mathscr{L}}{\partial \phi_0} = \pi \quad , \quad \mathscr{F}\mathscr{L}^* p^i_\phi = \frac{\partial \mathscr{L}}{\partial \phi_i} = 0 \quad , \quad \mathscr{F}\mathscr{L}^* p^\mu_\pi = \frac{\partial \mathscr{L}}{\partial \pi_\mu} = 0 \ , \tag{7.91}$$

and give the primary constraints $p^0_\phi - \pi = 0$, $p^i_\phi = 0$, and $p^\mu_\pi = 0$, which define the primary constraint submanifold $\jmath_\circ : P_\circ \hookrightarrow J^1\pi^*$ with local coordinates (x^μ, ϕ, π) and the restricted Legendre map $\mathscr{F}\mathscr{L}_\circ$ is given by $\mathscr{F}\mathscr{L} = \jmath_\circ \circ \mathscr{F}\mathscr{L}_\circ$ as before. Once again, the primary constraints can be used to compute the null vectors (7.76) by using equation (3.24) given in Proposition 3.2. The Lagrangian function $\mathscr{L}$ is almost-regular and the De Donder–Weyl Hamiltonian $\mathscr{H}_\circ \in \mathrm{C}^\infty(P_\circ)$ obtained from $E_{\mathscr{L}} = \mathscr{F}\mathscr{L}^*_\circ \mathscr{H}_\circ$ is

$$\mathscr{H}_\circ = \frac{1}{2}\pi^2 \ , \tag{7.92}$$

and the Hamilton-Cartan forms are

$$\begin{aligned}
\Theta^\circ_{\mathscr{H}} &= \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_0 - \frac{1}{2}\pi^2 \mathrm{d}^m x \in \mathit{\Omega}^m(P_\circ) \ , \\
\Omega^\circ_{\mathscr{H}} &= \mathrm{d}\phi \wedge \mathrm{d}\pi \wedge \mathrm{d}^{m-1}x_0 + \pi \mathrm{d}\pi \wedge \mathrm{d}^m x \equiv \widetilde{\Omega}^\circ_{\mathscr{H}} + \mathrm{d}\mathscr{H}_\circ \wedge \mathrm{d}^m x \in \mathit{\Omega}^{m+1}(P_\circ) \ .
\end{aligned}$$

The locally decomposable multivector fields $\mathbf{X}^\circ_{\mathscr{H}} = X_0 \wedge X_1 \wedge \cdots \wedge X_{m-1} \in \mathfrak{X}^m(P_\circ)$ satisfying the normalized transverse condition $i(\mathbf{X}^\circ_{\mathscr{H}})\mathrm{d}^m x = 1$ used to produce the field equations have the following components:

$$X_\mu = \frac{\partial}{\partial x^\mu} + A_\mu \frac{\partial}{\partial \phi} + B_\mu \frac{\partial}{\partial \pi} \ . \tag{7.93}$$

The field equations obtained from $i(\mathbf{X}^\circ_{\mathscr{H}})\Omega^\circ_{\mathscr{H}} = 0$ are

$$0 = B_0 \ , \tag{7.94}$$

$$0 = A_0 - \pi \ , \tag{7.95}$$

$$0 = A_\lambda B_0 - A_0 B_\lambda + \pi B_\lambda \ . \tag{7.96}$$

Once again, using (7.94) and (7.95), the third equation (7.96) is an identity. These equations do not produce any compatibility constraints. This again is in agreement with Proposition 3.1 which dictates that compatibility constraints may be produced by $i(Z)\mathrm{d}\mathscr{H}_\circ = 0$, for every $Z \in \ker \widetilde{\Omega}^\circ_{\mathscr{H}} \cap \ker \mathrm{d}^m x$, and as

$$\ker \widetilde{\Omega}^\circ_{\mathscr{H}} \cap \ker \mathrm{d}^m x = \{0\} \tag{7.97}$$

is obtained computationally, it is confirmed that no compatibility constraints exist.

Upon working on $(x^\mu, \phi(x), \pi(x))$ which are integral sections of the multivector fields $\mathbf{X}^\circ_{\mathscr{H}}$, and hence

$$A_\mu = \partial_\mu \phi \quad , \quad B_\mu = \partial_\mu \pi \ , \tag{7.98}$$

it follows that the field equations (7.94) and (7.95) take the form

$$\dot{\pi} = 0 \quad , \quad \dot{\phi} - \pi = 0 \ . \tag{7.99}$$

Notice that plugging in the second equation into the first above gives

$$\ddot{\phi} = 0 \ . \tag{7.100}$$

The equivalence between the Lagrangian and the Hamiltonian formalisms again is evident.

7.3.3 Symmetries

The spacetime symmetry of the Carroll spacetime geometry are the Carroll transformations [80]:

$$x'^0 = x^0 + b_k x^k \quad , \quad x'^i = x^i + \epsilon^i{}_j x^j \quad \Rightarrow \quad x'^\mu = C^\mu{}_\nu x^\nu \ , \tag{7.101}$$

where $C^0{}_0 = 1$, $C^0{}_k = b_k$, $C^k{}_0 = 0$, $(C^i{}_j) \in O(d)$. These transformations can be represented infinitesimally by

$$C^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu \ , \tag{7.102}$$

so that now $b_k = \epsilon^0{}_k$ is infinitesimal. Up to linear terms in $\phi'(x') = \phi(x)$ and $\pi'(x') = \pi(x)$, the infinitesimal transformations of the fields, $\delta\phi(x) \equiv \phi'(x) - \phi(x)$ and $\delta\pi(x) \equiv \pi'(x) - \pi(x)$, are

$$\delta\phi(x) = -\epsilon^\alpha{}_\beta x^\beta \partial_\alpha \phi = -b_k x^k \dot{\phi} - \epsilon^i{}_j x^j \partial_i \phi \ , \tag{7.103}$$

$$\delta\pi(x) = -\epsilon^\alpha{}_\beta x^\beta \partial_\alpha \pi = -b_k x^k \dot{\pi} - \epsilon^i{}_j x^j \partial_i \pi \ . \tag{7.104}$$

Furthermore

$$\phi'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} \phi_\nu(x) \iff \phi'_\mu(C \cdot x) = (C^{-1})^\nu{}_\mu \phi_\nu(x) \ , \tag{7.105}$$

where $C \equiv (C^\mu{}_\nu)$ and $C \cdot x \equiv C^\mu{}_\nu x^\nu$; hence,

$$\delta\phi_\mu(x) \equiv \phi'_\mu(x) - \phi_\mu(x) = -\epsilon^\nu{}_\mu \phi_\nu - \epsilon^\alpha{}_\beta x^\beta \partial_\alpha \phi_\mu \ , \tag{7.106}$$

$$\delta\pi_\mu(x) \equiv \pi'_\mu(x) - \pi_\mu = -\epsilon^\nu{}_\mu \pi_\nu - \epsilon^\alpha{}_\beta x^\beta \partial_\alpha \pi_\mu \ . \tag{7.107}$$

Component-wise,

$$\delta\phi_0(x) = -\epsilon^\alpha{}_\beta x^\beta \partial_\alpha \phi_0 = -b_k x^k \dot{\phi}_0 - \epsilon^k{}_j x^j \partial_k \phi_0 \ , \tag{7.108}$$

$$\delta\phi_i(x) = -\epsilon^\nu{}_i \phi_\nu - \epsilon^\alpha{}_\beta x^\beta \partial_\alpha \phi_i = -b_i \phi_0 - \epsilon^k{}_i \phi_k - b_k x^k \dot{\phi}_i - \epsilon^k{}_j x^j \partial_k \phi_i \ , \tag{7.109}$$

and similarly for the components of $\delta\pi_\mu(x)$. Under the transformations presented above, the Lagrangian transforms as $\delta\mathscr{L}(x) = -\dfrac{\partial}{\partial x^\alpha}\Big(\epsilon^\alpha{}_\beta x^\beta \mathscr{L}(x)\Big)$. These transformations are produced by the Lie derivatives of the local sections $\phi : M \to E$ and $j^1\phi : M \to J^1\pi$ in the direction of a vector field $\xi = -\xi^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ which generates the Carroll transformations, where now $\xi^\mu(x) = \epsilon^\mu{}_\nu x^\nu$ is given by (7.101) and (7.102). The vector field which generates the Carroll transformations on $J^1\pi$ is given by

$$\begin{aligned} X &= -\epsilon^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} + \epsilon^\nu{}_\mu \phi_\nu \frac{\partial}{\partial \phi_\mu} + \epsilon^\nu{}_\mu \pi_\nu \frac{\partial}{\partial \pi_\mu} \\ &= -b_i x^i \frac{\partial}{\partial x^0} - \epsilon^i{}_j x^j \frac{\partial}{\partial x^i} + \Big(b_i \phi_0 + \epsilon^j{}_i \phi_j\Big) \frac{\partial}{\partial \phi_i} + \Big(b_i \pi_0 + \epsilon^j{}_i \pi_j\Big) \frac{\partial}{\partial \pi_i} \in \mathfrak{X}(J^1\pi) \ . \end{aligned} \tag{7.110}$$

Evidently X is tangent to the constraint submanifold S_1 since $\mathrm{L}(X)(\pi - \phi_0) = 0$ and $\mathrm{L}(X)\pi_0 = 0$. Finally, since $\mathrm{L}(X)\mathbf{L} = 0 \Rightarrow \mathrm{L}(X)\Theta_{\mathscr{L}} = 0$, the momentum map $J_{\mathscr{L}}(X)$ corresponding to the Carroll spacetime transformations in the Lagrangian setting is given by

$$J_{\mathscr{L}}(X) = -i(X)\Theta_{\mathscr{L}} = -\epsilon^i{}_j x^j \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-2}x_{0i} - \frac{1}{2}\pi^2 \left(b_i x^i \mathrm{d}^{m-1}x_0 + \epsilon^i{}_j x^j \mathrm{d}^{m-1}x_i\right) \in \Omega^{m-1}(J^1\pi) \ . \tag{7.111}$$

In the Hamiltonian setting the momentum map $J^\circ_{\mathscr{H}}(Y) \in \Omega^{m-1}(P_\circ)$ is constructed using the vector field $Y \in \mathfrak{X}(P_\circ)$ given by

$$Y = \mathscr{F}\mathscr{L}_{\circ *}X = -\epsilon^\mu{}_\nu x^\nu \frac{\partial}{\partial x^\mu} = -b_i x^i \frac{\partial}{\partial x^0} - \epsilon^i{}_j x^j \frac{\partial}{\partial x^i} \ . \tag{7.112}$$

As in the Lagrangian setting, $\mathrm{L}(Y)\Theta^\circ_{\mathscr{H}} = 0$, it follows that $J^\circ_{\mathscr{H}}(Y)$ is given by,

$$J^\circ_{\mathscr{H}}(Y) = -i(Y)\Theta^\circ_{\mathscr{H}} = -\epsilon^i{}_j x^j \pi \, \mathrm{d}\phi \wedge \mathrm{d}^{m-2}x_{0i} - \frac{1}{2}\pi^2 \left(b_i x^i \mathrm{d}^{m-1}x_0 + \epsilon^i{}_j x^j \mathrm{d}^{m-1}x_i\right) \ . \tag{7.113}$$

7.4 Carrollian Magnetic Scalar Field Theory

We remind the reader that Carrollian field theories were first introduced in [77, 109]; For the state of the art, the reader is encouraged to see [18, 19, 64, 80] for instance.

Once again, consider m-dimensional spacetime M and two scalar fields, $\phi(x)$ and $\pi(x)$, given by the sections $\phi : M \to E : x^\mu \mapsto (x^\mu, \phi(x), \pi(x))$ of $\pi : E \to M : (\phi, \pi) \mapsto x^\mu$. The *magnetic Carrollian contraction* [80] of the canonical Klein–Gordon Lagrangian is performed by reinserting the factors of c (speed of light) into the Lagrangian (7.28) and taking the limit $c \to 0$. The Minkowski metric becomes degenerate as in the electric Carrollian scalar field theory, given by equation (7.73).

7.4.1 Lagrangian Formalism

The Lagrangian function $\mathscr{L} \in C^\infty(J^1\pi)$ obtained from taking the magnetic Carrollian contraction of the canonical Klein–Gordon Lagragian (7.28) is

$$\mathscr{L} = \pi\phi_0 - \frac{1}{2}\phi_i\phi^i \ . \tag{7.114}$$

This Lagrangian is singular since the components of the multi Hessian matrix are

$$\frac{\partial^2 \mathscr{L}}{\partial\phi_0\partial\phi_0} = 0 \quad , \quad \frac{\partial^2 \mathscr{L}}{\partial\phi_0\partial\phi_i} = 0 \quad , \quad \frac{\partial^2 \mathscr{L}}{\partial\phi_i\partial\phi_j} = -\delta^{ij} \quad , \quad \frac{\partial^2 \mathscr{L}}{\partial\pi_\mu\partial\pi_\nu} = 0 \quad , \quad \frac{\partial^2 \mathscr{L}}{\partial\pi_\mu\partial\phi_\nu} = 0 \ . \tag{7.115}$$

The null vectors of the Hessians $\dfrac{\partial^2 \mathscr{L}}{\partial\phi_\mu\partial\phi_\nu}$ and $\dfrac{\partial^2 \mathscr{L}}{\partial\pi_\mu\partial\pi_\nu}$ are

$$\gamma^{(\phi)}_\mu = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad \gamma^{(\pi)}_\mu = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \ , \tag{7.116}$$

respectively. The Lagrangian energy function and the Poincaré–Cartan forms are now

$$E_{\mathscr{L}} = -\frac{1}{2}\phi_i\phi^i \in C^\infty(J^1\pi) \ , \tag{7.117}$$

$$\Theta_{\mathscr{L}} = \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_0 - \phi^i \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i + \frac{1}{2}\phi_i\phi^i \mathrm{d}^m x \in \Omega^m(J^1\pi) \ , \tag{7.118}$$

$$\Omega_{\mathscr{L}} = \mathrm{d}\phi \wedge \mathrm{d}\pi \wedge \mathrm{d}^{m-1}x_0 + \mathrm{d}\phi^i \wedge \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i + \phi^i \mathrm{d}\phi_i \wedge \mathrm{d}^m x \in \Omega^{m+1}(J^1\pi), \tag{7.119}$$

and $\Omega_{\mathscr{L}}$ is premultisymplectic. Taking multivector fields $\mathbf{X}_{\mathscr{L}} = X_0 \wedge X_1 \wedge \cdots \wedge X_{m-1} \in \mathfrak{X}^m(J^1\pi)$ satisfying $i(\mathbf{X}_{\mathscr{L}})\mathrm{d}^m x = 1$, their components are given as

$$X_\mu = \frac{\partial}{\partial x^\mu} + A_\mu \frac{\partial}{\partial \phi} + B_\mu \frac{\partial}{\partial \pi} + C_{\mu 0}\frac{\partial}{\partial \phi_0} + C_{\mu i}\frac{\partial}{\partial \phi_i} + D_{\mu\nu}\frac{\partial}{\partial \pi_\nu} \ , \tag{7.120}$$

and the field equations obtained from $i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = 0$ are

$$C_i^i - B_0 = 0 \ , \tag{7.121a}$$
$$A_0 = 0 \ , \tag{7.121b}$$
$$A_\lambda B_0 - B_\lambda A_0 + C_\lambda^i A_0 + A_\lambda C_0^i + \phi^i C_{\lambda i} = 0 \ . \tag{7.121c}$$

Equations (7.121a) and (7.121b) are relations for the coefficients of the multivector fields while equation (7.121c) is an identity following from (7.121a) and (7.121b) as usual. There are no compatibility constraints in agreement with the geometric analysis presented in Proposition 3.1, which shows that

$$\ker \widetilde{\Omega}_{\mathscr{L}} \cap \ker \mathrm{d}^m x = \left\langle \frac{\partial}{\partial \phi_0}, \frac{\partial}{\partial \pi_\mu} \right\rangle \ , \tag{7.122}$$

and $i\left(\dfrac{\partial}{\partial \phi_0}\right)\mathrm{d}E_{\mathscr{L}} = 0 = i\left(\dfrac{\partial}{\partial \pi_\mu}\right)\mathrm{d}E_{\mathscr{L}}$ is satisfied without producing any constraints.

Then, upon imposing the SOPDE condition $A_\mu = \phi_\mu$, $B_\mu = \pi_\mu$, equations (7.121a) and (7.121b) become

$$C_i^i - \pi_0 = 0 \quad , \quad \phi_0 = 0 \ . \tag{7.123}$$

The first equation gives a relation for the coefficients C_i^i, and the second one is a SOPDE-constraint which defines the constraint submanifold $S_1 \hookrightarrow J^1\pi$.

Now impose the tangency condition of the multivector fields $\mathbf{X}_{\mathscr{L}}$ to S_1,

$$\mathrm{L}(X_\mu)(\phi_0) = C_{\mu 0} = 0 \quad (\text{on } S_1) \ . \tag{7.124}$$

These are not constraints, but new relations among the coefficient of $\mathbf{X}_{\mathscr{L}}$ and hence the final constraint submanifold in the Lagrangian formalism is S_1. Then, on $(x^\mu, \phi(x), \pi(x), \phi_\nu(x), \pi_\nu(x))$, the integral sections of the multivector fields $\mathbf{X}_{\mathscr{L}}$ satisfy equations (7.46) and the field equations take the form

$$\partial_i \partial^i \phi - \dot{\pi} = 0 \quad , \quad \dot{\phi} = 0 \quad ; \quad (\text{on } S_1) \ . \tag{7.125}$$

For further details on these equations see [19, 80]. Furthermore, from (7.124) it follows that, on S_1,

$$\partial_\mu \phi_0 \equiv \partial_\mu \dot{\phi} = 0 \ . \tag{7.126}$$

The equations above are fulfilled by the solutions to the field equations, in particular, as a consequence of the second equation in (7.125).

7.4.2 De Donder–Weyl Hamiltonian Formalism

The Legendre map $\mathscr{FL} : J^1\pi \to J^1\pi^*$ gives the multimomenta

$$\mathscr{FL}^* p_\phi^0 = \frac{\partial \mathscr{L}}{\partial \phi_0} = \pi \quad , \quad \mathscr{FL}^* p_\phi^i = \frac{\partial \mathscr{L}}{\partial \phi_i} = -\phi^i \quad , \quad \mathscr{FL}^* p_\pi^\mu = \frac{\partial \mathscr{L}}{\partial \pi_\mu} = 0 \ . \tag{7.127}$$

It follows that $p_\phi^0 - \pi = 0$ and $p_\pi^\mu = 0$ are primary constraints which define the primary constraint submanifold $P_\circ \hookrightarrow J^1\pi^*$ with local coordinates $(x^\mu, \phi, \pi, p_\phi^i)$; consequently, the Lagrangian is again almost-regular. As above, the null vectors (7.116) are given by the partial derivatives of the primary constraints with respect to the multimomenta according to (3.24) in Proposition 3.2. The restricted Legendre map $\mathscr{FL}_\circ$ is given by $\mathscr{FL} = \jmath_\circ \circ \mathscr{FL}_\circ$ as usual.

The De Donder–Weyl Hamiltonian $\mathscr{H}_\circ \in C^\infty(P_\circ)$ is given by

$$\mathscr{H}_\circ = -\frac{1}{2} p_{\phi i} p^i_\phi \ , \tag{7.128}$$

and the Hamilton–Cartan forms on $P_\circ$ are

$$\Theta^\circ_{\mathscr{H}} = \pi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_0 + p^i_\phi \mathrm{d}\phi \wedge \mathrm{d}^{m-1}x_i + \frac{1}{2} p_{\phi i} p^i_\phi \mathrm{d}^m x \in \Omega^m(P_\circ) \ , \tag{7.129}$$

$$\Omega^\circ_{\mathscr{H}} = \mathrm{d}\phi \wedge \mathrm{d}\pi \wedge \mathrm{d}^{m-1}x_\circ + \mathrm{d}\phi \wedge \mathrm{d}p^i_\phi \wedge \mathrm{d}^{m-1}x_i - p_{\phi i} \mathrm{d}p^i_\phi \wedge \mathrm{d}^m x \equiv \widetilde{\Omega}^\circ_{\mathscr{H}} + \mathrm{d}\mathscr{H}_\circ \wedge \mathrm{d}^m x \in \Omega^{m+1}(P_\circ) \, . \tag{7.130}$$

Now take a locally decomposable multivector field $\mathbf{X}^\circ_{\mathscr{H}} = X_0 \wedge X_1 \wedge \cdots \wedge X_{m-1} \in \mathfrak{X}^m(P_\circ)$ satisfying $i(\mathbf{X}^\circ_{\mathscr{H}})\mathrm{d}^m x = 1$, so their components are given as

$$X_\mu = \frac{\partial}{\partial x^\mu} + A_\mu \frac{\partial}{\partial \phi} + B_\mu \frac{\partial}{\partial \pi} + C^i_\mu \frac{\partial}{\partial p^i_\phi} \ . \tag{7.131}$$

The field equations obtained from $i(\mathbf{X}^\circ_{\mathscr{H}})\Omega^\circ_{\mathscr{H}} = 0$ are:

$$B_0 + C^i_i = 0 \ , \tag{7.132a}$$
$$A_0 = 0 \ , \tag{7.132b}$$
$$A_i + p_{\phi i} = 0 \ , \tag{7.132c}$$
$$A_\lambda B_0 - A_0 B_\lambda + A_\lambda C^i_i - A_i C^i_\lambda - p_{\phi i} C^i_\lambda = 0 \ . \tag{7.132d}$$

Equation (7.132d) is a combination of the equations (7.132a), (7.132b), and (7.132c) which give relations or determine some coefficients. Then, no compatibility constraints appear. Again, this is in agreement with the procedure given in Proposition 3.1 in which we set $i(Z)\mathrm{d}\mathscr{H}_\circ = 0$ for every $Z \in \ker \widetilde{\Omega}^\circ_{\mathscr{H}} \cap \ker \mathrm{d}^m x$; however,

$$\ker \widetilde{\Omega}^\circ_{\mathscr{H}} \cap \ker \, \mathrm{d}^m x = \{0\} \ , \tag{7.133}$$

and condition (3.20) from Proposition 3.1 holds.

Working on $(x^\mu, \phi(x), \pi(x), p^i_\phi(x))$ taken to be integral sections of $\mathbf{X}^\circ_{\mathscr{H}}$,

$$A_\mu = \partial_\mu \phi \quad , \quad B_\mu = \partial_\mu \pi \quad , \quad C^i_\mu = \partial_\mu p^i_\phi \ , \tag{7.134}$$

then (7.132a), (7.132b), and (7.132c) give

$$\dot{\pi} + \partial_i p^i_\phi = 0 \quad , \quad \dot{\phi} = 0 \quad , \quad \partial_i \phi + p_{\phi i} = 0 \ . \tag{7.135}$$

Plugging in the third equation into the first equation gives

$$\dot{\pi} - \partial_i \partial^i \phi = 0 \quad , \quad \dot{\phi} = 0 \ . \tag{7.136}$$

Once again, the equivalence between the Lagrangian and the Hamiltonian formalisms is evident as the resulting field equations are shown to be equivalent.

7.4.3 Symmetries

The Carroll transformations for the magnetic scalar field theory are written slightly differently than how they are written for the electric scalar field 7.3.3. The Carroll transformations which are symmetries of the magnetic scalar field theory are

$$\delta\phi(x) = -\epsilon^{\alpha}{}_{\beta}x^{\beta}\frac{\partial\phi}{\partial x^{\alpha}} = -b_k x^k\frac{\partial\phi}{\partial x^0} - \epsilon^{i}{}_{j}x^j\partial_i\phi \ , \tag{7.137}$$

$$\delta\pi(x) = -\epsilon^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha}\pi - b_i\phi^i = -b_k x^k\dot{\pi} - \epsilon^{i}{}_{j}x^j\partial_i\pi - b_i\phi^i \ , \tag{7.138}$$

$$\delta\phi_0(x) = -\epsilon^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha}\phi_0 = -b_k x^k\dot{\phi}_0 - \epsilon^{k}{}_{j}x^j\partial_k\phi_0 \ , \tag{7.139}$$

$$\delta\phi_i(x) = -\epsilon^{\nu}{}_{i}\phi_{\nu} - \epsilon^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha}\phi_i = -b_i\phi_0 - \epsilon^{k}{}_{i}\phi_k - b_k x^k\dot{\phi}_i - \epsilon^{k}{}_{j}x^j\partial_k\phi_i \ . \tag{7.140}$$

Under these transformations the Lagrangian transforms as $\delta\mathscr{L}(x) = -\partial_{\alpha}\Big(\epsilon^{\alpha}{}_{\beta}x^{\beta}\mathscr{L}(x)\Big)$. Notice that the field $\pi(x)$ no longer transforms as a Carrollian scalar; instead, $\pi(x)$ transforms under a Carroll spacetime transformation $-\epsilon^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha}\pi$ plus a term $-b_i\phi^i(x)$ which cannot be represented on E. The field transformations behave geometrically as the Lie derivatives of the local sections $j^1\phi : M \to J^1\pi$. The vector field generating the Carroll transformations on $J^1\pi$ is

$$\begin{aligned} X &= -\epsilon^{\mu}{}_{\nu}x^{\nu}\frac{\partial}{\partial x^{\mu}} + b_i\phi^i\frac{\partial}{\partial\pi} + \epsilon^{\nu}{}_{\mu}\phi_{\nu}\frac{\partial}{\partial\phi_{\mu}} + \epsilon^{\nu}{}_{\mu}\pi_{\nu}\frac{\partial}{\partial\pi_{\mu}} \\ &= -b_i x^i\frac{\partial}{\partial x^0} - \epsilon^{i}{}_{j}x^j\frac{\partial}{\partial x^i} + b_i\phi^i\frac{\partial}{\partial\pi} + \Big(b_i\phi_0 + \epsilon^{j}{}_{i}\phi_j\Big)\frac{\partial}{\partial\phi_i} + \epsilon^{\nu}{}_{\mu}\pi_{\nu}\frac{\partial}{\partial\pi_{\mu}} \in \mathfrak{X}(J^1\pi) \ , \end{aligned} \tag{7.141}$$

which is tangent to the constraint submanifold $S_1 \subset J^1\pi$, given by the constraint $\phi_0 = 0$, since $\mathrm{L}(X)\phi_0 = 0$. Furthermore, although the vector field X leaves the Lagrangian density invariant ($\mathrm{L}(X)\mathbf{L} = 0$), it does not produce an exact Noether symmetry as

$$\mathrm{L}(X)\Theta_{\mathscr{L}} = b^i\phi_0(\phi_i\mathrm{d}^m x - \mathrm{d}\phi\wedge\mathrm{d}^{m-1}x_i) \ . \tag{7.142}$$

Moreover, the vector field X as written above on all of $J^1\pi$ is not $\mathscr{FL}$-projectable onto $P_\circ \subset J^1\pi^*$. Instead, X on S_1 given by

$$X|_{S_1} = -b_i x^i\frac{\partial}{\partial x^0} - \epsilon^{i}{}_{j}x^j\frac{\partial}{\partial x^i} + b_i\phi^i\frac{\partial}{\partial\pi} + \epsilon^{j}{}_{i}\phi_j\frac{\partial}{\partial\phi_i} + \epsilon^{\nu}{}_{\mu}\pi_{\nu}\frac{\partial}{\partial\pi_{\mu}} \ , \tag{7.143}$$

is $\mathscr{FL}_\circ$-projectable onto $P_\circ$, giving

$$Y = \mathscr{FL}_{\circ *}X|_{S_1} = -b_i x^i\frac{\partial}{\partial x^0} - \epsilon^{i}{}_{j}x^j\frac{\partial}{\partial x^i} - b_i p^i_{\phi}\frac{\partial}{\partial\pi} + \epsilon^{i}{}_{j}p^j_{\phi}\frac{\partial}{\partial p^i_{\phi}} \in \mathfrak{X}(P_\circ) \ . \tag{7.144}$$

Now, letting $\widetilde{X}$ be the local extension of $X|_{S_1}$ to $J^1\pi$ whose coordinate expression is given by (7.143); then, it follows that $\mathrm{L}(\widetilde{X})\Theta_{\mathscr{L}} = 0$ on $J^1\pi$. It is thereby possible to define the momentum maps as $J_{\mathscr{L}}(\widetilde{X}) = -i(\widetilde{X})\Theta_{\mathscr{L}} \in \varOmega^{m-1}(J^1\pi)$ and $J^\circ_{\mathscr{H}}(Y) = -i(Y)\Theta^\circ_{\mathscr{H}} \in \varOmega^{m-1}(P_\circ)$, giving:

$$\begin{aligned} J_{\mathscr{L}}(\widetilde{X}) = &-\left(\epsilon^{i}{}_{j}x^j\pi + b_j x^j\phi^i\right)\mathrm{d}\phi\wedge\mathrm{d}^{m-2}x_{0i} + \phi^i\epsilon^{j}{}_{k}x^k\mathrm{d}\phi\wedge\mathrm{d}^{m-2}x_{ij} \\ &+ \frac{1}{2}\phi_i\phi^i\left(b_k x^k\mathrm{d}^{m-1}x_0 + \epsilon^{k}{}_{j}x^j\mathrm{d}^{m-1}x_k\right) \ , \end{aligned} \tag{7.145}$$

$$\begin{aligned} J^\circ_{\mathscr{H}}(Y) = &-\left(\epsilon^{i}{}_{j}x^j\pi - b_j x^j p^i_{\phi}\right)\mathrm{d}\phi\wedge\mathrm{d}^{m-2}x_{0i} - p^i_{\phi}\epsilon^{j}{}_{k}x^k\mathrm{d}\phi\wedge\mathrm{d}^{m-2}x_{ij} \\ &+ \frac{1}{2}p_{\phi i}p^i_{\phi}\left(b_k x^k\mathrm{d}^{m-1}x_0 + \epsilon^{k}{}_{j}x^j\mathrm{d}^{m-1}x_k\right) \ . \end{aligned} \tag{7.146}$$

8 The Nambu–Goto Action for Strings and p-Branes

For a textbook treatment of the Nambu–Goto action for p-branes see, for instance, [17, 74].

We now provide the multisymplectic formalism for bosonic p-branes by explicitly working out the case for string theory ($p = 1$) and then discussing the generalization to $p > 1$. The De Donder–Weyl formulation of string theory (and p-branes in general) from the Nambu–Goto action can be found in [100] without mention of the underlying multisymplectic structure.

The fields of interest are the embeddings of a brane worldvolume in spacetime. Spacetime M is a smooth $D = (d + 1)$-dimensional manifold with local coordinates x^μ ($\mu = 0, 1, ..., d$) and spacetime metric $G_{\mu\nu}$ with signature $(- + \cdots +)$. The p-brane worldvolume Σ is a smooth manifold with $\dim\Sigma = m = p + 1$ and local coordinates σ^a (with $a = 0, 1, ..., p$). Given the embedding $X : \Sigma \to M : \sigma^a \mapsto x^\mu(\sigma)$, the embedding maps $x^\mu(\sigma)$ are taken to be fields on Σ. The configuration bundle E over Σ is taken to be the locally trivial bundle $E = \Sigma \times M$ and comes with a surjective projection map $\pi : E \to \Sigma$ as usual and sections of E are given by $\phi : \Sigma \to \Sigma\times M : \sigma^a \mapsto (\sigma^a, x^\mu(\sigma))$. The coordinates on E are (σ^a, x^μ), while on $J^1\pi$ and $J^1\pi^*$ the local coordinates are $(\sigma^a, x^\mu, x^\mu_a)$ and $(\sigma^a, x^\mu, p^a_\mu)$ respectively. It follows that $\dim E = m + D$ and $\dim J^1\pi = \dim J^1\pi^* = m + D + mD$. The first-order jet prolongations are given as $j^1\phi : \Sigma \to J^1\pi : \sigma^a \mapsto \Big(\sigma^a, x^\mu(\sigma), \dfrac{\partial x^\mu}{\partial \sigma^a}(\sigma)\Big)$.

The Nabu–Goto Lagrangian is given as

$$\mathbf{L} = \mathscr{L}(\sigma^a, x^\mu, x^\mu_a)\mathrm{d}^m\sigma = -T_p\sqrt{-\mathrm{det}g}\ \mathrm{d}^m\sigma = -T_p\sqrt{-\mathrm{det}(G_{\mu\nu}x^\mu_a x^\nu_b)}\ \mathrm{d}^m\sigma, \tag{8.1}$$

where T_p is the p-brane tension which has units of $[T_p] = (mass)/(length)^p$. Furthermore, we have the induced metric on Σ:

$$g_{ab} = G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\ . \tag{8.2}$$

The Legendre map $\mathscr{FL} : J^1\pi \to J^1\pi^* : (\sigma^a, x^\mu, x^\mu_a) \mapsto (\sigma^a, x^\mu, p^a_\mu)$ gives

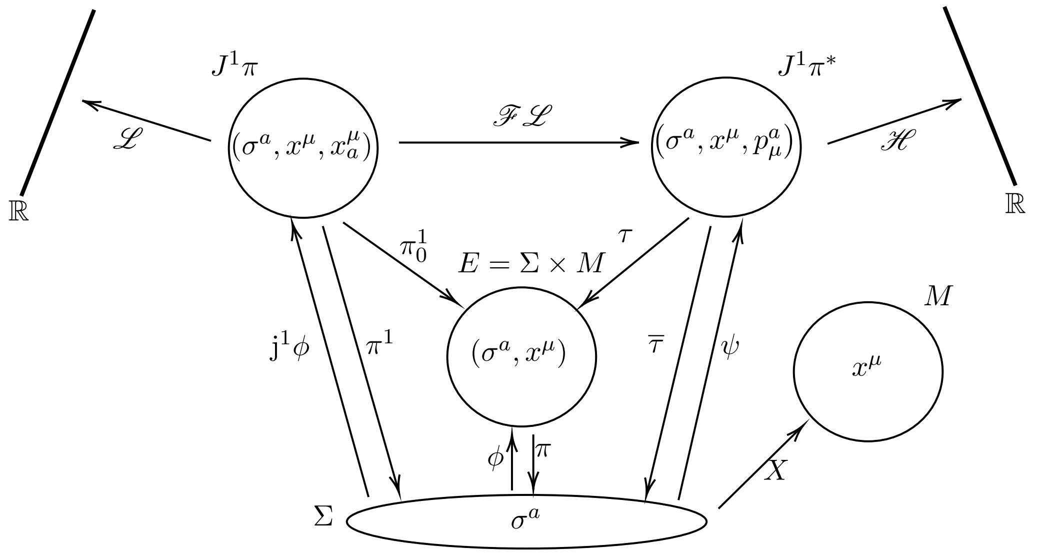
$$\mathscr{FL}^* p^a_\mu = \frac{\partial\mathscr{L}}{\partial x^\mu_a}\ ; \tag{8.3}$$

the Hessian matrix

$$\frac{\partial^2\mathscr{L}}{\partial x^\mu_a\partial x^\nu_b} = -T_p\sqrt{-\mathrm{det}g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i\left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]\ . \tag{8.4}$$

is non-singular and the Lagrangian is regular thusly.

The full geometric setting is illustrated in the following figure:



8.1 Lagrangian Formalism

The Lagrangian for the Nambu–Goto string ($p = 1$) is

$$\mathbf{L} = \mathscr{L}(\sigma^a, x^\mu, x^\mu_a)\mathrm{d}^2\sigma = -T\sqrt{-\det g}\ \mathrm{d}^2\sigma\ , \tag{8.5}$$

from which it follows that

$$\frac{\partial \mathscr{L}}{\partial x^\mu_a} = -T\sqrt{-\det g}\ G_{\mu\nu} g^{ba} x^\nu_b = \frac{T}{\sqrt{-\det g}}\ \epsilon^{bd}\epsilon^{ac} g_{dc} G_{\mu\nu} x^\nu_b\ , \tag{8.6}$$

where $g^{ba} \equiv (g^{-1})^{ba} = \frac{1}{\det g}\epsilon^{bc}\epsilon^{ad} g_{dc}$ and so it follows that the Hessian can be written as

$$\frac{\partial^2 \mathscr{L}}{\partial x^\mu_a \partial x^\nu_b} = -T\sqrt{-\det g}\left[G_{\mu\nu} g^{ba} - G_{\mu\alpha} G_{\beta\nu} x^\alpha_c x^\beta_i \left(g^{ba} g^{ci} + g^{cb} g^{ai} - g^{ca} g^{bi}\right)\right]\ . \tag{8.7}$$

The Lagrangian energy is

$$E_{\mathscr{L}} \equiv \frac{\partial \mathscr{L}}{\partial x^\mu_a} x^\mu_a - \mathscr{L} = -T\sqrt{-\det g}\ (g^{ba} g_{ab} - 1) = -T\sqrt{-\det g} \in C^\infty(J^1\pi)\ , \tag{8.8}$$

where the last equality follows as g_{ab} is a 2×2 matrix in the case of the string. Using (8.6) and (8.8), the Poincaré–Cartan 2-from and multisymplectic Poincaré–Cartan 3-form on $J^1\pi$ are

$$\Theta_{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial x^\mu_a}\ \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a - E_{\mathscr{L}} \wedge \mathrm{d}^2\sigma = -T\sqrt{-\det g}\left[G_{\mu\nu} g^{ba} x^\nu_b \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a - \mathrm{d}^2\sigma\right]\ , \tag{8.9}$$

$$\begin{aligned}
\Omega_{\mathscr{L}} = T\Bigg\{ &\sqrt{-\det g}\left[G_{\mu\nu} g^{ba} - G_{\mu\alpha} G_{\beta\nu} x^\alpha_c x^\beta_i \left(g^{ba} g^{ci} + g^{cb} g^{ai} - g^{ca} g^{bi}\right)\right] \mathrm{d}x^\nu_b \wedge \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a \\
&+ \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\, G_{\mu\nu} g^{ba} x^\nu_b\right) \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a \\
&- \sqrt{-\det g}\ \left[G_{\mu\nu} g^{ba} - G_{\mu\alpha} G_{\beta\nu} x^\alpha_c x^\beta_i \left(g^{ba} g^{ci} + g^{cb} g^{ai} - g^{ca} g^{bi}\right)\right] x^\mu_a \mathrm{d}x^\nu_b \wedge \mathrm{d}^2\sigma \\
&- \left[\frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det g}\, G_{\mu\nu} g^{ba} x^\nu_b\right) + \frac{1}{2}\sqrt{-\det g}\, g^{ba} x^\alpha_a x^\beta_b \partial_\mu G_{\alpha\beta}\right] \mathrm{d}x^\mu \wedge \mathrm{d}^2\sigma \Bigg\}\ .
\end{aligned} \tag{8.10}$$

The field equations, $i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = 0$, can be computed using a representative of a class of π^1-transverse and locally decomposable multivector fields $\mathbf{X}_{\mathscr{L}} \in \mathfrak{X}^2(J^1\pi)$, written as $\mathbf{X}_{\mathscr{L}} = X_0 \wedge X_1$, with components given by the following local expression

$$X_a = \frac{\partial}{\partial \sigma^a} + B^\mu_a \frac{\partial}{\partial x^\mu} + H^\nu_{ab} \frac{\partial}{\partial x^\nu_b}, \tag{8.11}$$

from which it follows that $i(\mathbf{X}_{\mathscr{L}})\mathrm{d}^2\sigma = 1$. Then, $i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = 0$ is given as

$$\begin{aligned} i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = T\Bigg\{ & \sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i \left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right](B^\mu_a - x^\mu_a)\,\mathrm{d}x^\nu_b \\ & + B^\rho_a\left[\frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b\right) - \frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\,G_{\rho\nu}g^{ba}x^\nu_b\right)\right]\mathrm{d}x^\mu \\ & + \sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i \left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]H^\nu_{ab}\mathrm{d}x^\mu \\ & + \left[\frac{1}{2}\sqrt{-\det g}\,g^{ba}x^\alpha_a x^\beta_b\frac{\partial G_{\alpha\beta}}{\partial x^\mu} + \frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b\right)\right]\mathrm{d}x^\mu \\ & - \left[\frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b b\right)x^\mu_a B^\rho_k\right. \\ & + \frac{1}{2}\sqrt{-\det g}\,g^{ba}x^\alpha_a x^\beta_b B^\rho_k\frac{\partial G_{\alpha\beta}}{\partial x^\rho} + \frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det g}\,G_{\rho\nu}g^{ba}x^\nu_b\right)B^\rho_k \\ & + \sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i \left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]\left(x^\mu_a H^\nu_{kb} - B^\mu_a H^\nu_{kb} + B^\mu_k H^\nu_{ab}\right) \\ & \left. - \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b\right)\left(B^\mu_a B^\rho_k - B^\mu_k B^\rho_a\right)\right]\mathrm{d}\sigma^k\Bigg\} = 0\ . \end{aligned} \tag{8.12}$$

Setting differential forms separately equal to zero produces the following equations:

$$\begin{aligned} 0 &= B^\rho_a\left[\frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b\right) - \frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\,G_{\rho\nu}g^{ba}x^\nu_b\right)\right] \\ & \quad + \sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i \left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]H^\nu_{ab} \\ & \quad + \left[\frac{1}{2}\sqrt{-\det g}\,g^{ba}x^\alpha_a x^\beta_b\frac{\partial G_{\alpha\beta}}{\partial x^\mu} + \frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b\right)\right] \end{aligned} \tag{8.13}$$

$$0 = \sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i \left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right](B^\mu_a - x^\mu_a)\ , \tag{8.14}$$

$$\begin{aligned} 0 &= \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b b\right)x^\mu_a B^\rho_k \\ & \quad + \frac{1}{2}\sqrt{-\det g}\,g^{ba}x^\alpha_a x^\beta_b B^\rho_k\frac{\partial G_{\alpha\beta}}{\partial x^\rho} + \frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det g}\,G_{\rho\nu}g^{ba}x^\nu_b\right)B^\rho_k \\ & \quad + \sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i \left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]\left(x^\mu_a H^\nu_{kb} - B^\mu_a H^\nu_{kb} + B^\mu_k H^\nu_{ab}\right) \\ & \quad - \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}x^\nu_b\right)\left(B^\mu_a B^\rho_k - B^\mu_k B^\rho_a\right)\ . \end{aligned} \tag{8.15}$$

Notice that plugging (8.14) into (8.15) implies (8.13), which means that (8.15) is an identity as expected. Equation (8.14) is the SOPDE condition for $\mathbf{X}_{\mathscr{L}}$, i.e. $B^\mu_a = x^\mu_a$. Furthermore, (8.13) becomes the Euler–Lagrange equations when working with $j^1\phi = \left(\sigma^a, x^\mu(\sigma), \dfrac{\partial x^\mu}{\partial \sigma^a}\right)$ that are holonomic sections of $\mathbf{X}_{\mathscr{L}}$ which satisfy

$$D^\mu_a = x^\mu_a = \frac{\partial x^\mu}{\partial \sigma^a}\quad , \quad H^\nu_{ab} = \frac{\partial x^\nu_b}{\partial \sigma^a} = \frac{\partial^2 x^\nu}{\partial \sigma^a \partial \sigma^b}\quad , \quad g_{ab} = G_{\mu\nu}\frac{\partial x^\mu}{\partial \sigma^a}\frac{\partial x^\nu}{\partial \sigma^b}\ . \tag{8.16}$$

Working with such sections that satisfy (8.16) it follows that (8.13) becomes

$$\begin{aligned} 0 = & -\sqrt{-\det g}\,g^{ba}\frac{\partial x^\alpha}{\partial \sigma^a}\frac{\partial x^\beta}{\partial \sigma^b}\frac{\partial G_{\alpha\beta}}{\partial x^\mu} + \frac{\partial}{\partial \sigma^a}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}\frac{\partial x^\nu}{\partial \sigma^b}\right) + \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\,G_{\mu\nu}g^{ba}\frac{\partial x^\nu}{\partial \sigma^b}\right)\frac{\partial x^\rho}{\partial \sigma^a} \\ & + \sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}\frac{\partial x^\alpha}{\partial \sigma^c}\frac{\partial x^\beta}{\partial \sigma^i}\left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]\frac{\partial^2 x^\nu}{\partial \sigma^a \partial \sigma^b}\ , \end{aligned} \tag{8.17}$$

which are the Euler–Lagrange equations for this field theory.

8.2 De Donder–Weyl Hamiltonian Formalism

The De Donder–Weyl Hamiltonian formalism is developed starting from the Legendre map $\mathscr{F}\mathscr{L}: J^1\pi \to J^1\pi^*$ which satisfies

$$\mathscr{F}\mathscr{L}^*\sigma^a = \sigma^a \quad , \quad \mathscr{F}\mathscr{L}^*x^\mu = x^\mu \quad , \quad \mathscr{F}\mathscr{L}^*p^a_\mu = -T\sqrt{-\mathrm{det}g}\; G_{\mu\nu}g^{ba}x^\nu_b \; . \tag{8.18}$$

This Legendre map is invertible as expected from the regularity of the generalized Hessian (8.4). The De Donder–Weyl Hamiltonian is defined as

$$\mathscr{H}(\sigma^a, x^\mu, p^a_\mu) \equiv p^a_\mu(\mathscr{F}\mathscr{L}^{-1})^*x^\mu_a - (\mathscr{F}\mathscr{L}^{-1})^*\mathscr{L} \in C^\infty(J^1\pi^*) \; . \tag{8.19}$$

This Hamiltonian can be expressed in terms of the variables on $J^1\pi^*$ as [100]

$$\Pi^{ab} \equiv G^{\mu\nu}p^a_\mu p^b_\nu \;\Rightarrow\; \mathscr{F}\mathscr{L}^*\Pi^{ab} = -T^2\mathrm{det}g\; g^{ba} \iff \mathscr{F}\mathscr{L}^*\mathrm{det}\Pi = (-T^2\mathrm{det}g)^2\mathrm{det}(g^{-1}) = T^4\mathrm{det}g \; , \tag{8.20}$$

from which it follows that

$$(\mathscr{F}\mathscr{L}^{-1})^*x^\nu_b = -\frac{1}{T}\sqrt{-\mathrm{det}\Pi}\; G^{\mu\nu}\Pi_{ab}p^a_\mu \implies \mathscr{H}(\sigma^a, x^\mu, p^a_\mu) = -\frac{1}{T}\sqrt{-\mathrm{det}\Pi} \; , \tag{8.21}$$

where

$$\Pi_{ab} = \frac{1}{\mathrm{det}\Pi}\epsilon_{cb}\epsilon_{da}\Pi^{cd} \; . \tag{8.22}$$

The Hamilton–Cartan 2–form and the multisymplectic Hamilton–Cartan 3-form on $J^1\pi^*$ are

$$\Theta_{\mathscr{H}} = p^a_\mu \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a - \mathscr{H} \wedge \mathrm{d}^2\sigma = p^a_\mu \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a + \frac{1}{T}\sqrt{-\mathrm{det}\Pi}\; \mathrm{d}^2\sigma \; , \tag{8.23}$$

$$\Omega_{\mathscr{H}} = -\mathrm{d}p^a_\mu \wedge \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a - \frac{\sqrt{-\mathrm{det}\Pi}}{T}\Pi_{ba}\left(\frac{1}{2}\partial_\mu G^{\rho\sigma}p^a_\rho p^b_\sigma \mathrm{d}x^\mu + G^{\mu\nu}p^b_\nu \mathrm{d}p^a_\mu\right) \wedge \mathrm{d}^2\sigma \; . \tag{8.24}$$

The field equations, $i(\mathbf{X}_{\mathscr{H}})\Omega_{\mathscr{H}} = 0$, can be computed using a representative of a class of π^1-transverse locally decomposable multivector fields $\mathbf{X}_{\mathscr{H}} = X_0 \wedge X_1 \in \mathfrak{X}^2(J^1\pi^*)$, with components given by the following local expression

$$X_a = \frac{\partial}{\partial\sigma^a} + D^\mu_a\frac{\partial}{\partial x^\mu} + H^c_{a\mu}\frac{\partial}{\partial p^c_\mu} \; . \tag{8.25}$$

Then,

$$\begin{aligned} 0 = i(\mathbf{X}_{\mathscr{H}})\Omega_{\mathscr{H}} = &-\left(H^a_{a\mu} - \frac{\sqrt{-\mathrm{det}\Pi}}{2T}\Pi_{ba}\frac{\partial G^{\rho\sigma}}{\partial x^\mu}p^a_\rho p^b_\sigma\right)\mathrm{d}x^\mu + \left(D^\mu_a + \frac{\sqrt{-\mathrm{det}\Pi}}{T}\Pi_{ba}G^{\mu\nu}p^b_\nu\right)\mathrm{d}p^a_\mu \\ &-\left[D^\mu_a H^a_{k\mu} - D^\mu_k H^a_{a\mu} + \frac{\sqrt{-\mathrm{det}\Pi}}{T}\Pi_{ba}\left(\frac{1}{2}D^\mu_k\frac{\partial G^{\rho\sigma}}{\partial x^\mu}p^a_\rho p^b_\sigma + H^a_{k\mu}G^{\mu\nu}p^b_\nu\right)\right]\mathrm{d}\sigma^k \; . \end{aligned} \tag{8.26}$$

Setting differential forms separately equal to zero produces the following field equations:

$$0 = H^a_{a\mu} - \frac{\sqrt{-\mathrm{det}\Pi}}{2T}\Pi_{ba}\partial_\mu G^{\rho\sigma}p^a_\rho p^b_\sigma = 0, \tag{8.27a}$$

$$0 = D^\mu_a + \frac{\sqrt{-\mathrm{det}\Pi}}{T}\Pi_{ba}G^{\mu\nu}p^b_\nu, \tag{8.27b}$$

$$0 = D^\mu_a H^a_{k\mu} - D^\mu_k H^a_{a\mu} + \frac{\sqrt{-\mathrm{det}\Pi}}{T}\Pi_{ba}\left(\frac{1}{2}D^\mu_k\partial_\mu G^{\rho\sigma}p^a_\rho p^b_\sigma + H^a_{k\mu}G^{\mu\nu}p^b_\nu\right) . \tag{8.27c}$$

Note that plugging (8.27b) into (8.27c) implies (8.27a), which means that (8.27c) is an identity as expected. Furthermore, the variational problem on $J^1\pi^*$ is solved by integral sections $\psi(\sigma) = (\sigma^a, x^\mu(\sigma), p^a_\mu(\sigma))$ of $\mathbf{X}_{\mathscr{H}}$ for which $D^\mu_a = \dfrac{\partial x^\mu}{\partial \sigma^a}$, $H^c_{a\mu} = \dfrac{\partial p^c_\mu}{\partial \sigma^a}$, and so the field equations become

$$0 = \partial_a p^a_\mu - \frac{\sqrt{-\det\Pi}}{2T}\Pi_{ba}\partial_\mu G^{\rho\sigma} p^a_\rho p^b_\sigma \ , \tag{8.28}$$

$$0 = \partial_a x^\mu + \frac{\sqrt{-\det\Pi}}{T}\Pi_{ba} G^{\mu\nu} p^b_\nu \ , \tag{8.29}$$

which are the corresponding Hamilton–De Donder–Weyl equations for the bosonic string.

8.3 Symmetries

Worldsheet diffeomorphisms:

Consider worldsheet diffeomorphisms produced by

$$\tilde{\sigma}^a = \sigma^a + \xi^a \ \Rightarrow \ \frac{\partial \tilde{\sigma}^a}{\partial \sigma^b} = \delta^a_b + \frac{\partial \xi^a}{\partial \sigma^b} \ . \tag{8.30}$$

It follows that

$$\det\tilde{g} = \left[\det\left(\frac{\partial \tilde{\sigma}^a}{\partial \sigma^b}\right)\right]^{-2} \det g \quad , \quad \mathrm{d}^2\tilde{\sigma} = \det\left(\frac{\partial \tilde{\sigma}^a}{\partial \sigma^b}\right)\mathrm{d}^2\sigma \ , \tag{8.31}$$

and hence

$$\mathbf{L} = -T\sqrt{-\det\tilde{g}}\ \mathrm{d}^2\tilde{\sigma} = -T\sqrt{-\det g}\ \mathrm{d}^2\sigma \ . \tag{8.32}$$

The vector field $\xi \in \mathfrak{X}(\Sigma)$ which generates the worldsheet diffeomorphisms (9.22) is given by

$$\xi = -\xi^a \frac{\partial}{\partial \sigma^a} \in \mathfrak{X}(\Sigma) \ . \tag{8.33}$$

The lift of the vector field $\xi \in \mathfrak{X}(\Sigma)$ to $E = \Sigma \times M$, obtained from (4.8), is

$$\xi_E = -\xi^a \frac{\partial}{\partial \sigma^a} \in \mathfrak{X}(E) \ , \tag{8.34}$$

and the canonical lift of ξ_E to $J^1\pi$ gives

$$X_\xi = -\xi^a \frac{\partial}{\partial \sigma^a} + x^\mu_b \frac{\partial \xi^b}{\partial \sigma^a}\frac{\partial}{\partial x^\mu_a} \ \in \mathfrak{X}(J^1\pi) \ . \tag{8.35}$$

The resulting field variation given by the Lie derivative (4.8) of the local sections $\phi : \Sigma \to E$ by the vector field ξ is

$$\delta X^\mu(\sigma) = -\xi^a \frac{\partial X^\mu}{\partial \sigma^a} \ . \tag{8.36}$$

It follows that

$$\mathrm{L}(X_\xi)\mathbf{L} = 0 \ \Leftrightarrow \ \mathrm{L}(X_\xi)\Theta_{\mathscr{L}} = 0 \ , \tag{8.37}$$

so the covariant momentum map is given as

$$J_{\mathscr{L}}(X_\xi) = -i(X_\xi)\Theta_{\mathscr{L}} = -T\sqrt{-\det g}\left(\epsilon_{ac}\xi^c G_{\mu\nu} g^{ba} x^\nu_b \mathrm{d}x^\mu + \xi^a \mathrm{d}^1\sigma_a\right) \ \in \varOmega^1(J^1\pi) \ . \tag{8.38}$$

In the Hamiltonian formalism, the vector field $Y_\xi = \mathscr{FL}_* X_\xi \in \mathfrak{X}(J^1\pi^*)$ which takes the form

$$Y_\xi = -\xi^a \frac{\partial}{\partial \sigma^a} - \frac{1}{T}\sqrt{-\det\Pi}\ G^{\mu\nu}\Pi_{bc} p^b_\nu \frac{\partial \xi^c}{\partial \sigma_a}\frac{\partial}{\partial p^a_\mu} \ , \tag{8.39}$$

generates the worldsheet diffeomorphisms on the multimomentum phase space $J^1\pi^*$ and is used to construct the covariant momentum map

$$J_{\mathscr{H}}(Y_\xi) = -i(Y_\xi)\Theta_{\mathscr{H}} = \epsilon_{ac}\xi^c p^a_\mu \mathrm{d}x^\mu - \frac{1}{T}\sqrt{-\mathrm{det}\Pi}\ \xi^a \mathrm{d}^1\sigma_a \in \Omega^1(J^1\pi^*)\ . \tag{8.40}$$

Spacetime isometries:

Spacetime symmetries, which arise from performing transformations on the coordinates of M, are gauge symmetries in string theory since they are generated by vector fields on the configuration manifold E which are vertical with respect to the projection onto the base space Σ.

Consider infinitesimal spacetime diffeomorphisms $x^\mu \to x^\mu + \zeta^\mu(x)$ generated by the vector field

$$\zeta = -\zeta^\mu \frac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)\ , \tag{8.41}$$

which is a gauge transformation as the corresponding vector field $\zeta_E \in \mathfrak{X}(E)$ is π-vertical:

$$\zeta_E = -\zeta^\mu \frac{\partial}{\partial x^\mu}\ . \tag{8.42}$$

The canonical lift to $J^1\pi$, given by the jet prolongation of ζ_E, is

$$X_\zeta = -\zeta^\mu \frac{\partial}{\partial x^\mu} - x^\nu_a \partial_\nu \zeta^\mu \frac{\partial}{\partial x^\mu_a} \in \mathfrak{X}(J^1\pi)\ . \tag{8.43}$$

The Lie derivative of $\mathbf{L} = -T\sqrt{-\mathrm{det}g}\ \mathrm{d}^2\sigma$ with respect to X_ζ is

$$\mathrm{L}(X_\zeta)\mathbf{L} = \frac{T}{2}\sqrt{-\mathrm{det}g}\ g^{ba} x^\nu_b x^\mu_a\, \mathrm{L}(\zeta) G_{\mu\nu} \mathrm{d}^2\sigma\ , \tag{8.44}$$

where $\mathrm{L}(\zeta)G_{\mu\nu}$ is the Lie derivative of $G_{\mu\nu}$ on M with respect to $\zeta = -\zeta^\mu \frac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$. Then, when ζ is a Killing vector field,

$$\mathrm{L}(X_\zeta)\mathbf{L} = 0\ \Rightarrow\ \mathrm{L}(X_\zeta)\Theta_{\mathscr{L}} = 0\ , \tag{8.45}$$

and it therefore follows that the covariant momentum map is given by

$$J_{\mathscr{L}}(X_\zeta) = -i(X_\zeta)\Theta_{\mathscr{L}} = T\sqrt{-\mathrm{det}g}\ G_{\mu\nu} g^{ba} x^\nu_b \zeta^\mu \mathrm{d}^1\sigma_a \in \Omega^1(J^1\pi)\ . \tag{8.46}$$

On the Hamiltonian side,

$$Y_\zeta = \mathscr{F}\mathscr{L}_* X_\zeta = -\zeta^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{T}\sqrt{-\mathrm{det}\Pi}\, G^{\mu\nu} \Pi_{ba}\, p^b_\mu \frac{\partial \zeta_\mu}{\partial x^\nu} \frac{\partial}{\partial p^a_\mu} \in \mathfrak{X}(J^1\pi^*)\ , \tag{8.47}$$

so the covariant momentum map is given by

$$J_{\mathscr{H}}(Y_\zeta) = -i(Y_\zeta)\Theta_{\mathscr{H}} = -p^a_\mu\, \zeta^\mu\, \mathrm{d}^1\sigma_a \in \Omega^1(J^1\pi^*)\ . \tag{8.48}$$

8.4 Generalization to p-Branes (p >1)

The generalization to p-branes ($m = p+1$) is straightforward to perform by using the general Lagrangian density (8.1) for which the corresponding Lagrangian energy $E_{\mathscr{L}} \in C^\infty(J^1\pi)$ is

$$E_{\mathscr{L}} \equiv \frac{\partial \mathscr{L}}{\partial x^\mu_a} x^\mu_a - \mathscr{L} = -T_p\sqrt{-\mathrm{det}g}\ (g^{ba}g_{ab} - 1) = -pT_p\sqrt{-\mathrm{det}g}\ , \tag{8.49}$$

where g_{ab} is now an $m \times m$ matrix, so $g^{ba}g_{ab} = m = p+1$. Then

$$\Theta_{\mathscr{L}} = -T_p\sqrt{-\det g}\ \left[G_{\mu\nu}g^{ba}x^\nu_b \mathrm{d}x^\mu \wedge \mathrm{d}^{m-1}\sigma_a - p\mathrm{d}^m\sigma\right], \tag{8.50}$$

$$\begin{aligned}\Omega_{\mathscr{L}} = T_p\Bigg\{ &\sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i\left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]\mathrm{d}x^b_\nu \wedge \mathrm{d}x^\mu \wedge \mathrm{d}^{m-1}\sigma_a \\ &+ \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\, G_{\mu\nu}g^{ba}x^\nu_b\right)\mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \wedge \mathrm{d}^{m-1}\sigma_a \\ &- \sqrt{-\det g}\ \left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i\left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]x^\mu_a \mathrm{d}x^\nu_b \wedge \mathrm{d}^m\sigma \\ &- \left[\frac{\partial}{\partial\sigma^a}\left(\sqrt{-\det g}\, G_{\mu\nu}g^{ba}x^\nu_b\right) + \frac{1}{2}\sqrt{-\det g}\, g^{ba}x^\alpha_a x^\beta_b \partial_\mu G_{\alpha\beta}\right]\mathrm{d}x^\mu \wedge \mathrm{d}^m\sigma\Bigg\} .\end{aligned} \tag{8.51}$$

Proceeding similarly as above, take an m-multivector field $\mathbf{X}_{\mathscr{L}} = X_0 \wedge X_1 \wedge \cdots \wedge X_{m-1}$, with components

$$X_a = \frac{\partial}{\partial\sigma^a} + B^\mu_a\frac{\partial}{\partial x^\mu} + H^\nu_{ab}\frac{\partial}{\partial x^\nu_b} , \tag{8.52}$$

to obtain the field equations, one of which is an identity, another gives the SOPDE condition $B^\mu_a = x^\mu_a$, and the third equation upon plugging in the SOPDE condition is

$$\begin{aligned}0 = \frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\right) - \frac{\partial}{\partial\sigma^a}\left(\sqrt{-\det g}\ G_{\mu\nu}g^{ba}x^\nu_b\right) - \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\ G_{\mu\nu}g^{ba}x^\nu_b\right)x^\rho_a \\ -\sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}x^\alpha_c x^\beta_i\left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]H^\nu_{ab} ,\end{aligned} \tag{8.53}$$

which becomes the Euler–Lagrange equations when working on integral sections of $\mathbf{X}_{\mathscr{L}}$:

$$\begin{aligned}0 = \frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g}\right) - \frac{\partial}{\partial\sigma^a}\left(\sqrt{-\det g}\ G_{\mu\nu}g^{ba}\frac{\partial x^\nu}{\partial\sigma^b}\right) - \frac{\partial}{\partial x^\rho}\left(\sqrt{-\det g}\ G_{\mu\nu}g^{ba}\frac{\partial x^\nu}{\partial\sigma^b}\right)\frac{\partial x^\rho}{\partial\sigma^a} \\ -\sqrt{-\det g}\left[G_{\mu\nu}g^{ba} - G_{\mu\alpha}G_{\beta\nu}\frac{\partial x^\alpha}{\partial\sigma^c}\frac{\partial x^\beta}{\partial\sigma^i}\left(g^{ba}g^{ci} + g^{cb}g^{ai} - g^{ca}g^{bi}\right)\right]\frac{\partial^2 x^\nu}{\partial\sigma^a\partial\sigma^b} .\end{aligned} \tag{8.54}$$

The Hamiltonian formulation is also carried out as in the string case with the Legendre map $\mathscr{FL}$ which, for p-branes, is

$$\mathscr{FL}^*\sigma^a = \sigma^a \quad , \quad \mathscr{FL}^*x^\mu = x^\mu \quad , \quad \mathscr{FL}^*p^a_\mu = \frac{\partial\mathscr{L}}{\partial x^\mu_a} = -T_p\sqrt{-\det g}\ G_{\mu\nu}g^{ba}x^\nu_b . \tag{8.55}$$

Next, define the matrix

$$\Pi^{ab} = G^{\mu\nu}p^a_\mu p^b_\nu \ \Rightarrow\ \mathscr{FL}^*\Pi^{ab} = -T_p^2\det g\ g^{ba}, \tag{8.56}$$

from which it follows that

$$\left(\mathscr{FL}^{-1}\right)^*\det g = -\frac{1}{T_p^2}\left(\frac{\det\Pi}{-T_p^2}\right)^{1/p} , \tag{8.57}$$

and hence

$$\left(\mathscr{FL}^{-1}\right)^* x^\nu_b = -\sqrt{\left(\frac{\det\Pi}{-T_p^2}\right)^{1/p}}\, G^{\mu\nu}\Pi_{ab}p^a_\mu. \tag{8.58}$$

The inverse matrix Π_{ab} is given by the following general version of the identity (8.22):

$$\Pi_{ab} = -\frac{1}{p!\det\Pi}\epsilon_{aa_2\cdots a_m}\epsilon_{bb_2\cdots b_m}\Pi^{b_2a_2}\cdots\Pi^{b_ma_m} . \tag{8.59}$$

Then the De Donder–Weyl Hamiltonian for the p-brane theory is

$$\mathscr{H}(\sigma^a, x^\mu, p^a_\mu) = -\sqrt{\left(\frac{\det\Pi}{-T_p^2}\right)^{1/p}} \in C^\infty(J^1\pi^*) \ . \tag{8.60}$$

The De Donder–Weyl field equations are:

$$\frac{\partial X^\mu}{\partial\sigma^a} = \left(\frac{\partial\mathscr{H}}{\partial p^a_\mu}\right)\circ\psi \quad , \quad \frac{\partial p^a_\mu}{\partial\sigma^a} = -\left(\frac{\partial\mathscr{H}}{\partial x^\mu}\right)\circ\psi \ , \tag{8.61}$$

however, the Hamiltonian is computationally more tedious to deal with than in the case of the string.

The multisymplectic symmetries are now worldvolume diffeomorphisms and spacetime isometires. The worldvolume diffeomorphsims are generated on $J^1\pi$ by the vector field

$$X_\xi = -\xi^a\frac{\partial}{\partial\sigma^a} + x^\mu_b\frac{\partial\xi^b}{\partial\sigma^a}\frac{\partial}{\partial x^\mu_a} \ \in \mathfrak{X}(J^1\pi) \ , \tag{8.62}$$

and the corresponding momentum map $J_{\mathscr{L}}(X_\xi) = -i(X_\xi)\Theta_{\mathscr{L}} \in \Omega^{m-1}(J^1\pi)$ is

$$J_{\mathscr{L}}(X_\xi) = T_p\sqrt{-\det g}\ \left[G_{\mu\nu}g^{ba}x^\nu_b\xi^c \mathrm{d}x^\mu\wedge\mathrm{d}^{m-2}\sigma_{ac} + p\xi^a\mathrm{d}^{m-1}\sigma_a\right] \ . \tag{8.63}$$

Furthermore, as in the string theory case, the spacetime isometries which produce symmetries of the multisymplectic form $\Omega_{\mathscr{L}}$ are generated on $J^1\pi$ by the following vector field:

$$X_\zeta = -\zeta^\mu\frac{\partial}{\partial x^\mu} - x^\nu_a\partial_\nu\zeta^\mu\frac{\partial}{\partial x^\mu_a} \in \mathfrak{X}(J^1\pi) \ . \tag{8.64}$$

The corresponding momentum map $J_{\mathscr{L}} = -i(X_\zeta)\Theta_{\mathscr{L}} \in \Omega^{m-1}(J^1\pi)$ is given as,

$$J_{\mathscr{L}}(X_\zeta) = -T_p\sqrt{-\det g}\ G_{\mu\nu}g^{ba}x^\nu_b\zeta^\mu\mathrm{d}^{m-1}\sigma_a \ . \tag{8.65}$$

8.5 The Nonrelativistic Particle Limit

The *nonrelativistic particle limit* of the bosonic string action (discussed in e.g. [14]) arises as a geometric limit of the multisymplectic phase spaces in the multisymplectic framework. The line element in the target space (spacetime) M is

$$\mathrm{d}s = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\left(\mathrm{d}x^0\right)^2 + \delta_{IJ}\mathrm{d}x^I\mathrm{d}x^J \ . \tag{8.66}$$

The nonrelativistic limit generally considered on M involves taking the temporal coordinate on M to go to infinity: $x^0 \to \omega x^0$, $\omega\to\infty$. This is the nonrelativistic particle limit of the Nambu-Goto string action [14].

The equivalent nonrelativistic limit on the multisymplectic phase space $J^1\pi$ is performed by taking

$$x^0 \to \omega x^0 \quad , \quad x^0_a \to \omega x^0_a \quad , \quad \omega\to\infty \quad . \tag{8.67}$$

Upon doing so the Nambu-Goto Lagrangian

$$\mathscr{L} = -T\sqrt{-\det\left(\eta_{\mu\nu}x^\mu_a x^\nu_b\right)} \ , \tag{8.68}$$

becomes

$$\mathscr{L} = -T\omega\sqrt{\epsilon^{ab}\epsilon^{cd}x_a^0 x_c^0 \delta_{IJ} x_b^I x_d^J} + O\left(\frac{1}{\omega}\right) , \tag{8.69}$$

and the nonrelativistic string Lagrangian is obtained by redefining the string tension as $T \equiv T\omega$ and setting $O\left(\frac{1}{\omega}\right) = 0$ so that

$$\mathscr{L} = -T\sqrt{\epsilon^{ab}\epsilon^{cd}x_a^0 x_c^0 \delta_{IJ} x_b^I x_d^J} \equiv -T\sqrt{\alpha} . \tag{8.70}$$

The corresponding Poincaré-Cartan form $\Theta_{\mathscr{L}} \in \Omega^2(J^1\pi)$ is

$$\Theta_{\mathscr{L}} = -\frac{T}{\sqrt{\alpha}}\left[\epsilon^{bc}\epsilon^{ad}\delta_{IJ}\left(x_b^0 x_d^I x_c^J \mathrm{d}x^0 + x_c^0 x_d^0 x_b^I \mathrm{d}x^J\right) \wedge \mathrm{d}^1\sigma_a - \alpha \mathrm{d}^2\sigma\right] . \tag{8.71}$$

This result can also be obtained by taking the limit (8.67) of the Poincaré-Cartan form corresponding to the relativistic Lagrangian (8.68) and then redefining $T \equiv T\omega$ and setting $O\left(\frac{1}{\omega}\right) = 0$ as before.

9 The Polyakov String Action

For a textbook treatment of the Polyakov action see, for instance, [17, 74].

In the this section, the premultisymplectic treatment of the relativistic string on a Minkowski background spacetime is carried out starting from the famous *Polyakov action*. The premultisymplectic constraint analysis of this theory has been carried out previously [41], using Ehresmann connections to write the field equations intrinsically. Here, we carry out the premultisymplectic constraint analysis using multivector fields.

9.1 Lagrangian Formalism

Consider the string worldsheet Σ with coordinates σ^a $(a = 0, 1)$ and auxiliary metric g_{ab}; partial worldsheet derivatives are denoted as $\partial_a = \partial/\partial\sigma^a$. The embedding of the string worldsheet in spacetime M (which has coordinates x^μ) is specified by local maps $\Sigma \to M$ which are denoted as $x^\mu = x^\mu(\sigma)$. The variational fields considered here are the embedding maps $x^\mu(\sigma)$ and the components of the auxiliary inverse metric $g^{ab}(\sigma)$. It follows that the configuration bundle is specified as $\pi : E \to \Sigma : (\sigma^a, x^\mu, g^{ab}) \mapsto \sigma^a$, and its first-order jet bundle $J^1\pi$ has local coordinates $(\sigma^a, x^\mu, g^{ab}, x_a^\mu, g_c^{ab})$.

The *Polyakov Lagrangian* function is written as

$$\mathscr{L} = -\frac{T}{2}\sqrt{-g}\,\eta_{\mu\nu}g^{ab}x_a^\mu x_b^\nu \in C^\infty(J^1\pi) . \tag{9.1}$$

The resulting Lagrangian energy is

$$E_{\mathscr{L}} = -\frac{T}{2}\sqrt{-g}\,\eta_{\mu\nu}g^{ab}x_a^\mu x_b^\nu \in C^\infty(J^1\pi) , \tag{9.2}$$

and the Poincaré–Cartan forms are:

$$\Theta_{\mathscr{L}} = -\frac{T}{2}\sqrt{-g}\,\eta_{\mu\nu}g^{ab}x_b^\nu \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a + \frac{T}{2}\sqrt{-g}\,\eta_{\mu\nu}g^{ab}x_a^\mu x_b^\nu \mathrm{d}^2\sigma , \tag{9.3}$$

$$\begin{aligned}\Omega_{\mathscr{L}} = {}& T\sqrt{-g}\,\eta_{\mu\nu}\left(g^{ab}\mathrm{d}x_b^\nu \wedge \mathrm{d}x^\mu + x_b^\nu \mathrm{d}g^{ab} \wedge \mathrm{d}x^\mu - \frac{1}{2}g^{ab}x_b^\nu g_{dc}\mathrm{d}g^{cd} \wedge \mathrm{d}x^\mu\right) \wedge \mathrm{d}^1\sigma_a \\ & - \frac{T}{2}\sqrt{-g}\,\eta_{\mu\nu}\left(x_a^\mu x_b^\nu \mathrm{d}g^{ab} + 2g^{ab}x_a^\mu \mathrm{d}x_b^\nu - \frac{1}{2}g^{ab}x_a^\mu x_b^\nu g_{dc}\mathrm{d}g^{cd}\right) \wedge \mathrm{d}^2\sigma .\end{aligned} \tag{9.4}$$

The multi-Hessian for the Polyakov Lagrangian (9.1) is given by,

$$\frac{\partial^2 \mathscr{L}}{\partial x^\mu_a \partial x^\nu_b} = -T\sqrt{-g}\,\eta_{\mu\nu} g^{ab} \quad , \quad \frac{\partial^2 \mathscr{L}}{\partial g^{ab}_i \partial g^{cd}_j} = \frac{\partial^2 \mathscr{L}}{\partial x^\mu_a \partial g^{cd}_j} = 0 \ , \tag{9.5}$$

which is clearly singular.

Now, we use a 2-multivector field,

$$\mathbf{X}_{\mathscr{L}} = \bigwedge_{a=0}^{1} X_a = \bigwedge_{a=0}^{1} \left(\frac{\partial}{\partial \sigma^a} + A^\mu_a \frac{\partial}{\partial x^\mu} + G^{bc}_a \frac{\partial}{\partial g^{bc}} + D^\nu_{ab} \frac{\partial}{\partial x^\nu_b} + K^{bc}_{ad} \frac{\partial}{\partial g^{bc}_d} \right) \in \mathfrak{X}^2(J^1\pi) \ , \tag{9.6}$$

to calculate $i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = 0$, and obtain

$$0 = \sqrt{-g}\,\eta_{\mu\nu} g^{ab}(x^\mu_a - A^\mu_a) \ , \tag{9.7}$$

$$0 = \sqrt{-g}\,\eta_{\mu\nu} \left(x^\mu_a x^\nu_b - 2x^\nu_b A^\mu_a - g^{cd} g_{ab} x^\nu_d A^\mu_c - \frac{1}{2} g^{cd} g_{ba} x^\mu_c x^\nu_d \right) \ , \tag{9.8}$$

$$0 = \sqrt{-g} \left(\eta_{\mu\nu} g^{ab} D^\nu_{ab} + \eta_{\mu\nu} x^\nu_b G^{ab}_a - \frac{1}{2} \eta_{\mu\nu} g^{ab} g_{cd} x^\nu_b G^{cd}_a \right) \ . \tag{9.9}$$

Equations (9.7) lead to one of the SOPDE conditions, $A^\mu_a = x^\mu_a$, and they need not be imposed. However, the remaining SOPDE condition, $G^{bc}_a = g^{bc}_a$, must be imposed *ad hoc*. It is evident that imposing this last SOPDE condition only affects equation (9.9). Upon combining equations (9.7) and (9.8) and imposing the aforementioned SOPDE condition, $G^{bc}_a = g^{bc}_a$, the equations become:

$$0 = \sqrt{-g}\,\eta_{\mu\nu} \left(x^\mu_a x^\nu_b - \frac{1}{2} g^{cd} g_{ba} x^\mu_c x^\nu_d \right) \ , \tag{9.10}$$

$$0 = \sqrt{-g} \left(\eta_{\mu\nu} g^{ab} D^\nu_{ab} + \eta_{\mu\nu} x^\nu_b g^{ab}_a - \frac{1}{2} \eta_{\mu\nu} g^{ab} g_{cd} x^\nu_b g^{cd}_a \right) \ . \tag{9.11}$$

No SOPDE constraints arise from (9.11). However, equation (9.10), which arises from combining equations (9.7) and (9.8), is a compatibility constraint. Similarly to Einstein–Cartan gravity, no stability (tangency) constraints arise from demanding that the Lie derivative of the constraint function (9.10) with respect to the vector fields X_a of the multivector field $\mathbf{X}_{\mathscr{L}}$ vanish on the submanifold defined by (9.10). It follows that the compatibility constraint submanifold is the final constraint submanifold $S_f \hookrightarrow J^1\pi^*$ of this Lagrangian system. The final field equations on integral sections of $\mathbf{X}_{\mathscr{L}}$ are written as,

$$0 = \sqrt{-g}\,\eta_{\mu\nu} \left(\partial_a x^\mu \partial_b x^\nu - \frac{1}{2} g^{cd} g_{ba} \partial_c x^\mu \partial_d x^\nu \right) \ , \tag{9.12}$$

$$0 = \sqrt{-g} \left(\eta_{\mu\nu} g^{ab} \partial_a \partial_b x^\nu + \eta_{\mu\nu} \partial_b x^\nu \partial_a g^{ab} - \frac{1}{2} \eta_{\mu\nu} g^{ab} g_{cd} \partial_b x^\nu \partial_a g^{cd} \right) \ . \tag{9.13}$$

Equation (9.13) is the Euler–Lagrange equation for x^μ while (9.12) is the Euler–Lagrange equation for g^{ab} which is well-known in the physics literature as the *Virasoro constraint* (see e.g. [17, 74]).

It should be noted that the *Hilbert stress–energy tensor* on $J^1\pi$ is given as

$$T_{ab} = -\frac{2}{T} \frac{1}{\sqrt{-g}} \frac{\partial \mathscr{L}}{\partial g^{ab}} = \eta_{\mu\nu} \left(x^\mu_a x^\nu_b - \frac{1}{2} g^{cd} g_{ba} x^\mu_c x^\nu_d \right) \ . \tag{9.14}$$

The vanishing of this stress–energy tensor gives the compatibility constraint (9.10). Notice that T_{ab} is traceless everywhere on $J^1\pi$ and not just on the compatibility constraint as expected. In conclusion, composing the compatibility constraint (9.10) with sections $j^1\phi : \Sigma \to J^1\pi$ gives the well-known Virasoro constraint and composing the *Hilbert stress–energy tensor* (9.14) with sections $j^1\phi$ gives the stress-energy tensor that is well-known in the string theory literature [17, 74]; this is in accordance with the reoccurring theme that we work on the premultisymplectic fiber bundle $J^1\pi$ while in the physics literature, field theory is carried out on the space of sections of $J^1\pi$ over the base space (which in this case is the string worldsheet Σ).

9.2 De Donder–Weyl Hamiltonian Formalism

Since the Polyakov Lagrangian is singular, the Hamilton–De Donder–Weyl formalism takes place on a primary constraint submanifold $P_\circ \hookrightarrow J^1\pi^*$. The local coordinates on $\mathcal{M}\pi$ and $J^1\pi^*$ are $(\sigma^a, x^\mu, p^a_\mu, \pi^c_{ab}, p)$ and $(\sigma^a, x^\mu, p^a_\mu, \pi^c_{ab})$ respectively. The Legendre map gives,

$$\mathscr{F}\mathscr{L}^* p^a_\mu = \frac{\partial \mathscr{L}}{\partial x^\mu_a} = -T\sqrt{-g}\,\eta_{\mu\nu} g^{ab} x^\nu_b \quad , \quad \mathscr{F}\mathscr{L}^* \pi^c_{ab} = \frac{\partial \mathscr{L}}{\partial g^{ab}_c} = 0 \ , \tag{9.15}$$

for the multimomenta. The first equation above can be inverted to solve for the x^ν_b in terms of the multimomenta and is therefore not a constraint. Furthermore, the second equation above specifies the primary constraint submanifold $P_\circ \hookrightarrow J^1\pi^*$.

The De Donder–Weyl Hamiltonian $\mathscr{H}_\circ \in C^\infty(P_\circ)$ which satisfies $E_{\mathscr{L}} = \mathscr{F}\mathscr{L}^*_\circ \mathscr{H}_\circ$ is given as,

$$\mathscr{H}_\circ = -\frac{1}{2T\sqrt{-g}}\eta^{\mu\nu} g_{ab} p^a_\mu p^b_\nu \ , \tag{9.16}$$

and the corresponding Hamiltonian section is given as $h_\circ = (\sigma^a, x^\mu, p^a_\mu, \pi^c_{ab}, p = -\mathscr{H}_\circ)$.

The corresponding Hamilton–Cartan forms are,

$$\Theta^\circ_{\mathscr{H}} = p^a_\mu \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a + \frac{1}{2T\sqrt{-g}}\eta^{\mu\nu} g_{ab} p^a_\mu p^b_\nu \mathrm{d}^2\sigma \ , \tag{9.17}$$

$$\Omega^\circ_{\mathscr{H}} = -\mathrm{d}p^a_\mu \wedge \mathrm{d}x^\mu \wedge \mathrm{d}^1\sigma_a - \frac{1}{2T\sqrt{-g}}\eta^{\mu\nu}\left[2g_{ab}p^b_\nu \mathrm{d}p^a_\mu + \left(p^a_\mu p^b_\nu - \frac{1}{2}p^d_\mu p^c_\nu g_{cd} g^{ba}\right)\mathrm{d}g_{ab}\right] \wedge \mathrm{d}^2\sigma \ . \tag{9.18}$$

Now use a 2-multivector field,

$$\mathbf{X}^\circ_{\mathscr{H}} = \bigwedge_{a=0}^{1} X_a = \bigwedge_{a=0}^{1}\left(\frac{\partial}{\partial\sigma^a} + A^\mu_a \frac{\partial}{\partial x^\mu} + G^{bc}_a \frac{\partial}{\partial g^{bc}} + D^b_{a\nu}\frac{\partial}{\partial p^b_\nu}\right) \in \mathfrak{X}^2(P_\circ) \ , \tag{9.19}$$

to calculate $i(\mathbf{X}^\circ_{\mathscr{H}})\Omega^\circ_{\mathscr{H}} = 0$; then,

$$0 = A^\mu_a + \frac{1}{T\sqrt{-g}}\eta^{\mu\nu} g_{ab} p^b_\nu \quad , \quad 0 = D^a_{a\mu} \quad , \quad 0 = \frac{1}{2T\sqrt{-g}}\eta^{\mu\nu}\left(p^a_\mu p^b_\nu - \frac{1}{2}g_{cd} g^{ba} p^d_\mu p^c_\nu\right) \ . \tag{9.20}$$

The third equation above is the Hamiltonian compatibility constraint which is the $\mathscr{F}\mathscr{L}$ projection of the Lagrangian compatibility constraint (9.10). As in the Lagrangian formalism, no (tangency) constraints arise since the tangency conditions (i.e., the Lie derivatives of this compatibility constraint function with respect to the vector fields X_a) provide equations for the coefficients $D^b_{a\nu}$. The third equation thereby defines the final constraint submanifold $P_f \hookrightarrow P_\circ \hookrightarrow J^1\pi^*$. Working on integral sections of the multivector field $\mathbf{X}^\circ_{\mathscr{H}}$ gives the Hamilton–De Donder–Weyl equations:

$$0 = \partial_a x^\mu + \frac{1}{T\sqrt{-g}}\eta^{\mu\nu} g_{ab} p^b_\nu \quad , \quad 0 = \partial_a p^a_\mu \quad , \quad 0 = \frac{1}{2T\sqrt{-g}}\eta^{\mu\nu}\left(p^a_\mu p^b_\nu - \frac{1}{2}g_{cd} g^{ba} p^d_\mu p^c_\nu\right) \ . \tag{9.21}$$

The first equation gives the relation for the partial derivatives of the fields and the multimomenta; this equation is the inverse of the equation obtained from the Legendre transform. The second equation above is equivalent to the Euler–Lagrange equation (9.13) in the Lagrangian formalism. And finally, the third equation is the De Donder–Weyl version of the Virasoro constraint (9.12).

9.3 Symmetries

The Polyakov string action exhibits three different symmetries: *worldsheet diffeomorphisms*, *space-time Poincaré transformations*, and *Weyl transformations*. Each of these transformations generate exact Noether symmetries.

Worldsheet diffeomorphisms:

The worldsheet diffeomorphisms are produced by a general change of coordinates on the string worldsheet as,

$$\tilde{\sigma}^a = \sigma^a + \xi^a \implies \partial_b \tilde{\sigma}^a = \delta^a_b + \partial_b \xi^a \ . \tag{9.22}$$

The fields $x^\mu(\sigma)$ and $g^{ab}(\sigma)$ transform under the worldsheet diffeomorphisms as,

$$x^\mu \to x^\mu - \xi^c \partial_c x^\mu \quad , \quad g^{ab} \to g^{ab} - \xi^c \partial_c g^{ab} + g^{ac} \partial_c \xi^b + g^{cb} \partial_c \xi^a \ . \tag{9.23}$$

The worldsheet diffeomorphisms are generated by the vector field $\xi_\Sigma \in \mathfrak{X}(\Sigma)$ given by

$$\xi_\Sigma = -\xi^a \frac{\partial}{\partial \sigma^a} \in \mathfrak{X}(\Sigma) \ , \tag{9.24}$$

whose canonical lift to the configuration bundle E is given by

$$\xi_E = -\xi^a \frac{\partial}{\partial \sigma^a} - \left(g^{ac} \partial_c \xi^b + g^{cb} \partial_c \xi^a \right) \frac{\partial}{\partial g^{ab}} \in \mathfrak{X}(E) \ . \tag{9.25}$$

The canonical lift $X_\xi \in \mathfrak{X}(J^1\pi)$ of ξ_E to $J^1\pi$, given by (4.20), is

$$X_\xi = -\xi^a \frac{\partial}{\partial \sigma^a} - \left(g^{ac} \partial_c \xi^b + g^{cb} \partial_c \xi^a \right) \frac{\partial}{\partial g^{ab}} + x^\mu_b \partial_a \xi^b \frac{\partial}{\partial x^\mu_a} - \left(g^{ac}_d \partial_c \xi^b + g^{cb}_d \partial_c \xi^a \right) \frac{\partial}{\partial g^{ab}_d} \in \mathfrak{X}(J^1\pi) \ , \tag{9.26}$$

which is tangent to the final Lagrangian constraint submanifold S_f, as a simple calculation shows. The canonical lifts to the multimomentum bundles are given by (4.36) and (4.40), respectively, and they are

$$\begin{aligned} Z_\xi = & -\xi^a \frac{\partial}{\partial \sigma^a} - \left(g^{ac} \partial_c \xi^b + g^{cb} \partial_c \xi^a \right) \frac{\partial}{\partial g^{ab}} + \left(p^a_\mu \partial_b \xi^b - p^b_\mu \partial_b \xi^a \right) \frac{\partial}{\partial p^a_\mu} \\ & + \left(\pi^c_{ab} \partial_d \xi^d - \pi^d_{ab} \partial_d \xi^c + \pi^c_{ad} \partial_b \xi^d + \pi^c_{db} \partial_a \xi^d \right) \frac{\partial}{\partial \pi^c_{ab}} + p \partial_a \xi^a \frac{\partial}{\partial p} \ \in \mathfrak{X}(\mathcal{M}\pi) \ , \end{aligned} \tag{9.27}$$

$$\begin{aligned} Y_\xi = & -\xi^a \frac{\partial}{\partial \sigma^a} - \left(g^{ac} \partial_c \xi^b + g^{cb} \partial_c \xi^a \right) \frac{\partial}{\partial g^{ab}} + \left(p^a_\mu \partial_b \xi^b - p^b_\mu \partial_b \xi^a \right) \frac{\partial}{\partial p^a_\mu} \\ & + \left(\pi^c_{ab} \partial_d \xi^d - \pi^d_{ab} \partial_d \xi^c + \pi^c_{ad} \partial_b \xi^d + \pi^c_{db} \partial_a \xi^d \right) \frac{\partial}{\partial \pi^c_{ab}} \ \in \mathfrak{X}(J^1\pi^*) \ . \end{aligned} \tag{9.28}$$

This vector field Y_ξ is tangent to $P_\circ$, since the constraints defining $P_\circ$ are $\pi^c_{ab} = 0$, and $\mathrm{L}(Y_\xi)\pi^c_{ab} = 0$ on $P_\circ$. Thus, following Definition 4.9, the canonical lift to the Hamiltonian multiphase space $P_\circ \subset J^1\pi^*$ is given as

$$Y^\circ_\xi = -\xi^a \frac{\partial}{\partial \sigma^a} - \left(g^{ac} \partial_c \xi^b + g^{cb} \partial_c \xi^a \right) \frac{\partial}{\partial g^{ab}} + \left(p^a_\mu \partial_b \xi^b - p^b_\mu \partial_b \xi^a \right) \frac{\partial}{\partial p^a_\mu} \in \mathfrak{X}(P_\circ) \ , \tag{9.29}$$

since $(\jmath_\circ)_* Y^\circ_\xi = Y_\xi|_{P_\circ}$. It can be shown that $(\mathscr{F}\mathscr{L}_\circ)_* X_\xi = Y^\circ_\xi$ is satisfied and that Y°_ξ is also tangent to the final Hamiltonian constraint submanifold P_f.

It is straightforward to show that $\mathrm{L}(X_\xi)\Omega_{\mathscr{L}} = 0$; then,

$$0 = \mathrm{L}(X_\xi)\Omega_{\mathscr{L}} = \mathrm{L}(X_\xi)\big(\mathscr{F}\mathscr{L}_\circ^{\,*}\,\Omega^\circ_{\mathscr{H}}\big) = \mathscr{F}\mathscr{L}_\circ^{\,*}\left(\mathrm{L}\left((\mathscr{F}\mathscr{L}_\circ)_* X_\xi\right)\Omega^\circ_{\mathscr{H}}\right) = \mathscr{F}\mathscr{L}_\circ^{\,*}\big(\mathrm{L}(Y^\circ_\xi)\Omega^\circ_{\mathscr{H}}\big) \ , \tag{9.30}$$

and, as $\mathscr{F}\mathscr{L}_\circ$ is a surjective submersion, we conclude that $\mathrm{L}(Y_\xi^\circ)\Omega^\circ_{\mathscr{H}} = 0$. Hence, both X_ξ and Y_ξ° are natural infinitesimal Noether symmetries. The multimomentum maps associated with X_ξ and Y_ξ° are given as,

$$J_{\mathscr{L}}(X_\xi) = -i(X_\xi)\Theta_{\mathscr{L}} = T\sqrt{-g}\,\xi^c\eta_{\mu\nu}g^{ab}x^\nu_b\left(\frac{1}{2}x^\mu_a\mathrm{d}^1\sigma_c - \epsilon_{ac}\mathrm{d}x^\mu\right) \in \Omega^1(J^1\pi)\ , \tag{9.31}$$

$$J^\circ_{\mathscr{H}}(Y_\xi^\circ) = -i(Y_\xi^\circ)\Theta^\circ_{\mathscr{H}} = -\epsilon_{ab}\xi^b p^a_\mu\mathrm{d}x^\mu - \frac{1}{2T\sqrt{-g}}\eta_{\mu\nu}g_{ab}p^a_\mu p^b_\nu\xi^c\mathrm{d}^1\sigma_c \in \Omega^1(P_\circ)\ . \tag{9.32}$$

Poincaré transformations:

Next, the infinitesimal Poincaré transformations of the background spacetime are given as $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu$ where both $\omega^\mu{}_\nu$ and a^μ are constants and the matrices $\omega^\mu{}_\nu$ are antisymmetric, similarly to the Einstein–Cartan theory of the previous section. The vector field which generates these transformations on the configuration bundle E is π-vertical and is given as,

$$\zeta_E = -\left(\omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu\right)\frac{\partial}{\partial x^\mu}\ . \tag{9.33}$$

Its canonical lift (4.20) to $J^1\pi$ is,

$$X_\zeta = j^1\zeta_E = -\left(\omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu\right)\frac{\partial}{\partial x^\mu} - x^\nu_a\,\omega^\mu_\nu\frac{\partial}{\partial x^\mu_a} \in \mathfrak{X}(J^1\pi)\ . \tag{9.34}$$

As expected, X_ζ is tangent to the final Lagrangian constraint submanifold S_f. The canonical lifts to the multimomentum bundles, given respectively by (4.36) and (4.40), are

$$Z_\zeta = -\left(\omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu\right)\frac{\partial}{\partial x^\mu} + p^a_\nu\,\omega^\nu_\mu\frac{\partial}{\partial p^a_\mu} \in \mathfrak{X}(\mathcal{M}\pi)\ , \tag{9.35}$$

$$Y_\zeta = -\left(\omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu\right)\frac{\partial}{\partial x^\mu} + p^a_\nu\,\omega^\nu_\mu\frac{\partial}{\partial p^a_\mu} \in \mathfrak{X}(J^1\pi^*)\ . \tag{9.36}$$

As $\mathrm{L}(Y_\zeta)\pi^c_{ab} = 0$ on $P_\circ$, it follows that Y_ζ is tangent to $P_\circ$ and, by Definition 4.9, the canonical lift to $P_\circ \subset J^1\pi^*$ is

$$Y_\zeta^\circ = -\left(\omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu\right)\frac{\partial}{\partial x^\mu} + p^a_\nu\,\omega^\nu_\mu\frac{\partial}{\partial p^a_\mu} \in \mathfrak{X}(P_\circ)\ . \tag{9.37}$$

As above, $(\mathscr{F}\mathscr{L}_\circ)_*X_\zeta = Y_\zeta^\circ$, and Y_ζ° is also tangent to the final Hamiltonian constraint submanifold P_f as expected.

Again $\mathrm{L}(X_\zeta)\Omega_{\mathscr{L}} = 0$ and $\mathrm{L}(Y_\zeta^\circ)\Omega^\circ_{\mathscr{H}} = 0$; X_ζ and Y_ζ° are therefore natural infinitesimal Noether symmetries. The corresponding multimomentum maps are:

$$J_{\mathscr{L}}(X_\zeta) = -i(X_\zeta)\Theta_{\mathscr{L}} = -T\sqrt{-g}\,\eta_{\mu\rho}g^{ab}x^\rho_b\left(\omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu\right)\mathrm{d}^1\sigma_a \in \Omega^1(J^1\pi)\ , \tag{9.38}$$

$$J^\circ_{\mathscr{H}}(Y_\zeta^\circ) = -i(Y_\zeta^\circ)\Theta^\circ_{\mathscr{H}} = p^a_\mu\left(\omega^\mu{}_\nu x^\nu + \mathrm{a}^\mu\right)\mathrm{d}^1\sigma_a \in \Omega^1(P_\circ)\ . \tag{9.39}$$

Weyl transformations:

Finally, the Weyl gauge transformations are given as: $g^{ab} \to e^{-\varphi(\sigma)}g^{ab}$. These transformations are generated on the configuration manifold E by the vector field $\varphi_E \in \mathfrak{X}(E)$ given as

$$\varphi_E = e^{-\varphi}g^{ab}\frac{\partial}{\partial g^{ab}}\ . \tag{9.40}$$

The canonical lift (4.20) of φ_E to the Lagrangian multiphase space $J^1\pi$ is,

$$X_\varphi = j^1\varphi_E = e^{-\varphi}g^{ab}\frac{\partial}{\partial g^{ab}} + e^{-\varphi}\left(g^{ab}_c - g^{ab}\partial_c\varphi\right)\frac{\partial}{\partial g^{ab}_c} \in \mathfrak{X}(J^1\pi)\ , \tag{9.41}$$

which is tangent to the final Lagrangian constraint submanifold S_f. Furthermore, the canonical lifts (4.36) and (4.40) to the multimomentum bundles are

$$Z_\varphi = -e^{-\varphi} g^{ab} \frac{\partial}{\partial g^{ab}} + e^{-\varphi} \pi^c_{ab} \frac{\partial}{\partial \pi^c_{ab}} - e^{-\varphi} g^{ab} \pi^c_{ab} \partial_c \varphi \frac{\partial}{\partial p} \in \mathfrak{X}(\mathcal{M}\pi) \; , \tag{9.42}$$

$$Y_\varphi = -e^{-\varphi} g^{ab} \frac{\partial}{\partial g^{ab}} + e^{-\varphi} \pi^c_{ab} \frac{\partial}{\partial \pi^c_{ab}} \in \mathfrak{X}(J^1\pi^*) \; . \tag{9.43}$$

Y_φ is tangent to $P_\circ$ so, by Definition 4.9, the canonical lift of φ_E to the primary constraint submanifold $P_\circ \subset J^1\pi^*$ is

$$Y^\circ_\varphi = e^{-\varphi} g^{ab} \frac{\partial}{\partial g^{ab}} \in \mathfrak{X}(P_\circ) \; . \tag{9.44}$$

Once again, it follows that $(\mathscr{F}\mathscr{L}_\circ)_* X_\varphi = Y^\circ_\varphi$, and Y°_φ is also tangent to the final Hamiltonian constraint submanifold P_f.

Once again, $\mathrm{L}(X_\varphi)\Omega_{\mathscr{L}} = 0$ and $\mathrm{L}(Y^\circ_\varphi)\Omega^\circ_{\mathscr{H}} = 0$, so it follows that X_φ and Y°_φ are natural infinitesimal Noether symmetries. In addition, it is straightforward to show that $i(X_\varphi)\Theta_{\mathscr{L}} = 0$ and $i(Y^\circ_\varphi)\Theta^\circ_{\mathscr{H}} = 0$, which means that their corresponding momentum maps are equal to zero. This is what is expected in Weyl invariant field theories (see e.g. [94]).

10 Yang–Mills Theory

For a standard textbook presentation of Yang–Mills theory, see e.g., [129].

Yang–Mills with non-Abelian gauge group $SU(N)$ takes place on 4-dimensional Minkowski spacetime manifold M endowed with the Minkowski metric $\eta_{\mu\nu}$ of signature $(-+++)$. The gauge connection is denoted as $A_\mu = A^a_\mu t_a$, where t_a are the generators (in some matrix representation) of the corresponding Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$. These generators satisfy $[t_b, t_c] = f^a_{bc} t_a$ $\big(a = 1, ..., \dim G\big)$; where f^a_{bc} are the structure constants. Then, $J^1\pi$ has local coordinates $(x^\mu, A^a_\nu, A^a_{\mu\nu})$ while on $J^1\pi^*$ they are $(x^\mu, A^a_\nu, \pi^{\mu\nu}_a)$.

10.1 Lagrangian Formalism

The Lagrangian density is

$$\mathbf{L} = \mathscr{L} \mathrm{d}^4x = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} \mathrm{d}^4x \in \Omega^4(J^1\pi) \; , \tag{10.1}$$

where $F^a_{\mu\nu} = A^a_{\mu\nu} - A^a_{\nu\mu} - f^a_{bc} A^b_\mu A^c_\nu$ corresponds to the non-Abelian field strength curvature tensor constructed from the covariant derivative,

$$D^a_{\mu b} = \delta^a{}_b \partial_\mu - f^a_{cb} A^c_\mu \quad , \quad D_{\mu b}{}^a = \delta_b{}^a \partial_\mu + f^b_{ca} A^c_\mu \quad . \tag{10.2}$$

The Yang–Mills Lagrangian is singular as the multi-Hessian is singular:

$$\frac{\partial^2 \mathscr{L}}{\partial A^a_{\mu\nu} \partial A^b_{\rho\sigma}} = \delta_{ab} \left(\eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\rho} \eta^{\nu\sigma} \right) \; . \tag{10.3}$$

Using the notation in which rank-2 tensor components split up into their symmetric part $V_{(\mu\nu)}$ and antisymmetric part $V_{[\mu\nu]}$ as $V_{\mu\nu} = V_{(\mu\nu)} + V_{[\mu\nu]}$, it is evident that the relevant null vectors of the Hessian matrix above form the set of symmetric spacetime matrices $\big\{V_{\mu\nu} = V_{(\mu\nu)}\big\}$.

The Lagrangian energy function $E_{\mathscr{L}} \in C^\infty(J^1\pi)$ and the Poincaré–Cartan forms on $J^1\pi$ are

$$E_{\mathscr{L}} = -F_a^{\mu\nu} A_{\mu\nu}^a + \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \ , \tag{10.4}$$

$$\Theta_{\mathscr{L}} = -F_a^{\mu\nu} \mathrm{d}A_\nu^a \wedge \mathrm{d}^3x_\mu + \left(F_a^{\mu\nu} A_{\mu\nu}^a - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \right) \mathrm{d}^4x \ , \tag{10.5}$$

$$\begin{aligned} \Omega_{\mathscr{L}} = &- \left(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}\right) \left(\mathrm{d}A_{a\rho\sigma} + f_{abc} A_\sigma^b \mathrm{d}A_\rho^c \right) \wedge \mathrm{d}A_\nu^a \wedge \mathrm{d}^3x_\mu - f_{bc}^a F_a^{\mu\nu} A_\mu^b \mathrm{d}A_\nu^c \wedge \mathrm{d}^4x \\ &+ \left(\eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma}\right) A_{a\rho\sigma} \left(\mathrm{d}A_{\mu\nu}^a \wedge \mathrm{d}^4x + f_{bc}^a A_\mu^b \mathrm{d}A_\nu^c \wedge \mathrm{d}^4x \right) \ . \end{aligned} \tag{10.6}$$

Upon contracting $\Omega_{\mathscr{L}}$ as usual, the sopde condition arises automatically (thus leading to no sopde constraints) and the resulting field equations are the Euler–Lagrange equations,

$$D_{\mu b}^a F_a^{\mu\nu} = 0 \ . \tag{10.7}$$

10.2 De Donder–Weyl Hamiltonian Formalism

The Legendre map $\mathscr{FL} : J^1\pi \to J^1\pi^*$ is almost-regular and its image is the primary constraint submanifold $P_\circ \subset J^1\pi^*$ defined by

$$\mathscr{FL}_* \pi_a^{\mu\nu} = -F_a^{\mu\nu} \ . \tag{10.8}$$

It follows that, as $F_a^{\mu\nu} = F_a^{[\mu\nu]}$, the image of $\mathscr{FL}$ gives $\pi^{(\mu\nu)} = 0$. The Hamilton–Cartan forms are

$$\Theta_{\mathscr{H}}^\circ = \pi_a^{[\mu\nu]} \mathrm{d}A_\nu^a \wedge \mathrm{d}^3x_\mu - \left(\frac{1}{2} f_{bc}^a \pi_a^{[\mu\nu]} A_\mu^b A_\nu^c - \frac{1}{4} \pi_a^{[\mu\nu]} \pi_{[\mu\nu]}^a \right) \mathrm{d}^4x \ , \tag{10.9}$$

$$\Omega_{\mathscr{H}}^\circ = -\mathrm{d}\pi_a^{[\mu\nu]} \wedge \mathrm{d}A_\nu^a \wedge \mathrm{d}^3x_\mu + \left[\frac{1}{2} \left(f_{bc}^a A_\mu^b A_\nu^c - \pi_{[\mu\nu]}^a \right) \mathrm{d}\pi_a^{[\mu\nu]} + f_{bc}^a \pi_a^{[\mu\nu]} A_\mu^b \mathrm{d}A_\nu^c \right] \wedge \mathrm{d}^4x \ . \tag{10.10}$$

There are no Hamiltonian constraints other than the primary constraints (10.8), hence $P_f = P_\circ$. The (Hamilton)–De Donder–Weyl field equations for this theory are

$$D_{\mu b}^a \pi_a^{[\mu\nu]} = 0 \quad , \quad \partial_\mu A_\nu^a + \frac{1}{2} \left(\pi_{[\mu\nu]}^a - f_{bc}^a A_\mu^b A_\nu^c \right) = 0 \ . \tag{10.11}$$

The second equation above is essentially equivalent to equation (10.8).

10.3 Symmetries

$SU(N)$ gauge transformations:

The $SU(N)$ gauge fields transform as

$$\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda] \equiv D_\mu \Lambda \quad , \quad \delta A_\mu^a = \delta^a{}_b \partial_\mu \Lambda^b - f_{cb}^a \Lambda^b A_\mu^c \equiv D_{\mu b}^a \Lambda^b \ , \tag{10.12}$$

The vector field which generates the $SU(N)$ transformations on E is

$$X_\Lambda^E = -D_{\mu b}^a \Lambda^b \frac{\partial}{\partial A_\nu^a} \in \mathfrak{X}(E) \ , \tag{10.13}$$

the canonical lift of X_Λ^E to $J^1\pi$ is given by

$$X_\Lambda = j^1 X_\Lambda^E = -D_{\mu b}^a \Lambda^b \frac{\partial}{\partial A_\mu^a} - \left(\frac{\partial D_{\nu b}^a \Lambda^b}{\partial x^\mu} - f_{cb}^a A_{\mu\nu}^c \Lambda^b \right) \frac{\partial}{\partial A_{\mu\nu}^a} \in \mathfrak{X}(J^1\pi) \ . \tag{10.14}$$

This symmetry can be pushed forward to the De Donder–Weyl Hamiltonian side as $Y_\Lambda = \mathscr{F}\mathscr{L}_* X_\Lambda$. The corresponding multimomentum maps are

$$J_{\mathscr{L}}(X_\Lambda) = -i(X_\Lambda)\Theta_{\mathscr{L}} = -F_a^{\mu\nu} D_{\nu b}^a \Lambda^b \mathrm{d}^3 x_\mu \in \Omega^3(J^1\pi) \ , \tag{10.15}$$

$$J_{\mathscr{H}}^\circ(Y_\Lambda) = -i(Y_\Lambda)\Theta_{\mathscr{H}}^\circ = \pi_a^{[\mu\nu]} D_{\nu b}^a \Lambda^b \mathrm{d}^3 x_\mu \in \Omega^3(P_\circ) \ . \tag{10.16}$$

As mentioned in our discussion of the infinite-dimensional covariant phase space formulation of Yang–Mills, the Noether charge which gives the constraint (6.67) is in agreement with our expression for $J_{\mathscr{H}}^\circ(Y_\Lambda)$ up to a total derivative.

Lorentz transformations:

Yang-Mills also under Lorentz transformations, written infinitesimally as

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu = \left(\delta^\mu{}_\nu - \omega^\mu{}_\nu\right) x^\nu \ , \tag{10.17}$$

recalling that the $\omega^\mu{}_\nu$ are the infinitesimal Lorentz matrices which are antisymmetric. The vector field on M which generates the Lorentz transformations is therefore given as

$$\xi_M = -\xi^\mu \frac{\partial}{\partial x^\mu} \in \mathfrak{X}(M) \ , \tag{10.18}$$

where $\xi^\mu = \omega^\mu{}_\nu x^\nu$. Then, the canonical lift to the configuration bundle is

$$\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} + A_\nu^a \partial_\mu \xi^\nu \frac{\partial}{\partial A_\mu^a} \in \mathfrak{X}(E) \ . \tag{10.19}$$

It follows that the canonical lift to $J^1\pi$ is given as

$$X_\xi = j^1\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} + A_\nu^a \partial_\mu \xi^\nu \frac{\partial}{\partial A_\mu^a} + \left(A_{\nu\mu}^a \partial_\rho \xi^\nu + A_{\rho\nu}^a \partial_\mu \xi^\nu\right) \frac{\partial}{\partial A_{\rho\mu}^a} \ , \tag{10.20}$$

and the corresponding multimomentum map on $J^1\pi$ is

$$J_{\mathscr{L}}(X_\xi) = -i(X_\xi)\Theta_{\mathscr{L}} = F_a^{\mu\nu} A_\rho^a \partial_\nu \xi^\rho \mathrm{d}^3 x_\mu + F_a^{\mu\nu} \xi^\rho \mathrm{d}A_\nu^a \wedge \mathrm{d}^2 x_{\mu\rho} + \left(F_a^{\mu\nu} A_{\mu\nu}^a - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}\right) \xi^\rho \mathrm{d}^3 x_\rho \ . \tag{10.21}$$

The the $\mathscr{F}\mathscr{L}$ projection of X_ξ is $Y_\xi = \mathscr{F}\mathscr{L}_* X_\xi$ and the corresponding multimomentum map on $P_\circ \subset J^1\pi^*$ is calculated accordingly.

11 Chern–Simons Theory

For the multisymplectic treatment of Chern–Simons theory with the Abelian gauge group $U(1)$ see [67]. Consider Chern–Simons theory in $2+1$ dimensions with gauge group $G = SU(N)$, $\dim G = N^2 - 1$. The gauge connection is denoted as $A_\mu = A_\mu^a t_a$ where t_a are the generators (in some matrix representation) of the corresponding Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$. The generators satisfy $[t_b, t_c] = f_{bc}^a t_a$, $a = 1, ..., \dim G$, where f_{bc}^a are structure constants. Then, $J^1\pi$ has local coordinates $(x^\mu, A_\nu^a, A_{\mu\nu}^b)$. Now, let $A = A_\mu \mathrm{d}x^\mu \in \Omega^1(J^1\pi)$ and $F = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \in \Omega^2(J^1\pi)$, with $F_{\mu\nu} \equiv 2A_{[\mu\nu]} + \dfrac{1}{2}[A_\mu, A_\nu]$.

11.1 Lagrangian Formalism

The Chern–Simons Lagrangian density is

$$\begin{aligned}\mathbf{L} &= \mathscr{L}(x^\mu, A_\nu, A_{\mu\nu})\mathrm{d}^3x = \mathrm{Tr}\left[F\wedge A - \frac{1}{3}A\wedge A\wedge A\right] = \epsilon^{\mu\nu\rho}\mathrm{Tr}\left[A_{\mu\nu}A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right]\mathrm{d}^3x \\ &= \epsilon^{\mu\nu\rho}g_{ab}\left(-\frac{1}{4}F^a_{\mu\nu}A^b_\rho + \frac{1}{6}f^b_{cd}A^a_\mu A^c_\nu A^d_\rho\right)\mathrm{d}^3x = -\frac{1}{2}\epsilon^{\mu\nu\rho}g_{ab}\left(A^a_{\mu\nu}A^b_\rho + \frac{2}{3}f^b_{cd}A^a_\mu A^c_\nu A^d_\rho\right)\mathrm{d}^3x\ ,\end{aligned} \tag{11.1}$$

where the trace shown above is ad-invariant on the Lie algebra and can thereby be taken using the *Cartan-Killing metric* $g_{ab} = -2\mathrm{Tr}(t_a t_b)$ on $\mathfrak{g}$. This theory is singular as the Hessian is

$$\frac{\partial^2\mathscr{L}}{\partial A^a_{\mu\nu}\partial A^b_{\rho\sigma}} = 0\ . \tag{11.2}$$

The Lagrangian energy and the Poincaré-Cartan forms are given by

$$E_{\mathscr{L}} = \frac{1}{3}\epsilon^{\mu\nu\rho}g_{ab}f^b_{cd}A^a_\mu A^c_\nu A^d_\rho \in C^\infty(J^1\pi)\ , \tag{11.3}$$

$$\Theta_{\mathscr{L}} = -\epsilon^{\mu\nu\rho}g_{ab}\left[\frac{1}{2}A^b_\rho \mathrm{d}A^a_\nu\wedge\mathrm{d}^2x_\mu + \frac{1}{3}f^b_{cd}A^a_\mu A^c_\nu A^d_\rho\mathrm{d}^3x\right] \in \Omega^3(J^1\pi)\ , \tag{11.4}$$

$$\Omega_{\mathscr{L}} = \epsilon^{\mu\nu\rho}g_{ab}\left[\frac{1}{2}\mathrm{d}A^b_\rho\wedge\mathrm{d}A^a_\nu\wedge\mathrm{d}^2x_\mu + f^b_{cd}A^a_\mu A^c_\nu\mathrm{d}A^d_\rho\wedge\mathrm{d}^3x\right] \in \Omega^4(J^1\pi)\ . \tag{11.5}$$

The Lagrangian constraint in this theory arises from imposing the SOPDE condition on the multivector field solutions to the Lagrangian field equations, giving the following constraint

$$\epsilon^{\mu\nu\rho}\left(g_{ab}A^b_{\mu\nu} + g_{db}f^b_{ca}A^d_\mu A^c_\nu\right) = 0\ , \tag{11.6}$$

which defines the submanifold $\mathcal{S}_f \subset J^1\pi$. The tangency condition on this submanifold gives no new constraints.

11.2 De Donder–Weyl Hamiltonian Formalism

The multimomentum phase space $J^1\pi^*$ has local coordinates $(x^\mu, A^a_\nu, \pi^{\mu\nu}_a)$ and the Legendre map $\mathscr{F}\mathscr{L}: J^1\pi\to J^1\pi^* : (x^\mu, A^a_\nu, A^a_{\mu\nu}) \mapsto (x^\mu, A^a_\nu, \pi^{\mu\nu}_a)$ satisfies

$$\mathscr{F}\mathscr{L}^*\pi^{\mu\nu}_a = \frac{\partial\mathscr{L}}{\partial A^a_{\mu\nu}} = -\frac{1}{2}\epsilon^{\mu\nu\rho}g_{ab}A^b_\rho\ , \tag{11.7}$$

which means that the image of $\mathscr{F}\mathscr{L}$ is the submanifold $P_\circ \subset J^1\pi^*$ defined by the constraints

$$\pi^{\mu\nu}_a + \frac{1}{2}\epsilon^{\mu\nu\rho}g_{ab}A^b_\rho = 0\ . \tag{11.8}$$

Thus, local coordinates on $P_\circ$ are (x^μ, A^a_ν). The De Donder–Weyl Hamiltonian function is

$$\mathscr{H}_\circ = \frac{1}{3}\epsilon^{\mu\nu\rho}g_{ab}f^b_{cd}A^a_\mu A^c_\nu A^d_\rho \in C^\infty(P_\circ)\ , \tag{11.9}$$

and the Hamilton-Cartan forms on $P_\circ$ are

$$\Theta^\circ_{\mathscr{H}} = -\epsilon^{\mu\nu\rho}g_{ab}\left[\frac{1}{2}A^b_\rho\mathrm{d}A^a_\nu\wedge\mathrm{d}^2x_\mu + \frac{1}{3}f^b_{cd}A^a_\mu A^c_\nu A^d_\rho\mathrm{d}^3x\right]\ , \tag{11.10}$$

$$\Omega^\circ_{\mathscr{H}} = \epsilon^{\mu\nu\rho}g_{ab}\left[\frac{1}{2}\mathrm{d}A^b_\rho\wedge\mathrm{d}A^a_\nu\wedge\mathrm{d}^2x_\mu + f^b_{cd}A^a_\mu A^c_\nu\mathrm{d}A^d_\rho\wedge\mathrm{d}^3x\right]\ . \tag{11.11}$$

There are no further Hamiltonian constraints other than the primary constraints and $\mathscr{F}\mathscr{L}(\mathcal{S}_f) = P_\circ$. The field equations in both the Langrangian and Hamiltonian settings are

$$\epsilon^{\mu\nu\nu}g_{ab}\left(\partial_\mu A^b_\nu + f^b_{dc}A^d_\mu A^c_\nu\right) = 0 \ \Rightarrow\ F^b_{\mu\nu} = 0 \ \Rightarrow\ F_{\mu\nu} = 0\ . \tag{11.12}$$

11.3 Symmetries

$SU(N)$ gauge transformations:

The gauge symmetry group $SU(N)$ of the Chern–Simons theory produces infinitesimal transformations

$$\delta A_\mu = D_\mu \chi = \partial_\mu \chi + [A_\mu, \chi] \quad , \quad \delta A^a_\mu = D^a_{\mu b}\chi^b = \delta^a{}_b \partial_\mu \chi^b - f^a_{cb}\chi^b A^c_\mu \ , \tag{11.13}$$

where $\chi = \chi^a t_a \in \mathrm{C}^\infty(M)$. The vector field which generates the $SU(N)$ transformations on E is

$$X^E_\chi = -D^a_{\mu b}\chi^b \frac{\partial}{\partial A^a_\nu} \in \mathfrak{X}(E) \ , \tag{11.14}$$

whose canonical lift to $J^1\pi$ is

$$X_\chi = j^1 X^E_\chi = -D^a_{\mu b}\chi^b \frac{\partial}{\partial A^a_\mu} - \left(\frac{\partial D^a_{\nu b}\chi^b}{\partial x^\mu} - f^a_{cb} A^c_{\mu\nu}\chi^b \right) \frac{\partial}{\partial A^a_{\mu\nu}} \in \mathfrak{X}(J^1\pi) \ . \tag{11.15}$$

and its $\mathscr{F}\mathscr{L}$-projection onto $P_\circ \subset J^1\pi^*$ is

$$Y_\chi = \mathscr{F}\mathscr{L}_* X_\chi = -D^a_{\mu b}\chi^b \frac{\partial}{\partial A^a_\mu} \in \mathfrak{X}(P_\circ) \ . \tag{11.16}$$

Finally, the corresponding multimomentum maps are

$$J_{\mathscr{L}}(X_\chi) = -i(X_\chi)\Theta_{\mathscr{L}} = -\frac{1}{2}\epsilon^{\mu\nu\rho} g_{ab} A^b_\rho D^a_{\nu b}\chi^b \mathrm{d}^2 x_\mu \in \Omega^2(J^1\pi) \ , \tag{11.17}$$

$$J^\circ_{\mathscr{H}}(Y_\chi) = -i(Y_\chi)\Theta^\circ_{\mathscr{H}} = -\frac{1}{2}\epsilon^{\mu\nu\rho} g_{ab} A^b_\rho D^a_{\nu b}\chi^b \mathrm{d}^2 x_\mu \in \Omega^2(J^1\pi) \in \Omega^2(P_\circ) \ . \tag{11.18}$$

Lorentz transformations:

As in Yang–Mills, Lorentz transformations generate a Noether symmetry which we denote as $\xi = -\xi^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ where $\xi^\mu = \omega^\mu{}_\nu x^\nu$ and the $\omega^\mu{}_\nu$ are the infinitesimal antisymmetric Lorentz matrices as in previous examples. The canonical lift to the configuration bundle E is

$$\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} + A^a_\nu \partial_\mu \xi^\nu \frac{\partial}{\partial A^a_\nu} \in \mathfrak{X}(E) \ , \tag{11.19}$$

which can be lifted again to $J^1\pi$, giving

$$X_\xi = j^1\xi_E = -\xi^\mu \frac{\partial}{\partial x^\mu} + A^a_\nu \partial_\mu \xi^\nu \frac{\partial}{\partial A^a_\mu} + \left(A^a_{\nu\mu}\partial_\rho \xi^\nu + A^a_{\rho\nu}\partial_\mu \xi^\nu \right) \frac{\partial}{\partial A^a_{\rho\mu}} \in \mathfrak{X}(J^1\pi) \ , \tag{11.20}$$

as before. The corresponding multimomentum map $J_{\mathscr{L}}(X_\xi) \in \Omega^2(J^1\pi)$ is

$$J_{\mathscr{L}}(X_\xi) = -\, i(X_\xi)\Theta_{\mathscr{L}} = \epsilon^{\mu\nu\rho} g_{ab} \left[\frac{1}{2} A^b_\rho \left(A^a_\sigma \partial_\nu \xi^\sigma \mathrm{d}^2 x_\mu + \xi^\sigma \mathrm{d}A^a_\nu \wedge \mathrm{d}^1 x_{\mu\sigma} \right) - \frac{1}{3} f^b_{cd} \xi^\sigma A^a_\mu A^c_\nu A^d_\rho \mathrm{d}^2 x_\sigma \right] \ . \tag{11.21}$$

Furthermore, the projection of $X_\xi \in \mathfrak{X}(J^1\pi)$ to $P_\circ \subset J^1\pi^*$ is

$$Y_\xi = \mathscr{F}\mathscr{L}_* X_\xi = -\xi^\mu \frac{\partial}{\partial x^\mu} + A^a_\nu \partial_\mu \xi^\nu \frac{\partial}{\partial A^a_\mu} \in \mathfrak{X}(P_\circ) \ . \tag{11.22}$$

The resulting multimomentum map $J^\circ_{\mathscr{H}}(Y_\xi) \in \Omega^2(P_\circ)$ is

$$J^\circ_{\mathscr{H}}(Y_\xi) = -i(Y_\xi)\Theta^\circ_{\mathscr{H}} = \epsilon^{\mu\nu\rho} g_{ab} \left[\frac{1}{2} A^b_\rho \left(A^a_\sigma \partial_\nu \xi^\sigma \mathrm{d}^2 x_\mu + \xi^\sigma \mathrm{d}A^a_\nu \wedge \mathrm{d}^1 x_{\mu\sigma} \right) - \frac{1}{3} f^b_{cd} \xi^\sigma A^a_\mu A^c_\nu A^d_\rho \mathrm{d}^2 x_\sigma \right] \ . \tag{11.23}$$

12 Einstein–Cartan Gravity in $2+1$ Dimensions

Now we study gravity with cosmological constant λ in $2+1$ dimensions is developed in the tetrad formalism using the vielbein e^a_μ given by $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ ($\mu = 0,1,2,\ a = 0,1,2$), and the Hodge dual spin connection $\omega^a_\mu = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}$, which are treated as the variational fields of the theory as in Witten's original work [147].

Similar treatments of Einstein–Cartan gravity in the multisymplectic setting exist, for instance, in [57, 142]; a multisymplectic analysis of Einstein–Cartan gravity, including the study of boundary terms can be found in [91] and in the doctoral thesis [106]. Furthermore, see [56] for the multisymplectic treatment of the second-order Einstein–Hilbert action.

It is also worth noting that there exists a different geometric treatment, sometimes referred to as the *polysymplectic formalism* [95]; for a presentation of vielbein gravity in this formalism, see [96].

12.1 Lagrangian Formalism

The configuration bundle E has coordinates $(x^\mu, e^a_\mu, \omega^c_\rho)$; the induced coordinates on $J^1\pi$ are $(x^\mu, e^a_\mu, \omega^c_\rho, e^a_{\sigma\mu}, \omega^c_{\sigma\rho})$ and $j^1\phi : M \to J^1\pi : x^\mu \mapsto \big(x^\mu, e^a_\mu(x), \omega^c_\rho(x), \partial_\sigma e^a_\mu(x), \partial_\sigma \omega^c_\rho(x)\big)$ are the first-order jet prolongations. The Lagrangian density is given by

$$\mathbf{L} = \mathscr{L}(x^\mu, e^a_\mu, \omega^c_\rho, e^a_{\sigma\mu}, \omega^c_{\sigma\rho})\mathrm{d}^3x = \epsilon^{\mu\nu\rho}\left[2\eta_{ac}e^a_\mu\omega^c_{\nu\rho} + \epsilon_{abc}\left(e^a_\mu\omega^b_\nu\omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\right)\right]\mathrm{d}^3x \ . \tag{12.1}$$

It is well-known that this Lagrangian density can be written as a *Chern–Simons theory* [147],

$$\mathbf{L} = \frac{1}{2}\Big\langle A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\Big\rangle \ , \tag{12.2}$$

with gauge field $A = \big(e^a_\mu P_a + \omega^a_\mu J_a\big)\,\mathrm{d}x^\mu$, where $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$ are the Lorentz generators in the dual form and P^a are the translation generators which satisfy

$$[J_a, J_b] = \epsilon_{abc}J^c \quad , \quad [J_a, P_b] = \epsilon_{abc}P^c \quad , \quad [P_a, P_b] = \lambda\epsilon_{abc}J^c \ . \tag{12.3}$$

The invariant bilinear form on the Lie algebra of the gauge group is, in general, given as

$$\langle J_a, P_b\rangle = c_0\eta_{ab} \quad , \quad \langle J_a, J_b\rangle = c_1\eta_{ab} \quad , \quad \langle P_a, P_b\rangle = \lambda c_1\eta_{ab} \ . \tag{12.4}$$

In this work, the choice $c_0 = 1,\ c_1 = 0$ is made and hence,

$$\langle J_a, P_b\rangle = \eta_{ab} \quad , \quad \langle J_a, J_b\rangle = \langle P_a, P_b\rangle = 0 \ . \tag{12.5}$$

This field theory is singular which can be seen directly from the following components of the Hessian matrix:

$$\frac{\partial^2\mathscr{L}}{\partial e^a_{\mu\nu}\partial e^b_{\rho\sigma}} = 0 \quad , \quad \frac{\partial^2\mathscr{L}}{\partial \omega^a_{\mu\nu}\partial \omega^b_{\rho\sigma}} = 0 \quad , \quad \frac{\partial^2\mathscr{L}}{\partial e^a_{\mu\nu}\partial \omega^b_{\rho\sigma}} = 0 \ . \tag{12.6}$$

The null vectors of the Hessian matrix above are given as

$${}_{(1)}\gamma^a_{\mu\nu} \ = \ {}_{(2)}\gamma^a_{\mu\nu} \ = \ \left(\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \ \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \ \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right)^{\mathbf{T}} \ , \tag{12.7}$$

respectively. The null vector above is expressed as a column vector in the latin (locally Minkowski tangent space) upper index while, at each row of the column vector, the 3×3 matrices are indexed by the roman lower indices.

The Lagrangian energy $E_{\mathscr{L}} \in C^\infty(J^1\pi)$, the Poincaré–Cartan 3-form $\Theta_{\mathscr{L}} \in \Omega^3(J^1\pi)$, and the premultisymplectic Poincaré–Cartan 4-form $\Omega_{\mathscr{L}} \in \Omega^4(J^1\pi)$ are given by

$$E_{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial e^a_{\sigma\mu}} e^a_{\sigma\mu} + \frac{\partial \mathscr{L}}{\partial \omega^c_{\sigma\rho}} \omega^c_{\sigma\rho} - \mathscr{L} = -\epsilon^{\mu\nu\rho}\epsilon_{abc}\Big(e^a_\mu \omega^b_\nu \omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\Big) \ . \tag{12.8}$$

$$\Theta_{\mathscr{L}} = 2\epsilon^{\mu\nu\rho}\eta_{ac} e^a_\mu \mathrm{d}\omega^c_\rho \wedge \mathrm{d}^2 x_\nu + \epsilon^{\mu\nu\rho}\epsilon_{abc}\Big(e^a_\mu \omega^b_\nu \omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\Big)\mathrm{d}^3 x \ , \tag{12.9}$$

$$\begin{aligned} \Omega_{\mathscr{L}} &= -2\epsilon^{\mu\nu\rho}\eta_{ac} \mathrm{d}e^a_\mu \wedge \mathrm{d}\omega^c_\rho \wedge \mathrm{d}^2 x_\nu - \epsilon^{\mu\nu\rho}\epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho \mathrm{d}e^a_\mu + 2e^a_\mu\omega^b_\nu \mathrm{d}\omega^c_\rho + \lambda e^b_\nu e^c_\rho \mathrm{d}e^a_\mu\right) \wedge \mathrm{d}^3 x \\ &= \widetilde{\Omega}_{\mathscr{L}} + \mathrm{d}E_{\mathscr{L}} \wedge \mathrm{d}^3 x \ . \end{aligned} \tag{12.10}$$

The field equations are obtained using multivector fields written as $\mathbf{X}_{\mathscr{L}} = X_0 \wedge X_1 \wedge X_2 \in \mathfrak{X}^3(J^1\pi)$ with components given by

$$X_\nu = \frac{\partial}{\partial x^\nu} + B^a_{\nu\mu}\frac{\partial}{\partial e^a_\mu} + C^c_{\nu\rho}\frac{\partial}{\partial \omega^c_\rho} + D^a_{\nu\sigma\mu}\frac{\partial}{\partial e^a_{\sigma\mu}} + H^c_{\nu\sigma\rho}\frac{\partial}{\partial \omega^c_{\sigma\rho}} \ . \tag{12.11}$$

Then,

$$\begin{aligned} i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = &-2\epsilon^{\mu\nu\rho}\eta_{ac}\left(B^a_{\nu\mu}\mathrm{d}\omega^c_\rho - C^c_{\nu\rho}\mathrm{d}e^a_\mu\right) + \epsilon^{\mu\nu\rho}\epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho \mathrm{d}e^a_\mu + 2e^a_\mu\omega^b_\nu\mathrm{d}\omega^c_\rho + \lambda e^b_\nu e^c_\rho \mathrm{d}e^a_\mu\right) \\ &- \epsilon^{\mu\nu\rho}\Big[2\eta_{ac}\left(B^a_{\lambda\mu}C^c_{\nu\rho} - B^a_{\nu\mu}C^c_{\lambda\rho}\right) - \epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho B^a_{\lambda\mu} + 2e^a_\mu\omega^b_\nu C^c_{\lambda\rho} + \lambda e^b_\nu e^c_\rho B^a_{\lambda\mu}\right)\Big]\,\mathrm{d}x^\lambda = 0 \ . \end{aligned} \tag{12.12}$$

Setting differential forms separately equal to zero produces the following two independent field equations:

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}C^c_{\nu\rho} + \frac{1}{2}\epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho + \lambda e^b_\nu e^c_\rho\right)\right] \ , \tag{12.13}$$

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}B^a_{\nu\mu} - \epsilon_{abc}e^a_\mu\omega^b_\nu\right] \ . \tag{12.14}$$

These equations are compatible on all of $J^1\pi$ as no compatibility constraints arise. Note that, as

$$\ker \widetilde{\Omega}_{\mathscr{L}} \cap \ker \mathrm{d}^3 x = \left\langle \frac{\partial}{\partial e^a_{\rho\mu}}, \frac{\partial}{\partial \omega^a_{\rho\mu}} \right\rangle \ , \tag{12.15}$$

it follows from Proposition 3.1 that $i\left(\dfrac{\partial}{\partial e^a_{\rho\mu}}\right)\mathrm{d}E_{\mathscr{L}} = i\left(\dfrac{\partial}{\partial \omega^a_{\rho\mu}}\right)\mathrm{d}E_{\mathscr{L}} = 0$ is satisfied automatically. Observe also that, as $\dfrac{\partial}{\partial e^a_{\rho\mu}}, \dfrac{\partial}{\partial \omega^a_{\rho\mu}} \in \ker \Omega_{\mathscr{L}}$, the coefficients $D^a_{\nu\sigma\mu}, H^c_{\nu\sigma\rho}$ in (12.11) remain undetermined.

Nevertheless, solutions to the field equations must be holonomic multivector fields $\mathbf{X}_{\mathscr{L}}$ which thereby have components (12.11) for which $B^a_{\nu\mu} = e^a_{\nu\mu}$ and $C^c_{\nu\rho} = \omega^c_{\nu\rho}$; it follows that the field equations (12.13) and (12.14) become:

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}\omega^c_{\nu\rho} + \frac{1}{2}\epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho + \lambda e^b_\nu e^c_\rho\right)\right] \ , \tag{12.16}$$

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}e^a_{\nu\mu} - \epsilon_{abc}e^a_\mu\omega^b_\nu\right] \ . \tag{12.17}$$

Both of these equations are SOPDE constraints and define the submanifold $S_1 \hookrightarrow J^1\pi$. As usual, it is necessary to ensure the tangency of the multivector field $\mathbf{X}_{\mathscr{L}}$ to the submanifold S_1. Recall that this is done by taking the Lie derivative of the constraints with respect to the multivector field components (12.11), where $B^a_{\nu\mu} = e^a_{\nu\mu}$ and $C^c_{\nu\rho} = \omega^c_{\nu\rho}$, giving:

$$0 = \mathrm{L}(X_\sigma)\left[\epsilon^{\mu\nu\rho}\Big(\eta_{ac}\omega^c_{\nu\rho} + \frac{1}{2}\epsilon_{abc}(\omega^b_\nu\omega^c_\rho + \lambda e^b_\nu e^c_\rho)\Big)\right] = \epsilon^{\mu\nu\rho}\Big[\eta_{ac}H^c_{\sigma\nu\rho} + \epsilon_{abc}\left(\omega^b_\nu\omega^c_{\sigma\rho} + \lambda e^b_\nu e^c_{\sigma\rho}\right)\Big] \, , \tag{12.18}$$

$$0 = \mathrm{L}(X_\sigma)\Big[\epsilon^{\mu\nu\rho}(\eta_{ac}e^a_{\nu\mu} - \epsilon_{abc}e^a_\mu\omega^b_\nu)\Big] = \epsilon^{\mu\nu\rho}\Big[\eta_{ac}D^a_{\sigma\nu\mu} - \epsilon_{abc}e^a_{\sigma\mu}\omega^b_\nu - \epsilon_{abc}e^a_\mu\omega^b_{\sigma\nu}\Big] \, . \tag{12.19}$$

These equations are relations for the coefficients $D^a_{\sigma\nu\mu}$ and $H^c_{\sigma\nu\rho}$ and are not constraints; it follows that there are no tangency constraints in this field theory.

Furthermore, the integral sections $(x^\mu, e^a_\mu(x), \omega^c_\rho(x), e^a_{\sigma\mu}(x), \omega^c_{\sigma\rho}(x))$ of these holonomic multivector fields satisfy that $B^a_{\nu\mu} = \partial_\nu e^a_\mu$ and $C^c_{\nu\rho} = \partial_\nu\omega^c_\rho$, and therefore the equations (12.13) and (12.14) on S_1 become:

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}\left(\partial_\nu\omega^c_\rho - \partial_\rho\omega^c_\nu\right) + \epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho + \lambda e^b_\nu e^c_\rho\right)\right] \, , \tag{12.20}$$

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right) + \epsilon_{abc}\left(e^a_\mu\omega^b_\nu + \omega^a_\mu e^b_\nu\right)\right] \, . \tag{12.21}$$

These are the well-known field equations for gravity in $2+1$ dimensions; the first equation is Einstein's equation with cosmological constant while the second guarantees that the spin connection is torsion free, thereby fully specifying the spin connection in terms of the dreibein e^a_μ.

12.2 De Donder–Weyl Hamiltonian Formulism

The Legendre map $\mathscr{FL} : J^1\pi \to J^1\pi^*$ associated to $\mathscr{L}$ gives the multimomenta

$$\mathscr{FL}^* p^{\mu\nu}_i = \frac{\partial\mathscr{L}}{\partial e^a_{\mu\nu}} = 0 \quad , \quad \mathscr{FL}^*\pi^{\mu\nu}_a = \frac{\partial\mathscr{L}}{\partial\omega^a_{\mu\nu}} = 2\epsilon^{\mu\nu\rho}\eta_{ab}e^b_\rho \, . \tag{12.22}$$

The relations above give the constraints $p^{\mu\nu}_i = 0$ and $\pi^{\mu\nu}_c - 2\epsilon^{\mu\nu\rho}\eta_{ab}e^b_\rho = 0$, which define the primary constraint submanifold $\jmath_\circ : P_\circ \hookrightarrow J^1\pi^*$ on which the Hamiltonian formulation takes place. As expected from Proposition 3.2, the null vectors (12.7) are given by the partial derivatives of the primary constraints with respect to the multimomenta according to (3.24):

$$_{(1)}\gamma^a_{\mu\nu} = \mathscr{FL}^*\frac{\partial}{\partial p^{\mu\nu}_a}\left(p^{\mu\nu}_a\right) \quad , \quad _{(2)}\gamma^a_{\mu\nu} = \mathscr{FL}^*\frac{\partial}{\partial\pi^{\mu\nu}_a}\left(\pi^{\mu\nu}_a - 2\epsilon^{\mu\nu\rho}\eta_{ab}e^b_\rho\right) \, . \tag{12.23}$$

The restricted Legendre map $\mathscr{FL}_\circ$ is given as $\mathscr{FL} = \jmath_\circ \circ \mathscr{FL}_\circ$ and the primary constraint submanifold $P_\circ$ is locally diffeomorphic to E and the system is almost-regular. Then, the local coordinates on $P_\circ$ are $(x^\mu, e^a_\mu, \omega^c_\rho)$.

The De Donder–Weyl Hamiltonian on $P_\circ$, which satisfies $E_\mathscr{L} = \mathscr{FL}^*_\circ\mathscr{H}_\circ$, is given by

$$\mathscr{H}_\circ = -\epsilon^{\mu\nu\rho}\epsilon_{abc}\Big(e^a_\mu\omega^b_\nu\omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\Big) \, . \tag{12.24}$$

The Hamilton–Cartan 3-form $\Theta^\circ_\mathscr{H} \in \varOmega^3(P_\circ)$ and the premultisymplectic Hamilton–Cartan 4-form $\Omega^\circ_\mathscr{H} \in \varOmega^4(P_\circ)$ are given by

$$\Theta^\circ_\mathscr{H} = 2\epsilon^{\mu\nu\rho}\eta_{ac}e^a_\mu \mathrm{d}\omega^c_\rho \wedge \mathrm{d}^2x_\nu + \epsilon^{\mu\nu\rho}\epsilon_{abc}\Big(e^a_\mu\omega^b_\nu\omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\Big)\mathrm{d}^3x \, , \tag{12.25}$$

$$\Omega^\circ_\mathscr{H} = -2\epsilon^{\mu\nu\rho}\eta_{ac}\mathrm{d}e^a_\mu \wedge \mathrm{d}\omega^c_\rho \wedge \mathrm{d}^2x_\nu - \epsilon^{\mu\nu\rho}\epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho\mathrm{d}e^a_\mu + 2e^a_\mu\omega^b_\nu\mathrm{d}\omega^c_\rho + \lambda e^b_\nu e^c_\rho\mathrm{d}e^a_\mu\right) \wedge \mathrm{d}^3x \, . \tag{12.26}$$

As it is usual, the field equations are obtained using a multivector field $\mathbf{X}_{\mathscr{H}} = X_0 \wedge X_1 \wedge X_2 \in \mathfrak{X}^3(P_\circ)$ with components given by

$$X_\nu = \frac{\partial}{\partial x^\nu} + B^a_{\nu\mu}\frac{\partial}{\partial e^a_\mu} + C^c_{\nu\rho}\frac{\partial}{\partial \omega^c_\rho} \ . \tag{12.27}$$

Now the geometric field equation (3.11) gives

$$\begin{aligned} i(\mathbf{X}_{\mathscr{H}})\Omega_{\mathscr{H}} =& 2\epsilon^{\mu\nu\rho}\eta_{ac}\left(B^a_{\nu\mu}\mathrm{d}\omega^c_\rho - C^c_{\nu\rho}\mathrm{d}e^a_\mu\right) - \epsilon^{\mu\nu\rho}\epsilon_{abc}\Big(\omega^b_\nu\omega^c_\rho\mathrm{d}e^a_\mu + 2e^a_\mu\omega^b_\nu\mathrm{d}\omega^c_\rho + \lambda e^b_\nu e^c_\rho \mathrm{d}e^a_\mu\Big) + \\ & \epsilon^{\mu\nu\rho}\left[2\eta_{ac}\left(B^a_{\lambda\mu}C^c_{\nu\rho} - B^a_{\nu\mu}C^c_{\lambda\rho}\right) + \epsilon_{abc}\Big(\omega^b_\nu\omega^c_\rho B^a_{\lambda\mu} + 2e^a_\mu\omega^b_\nu C^c_{\lambda\rho} + \lambda e^b_\nu e^c_\rho B^a_{\lambda\mu}\Big)\right]\mathrm{d}x^\lambda = 0 \ . \end{aligned} \tag{12.28}$$

Setting differential forms separately equal to zero yields

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}C^c_{\nu\rho} + \frac{1}{2}\epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho + \lambda e^b_\nu e^c_\rho\right)\right] \ , \tag{12.29}$$

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}B^a_{\nu\mu} - \epsilon_{abc}e^a_\mu\omega^b_\nu\right] \ . \tag{12.30}$$

Similarly to the Lagrangian formalism, no compatibility constraints arise. Again, observe that

$$\ker\widetilde{\Omega}^\circ_{\mathscr{H}} \cap \ker \mathrm{d}^3x = \left\langle \frac{\partial}{\partial p^{\rho\mu}_i}, \frac{\partial}{\partial \pi^{\rho\mu}_a} \right\rangle \ , \tag{12.31}$$

from which it follows that $i\left(\frac{\partial}{\partial p^{\rho\mu}_i}\right)\mathrm{d}\mathscr{H}_\circ = i\left(\frac{\partial}{\partial \pi^{\rho\mu}_a}\right)\mathrm{d}\mathscr{H}_\circ = 0$ automatically. Then, by taking $(x^\mu, e^a_\mu(x), \omega^{bc}_\rho(x))$ to be integral sections of $\mathbf{X}_{\mathscr{H}}$, for which $B^a_{\nu\mu} = \partial_\nu e^a_\mu$ and $C^c_{\nu\rho} = \partial_\nu\omega^c_\rho$, the field equations become again

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}\left(\partial_\nu\omega^c_\rho - \partial_\rho\omega^c_\nu\right) + \epsilon_{abc}\left(\omega^b_\nu\omega^c_\rho + \lambda e^b_\nu e^c_\rho\right)\right] \ , \tag{12.32}$$

$$0 = \epsilon^{\mu\nu\rho}\left[\eta_{ac}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right) + \epsilon_{abc}\left(e^a_\mu\omega^b_\nu + \omega^a_\mu e^b_\nu\right)\right] \ , \tag{12.33}$$

which are the same field equation obtained in the Lagrangian formulation.

12.3 Symmetries

Spacetime diffeomorphisms:

Spacetime diffeomorphisms are generated by the infinitesimal parameter ξ^μ via the following transformations:

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \ \Rightarrow \ e'^a_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}e^a_\nu(x) \ , \ \omega^a_\mu(x) \to \omega'^a_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}\omega^a_\nu(x) \ , \tag{12.34}$$

hence,

$$\begin{aligned} \delta e^a_\mu(x) &= e'^a_\mu(x) - e^a_\mu(x) = -\xi^\nu\partial_\nu e^a_\mu - e^a_\nu\partial_\mu\xi^\nu \ , \\ \delta\omega^a_\mu(x) &= \omega'^a_\mu(x) - \omega^a_\mu(x) = -\xi^\nu\partial_\nu\omega^a_\mu - \omega^a_\nu\partial_\mu\xi^\nu \ . \end{aligned} \tag{12.35}$$

Once again, these field variations are given by the Lie derivative of the local sections $\phi : M \to E$ generated by the vector field $\xi = -\xi^\mu\frac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ as defined in (4.8). The lift of the vector field ξ to E is given by

$$\xi_E = -\xi^\mu\frac{\partial}{\partial x^\mu} + e^a_\nu\partial_\mu\xi^\nu\frac{\partial}{\partial e^a_\mu} + \omega^a_\nu\partial_\mu\xi^\nu\frac{\partial}{\partial\omega^a_\mu} \in \mathfrak{X}(E) \ . \tag{12.36}$$

The canonical lift $X_\xi \in \mathfrak{X}(J^1\pi)$ of ξ_E to $J^1\pi$ is given as

$$X_\xi = -\xi^\mu \frac{\partial}{\partial x^\mu} + e^a_\nu \partial_\mu \xi^\nu \frac{\partial}{\partial e^a_\mu} + \omega^a_\nu \partial_\mu \xi^\nu \frac{\partial}{\partial \omega^a_\mu} + \left(e^a_{\sigma\rho}\partial_\mu \xi^\rho - e^a_{\rho\mu}\partial_\sigma \xi^\rho\right) \frac{\partial}{\partial e^a_{\sigma\mu}} + \left(\omega^a_{\sigma\rho}\partial_\mu \xi^\rho - \omega^a_{\rho\mu}\partial_\sigma \xi^\rho\right) \frac{\partial}{\partial \omega^a_{\sigma\mu}} \; . \tag{12.37}$$

It follows that $\mathrm{L}(X_\xi)\Theta_{\mathscr{L}} = 0$ so the momentum map $J_{\mathscr{L}}(X_\xi) \in \Omega^2(J^1\pi)$ is given by

$$\begin{aligned} J_{\mathscr{L}}(X_\xi) &= -i(X_\xi)\Theta_{\mathscr{L}} \\ &= -2\epsilon^{\mu\nu\rho}\eta_{ac}e^a_\mu \left(\xi^\sigma \mathrm{d}\omega^c_\rho \wedge \mathrm{d}^1 x_{\nu\sigma} + \omega^c_\sigma \partial_\rho \xi^\sigma \mathrm{d}^2 x_\nu\right) + \epsilon^{\mu\nu\rho}\epsilon_{abc}\left(e^a_\mu \omega^b_\nu \omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\right)\xi^\sigma \mathrm{d}^2 x_\sigma \; . \end{aligned} \tag{12.38}$$

Furthermore, since $P_\circ$ is locally diffeomorphic to E, the vector field $Y_\xi = \mathscr{F}\mathscr{L}_{\circ *}X_\xi$ which generates the spacetime diffeomorphisms on $P_\circ$ is written as

$$Y_\xi = -\xi^\mu \frac{\partial}{\partial x^\mu} + e^a_\nu \partial_\mu \xi^\nu \frac{\partial}{\partial e^a_\mu} + \omega^a_\nu \frac{\partial \xi^\nu}{\partial x^\mu}\frac{\partial}{\partial \omega^a_\mu} \in \mathfrak{X}(P_\circ) \; , \tag{12.39}$$

and since $\mathrm{L}(Y_\xi)\Theta^\circ_{\mathscr{H}} = 0$, the momentum map $J_{\mathscr{H}}(Y_\xi) \in \Omega^2(P_\circ)$ is given by

$$\begin{aligned} J^\circ_{\mathscr{H}}(\xi) &= -i(Y_\xi)\Theta^\circ_{\mathscr{H}} \\ &= -2\epsilon^{\mu\nu\rho}\eta_{ac}e^a_\mu \left(\xi^\sigma \mathrm{d}\omega^c_\rho \wedge \mathrm{d}^1 x_{\nu\sigma} + \omega^c_\sigma \partial_\rho \xi^\sigma \mathrm{d}^2 x_\nu\right) + \epsilon^{\mu\nu\rho}\epsilon_{abc}\left(e^a_\mu \omega^b_\nu \omega^c_\rho + \frac{1}{3}\lambda e^a_\mu e^b_\nu e^c_\rho\right)\xi^\sigma \mathrm{d}^2 x_\sigma \; . \end{aligned} \tag{12.40}$$

Local Poincaré transformations:

Local Poincaré transformations of the gauge field are constructed using the infinitesimal parameter $\Lambda(x) = P_a\rho^a + J_a\tau^a$ as

$$\delta A_\mu(x) = -D_\mu\Lambda(x) = -\partial_\mu\Lambda - [A_\mu, \Lambda], \tag{12.41}$$

where $\tau^a(x)$ are the infinitesimal generators of local Lorentz transformations and $\rho^a(x)$ are the infinitesimal generators of local translations. It follows from the Lie algebra (12.3) that

$$\delta e^a_\mu(x) = -\frac{\partial \rho^a}{\partial x^\mu} - \epsilon_{abc}e^b_\mu\tau^c - \epsilon_{abc}\omega^b_\mu\rho^c \quad , \quad \delta\omega^a_\mu(x) = -\frac{\partial \tau^a}{\partial x^\mu} - \epsilon_{abc}\omega^b_\mu\tau^c - \lambda\epsilon_{abc}e^b_\mu\rho^c \; . \tag{12.42}$$

The vector field on the configuration bundle E which generates the local Poincaré transformations is

$$\zeta_E = \left(\partial_\mu\rho^a + \epsilon^a{}_{bc}e^b_\mu\tau^c + \epsilon^a{}_{bc}\omega^b_\mu\rho^c\right)\frac{\partial}{\partial e^a_\mu} + \left(\partial_\mu\tau^a + \epsilon^a{}_{bc}\omega^b_\mu\tau^c + \lambda\epsilon^a{}_{bc}e^b_\mu\rho^c\right)\frac{\partial}{\partial \omega^a_\mu} \; . \tag{12.43}$$

The canonical lift of ζ_E to $J^1\pi$ is

$$\begin{aligned} X_\zeta = &\left(\partial_\mu\rho^a + \epsilon^a{}_{bc}e^b_\mu\tau^c + \epsilon^a{}_{bc}\omega^b_\mu\rho^c\right)\frac{\partial}{\partial e^a_\mu} + \left(\partial_\mu\tau^a + \epsilon^a{}_{bc}\omega^b_\mu\tau^c + \lambda\epsilon^a{}_{bc}e^b_\mu\rho^c\right)\frac{\partial}{\partial \omega^a_\mu} \\ &+ \epsilon^a{}_{bc}\left(e^b_\mu\partial_\sigma\tau^c + \omega^b_\mu\partial_\sigma\rho^c + e^b_{\sigma\mu}\tau^c\right)\frac{\partial}{\partial e^a_{\sigma\mu}} + \epsilon^a{}_{bc}\left(\omega^b_\mu\partial_\sigma\tau^c + \lambda e^b_\mu\partial_\sigma\rho^c + \omega^b_{\sigma\mu}\tau^c\right)\frac{\partial}{\partial \omega^a_{\sigma\mu}} \; , \end{aligned} \tag{12.44}$$

while the projection $\mathscr{F}\mathscr{L}_{\circ *}X_\zeta \equiv Y_\zeta$ onto $P_\circ$ is

$$Y_\zeta = \left(\partial_\mu\rho^a + \epsilon^a{}_{bc}e^b_\mu\tau^c + \epsilon^a{}_{bc}\omega^b_\mu\rho^c\right)\frac{\partial}{\partial e^a_\mu} + \left(\partial_\mu\tau^a + \epsilon^a{}_{bc}\omega^b_\mu\tau^c + \lambda\epsilon^a{}_{bc}e^b_\mu\rho^c\right)\frac{\partial}{\partial \omega^a_\mu} \; . \tag{12.45}$$

The momentum map $J_{\mathscr{L}}(X_\zeta) \in \Omega^2(J^1\pi)$ on $J^1\pi$ is given by

$$J_{\mathscr{L}}(X_\zeta) = -i(Y_\zeta)\Theta_{\mathscr{L}} = -2\epsilon^{\mu\nu\rho}\eta_{ac}e^a_\mu\left(\partial_\mu\tau^c + \epsilon^c{}_{db}\omega^d_\mu\tau^b + \lambda\epsilon^c{}_{db}e^d_\mu\rho^b\right)\mathrm{d}^2 x_\nu \; . \tag{12.46}$$

and the momentum map $J_{\mathscr{H}}(Y_\zeta) \in \Omega^2(P_\circ)$ on the multimomentum phase space is given by the same expression

$$J^\circ_{\mathscr{H}}(Y_\zeta) = -i(Y_\zeta)\Theta^\circ_{\mathscr{H}} = -2\epsilon^{\mu\nu\rho}\eta_{ac}e^a_\mu\left(\partial_\mu\tau^c + \epsilon^c{}_{db}\omega^d_\mu\tau^b + \lambda\epsilon^c{}_{db}e^d_\mu\rho^b\right)\mathrm{d}^2 x_\nu \; . \tag{12.47}$$

13 The Dirac Field

We now study the Dirac Lagrangian for Dirac spinors (see e.g., [129]). The De Donder–Weyl construction of the Dirac equation first appeared in [4] [133]. Here we give the premultisymplectic construction of the variational calculus in both the Lagrangian and De Donder–Weyl Hamiltionian setting.

Recall that tensor fields are not, in general, the only objects which transform under Lorentz transformations which can be used to construct Lorentz invariant Lagrangians. The Lorentz group $SO(3,1)$ is isomorphic to $SU(2)\times SU(2)$. The Dirac spinor ψ is constructed by taking a left-handed Weyl spinor ψ_L which transforms in the $\left(\frac{1}{2},0\right)$ representation of $SU(2)\times SU(2)$ and stacking it on top of a right-handed Weyl spinor ψ_R which transforms in the $\left(0,\frac{1}{2}\right)$ representation of $SU(2)\times SU(2)$ so that

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix}\psi_a\\ \bar{\chi}^{\dot{a}}\end{pmatrix} \ , \tag{13.1}$$

transforms in the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation of $SU(2)\times SU(2)$. The undotted indices $\{a\}$ above are indices belonging to $SU(2)_L$ of $SU(2)_L\times SU(2)_R$ while the dotted indices $\{\dot{a}\}$ belong to $SU(2)_R$. Furthermore, the generators of the spinor representation of the Lorentz group are given as

$$\Sigma^{\mu\nu} = i[\gamma^\mu,\gamma^\nu] \ , \tag{13.2}$$

where the γ^μ are the standard Dirac matrices which satisfy the Clifford algebra $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{I}$. Then, given the Lorentz matrices in infinitesimal form as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \ , \tag{13.3}$$

where $\omega^\mu{}_\nu$ are the infinitesimal Lorentz matrices which are antisymmetric, it follows that the Dirac spinors transform as,

$$\psi \to \psi'(x) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\psi(\Lambda^{-1}x) \ . \tag{13.4}$$

Now, consider the configuration bundle $\pi : E \to M$, in which the configuration manifold E is locally trivial $E = M\times F$ and F is $\mathbb{Z}_2$ graded (see Appendix B). Denote the local coordinates on E as $(x^\mu,\psi^a,\bar{\psi}^a)$ where now we have fermionic coordinates $\epsilon(\psi^a) = \epsilon(\bar{\psi}^a) = 1$. Then, the vertical endomorphism on $J^1\pi$ is given as (see [33, 120])

$$\mathcal{S} = \theta^a \wedge \mathrm{d}^{m-1}x_\mu \otimes \frac{\partial}{\partial\psi^a_\mu} + \bar{\theta}^a \wedge \mathrm{d}^{m-1}x_\mu \otimes \frac{\partial}{\partial\bar{\psi}^a_\mu} \ , \tag{13.5}$$

and we use the following contact forms on $J^1\pi$:

$$\theta^a \equiv \mathrm{d}\psi^a - \psi^a_\mu \mathrm{d}x^\mu \quad , \quad \bar{\theta}^a \equiv \mathrm{d}\bar{\psi}^a - \bar{\psi}^a_\mu \mathrm{d}x^\mu \ . \tag{13.6}$$

Now, defining the Poincaré–Cartan m-form $\Theta_{\mathscr{L}}$ on the extended phase space $J^1\pi$ as before (1.24),

$$\Theta_{\mathscr{L}} \equiv \mathscr{L}\mathrm{d}^m x + \mathcal{S}(\mathrm{d}\mathscr{L}) \ , \tag{13.7}$$

we obtain the following coordinate expression:

$$\Theta_{\mathscr{L}} = \left(\theta^a \frac{\partial\mathscr{L}}{\partial\psi^a_\mu} + \bar{\theta}^a \frac{\partial\mathscr{L}}{\partial\bar{\psi}^a_\mu}\right) \wedge \mathrm{d}^{m-1}x_\mu + \mathscr{L}\mathrm{d}^m x \ , \tag{13.8}$$

where we take derivatives from the left and we now must be careful with the ordering of the variables which are not commutative. The Poincaré–Cartan $(m+1)$-form is given by $\Omega_{\mathscr{L}} = -\mathrm{d}\Theta_{\mathscr{L}}$ as usual.

13.1 Lagrangian Formalism

The famous Dirac equation,

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0\ , \tag{13.9}$$

can be derived from the Dirac Lagrangian,

$$\mathscr{L} = \bar{\psi}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi\ , \tag{13.10}$$

which can be taken to be a functional of holonomic sections which map from a 4-dimensional Minkowski spacetime M (with coordinates x^{μ}) to the first jet bundle $J^1\pi$ of the Grassmannian configuration manifold $\pi : E \to M$ with coordinates $\left(x^{\mu}, \psi, \bar{\psi}\right)$. It follows that $J^1\pi$ has the local coordinates $\left(x^{\mu}, \psi, \bar{\psi}, \psi_{\mu}, \bar{\psi}_{\mu}\right)$. The dotted and undotted indices of $SU(2) \times SU(2)$ will be suppressed; however, it should be noted that the corresponding rules of matrix multiplication apply throughout all algebraic manipulations. Notice that we are taking ψ and $\bar{\psi}$ to be variationally independent fields even though they are related by Dirac conjugation: $\bar{\psi} \equiv \psi^{\dagger}\gamma^{\circ}$ (and $\psi^{\dagger}$ is the Hermitian conjugate).

Here we would like to perform the variational calculus on $J^1\pi$ using the relevant Poincaré–Cartan 5-form which for this field theory is premultisymplectic as the Dirac Lagrangian is singular in the De Donder–Weyl sense. The Dirac Lagrangian function on $J^1\pi$ is written as,

$$\mathscr{L} = i\bar{\psi}\gamma^{\mu}\psi_{\mu} - m\bar{\psi}\psi \in C^{\infty}(J^1\pi)\ , \tag{13.11}$$

giving the Lagrangian energy $E_{\mathscr{L}} = m\bar{\psi}\psi \in C^{\infty}(J^1\pi)$. The resulting Poincaré–Cartan forms on $J^1\pi$ are:

$$\Theta_{\mathscr{L}} = i\bar{\psi}\gamma^{\mu}\mathrm{d}\psi \wedge \mathrm{d}^3x_{\mu} - m\bar{\psi}\psi\mathrm{d}^4x\ , \tag{13.12}$$

$$\Omega_{\mathscr{L}} = -i\mathrm{d}\bar{\psi}\gamma^{\mu} \wedge \mathrm{d}\psi \wedge \mathrm{d}^3x_{\mu} + m\left(\bar{\psi}\mathrm{d}\psi + \mathrm{d}\bar{\psi}\psi\right) \wedge \mathrm{d}^4x\ . \tag{13.13}$$

Now use a 4-multivector field $\mathbf{X}_{\mathscr{L}} = X_0 \wedge X_1 \wedge X_2 \wedge X_3$, whose components are written as,

$$X_{\nu} = \partial_{\nu} + A_{\nu}\frac{\partial}{\partial\psi} + B_{\nu}\frac{\partial}{\partial\bar{\psi}} + C_{\nu\mu}\frac{\partial}{\partial\psi_{\mu}} + D_{\nu\mu}\frac{\partial}{\partial\bar{\psi}_{\mu}}\ , \tag{13.14}$$

to compute $i(\mathbf{X}_{\mathscr{L}})\Omega_{\mathscr{L}} = 0$ and obtain:

$$i\gamma^{\mu}A_{\mu} - m\psi = 0\ , \tag{13.15a}$$

$$iB_{\mu}\gamma^{\mu} + m\bar{\psi} = 0\ . \tag{13.15b}$$

No compatiblity constraints arise from the equations above. However, the SOPDE condition ($A_{\mu} = \psi_{\mu}$, $B_{\mu} = \bar{\psi}_{\mu}$) needs to be imposed on the equations above, giving the following SOPDE constraints:

$$i\gamma^{\mu}\psi_{\mu} - m\psi = 0\ , \tag{13.16a}$$

$$i\bar{\psi}_{\mu}\gamma^{\mu} + m\bar{\psi} = 0\ . \tag{13.16b}$$

Furthermore, it is necessary to that the tangency condition for $\mathbf{X}_{\mathscr{L}}$ is satisfied since the equations above are constraints.

$$0 = \mathrm{L}(X_{\nu})\,(i\gamma^{\mu}\psi_{\mu} - m\psi) = i\gamma^{\mu}C_{\nu\mu} - mA_{\nu}\ , \tag{13.17a}$$

$$0 = \mathrm{L}(X_{\nu})\left(i\bar{\psi}_{\mu}\gamma^{\mu} + m\bar{\psi}\right) = iD_{\nu\mu}\gamma^{\mu} + mB_{\nu}\ . \tag{13.17b}$$

These equations are relations for coefficients of $\mathbf{X}_{\mathscr{L}}$ and give no new constraints. Consequently, the final constraint submanifold is given by the SOPDE constraints (13.16). The final field equations are written on integral sections of $\mathbf{X}_{\mathscr{L}}$ as,

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \ , \tag{13.18a}$$
$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \ . \tag{13.18b}$$

The first equation is evidently the Dirac equation (13.9) which is the Euler–Lagrange equation for ψ while the second equation is the Euler–Lagrange equation for $\bar{\psi}$ and is equivalent to the first equation by Dirac conjugation. It is interesting to notice that, on integral curves, the tangency equations are specified as

$$i\gamma^\mu \partial_\nu \psi_\mu - m\partial_\nu \psi = 0 \ , \tag{13.19a}$$
$$i\partial_\nu \bar{\psi}_\mu \gamma^\mu + m\partial_\nu \bar{\psi} = 0 \ , \tag{13.19b}$$

which are the first spacetime derivatives of the field equations.

13.2 De Donder–Weyl Hamiltonian Formalism

The dual jet bundle $J^1\pi^*$ has local coordinates denoted as $(x^\mu, \psi, \bar{\psi}, \pi^\mu, \bar{\pi}^\mu)$. The Legendre transform $\mathscr{F}\mathscr{L} : J^1\pi \to J^1\pi^*$ satisfies,

$$\mathscr{F}\mathscr{L}^*\pi^\mu = \frac{\partial \mathscr{L}}{\partial \psi_\mu} = i\bar{\psi}\gamma^\mu \quad , \quad \mathscr{F}\mathscr{L}^*\bar{\pi}^\mu = \frac{\partial \mathscr{L}}{\partial \bar{\psi}_\mu} = 0 \ , \tag{13.20}$$

which defines the primary constraint submanifold $j_\circ : P_\circ \hookrightarrow J^1\pi^*$. Furthermore, the Hamiltonian function $\mathscr{H}_\circ = m\bar{\psi}\psi \in C^\infty(P_\circ)$ given by $E_{\mathscr{L}} = \mathscr{F}\mathscr{L}_\circ \mathscr{H}_\circ$ where $\mathscr{F}\mathscr{L} = j_\circ \circ \mathscr{F}\mathscr{L}_\circ$.

The Hamilton–Cartan forms are

$$\Theta^\circ_{\mathscr{H}} = i\bar{\psi}\gamma^\mu \mathrm{d}\psi \wedge \mathrm{d}^3x_\mu - m\bar{\psi}\psi \mathrm{d}^4x \ , \tag{13.21a}$$
$$\Omega^\circ_{\mathscr{H}} = -i\mathrm{d}\bar{\psi}\gamma^\mu \wedge \mathrm{d}\psi \wedge \mathrm{d}^3x_\mu + m\left(\bar{\psi}\mathrm{d}\psi + \mathrm{d}\bar{\psi}\psi\right) \wedge \mathrm{d}^4x \ . \tag{13.21b}$$

Using 4-multivector fields $\mathbf{X}_{\mathscr{H}} \in \mathfrak{X}^4(P_\circ)$ with components,

$$X_\nu = \partial_\nu + A_\nu \frac{\partial}{\partial \psi} + B_\nu \frac{\partial}{\partial \bar{\psi}} + C^\mu_\nu \frac{\partial}{\partial \pi^\mu} + D^\mu_\nu \frac{\partial}{\partial \bar{\pi}^\mu} \ , \tag{13.22}$$

to compute $i(\mathbf{X}_{\mathscr{H}})\Omega^\circ_{\mathscr{H}} = 0$ which gives

$$i\gamma^\mu A_\mu - m\psi = 0 \ , \tag{13.23a}$$
$$iB_\mu \gamma^\mu + m\bar{\psi} = 0 \ , \tag{13.23b}$$

from which no compatibility constraints arise. It follows that the final constraint submanifold in the De Donder–Weyl Hamiltonian formulation is the primary constraint submanifold $P_\circ \subset J^1\pi^*$. The field equations which solve the variational problem are written on integral sections of $\mathbf{X}_{\mathscr{H}}$ as,

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \ , \tag{13.24a}$$
$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \ , \tag{13.24b}$$

which are the same as in the Lagrangian setting (13.18).

The study of Noether symmetries in this setting follows along the same lines as in the other examples. The explicit calculation of the multimomentum maps is left for future work.

Part IV

Cohomological Field Theory

14 The Variational Bicomplex

In this section we give a brief review of some key aspects of the variational complex. For a full presentation of the variational complex, see [6, 8]. The relation between the variational bicomplex and multisymplectic geometry is explained in [23].

Recall from Section 1.4 that the exterior derivative on $J^\infty\pi$ naturally splits into horizontal and vertical parts as

$$\mathrm{d} = \mathrm{d_h} + \mathrm{d_v} \ . \tag{14.1}$$

Since $\mathrm{d}^2 = 0$, it follows that,

$$\mathrm{d_h^2} = 0 \quad , \quad \mathrm{d_h}\mathrm{d_v} + \mathrm{d_v}\mathrm{d_h} = 0 \quad , \quad \mathrm{d_v^2} = 0 \ . \tag{14.2}$$

Furthermore,

$$\mathrm{d_h}(\mathrm{d}x^\mu) = 0 = \mathrm{d_v}(\mathrm{d}x^\mu) \quad , \quad \mathrm{d_h}\theta^A_{\mathbf{J}} = \mathrm{d}x^\mu \wedge \theta^A_{\mu\mathbf{J}} \quad , \quad \mathrm{d_v}\theta^A_{\mathbf{J}} = 0 \ . \tag{14.3}$$

In the same way, an arbitrary differential form $\omega \in J^\infty\pi$ can also be split up into its horizontal and vertical parts as,

$$\omega = \omega^{p,q} + \omega^{p-1,q+1} + \cdots + \omega^{0,p+q} \ . \tag{14.4}$$

The space of differential forms on $J^\infty\pi$ with p horizontal components and q vertical components will be denoted $\Omega^{p,q}(J^\infty\pi)$. It follows that,

$$\Omega^k(J^\infty\pi) = \bigoplus_{p+q=k} \Omega^{p,q}(J^\infty\pi) \ , \tag{14.5}$$

and

$$\mathrm{d_h} : \Omega^{p,q} \to \Omega^{p+1,q} \quad , \quad \mathrm{d_v} : \Omega^{p,q} \to \Omega^{p,q+1} \quad . \tag{14.6}$$

Furthermore, differential forms on the base space M pull back via the projection map π^∞ to horizontal forms on $J^\infty\pi$ as,

$$\Omega^{p,0}(J^\infty\pi) = (\pi^\infty)^* \, \Omega^p(M) \ . \tag{14.7}$$

We also have the following projection map,

$$\pi^{p,q} : \Omega^k(J^\infty\pi) \to \Omega^{p,q}(J^\infty\pi) \ , \tag{14.8}$$

which is computed by substituting $\mathrm{d}y^i_{\mathbf{J}} = \theta^i_{\mathbf{J}} + y^i_{\mu\mathbf{J}}\mathrm{d}x^\mu$ from the definition of the contact forms.

Not that Lagrangian forms are maximal horizontal forms as $\mathbf{L} \in \Omega^{m,0}(J^\infty\pi)$. Now, the Poincaré–Cartan form $\Theta_{\mathscr{L}} \in \Omega^m(J^1\pi)$ for a first order field theory can be lifted to $J^\infty\pi$ as $\Theta_{\mathscr{L}} = (\pi^\infty_1)^*\Theta_{\mathscr{L}} \in \Omega^m(J^\infty\pi)$. Recalling the local expression (1.24) for the Poincaré–Cartan form, it follows that

$$\Theta_{\mathscr{L}} = \mathbf{L} + \theta_{\mathscr{L}} \ , \tag{14.9}$$

where, in the case of first-order field theories, $\theta_{\mathscr{L}}$ is given as in equation (2.14):

$$\theta_{\mathscr{L}} = \frac{\partial \mathscr{L}}{\partial y^A_\mu}\theta^A \wedge \mathrm{d}^{m-1}x_\mu = \frac{\partial \mathscr{L}}{\partial y^A_\mu}\mathrm{d_v}y^A \wedge \mathrm{d}^{m-1}x_\mu \in \Omega^{m-1,1}(J^\infty\pi) \ . \tag{14.10}$$

This is the same boundary term as in Remark 2.13 from which we recall that

$$\mathrm{d_v}\mathbf{L} = \hat{E}(\mathbf{L}) - \mathrm{d_h}\theta_{\mathscr{L}} \ . \tag{14.11}$$

The expression above, which is given on $J^\infty\pi$, is in direct correspondence with the variation of the Lagrangian on the space of holonomic sections $\mathcal{J}^\infty\pi$ as

$$\delta\mathscr{L}\mathrm{d}^m x = \frac{\delta\mathscr{L}}{\delta\phi^A}\mathrm{d}^m x + \partial_\mu\theta^\mu\mathrm{d}^m x \ , \tag{14.12}$$

where we are now using the slight abuse of notation $\mathscr{L} = \mathscr{L}\circ j^\infty\phi$ and we remind the reader that $\dfrac{\delta\mathscr{L}}{\delta\phi^A}$ is the Euler–Lagrange derivative (1.46) on $\mathcal{J}^\infty\pi$. Also recall that the boundary term above, which is in direct correspondence with $\theta_{\mathscr{L}}$, yields presymplectic potential (6.3) of the infinite-dimensional covariant phase space formalism upon performing integration over a Cauchy surface $\Sigma\subset M$. Furthermore, we can define

$$\omega_{\mathscr{L}} \equiv -\mathrm{d}_\mathrm{v}\theta_{\mathscr{L}} \in \varOmega^{m-1,2}(J^\infty\pi) \ , \tag{14.13}$$

which is, in the same sense, in direct correspondence with the presymplectic form (6.4) of the covariant phase space formalism. It is straightforward to show that

$$\Omega_{\mathscr{L}} = \omega_{\mathscr{L}} - \hat{E}(\mathbf{L}) \ . \tag{14.14}$$

It follows that there is a direct correspondence between the operator δ on $\mathcal{J}^\infty\pi$ and the vertical derivative d_v on $J^\infty\pi$. We also have a correspondence between the horizontal derivative d_h on $J^\infty\pi$ and the exterior derivative on M, which acts naturally on $\mathcal{J}^\infty\pi$ as in equation (1.42). This allows us to pass from the language of functionals to the language of differential forms. We are then able to study field theories in cohomological terms on the *variational bicomplex* constructed from the horizontal and vertical derivatives on $J^\infty\pi$.

The vertical exterior derivative $\mathrm{d}_\mathrm{v}:\Omega^{p,q}\to\Omega^{p,q+1}$ provides us with the ***vertical complex***:

$$\Omega^{\bullet,0}\xrightarrow{\mathrm{d}_v}\Omega^{\bullet,1}\xrightarrow{\mathrm{d}_v}\cdots$$

from which we construct the ***vertical cohomology***:

$$H^q\left(\mathrm{d}_\mathrm{v},\varOmega^{\bullet,q}(J^\infty\pi)\right) = \frac{\mathrm{Ker}\ \mathrm{d}_\mathrm{v}}{\mathrm{Im}\ \mathrm{d}_\mathrm{v}} \ . \tag{14.15}$$

Similarly, the horizontal exterior derivative $\mathrm{d}_\mathrm{h}:\Omega^{p,q}\to\Omega^{p+1,q}$ provides us with the ***horizontal complex***:

$$0\longrightarrow\mathbb{R}\longrightarrow\Omega^{0,0}\xrightarrow{\mathrm{d}_\mathrm{h}}\Omega^{1,0}\xrightarrow{\mathrm{d}_\mathrm{h}}\cdots\xrightarrow{\mathrm{d}_\mathrm{h}}\Omega^{m,0}\longrightarrow 0\ , \tag{14.16}$$

$$0\longrightarrow\Omega^{0,q}\xrightarrow{\mathrm{d}_\mathrm{h}}\Omega^{1,q}\xrightarrow{\mathrm{d}_\mathrm{h}}\cdots\xrightarrow{\mathrm{d}_\mathrm{h}}\Omega^{m,q}\longrightarrow 0\ ,\quad (q{>}0)\ , \tag{14.17}$$

and we then have the ***horizontal cohomology***:

$$H^p\left(\mathrm{d}_\mathrm{h},\varOmega^{p,\bullet}(J^\infty\pi)\right) = \frac{\mathrm{Ker}\ \mathrm{d}_\mathrm{h}}{\mathrm{Im}\ \mathrm{d}_\mathrm{h}} \ . \tag{14.18}$$

In Section 15.2 we will discuss the on-shell horizontal cohomology on $\mathcal{J}^k\pi$ known as the *characteristic cohomology* which classifies the conserved currents associated with Noether symmetries.

Now define the ***interior Euler operator*** $\mathcal{I}:\Omega^{p,q}(J^\infty\pi)\to\Omega^{p,q}(J^\infty\pi)$ as,

$$\mathcal{I}(\omega) = \frac{1}{q}\theta^A\wedge(-1)^{|\mathbf{J}|}\partial_\mathbf{J}\left[i\left(\frac{\partial}{\partial y^A_\mathbf{J}}\right)\omega\right]\quad,\quad \omega\in\Omega^{p,q}(J^\infty\pi)\ . \tag{14.19}$$

It follows that $\mathcal{I} \circ \mathrm{d_h} = 0$ as $\mathcal{I}^2 = \mathcal{I}$ and $\mathrm{Ker}(\mathcal{I}) = \mathrm{Im}(\mathrm{d_h})$. The space of ***functional forms*** $\mathcal{F}$ is defined as $\mathcal{F}^q = \mathcal{I}\left(\Omega^{p,q}(J^\infty\pi)\right)$ so that $\mathcal{I} : \Omega^{\bullet,q}(J^\infty\pi) \to \mathcal{F}^q$. The Euler–Lagrange operator $\hat{E}$ defined in equation (1.47) is given as,

$$\hat{E} = \mathcal{I} \circ \mathrm{d_v} : \Omega^{m,0}(J^\infty\pi) \to \mathcal{F}^1 \ . \tag{14.20}$$

A Lagrangian $\mathbf{L}$ is called ***(variationally) trivial*** if,

$$\hat{E}(\mathbf{L}) = 0 \ , \tag{14.21}$$

everywhere on $J^\infty\pi$. Furthermore, a Langrangian is called an ***exact Lagrangian***

$$\mathbf{L} = \mathrm{d_h}\eta \ , \quad \text{for some } \eta \in \Omega^{m-1,0} \ . \tag{14.22}$$

Note that exact Lagrangians are always trivial while the converse is not necessarily true [6]. The cohomology $H^m\left(\Omega^{m,0}\right)$ is of special importance:

$$H^m\left(\Omega^{m,0}\right) = \frac{\mathrm{Ker}\{\hat{E} : \Omega^{m,0} \to \Omega^{m+1,0}\}}{\mathrm{Im}\{\mathrm{d_h} : \Omega^{m-1,0} \to \Omega^{m,0}\}} = \frac{\{\text{trivial Lagrangians}\}}{\{\text{exact Lagrangians}\}} \ . \tag{14.23}$$

Furthermore, we have the following vertical differential

$$\delta_\mathrm{v} \equiv \mathcal{I} \circ \mathrm{d_v} : \mathcal{F}^q \to \mathcal{F}^{q+1} \ . \tag{14.24}$$

The differential map $\delta_\mathrm{v} : \mathcal{F}^1 \to \mathcal{F}^2$ is called the ***Helmholtz operator*** in the inverse problem of the calculus of variations. Using the spaces of functional forms $\mathcal{F}^q$ and the operator δ_v, we can then be used to augment the horizontal complex to obtain the ***Euler–Lagrange complex***:

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^{0,0} \xrightarrow{\mathrm{d_h}} \Omega^{1,0} \xrightarrow{\mathrm{d_h}} \cdots \Omega^{m,0} \xrightarrow{\hat{E}} \mathcal{F}^1 \xrightarrow{\delta_\mathrm{v}} \mathcal{F}^2 \xrightarrow{\delta_\mathrm{v}} \cdots$$

which can be added as an augmentation to the variational bicomplex.

The full variational bicomplex including the Euler–Lagrange complex is depicted in the following diagram.

$$\begin{tikzcd}
& \vdots & \vdots & & \vdots & \vdots & \\
0 \arrow[r] & \Omega^{0,2} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathrm{d_h}"] & \Omega^{1,2} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathrm{d_h}"] & \cdots \arrow[r, "\mathrm{d_h}"] & \Omega^{m,2} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathcal{I}"] & \mathcal{F}^2 \arrow[u, "\delta_\mathrm{v}"] \arrow[r] & 0 \\
0 \arrow[r] & \Omega^{0,1} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathrm{d_h}"] & \Omega^{1,1} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathrm{d_h}"] & \cdots \arrow[r, "\mathrm{d_h}"] & \Omega^{m,1} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathcal{I}"] & \mathcal{F}^1 \arrow[u, "\delta_\mathrm{v}"] \arrow[r] & 0 \\
0 \arrow[r] \ \mathbb{R} \arrow[r] & \Omega^{0,0} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathrm{d_h}"] & \Omega^{1,0} \arrow[u, "\mathrm{d_v}"] \arrow[r, "\mathrm{d_h}"] & \cdots \arrow[r, "\mathrm{d_h}"] & \Omega^{m,0} \arrow[u, "\mathrm{d_v}"] \arrow[ur, "\hat{E}"'] & &
\end{tikzcd}$$

We have shown how the vertical derivative $\mathrm{d_v}$ classifies solutions to the variational problem in the variational bicomplex. We also wish to control control gauge symmetries in similar terms by cohomological classification. The horizontal complex classifies the multimomentum maps associated with symmetries that are infinitesimally generated by evolutionary vector fields (5.10) as follows.

Suppose we have an evolutionary vector field $X_\xi \in \mathfrak{X}^{V(\pi^\infty)}(J^\infty\pi)$ which generates a Noether symmetry. Since evolutionary vector fields are π^∞-vertical and $\mathbf{L} \in \Omega^{m,0}(J^\infty\pi)$, it follows that $i(X_\xi)\mathbf{L} = 0$ and $i(X_\xi)\mathrm{d_h}\mathbf{L} = 0$. Then,

$$\mathrm{L}(X_\xi)\mathbf{L} = i(X_\xi)\mathrm{d_v}\mathbf{L} \ . \tag{14.25}$$

Now, by applying equation (14.11), obtain

$$\mathrm{L}(X_\xi)\mathbf{L} = i(X_\xi)\hat{E}(\mathbf{L}) - i(X_\xi)\mathrm{d_h}\theta_{\mathscr{L}} \ . \tag{14.26}$$

Furthermore, since X_ξ is evolutionary, it follows that $\mathrm{d_h}$ anti-commutes with the inner product $i(X_\xi)$, thus

$$\mathrm{L}(X_\xi)\mathbf{L} = i(X_\xi)\hat{E}(\mathbf{L}) + \mathrm{d_h}i(X_\xi)\theta_{\mathscr{L}} \ . \tag{14.27}$$

This expression can be rewritten in terms of the Poincaré–Cartan m-form by using equation (14.9) as

$$\mathrm{L}(X_\xi)\mathbf{L} = i(X_\xi)\hat{E}(\mathbf{L}) + \mathrm{d_h}i(X_\xi)\Theta_{\mathscr{L}} \ . \tag{14.28}$$

Now since X_ξ generates a Noether symmetry, it follows that $\mathrm{L}(X_\xi)\mathbf{L} = \mathrm{d_h}\beta_\xi$ for some $\beta_\xi \in \Omega^{m-1,0}(J^\infty\pi)$. Consequently, the multimomentum map $\in \Omega^{m-1}(J^\infty\pi)$ associated with X_ξ satisfies

$$\mathrm{d_h}J_\xi = i(X_\xi)\hat{E}(\mathbf{L}) \ , \tag{14.29}$$

so J_ξ is thereby conserved on the stationary surface $\Sigma \subset J^\infty\pi$:

$$\mathrm{d_h}J_\xi \underset{\Sigma}{=} 0 \ . \tag{14.30}$$

As we will discuss in Section 15.2, Noether currents are also classified in cohomological terms on the stationary surface $\Sigma_\phi \subset \mathcal{J}^k\pi$, which can be lifted to $\mathcal{J}^\infty\pi$. The calculation just given thereby puts multimomentum maps associated with evolutionary vector fields in direct correspondence with the characteristic cohomology of Noether currents.

However, there is no natural classification in cohomological terms of solutions to the field equations modulo gauge transformations in the variational bicomplex. Moreover, the variational bicomplex cannot control more intricate gauge structures such as those with reducible or open algebras. In order to fully control these situations which the variational bicomplex does not address, we need to use the *BRST–BV (antifield) formalism*, which we discuss in the following chapter.

15 The BRST–BV Antifield Formalism

This section follows [12] and [65] rather closely (see also [22, 84]). For a succinct presentation of the BRST–BV antifield formalism in the canonical Hamiltonian setting, see [13]. We also make a few notation changes: now M is taken to be D-dimensional and the natural coordinates of $J^1\pi$ are now (x^μ, y^i, y^i_μ).

We now tun our attention to constructing the BRST differential and its cohomology in the BRST–BV (antifield) formalism which is the most general version of the BRST framework. The BRST cohomology is constructed in classical field theory to find the *classical observables*, which are constructed from gauge invariant functions on the stationary surface Σ_ϕ. The BRST differential is a coboundary operator on a $\mathbb{Z}$-graded vector space with grading called ***(total) ghost number***. More specifically, the BRST differential is constructed on $J^k\pi$ with $\mathbb{Z}$-grading, while the BRST cohomology is constructed from the application of the BRST differential to sections of the jet bundle (i.e. to fields and the derivatives of the fields). For work on graded jet manifolds see, e.g., [119, 120, 135]; for details on anomalies and the relation between the BRST formalism and the variational bicomplex, see e.g., [128, 137].

The BRST differential is given as the sum of two differentials, δ known as the ***Koszul–Tate differential***, and γ known as the ***longitudinal derivative***. The differential δ is a boundary operator on a $\mathbb{N}$-graded space with grading called ***antifield*** (or ***antighost***) ***number*** while γ is a coboundary operator on a $\mathbb{N}$-graded space with grading called ***pure ghost number***. As $s = \delta + \gamma$, the total ghost number is given as,

$$gh = puregh - antifd \ . \tag{15.1}$$

The BRST symmetry arises from enlarging the configuration space E with ghost fields and antifields. Given a field theory with a gauge symmetry group that is L^{th}-stage reducible, the set of fields $\{\phi^i\}$ is augmented by ghosts fields so that the new set of fields is,

$$\Phi^A = \left\{\phi^i, C^{\alpha_s} | s = 0, \ldots, L;\ \alpha_s = 1, \ldots, m_s\right\} \ , \tag{15.2}$$

where m_s denotes the number of independent generators at the s^{th} stage. The corresponding antifields will be denoted as Φ^*_A. The $\mathbb{Z}_2$-grading which determines the parity (bosonic/fermionic statistics) of each antifield is given as $\epsilon(\Phi^*_A) = \epsilon(\Phi^A) + 1$ and the ghost number is given as $\text{gh}(\Phi^*_A) = -\text{gh}(\Phi^A) - 1$. The various gradings of the fields and antifields are listed in the following table.

field/antifield	ϕ^i	C^{α_s}	ϕ^*_i	$C^*_{\alpha_s}$
puregh	0	$s+1$	0	0
antifd	0	0	1	$s+2$
gh	0	$s+1$	-1	$-(s+2)$
ϵ	$\epsilon(\phi^i)$	$\epsilon(\varepsilon^{\alpha_s})+s+1$	$\epsilon(\phi^i)+1$	$\epsilon(\varepsilon^{\alpha_s})+s$

As we will soon discuss, in order to construct the BRST symmetry, it is necessary to appropriately augment the original Lagrangian $\mathscr{L}_\circ \in C^\infty(J^k\pi)$ of the theory under investigation. First, we discuss how to construct the BRST differential for irreducible gauge theories.

15.1 The Structure of Gauge Transformations on $\mathcal{J}^k\pi$

Much of the discussion in this section can be found in [65] and [84].

Gauge transformations are, in general, produced on the space of holonomic sections $\mathcal{J}^k\pi$ as,

$$\delta_\varepsilon \phi^i(x) = R^i{}_\alpha \varepsilon^\alpha \equiv \int_M \mathrm{d}^D x' R^i{}_\alpha(x, x')\varepsilon^\alpha(x') \ , \tag{15.3}$$

where the ε^α are the gauge parameters and the $R^i{}_\alpha$ are the generators of the gauge transformations which are functionals with $\alpha \in \{1, \ldots, m\}$. On $J^k\pi$,

$$R^i{}_\alpha(x, x') = \bar{R}^i{}_\alpha \delta(x, x') + \bar{R}^{i\mu}_\alpha \partial_\mu \delta(x, x') + \ldots + \bar{R}^{i\mu_1\ldots\mu_k}_\alpha \partial_{\mu_1\ldots\mu_k}\delta(x, x') \ , \tag{15.4}$$

so that the gauge transformations (15.3) on $J^k\pi$ are written explicitly as,

$$\delta_\varepsilon \phi^i(x) = R^i{}_\alpha \varepsilon^\alpha + R^{i\mu}_\alpha \partial_\mu \varepsilon^\alpha + \ldots + R^{i\mu_1\ldots\mu_k}_\alpha \partial_{\mu_1\ldots\mu_k}\varepsilon^\alpha \ , \tag{15.5}$$

where integration over M is implied as in equation (15.3). When the gauge parameters are bosonic, $\epsilon(\varepsilon^\alpha) = 0$, the gauge symmetry is a standard bosonic symmetry; conversely, when the gauge parameters are fermionic, $\epsilon(\varepsilon^\alpha) = 1$, the gauge symmetry is a *supersymmetry*. The parity of the gauge generators is given as $\epsilon(R^i{}_\alpha) = \epsilon(\phi^i) + \epsilon(\varepsilon^\alpha)$. Furthermore, $\delta_\varepsilon \partial_{\mu_1\ldots\mu_k}\phi^i = \partial_{\mu_1\ldots\mu_k}\delta_\varepsilon\phi^i$, using the total derivative ∂_μ on the space of sections $\mathcal{J}^k\pi$ as in equation (1.42). The action transforms under the gauge transformations as

$$\delta_\varepsilon S_\circ = \int_M \mathrm{d}^D x \left[\frac{\delta \mathscr{L}_\circ}{\delta \phi^i}\delta_\varepsilon \phi^i + \partial_\mu \theta^\mu_\varepsilon\right] \ , \tag{15.6}$$

where $\delta\mathscr{L}_\circ/\delta\phi^i$ are the Euler–Lagrange equations on $\mathcal{J}^k\pi$ as before and the second term is the boundary term. As usual, a gauge transformation produces a ***gauge symmetry*** when $\delta_\varepsilon S_\circ = 0$, which occurs when,

$$\delta_\varepsilon\left(\mathscr{L}_\circ \circ j^k\phi\right) = \partial_\mu \mathcal{F}^\mu \ , \tag{15.7}$$

for some $\mathcal{F}^\mu(\phi) \in C^\infty(\mathcal{J}^k\pi)$. It follows that there exists a ***Noether current***,

$$\mathcal{J}^\mu_\varepsilon = \frac{\partial\mathscr{L}_\circ}{\partial\phi^i_\mu}\delta_\varepsilon\phi^i - \mathcal{F}^\mu \ , \tag{15.8}$$

which is conserved on-shell:

$$\partial_\mu\mathcal{J}^\mu_\varepsilon \underset{\Sigma_\phi}{=} 0 \ , \tag{15.9}$$

recalling the definition (1.50) of the stationary surface Σ_ϕ; this is the statement of ***Noether's first theorem***. We will see in Section 15.2 how Noether currents are classified by the *characteristic cohomology* of degree $D-1$. Furthermore, by imposing appropriate boundary conditions, we can extend the analysis off-shell so that,

$$\delta_\varepsilon S_\circ = 0 \ \Rightarrow \ \frac{\delta\mathscr{L}_\circ}{\delta\phi^i}\delta_\varepsilon\phi^i = \frac{\delta\mathscr{L}_\circ}{\delta\phi^i}R^i{}_\alpha\varepsilon^\alpha = 0 \ , \tag{15.10}$$

which provides us with the ***Noether identities***:

$$\frac{\delta\mathscr{L}_\circ}{\delta\phi^i}R^i{}_\alpha = 0 \ ; \tag{15.11}$$

this is the statement of ***Noether's second theorem***.

Now, in order to study the behavior of local functions and local functionals on the stationary surfaces $\Sigma \subset J^k\pi$ and $\Sigma_\phi \subset \mathcal{J}^k\pi$, it is necessary to impose appropriate *regularity conditions* which we now briefly discuss (see e.g., [65] for more details). Note that if a local function $f \in C^\infty(J^k\pi)$ vanishes on the stationary surface Σ, then it can be written as a linear combination of the Euler–Lagrange equations and their total derivatives:

$$f|_\Sigma = 0 \quad \Leftrightarrow \quad f = \lambda^{i\mathbf{J}}\partial_{\mathbf{J}}\left(\frac{\delta\mathscr{L}_\circ}{\delta y^i}\right) \ , \quad \text{for some } \lambda^{i\mathbf{J}} \in C^\infty(J^k\pi) \ . \tag{15.12}$$

The ***regularity condition*** for the stationary surface Σ_ϕ is given as

$$F(\phi)|_{\Sigma_\phi} = 0 \ \Rightarrow \ F(\phi) = \int_M \frac{\delta\mathscr{L}_\circ}{\delta\phi^i}\lambda^i \ , \tag{15.13}$$

for some $\lambda^i = \lambda^i(\phi)$. Assuming that the regularity condition for the stationary surface Σ_ϕ is satisfied, solutions to the Noether identities form a complete (generating) set:

$$\delta\phi^i = R^i{}_\alpha\lambda^\alpha + \frac{\delta\mathscr{L}_\circ}{\delta\phi^j}T^{ji} \ , \tag{15.14}$$

where $T^{ij} = -(-1)^{\epsilon(\phi^i)\epsilon(\phi^j)}T^{ji}$.

Moreover, gauge transformations do not, in general, form a Lie algebra. The action by two gauge transformations of some group algebra satisfies,

$$[\delta_1,\delta_2]\phi^i = R^i{}_\gamma f^\gamma_{\alpha\beta}\varepsilon^\beta_1\varepsilon^\alpha_2 - \frac{\delta\mathscr{L}_\circ}{\delta\phi^j}E^{ji}_{\alpha\beta}\varepsilon^\beta_1\varepsilon^\alpha_2 \ , \tag{15.15}$$

where, in general, $f^\gamma_{\alpha\beta} = f^\gamma_{\alpha\beta}(\phi)$ and $E^{ji}_{\alpha\beta} = E^{ji}_{\alpha\beta}(\phi)$ are gauge structure tensors. The gauge algebra is called a ***closed algebra*** if $E^{ji}_{\alpha\beta} = 0$ (i.e. the algebra *closes* off-shell); if $E^{ji}_{\alpha\beta} \neq 0$ (i.e. the algebra does not *close* off-shell), then the gauge algebra is called an ***open algebra***. In order for a gauge algebra to be a Lie algebra, the algebra must be closed, the gauge structure tensors $f^{\gamma}_{\alpha\beta}$ must be constants (and are called *structure constants*), and the Jacobi identity must be satisfied:

$$[\delta_1, [\delta_2, \delta_3]] + [\delta_3, [\delta_1, \delta_2]] + [\delta_2, [\delta_3, \delta_1]] = 0 \ . \tag{15.16}$$

A gauge theory is called ***irreducible*** when the generators $R^i{}_\alpha$ are full-rank on-shell, and hence, all of the gauge transformations (15.3) are linearly independent. Then,

$$\operatorname{rank}\left(R^i{}_\alpha\big|_\Sigma\right) = m \ , \tag{15.17}$$

where m is the number of gauge transformations in the theory (i.e. the number of gauge degrees of freedom), recalling that $\alpha \in \{1, \ldots, m\}$. Then,

$$\operatorname{rank}\left(\frac{\partial_L \partial_R \mathscr{L}_\circ}{\partial\phi^i \partial\phi^j}\bigg|_\Sigma\right) = n - m = n_{\text{dof}} \ , \tag{15.18}$$

recalling that n is the number of fields in the theory under investigation.

A gauge theory with gauge transformations $\delta_\varepsilon \phi^A = R^i{}_{\alpha_0}\varepsilon^{\alpha_0}$ (where $\alpha_0 \in \{1, \ldots, m_0\}$) satisfying the Noether identities $\dfrac{\delta\mathscr{L}_\circ}{\delta\phi^A} R^i{}_{\alpha_0} = 0$ is called ***reducible*** when the gauge generators $R^i{}_{\alpha_0}$ are not full-rank on-shell, and hence $\operatorname{rank}\left(R^i{}_{\alpha_0}\big|_\Sigma\right) < m_0$, so that the gauge transformations are not all linearly independent. A gauge theory is ***first-stage reducible*** if $m_0 - m_1$ of the $R^i{}_{\alpha_0}$ are independent on-shell and there are m_1 on-shell independent generators $R^{\alpha_0}_{\alpha_1}$,

$$\operatorname{rank}\left(R^{\alpha_0}_{\alpha_1}\big|_\Sigma\right) = m_1 \quad , \quad \alpha_1 \in \{1, \ldots, m_1\} \ , \tag{15.19}$$

where $\epsilon\left(R^{\alpha_0}_{\alpha_1}\right) = \epsilon(\varepsilon^{\alpha_0}) + \epsilon(\varepsilon^{\alpha_1})$ and the ε^{α_1} are the first-level gauge parameters. It follows that,

$$R^i{}_{\alpha_0} R^{\alpha_0}_{\alpha_1} = \frac{\delta\mathscr{L}_\circ}{\delta\phi^j} V^{ji}_{\alpha_1} \ , \tag{15.20}$$

for some $V^{ij}_{\alpha_1} = -(-1)^{\epsilon(\phi^A)\epsilon(\phi^j)} V^{ji}_{\alpha_1}$. The equation above indicates that the functionals $R^{\alpha_0}_{\alpha_1}$ are the on-shell null vectors of the $R^i{}_{\alpha_0}$ so that $R^i{}_{\alpha_0} R^{\alpha_0}_{\alpha_1}\big|_\Sigma = 0$. Now we have that,

$$\operatorname{rank}\left(\frac{\partial_L \partial_R \mathscr{L}_\circ}{\partial\phi^i \partial\phi^j}\bigg|_\Sigma\right) = n - m_0 + m_1 = n_{\text{dof}} \ . \tag{15.21}$$

A gauge theory is ***L^{th}-stage reducible*** if there exist functionals $R^{\alpha_{s-1}}_{\alpha_s}$ with $\alpha_s \in \{1, \ldots, m_s\}$, we have

$$R^{\alpha_{s-2}}_{\alpha_{s-1}} R^{\alpha_{s-1}}_{\alpha_s} = \frac{\delta\mathscr{L}_\circ}{\delta\phi^i} V^{i\alpha_{s-2}}_{\alpha_s} \quad , \quad s \in \{1, \ldots, L\} \ , \tag{15.22}$$

$$\operatorname{rank}\left(R^{\alpha_{s-1}}_{\alpha_s}\big|_\Sigma\right) = \sum_{l=s}^{L} (-1)^{l-s} m_l \quad , \quad s \in \{0, \ldots, L\} \ , \tag{15.23}$$

$$\operatorname{rank}\left(\frac{\partial_L \partial_R \mathscr{L}_\circ}{\partial\phi^i \partial\phi^j}\bigg|_\Sigma\right) = n - \sum_{s=0}^{L} (-1)^s m_s = n_{\text{dof}} \ . \tag{15.24}$$

Now, we make some identifications between the structures on $\mathscr{J}^k\pi$ and $J^k\pi$. First, the gauge transformations (15.3) yield,

$$\mathrm{L}(X_\varepsilon) y^i = R^i{}_\alpha \varepsilon^\alpha \quad \leftrightarrow \quad \delta_\varepsilon \phi^i(x) = \int_M \left(R^i{}_\alpha \varepsilon^\alpha\right) \circ j^k\phi \ , \tag{15.25}$$

and hence $X_\varepsilon \in \mathfrak{X}(J^k\pi)$ is given as,

$$X_\varepsilon = \partial_{\mathbf{J}} \left(R^i{}_\alpha \varepsilon^\alpha \right) \frac{\partial}{\partial y^i_{\mathbf{J}}} \ . \tag{15.26}$$

If X_ε generates a gauge symmetry on $J^k\pi$, then we make the following identification with equation (15.7):

$$\delta_\varepsilon \left(\mathscr{L}_\circ \circ j^k\phi \right) = \partial_\mu \mathcal{F}^\mu(\phi) = \partial_\mu \left(f^\mu \circ j^k\phi \right) \quad \leftrightarrow \quad \mathrm{L}(X_\varepsilon)\mathscr{L}_\circ = \partial_\mu f^\mu \tag{15.27}$$

Moreover, the identification with he Noether identities (15.11) is given as

$$\frac{\delta \mathscr{L}_\circ}{\delta y^i} R^i{}_\alpha = 0 \quad \leftrightarrow \quad \int_M \frac{\delta \mathscr{L}_\circ}{\delta \phi^i} R^i{}_\alpha(\phi) = \int_M \left(\frac{\delta \mathscr{L}_\circ}{\delta y^i} R^i{}_\alpha \right) \circ j^k\phi = 0 \ . \tag{15.28}$$

The following correspondence between the multi-Hessians is also made:

$$\left. \frac{\delta^2 \mathscr{L}_\circ}{\delta y^i \delta y^j} \right|_\Sigma \quad \leftrightarrow \quad \left. \frac{\delta^2 \mathscr{L}_\circ}{\delta \phi^i \delta \phi^j} \right|_{\Sigma_\phi} \ . \tag{15.29}$$

These identifications allow us to pass from the infinite-dimensional setting given on the space of holonomic sections $\mathscr{J}^k\pi$, to the finite-dimensional setting on $J^k\pi$.

15.2 Characteristic Cohomology of Conserved Currents

We now wish to recall the horizontal cohomology of the variational bicomplex. When the cocycle and coboundary conditions weakly satisfied,

$$\begin{aligned} \mathrm{d_h}\omega_h &\underset{\Sigma}{=} 0 \quad \text{(weak cocycle condition)} \ , \\ \omega_h &\underset{\Sigma}{=} \mathrm{d_h}\eta \quad \text{(weak coboundary condition)} \ , \end{aligned} \tag{15.30}$$

we obtain,

$$H^\bullet \left(\mathrm{d_h}, \Omega^{\bullet,0}(\Sigma) \right) = \frac{\mathrm{Ker}\ \mathrm{d_h}|_\Sigma}{\mathrm{Im}\ \mathrm{d_h}|_\Sigma} \ , \tag{15.31}$$

which is known as the ***characteristic cohomology***. We remind the reader that the multimomentum maps associated with Noether symmetries generated by evolutionary vector fields are classified by the characteristic cohomology, as seen in equation (14.30).

The characteristic cohomology can be extended by the Koszul–Tate differential δ as

$$H^p(\mathrm{d_h}, \Omega^{p,0}(\Sigma)) \cong H^p_0(\mathrm{d_h}|\delta) \ , \tag{15.32}$$

where the subscript in the extended cohomology $H^p_0(\mathrm{d_h}|\delta)$ indicates the antifield number and now the superscript indicates the local form degree on all of $J^k\pi$ (i.e. $\omega^p \in \Omega^{p,0}(J^k\pi)$). Local forms which vanish on-shell can be written as a linear combination of the equations of motion (similarly to equation (15.12)). Then,

$$\mathrm{d_h}\omega^p_0 \underset{\Sigma}{=} 0 \quad \Rightarrow \quad \mathrm{d_h}\omega^p_0 = \lambda^{i\mathbf{J}} \partial_{\mathbf{J}} \left(\frac{\delta \mathscr{L}_\circ}{\delta y^i} \right) \ . \tag{15.33}$$

Furthermore, since $\delta y^*_i = \delta \mathscr{L}_\circ / \delta y^i$ we have

$$\omega_1 = -\lambda^{i\mathbf{J}} y^*_i \quad \Rightarrow \quad \delta\omega_1 = -\lambda^{i\mathbf{J}} \delta y^*_{i\mathbf{J}} = -\lambda^{i\mathbf{J}} \partial_{\mathbf{J}} (\delta y^*_i) = -\lambda^{i\mathbf{J}} \partial_{\mathbf{J}} \left(\frac{\delta \mathscr{L}_\circ}{\delta y^i} \right) \ , \tag{15.34}$$

and it follows that the cocycle condition of $H^\bullet\left(\mathrm{d_h}, \Omega^{\bullet,0}(\Sigma)\right)$ in equation (15.30) is extended off-shell by δ, giving $\mathrm{d_h}\omega_0^p + \delta\omega_1^{p+1} = 0$; the same analysis can be performed for the coboundary condition in equation (15.30). Then, the cocycle and coboundary conditions for $H_0^p(\mathrm{d_h}|\delta)$ are given as,

$$\begin{aligned} &\mathrm{d_h}\omega_0^p + \delta\omega_1^{p+1} = 0 \quad \text{(cocycle condition)} \ , \\ &\omega_0^p = \mathrm{d_h}\eta_0^{p-1} + \delta\eta_1^p \quad \text{(coboundary condition)} \ . \end{aligned} \tag{15.35}$$

The characteristic cohomology of degree $D-1$ is of special importance as it classifies the conserved currents associated with Noether symmetries (see, e.g., [12]) as follows.

Given a Lagrangian system with a multimomentum map $J_\xi \in \Omega^{D-1}(J^k\pi)$, the ***Noether current***

$$\mathrm{j}_\xi = j^k\phi^* J_\xi \in \Omega^{D-1}(M) \ , \tag{15.36}$$

associated with a symmetry of the action $S_\circ$ vanishes on shell by ***Noether's theorem***:

$$\mathrm{d}\mathrm{j}_\xi \underset{\Sigma_\phi}{=} 0 \ . \tag{15.37}$$

On-shell exact Noether currents,

$$\mathrm{j}_\xi \underset{\Sigma_\phi}{=} \mathrm{d}\mathrm{k}_\xi \ , \quad \text{for some } \mathrm{k}_\xi \in \Omega^{D-2}(M) \ , \tag{15.38}$$

are called ***trivial***. The corresponding on–shell de Rham cohomology on M is obtained from pulling back by holonomic sections as,

$$j^k\phi^*\left(\mathrm{d_h}\alpha\right) = \mathrm{d}\left(j^k\phi^*\alpha\right) \ , \quad \forall\alpha \in \Omega^p(J^k\pi) \ , \tag{15.39}$$

so the Noether currents therefore belong to $H^{m-1}(\mathrm{d}, \Omega^{m-1}(\Sigma_\phi))$.

15.3 The Koszul–Tate Differential and the Longitudinal Derivative

The equations of motion and Noether identities are implemented in the BRST formalism by means of the Koszul–Tate differential δ, which is nilpotent $\delta^2 = \frac{1}{2}[\delta,\delta] = 0$ and is chosen such that $\delta(\mathrm{d}x^\mu) = 0$ and hence $\delta\partial_\mu = \partial_\mu\delta \Rightarrow \delta\mathrm{d} + \mathrm{d}\delta = 0$. Furthermore, δ is a boundary operator on the $\mathbb{N}$-graded algebra A which has grading called ***antifield number***.

On $J^k\pi$, the Koszul–Tate differential δ is given as,

$$\delta = -\partial_{\mathbf{J}}\left(\frac{\delta\mathscr{L}_\circ}{\delta y^i}\right)\frac{\partial}{\partial y^*_{i\mathbf{J}}} + \partial_{\mathbf{J}}\left(y_i^* R^i{}_\alpha\right)\frac{\partial}{\partial C^*_{\alpha\mathbf{J}}} \ . \tag{15.40}$$

The Koszul–Tate differential δ provides a *homological resolution* of the algebra A, which is a $\mathbb{N}$-graded differential algebra $\overline{A}$ such that:

$$\mathrm{antifd}(\delta) = -1 \quad ; \quad \delta(x) = 0 \ , \ \forall x \in \overline{A}_0 \tag{15.41}$$

whose homology is nontrivial only at degree (antifield number) zero and is isomorphic to the associative algebra A:

$$\begin{aligned} &H_k(\delta) = 0 \ , \quad k>0 \ ; \\ &H_0(\delta) = A \ . \end{aligned} \tag{15.42}$$

It follows that the homology of δ is given as,

$$H(\delta) \equiv \frac{\mathrm{Ker}(\delta)}{\mathrm{Im}(\delta)} = \bigoplus_k H_k = A \ . \tag{15.43}$$

Next, we construct the exterior derivative along the gauge orbits, known as the *longitudinal derivative*. Given an arbitrary local function $f \in C^\infty(J^k\pi)$, local forms $\alpha \in \Omega^p(J^k\pi)$ and $\beta \in \Omega^q(J^k\pi)$, and a gauge symmetry generated by an infinitesimal vector field $X \in \mathfrak{X}(J^k\pi)$, the ***longitudinal derivative*** γ is defined by the following properties [84]:

$$\begin{aligned} (i) &\quad (\gamma f)(X) = \mathrm{L}(X) f \ , \\ (ii) &\quad \gamma^2 = 0 \ , \\ (iii) &\quad \gamma(\alpha\beta) = \alpha\gamma\beta + (-1)^q(\gamma\alpha)\beta \ . \end{aligned} \tag{15.44}$$

Furthermore, γ has pure ghost grading puregh$(\gamma) = 1$, and

$$\gamma\partial_\mu = \partial_\mu\gamma \Longrightarrow \gamma\mathrm{d} + \mathrm{d}\gamma = 0 \ , \tag{15.45}$$

$$\delta\gamma + \gamma\delta = 0 \ . \tag{15.46}$$

The longitudinal derivative is given on $J^k\pi$ explicitly as,

$$\gamma = \partial_{\mathbf{J}}\left(R^i{}_\alpha C^\alpha\right)\frac{\partial}{\partial y^i_{\mathbf{J}}} + \partial_{\mathbf{J}}\left(\frac{1}{2}f^\alpha_{\beta\gamma}C^\gamma C^\beta\right)\frac{\partial}{\partial C^\alpha_{\mathbf{J}}} - \partial_{\mathbf{J}}\left(y^*_j R^j{}_{\alpha,i}C^\alpha\right)\frac{\partial}{\partial y^*_{i\mathbf{J}}} + \partial_{\mathbf{J}}\left(f^\gamma_{\alpha\beta}C^*_\gamma C^\beta\right)\frac{\partial}{\partial C^*_{\alpha\mathbf{J}}} \ , \tag{15.47}$$

where $R^j{}_{\alpha,i} \equiv \partial R^j{}_\alpha/\partial y^i$.

15.4 The BRST Differential and the Antibracket

As mentioned earlier, the BRST differential given as $s = \delta + \gamma$ is nilpotent $s^2 = 0$ and acts as a coboundary operator on the $\mathbb{Z}$-graded vector space whose grading is called total ghost number: $\mathrm{gh}(s) = +1$. Furthermore, s satisfies the graded Leibniz rule:

$$s(AB) = A(sB) + (-1)^{\epsilon_B}(sA)B \ , \tag{15.48}$$

where A and B are local functionals of the fields and antifields. The local expression for s on $J^k\pi$ is given as

$$\begin{aligned} s &= \partial_{\mathbf{J}}\left(s\Phi^A\right)\frac{\partial}{\partial\Phi^A_{\mathbf{J}}} + \partial_{\mathbf{J}}\left(s\Phi^*_A\right)\frac{\partial}{\partial\Phi^*_{A\mathbf{J}}} \\ &= \partial_{\mathbf{J}}\left(R^i{}_\alpha C^\alpha\right)\frac{\partial}{\partial y^i_{\mathbf{J}}} + \partial_{\mathbf{J}}\left(\frac{1}{2}f^\alpha_{\beta\gamma}C^\gamma C^\beta\right)\frac{\partial}{\partial C^\alpha_{\mathbf{J}}} \\ &\quad - \partial_{\mathbf{J}}\left(\frac{\delta\mathscr{L}_\circ}{\delta\phi^i} + y^*_j R^j{}_{\alpha,i}C^\alpha\right)\frac{\partial}{\partial y^*_{i\mathbf{J}}} + \partial_{\mathbf{J}}\left(y^*_i R^i{}_\alpha + f^\gamma_{\alpha\beta}C^*_\gamma C^\beta\right)\frac{\partial}{\partial C^*_{\alpha\mathbf{J}}} \ . \end{aligned} \tag{15.49}$$

Then, the local BRST cohomology is constructed to classify functionals. Given two local functionals $A = \int a = \int j^k\phi^*\alpha$ and $B = \int b = \int j^k\phi^*\beta$ such that

$$sA = 0 \quad \Rightarrow \quad sa + \mathrm{d}m = 0 \quad \text{(cocycle condition)} \ , \tag{15.50}$$

$$A = sB \quad \Rightarrow \quad a = sb \quad \text{(coboundary condition)} \ . \tag{15.51}$$

The cohomology of local functionals defined from the cocycle and coboundary conditions above is the ***BRST cohomology of local functionals***: $H(s|\mathrm{d})$. Furthermore,

$$\begin{aligned} sa &= 0 \quad \text{(cocycle condition)} \ , \\ a &= sb \quad \text{(coboundary condition)} \ , \end{aligned} \tag{15.52}$$

gives the ***BRST cohomology of local forms***: $H(s)$,

$$H(s) = \frac{\mathrm{Ker}\ s}{\mathrm{Im}\ s} \ . \tag{15.53}$$

This cohomology is also known as the *BRST cohomology of local functions*. Then (see e.g. [84]),

$$H^g(s) = \begin{cases} 0 \ , & g<0 \ , \\ H^g(\gamma) \ , & g \geq 0 \ . \end{cases} \tag{15.54}$$

Furthermore, the ***antibracket*** of two functionals X and Y is defined as,

$$(X,Y) \equiv \frac{\partial_R X}{\partial \Phi^A}\frac{\partial_L Y}{\partial \Phi^*_A} - \frac{\partial_R X}{\partial \Phi^*_A}\frac{\partial_L Y}{\partial \Phi^A} \ , \tag{15.55}$$

using the left and right derivatives (see Appendix E). It follows that,

$$\mathrm{gh}(X,Y) = \mathrm{gh}X + \mathrm{gh}Y + 1 \quad , \quad \epsilon((X,Y)) = \epsilon_X + \epsilon_Y + 1 \ . \tag{15.56}$$

Moreover,

$$\begin{aligned} &(X,Y) = -(-1)^{(\epsilon_X+1)(\epsilon_Y+1)}(Y,X) \ , \\ &((X,Y),Z) + (-1)^{(\epsilon_X+1)(\epsilon_Y+\epsilon_Z)}((Y,Z),X) + (-1)^{(\epsilon_Z+1)(\epsilon_X+\epsilon_Y)}((Z,X),Y) = 0 \ . \end{aligned} \tag{15.57}$$

The second equation above is the graded Jacobi identity; it follows that the antibracket is a graded Poisson bracket. For more properties of the antibracket, see Appendix F.

The fields and antifields are canonically conjugate with respect to the antibracket as,

$$(\Phi^A, \Phi^*_B) = \delta^A_B \ ; \tag{15.58}$$

see e.g. [65] for more details. The BRST transformation of some functional X is generated by the extended action functional S via the antibracket as,

$$sX = (X,S) \ . \tag{15.59}$$

Then, the BRST transformations of the fields and antifields are given explicitely as,

$$s\Phi^A = (-1)^{\epsilon_A}(\Phi^A, S) = \frac{\partial_L S}{\partial \Phi^*_A} \quad , \quad s\Phi^*_A = (-1)^{\epsilon_A+1}(\Phi^*_A, S) = -\frac{\partial_L S}{\partial \Phi^A} \ . \tag{15.60}$$

Furthermore, the nilpotence of the BRST differential is therefore expressed in terms of the antibracket as,

$$(S,S) = 0 \ . \tag{15.61}$$

This equation is known as the ***master equation*** and can be written equivalently as,

$$\mathrm{L}(s)\mathbf{L} = (s\mathscr{L})\mathrm{d}^D x = \partial_{\mathbf{J}}\big(s\Phi^A\big)\frac{\partial \mathscr{L}}{\partial \Phi^A_{\mathbf{J}}} + \partial_{\mathbf{J}}\big(s\Phi^*_A\big)\frac{\partial \mathscr{L}}{\partial \Phi^*_{A\mathbf{J}}} = \mathrm{d_h}\beta \ , \tag{15.62}$$

for some $\beta = \beta^\mu \mathrm{d}^{D-1}x_\mu \in \Omega^{D-1}(J^k\pi)$. The extended action functional S which solves this equation is known as the ***proper solution*** to the master equation and, in general, it is given explicitly as

$$S = S_\circ + \int_M \left(\phi^*_i R^i{}_\alpha C^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} C^*_\alpha C^\gamma C^\beta\right) \ , \tag{15.63}$$

for irreducible gauge theories.

As mentioned earlier, the BRST cohomology is usually found by using the natural application of the BRST differential to functionals on the space of holonomic sections $\mathcal{J}^k\pi$. Our goal in the following sections is instead to investigate some of the properties of the BRST differential acting on differential forms on $J^k\pi$.

15.5 The Multisymplectic BRST Symmetry

Finally, we turn our attention to the (pre)multisymplectic characterization of the BRST symmetry in first-order field theories. We remind the reader that we now adjust the vertical endomorphism as done in [120] which is now on the graded jet manifold $J^1\pi$.

Recall that now the extended configuration bundle is $\pi : E \to M : (x^\mu, \Phi^A, \Phi^*_A) \mapsto x^\mu$. Then, the vertical endomorphism takes the following form:

$$\mathcal{S} = \theta^A \wedge \mathrm{d}^{D-1}x_\mu \otimes \frac{\partial}{\partial \Phi^A_\mu} + \theta^*_A \wedge \mathrm{d}^{D-1}x_\mu \otimes \frac{\partial}{\partial \Phi^*_{A\mu}} \ , \tag{15.64}$$

where now we have the augmented contact forms:

$$\theta^A \equiv \mathrm{d}\Phi^A - \Phi^A_\mu \mathrm{d}x^\mu \quad , \quad \theta^*_A \equiv \mathrm{d}\Phi^*_A - \Phi^*_{A\mu}\mathrm{d}x^\mu \ . \tag{15.65}$$

Now, defining the Poincaré–Cartan D-form $\Theta_{\mathscr{L}}$ on the extended phase space $J^1\pi$ as before,

$$\Theta_{\mathscr{L}} \equiv \mathscr{L}\mathrm{d}^D x + \mathcal{S}(\mathrm{d}\mathscr{L}) \ , \tag{15.66}$$

yields the following coordinate expression:

$$\begin{aligned} \Theta_{\mathscr{L}} &= \left(\theta^A \frac{\partial \mathscr{L}}{\partial \Phi^A_\mu} + \theta^*_A \frac{\partial \mathscr{L}}{\partial \Phi^*_{A\mu}}\right) \wedge \mathrm{d}^{D-1}x_\mu + \mathscr{L}\mathrm{d}^D x \ , \\ &= \left(\mathrm{d}y^i \frac{\partial \mathscr{L}}{\partial y^i_\mu} + \mathrm{d}C^\alpha \frac{\partial \mathscr{L}}{\partial C^\alpha_\mu}\right) \mathrm{d}^{D-1}x_\mu - \left(y^i_\mu \frac{\partial \mathscr{L}}{\partial y^i_\mu} + C^\alpha_\mu \frac{\partial \mathscr{L}}{\partial C^\alpha_\mu} - \mathscr{L}\right) \mathrm{d}^D x \ . \end{aligned} \tag{15.67}$$

The Poincaré–Cartan $(D+1)$-form is given by $\Omega_{\mathscr{L}} = -\mathrm{d}\Theta_{\mathscr{L}}$ as usual. Then, the multimomentum map for first–order field theories associated with the BRST symmetry is given as

$$\begin{aligned} J_{BRST} = -i(s)\Theta_{\mathscr{L}} + \beta &= (-1)^{\epsilon(\Phi^A)} \left(s\Phi^A \frac{\partial \mathscr{L}}{\partial \Phi^A_\mu} - s\Phi^*_A \frac{\partial \mathscr{L}}{\partial \Phi^*_{A\mu}}\right) \mathrm{d}^{D-1}x_\mu + \beta \ , \\ &= \left[(-1)^{\epsilon(\phi^i)} R^i{}_\alpha C^\alpha \frac{\partial \mathscr{L}}{\partial \phi^i_\mu} + (-1)^{\epsilon(C^\alpha)} \frac{1}{2} f^\alpha_{\beta\gamma} C^\gamma C^\beta \frac{\partial \mathscr{L}}{\partial C^\alpha_\mu} + \beta^\mu\right] \mathrm{d}^{D-1}x_\mu \ . \end{aligned} \tag{15.68}$$

The standard BRST charge is then obtained by taking holonomic sections restricted to some Cauchy surface $j^1\phi_\Sigma : \Sigma \subset M \to J^1\pi$ to compute

$$Q_{BRST} = \int_\Sigma (j^1\phi_\Sigma)^* J_{BRST} \ . \tag{15.69}$$

15.6 Yang–Mills Theory

Recall that the Yang–Mills Lagrangian prior to ghosts and antifields is written as,

$$\mathscr{L}_\circ = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \ , \tag{15.70}$$

with covariant derivative,

$$D^a_{\mu b} = \delta^a{}_b \partial_\mu - f^a_{cb} A^c_\mu \quad , \quad D_{\mu b}{}^a = \delta_b{}^a \partial_\mu + f^b_{ca} A^c_\mu \quad . \tag{15.71}$$

The extended Lagrangian which possesses the desired BRST symmetry includes one ghost field and the corresponding antifields is,

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + A^{*\mu}_a \left(C^a_\mu - f^a_{cb} A^c_\mu C^b\right) + \frac{1}{2} f^c_{ab} C^*_c C^b C^a \ . \tag{15.72}$$

The Poincaré–Cartan D-form on the graded jet manifold is given as,

$$
\begin{aligned}
\Theta_{\mathscr{L}} = &- F_a^{\mu\nu}\mathrm{d}A_\nu^a \wedge \mathrm{d}^3x_\mu + A_a^{*\mu}\mathrm{d}C^a \wedge \mathrm{d}^3x_\mu \\
&+ \left(-F_a^{\mu\nu}A_{\mu\nu}^a + \frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + f_{cb}^a A_a^{*\mu}A_\mu^c C^b - \frac{1}{2}f_{ab}^c C_c^* C^b C^a\right)\mathrm{d}^4x \ ,
\end{aligned}
\tag{15.73}
$$

The BRST transformations are given as,

$$
\begin{aligned}
sA_\mu^a &= D_{\mu b}^a C^b \ , \\
sC^a &= \frac{1}{2}f_{bc}^a C^c C^b \ , \\
sA_a^{*\mu} &= D_{\nu a}{}^b F_b^{\mu\nu} + f_{ab}^c A_{c\mu}^* C^b \ , \\
sC_a^* &= -D_{\mu a}{}^b A_b^{*\mu} + C_c^* f_{ab}^c C^b \ .
\end{aligned}
\tag{15.74}
$$

The BRST multimomentum map which arises from the BRST symmetry is,

$$
J_{BRST} = \left(-F_a^{\mu\nu}D_{\nu b}^a C^b - \frac{1}{2}f_{bc}^a A_a^{*\mu}C^c C^b\right)\mathrm{d}^3x_\mu \ , \tag{15.75}
$$

which up to the total derivative term $-D_{\nu b}^a\left(C^b F_a^{\mu\nu}\right)$ is equal to

$$
J_{BRST} = \left(C^b D_{\nu b}^a F_a^{\mu\nu} - \frac{1}{2}f_{bc}^a A_a^{*\mu}C^c C^b\right)\mathrm{d}^3x_\mu \ . \tag{15.76}
$$

Now we can compute the BRST charge using equation (15.69) to obtain

$$
Q_{BRST} = \int_\Sigma \mathrm{d}^3x_0 \left(C^b D_{\nu b}^a F_a^{0\nu} - \frac{1}{2}f_{bc}^a A_a^{*0}C^c C^b\right) \ , \tag{15.77}
$$

which is the standard BRST charge for Yang–Mills theory [13].

16 Conclusions and Outlook

16.1 Conclusions and Overview

The original work of Théophile De Donder in 1930 [36] and Hermann Weyl in 1935 [143] known today as the De Donder–Weyl Hamiltonian formalism is advantageous in the study of relativistic field theories since it treats space and time on the same footing. This makes the De Donder–Weyl framework especially appealing for the study of gravitational theories which are generally covariant. Nevertheless, the understanding of De Donder–Weyl constraints did not exist at the time when the formalism was first constructed. The De Donder–Weyl constraints were understood formally later [37, 39]; moreover, understanding the De Donder–Weyl constraint structure required a much more formal understanding of the differential geometry underlying the variational calculus. The geometric picture in which constraints are controlled is, in fact, a generalization of the geometric constraint analysis of time-dependent singular mechanics [28, 38]. However, the constraint analysis developed in [37, 39] is given in terms of the intrinsic global geometry of jet bundles and (extended and restricted) multimomentum bundles.

This work offers new geometric insights into the structure of De Donder–Weyl constraints by working locally. The geometric results we have obtained are rigorously stated in the following lemma and propositions: Lemma 3.17 Proposition 3.1 Proposition 3.2. We have published these results in [63].

This thesis also provides a variety of novelties by explicitly developing the full (pre)multisymplectic analysis of several physical models in both the Lagrangian and De Donder–Weyl Hamiltonian settings. Our original findings obtained through the analysis of specific field theories are the following:

- We present the first premultisymplectic treatment of Carrollian field theories [18, 19, 64, 80] which arise from taking the limit $c \to 0$ for the speed of light as first introduced in [77, 109]. The De Donder–Weyl formalism and its (pre)multisymplectic interpretation prove to be geometrically powerful enough to handle non-relativistic field theories.
- The multisymplectic formulation of the Nambu–Goto action for string theory and p-branes on the multimomentum phase space $J^1\pi^*$ found in [11] is extended in this work by additionally providing the multisymplectic Lagrangian formulation on $J^1\pi$ and a full discussion on the Noether symmetries on the multisymplectic phase spaces. We detail how the Nambu–Goto action is regular in the De Donder–Weyl sense. We also provide a brief discussion on the non-relativistic particle limit.
- We also analyze the bosonic string constructed from the Polyakov action. We provide a new geometrical interpretation of the well-known *Virasoro constraint* by finding from the premultisymplectic constraint analysis that the Virasoro constraint arises as a *compatibility constraint* in both the Lagrangian and De Donder–Weyl Hamiltonian settings.
- We provide the full premultisymplectic constraint analysis of $2+1$ Einstein–Cartan gravity which is a Chern–Simons theory [147]. We show that the field equations, Einstein's equations and the vanishing torsion, are given as *Second Order Partial Differential Equation (*SOPDE*)* constraints in the Lagrangian setting.
- Finally, we provide the premultisymplectic construction of the fermionic Dirac Lagrangian and find that the Dirac equation arises as a SOPDE constraint in the Lagrangian setting.

It should be noted that we have also the Yang–Mills and Chern–Simons theories with gauge group $SU(N)$. We have omitted the exposition of $3+1$ Einstein–Cartan gravity as its premultisymplectic geometry is virtually identical to the $2+1$ case; the explicit exposition of $3+1$ Einstein–Cartan gravity can be found in our latest publication [76].

Additionally, we present a formal mathematical presentation of the canonical lifts from the configuration bundle to the extended and restricted multimomentum bundles $\mathcal{M}\pi$ and $J^1\pi^*$ of the action of diffeomorphisms and their corresponding infinitesimal vector fields. The notion of natural symmetries in the multisymplectic De Donder–Weyl framework thereby follows, along with a variety of properties and new theorems. Hints of the results we provide are given as peripheral ideas in [67] and are the natural generalization of those in mechanics. The definitions we present for canonical lifts to the extended and restricted multimomentum bundles are given in [67]. Here, our exposition of these definitions is more precise and we explicitly prove a variety of results, some of which are hinted at but not proved explicitly in [67]. We also give a preliminary definition of canonical lifts to embedded submanifolds of $J^1\pi^*$ for field theories that are almost-regular in the De Donder–Weyl sense. However, our definition requires invariance of the primary constraint submanifold under the action of such lifts; this is a highly restrictive condition and a more general definition would be ideal. Nevertheless, the results obtained from our definitions for canonical lifts are the following: Proposition 4.1, Proposition 4.2, Proposition 4.3, Proposition 4.4, Proposition 4.6, Proposition 5.1.

We have also proposed in Chapter 6 a (pre)multisymplectic interpretation of the De Donder–Weyl Poisson bracket of functionals first introduced by Marsden *et al* [116]). This allows us to pass from the language of local functionals, to the language of vector fields and differential forms on $J^1\pi^*$. Our (pre)multisymplectic construction of the De Donder–Weyl Poisson bracket includes the derivation of the Hamilton–De Donder–Weyl field equations, analogous to how they are obtained from the Poisson bracket of the action functional with some arbitrary functional. Additionally, we have shown how this Poisson bracket can be derived from the (pre)symplectic structure of the infinite-dimensional covariant phase space formalism.

Furthermore, we have also shown in Chapter 6 how the multimomentum maps in Yang–Mills associated with the $SU(N)$ gauge symmetry are related, up to a total derivative, to the first-class constraints in the infinite-dimensional covariant phase space formalism; it thereby follows that the Poisson bracket of first-class constraints is directly related to the (pre)multisymplectic bracket (5.21) of multimomentum maps which, in turn, constitutes the bracket of observables in the (pre)multisymplectic setting. This new perspective allows us to understand the first-class constraints of a gauge theory in the canonical (instantaneous) Hamiltonian formalism in a covariant manner on $J^1\pi^*$. To our knowledge, this perspective does not exist in prior literature; prior to this work, the first-class constraints of a gauge theory in the canonical Hamiltonian formalism were understood in terms of the multimomentum maps on $J^1\pi^*$ by taking their zero level sets after pulling back onto Cauchy surfaces [69], thus breaking the covariance of the analysis. We intend to present our findings along with further developments in future publications.

Finally, we provide a first step towards understanding the Poincaré–Cartan geometry of jet bundles in the BRST–BV analysis of first-order field theories. After constructing the BRST symmetry, in Section 15.5 we find the corresponding multimomentum map $(m-1)$-form on $J^1\pi$. We find that the Noether charge obtained by pulling back by holonomic sections and integrating over a Cauchy surface agrees with the standard BRST charge up to a total derivative, working out the case for Yang–Mills explicitly in Section 15.6.

16.2 Outlook

As mentioned in the Introduction, one of the central challenges in quantum gravity is understanding the role of time. More precisely, the general covariance of classical gravitational theories is at odds with the preferential time direction that naturally emerges from the canonical Hamiltonian formulation of quantum field theory. It has long been known that the *ADM formalism* which foliates spacetime into Cauchy surfaces as $M = \mathbb{R} \times \Sigma$ provides the canonical Hamiltonian formulation of General Relativity, equivalent to the Einstein–Hilbert Lagrangian formulation. Upon naively performing canonical quantization using the ADM Hamiltonian, one obtains the famous Wheeler–DeWitt equation as the quantized Hamiltonian constraint [43], which serves as the analogue to the Schrödinger equation. It was shown in [126] that the classical Hamilton–Jacobi equation of the ADM formalism is the leading-order WKB semiclassical approximation $\hbar \to 0$ of the Wheeler–DeWitt equation for pure gravity.

However, the canonical quantization of Einstein gravity by using the ADM Hamiltonian occurs on very shaky grounds as the gauge algebra of first-class constraints does not close properly. Although the Poisson bracket of momentum constraints corresponding to spatial diffeomorphisms closes properly and forms an honest Lie algebra, the Poisson bracket of Hamiltonian constraints corresponding to timelike diffeomorphisms does not close and instead the algebra has "structure functions" (instead of structure constants) which depend on the dynamical fields of the theory, as pointed out by Lee and Wald in the infinite-dimensional covariant phase space formalism [108]. It thereby follows that the Poisson bracket of first-class constraints does not give a Lie algebra representation for the full spacetime diffeomorphisms under which General Relativity is gauge invariant. This is severely problematic in the quantum theory, leading to gauge anomalies and ambiguity in the ordering of operators, among other problems. However, the lack of a Lie algebra representation of spacetime diffeomorphisms in terms of first-class constraints (even in the covariant phase space formalism) is likely an artifact of the ADM formalism which sacrifices the general covariance of General Relativity by splitting space and time. Yet, we know from the Feynman (functional) path integral formulation of quantum field theory that the Einstein–Hilbert action is nonrenormalizable in dimensions higher than $2 + 1$, which shows that the challenge of quantizing General Relativity runs deeper than the splitting introduced in the ADM Hamiltonian formalism.

Nevertheless, the role that constraints can play in a quantum theory of gravity should not be overlooked, as the Wheeler–DeWitt equation suggests. Moreover, trying to understand classical gravity as a semiclassical approximation to some unknown quantum theory may be one of the most promising methods in the search for a consistent theory of quantum gravity. Bearing in mind such lessons, one might ask what role could the De Donder–Weyl formalism play in quantum gravity. A first effort to address this question was made in 1990 by Hořava who derived the Hamilton–Jacobi equation within the De Donder–Weyl formalism, first for pure gravity which works for arbitrary spacetime topology [86] and then for Einstein–Maxwell theory [87]. As Hořava notes, it is an intriguing task to investigate which quantum theory admits the Hamilton–Jacobi equation of the De Donder–Weyl formalism for pure gravity as its WKB approximation. In this light, one may naturally suspect that De Donder–Weyl Hamiltonian constraints of some unknown theory could play some role similar to that of the Wheeler–DeWitt equation. However, Hořava's work does not work with De Donder–Weyl constraints; instead, he circumvents the singular nature of the Einstein–Hilbert Lagrangian by constructing a Lepage equivalent Lagrangian which is regular in the De Donder–Weyl sense.

As we have already noted, the structure of De Donder–Weyl constraints has only been understood much later. Moreover, our understanding of De Donder–Weyl constraints is still lacking in the sense that we do not yet know if the constraint structure admits an interpretation *a la* Dirac in terms of Poisson brackets of first and second class constraints. Nevertheless, we have a solid understanding of De Donder–Weyl constraints in terms of the Hamilton–Cartan geometry on $J^1\pi^*$, and equivalently in terms of the Poincaré–Cartan geometry on $J^1\pi$ which provides us with the corresponding Lagrangian constraints. Furthermore, the gauge algebra of spacetime diffeomorphisms has recently been represented as a Lie algebra in the infinite-dimensional covariant phase space formalism [32]; other modern approaches to understanding the gauge algebra of spacetime diffeomorphisms involve Lie-infinity algebras (see [20, 21] and references therein). Both of the aforementioned approaches to the gauge algebra of General Relativity involve the multisymplectic geometry of (dual) jet bundles at the foundational level. Such progress is interesting since having a complete understanding of the Poisson brackets of observables is likely essential for quantization in the covariant De Donder–Weyl Hamiltonian formalism.

Quantization in the Lagrangian framework is well understood via Feynman's functional integral on the infinite-dimensional space of field configurations. In the path integral quantization of gauge theories, the standard approach for controlling the infinities which arise in the path integral from gauge redundancies involves constructing the gauge-invariant measure for the path integral by means of the well-known BRST analysis. However, it is unclear at this point if understanding the quantum and semiclassical properties of De Donder–Weyl constraints would be easier in the Lagrangian setting since understanding Feynman's path integral in formal geometric terms is likely a far greater challenge than understanding constraints and observables *a la* Dirac in the De Donder–Weyl Hamiltonian setting. Perhaps it is sufficient to understand the constraints and BRST analysis in the Lagrangian setting in terms of the Poincaré–Cartan geometry without having to make groundbreaking insights into the formal understanding of the path integral in order to understand the relation between the quantized constraints and their semiclassical approximations.

As suggested in [87], it is interesting to study the case of pure gravity in $2 + 1$ dimensions, as it has been proven for many years now that the Einstein–Hilbert action is fully quantizable [147]. Moreover, $2 + 1$ gravity admits the BTZ black hole solution, making it an even more interesting model, since we know that understanding the nature of black holes is deeply connected to the foundations of quantum gravity. Perhaps a next important step in understanding black holes involves developing the finite-dimensional De Donder–Weyl construction of Wald's black hole thermodynamics calculation, which he performed in the ADM formalism using the infinite-dimensional covariant phase space framework. Nevertheless, $2 + 1$ gravity is not the only coherent model of quantum gravity.

In fact, the first mathematically consistent version of quantum gravity was String Theory, whose renormalization produces the beta function which, upon vanishing, yields Einstein's equations plus higher-order corrections in the low-energy limit. This means that the Einstein–Hilbert action (plus corrections) can be interpreted as a semiclassical approximation of String Theory. This allowed us to think of gravity as emergent from quantum phenomena since String Theory starts classically from a fixed and flat background spacetime, thus changing the thought process from attempting to quantize a classical theory of gravity, to trying to obtain a classical theory of gravity as the semiclassical approximation of some quantum theory. The gravitational dynamics in String Theory emerges from the quantum spectrum as the propagating spin-2 massless particle state, the graviton. The classical geometry of spacetime is reconstructed by taking a high occupation number of coherent states of gravitons, in direct analogy with how classical electromagnetic fields are reconstructed from photons in electrodynamics. Furthermore, one can formulate String Theory classically on a non-trivial, dynamical, background spacetime and upon quantization obtain the corresponding quantum mechanics; this allows us to pass back and forth between classical and quantum gravity in precise mathematical terms. Another extraordinary feature is the cancellation of gravitational anomalies, which occurs uniquely in the case of strings within the quantum field theory of extended objects. Furthermore, the first mathematically precise construction of the holographic principle was the AdS/CFT correspondence [112][7] in which the large N limit of $\mathcal{N} = 4$ Super Yang–Mills is dual to the semiclassical limit of quantum gravity as given by Type IIB string theory on $AdS_5 \times S^5$. Additionally, the AdS/CFT correspondence provides us with an understanding of black hole thermodynamics in terms of the statistical mechanics of microstates. However, despite decades of immense effort, we have yet to find a version of String Theory which has a phenomenologically tenable quantum spectrum that resembles the Standard Model of particle physics.

[7] Today, this is the most cited paper in theoretical high energy physics with over 20,000 citations, with applications extending outside the research field of String Theory.

Most other known physical (though non-phenomenological)[8] models of quantum gravity, including the Einstein–Hilbert model in $1+1$ and $2+1$ dimensions, are topological quantum field theories in which all degrees of freedom are gauge degrees of freedom. This makes such theories of quantum gravity quite interesting to the mathematics community, as the usefulness of topological quantum field theories has been widely accepted, opening a new field of research in mathematics. In particular, topological quantum field theories of the cohomological type [146] which are constructed from the BRST framework, have proven to be interesting to both the theoretical physics and mathematics communities; the first of such theories include topological Yang–Mills in $3+1$ dimensions [150], nonlinear sigma models [151], and the first version of topological quantum gravity [149]. More recently, a topological quantum field theory of the (BRST) cohomological class was used to obtain Perelman's Ricci flow from non-relativistic quantum gravity with Lifshitz scaling [53]. We recall that the geometric advantages of studying the (pre)multisymplectic geometry of (dual) jet bundles are not limited to relativistic field theories. Moreover, understanding the BRST–BV analysis of field theories in terms of the differential geometry of jet bundles may offer further insight into the structure of topological quantum field theories.

Nevertheless, the semiclassical limit $\hbar \to 0$ is used in quantum mechanics to pass from the quantum commutator of operators to the Poisson bracket of classical observables. One then reconstructs the classical phase space by appropriately preparing coherent stats so that they become the points of phase space in the semiclassical limit; this process is straightforward in quantum mechanics since the classical phase space is finite dimensional. On the other hand, the classical phase space for field theories that is used in the physics literature is typically infinite-dimensional as we have previously discussed. Moreover, coherent states become functionals, making them much more difficult to control. Consequently, reconstructing such classical phase spaces from a semiclassical limit quickly becomes unmanageable, with some obstructions arising from the presence of gauge symmetries and constraints. Furthermore, the semiclassical limit of quantum field theories does not always exist, let alone the reconstruction of classical phase space. In the case of Quantum Electrodynamics, the reconstruction of the infinite-dimensional classical phase space of Electromagnetism is possible as a result of the abelian structure of the gauge algebra and, more importantly, due to the fact that gauge fixing can be done globally and the Marsden–Weinstein *symplectic reduction* is performed in a straightforward manner. In the case of topological field theories, the reduced covariant phase space has the exceptional quality of being finite-dimensional; as an example (mentioned in the Introduction), the global structure of the covariant phase space for JT gravity was uncovered in the covariant phase space quantization of the theory [123, 124]. Now, in the case of non-abelian Yang–Mills theory, we do not know the global structure of the infinite-dimensional classical phase space due to a variety of complications. Firstly, gauge fixation is not globally possible due to the Gribov ambiguities. Furthermore, symplectic reduction is not straightforward. Instead, we control the gauge symmetry of Yang–Mills by means of the local cohomology in the BRST–BV analysis. Nevertheless, the semiclassical limit of Yang–Mills is understood in certain regimes, such as in the perturbative expansion of the path integral around saddle points, or non-perturbatively in the expansion around instantons.

Understanding the Hamilton–Jacobi equation of the De Donder–Weyl formalism as the semiclassical approximation of some quantum system is likely not an easy task. The full global reconstruction of classical phase space for more complicated gauge theories such as Yang–Mills or General Relativity in dimensions greater than $2+1$ is an even more challenging task. Perhaps the finite-dimensional structure of (dual) jet bundles, together with their canonical geometric properties, may offer new insights into the understanding of semiclassical approximations and the reconstruction of classical phase spaces. Furthermore, the understanding of observables, local symmetries, and constraints in terms of Poisson brackets in the De Donder–Weyl formalism is becoming an increas-

[8]That is, theories which are renormalizable and unitary but whose characteristics do not resemble the phenomenological data taken from observations in our universe.

ingly reachable objective. Perhaps the cohomological classification of multimomentum maps and the BRST–BV analysis of field theories on $J^1\pi$ and $J^1\pi^*$ could play key roles in the understanding of gauge symmetries and observables in the De Donder–Weyl formalism.

Most of the modern literature on the De Donder–Weyl formalism comes from the differential geometry community, although there has been a reappearance occurring recently in the physics literature (see e.g. [100]-[103]). The applications of the infinite-dimensional covariant phase space formalism is far more abundant in the physics literature (e.g. [5, 82, 83]); it would be interesting to study some of these applications using the geometric De Donder–Weyl program. The fact that space and time are treated on the same footing in the De Donder–Weyl Hamiltonian formalism makes it interesting for the study of relativistic theories, especially gravitational theories. Moreover, the geometric formulation of the De Donder–Weyl formalism makes it even more interesting, as it is a generalization of the variational calculus in mechanics, and the finite-dimensional covariant phase spaces (the jet and multimomentum bundles) are equipped with canonical geometric structures that serve as powerful tools in the analysis of field theories.

A Multivector Fields

Let $\mathcal{M}$ be an n-dimensional manifold and $m < n$; sections of $\Lambda^m(\mathrm{T}\mathcal{M}) \equiv \overbrace{\mathrm{T}\mathcal{M} \wedge \ldots \wedge \mathrm{T}\mathcal{M}}^{(m)}$ are called ***m*-multivector fields** on $\mathcal{M}$ and they are the contravariant skew-symmetric tensor fields of order m on $\mathcal{M}$. The set of m-multivector fields on $\mathcal{M}$ is denoted $\mathfrak{X}^m(\mathcal{M})$. For more details on multivector fields and all the related topics presented here see, for instance, [24, 26, 47, 48, 89].

For every m-multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ and $p \in \mathcal{M}$, there exists an open neighborhood $U_p \subset \mathcal{M}$ and $X_1, ..., X_r \in \mathfrak{X}(\mathcal{M})$ such that

$$\mathbf{X}|_{U_p} = \sum_{1 \leq i_1 \ldots i_m \leq r} f^{i_1 \ldots i_m} X_{i_1} \wedge \ldots \wedge X_{i_m} \ , \tag{A.1}$$

where $f^{i_1 \ldots i_m} \in C^\infty(U_p)$ and $m \leq r \leq \dim\mathcal{M}$. Then, $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is ***locally decomposable*** if, for every $p \in \mathcal{M}$, there is an open neighborhood $U_p \subset \mathcal{M}$ and $X_1, ...X_m \in \mathfrak{X}(U_p)$ such that

$$\mathbf{X}\big|_{U_p} = X_1 \wedge ... \wedge X_m \ . \tag{A.2}$$

All multivector fields used in this paper are assumed to be locally decomposable.

The contraction between multivector fields $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ and differentiable k-forms $\xi \in \mathit{\Omega}^k(\mathcal{M})$ is

$$i(\mathbf{X})\xi|_{U_p} \equiv \sum_{1 \leq i_1 \ldots i_m \leq r} f^{i_1 \ldots i_m} i(X_{i_1} \wedge ... \wedge X_{i_m})\xi = \sum_{1 \leq i_1 \ldots i_m \leq r} f^{i_1 \ldots i_m}\, i(X_{i_m}) \ldots i(X_{i_1})\xi \ , \tag{A.3}$$

if $k \geq m$, and $i(\mathbf{X})\xi|_{U_p} = 0$, if $k < m$. Furthermore, the ***Lie derivative*** of ξ with respect to $\mathbf{X}$ is the graded bracket (of degree $m-1$)

$$\mathrm{L}(\mathbf{X})\xi \equiv [\mathrm{d}, i(\mathbf{X})]\xi = (\mathrm{d}\, i(\mathbf{X}) - (-1)^m\, i(\mathbf{X})\mathrm{d})\xi \ . \tag{A.4}$$

A ***distribution*** $\mathcal{D}$ of rank k on $\mathcal{M}$ is a sub-bundle of $\mathrm{T}\mathcal{M}$ of rank k (and an m-dimensional distribution on $\mathcal{M}$ is an m-dimensional sub-bundle of $\mathrm{T}\mathcal{M}$). For every $p \in \mathcal{M}$ there is a linear subspace $\mathcal{D}_p \subseteq \mathrm{T}_p\mathcal{M}$ such that $\mathcal{D} = \bigcup_{p \in \mathcal{M}} \mathcal{D}_p$. Given a distribution $\mathcal{D} \subseteq \mathrm{T}\mathcal{M}$, a nonempty immersed submanifold $W \subseteq \mathcal{M}$ is a an ***integral manifold of*** $\mathcal{D}$ if $\mathrm{T}_pW = \mathcal{D}_p$. The distribution $\mathcal{D} \subseteq \mathrm{T}\mathcal{M}$ is an ***integrable distribution*** if at every $p \in \mathcal{M}$ there is an integral manifold of $\mathcal{D}$ containing p. A multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ and an m-dimensional distribution $\mathcal{D}$ on $\mathcal{M}$ are ***locally associated*** if there exists an open set $U \subseteq \mathcal{M}$ such that $\mathbf{X}\big|_U$ is a section of $\Lambda^m\mathcal{D}\big|_U$.

Locally decomposable m-multivector fields are locally associated to m-dimensional distributions. Multivector fields associated with the same distribution form an equivalence class $\{\mathbf{X}\}$ in $\mathfrak{X}^m(\mathcal{M})$. If $\mathbf{X}_1,\ \mathbf{X}_2 \in \mathfrak{X}^m(\mathcal{M})$ are two nonvanishing multivector fields, both locally associated with the same distribution on the open set $U \subseteq \mathcal{M}$, then there exists $f \in C^\infty(U)$ such that $\mathbf{X}_1 = f\mathbf{X}_2$ (on U), and this is precisely the equivalence relation with equivalence class denoted $\{\mathbf{X}\}_U$. A multivector field is ***integrable*** if its locally associated distribution is integrable. An m-dimensional submanifold $W \hookrightarrow \mathcal{M}$ is an ***integral manifold*** of $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ if, and only if, for every point $p \in W$, the multivector $\mathbf{X}_p$ spans $\Lambda^m\mathrm{T}_pW$.

Now, let $\tau\colon \mathcal{M} \to M$ be a fiber bundle with $\dim M = m$ and $\dim \mathcal{M} = m + n$. Vector fields on $\mathcal{M}$ are called ***τ-vertical vector fields*** when they are tangent to the fibers of τ. The set of such vector fields will be denoted as $\mathfrak{X}^{\mathrm{V}(\tau)}(\mathcal{M})$. A multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is ***τ-transverse*** if at every point $p \in Y$, $i(\mathbf{X})(\tau^*\beta)\big|_p \neq 0$ for every $\beta \in \mathit{\Omega}^m(Y)$ with $\beta(\tau(p)) \neq 0$.

If M is an orientable manifold, the condition that a multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is $\tau-$transverse can be written as $i(\mathbf{X})\mathrm{d}^m x \neq 0$, where $\mathrm{d}^m x \equiv \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{m-1}$ is the local coordinate expression (x^μ) on M $(\mu = 0, ..., m-1)$ of the volume form ω. In particular, it is possible to take a representative $\mathbf{X}$ in the class of π-transverse multivector fields such that $f = 1$ so that

$$i(\mathbf{X})\mathrm{d}^m x = 1 \ . \tag{A.5}$$

The ***canonical prolongation*** of a section $\psi\colon U \subset M \to \mathcal{M}$ to $\Lambda^m \mathrm{T}\mathcal{M}$ is the section $\psi^{(m)}\colon U \subset M \to \Lambda^m \mathrm{T}\mathcal{M}$ defined as

$$\psi^{(m)} \equiv \Lambda^m \mathrm{T}\psi \circ \mathbf{Y}_\omega \ , \tag{A.6}$$

where $\Lambda^m \mathrm{T}\psi\colon \Lambda^m \mathrm{T}M \to \Lambda^m \mathrm{T}\mathcal{M}$ is the natural extension of ψ to the corresponding multitangent bundles and $\mathbf{Y}_\omega \in \mathfrak{X}^m(M)$ is the unique m-multivector field on M such that $i(\mathbf{Y}_\omega)\omega = 1$. Then, ψ is an integral section of $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ if, and only if, $\psi^{(m)} = \mathbf{X} \circ \psi$.

Consider a local section of τ, $\phi\colon U_x \subset M \to \mathcal{M} : x \mapsto y = \phi(x)$ where $x \in M$ and $y \in \mathcal{M}$; if $\phi(U_x)$ is the integral manifold of the multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ at y, then ϕ is called an ***integral section*** of $\mathbf{X}$. If a multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable, then it is τ-transverse if, and only if, its integral manifolds are local sections of π; that is, the local sections of π are integral sections of $\mathbf{X}$.

B Graded Algebras

This appendix closely follows [84].

$\mathbb{Z}_2$-graded (Supercommutative) Algebras:

An ***associative algebra*** A is a vector space over $\mathbb{C}$ such that, for any $x, y, \in A$, the following properties hold:

$$\begin{aligned} (x+y)z &= xz + yz \quad , \quad x(y+z) = xy + xz \ ; \\ \alpha(xy) &= (\alpha x)y = x(\alpha y) \ , \quad \alpha \in \mathbb{C} \ ; \\ (xy)z &= x(yz) \ . \end{aligned} \tag{B.1}$$

The algebra A is $\mathbb{Z}_2$***-graded (supercommutative)*** if it is the direct sum of two associative algebras,

$$A = A_0 \oplus A_1 \ , \tag{B.2}$$

such that,

$$xy = (-1)^{\epsilon_x \epsilon_y} yx \ , \tag{B.3}$$

where $\epsilon_z = 0$ if $z \in A_0$ and $\epsilon_z = 1$ if $z \in A_1$. It therefore follows that,

$$A_0 A_0 \subset A_0 \quad , \quad A_0 A_1 \subset A_1 \quad , \quad A_1 A_1 \subset A_0 \ . \tag{B.4}$$

$\mathbb{Z}$-graded Algebras:

An algebra A is a $\mathbb{Z}$-graded algebra if $A = \bigoplus_{\mathbb{Z}} A_n$ with multiplication rules: $A_n A_m \subset A_{n+m}$. Furthermore, an element $x \in A$ has degree given as $\deg(x) = n$ if and only if $x \in A_n$.

C Differentials and (Co)homology

This appendix also closely follows [84].

A graded derivation D is a linear transformation with $\mathbb{Z}_2$ parity grading which satisfies the Leibnitz rule:

$$D(xy) = xDy + (-1)^{\epsilon_D \epsilon_y}(Dx)y \ . \tag{C.1}$$

If $\epsilon_D = 0$ then D is an ***even derivation*** and if $\epsilon_D = 1$ then D is an ***odd derivation***.

A ***differential*** D is a nilpotent odd derivation:

$$D^2 = \frac{1}{2}[D, D] = 0 \ , \tag{C.2}$$

$$\epsilon(D) = 1 \ . \tag{C.3}$$

The differential D may also have a $\mathbb{Z}$-grading $\deg(D) = \pm 1$ on some $\mathbb{Z}$-graded algebra A. If $\deg(D) = +1$ then D is a coboundary operator forming the cochain complex:

$$\cdots \xrightarrow{D} A_n \xrightarrow{D} A_{n+1} \xrightarrow{D} A_{n+2} \xrightarrow{D} \cdots$$

and the ***cohomology of*** D, denoted $H^\bullet(D)$, is defined as

$$H^\bullet(D) \equiv \frac{\mathrm{Ker}D}{\mathrm{Im}D} \ , \tag{C.4}$$

where,

$$x \in \mathrm{Ker}D \Leftrightarrow Dx = 0 \quad \text{and} \quad x \in \mathrm{Im}D \Leftrightarrow \exists y \in A : Dy = x \ , \tag{C.5}$$

and an element $x \in \mathrm{Ker}D$ for which $x \notin \mathrm{Im}D$ is called a ***cocycle***. Furthermore,

$$H^k(D) \equiv \frac{\{x \in \mathrm{Ker}D : \deg(x) = k\}}{\{x \in \mathrm{Im}D : \deg(x) = k\}} \ , \tag{C.6}$$

and,

$$H^\bullet(D) = \bigoplus_k H^k(D) \ . \tag{C.7}$$

Furthermore, given a smooth differentiable manifold P with exterior derivative d, the ***de Rham cohomology*** of P is defined as

$$H^\bullet(\mathrm{d}; P) \equiv \frac{\mathrm{Ker(d)}}{\mathrm{Im(d)}} \ . \tag{C.8}$$

Now, if $\deg(D) = -1$, then we have the chain complex:

$$\cdots \xleftarrow{D} A_n \xleftarrow{D} A_{n+1} \xleftarrow{D} A_{n+2} \xleftarrow{D} \cdots$$

and the ***homology of D*** is given as,

$$H_\bullet(D) \equiv \frac{\mathrm{Ker}D}{\mathrm{Im}D} = \bigoplus_k H_k(D) \ , \tag{C.9}$$

and an element $x \in \mathrm{Ker}D$ for which $x \notin \mathrm{Im}D$ is called a ***cycle***.

D Lie Algebra Cohomology and Central Extensions

The classic reference on this subject is [27].

Consider a Lie group G with Lie algebra $\mathfrak{g}$ and a representation M. The ***k-th Lie algebra homology differential of*** $\mathfrak{g}$, denoted as $\delta_k : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g}$, is defined as,

$$\delta_k : \xi_1, \cdots \wedge \ldots \wedge \xi_{k+1} \mapsto \sum_{1 \le i < j \le k+1} (-1)^{i+j}[\xi_i, \xi_j] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_i, \cdots \wedge \hat{\xi}_j \wedge \ldots, \xi_k \ . \tag{D.1}$$

The space of linear maps $C^k(\mathfrak{g}; M) : \Lambda^k \mathfrak{g} \to M$ is given as

$$C^k(\mathfrak{g}; M) \equiv H(\Lambda^k \mathfrak{g}, M) \cong \Lambda^k \mathfrak{g}^* \otimes M \ . \tag{D.2}$$

The ***Chevalley-Eilenberg complex*** is given as,

$$\cdots \xrightarrow{\mathrm{d}_\mathfrak{g}} C^{k-1}(\mathfrak{g}; M) \xrightarrow{\mathrm{d}_\mathfrak{g}} C^k(\mathfrak{g}; M) \xrightarrow{\mathrm{d}_\mathfrak{g}} C^{k+1}(\mathfrak{g}; M) \xrightarrow{\mathrm{d}_\mathfrak{g}} \cdots \tag{D.3}$$

where $\mathrm{d}_\mathfrak{g}$ is the ***Chevalley–Eilenberg differential*** defined as,

$$\begin{aligned} \mathrm{d}_\mathfrak{g}\alpha(\xi_1, \ldots, \xi_{k+1}) \equiv & \sum_{i=1}^{k+1} (-1)^{i+1} \xi^A \cdot \alpha(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{k+1}) \\ & + \sum_{i<j} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{k+1}) \ . \end{aligned} \tag{D.4}$$

The cohomology of the Chevalley–Eilenberg differential,

$$H^\bullet(\mathfrak{g}; M) = \frac{\mathrm{Ker}(\mathrm{d}_\mathfrak{g})}{\mathrm{Im}(\mathrm{d}_\mathfrak{g})} \ , \tag{D.5}$$

is called the ***Lie algebra cohomology of*** $\mathfrak{g}$ ***with values in*** M.

A Lie algebra $\overline{\mathfrak{g}}$ is called an ***extension*** of $\mathfrak{g}$ by $\mathfrak{a}$ if the following short exact sequence of Lie algebra homomorphisms exists:

$$0 \longrightarrow \mathfrak{a} \longrightarrow \overline{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0 \ . \tag{D.6}$$

If $\mathfrak{a}$ is in the center of $\overline{\mathfrak{g}}$, i.e. $[\mathfrak{a}, \overline{\mathfrak{g}}] = 0$, then $\overline{\mathfrak{g}}$ is called the ***central extension*** of $\mathfrak{g}$ by $\mathfrak{a}$ as $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$.

E Left and Right Derivatives

See, e.g., [65] for more details.

Given a functional X of ϕ, the left and right derivatives are defined as

$$\frac{\partial_L X}{\partial \phi} \equiv \frac{\overrightarrow{\partial}}{\partial \phi} X \quad , \quad \frac{\partial_R X}{\partial \phi} \equiv \frac{\overleftarrow{\partial}}{\partial \phi} X \quad . \tag{E.1}$$

Then,

$$\mathrm{d}X = \mathrm{d}\phi \frac{\partial_L X}{\partial \phi} = \frac{\partial_R X}{\partial \phi} \mathrm{d}\phi \ , \tag{E.2}$$

and similarly

$$\delta X = \delta\phi \frac{\partial_L X}{\partial \phi} = \frac{\partial_R X}{\partial \phi} \delta\phi \ . \tag{E.3}$$

Furthermore,

$$\frac{\partial_L X}{\partial \phi} = (-1)^{\epsilon(\phi)(\epsilon(X)+1)} \frac{\partial_R X}{\partial \phi} \ . \tag{E.4}$$

F Properties of the Antibracket

See, e.g., [65] for more details.

$$\begin{aligned}(XY,Z) &= X(Y,Z) + (-1)^{\epsilon_Y \epsilon_Z}(X,Z)Y \ , \\ (X,YZ) &= (X,Y)Z + (-1)^{\epsilon_X \epsilon_Y} Y(X,Z) \ .\end{aligned} \tag{F.1}$$

$$\begin{aligned}((X,Y),Z) &= X(Y,Z) + (-1)^{\epsilon_Y(\epsilon_Z+1)}(X,Z)Y \ , \\ (X,(Y,Z)) &= (X,Y)Z + (-1)^{\epsilon_Y(\epsilon_X+1)} Y(X,Z) \ .\end{aligned} \tag{F.2}$$

$$(B,B) = \frac{\partial_R B}{\partial \Phi^A}\frac{\partial_L B}{\partial \Phi^*_A} \quad , \quad B \text{ is a boson} \ , \tag{F.3}$$

$$(F,F) = 0 \quad , \quad F \text{ is a fermion} \ , \tag{F.4}$$

$$((X,X),X) = 0 \quad , \quad \forall X \ . \tag{F.5}$$

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