Algebraic Geometry 12 (6) (2025) 869–887
doi:10.14231/AG-2025-026

Families of simple Jacobians with many automorphisms

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To Gerard van der Geer, with admiration and friendship, on the occasion of his 75th birthday

Abstract

We study an explicit $(2g-1)$-dimensional family of Jacobian varieties of dimension $\frac{1}{2}(d-1)(g-1)$, arising from quotient curves of unramified cyclic coverings of prime degree d of hyperelliptic curves of genus $g \geqslant 2$. By using a deformation argument, we prove that the generic element of the family is simple. Furthermore, we completely describe their endomorphism algebra, and we show that they admit a rank $\frac{1}{2}(d-1)-1$ group of non-polarized automorphisms. As an application of these results, we prove the generic injectivity of the Prym map for étale cyclic coverings of hyperelliptic curves of odd prime degree under some slight numerical restrictions. This result generalizes in several directions previous results on genus 2.

1. Introduction

The study of subvarieties of the moduli space of curves parametrizing Jacobians with non-trivial endomorphisms $\mathrm{End}(JC) \supsetneq \mathbb{Z}$ goes back to the work of Shimura, who constructed the moduli of complex abelian varieties with this property, and many other authors. Notably, Ciliberto, van der Geer, and Teixidor [CGT92] showed that any irreducible subvariety of $\mathcal{M}_{g_0}$, whose generic point classifies a curve whose Jacobian has non-trivial endomorphisms, has codimension at least $g_0 - 1$. So explicit high-dimensional families of Jacobians of curves with many non-polarized automorphisms are difficult to find. Many natural examples come from curves which map to elliptic curves. Hence, the corresponding Jacobians are far from being simple. There are a few other explicit examples discussed in [Ell01]: they are obtained from branched covers of the projective line, and they admit real multiplication by a subfield of even index at most 10 in a primitive cyclotomic field. In [Ell01] (see reference therein) it is explained how, via this construction, one can obtain almost all the other examples discovered by Brumer, Mestre, and

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Received 5 December 2024, accepted in final form 5 September 2025.
2020 Mathematics Subject Classification 14H40, 14H10, 14K12, 14D07.
Keywords: Jacobian variety, algebraic curve, Prym map.

The first and the fourth authors were partially supported by the Departament de Recerca i Universitats de la Generalitat de Catalunya (2021 SGR 00697), the Proyecto de Investigación PID2023-147642NB-100 and the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). The third author is partially supported by PRIN project *Moduli spaces and special varieties* (2022). The third and the fourth authors are members of GNSAGA (INdAM). The fourth author was partially supported by INdAM-GNSAGA project CUP E55F22000270001.

Tautz, Top, and Verberkmoes [TTV91]. The geometry of such examples is so rich that they have been exploited in other interesting contexts [AP16, LO16, Spe22].

In this paper, we deal with a higher-dimensional generalization of one of the examples presented in [Ell01]. More precisely, we consider cyclic unramified coverings $f\colon \widetilde{C} \to C$ of irreducible complex smooth hyperelliptic curves C, of degree an odd prime number $d = 2k+1$, fitting in the commutative diagram

$$\begin{tikzcd}
 & \widetilde{C} \arrow[dl, "\pi_0"'] \arrow[dr, "f"] & \\
C_0 \arrow[dr, "d:1"'] & & C . \arrow[dl, "2:1"] \\
 & \mathbb{P}^1 &
\end{tikzcd} \tag{1.1}$$

The curve $\widetilde{C}$ admits a group of automorphisms isomorphic to the dihedral group D_d generated by an automorphism σ of order d such that $C = \widetilde{C}/\langle\sigma\rangle$ and the lift j of the hyperelliptic involution on C. The quotient curve $C_0 := \widetilde{C}/\langle j\rangle$ is then of genus $g_0 := k(g-1)$, and by varying the hyperelliptic curve C, we obtain a $(2g-1)$-dimensional family of curves in $\mathcal{M}_{g_0}$. One of the main results in this paper is that the Jacobian of a general curve in this family is simple. We show this using monodromy arguments on the moduli space $\widetilde{\mathcal{RH}_g[d]}$ of classes of pairs (C, η) with C a hyperelliptic curve of genus g and $\eta \in JC[d] \smallsetminus \{0\}$.

In diagram (1.1), the quotient map $\pi_0\colon \widetilde{C} \to C_0$ is ramified, so the pullback map π_0^* is injective. Thus, we identify JC_0 with its image in $J\widetilde{C}$. It is known [Rie83] that the automorphisms $\sigma^i + \sigma^{-i}$, for $i = 1, \dots, k$, of $J\widetilde{C}$ restrict to automorphisms of JC_0, denoted by β_i. It turns out that these automorphisms of JC_0 do not preserve the polarization (see [NOS24, Proposition 3.1]). Our main theorem is the following.

Theorem 1.1. *For a generic cyclic unramified covering $f\colon \widetilde{C} \to C$ of odd prime degree d, the Jacobian JC_0 is simple, and for $d \geqslant 5$ the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(JC_0)$ of JC_0 is the totally real field $\mathbb{Q}(\xi + \xi^{-1})$, where ξ is a primitive dth root of the unity. Moreover, the group of automorphisms has rank $k-1$ and is generated by any $k-1$ elements in $\{\beta_1, \dots, \beta_k\}$.*

The idea of the proof is to show that the infinitesimal deformations of $(\widetilde{C}, C)$ which preserve the hyperelliptic structure "rigidify" the cotangent space to JC_0. In this way, we deduce the simplicity, as well as the description of the endomorphism algebra. Hence, by means of Dirichlet's unit theorem, we obtain the description of the automorphism group.

Theorem 1.1 has a direct application to Prym theory. In order to state our second result, we need to introduce some notation. Let $\mathcal{RH}_g[d]$ denote the coarse moduli space of cyclic unramified degree d covers of hyperelliptic genus g curves. As before, d is an odd prime number. It is well known that every element $f\colon \widetilde{C} \to C$ in $\mathcal{RH}_g[d]$ can be described by a pair $(C, \langle\eta\rangle)$, where C is a hyperelliptic genus g curve and $\eta \in JC \smallsetminus \{0\}$ is a non-trivial d-torsion point. The Prym variety $P(\widetilde{C}, C)$ of the covering is defined as the connected component containing the origin of the kernel of the Norm map $J\widetilde{C} \to JC$, and thus $\dim P(\widetilde{C}, C) = (d-1)(g-1)$. Moreover, the principal polarization on $J\widetilde{C}$ induces a polarization τ on $P(\widetilde{C}, C)$ of type $\delta = (1, \dots, 1, d, \dots, d)$, with d repeated $g-1$ times. In this setting one defines the Prym map

$$\mathcal{P}_g[d]\colon \mathcal{RH}_g[d] \longrightarrow \mathcal{A}^{\delta}_{(g-1)(d-1)}$$

as the map sending the isomorphism class of the covering $f\colon \widetilde{C} \to C$ to the isomorphism class of its Prym variety $(P(\widetilde{C}, C), \tau)$.

The case of double coverings has been treated by Mumford in [Mum74], where it is shown that $\mathcal{RH}_g[2]$ consists of several irreducible components. In [AP16], the authors proved that the generic fiber of the Prym map is positive dimensional for $d = 3$ and $g \leqslant 4$, as well as for $d = 5$ and $g = 2$. When $d = 7$ and $g = 2$, the map $\mathcal{P}_2[7]$ is dominant onto its image, and it is generically finite of degree 10 (see [LO16]). Recently, the generic finiteness of $\mathcal{P}_g[d]$ for $d \geqslant 6$ and $g = 2$ was proved in [Ago20]. This has been extended in [NOS24] by proving the generic Torelli theorem for d prime and such that $\frac{1}{2}(d-1)$ is also a prime (the so-called Sophie Germain primes). As a consequence of Theorem 1.1, we prove the following.

THEOREM 1.2. *Assume that d is a prime number with $(d-1)(g-1) \geqslant 7$ and $d \geqslant 3$, and that $g \geqslant 2$ is not congruent to 3 modulo d. Then the Prym map $\mathcal{P}_g[d]$ is generically injective.*

This improves the main theorem in [NOS24] in two directions: it removes the assumption of $\frac{1}{2}(d-1)$ being prime in genus 2, and it extends to hyperelliptic curves of genus $g \geqslant 3$ (under some slight numerical constrains). One can also remove the congruence condition if $g > 2d - 1$ (see Remark 4.5).

Roughly speaking, the proof is based on the existence of a (non-polarized) isomorphism $P \to JC_0 \times JC_0$, where C_0 is, as before, the quotient of $\widetilde{C}$ by a lifting j of the hyperelliptic involution on C. By using the endomorphisms on P and the polarization τ, one can intrinsically recover the curve C_0 and the automorphisms β_i of the Jacobian JC_0 from (P, τ). This part uses the simplicity of JC_0 and some properties of the endomorphism ring $\operatorname{End}(JC_0)$. The last part of the proof relies on a “theta-duality” procedure inspired by the properties of the Fourier–Mukai transform.

Furthermore, under a more general assumption on the degree d, we prove the following.

THEOREM 1.3. *For $g \geqslant 3$ and $d \geqslant 5$, and for $g = 2$ and $d \geqslant 6$, the Prym map $\mathcal{P}_g[d]$ is generically finite onto its image.*

The generic finiteness of the Prym map was proved for $g = 2$ in [Ago20]. The argument for $g \geqslant 3$ in Section 3 is independent of the rest of the paper. The expectation is that for all the cases of Theorem 1.3, the Prym map $\mathcal{P}_g[d]$ is actually generically injective.

2. The simplicity of JC_0

In this section, we introduce some notation and prove Theorem 1.1. In the rest of the paper, d is an odd prime number, and we set $k := \frac{1}{2}(d-1)$. Let $f\colon \widetilde{C} \to C$ be a cyclic étale cover of a hyperelliptic curve C of genus g. Let $\eta \in JC$ be a non-trivial d-torsion point such that $\langle \eta \rangle$ equals the kernel of $f^*\colon JC \to J\widetilde{C}$. Let $\mathcal{RH}_g[d]$ be the coarse moduli space of cyclic étale covers of hyperelliptic curves of degree d. In order to use a monodromy argument, we will need the irreducibility of $\mathcal{RH}_g[d]$ and of the moduli space $\widetilde{\mathcal{RH}}_g[d]$ of classes of pairs $[C, \rho]$, where $\rho \in JC[d] \smallsetminus \{0\}$. Notice that this is not true for $d = 2$ by [Mum74, Section 7] (see also [BF86] and [Nar96]).

PROPOSITION 2.1. *The moduli spaces $\mathcal{RH}_g[d]$ and $\widetilde{\mathcal{RH}}_g[d]$ are irreducible for $d \geqslant 3$.*

Proof. The irreducibility of the moduli space $\mathcal{RH}_g[d]$ can be deduced from the proof of [BF86, Theorem 1]. The theorem is proved for m-gonal curves with $m \geqslant 3$ and cyclic étale coverings of degree $n \geqslant 2$, but its proof can be easily extended to hyperelliptic curves ($m = 2$) and n odd. Following the notation of [BF86], it is enough to show that the conjugacy class of $(\alpha_{ij}; (i\ j))$ contains the element $(\mathbf{0}; (1\ 2)) \in \mathbb{Z}_n^2 \rtimes S_2$. Setting $\tau = ((-\alpha, 0, \dots, \alpha, 0, \dots, 0); (1\ 2))$, one checks that $\tau \cdot (\alpha_{ij}; (i\ j)) \cdot \tau^{-1} = (\mathbf{0}; (1\ 2))$. For the sake of clarity, we include a more intuitive argument by induction. Observe that there is a natural finite unramified map $\widetilde{\mathcal{RH}_g}[d] \to \mathcal{RH}_g[d]$; hence it is enough to prove the irreducibility of $\widetilde{\mathcal{RH}_g}[d]$. On the other hand, there is nothing to prove for $g = 1, 2$ since in this case all curves are hyperelliptic. We work in the Deligne–Mumford compactification $\overline{\mathcal{H}}_{g,d}$ of $\widetilde{\mathcal{RH}_g}[d]$, which has a finite map on the closure of the hyperelliptic locus $\overline{\mathcal{H}}_g$ in $\overline{\mathcal{M}}_g$. Let us consider a hyperelliptic curve H_0 of genus $g-2$ with two marked Weierstrass points $w, w' \in H_0$. Let E_1, E_2 be two elliptic curves with marked origins O, O'. By gluing w with O and w' with O', we obtain a semistable hyperelliptic curve $H \in \overline{\mathcal{H}}_g$ whose normalization is the disjoint union $H_0 \sqcup E_1 \sqcup E_2$. Observe that $JH = JH_0 \times E_1 \times E_2$. Let $(a, b, c) \in JH[d]$ be any d-torsion point in JH. We want to connect $(H, (a, b, c)) \in \overline{\mathcal{H}}_{g,d}$ with $(H, (0, p, 0))$, where p is a fixed d-torsion point in E_1. First, we observe that, fixing H_0 and E_2, we can use the irreducibility in genus 1 to connect $(0, p, 0)$ with $(0, b, 0)$. If $a = c = 0$, we are done. Otherwise, we use the irreducibility of $\overline{\mathcal{H}}_{g-1,d}$ (fixing E_1) to connect (a, b, c) with $(a', b, 0)$. Now fixing E_2 and applying induction again, we can connect $(a', b, 0)$ with $(0, b, 0)$, and we are done. □

Let σ be a generator of the Galois group of f and ι the hyperelliptic involution of C. We have $\langle\sigma\rangle \cong \mathbb{Z}/d$. We denote by σ as well the induced automorphism on $J\widetilde{C}$. Since the covering $\widetilde{C}$ is defined as $\operatorname{Spec}(\mathcal{O}_C \oplus \eta \oplus \cdots \oplus \eta^{d-1})$ and $\iota^*\eta^i = \eta^{d-i}$, the hyperelliptic involution of C lifts to an involution j on $\widetilde{C}$. The resulting covering $\widetilde{C} \to \mathbb{P}^1$ is Galois with Galois group the dihedral group $D_d = \langle j, \sigma \mid j^2 = \sigma^d = 1,\ j\sigma j = \sigma^{-1}\rangle$. Notice that all the involutions $j\sigma^i \in D_d$, with $i = 0, \dots, d-1$, yield intermediate quotient curves $C_i := \widetilde{C}/\langle j\sigma^i\rangle$. Since all the involutions are conjugate, the corresponding curves C_i are all isomorphic. Hence, from now on we focus on C_0.

Let χ be a generating character of $\langle\sigma\rangle$; that is, $\chi(\sigma) = \xi$, where ξ is a primitive dth root of the unity that we fix from now on. By means of the projection formula, we obtain the decomposition

$$H^0(\widetilde{C}, \omega_{\widetilde{C}}) = \bigoplus_{i=0}^{d-1} H^0\big(C, \omega_C \otimes \eta^i\big)\,, \tag{2.1}$$

where $H^0(C, \omega_C \otimes \eta^i)$ is the eigenspace associated with the character χ^i. By the Riemann–Roch formula, we have $h^0(C, \omega_C \otimes \eta^i) = g - 1$ for $i > 0$. Notice that the g-dimensional vector space $H^0(C, \omega_C)$ corresponds to the eigenspace with eigenvalue 1; thus it is the only one which is invariant under the action of σ on $H^0(\widetilde{C}, \omega_{\widetilde{C}})$.

Let $P = P(\widetilde{C}, C)$ be the Prym variety of the covering f. It is an abelian variety of dimension $(d-1)(g-1)$ and polarization of type $(1, \dots, 1, d, \dots, d)$ with d appearing $g-1$ times. The dual of the tangent space of P at 0 decomposes as follows:

$$T_0P^* = \bigoplus_{i=1}^{d-1} H^0\big(C, \omega_C \otimes \eta^i\big)\,. \tag{2.2}$$

Now we describe the tangent space to JC_0. Let s be a section in $H^0(C, \omega_C \otimes \eta^i) \subset H^0(\widetilde{C}, \omega_{\widetilde{C}})$. We have $\sigma(s) = \xi^i s$. Furthermore, the relations in the dihedral group imply the equalities

$$\sigma j(s) = j\sigma^{-1}(s) = j(\xi^{-i}s) = \xi^{-i}j(s)\,. \tag{2.3}$$

Hence, if $s \in H^0(C, \omega_C \otimes \eta^i)$, then $js \in H^0(C, \omega_C \otimes \eta^{-i})$. Let us define the vector spaces

$$V(i) := \big\{s + js \mid s \in H^0\big(C, \omega_C \otimes \eta^i\big)\big\} \subset H^0\big(C, \omega_C \otimes \eta^i\big) \oplus H^0\big(C, \omega_C \otimes \eta^{-i}\big)$$

for $i = 1, \dots, k$.

We have the following.

PROPOSITION 2.2. *The cotangent space of JC_0 at the origin is*

$$T_0 JC_0^* = \bigoplus_{i=1}^{k} V(i)\,. \tag{2.4}$$

Proof. This follows from $H^0(C_0, \omega_{C_0}) = H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\langle j\rangle}$ and equalities (2.1) and (2.3). ☐

Remark 2.3. Let us observe that

$$\big(\sigma^i + \sigma^{-i}\big)(s + js) = (1 + j)\big(\sigma^i + \sigma^{-i}\big)(s) \quad \text{for all } s \in H^0\big(C, \omega_C \otimes \eta^l\big)\,.$$

It follows that the action of $\beta_i = (\sigma^i + \sigma^{-i})_{|JC_0}$ on $T_0 JC_0^*$ is diagonalizable with eigenspaces $V(l)$ and corresponding eigenvalues $\xi^{il} + \xi^{-il}$ for $l = 1, \dots, k$.

Now, let ϵ be in $H^1(C, T_C)^{\langle\iota\rangle}$, that is, let ϵ be an infinitesimal deformation of C preserving the hyperelliptic structure. Such deformations come from deformations of $(C, \langle\eta\rangle)$; in other words, we see these deformations as deformations of $\widetilde{C}$. Therefore, we have $H^1(C, T_C)^{\langle\iota\rangle} \subset H^1(\widetilde{C}, T_{\widetilde{C}})$. The equivariance of the multiplication map yields the following.

PROPOSITION 2.4. *The multiplication maps*

$$\begin{aligned} m_i \colon H^1(C, T_C) \cdot \big(H^0\big(C, \omega_C \otimes \eta^i\big) \oplus H^0\big(C, \omega_C \otimes \eta^{-i}\big)\big) \\ \longrightarrow H^1\big(C, \eta^i\big) \oplus H^1\big(C, \eta^{-i}\big) \cong \big(H^0\big(C, \omega_C \otimes \eta^{-i}\big) \oplus H^0\big(C, \omega_C \otimes \eta^i\big)\big)^* \end{aligned}$$

restrict as follows:

$$H^1(C, T_C)^{\langle\iota\rangle} \cdot V(i) \longrightarrow V(i)^*\,. \tag{2.5}$$

In particular, we consider the "diagonal" multiplication maps $\overline{m}_i \colon V(i) \to H^0(C, \omega_C^2)^{\langle\iota\rangle}$ sending $s + js$ to $s \cdot js$ for $i = 1, \dots, k$. Let L_i be the image of $\overline{m}_i$. We have the following.

LEMMA 2.5. *Let C be a general hyperelliptic curve. Then $L_i \cap L_l = \{0\}$ for all $i \neq l$.*

Proof. Towards a contradiction, let us assume that $L_i \cap L_l \neq \{0\}$ for some $i \neq l$. Then we have $s \cdot js = t \cdot jt$ for some non-trivial $s \in H^0(C, \omega_C \otimes \eta^i)$ and $t \in H^0(C, \omega_C \otimes \eta^l)$. We set $s' = js$ and $t' = jt$, and we denote by $D(s) = p_1 + \dots + p_{2g-2}$ and $D(t) = q_1 + \dots + q_{2g-2}$ the divisors of s and t, respectively. Under our assumption, we obtain an equality of divisors

$$p_1 + \dots + p_{2g-2} + \iota(p_1) + \dots + \iota(p_{2g-2}) = q_1 + \dots + q_{2g-2} + \iota(q_1) + \dots + \iota(q_{2g-2})\,.$$

Now we set

$$D(s) = A + \iota B \quad \text{and} \quad D(s') = B + \iota A \quad \text{with } A, B \in \operatorname{Div}(C) \tag{2.6}$$

in such a way that $A + B = D(t)$. Exchanging η with η^{-1} if necessary, we may assume that $\deg B \geqslant g - 1$. Thus,

$$\mathcal{O}_C(D(t) - D(s')) = \mathcal{O}_C(A - \iota A) = \eta^{l-i} \neq \mathcal{O}_C\,,$$

with $\deg A \leqslant g - 1$.

Now, we claim that $A-\iota A$ cannot be a d-torsion point so that we get a contradiction and can conclude. In order to prove the claim, let us fix $A=p_1+\dots+p_{g-1}$ and assume that $d\,A\sim d\,\iota A$. So we can construct a map $u\colon C\to\mathbb{P}^1$ of degree $d(g-1)$ whose fibers over 0 and over the point at infinity are given by dA and $d\,\iota A$, respectively. Denoting by the h the $2:1$ map defined by the hyperelliptic involution on C, we get the following diagram:

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, "h"'] & \mathbb{P}^1 \arrow[d, "g"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd},$$

where g is the quotient map under the action $z\mapsto z^{-1}$. The commutativity of the diagram follows from the fact that $z\mapsto z^{-1}$ exchanges the zeros and poles of u. The Riemann–Hurwitz formula tells us that

$$2g-2=-2d(g-1)+2(d-1)(g-1)+r',$$

where $r'=4(g-1)$ is the degree contribution of the ramification divisor over $\mathrm{Branch}(u)\smallsetminus\{0,\infty\}$, where $\mathrm{Branch}(u)$ is the branch locus of u. The commutativity of the diagram forces $\mathrm{Branch}(u)$ to be of the form

$$\{b_1,\dots,b_{2(g-1)},b_1^{-1},\dots,b_{2(g-1)}^{-1}\}\subset\mathbb{C}^*.$$

By a moduli count, we get that the set of hyperelliptic curves with such a divisor A depends on strictly less than $2g-1$ parameters; hence C cannot be general. Notice that we are assuming that the divisor A is simple and does not contain Weierstrass points. If this is not the case, then we simply run the same argument with a map u of smaller degree. The same conclusion is then obtained. □

In order to better describe the multiplication map in (2.5), we first analyze how we can exhibit special deformations lying in $H^1(C,T_C)^{\langle\iota\rangle}$. We have the following.

LEMMA 2.6. *Let $D\in\mathrm{Div}(C)$ be an effective divisor (of degree less than or equal to $2g-2$), and assume that $D=F+A+\iota A+W$, where W is a combination of Weierstrass points and F does not contain divisors of the form $x+\iota(x)$ for $x\in C$. Then the intersection*

$$H^0(C,T_C(D)_{|D})\cap H^1(C,T_C)^{\langle\iota\rangle}\subset H^1(C,T_C)$$

is generated by the Schiffer deformations $\{\epsilon_{a_i}+\epsilon_{\iota(a_i)},\epsilon_{w_l}\}_{i,l}$, where $A=\sum a_i$ and $W=\sum w_l$. In particular, $\dim H^0(C,T_C(D)_{|D})\cap H^1(C,T_C)^{\langle\iota\rangle}\leqslant\deg(A)+\deg(W)$.

Proof. From the exact sequence

$$0\longrightarrow T_C\longrightarrow T_C(D)\longrightarrow T_C(D)_{|D}\longrightarrow 0,\qquad(2.7)$$

we deduce that $H^0(C,T_C(D)_{|D})$ injects inside $H^1(C,T_C)$; by definition, the image is generated by the Schiffer deformations ϵ_p for $p\in D$. By considering the elements left invariant by the action of ι, we get that the intersection $H^0(C,T_C(D)_{|D})\cap H^1(C,T_C)^{\langle\iota\rangle}$ is as claimed. The statement on the dimension follows easily by recalling that we are admitting divisors D with non-reduced support. □

This allows us to state the following.

LEMMA 2.7. *For every $s\in H^0(C,\omega_C\otimes\eta^i)$, the map $H^1(C,T_C)^{\langle\iota\rangle}\to V(i)^*$ sending ϵ to $\epsilon s+\epsilon js$ is surjective.*

Proof. Let D be the divisor which is the intersection of the divisors attached to s and js. As in Lemma 2.6, we put $D = F + A + \iota A + W$, and we consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_C(D) \xrightarrow{(\cdot s, \cdot js)} (\omega_C \otimes \eta^i) \oplus (\omega_C \otimes \eta^{-i}) \longrightarrow \omega_C^2(-D) \longrightarrow 0\,.$$

After tensoring with T_C and combining with the exact sequence in (2.7), in cohomology we have

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^0(C, T_C(D)_{|D}) \arrow[d] & & \\
 & & H^1(C, T_C) \arrow[d] \arrow[dr, "\alpha"] & & \\
0 \arrow[r] & H^0(C, \omega_C(-D)) \arrow[r] & H^1(C, T_C(D)) \arrow[r] \arrow[d] & H^1(C, \eta^i) \oplus H^1(C, \eta^{-i}) \arrow[r] & \cdots\,. \\
 & & 0 & &
\end{tikzcd}$$

We claim that $\dim(\ker\alpha) \cap H^1(C, T_C)^{\langle\iota\rangle} \leqslant g$. Indeed, the intersection $(\ker\alpha) \cap H^1(C, T_C)^{\langle\iota\rangle}$ applies to $H^0(C, \omega_C(-D))$, and therefore there is an exact sequence

$$0 \longrightarrow H^0(C, T_C(D)_{|D})^{\langle\iota\rangle} \longrightarrow \ker\alpha \cap H^1(C, T_C)^{\langle\iota\rangle} \longrightarrow H^0(C, \omega_C(-D)) \longrightarrow \cdots\,.$$

By Lemma 2.6, we know that $\dim H^0(C, T_C(D)_{|D})^{\langle\iota\rangle} \leqslant \deg(A) + \deg(W)$. On the other hand, since $h^0(C, \omega_C(-D)) \leqslant g - (\deg(A) + \deg(W))$, we have

$$\dim\ker\alpha \cap H^1(C, T_C)^{\langle\iota\rangle} \leqslant \dim H^0(C, T_C(D)_{|D})^{\langle\iota\rangle} + H^0(C, \omega_C(-D)) \leqslant g\,.$$

Since the image of α restricted to the invariant part $H^1(C, T_C)^{\langle\iota\rangle}$ (of dimension $2g-1$) is necessarily a subset of $V(i)^*$, which has dimension $g-1$, we have shown the surjectivity. $\square$

From now on, we assume that $(C, \langle\eta\rangle)$ is general, and we focus on the simplicity of JC_0. Let us assume that there exists an abelian subvariety $A \subset JC_0$. We shall show that it is trivial; that is, either $A = \{0\}$ or $A = JC_0$. Let $W := T_0A^*$. By using the polarizations on JC_0 and A, we can think of W as a subspace of $\oplus_{i=1}^k V(i)$. Let ϵ be in $H^1(C, T_C)^{\langle\iota\rangle}$. By (2.5), we have $\epsilon \cdot \bigoplus V(i) \subset \bigoplus V(i)^*$. In the same way, since we assume that A is an abelian subvariety of JC_0 deforming with $(C, \langle\eta\rangle)$, we have $\epsilon \cdot W \subset W^*$. Using the existence of a polarization, we identify W^* with $\overline{W}$. First, we observe the following.

Proposition 2.8. *If $W \cap V(i) \neq \{0\}$, then $V(i) \subseteq W$.*

Proof. Indeed, let $s_i + js_i$ be an element in this intersection. The map $H^1(C, T_C)^{\langle\iota\rangle} \to V(i)^*$ sending ϵ to $\epsilon s_i + \epsilon j s_i$ is surjective by Lemma 2.7. Therefore, for every $t_i + jt_i \in V(i)^*$, there exists an $\epsilon \in H^1(C, T_C)^{\langle\iota\rangle}$ such that $\epsilon(s_i + js_i) = t_i + jt_i$. By the invariance of W by deformations and conjugation, we obtain $V(i) \subseteq W$, as desired. $\square$

Now, let $0 \neq \Omega$ be an element of W. By equality (2.4), we have a decomposition $\Omega = \sum_{i=1}^k \Omega_i$ with $\Omega_i = s_i + js_i \in V(i)$ and $s_i \in H^0(C, \omega_C \otimes \eta^i)$. We have the following "killing lemma".

LEMMA 2.9. *Let $1 \leqslant i, l \leqslant k$ be two different indices, and let $\Omega_i \in V(i)$ and $\Omega_l \in V(l)$ be two non-trivial elements. Then, there exists an $\epsilon \in H^1(C, T_C)^{\langle \iota \rangle}$ such that $\epsilon\Omega_i = 0$ and $\epsilon(\Omega_l) \neq 0$.*

Proof. Let $s_i \in H^0(C, \omega_C \otimes \eta^i)$ be such that $\Omega_i = s_i + js_i$. The vanishing of $\epsilon\Omega_i$ for some $\epsilon \in H^1(C, T_C)^{\langle \iota \rangle}$ is equivalent to the conditions $\epsilon s_i = \epsilon j s_i = 0$. Indeed, ϵs_i belongs to $H^1(C, \eta^i)$ while $\epsilon j s_i$ belongs to $H^1(C, \eta^{-i})$. Since ϵ and j commute, it is enough to look for an ϵ such that $\epsilon s_i = 0$.

Furthermore, let us observe that we can apply a deformation $\epsilon \in H^1(C, T_C)^{\langle \iota \rangle}$ to the initial Ω_i and Ω_l and then conjugate to obtain new elements in the same subspaces. By Lemma 2.7 applied to the section s_i, we can then assume that s_i is "generic"; namely, $R = (s_i)_0$ does not contain Weierstrass points or divisors of the form $x + \iota x$ for $x \in C$. Then, we consider the short exact sequence attached to s_i:

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\cdot s_i} \omega_C \otimes \eta^i \longrightarrow \omega_C \otimes \eta^i_{|R} \longrightarrow 0\,.$$

Tensoring with T_C, we obtain

$$0 \longrightarrow T_C \longrightarrow \eta^i \longrightarrow \eta^i_{|R} \longrightarrow 0\,,$$

which, in cohomology, yields

$$0 \longrightarrow \mathbb{C}^{2g-2} \longrightarrow H^1(C, T_C) \xrightarrow{\alpha} H^1(C, \eta^i) \longrightarrow 0\,. \tag{2.8}$$

Notice that $\alpha(\epsilon) = \epsilon s_i \in H^0(C, \omega_C \otimes \eta^{-i})^* \cong H^1(C, \eta^i)$. By Grassmann's formula, we have

$$\dim \operatorname{Ker}(\alpha) \cap H^1(C, T_C)^{\langle \iota \rangle} \geqslant 2g - 2 + \dim H^1(C, T_C)^{\langle \iota \rangle} - \dim H^1(C, T_C) = g\,. \tag{2.9}$$

We now put $\Omega_l = s_l + js_l \neq 0$. Let D be the divisor which is the intersection of the divisors attached to s_i and s_l. Recall that, under our assumption of generality, D does not contain Weierstrass points or divisors of the form $x + \iota x$. We consider the sequence

$$0 \longrightarrow T_C(D) \longrightarrow \eta^i \oplus \eta^l \longrightarrow \omega_C \otimes \eta^{i+l}(-D) \longrightarrow 0\,,$$

where the first map is given by the multiplications with the sections s_i and s_l on each summand and the second sends (x, y) to $s_l x - s_i y$. We can assume that $\deg D \leqslant g - 2$. Indeed, if $\deg D \geqslant g$, we can use the same argument replacing s_i with js_i, while the case $\deg D = g - 1$ is excluded by Lemma 2.5 since, in this case, up to a constant, $s_i j(s_i) = s_l j(s_l)$. Hence, we consider the following diagram in cohomology:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^0(C, T_C(D)_{|D}) \arrow[d] & & \\
 & & H^1(C, T_C) \arrow[d] \arrow[dr, "\beta"] & & \\
0 \arrow[r] & H^0(C, \omega_C \otimes \eta^{i+l}(-D)) \arrow[r] & H^1(C, T_C(D)) \arrow[r] \arrow[d] & H^1(C, \eta^i) \oplus H^1(C, \eta^l) \arrow[r] & \cdots\,. \\
 & & 0 & &
\end{tikzcd}$$

In order to prove our statement, we need to show that there exists an $\epsilon \in H^1(C, T_C)^{\langle \iota \rangle}$ such that $\epsilon \cdot s_i = 0$ and $\epsilon \cdot s_l \neq 0$ for $l \neq i$ and $\Omega_l \neq 0$. In other words, we search for an ϵ in $\ker(\alpha) \cap H^1(C, T_C)^{\langle\iota\rangle}$ such that $\beta(\epsilon) \neq 0$. As in Lemma 2.7, we have

$$\dim \ker(\beta) \cap H^1(C, T_C)^{\langle\iota\rangle} \leqslant \dim H^0(C, T_C(D)_{|D})^{\langle\iota\rangle} + H^0\big(C, \omega_C \otimes \eta^{i+l}(-D)\big)\,.$$

Under our assumption of generality on D, by Lemma 2.6, we have $\dim H^0(C, T_C(D)_{|D})^{\langle\iota\rangle} = 0$, while $h^0(C, \omega_C \otimes \eta^{i+l}(-D)) \leqslant g-1$. Using (2.9), we conclude. □

Now we are ready to describe completely the cotangent space W.

PROPOSITION 2.10. *We have* $W = \bigoplus_{i\in I} V(i)$ *for some* $I \subset \{1, \ldots, k\}$.

Proof. Let $\Omega = \sum_{i\in I} \Omega_i \in W$ for a certain subset of indices $I \subset \{1, \ldots, k\}$, and assume that $\Omega_i \neq 0$ for $i \in I$. We can assume without lost of generality that $I = \{1, \ldots, l\}$ for $1 \leqslant l \leqslant k$.

We claim that $W \cap V(i) \neq \{0\}$ for all $i = 1, \ldots, l$. If $l = 1$, there is nothing to prove; hence we assume that $l \geqslant 2$. We will prove that $W \cap V(1) \neq \{0\}$; the same argument works for the other indices. By Lemma 2.9, there exists a deformation $\epsilon \in H^1(C, T_C)^{\langle\iota\rangle}$ sending Ω_2 to zero and such that $\epsilon\Omega_1$ is not zero. This means that $\epsilon\Omega = \sum_{i\in I\setminus\{2\}} \epsilon\Omega_i$. As already remarked, we have $\epsilon\Omega \in \overline{W}$; thus $\overline{\epsilon\Omega} \in W$.

Notice that $\epsilon\Omega_i = \epsilon s_i + \epsilon j s_i$ belongs to $\overline{V(i)}$. Hence, using conjugation again, we obtain that $\overline{\epsilon\Omega_i}$ belongs to $V(i)$. Now we start again with the new element $\overline{\epsilon\Omega}$ which has at least one index less. Iterating, we obtain an element in W which belongs to $V(1)$; this proves the claim.

All in all, using Proposition 2.8, we obtain $V(1) \oplus \cdots \oplus V(l) \subseteq W$. If the inclusion were not an equality, we could choose a new Ω in $W \setminus \bigoplus_{i=1}^{l} V(i)$ and start again. Since $\dim W$ is finite, this procedure ends, and we get the statement. □

Now we are ready for the proof of Theorem 1.1. Let $\widetilde{C} \to C$ be a general element in $\mathcal{RH}_g[d]$. We have the following.

THEOREM 2.11. *The Jacobian* JC_0 *is simple.*

Proof. We proceed by contradiction. Let us assume that there exists a non-trivial abelian subvariety $A \subset JC_0$ and set $B = JC_0/A$. By Proposition 2.10, we have $W := T_0A^* = \bigoplus_{i\in I} V(i)$ for a set of indices I that we can assume to be $I = \{1, \ldots, l\}$.

Observe that the case $l = 1$ is not possible. Indeed, this would give $T_0A^* = V(1)$. By Remark 2.3, the automorphism β_1 leaves A invariant, and the restriction $\beta_{1|A}$ would determine an endomorphism of A which is equal to $(\xi + \xi^{-1})\,\mathrm{Id}$. Since $(\xi + \xi^{-1}) \in \mathbb{R}$ is not an integer, this is impossible. Hence, let us assume $l \geqslant 2$. We now work in the irreducible moduli space $\widetilde{\mathcal{RH}_g}[d]$ of classes of pairs $[C, \rho]$, where $\rho \in JC[d] \smallsetminus \{0\}$ (see Proposition 2.1).

The isogeny decomposition of JC_0 must be preserved under monodromy; that is, it remains the same when we replace $[C, \eta] \in \widetilde{\mathcal{RH}_g}[d]$ with $[C, \eta^l]$ for any $l = 1, \ldots, d-1$. The action $\eta \mapsto \eta^l$ is given by an automorphism belonging to the Galois group $G := \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. Since d is prime, we have $G \cong \mathbb{Z}/(d-1)\mathbb{Z}$. Since by definition $V(i) = V(d-i)$, we take into account only the automorphisms $\eta \mapsto \eta^l$ for $l \in \{2, \ldots, k\}$.

In order to investigate the possible configurations for W, we have to study the Galois orbits of the units $\xi, \ldots, \xi^{k-1}, \xi^k$. Indeed, these orbits precisely provide the only possible partitions of $\{1, \ldots, k\}$ that remain fixed under the action of the automorphisms of G. More precisely, let $\eta \mapsto \eta^l$ be one of these automorphisms. If l had order equal to k or $2k$, then the automorphism would act as a permutation of the set $\{1, \ldots, k\}$. This would mean that there is a single Galois orbit, namely $W = \bigoplus_{i=1}^k V(i)$, in other words, that $A = JC_0$, which is incompatible with our assumptions. Therefore, let us assume that $m := \mathrm{ord}(l) < k$. Let $I_1 \cup \cdots \cup I_{k/m}$ be a partition of the set $\{1, \ldots, k\}$ given by the Galois orbits of $\xi, \ldots, \xi^k$ under the automorphism $\eta \mapsto \eta^l$. Notice that all the Galois orbits are of length equal to m. Then $W = \bigoplus_{i \in I_r} V(i)$ for a certain I_r among $I_1, \ldots, I_{k/m}$. We consider the endomorphism $\sum_{i \in I_r} \beta_i$. By construction, it restricts to an endomorphism of W which equals $\lambda \mathrm{Id}$, with $\lambda = (\sum_{i \in I_r} \xi^i + \xi^{-i})$. Again, since $\lambda \in \mathbb{R}$ is not an integer, we obtain a contradiction. □

Let $\mathrm{End}^0(JC_0) := \mathrm{End}(JC_0) \otimes \mathbb{Q}$ be the endomorphism algebra of JC_0. For simplicity, we set $D := \mathrm{End}^0(JC_0)$, and we let Z be the centre. Furthermore, if $d \mapsto d'$ is the positive Rosati involution on D, we denote by Z_0 the fixed field of such an involution restricted to Z. Since JC_0 is simple, by Albert's classification, D is a division algebra with $Z_0 \subseteq Z \subseteq D$ satisfying one of the following:

(1) $Z_0 = Z = D$ is a totally real field.
(2) D is a quaternion algebra (totally definite/totally indefinite) over a totally real field $Z = Z_0$.
(3) D is a division algebra over the CM-field Z (over the totally real subfield Z_0).

Under our assumptions of genericity, the deformation argument exploited in Proposition 2.10 allows us to exclude the last two possibilities. Indeed, we have the following.

THEOREM 2.12. *The endomorphism algebra D of JC_0 is the totally real field $D = \mathbb{Q}(\xi + \xi^{-1})$.*

Proof. We claim that it is enough to show that $\mathbb{Q}(\xi + \xi^{-1})$ is the maximal field contained in D. Indeed, this would automatically exclude cases (2) and (3), and the maximality would yield D as in the statement.

In order to prove the claim, let us assume towards a contradiction that there exists an intermediate field extension $\mathbb{Q}(\xi + \xi^{-1}) \subsetneq L \subseteq D$. By the primitive element theorem, there exists an α such that $L = \mathbb{Q}(\alpha)$. Since $\mathbb{Q}(\xi + \xi^{-1)} \neq L$, we have

$$[L : \mathbb{Q}] > \left[\mathbb{Q}\left(\xi + \xi^{-1}\right) : \mathbb{Q}\right] = k\,. \tag{2.10}$$

By an abuse of notation, we will denote by the same α the endomorphism induced on $T_0 JC_0^*$. This is diagonalizable: indeed, its minimal polynomial divides the polynomial of α over $\mathbb{Q}$, which is irreducible and separable. Therefore, let $\bigoplus_{l=1}^r W_l$ be the eigenspace decomposition of α. We apply the same deformation argument used in Proposition 2.10: we know that each of the subspaces of $H^{1,0}(JC_0)$ that remains invariant under deformations by $\epsilon \in H^1(C, T_C)$ is either one of the $V(i)$ for $i = 1, \ldots, k$, or a direct sum of several of them. Thus, we get $W_l = \bigoplus_{i \in I_l} V(i)$ for some $I_l \subset \{1, \ldots, k\}$. This yields a contradiction: indeed, by (2.10), we get $r > k$, which is impossible since there are exactly k subspaces $V(i)$. □

COROLLARY 2.13. *The group of automorphisms of JC_0 has rank $k-1$ and is generated by any subset of cardinality $k-1$ of $\{\beta_1, \ldots, \beta_k\}$.*

Proof. The statement about the rank is a consequence of the Dirichlet theorem on units of a field. In [Was97, Theorem 8.1], it is stated that the (cyclotomic) units of $\mathbb{Q}(\xi + \xi^{-1})$ are generated by -1 and by

$$\xi^{(1-a)/2} \frac{1 - \xi^a}{1 - \xi} \quad \text{for } 1 < a < \tfrac{1}{2}d\,,\ (a, d) = 1\,. \tag{2.11}$$

The second claim follows from combining (2.11) with the fact that $\beta_1 + \cdots + \beta_k = -1$. The latter is straightforward since $\beta_i = \sigma^i + \sigma^{-i}$ and $\sum_{j=0}^d \sigma^j = 0$ on P by definition. □

Observe that Theorems 2.11 and 2.12, together with Corollary 2.13, give Theorem 1.1 in the introduction.

3. Generic finiteness of the Prym map

The aim of this section is to prove that the Prym map

$$\mathcal{P}_g[d]\colon \mathcal{RH}_g[d] \longrightarrow \mathcal{A}^{\delta}_{(g-1)(d-1)}$$

is generic finite as soon as the dimensions of the source and the target allow it. This is equivalent to the surjectivity of the codifferential map

$$d\mathcal{P}^*_g[d]\colon T^*_{[P,\tau]}\mathcal{A}^{\delta}_{(g-1)(d-1)} \longrightarrow T^*_{[C,\langle\eta\rangle]}\mathcal{RH}_g[d]$$

at the general points $[C,\langle\eta\rangle]$ in $\mathcal{RH}_g[d]$ and $[P,\tau]$ in the image of the Prym map. Notice that, combining [Ago20, Remark 2.5] with the isomorphism $T^*_{[C,\langle\eta\rangle]}\mathcal{RH}_g[d] \cong H^0(C,\omega_C^2)^{\langle\iota\rangle}$, we can equivalently describe $d\mathcal{P}^*_g[d]$ as a multiplication map followed by a projection as follows:

$$\bigoplus_{i=1}^{k} H^0\big(C,\omega_C\otimes\eta^i\big)\otimes H^0\big(C,\omega_C\otimes\eta^{-i}\big) \xrightarrow{m} H^0\big(C,\omega_C^2\big) \xrightarrow{p} H^0\big(C,\omega_C^2\big)^{\langle\iota\rangle}. \tag{3.1}$$

We start by showing the following.

PROPOSITION 3.1. *Let L be the line bundle corresponding to the hyperelliptic linear series on C. We assume that $g \geqslant 5$. Let $\eta\in\operatorname{Pic}^0(C)$ be such that η^2 is not of the form $L^2(-D)$, where D is an effective divisor of degree 4. Then, the map*

$$m_\eta\colon H^0(C,\omega_C\otimes\eta)\otimes H^0\big(C,\omega_C\otimes\eta^{-1}\big) \longrightarrow H^0\big(C,\omega_C^2\big)$$

is surjective.

Remark 3.2. The assumption on η in Proposition 3.1 includes both $\eta\neq 0$ and that $\omega_C\otimes\eta$ is generated by its sections. Indeed, this last statement is equivalent to saying that $\eta\neq\mathcal{O}(p-q)$ for any two points p, q, since if there exists a base point p for $\omega_C\otimes\eta$, then $h^0(C,\omega_C\otimes\eta(-p)) = g-1$, that is, $h^0(\eta^{-1}(p)) = 1$. Let us see that our hypothesis implies that η is not of the form $\mathcal{O}(p-q)$: if it were so, then since the hyperelliptic involution ι acts as $-\mathrm{Id}$ on $\operatorname{Pic}^0(C)$, we would also have $\eta=\mathcal{O}(\iota q-\iota p)$. This implies $\eta^2=\mathcal{O}_C(p+\iota q-q-\iota p)=\mathcal{O}_C(p+\iota p+q+\iota q-2q-2\iota p))=L^2(-2q-2\iota p)$, which contradicts the assumption.

Proof. Assume towards a contradiction that m_η is not surjective. The dual of the multiplication map is given by

$$\varphi\colon H^1(C,T_C)\longrightarrow \operatorname{Hom}^s\big(H^0(C,\omega_C\otimes\eta), H^0\big(C,\omega_C\otimes\eta^{-1}\big)^*\big),$$

which can be described as follows: let $0\neq\epsilon\in H^1(C,T_C)=\operatorname{Ext}^1(\omega_C,\mathcal{O}_C)$ correspond to a non-split extension

$$0\longrightarrow\mathcal{O}_C\longrightarrow E_\epsilon\longrightarrow\omega_C\longrightarrow 0. \tag{3.2}$$

We tensor this sequence with η,

$$0\longrightarrow\eta\longrightarrow E_\epsilon\otimes\eta\longrightarrow\omega_C\otimes\eta\longrightarrow 0,$$

and we consider the coboundary map

$$\partial_\epsilon\colon H^0(C,\omega_C\otimes\eta)\longrightarrow H^1(C,\eta)\cong H^0\big(C,\omega_C\otimes\eta^{-1}\big)^*.$$

Then, for any $s \in H^0(C, \omega_C \otimes \eta)$, we have $\varphi(\epsilon)(s) = \partial_\epsilon(s)$. Therefore, the non-injectivity of the multiplication map implies the existence of a non-split extension $[E_\epsilon]$ as above such that $\partial_\epsilon(s) = 0$ for all s. This is equivalent to the condition $h^0(C, E_\epsilon \otimes \eta) = g - 1$. We use the following result by Segre–Nagata and Ghione (see [Laz04, Example 7.2.14]).

Lemma 3.3. *Let C be a curve and E a rank v, degree p vector bundle on it. Then there exists a line subbundle $M \subset E$ such that*

$$\deg(M) \geqslant \frac{p}{v} - \frac{v-1}{v}(g-1)\,.$$

In our case, we have $v = 2$ and $p = 2g - 2$; thus Lemma 3.3 yields the existence of a line subbundle M such that $\deg(M) \geqslant \frac{1}{2}(g-1)$. Since by assumption $g \geqslant 5$, we have $\deg M \geqslant 2$. So, up to saturation (which could increase the degree of M), we can assume the existence of a sequence of locally free sheaves

$$0 \longrightarrow M \longrightarrow E_\epsilon \otimes \eta \longrightarrow N \longrightarrow 0 \tag{3.3}$$

and a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & M \arrow[d] \arrow[dr, "\tau"] & & \\
0 \arrow[r] & \eta \arrow[r] & E_\epsilon \otimes \eta \arrow[r] \arrow[d] & \omega_C \otimes \eta \arrow[r] & 0\,. \\
 & & N \arrow[d] & & \\
 & & 0 & &
\end{tikzcd} \tag{3.4}$$

Notice that the map τ is necessarily non-zero, since otherwise there would be an injective map $M \to \eta$, which is impossible since the degree of M is positive. Observe that $N \cong \omega_C \otimes \eta^2 \otimes M^{-1}$ and that the vertical exact sequence implies that

$$h^0(N) + h^0(M) \geqslant g - 1\,, \quad \deg N + \deg M = 2g - 2\,.$$

Now we discuss the possible values for the degrees and the dimensions of the space of sections of N and M. We first prove several claims.

Claim 1. *We have $\deg(M) < 2g - 2$ and therefore $\deg(N) > 0$.*

Indeed, otherwise τ is a non-trivial map between two line bundles of the same degree, so $M = \omega_C \otimes \eta$, $N = \eta$, and the map τ must be multiplication by a non-zero constant. The inverse of τ would give a section of $E_\epsilon \otimes \eta \to \omega_C \otimes \eta$, and therefore the extension (3.2) would split.

Claim 2. *We have $1 \leqslant h^0(C, N) \leqslant g - 2$, which yields $h^0(C, M) \geqslant 1$.*

Indeed, if $h^0(N) = g - 1$, then by the Clifford inequality, $g - 1 \leqslant \frac{1}{2}\deg N + 1 \leqslant g - 2 + 1$. Then N is a power of the line bundle L corresponding to the hyperelliptic linear series: $N = L^{(g-2)}$. Thus $N = \omega_C \otimes L^{-1}$ and $M = L \otimes \eta^2$. In particular, $\deg(M) = 2$. By Lemma 3.3, we have $\deg M \geqslant \frac{1}{2}(g-1)$, which is only possible for $g = 5$. In this case, we can assume $h^0(C, L \otimes \eta^2) = 0$;

otherwise, we would have $\eta^2 = \mathcal{O}_C(p-q)$ for certain $p, q \in C$, which contradicts our assumption. Thus $E_\epsilon \otimes \eta$ fits in the short exact sequence

$$0 \longrightarrow L \otimes \eta^2 \longrightarrow E_\epsilon \otimes \eta \longrightarrow L^3 \longrightarrow 0$$

with zero coboundary map $H^0(L^3) \to H^1(L \otimes \eta^2)$. Equivalently, the multiplication map

$$H^0\big(L^3\big) \otimes H^0\big(L^3 \otimes \eta^{-2}\big) \longrightarrow H^0\big(L^6 \otimes \eta^{-2}\big)$$

is not surjective. Note that $L^3 \otimes \eta^{-2}$ is base-point-free by our hypothesis. Since both sides of the map are of the same dimension, the base-point-free pencil trick shows that the multiplication map is an isomorphism, which gives a contradiction.

Moreover, $h^0(C, N)$ must be positive since otherwise $h^0(C, M) = g-1$, and then the map τ induces an isomorphism $H^0(C, M) \to H^0(C, \omega_C \otimes \eta)$. Since τ can be seen as a section of $\omega_C \otimes \eta \otimes M^{-1}$, this section cannot have zeros. Then $M \cong \omega_C \otimes \eta$ and, as before, the initial extension (3.2) splits.

CLAIM 3. *We have* $\deg N > 1$.

Otherwise, by Claim 2, we would have $h^0(C, N) = 1$, so either N is trivial, or $N = \mathcal{O}_C(p)$ for some point $p \in C$. The trivial case implies $M \cong \omega_C \otimes \eta^2$ and, as in the proof of Claim 1, this yields a contradiction to the non-splitness of the extension (3.2). Assume $N = \mathcal{O}_C(p)$; then $M = \omega_C \otimes \eta^2(-p)$. The inclusion $\omega_C \otimes \eta^2(-p) \hookrightarrow \omega_C \otimes \eta$ implies $\eta^2(-p) \hookrightarrow \eta$, and thus $\eta^{-1}(p)$ has a section, so η is of the form $\mathcal{O}_C(p-q)$, and we have a contradiction.

Summarizing, we have found the following restrictions on the degrees on M, N (with $M \otimes N = \omega_C \otimes \eta^2$):

$$\deg(N) + \deg(M) = 2g-2\,, \quad \deg(M) \geqslant 2\,, \quad \deg(N) \geqslant 2\,.$$

On the dimensions of the spaces of global sections, we have the restrictions

$$h^0(C,N) + h^0(C,M) \geqslant g-1\,, \quad 1 \leqslant h^0(C,N) \leqslant g-2\,, \quad 1 \leqslant h^0(C,M) \leqslant g-1\,.$$

LEMMA 3.4. *With these restrictions, we have one of the following possibilities:*

(i) *The line bundles M and N are special; that is, $h^1(C, M) > 0$ and $h^1(C, N) > 0$.*

(ii) *We have $N = L$ and $M = \omega_C \otimes \eta^2 \otimes L^{-1}$,*

(iii) *We have $M = L$ and $N = \omega_C \otimes \eta^2 \otimes L^{-1}$.*

Proof. If one of the line bundles, let us say N, is not special, then $h^0(C, \omega_C \otimes N^{-1}) = 0$ and therefore N, which can be represented by an effective divisor, must have degree $g + k$ with $k \geqslant 0$. Notice that $g + k \leqslant 2g - 4$, that is, $k \leqslant g - 4$. By the Riemann–Roch theorem, we have $h^0(C, N) = k + 1$. On the other hand, M has degree $g - 2 - k$, so it is special. By the Clifford inequality, $h^0(C, M) \leqslant \frac{1}{2} \deg M + 1$. Then

$$g-1 \leqslant h^0(N) + h^0(M) \leqslant k + 1 + \tfrac{1}{2}\deg M + 1 = k + 2 + \tfrac{1}{2}g - 2 - k\,.$$

This implies that $k \geqslant g-4$ and M satisfies the equality in Clifford's theorem; hence $M = L$ and $N = \omega_C \otimes \eta^2 \otimes L^{-1}$. The same proof applies verbatim if M is not special. ☐

Now we can finish the proof of Proposition 3.1. Assuming that the multiplication map is not surjective, we have found the existence of line bundles M and N satisfying a series of numerical constrains. Our aim now is to prove that these conditions lead to a contradiction with the hypothesis of the proposition. To do so, we analyze the three possibilities that appear in Lemma 3.4.

Let us first assume that we are in case (i), namely that M and N are special. Then, since C is hyperelliptic, we get that $M = L^r(D_1)$ and $N = L^s(D_2)$, where L is, as before, the g^1_2 on C, and D_1 and D_2 are effective. Moreover, we have $h^0(M) = r+1$ and $h^0(N) = s+1$ with $r+s \geqslant g-3$. Then, we have

$$\omega_C \otimes \eta^2 = M \otimes N = L^{r+s}(D_1 + D_2) = L^{g-3}(E)\,,$$

where E is an effective divisor of degree 4. Since $\omega_C = L^{g-1}$, we obtain $L^2 \otimes \eta^2 = \mathcal{O}_C(E) = L^4(-\iota E)$. Therefore, $\eta^2 = L^2(-\iota E)$, which contradicts our hypothesis.

Case (ii) easily yields a contradiction. Indeed, if $N = L$ and $M = \omega_C \otimes \eta^2 \otimes L^{-1}$, the map τ gives a section of $L \otimes \eta^{-1}$. Then η is of the form $L(-p-q) = \mathcal{O}_C(\iota(p) - q)$.

Thus, it only remains to handle case (iii), that is, $M = L$. Let us first observe that, as in Claim 2, we have to study the combination $M = L$, $N = \omega_C \otimes \eta^2 \otimes L^{-1} = L^3 \otimes \eta^2$ for $g = 5$. The strategy is the same as in Claim 2. Indeed, we can assume that $L^3 \otimes \eta^2$ is not special, in other words, that $E_\epsilon \otimes \eta$ fits in the short exact sequence

$$0 \longrightarrow L \longrightarrow E_\epsilon \otimes \eta \longrightarrow L^3 \otimes \eta^2 \longrightarrow 0$$

with zero coboundary map $H^0(L^3 \otimes \eta^2) \to H^1(L)$. Equivalently, the multiplication map

$$H^0(L^3) \otimes H^0(L^3 \otimes \eta^2) \longrightarrow H^0(L^6 \otimes \eta^2)$$

is not surjective. Note that $L^3 \otimes \eta^2$ is a base-point-free pencil by our hypothesis. Since both sides of the map are of the same dimension, the base-point-free pencil trick shows that the multiplication map is an isomorphism, which gives a contradiction. $\square$

COROLLARY 3.5. *Let (C, η) be a general element in $\mathcal{RH}_g[d]$ with $g \geqslant 5$. Then the map*

$$m_\eta\colon H^0(C, \omega_C \otimes \eta) \otimes H^0(C, \omega_C \otimes \eta^{-1}) \longrightarrow H^0(C, \omega_C^2)$$

is surjective.

Proof. It is enough to prove that such a generic element (C, η) satisfies the hypothesis of the Proposition 3.1. First suppose that $g \geqslant 5$ is odd; then $\kappa = L^{(g-1)/2}$ is a theta characteristic. Let Θ_κ be the corresponding symmetric theta divisor. One verifies that $C^{(2)} - C^{(2)} \subset \Theta_\kappa$. So, if for all $\eta \in JC[d]$, we had $\eta = \mathcal{O}(2L - D) \in C^{(2)} - C^{(2)}$ (with $D \in C^{(4)}$), then all the d-torsion points would be contained in Θ_κ, which contradicts the bound in [Par21]. In the case where $g > 5$ is even, one can use the same argument with the theta characteristic $\kappa = L^{(g-2)/2} \otimes \mathcal{O}(w)$, where w is a Weierstrass point. $\square$

Notice that Corollary 3.5 is no longer true for genus $g \leqslant 3$ by dimension reasons. It is also false for $g = 4$, where, interestingly enough, the image of the multiplication map is of codimension one. So, in order to get a surjective map onto $H^0(C, \omega_C)^{\langle \iota \rangle}$, one needs to involve another summand of (3.1). For this reason, we will treat cases $g = 3$ and $g = 4$ with direct methods.

PROPOSITION 3.6. *Let $g \in \{3, 4\}$ and $d \geqslant 5$. For any d-torsion element $\eta \in JC[d] \smallsetminus \{0\}$, the map $p \circ m$ is surjective.*

Proof. Since $d \geqslant 5$, it is enough to check that the restriction of $p \circ m$ to the first two summands,

$$H^0(C, \omega_C \otimes \eta) \otimes H^0(C, \omega_C \otimes \eta^{-1}) \oplus H^0(C, \omega_C \otimes \eta^2) \otimes H^0(C, \omega_C \otimes \eta^{-2}) \longrightarrow H^0(\omega_C^2)^{\langle \iota \rangle}\,, \tag{3.5}$$

is surjective.

Case $g = 4$, $d \geqslant 5$. Consider the surjective difference map

$$C^{(2)} \times C^{(2)} \longrightarrow JC\,, \quad (z,w) \longmapsto [z-w]\,.$$

So $\eta = D_1 - D_2$ for some divisors $D_i \in C^{(2)}$. Therefore, setting $D_2 = x + y$ for $x, y \in C$, we get $\omega_C \otimes \eta = L^3(D_1 - D_2) = L(D_1 + \iota x + \iota y)$. Thus, we have the inclusion $H^0(C,L) \subset H^0(C, \omega_C \otimes \eta)$ which yields an isomorphism

$$H^0(C, \omega_C \otimes \eta) \cong H^0(C,L) \oplus \langle u \rangle\,,$$

where $\langle u \rangle = H^0(C, \mathcal{O}_C(\iota x + \iota y))$. Similarly, we have

$$H^0\big(C, \omega_C \otimes \eta^{-1}\big) \cong H^0(C,L) \oplus \langle ju \rangle\,.$$

Hence, under the above isomorphisms, the multiplication map

$$m_\eta \colon H^0(C, \omega_C \otimes \eta) \otimes H^0\big(C, \omega_C \otimes \eta^{-1}\big) \longrightarrow H^0\big(C, \omega_C^2\big)$$

can be identified with the multiplication map

$$\big(H^0(C,L) \oplus \langle u \rangle\big) \otimes \big(H^0(C,L) \oplus \langle ju \rangle\big) \longrightarrow H^0\big(C, \omega_C^2\big)\,,$$

which clearly contains $\Lambda^2 H^0(C,L)$ in its kernel. Since the source and the target of m_η have the same dimension, m_η is not surjective. We set $H^0(C,L) = \langle s_1, s_2 \rangle$. It is easy to show that $u \cdot ju \in \langle s_1^2, s_2^2, s_1 s_2 \rangle$. Thus, taking the second summand in (3.5) and considering the analogous decompositions

$$H^0\big(C, \omega \otimes \eta^2\big) \cong H^0(C,L) \oplus \langle u' \rangle\,, \quad H^0\big(C, \omega \otimes \eta^{-2}\big) \cong H^0(C,L) \oplus \langle ju' \rangle\,,$$

we prove that the elements

$$s_1^2\,,\ s_2^2\,,\ s_1 s_2\,,\ s_1 u + s_1 ju\,,\ s_2 u + s_2 ju\,,\ s_1 u' + s_1 ju'\,,\ s_2 u' + s_2 ju'$$

are linearly independent in $H^0(C, \omega_C^2)^{\langle \iota \rangle}$ and can be used to generate $\operatorname{Im} p \circ m$. Therefore, since $h^0(C, \omega_C^2)^{\langle \iota \rangle} = 7$, we can conclude the surjectivity of the map. The check on the dimension is done using computer calculations in SAGE.[1]

Case $g = 3$, $d \geqslant 5$. Notice that in this case we have $h^0(C, \omega_C \otimes \eta^i) = 2$ for every i, while $h^0(C, \omega_C^2)^{\langle \iota \rangle} = 5$. Moreover, by the base-point-free-pencil trick, the restricted maps m_η and m_{η^2} are injective. We fix the following bases:

$$H^0(C, \omega_C \otimes \eta) = \langle t_1, t_2 \rangle\,, \quad H^0\big(C, \omega_C \otimes \eta^2\big) = \langle w_1, w_2 \rangle\,.$$

Again by means of computer calculations in SAGE (see the link in the footnote), we can show that the elements

$$t_1 jt_1\,,\ t_2 jt_2\,,\ w_1 jw_1\,,\ w_2 jw_2\,,\ t_1 jt_2 + t_2 jt_1$$

in $H^0(C, \omega_C^2)^{\langle \iota \rangle}$ are linearly independent. Since by definition they belong to $\operatorname{Im} d\mathcal{P}^*$, the map $p \circ m$ is surjective. □

THEOREM 3.7. *Assume that either $g = 2$ and $d \geqslant 6$, or $g \geqslant 3$ and $d \geqslant 5$, or $g \geqslant 5$ and $d \geqslant 3$. Then the Prym map $\mathcal{P}_g[d]$ is generically finite.*

Proof. For $g = 2$, this is the main theorem in the paper [Ago20] by Agostini.

The generic finiteness of $\mathcal{P}_g[d]$ is equivalent to the surjectivity of the composition map (3.1) for a general element $[C, \langle \eta \rangle] \in \mathcal{RH}_g[d]$. Since the projection is surjective, it is enough to show that the multiplication map m is surjective. Corollary 3.5 shows that m is surjective for $g \geqslant 5$. The cases $g \in \{3,4\}$ are solved in Proposition 3.6. □

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[1] See `https://amor.cms.hu-berlin.de/~speltair/SageCasesg3g4.html`.

4. Application to the generic injectivity of the Prym map

In this section, we will use Theorem 2.11 to show that, under some numerical conditions, the Prym map

$$\mathcal{P}_g[d]\colon \mathcal{RH}_g[d] \longrightarrow \mathcal{A}^{\delta}_{(g-1)(d-1)}$$

is generically injective. First, we collect some properties of JC_0.

Proposition 4.1 (Ries [Rie83], Ortega [Ort03]). *The following properties hold:*

- *We have $P = \operatorname{Im}(1-\sigma)$ and $\sigma^*\tau = \tau$. Thus σ restricts to an automorphism of P as polarized abelian variety.*
- *For $i \in \{1,\dots,d-1\}$, we have $JC_i = \operatorname{Im}(1+j\sigma^i) \subset P$. Moreover, the equality $\sigma(1+j\sigma^i) = (1+j\sigma^{i-2})\sigma$ gives that σ maps JC_i to JC_{i-2}. In particular, $\sigma^i\colon JC_0 \to JC_{d-2i}$ is an isomorphism.*
- *For $i \in \{1,\dots,k\}$, the maps $\sigma^i + \sigma^{-i}$ restrict to automorphisms of JC_0 (not preserving its principal polarization); we call the restriction β_i. Moreover, $\beta_i' = \beta_i$; that is, the restriction is symmetric with respect to the Rosati involution.*
- *For $i \in \{1,\dots,k\}$, the map $\psi_i\colon JC_0 \times JC_0 \to P$, $(x,y) \mapsto x + \sigma^i(y)$, is an isomorphism such that*
  - (i) $\psi_i^*(\tau) = \left(\begin{smallmatrix} 2\lambda_0 & \lambda_0\beta_i \\ \lambda_0\beta_i & 2\lambda_0 \end{smallmatrix}\right)$, *where $\lambda_0\colon JC_0 \to \widehat{JC}_0$ is the natural principal polarization;*
  - (ii) $\sigma^i \circ \psi_i = \psi_i \circ \left(\begin{smallmatrix} 0 & -1 \\ 1 & \beta_i \end{smallmatrix}\right)$.

As in [NOS24], we factor the Prym map as the composition of two maps

$$\mathcal{P}_g[d]\colon \mathcal{RH}_g[d] \xrightarrow{\mathcal{P}_1[d]} \mathcal{D}_k \xrightarrow{\mathcal{P}_2[d]} \mathcal{A}^{\delta}_{(g-1)(d-1)}\,, \tag{4.1}$$

where $\mathcal{D}_k$ parametrizes classes of objects $(C_0,\beta_1,\dots,\beta_k)$, $\mathcal{P}_1[d]$ sends (C,η) to $(C_0,\beta_1,\dots,\beta_k)$, and $\mathcal{P}_2[d]$ sends $(C_0,\beta_1,\dots,\beta_k)$ to $\left(JC_0 \times JC_0, \left(\begin{smallmatrix} 2\lambda_0 & \lambda_0\beta_i \\ \lambda_0\beta_i & 2\lambda_0 \end{smallmatrix}\right)\right)$, which, by Proposition 4.1, lies in $\operatorname{Im}\mathcal{P}_g[d]$.

First, we prove that $\mathcal{P}_2[d]$ is generically injective. Let (P,τ) be a general element in $\operatorname{Im}\mathcal{P}_g[d]$. Then, by [NOS24, Proposition 3.1], we have

$$\{\alpha \in \operatorname{Aut}(P,\tau) \mid \alpha(x) = x,\ \forall x \in K(\tau)\} \cong \langle\sigma\rangle\,. \tag{4.2}$$

This is shown by using the assumptions on α to construct an $\tilde\alpha$ acting on $J\widetilde{C}$, then proving that $\tilde\alpha$ is compatible with the principal polarization on $J\widetilde{C}$ and that it acts as the identity on f^*JC.

Hence, for general $(P,\tau) \in \operatorname{Im}\mathcal{P}_g[d]$, the set $\langle\sigma\rangle$ is uniquely determined. Therefore, we now work with the triplet $(P,\tau,\langle\sigma\rangle)$.

The following proposition is proved in [NOS24] by means of arithmetic arguments that require the extra conditions $g=2$ and k prime. We will use instead Theorem 1.1 to prove this more general version. The strategy is the same; we recall the main facts for the convenience of the reader.

Proposition 4.2. *The map $\mathcal{P}_2[d]$ is generically injective. In other words, we can recover the element $(C_0,\beta_1,\dots,\beta_k) \in \mathcal{D}_k$ from the general element (P,τ) lying in the image of $\mathcal{P}_g[d]$.*

Proof. As in [NOS24], we consider isomorphisms $\varphi\colon JN \times JN \to P$, where N is a smooth genus $k(g-1)$ curve, such that

$$\hat\varphi \circ \lambda_\tau \circ \varphi = \begin{pmatrix} 2\lambda_0 & \lambda_0\gamma \\ \lambda_0\gamma & 2\lambda_0 \end{pmatrix} \tag{$\star$}$$

for a certain $\gamma \in \mathrm{Aut}(JN)$, and such that

$$\sigma^i \circ \varphi = \varphi \circ \begin{pmatrix} 0 & -1 \\ 1 & \beta_i \end{pmatrix} \tag{$\star\star$}$$

for a certain $i \in \{1, \ldots, k\}$. That is, we are considering all the automorphisms ϕ_i of Proposition 4.1, plus possibly other ones with the same behaviour. Namely, we are defining the set

$$\Lambda(P, \tau, \langle\sigma\rangle) := \{(N, \varphi) \mid \varphi\colon JN \times JN \cong P \text{ satisfies } (\star) \text{ and } (\star\star) \text{ for the same } \gamma \in \mathrm{Aut}(JN)\}\,.$$

Our aim is to prove that for a generic $(P, \tau, \langle\sigma\rangle)$ in $\mathrm{Im}\,\mathcal{P}_g[d]$, we have the equality

$$\Lambda(P, \tau, \langle\sigma\rangle) = \{(C_0, \beta_i),\ i = 1, \ldots, k\}\,.$$

Let $(N, \varphi) \in \Lambda(P, \tau, \langle\sigma\rangle)$, and let i be the corresponding exponent of property $(\star\star)$. The composition

$$F\colon JN \times JN \xrightarrow{\varphi} P \xrightarrow{\phi_i^{-1}} JC_0 \times JC_0$$

gives an isomorphism that pulls back the polarization of $JC_0 \times JC_0$ to that of $JN \times JN$; that is,

$$\begin{pmatrix} 2\lambda_0 & \lambda_0\gamma \\ \lambda_0\gamma & 2\lambda_0 \end{pmatrix} = \hat{F} \circ \begin{pmatrix} 2\lambda_0 & \lambda_0\beta_i \\ \lambda_0\beta_i & 2\lambda_0 \end{pmatrix} \circ F\,.$$

As in [NOS24], this gives immediately that $N \cong C_0$.

Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be the matrix associated with F. It is easy to see that $A^*(\lambda_{C_0}) = \lambda_{C_0}$ and that an analogous argument works for D. Thus, A and D are automorphisms of JC_0 compatible with the principal polarization. Hence, they come from automorphisms of C_0.

On the other hand, property $(\star\star)$ determines the following equality:

$$\begin{pmatrix} 0 & -1 \\ 1 & \beta_i \end{pmatrix} \circ F = F \circ \begin{pmatrix} 0 & -1 \\ 1 & \gamma \end{pmatrix}.$$

An easy computation shows that it yields $\gamma = D^{-1}\beta_i D$. By Theorem 1.1, the automorphism D commutes with any β_i. Hence, $\gamma = \beta_i$ and we are done. □

The next step is to read from $(C_0, \beta_1, \ldots, \beta_k) \in \mathcal{D}_k$ the correct information to obtain the element $(C, \eta) \in \mathcal{RH}_g[d]$. For this, we can apply a similar argument to that in [NOS24, Section 3, Step 4], with some minor changes. First, we observe that the map $h_0\colon C_0 \to \mathbb{P}^1$ determines (C, η). Then we apply the theta-duality construction to the curves $\beta_i(C_0)$; that is, we look for the set of translates of the theta divisor that contain each of these curves. These sets determine the map h_0 under some numerical restrictions. To be more precise, we fix a point $x \in C_0$ consider the injection $\iota_x\colon C_0 \hookrightarrow JC_0$, $p \mapsto [p - x]$. By Riemann's parametrization theorem, the canonical theta divisor Θ^{can} in $\mathrm{Pic}^{k(g-1)-1}(C_0)$ is the Brill–Noether locus $W_{k(g-1)-1}(C_0)$. We define

$$T_i' := \left\{\xi \in \mathrm{Pic}^{k(g-1)-1}(C_0) \mid \beta_i(C_0) + \xi \subset \Theta^{\mathrm{can}} = W_{k(g-1)-1}(C_0)\right\}.$$

Observe that to use the definition of β_i, we need to see $JC_0 = \pi_0^*(JC_0)$ as a subvariety of P. Given a point p in C_0, its image $\iota_x(p) = [p - x]$ appears as $[p' + j(p') - x' - j(x')] \in P$, where $x', p' \in \widetilde{C}$ are preimages of x and p, respectively. We denote by

$$p'\,,\ p_1' := \sigma(p')\,,\ \ldots\,,\ p_{d-1}' = p_{2k}' := \sigma^{d-1}(p')$$

the whole fiber $f^{-1}(f(p'))$, and analogously for x'. We denote by p_i (respectively, x_i) the image of p_i' (respectively, x_i') in C_0.

Proposition 4.3. *Assume that* $(d-1)(g-1) \geqslant 7$ *and that* $g \geqslant 2$ *is not congruent to* 3 *modulo* d. *Then* $T_i' = x_i + x_{d-i} + W_{k(g-1)-3}(C_0)$.

Notice that the assumption $(d-1)(g-1) \geqslant 7$ is equivalent to $\dim W_{k(g-1)-3}(C_0) > 0$.

Proof. As in the proof of [NOS24, Proposition 3.13], we compute the action of $\beta_i = \sigma^i + \sigma^{-i}$ on $[p' + j(p') - x' - j(x')]$ to obtain that, as elements in JC_0,

$$\beta_i([p-x]) = [p_i + p_{d-i} - x_i - x_{d-i}]\,.$$

Observe that these points describe the fiber $h_0^{-1}(h_0(x))$. More precisely, as divisors, we have

$$h_0^{-1}(h_0(x)) = x + x_1 + \cdots + x_{d-1}\,. \tag{4.3}$$

By definition, $\xi \in T_i'$ means that

$$h^0(C_0, \xi + p_i + p_{d-i} - x_i - x_{d-i}) > 0 \quad \text{for all } p \in C_0\,.$$

If ξ is of the form $x_i + x_{d-i} + E$ for some effective divisor E, the condition is satisfied trivially; hence $x_i + x_{d-i} + W_{k(g-1)-3}(C_0) \subset T_i'$. To prove the opposite inclusion, we consider $\xi \in T_i'$. If $h^0(C_0, \xi - x_i - x_{d-i}) > 0$, then we are done. So assume $h^0(C_0, \xi - x_i - x_{d-i}) = 0$. Set $L := K_{C_0} - (\xi - x_i - x_{d-i})$. Arguing as in loc. cit., we obtain that L gives a $g^1_{k(g-1)+1}$ whose associated map φ_L satisfies that $h_0^{-1}(h_0(p))$ is contained in the fiber of φ_L. Hence, φ_L factorizes through h_0, and $\deg(h_0) = d$ divides $\deg(L) = k(g-1)+1$. For $g = 2$, this is impossible since $k+1 < d = 2k+1$. Assume that g is at least 3, and put $dr = k(g-1)+1 = \frac{1}{2}(d-1)(g-1)+1$. Then $2rd = (d-1)(g-1)+2$, so g is congruent to 3 modulo d, and we have a contradiction. □

Remark 4.4. Proposition 4.3 allows recovering intrinsically the class of the divisors $x_i + x_{d-i}$ since there are no translations leaving invariant $W_{k(g-1)-3}(C_0)$. It may well happen that $x_i + x_{d-i}$ belong to a g^1_2-linear series since we have not excluded the possibility of C_0 being hyperelliptic. Assume that this is the case for a generic point in C_0; that is, for a generic p, there is an index i such that $h^0(C, \mathcal{O}(p_i + p_{d-i})) = 2$. We can assume that $h^0(C, \mathcal{O}(x_1 + x_{d-i})) = 2$ for the fixed point x we used to embed the curve. Then $\beta_1(p-x) = p_1 + p_{d-1} - x_i - x_{d-1}$. Replacing p with a convenient point p' in the same fiber, we obtain $\beta_1(p'-x) = p_i + p_{d-i} - x_i - x_{d-1} = 0$ since both divisors represent the hyperelliptic linear series. This contradicts that β_1 is an automorphism.

Since we have recovered the whole fiber $h_0^{-1}(h_0(x))$ for a generic point in C_0 and the curve is smooth, the whole map h_0 is obtained, and therefore we have finished the proof of Theorem 1.2.

Remark 4.5. According to the Castelnuovo–Severi inequality [Acc94], the curve C_0 cannot admit more than two maps to $\mathbb{P}^1$ of degree d if its genus satisfies $g(C_0) = (g-1)(d-1) > (d-1)^2$, that is, if $g > 2d-1$. This implies that even in this range the map $\mathcal{P}_2[d]$ is injective. Therefore, one can conclude that the generic injectivity of $\mathcal{P}_g[d]$ holds as well.

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