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SOME RESULTS ON APPROXIMATE CONTINUOUS SELECTIONS, FIXED POINTS AND MINIMAX INEQUALITIES

by

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SOME RESULTS ON APPROXIMATE CONTINUOUS SELECTIONS, FIXED POINTS AND MINIMAX INEQUALITIES

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ABSTRACT

We introduce the class of multi-valued D-mappings from a non-empty compact subset K of a locally convex Hausdorff topological vector space into a vector space (which, for K convex, includes the class of convex mappings) and we prove that they have a special approximate continuous selection property, from which an approximate fixed point theorem is derived. These results are applied, in particular, to Darboux continuous functions defined on a closed real interval. Furthermore, we extend some game theoretic statements concerning continuous decision rules to the case when the strategies of one player are constrained by those of the other and we obtain some results related to Ky Fan's inequality involving several functions.

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INTRODUCTION

The existence of continuous selections of multi–valued mappings is an important issue in nonlinear analysis. A useful result in this respect is the famous Michael selection theorem [2, p. 82, Thm. 1], valid for lower semicontinuous mappings. Instead, the class of upper semicontinuous mappings fails to have this property, although another celebrated theorem, due to Cellina, asserts the existence of approximate continuous selections for mappings in this class [2, p. 84, Thm. 1]. In Section 1, we introduce the class of D–mappings from a non–empty compact subset K of a locally convex Hausdorff topological vector space into another vector space and prove that they satisfy an approximate continuous selection theorem. When K is convex, the class of D–mappings includes that of convex mappings. We also obtain the existence of continuous selections of ε–solutions of parametric constrained quasiconcave maximization problems as well as equilibrium results for two–person games in which the strategies of one player are constrained by those of the other, in terms of continuous decision rules.

The Brouwer fixed point theorem in connection with the existence of continuous selections is useful to obtain fixed point theorems for multi–valued mappings as, e.g., in the case of the Browder fixed point theorem [4, Thm. 1] for mappings from a non–empty compact convex subset of a topological vector space into itself which have non–empty convex images and open inverse images. In Section 2, we employ this technique to derive approximate fixed point theorems from the approximate continuous selection results given in the previous section.

It is well known, and easy to prove, that the Browder fixed point theorem is equivalent to Ky Fan's inequality [5]. The latter is an useful tool in nonlinear analysis and game theory. In Section 3, among other related results, we present a vector version of Ky Fan's inequality in terms of weakly maximal elements.

For the terminology and notation concerning multi–valued mappings which we will adopt, we refer to Chapert 1 of the book of Aubin and Cellina [2]. All fixed point theorems we will use are classical and well known: those of Brouwer, Kakutani and Browder. A good introduction to fixed point theory and its applications can be found in the book by Istrătescu [6]. All topological vector spaces will be assumed to be real; see the book of Köthe for the basic properties of them which will be needed. The notation employed in the paper is standard, e.g., co will denote convex hull, $c\ell$ topological closure and $\Pi_{i\in I}$ topological product. By a polytope in a vector space we mean the convex hull of a finite set. Note that, although the vector space is no endowed with any topology, as a polytope is contained in a finite dimensional subspace it has a natural topology, namely, the Euclidean one; moreover, if the vector space is topological, the topology induced on any finite dimensional subspace, and hence on any polytope, is just the Euclidean one. For a mapping $G : K \to 2^C$ from a topological space K into a convex subset C of a vector space L, we will denote by $S(G)$ the set of continuous selections of G taking values in some polytope, i.e., $p \in S(G)$ means that p is a continuous function from K into some polytope $P \subset C$ and $p(x) \in G(x)$ for all $x \in K$. From the preceding considerations, it follows that if L has a compatible topology, then any $p \in S(G)$, viewed as a function from K into L, is continuous; in other words, p is a continuous selection of G in the usual sense.

1. APPROXIMATE CONTINUOUS SELECTIONS

The results presented in this Section are based on the following lemma, which is inspired in the proof of the Browder fixed point theorem [4, Thm. 1]:

Lemma 1.1. Let K be a non–empty compact Hausdorff topological space, L a vector space and $T : K \to 2^L$ a mapping with non–empty values such that for each y in L, $T^{-1}(y)$ is open in K. Then there exists a finite subset Y of $T(K)$ and a continuous function $p : K \to co\,Y$ such that $p(x) \in co\big(Y \cap T(x)\big)$ for all $x \in K$.

Proof: By the assumptions, the family $\{T^{-1}(y)\}_{y \in L}$ is an open cover of K. Since K is compact, there exists a finite set $Y \subset L$ such that $\{T^{-1}(y)\}_{y \in Y}$ is still a cover of K. Let $\{\beta_y\}_{y \in Y}$ be a continuous partition of unity corresponding to this covering and define $p : K \to co\,Y$ by

$$p(x) = \sum_{y \in Y} \beta_y(x) y \ .$$

Then, p is continuous and, as for each $y \in Y$ and $x \in K$ we have $\beta_y(x) > 0$ only if $x \in T^{-1}(y)$, that is, $y \in T(x)$, we obtain

$$p(x) \in co\big(Y \cap T(x)\big) \ .$$

From this lemma, we get the following theorem which states the existence of continuous approximations, in some sense, to arbitrary functions.

Theorem 1.1. Let K be a non–empty compact subset of a locally convex Hausdorff topological vector space E, D a dense subset of K, L a vector space and $f : D \to L$. Then for each neighborhood V of 0 in E there exists a finite subset Y_V of $f(D)$ and a continuous function $p_V : K \to co\,Y_V$ such that $p_V(x) \in co\Big(Y_V \cap f\big((x+V) \cap D\big)\Big)$ for all $x \in K$.

Proof: Since E is locally convex, without loss of generality we can assume that V is open and absolutely convex. Define $T_V : K \to 2^L$ by

$$T_V(x) = f\big((x+V) \cap D\big) \ .$$

As D is dense in K, $T_V(x)$ is non–empty for any $x \in K$. Let $y \in L$ and $x \in T_V^{-1}(y)$. Then, there exists $x' \in (x+V) \cap D$ such that $f(x') = y$. Since V is open, $x' \in x + \lambda V$ for some $\lambda \in (0,1)$. For any $x'' \in \big(x + (1-\lambda)V\big) \cap K$ we have, using the absolute convexity of V,

$$x' \in x + \lambda V \subset x'' - (1-\lambda)V + \lambda V = x'' + V \ ,$$

whence

$$y = f(x') \in f\big((x''+V) \cap D\big) = T_V(x'') \ .$$

Therefore, $(x+(1-\lambda)V) \cap K \subset T_V^{-1}(y)$, which proves that $T_V^{-1}(y)$ is open in K. Hence, applying Lemma 1.1, we get the existence of Y_V and p_V as required.

It may be illustrating to state the particular case of the preceding theorem corresponding to the case when L is the real line. It says that, given $f : D \to R$, for any neighborhood V of 0 in E there exists a continuous function $p_V : K \to R$ such that for any $x \in K$ one can find $x', x'' \in (x+V) \cap D$ for which $f(x') \le p_V(x) \le f(x'')$.

In order to obtain more precise results, we introduce the following class of multi–valued mappings:

Definition 1.1. Given a non–empty compact subset K of a locally convex Hausdorff topological vector space E and a vector space L, we say that $T : K \to 2^L$ is a D–mapping if $Dom\, T$ is dense in K and there exists a base $\mathcal{B}$ of neighborhoods of 0 in E such that $T\big((x+V) \cap K\big)$ is convex for all $x \in K$ and $V \in \mathcal{B}$.

The class of D–mappings includes that of convex multi–valued mappings (on convex domains), i.e., those whose graph is convex or, equivalently, those for which the inclusion

$$(1-\lambda)T(x_1) + \lambda T(X_2) \subset T\big((1-\lambda)X_1 + \lambda x_2\big)$$

holds for any x_1, $x_2 \in K$ and $\lambda \in [0,1]$. In the particular case when K is a closed real interval, $L = R$ and T is single–valued, a sufficient condition for T being a D–mapping is that T be Darboux continuous. This means that for any p, q with $a \le p < q \ge b$ and any $c \in R$ between $T(p)$ and $T(p)$ there is an $s \in [p,q]$ such that $f(s) = c$ (see, e.g., [11, p. 55]).

Theorem 1.2. Let E, K and L be as in Theorem 1.1 and let $T : K \to 2^L$ be a D–mapping. Then for each nieghborhood V of 0 in E there exist a polytope P_V in $co\,T(K)$, a continuous function $p_V : K \to P_V$ and $\delta_V : K \to V$ such that $p_V(x) \in T\big(x + \delta_V(x)\big)$ for all $x \in K$.

Proof: Let $f : Dom\,T \to L$ be any selection of T and take $V' \in \mathcal{B}$ such that $V' \subset V$. By Theorem 1.1, there exists a polytope P_V in $co\,f(Dom\,T) \subset co\,T(K)$ and a continuous function $p_V : K \to P_V$ such that, for any $x \in K$, one has

$$p_V(x) \in co\,f\big((x+V') \cap Dom\,T\big) \subset co\,T(x+V') = T(x+V') \subset T(x+V).$$

This implies the existence of δ_V as in the statement.

The existence of p_V and δ_V satisfying the conditions of Theorem 1.2 is not sufficient for T being a D–mapping; consider, e.g., the restriction to any closed interval of the characteristic function of the set of rational numbers. Nevertheless, this is not a so weak condition as to be satisfied for arbitrary functions, since the restriction of the characteristic function of a closed interval to a bigger closed interval does not verify it.

Since, as we have already observed, any Darboux continuous function can be regarded as a D–mapping, we obtain the following approximability result for Darboux continuous functions by continuous functions.

Corollary 1.1. Let $f : [a,b] \to R$ be a Darboux continuous function. Then for each $\varepsilon > 0$ there exists $\delta_\varepsilon : [a,b] \to [-\varepsilon, \varepsilon]$ such that the function $p_\varepsilon : [a,b] \to R$ defined by $p_\varepsilon(x) = f\big(x + \delta_\varepsilon(x)\big)$ is continuous.

In other words, the preceding corollary says that any Darboux continuous function can be made continuous by an arbitrarily small perturbation of the independent variable.

Our next result, which is based on Lemma 1.1, can be interpreted in terms of continuous selections of ε–solutions of parametric constrained optimization problems.

Theorem 1.3. Let K be a non–empty compact Hausdorff topological space, F a Hausdorff topological vector space, C a convex subset of F, $G : K \to 2^C$ a continuous mapping with non–empty compact convex values and $f : K \times C \to R$ an upper semicontinuous function such that, for any $x \in K$, the partial mapping $f(x,.)$ is quasiconcave and, for any $y \in C$, $f(.,y)$ is continuous. Then

$$\inf_{p \in S(G)} \sup_{x \in K} \Big\{ \max_{y \in G(x)} f(x,y) - f\big(x, p(x)\big) \Big\} = 0.$$

Proof: The inequality $\geq$ being obvious, we only have to prove the opposite one. For any $\varepsilon > 0$, we define $T_\varepsilon : K \to 2^C$ by

$$T_\varepsilon(x) = \Big\{ z \in G(x) \,\Big|\, \max_{y \in G(x)} f(x,y) - f(x,z) < \varepsilon \Big\}.$$

The sets $T_\varepsilon(x)$ are non-empty and, by our assumptions on G and f, convex. On the other hand, by the upper semicontinuity of the function $\max_{y\in G(.)} f(.,y)$, (see [2, p. 52, Thm. 5]) and the continuity of the partial mapping $f(.,y)$, the sets $T_\varepsilon^{-1}(y)$ are open in K. Hence, by Lemma 1.1, there exist a polytope $P_\varepsilon \subset C$ and a continuous function $p_\varepsilon : K \to P_\varepsilon$ such that $p_\varepsilon(x) \in co\,T_\varepsilon(x) = T_\varepsilon(X)$ for all $x \in K$. But this implies that $p_\varepsilon \in S(G)$ and

$$\sup_{x\in K}\Big\{\max_{y\in G(x)} f(x,y) - f\big(x,p_\varepsilon\big)\Big\} \le \varepsilon\ ;$$

Since ε can be made arbitrarily small, we get the desired inequality.

From Theorem 1.3, we deduce the following result, which is related to a well known theorem in game theory (see, e,g., [1, Thm. 8.4]) concerning optimal continuous decision rules.

Corollary 1.2. Under the assumptions of Theorem 1.3,

$$\sup_{p\in S(G)}\inf_{x\in K} f\big(x,p(x)\big) = \inf_{x\in K}\max_{y\in G(x)} f(x,y)\,.$$

Proof: The inequality $\le$ is evident, while $\ge$ follows from Theorem 1.3:

$$\begin{aligned}
&\inf_{x\in K}\max_{y\in G(x)} f(x,y) - \sup_{p\in S(G)}\inf_{x\in K} f\big(x,p(x)\big) = \\
&= \inf_{p\in S(G)}\Big\{\inf_{x\in K}\max_{y\in G(x)} f(x,y) - \inf_{x\in K} f\big(x,p(x)\big)\Big\} \le \\
&\le \inf_{p\in S(G)}\sup_{x\in K}\Big\{\max_{y\in G(x)} f(x,y) - f\big(x,p(x)\big)\Big\} = 0\,;
\end{aligned}$$

note that the preceding expressions do not make sense only when the equality we are proving holds as $-\infty = -\infty$.

The conclusion of Corollary 1.2 is also satisfied under the following assumptions, which are weaker than those of Corollary 1.2 except in the requirement of the sets $G^{-1}(y)$ to be open in K .

Theorem 1.4. Let K and L be as in Lemma 1.1, C a convex subset of $L, G : K \to 2^C$ a mapping with non-empty convex values such that $G^{-1}(y)$ is open in K for all $y \in C$ and $f : K \times C \to R$ a function which is lower semicontinuous in its first variable and quasiconcave in its second variable. Then

$$\sup_{p\in S(G)}\inf_{x\in K} f\big(x,p(x)\big) = \min_{x\in K}\sup_{y\in G(x)} f(x,y)\,.$$

Proof: We first observe that our assumptions on G and f imply that the function $\sup_{y\in G(.)} f(.,y)$ is lower semicontinuous and therefore attains its minimum on K. The inequality $\leq$ being obvious, we shall prove the opposite one. Let $\alpha = \min_{x\in K} \sup_{y\in G(x)} f(x,y)$. For any $\varepsilon > 0$, we define the mapping $T_\varepsilon : K \to 2^C$ by

$$T_\varepsilon(x) = \{y \in G(x) \mid f(x,y) > \alpha - \varepsilon\}.$$

Clearly, the sets $T_\varepsilon(x)$ are non–empty and convex and the sets $T_\varepsilon^{-1}(y)$, $y \in C$, are open. Hence, by Lemma 1.1, there exist a polytope $P_\varepsilon \subset C$ and a continuous function $p_\varepsilon : K \to P_\varepsilon$ such that $p_\varepsilon(x) \in co\,T_\varepsilon(x) = T_\varepsilon(x)$ for all $x \in K$. But this implies that $p_\varepsilon \in S(G)$ and $\inf_{x\in k} f(x,p_\varepsilon(x)) \geq \alpha - \varepsilon$; as ε can be taken arbitrary close to zero, we obtain the required inequality.

Corollary 1.3. Under the assumptions of either Theorem 1.3 or Theorem 1.4,

$$\sup_{p\in S(G)} \inf_{x\in K} f(x,p(x)) = \inf_{x\in K} \sup_{p\in S(G)} f(x,p(x)).$$

Proof: It is an immediate consequence of theorems 1.3 and 1.4 and the obvious inequalities

$$\sup_{p\in S(G)} \inf_{x\in K} f(x,p(x)) \leq \inf_{x\in K} \sup_{p\in S(G)} f((x,p(x)) \leq \inf_{x\in K} \sup_{y\in G(x)} f(x,y).$$

In order to derive a counterpart to Theorem 1.4 in terms of continuous decision rules of the first player, we need the following lemma (related to Proposition 8.2 in [1]):

Lemma 1.2. Under the assumptions of Theorem 1.4,

$$\min_{x\in K} \sup_{y\in G(x)} f(x,y) = \sup_{H\in\Sigma(G)} \min_{x\in K} \max_{y\in H\cap G(x)} f(x,y),$$

where $\Sigma(G)$ denotes de family of finite subsets H of C such that $H \cap G(x) \neq \emptyset$ for all $x \in K$.

Proof: As at the beginning of the preceding proof, we note that our assumptions on G and f imply the lower semicontinuity of $\max_{y\in H\cap G(.)} f(.,y)$ for any $H \in \Sigma(G)$ and hence the correctness of the "min" appearing in the right hand side of the equality we are going to prove. The inequality $\geq$ is obvious. To prove the opposite one, let

$$\hat{v} = \sup_{H\in\Sigma(G)} \min_{x\in K} \max_{y\in H\cap G(X)} f(x,y)$$

and, for any $y \in C$, define

$$S_y = \{x \in K \mid f(x,y) \leq \hat{v}\} \cup (K\backslash G^{-1}(y)).$$

Clearly, the sets S_y are closed; moreover, they satisfy the finite intersection property. Indeed, let H be a finite subset of C. If $H \in \Sigma(G)$, from the definition of $\hat{v}$ it follows the existence of $\overline{x} \in K$ such that $f(\overline{x}, y) \leq \hat{v}$ for all $y \in H \cap G(\overline{x})$. Evidently, $\overline{x} \in \cap_{y \in H} S_y$. If, instead, $H \notin \Sigma(G)$, then there exists $\overline{x} \in K$ such that $H \cap G(\overline{x}) = \emptyset$. But then $\overline{x} \in \cap_{y \in H}(K \backslash G^{-1}(y)) \subset \cap_{y \in H} S_y$. Since K is compact and the sets S_y have the finite intersection property, we have $\cap_{y \in C} S_y \neq \emptyset$. It is easy to check that any $\overline{x} \in \cap_{y \in C} S_y$ satisfies $\sup_{y \in G(\overline{x})} f(\overline{x}, y) \leq \hat{v}$, whence the inequality $\leq$ in the statement immediately follows.

The following generalization of Ky Fan's inequality will be also used:

Lemma 1.3. Let K be a non–empty compact convex subset of a Hausdorff topological vector space $E, G : K \to 2^K$, $f : K \times K \to R$ and $\alpha = \sup_{x \in G(x)} f(x, x)$. Assume that, for each $x \in K$, the set $\{y \in G(x) \mid f(x, y) > \alpha\}$ is convex and, for each $y \in K$, the set $\{x \in G^{-1}(y) \mid f(x, y) > \alpha\}$ is open in K. Then, there exists $\overline{x} \in K$ such that

$$\sup_{y \in G(\overline{x})} f(\overline{x}, y) \leq \sup_{x \in G(x)} f(x, x).$$

Proof: Let $T : K \to 2^K$ be the mapping defined by

$$T(x) = \{y \in G(x) \mid f(x, y) > \alpha\}.$$

Let us suppose that $T(x) \neq \emptyset$ $(x \in K)$. Then, by Browder's theorem [4, Thm. 1], T has a fixed point $x_0 \in K$, i.e., a point such that $x_0 \in G(x_0)$ and $f(x_0, x_0) > \alpha$, thus contradicting the definition of α. Hence, there exists $\overline{x} \in K$ with $T(\overline{x}) = \emptyset$, which means that $\overline{x}$ satisfies the required conditions.

The assumptions of the preceding lemma hold, in particular, when G satisfies the hypotheses of Browder's theorem (i.e., when G has convex images and open inverse images) and the sets $\{y \in K \mid f(x, y) > \alpha\}$ $(x \in K)$ and $\{x \in K \mid f(x, y) > \alpha\}$ $(y \in K)$ are convex and open, respectively (hence, when the partial mappings $f(x, .)$ and $f(., y)$ are quasiconcave and lower semicontinuous, respectively). When $G(x) = K$ $(x \in K)$, Lemma 1.3 yields the usual Ky Fan's inequality.

Theorem 1.5. Under the assumptions of Theorem 1.4,

$$\inf_{q \in \Gamma(C,K)} \sup_{y \in G(q(y))} f(q(y), y) = \min_{x \in K} \sup_{y \in G(x)} f(x, y),$$

where $\Gamma(C, K)$ denotes the set of functions $q : C \to K$ whose restrictions to any polytope contained in C are continuous.

Proof: The inequality $\leq$ is obvious. To prove that $\geq$ also holds, let $q \in \Gamma(C,K)$. Using Lemma 1.2, we get:

$$\begin{aligned}
\min_{x\in K}\sup_{y\in G(x)} f(x,y) &= \sup_{H\in\Sigma(G)}\min_{x\in K}\max_{y\in H\cap G(x)} f(x,y) \leq \\
&\leq \sup_{H\in\Sigma(G)}\inf_{x\in K}\sup_{y\in co\,(H\cap G(x))} f(x,y) \leq \\
&\leq \sup_{H\in\Sigma(G)}\inf_{y'\in C}\sup_{y\in co\,\left(H\cap G(q(y'))\right)} f(q(y'),y) \leq \\
&\leq \sup_{H\in\Sigma(G)}\inf_{y'\in co\,H}\sup_{y\in co\,\left(H\cap G(q(y'))\right)} f(q(y'),y).
\end{aligned}$$

For any $H \in \sum(G)$ let $T_H : co\,H \to 2^{co\,H}$ be the mapping defined by $T_H(y') = co\left(H \cap G(q(y'))\right)$. Since $H \in \sum(G)$, the sets $T_H(y')$ are non–empty (and convex). Let $z \in co\,H$ and $y' \in T_H^{-1}(z)$. Then, by $z \in T_H(y')$, there exist $h_1,\ldots,h_p \in H \cap G(q(y'))$ and nonnegative numbers $\lambda_1,\ldots,\lambda_p$ such that $\sum_{i=1}^p \lambda_i = 1$ and $\sum_{i=1}^p \lambda_i h_i = z$. Since $q(y') \in G^{-1}(h_i)$ $(i = 1,\ldots,p)$, $q|_{co\,H}$ is continuous and the sets $G^{-1}(h_i)$ are open in K, there are neighborhoods N_i $(i = 1,\ldots,p)$ of y' in K such that $q(N_i) \subset G^{-1}(h_i)$ $(i = 1,\ldots,p)$. Taking their intersection we can assume that all the $N_i's$ coincide, so that we shall omit the subscript i. Thus, we have $h_i \in G(q(N))$ $(i = 1,\ldots,p)$ and hence, for any $y \in N \cap co\,H$, $z = \sum_{i=1}^p \lambda_i h_i \in co\left(H \cap G(q(y))\right) = T_H(y)$, i.e., $y \in T_H^{-1}(z)$. Therefore, $N \subset T_H^{-1}(z)$, which proves that the sets $T_H^{-1}(z)$ are open in $co\,H$. Then, by Lemma 1.3, there exists $y_H \in co\,H$ such that

$$\sup_{y\in co\,\left(H\cap G(q(y_H))\right)} f(q(y_H),y) \leq \sup_{y\in co\,\left(H\cap G(q(y))\right)} f(q(y),y).$$

Thus,

$$\begin{aligned}
\min_{x\in K}\sup_{y\in G(x)} f(x,y) &\leq \sup_{H\in\Sigma(G)}\sup_{y\in co\,\left(H\cap G(q(y_H))\right)} f(q(y_H),y) \leq \\
&\leq \sup_{H\in\Sigma(G)}\sup_{y\in co\,\left(H\cap G(q(y))\right)} f(q(y),y) \leq \\
&\leq \sup_{y\in G(q(y))} f(q(y),y).
\end{aligned}$$

Since $q \in \Gamma(C,K)$ is arbitrary, it follows that

$$\min_{x\in K}\sup_{y\in G(x)} f(x,y) \leq \inf_{q\in\Gamma(C,K)}\sup_{y\in G(q(y))} f(q(y),y).$$

The proof is complete.

When L is a topological vector space, the preceding theorem remains valid by replacing $\Gamma(C, K)$ by the set of continuous functions from C into K; the same proof applies.

The key result in proving Theorem 1.5 has been our generalization of Ky Fan's inequality (Lemma 1.3); in fact, both results are equivalent, since, when $C = K$, letting $q =$ the identity mapping in Theorem 1.5, we get

$$\sup_{y \in G(y)} f(y, y) \geq \min_{x \in K} \sup_{y \in G(x)} f(x, y),$$

i.e., Lemma 1.3.

2. APPROXIMATE FIXED POINTS

In this Section, by approximate fixed points of a multi–valued mappging from a topological space into itself we mean two points which are close one to the other and such that one of them belongs to the image of the other one. The results we will derive, following the pattern of the proof of the Browder fixed point theorem [4, Thm. 1], are based on the approximate selection theorems obtained in Section 1 and on the Brouwer fixed point theorem.

We start with the following consequence of Theorem 1.1, which is valid for arbitrary functions.

Theorem 2.1. Let K be a non–empty compact convex subset of a locally convex Hausdorff topological vector space E, D a dense subset of K and $f : D \to K$. Then for each neighborhood V of 0 in E there exists $x_V \in K$ such that $x_V \in co\, f\big((x_V + V) \cap D\big)$.

Proof: For Y_V and p_V as given by Theorem 1.1, the restriction of p_V to the polytope $co\, Y_V$ applies $co\, Y_V$ into itself whence, by the Brouwer fixed point theorem, there exists $x_V \in co\, Y_V \subset co\, f(D) \subset K$ such that

$$x_V = p_V(x_V) \in co\Big(Y_V \cap f\big((x + V) \cap D\big)\Big) \subset co\, f\big((x + V) \cap D\big).$$

Let us note that, in the particular case when K is a closed real interval $[a, b]$, Theorem 2.1 says that for any $\varepsilon > 0$ one can find $x_\varepsilon \in [a, b]$ and x'_ε, $x''_\varepsilon \in D$ such that $|x'_\varepsilon - x_\varepsilon| < \varepsilon$ and $|x''_\varepsilon - x_\varepsilon| < \varepsilon$.

The preceding theorem could be regarded as two weak in view of its great generality, due to the absence of hypoteses on f; in order to demonstrate that this is not so, we are now going to derive from it, as a corollary, the finite–dimensional version of the Kakutani fixed point theorem (sse, e.g., [6, Cor. 10.3.10]):

Corollary 2.1 (Kakutani's theorem). Let K be a non–empty compact convex set in R^n and $T : K \to 2^K$ an upper semicontinuous mapping with non–empty compact convex values. Then T has a fixed point.

Proof: Take any selection $f : K \to K$ of T. For $k = 1, 2, ...,$ applying Theorem 2.1 with $V = \{v \in R^n \mid \|v\| < \frac{1}{k}\}$, where $\| \ \|$ is any norm in R^n, by Carathéodory's theorem [10, Thm. 17.1] there exist $x^k, x_1^k, \ldots, x_{n+1}^k \in K$ and nonnegative $\lambda_1^k, \ldots, \lambda_{n+1}^k \in R$ such that $\|x_i^k - x^k\| < \frac{1}{k}(i = 1, \ldots, n+1)$, $\sum_{i=1}^{n+1} \lambda_i^k = 1$ and $x^k = \sum_{i=1}^{n+1} \lambda_i^k f(x_i^k)$. By a standard compactness argument, we know the existence of a sequence $k_1 < k_2 < \ldots < k_j < \ldots$ such that, for any $i = 1, \ldots, n+1$, we have $\lim_{j\to\infty} \lambda_i^{k_j} = \lambda_i$ and $\lim_{j\to\infty} f(x_i^{k_j}) = y_i$ for some $\lambda_i \geq 0$ and $y_i \in K$ and also $\lim_{j\to\infty} x^{k_j} = x$ for some $x \in K$. As $\|x_i^k - x^k\| < \frac{1}{k}$, the last equality implies that $\lim_{j\to\infty} x_i^{k_j} = x$ $(i = 1, \ldots, n+1)$ whence, as $f(x_i^{k_j}) \in T(x_i^{k_j})$ for all i, j and the graph of T is closed [4, p. 185], $y_i \in T(x)$. Therefore, using that $\sum_{i=1}^{n+1} \lambda_i = 1$, $x = \sum_{i=1}^{n+1} \lambda_i y_i$ and $T(x)$ is convex, we get $x \in T(x)$.

From Theorem 1.2 we obtain an approximate fixed point theorem for D–mappings:

Theorem 2.2. Let E and K be as in Theorem 2.1 and let $T : K \to 2^K$ be a D–mapping. Then for each neighborhood V of 0 in E there exist $x_V, x'_V \in K$ such that $x'_V - x_V \in V$ and $x_V \in T(x'_V)$.

Proof: By Theorem 1.2, there exist a polytope $P_V \subset co\,T(K) \subset K$, a continuous function $p_V : K \to P_V$ and $\delta_V : K \to V$ such that $p_V(x) \in T(x + \delta_V(x))$ for all $x \in K$. Then, applying the Brouwer fixed point theorem to the restriction of p_V to P_V, we get the existence of $x_V \in P_V \subset K$ such that

$$x_V = p_V(x_V) \in T(x_V + \delta_V(x_V));$$

thus, we can take $x'_V = x_V + \delta_V(x_V)$.

Corollary 2.2. Let $f : [a, b] \to [a, b]$ be a Darboux continuous function. Then for each $\varepsilon > 0$ there exist $x_\varepsilon, x'_\varepsilon \in [a, b]$ such that $|x'_\varepsilon - x_\varepsilon| < \varepsilon$ and $x_\varepsilon = f(x'_\varepsilon)$.

Corollary 2.3. Let E, K and T be as in Theorem 2.2 and let $\overline{T} : K \to 2^K$ be the smallest upper semicontinuous mapping with closed values which contains $\overline{T}$ pointwise. The T has a fixed point.

Proof: Let $\mathcal{N}$ denote the filter of neighborhoods of 0 in E. By Theorem 2.2 we can construct two nets $\{x_V\}_{V\in\mathcal{N}}$, $\{x'_V\}_{V\in\mathcal{N}}$ in K whose elements satisfy $x'_V - x_V \in V$ and $x_V \in T(x'_V)$ for all $V \in \mathcal{N}$. Since K is compact, $\{x_V\}_{V\in\mathcal{N}}$ has at least an accumulation point $x \in K$ (see, e.g., [6, Thm. 1.4. 33]). Let $N \in \mathcal{N}$. There exists $N_0 \in \mathcal{N}$ sych that $N_0 + N_0 \subset N$. Take $V \in \mathcal{N}$ such that $V \subset N_0$ and $x_V \in x + N_0$. We have

$$x'_V \in x_V + V \subset x + N_0 + N_0 \subset x + N$$

and $x_V \in x + N$, whence $(x'_V, x_V) \in (x, x) + N \times N$. Since $(x'_V, x_V) \in \operatorname{Graph} T$ and V is arbitrary, this shows that $(x, x) \in c\ell \operatorname{Graph} T$ whence, as $c\ell \operatorname{Graph} T = \operatorname{Graph} \overline{T}$ (see [4, p. 285]), $x \in \overline{T}(x)$.

Corollary 2.4. Any upper semicontinuous D–mapping with closed values from a non–empty compact convex subset of a locally convex Hausdorff topological vector space into itself has a fixed point.

3. MINIMAX INEQUALITIES AND RELATED RESULTS

In this Section, we give some minimax inequalities and similar results related to Ky Fan's inequality, involving several functions. Our first theorem generalizes Lemma 1 in [9]. We shall use the following lemma:

Lemma 3.1. Let X be a topological space, I any index set, Y_i a compact space and $T_i : X \to 2^{Y_i}$ an upper semicontinuous mapping with non–empty compact values for each $i \in I$. Then, the mapping $T : X \to 2^Y$, where $Y = \Pi_{i \in I} Y_i$, defined by $T(x) = \Pi_{i \in I} T_i(x)$, is upper semicontinuous, too.

Proof: By Tychonoff's theorem [8, p. 18], Y and the sets $T(x)$ are compact. Moreover, since the sets $\operatorname{Graph} T_i$ are closed [4, p. 285] and

$$\operatorname{Graph} T = \left\{ (x, y) \in X \times Y \mid (x, y_i) \in \operatorname{Graph} T_i \text{ for each } i \in I \right\},$$

one can easily prove that $\operatorname{Graph} T$ is also closed, whence T is upper semicontinuous.

Theorem 3.1. Let I be any index set and for each $i \in I$ let E_i be a locally convex Hausdorff topological vector space, K_i a non–empty compact convex subset of E_i, $G_i : K \to 2^{K_i}$ a continuous mapping with non–empty compact convex values, where $K = \Pi_{j \in I} K_j$ and $f_i : K \times K_i \to R$ a continuous function such that $f_i(x, .)$ is quasiconcave for any $x \in K$. Then there exists a fixed point $\overline{x} \in K$ of the mapping $G : K \to 2^K$ defined by $G(x) = \Pi_{i \in I} G_i(x)$ such that

$$f_i(\overline{x}, \overline{x}_i) = \max_{y \in G_i(\overline{x})} f_i(\overline{x}, y) \qquad (i \in I)$$

Proof: The topological product $\Pi_{j \in i} E_j$ is also Hausdorff and locally convex [8, p. 207] and K is compact. For each $i \in I$, let $T_i : K \to 2^{K_i}$ be the mapping defined by

$$T_i(x) = \left\{ z \in G_i(x) \mid f_i(x, z) = \max_{y \in G_i(x)} f_i(x, y) \right\}.$$

It is easy to prove that T_i is upper semicontinuous with non–empty compact convex values, whence, by Lemma 3.1, the mapping $T : K \to 2^K$ given by $T(x) = \Pi_{i \in I} T_i(x)$ also has these properties. Therefore, by Kakutani fixed point theorem [6, Cor. 10.3.10], T has a fixed point $\overline{x} \in K$. Clearly, $\overline{x}$ satisfies the required condition.

For some applications, the quasiconcavity assumption on the functions $f_i(x, .)$ that we have made in the preceding theorem may be too strong (see the comments following Corollary 3.1). In fact, Theorem 3.1 remains valid under the weaker assumption of convexity of the sets $T_i(x)$.

The next corollary generalizes Theorem 1.5 in [4], which is in fact the infinite version, in locally convex spaces, of the Nash Theorem on the existence of equilibrium points in non–cooperative games:

Corollary 3.1. Let I, E_i, K_i and K be as in Theorem 3.1 and for each $i \in I$ let $K_{\hat{i}} = \Pi_{j \in I \setminus \{i\}} K_j$ (so that we can identify any point $x \in K$ with a pair $(x_{\hat{i}}, x_i) \in K_{\hat{i}} \times K_i$ in a natural way), $C_i : K_{\hat{i}} \to 2^{K_i}$ a continuous mapping with non–empty compact convex values and $g_i : K \to R$ a continuous function such that $g_i(v, .)$ is quasiconcave for all $v \in K_{\hat{i}}$. Then there exists $\overline{x} \in K$ such that

$$\overline{x}_i \in C_i(\overline{x}_{\hat{i}}), \; g_i(\overline{x}) = \max_{y \in C_i(\overline{x}_{\hat{i}})} g_i(\overline{x}_{\hat{i}}, y) \qquad (i \in I).$$

Proof: Applying Theorem 3.1 to the mappings $G_i : K \to 2^{K_i}$ defined by $G_i(x) = C_i(x_{\hat{i}})$ and the functions $f_i : K \times K_i \to R$ given by $f_i(x, y) = g_i(x_{\hat{i}}, y) - g_i(x)$, we obtain the existence of $\overline{x} \in k$ such that $\overline{x}_i \in C_i(\overline{x}_{\hat{i}})$ and $f_i(\overline{x}, \overline{x}_i) = \max_{y \in G_i(\overline{x})} f_i(\overline{x}, y)$ for all $i \in I$; the last condition yields the desired quality for the $g_i's$.

According to what we have observed after Theorem 3.1, the assumption we have made in Corollary 3.1 on the quasiconcavity of the functions $g_i(v, .)$ can be relaxed by requiring only the sets

$$\left\{ z \in C_i(v) \mid g_i(v, z) = \max_{y \in C_i(v)} g_i(v, y) \right\} \qquad (i \in I, \; v \in K_{\hat{i}})$$

to be convex. This condition holds, e.g., in the case of generalized rational games [3], i.e., when each g_i is the ratio $\frac{M_i}{N_i}$ of two continuous functions $M_i, N_i : K \to R$ such that the partial mappings $M_i(v,.)$ and $N_i(v,.)$ are concave and convex, respectively, for all $v \in K_{\hat{i}}$, N_i is strictly positive and for each $v \in K_{\hat{i}}$ there exists $z \in C_i$ such that $M_i(v,z) \geq 0$. In [3], equilibrium points of a game of this type are related to the optimal solution of an open expanding economy model.

Theorem 3.1 has been proved using the Kakutani fixed point theorem. In a similar way, the following theorem is a consequence of Browder's theorem [4, Thm. 1].

Theorem 3.2. For each $i = 1, \dots, n$, let E_i be a Hausdorff topological vector space, K_i a non–empty compact convex subset of E_i, $G_i : K \to 2^{K_i}$, where $K = \Pi_{j=1}^n K_j$, a continuous mapping with non–empty compact convex values and $f_i : K \times K_i \to R$ an upper semicontinuous function such that, for each $x \in K$, $f_i(x,.)$ is quasiconcave and, for each $y \in K$, $f_i(.,y)$ is continuous. Then

$$\inf_{x \in G(x)} \max_{i=1,\dots,n} \left\{ \max_{y \in G_i(x)} f_i(x,y) - f_i(x,x_i) \right\} = 0 \,,$$

where $G : K \to 2^K$ is the mapping defined by $G(x) = \Pi_{i=1}^n G_i(x)$.

Proof: We only have to prove the inequality ≤ 0, since the opposite one is immediate. For any $\varepsilon > 0$ and $i \in \{1, \dots, n\}$, let $T_{\varepsilon,i} : K \to 2^{K_i}$ be the mapping defined by

$$T_{\varepsilon,i}(x) = \left\{ z \in G_i(x) \;\middle|\; \max_{y \in G_i(x)} f_i(x,y) - f_i(x,z) < \varepsilon \right\}$$

and denote by $T_\varepsilon : K \to 2^K$ the product mapping given by

$$T_\varepsilon(x) = \Pi_{i=1}^n T_{\varepsilon,i}(x) \,.$$

Our assumptions on the $G_i's$ and the $f_i's$ imply the convexity of the sets $T_\varepsilon(x)$ and the upper semicontinuity of the functions $\max_{y \in G_i(.)} f_i(.,y)$ (see, e.g., [2, p. 52, Thm. 5]). From the latter property, it follows that the sets $T_\varepsilon^{-1}(z) = \cap_{i=1}^n T_{\varepsilon,i}^{-1}(z_i)$ are open. Since, furthermore, $T_\varepsilon(x) \neq \emptyset$ for all $x \in K$, by the Browder fixed point theorem [4, Thm. 1] the mapping T_ε has a fixed point x^ε, i.e., a point x^ε which satisfies $x^\varepsilon \in G(x^\varepsilon)$ and

$$\max_{i=1,\dots,n} \left\{ \max_{y \in G_i(x^\varepsilon)} f_i(x^\varepsilon,y) - f_i(x^\varepsilon,x_i^\varepsilon) \right\} < \varepsilon \,.$$

Since ε can be made arbitrarily small, we easily get the required result.

Corollary 3.2. Under the assumptions of Theorem 3.2,

$$\inf_{x \in G(x)} \max_{y \in G_i(x)} f_i(x,y) \leq \sup_{x \in G(x)} f_i(x,x_i) \qquad i = 1, \dots, n$$

Finally, from Theorem 1.4 we are going to derive a vector version of Ky Fan's inequality [4]. We recall that a point $a \in R^n$ is said to be a weakly maximal element of a set $A \subset R^n$ (with respect to the usual componentwise partial ordering of R^n) if $a \in A$ and there is no $a' \in A$ with $a'_i > a_i (i = 1, \ldots, n)$ (see, e.g., [7, p. 105]).

Theorem 3.3. For each $i = 1, \ldots, n,$ let E_i and K_i be as in Theorem 3.2, $G_i : K \to 2^{K_i}$, where $K = \Pi_{j=1}^n K_j$, a mapping with non–empty convex images and open inverse images and $f_i : K \times K_i \to R$ a function such that $f_i(x, .)$ is quasiconvex for all $x \in K$ and $f_i(., y)$ is lower semicontinuous for all $y \in K_i$. Denote by $F : K \to R^n$ the vector function $F(x) = \big(f_1(x, x_1), \ldots, f_n(x, x_n)\big)$ and by $G : K \to 2^K$ the mapping $G(x) = \Pi_{i=1}^n G_i(x)$. Then, for any weakly maximal element $a \in R^n$ of $c\ell\ F\big(\{x \in K | x \in G(x)\}\big)$ there is an index $i = i(a) \in \{1, \ldots, n\}$ such that

$$\min_{x \in K} \sup_{y \in G_i(x)} f_i(x, y) \le a_i\, .$$

Proof: As in the proof of Theorem 1.4, we observe that our assumptions on the $G_i's$ and the $f_i's$ imply the lower semicontinuity of the functions $\sup_{y \in G_i(.)} f_i(., y)$, whence each of them attains its minimum on K. Let us suppose that $\min_{x \in K} \sup_{y \in G_i(x)} f_i(x, y) > a_i$ holds for all $i = 1, \ldots, n$. Then, by Theorem 1.4, we also have $\sup_{q \in S(G_i)} \inf_{x \in K} f_i\big(x, q(x)\big) > a_i \ (i = 1, \ldots, n)$, and hence for each i we can find $p_i \in S(G_i)$ such that $\inf_{x \in K} f_i\big(x, p_i(x)\big) > a_i$. Each p_i is a continuous function from K into a polytope $P_i \subset K_i$ and satisfies the condition $p_i(x) \in G_i(x)$ for all $x \in K$. Let $P = \Pi_{i=1}^n P_i$ and define $p : P \to P$ by $p(x) = \big(p_1(x), \ldots, p_n(x)\big)$. By the Brouwer fixed point theorem, there is a fixed point $\overline{x} \in P \subset K$ of p. But this point satisfies $\overline{x}_i = p_i(\overline{x}) \in G_i(\overline{x}) \ (i = 1, \ldots, n)$, i.e., $\overline{x} \in G(\overline{x})$, and

$$f_i(\overline{x}, \overline{x}_i) = f_i\big(\overline{x}, p_i(\overline{x})\big) \ge \inf_{x \in K} f_i\big(x, p_i(x)\big) > a_i \qquad (i = 1, \ldots, n),$$

which contradicts the weak maximality of a.

The particular case of Theorem 3.3 corresponding to $n = 1$ coincides with Lemma 1.3; if, moreover, $G(x) = K$ for all $x \in K$, then we obtain the classical Ky Fan's inequality. Let us also observe that, in the general case, the multi–valued mapping G satisfies the assumptions of the Browder fixed point theorem [4, Thm. 1] and therefore it has a fixed point, i.e., the set $c\ell\, F\big(\{x \in K \,|\, x \in G(x)\}\big)$ is non–empty. On the other hand, it is easy to check that any weakly maximal element of a set is also a weakly maximal element of its closure, but the converse is false; note that, when $n = 1$, the only weakly maximal element of a bounded from above set is its maximum (provided that it exists) and hence the weakly maximal element of its closure is the supremum of the set.

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