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Treball final de grau

Grothendieck's vanishing theorems for affine schemes

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L'algèbre n'est qu'une
géométrie écrite; la géométrie
n'est qu'une algèbre figurée.[7]

Sophie Germain

Abstract

This work aims to introduce modern algebraic geometry and present two of Grothendieck's vanishing theorems for affine schemes.

The first part covers fundamental concepts of sheaves and scheme theory, with a focus on affine schemes. In the second part, the reader is introduced to sheaf cohomology, particularly emphasizing Čech cohomology, leading to the presentation of two Grothendieck's vanishing theorems for affine schemes.

Resum

L'objectiu d'aquest treball és fer una introducció a la geometria algebraica moderna i presentar dos dels teoremes d'anul·lació de Grothendieck per esquemes afins.

La primera part d'aquest treball introdueix conceptes fonamentals sobre feixos i teoria d'esquemes, centrant-nos en esquemes afins. A la segona part, el lector és introduït a la cohomologia de feixos, emfatitzant particularment en la cohomologia de Čech, finalment es presenten dos teoremes d'anul·lació de Grothendieck per esquemes afins.

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Introduction

This work arises from the course "Anells de polinomis en diverses variables", in which the classical correspondence of algebraic geometry, between reduced algebraic varieties over an algebraically closed field k and finitely generated k -algebras without nilpotent elements, is explained. The professor of the course encouraged us to read about scheme theory, and this is the origin of this work.

In the late 19th century, David Hilbert proved the well-known Hilbert's Nullstellensatz, which establishes an intrinsic relationship between algebra and geometry. This result, together with the commutative algebra of Emmy Noether and the contributions of mathematicians such as Oscar Zariski and André Weil, laid the foundations of classical algebraic geometry in the early decades of the 20th century. Scheme theory has its origins a few years later, around the 1950s. Sheaf theory had been introduced as an abstract tool to study local-to-global properties of geometric objects, although it has its origins in analysis. It was Alexander Grothendieck (1928-2014), who in the 1950s defined the concept of scheme, a topological space together with a sheaf on it, that locally is like an affine scheme, the generalization of an affine variety. This paradigm shift, from concrete geometric objects to abstract structures, led to a revolution in algebraic geometry. He published from 1960 through 1967, *Éléments de Géométrie Algébrique*, an eight-part work in which the theory of schemes and the cohomology of sheaves are rigorously developed. Nowadays, scheme theory is a fundamental tool while working on algebraic geometry, in addition to the importance it has in other areas such as number theory and commutative algebra.

We propose two main objectives. The first is to provide an introduction to scheme theory, for which we will also introduce the initial definitions and properties of abelian sheaves. This first part will be developed in the first two chapters. The second goal is to use this introduction to understand the statement and implications of two cohomology vanishing theorems for sheaves on affine schemes. To understand these results, we will define concepts such as quasicoherent sheaves and the dimension of a scheme. This part will be mostly explained in the last two chapters.

The text is structured in four chapters.

In Chapter 1, we will define the basic concepts of sheaf theory. We will discuss sheafification, a process by which a sheaf can be associated with a given presheaf. This construction, at least its existence, will be of vital importance throughout the work. To introduce some concepts, we will need to talk about the concept of direct

limit. We will define the kernel and image of a sheaf morphism and we will see that to check the injectivity of a sheaves morphism, it is enough to verify it for the stalks, but this is not the case for surjectivity. This loss of surjectivity is what will later motivate sheaf cohomology in Chapter 4.

In the second chapter, we will discuss schemes. The first section aims to define a scheme. To do this, we will first introduce the spectrum of a ring as a set. Subsequently, we will endow this set with a topology, Zariski's topology. Then, we will construct a sheaf of rings over this topological space. And, finally give the definitions of affine scheme, scheme and scheme of finite type. In the second section, we will define open subscheme and closed subscheme. In the third section we will give some important properties of schemes, such as being noetherian and the notion of dimension. In the last section we will explain how to glue two schemes together using two isomorphic open subsets. With this, we will look at the construction of the projective line and a family of sheaves on it.

Chapter 3 is dedicated to sheaves of modules. We will focus on defining quasicohherent sheaves. Quasicohherent sheaves generalize the notion of modules over a ring to the setting of schemes. In order to illustrate their importance, we will show that an exact sequence of sheaves, the first of them quasicohherent, preserves the exactness when taking global sections. Finally we will explain the relationship between closed subschemes and quasicohherent sheaves of ideals.

The final chapter is an introduction to sheaf cohomology. In particular, we are going to define Čech cohomology. First, we will define Čech cohomology of covering. This will allow us to explicitly calculate the cohomology of some examples we have seen in the previous chapters. Subsequently, we will see that by refining the covering, we can define the cohomology of sheaves in general. Finally, we will conclude with some cohomology vanishing results. The first one will tell us that the cohomology of quasicohherent sheaves on affine schemes vanishes. The second, and more general result, is a vanishing theorem above the dimension of the scheme. With these two results we will be able to compute the cohomology groups of an important family of sheaves on the n -projective space. To finish the work, we present a proof of Bézout's theorem for projective plane curves, in which we will use almost everything we will have explained in the text.

Previous results

We will provide some of the basic definitions of ring localizations that we will assume to be known at the beginning of this work, as well as the notation we will introduce now. Some of the properties of localizations will also be assumed to

be known, although they will be stated when they are used. All of the results of localizations that will be used through the work can be found in [1, Chapter 11].

Definition 0.1. Let R be a commutative ring with unity. We say $S \subseteq R$ is a **multiplicative subset** if $1 \in S$ and $x, y \in S$ implies $xy \in S$.

Definition 0.2. Let R be a commutative ring with unity and S a multiplicative subset. We define the equivalence relation $\sim$ in $R \times S$ as follows: $(x, s) \sim (y, t)$ if $\exists u \in S$, such that, $xtu = ysu$. We denote by $S^{-1}R = R \times S / \sim$, and by $\frac{x}{s}$ the equivalence class of $(x, s) \in R \times S$. Let us define:

$$\frac{x}{s} \cdot \frac{y}{t} := \frac{xy}{st}, \quad \frac{x}{s} + \frac{y}{t} := \frac{tx + ys}{st}, \quad 0_{S^{-1}R} := \frac{0}{1} \quad 1_{S^{-1}R} := \frac{1}{1}$$

$S^{-1}R$ with this structure is a ring, named **the localization at of R at S** .

Definition 0.3. Let R be a commutative ring with unity and $f \in R$. $S_f := \{f^n \mid n \geq 0\}$. The **localization of R at f** is defined to be $S_f^{-1}R$, and we will denote it by R_f .

Definition 0.4. Let R be a commutative ring with unity and $\mathfrak{p} \subseteq R$ a prime ideal. The, $S_{\mathfrak{p}} = R \setminus \mathfrak{p}$ is a multiplicative subset. The **localization of R at $\mathfrak{p}$** is $S_{\mathfrak{p}}^{-1}R$ and it is denoted by $R_{\mathfrak{p}}$.

Chapter 1

Sheaves

1.1 Introduction

Definition 1.1. Let X be a topological space, a **presheaf** $\mathcal{F}$ of abelian groups on X is:

- For every open subset $U \subseteq X$ an abelian group $\mathcal{F}(U)$.
- For every pair of open subsets $V \subseteq U$ a morphism $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ of abelian groups, named **restriction maps**, such that:

1. $\rho_{UU} = \text{id}_{\mathcal{F}(U)}$
2. For every three open subsets such that $W \subseteq V \subseteq U$ then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Definition 1.2. If $\mathcal{F}$ is a presheaf (of abelian groups) on X and $U \subseteq X$ an open subset, the **sections** of the presheaf over U are the elements of $\mathcal{F}(U)$. If $s \in \mathcal{F}(U)$ is a section over U , and $V \subseteq U$, it is usual to write $s|_V$ in spite of $\rho_{UV}(s)$

Definition 1.3. A **sheaf** $\mathcal{F}$ on X (of abelian groups) is a presheaf on X that satisfies the following conditions:

1. (Locality axiom) If U is an open subset of X , $\{U_i\}_{i \in I}$ an open covering of U and $s \in \mathcal{F}(U)$. If $s|_{U_i} = 0$ for all $i \in I$, then, $s = 0$.
2. (Gluing axiom) If U is an open subset of X , $\{U_i\}_{i \in I}$ an open covering of U and $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each i, j . Then, $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Notice that the same definition of presheaf of abelian groups can be rewritten to define a presheaf with values in a category $\mathcal{C}$, by replacing, in the definition above, abelian groups for elements of $\text{Ob}(\mathcal{C})$, and morphism of abelian groups for an element of $\text{Hom}(X, Y)$ with $X, Y \in \text{Ob}(\mathcal{C})$. Sheaves of abelian groups are also named **abelian sheaves**.

Example 1.4. Let X be a topological space. We define the **sheaf of continuous functions** $\mathcal{C}(X, \mathbb{R})$ as follows:

1. For all $U \subseteq X$ open subset $\mathcal{C}(X, \mathbb{R})(U) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$
2. Take the usual restriction maps of continuous functions.

Any continuous function that restricts to zero on an open covering of X is the zero function so the locality axiom holds. If we have a family of continuous functions $\{f_i : U_i \rightarrow \mathbb{R}\}$ that matches on the overlaps (ie: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$) we can define a continuous function f as $f(x) = f_i(x)$ if $x \in U_i$, so the gluing axiom also holds. Hence we have defined a sheaf of rings on X .

Notice that for the same reason we can define the sheaves $\mathcal{C}^m(X, \mathbb{R})$ of functions m -times continuously differentiable on a differentiable manifold X .

Example 1.5. Let V be an algebraic variety over a field k (the set of solutions of a system of polynomial equations). Considering Zariski topology, for each $U \subseteq V$ open subset let $\mathcal{O}(U)$ be the ring of regular functions from U to k (functions defined locally by rational functions as a well defined quotient of polynomials), and take the restriction maps as usual. It is clear that $\mathcal{O}$ is a presheaf of rings on V , this is indeed a sheaf: it satisfies the locality axiom: given that a function that is the 0 function locally is the 0 function, and it satisfies the gluing axiom since a function that is locally regular is indeed regular by definition.

Example 1.6. Let X be a topological space and A an abelian group. If one wants to define a **constant sheaf** on X the first attempt to do it will be to define $A(U) = A$. The matter is that in general it is not a sheaf. For example if $X = U_1 \sqcup U_2$ a disjoint union, for any choice of $s_1 \neq s_2$ elements of A they always match on the intersections since it is $\emptyset$ so they cannot be glued. This problem can be solved by defining $A_X(U) = \{f : U \rightarrow A \mid f \text{ locally constant}\}$, this is f continuous taking on A the discrete topology. With this we have $A_X(U) = A$ if U is connected, indeed we have $A_X(U)$ is the product of copies of A one for each connected component of U . It is easy to check that this is a sheaf.

Definition 1.7. Let $\mathcal{F}$ be a presheaf on X , a **presubsheaf** $\mathcal{G}$ of $\mathcal{F}$ is a presheaf on X such that for every $U \subseteq X$ open subset, $\mathcal{G}(U) \subseteq \mathcal{F}(U)$, and the restriction maps are the induced by the restriction maps of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves then $\mathcal{G}$ is called a **subsheaf** of $\mathcal{F}$.

Definition 1.8. Let X be a topological space and $\mathcal{F}, \mathcal{G}$ (pre)sheaves on X . A **morphism of (pre)sheaves** $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is, for every open set $U \subseteq X$, a morphism $\phi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ compatible with the restriction maps. This is, for every pair of open subsets $V \subseteq U \subseteq X$

the following is commutative:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \rho_{UV}^{\mathcal{F}} \downarrow & & \downarrow \rho_{UV}^{\mathcal{G}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

Example 1.9. Let X be an open subset of $\mathbb{R}^m$ and take the sheaves $\mathcal{C}^n(X, \mathbb{R})$ and $\mathcal{C}^{n-1}(X, \mathbb{R})$ (see 1.4). The morphism $D : \mathcal{C}^n(X, \mathbb{R}) \rightarrow \mathcal{C}^{n-1}(X, \mathbb{R})$ defined by the operator $D = \frac{d}{dx}$ is a morphism of sheaves.

Definition 1.10. A map $\phi : \mathcal{F} \rightarrow \mathcal{G}$ between (pre)sheaves is an **isomorphism** if there exists a map $\psi : \mathcal{G} \rightarrow \mathcal{F}$ such that $\phi \circ \psi = \text{id}_{\mathcal{G}}$ and $\psi \circ \phi = \text{id}_{\mathcal{F}}$.

Definition 1.11. We denote by $\text{Sh}(-)$ the **category of sheaves** (of abelian groups) on a topological space.

1.2 Sheafification

When one has a presheaf $\mathcal{F}$ on X a topological space there is a natural way to associate to it a sheaf $\mathcal{F}^{\text{sh}}$ that approximates $\mathcal{F}$ the best possible way as a sheaf. This procedure is known as sheafification.

Definition 1.12. Let $\mathcal{F}$ be a presheaf on X . A **sheafification** of $\mathcal{F}$ is a sheaf $\mathcal{F}^{\text{sh}}$ on X and a presheaves morphism $sh : \mathcal{F} \rightarrow \mathcal{F}^{\text{sh}}$ such that:

- (a) For every $\mathcal{G}$ presheaf on X and $\phi : \mathcal{F} \rightarrow \mathcal{G}$ presheaves morphism there is an induced sheaf morphism $\phi^{\text{sh}} : \mathcal{F}^{\text{sh}} \rightarrow \mathcal{G}^{\text{sh}}$ in which:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ sh \downarrow & & \downarrow sh \\ \mathcal{F}^{\text{sh}} & \xrightarrow{\phi^{\text{sh}}} & \mathcal{G}^{\text{sh}} \end{array}$$

- (b) **Universal property:** If $\mathcal{G}$ is a sheaf on X and $\phi : \mathcal{F} \rightarrow \mathcal{G}$ presheaves morphism then:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ sh \downarrow & \nearrow \exists! \varphi & \\ \mathcal{F}^{\text{sh}} & & \end{array}$$

- (c) The morphism sh induces an isomorphism on the stalks, $\mathcal{F}_x \simeq \mathcal{F}_x^{\text{sh}}$, $\forall x \in X$.

Proposition 1.13. *Assuming the existence of the sheafification. Given a presheaf $\mathcal{F}$ presheaf on X , $\mathcal{F}^{sh}$ is unique up to isomorphism.*

Proof. In general, objects with a universal property are unique up to isomorphism. Nevertheless, let us see the proof in this case.

Let $\mathcal{F}$ be a presheaf on X a topological space, and $\mathcal{F}_1, \mathcal{F}_2$ two sheafifications of $\mathcal{F}$, let sh_1, sh_2 denote the morphisms of sheafification.

$$sh_1 : \mathcal{F} \rightarrow \mathcal{F}_1 \quad sh_2 : \mathcal{F} \rightarrow \mathcal{F}_2$$

Applying the second property of sh_1 with $\mathcal{F}_2$ sheaf on X and the morphism of presheaves $sh_2 : \mathcal{F} \rightarrow \mathcal{F}_2$, there exists a unique $\varphi_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ such that $\varphi_1 \circ sh_1 = sh_2$.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{sh_2} & \mathcal{F}_2 \\ sh_1 \downarrow & \nearrow \varphi_1 & \\ \mathcal{F}_1 & & \end{array}$$

By symmetry, there exists a unique $\varphi_2 : \mathcal{F}_2 \rightarrow \mathcal{F}_1$, such that, $\varphi_2 \circ sh_2 = sh_1$.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{sh_1} & \mathcal{F}_1 \\ sh_2 \downarrow & \nearrow \varphi_2 & \\ \mathcal{F}_2 & & \end{array}$$

Therefore, $\varphi_1 \circ (\varphi_2 \circ sh_2) = sh_2$ and $\varphi_2 \circ (\varphi_1 \circ sh_1) = sh_1$, so φ_1 is an isomorphism from $\mathcal{F}_1$ to $\mathcal{F}_2$. $\square$

Theorem 1.14. *Given a presheaf $\mathcal{F}$ presheaf on X there exists the sheafification.*

Proof. See [5, pg 31-34] $\square$

1.3 Direct image, inverse image and stalks

The next natural question to ask is, given a continuous function between two topological spaces $f : X \rightarrow Y$, how to use f to define a sheaf on Y when one has a sheaf defined on X . Conversely, if one has a sheaf defined on Y , how to use f to define one on X .

We are going to see that the first case is quite straightforward; it suffices to take the inverse image of f to define a sheaf on Y . However, the second case is not going to be as simple because not every continuous function is necessarily open, and this will require to introduce some more technical concepts.

Definition 1.15. Let X, Y be topological spaces and $f : X \rightarrow Y$ a continuous map. The **direct image functor** $f_* : \text{Sh}(X) \rightarrow \text{Sh}(Y)$ is defined on open subset $U \subseteq Y$ by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$$

So, f_* sends a sheaf on X to $f_*(\mathcal{F})$ that turns out to be a sheaf on Y . We denote $f_*\mathcal{F}$ as **direct image sheaf** or **pushforward sheaf** of $\mathcal{F}$ along f .

As we have said before, the definition of the inverse image functor is not that simple, since the image $f(U)$ of an open set does not need to be open. This is the reason why we should introduce what a direct limit is:

Definition 1.16. Consider a directed set $(I, \leq)$, a non-empty set I with a preorder $\leq$ with the property that for every pair $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. A **directed system** over I is a pair (A_i, f_{ij}) where:

1. $\{A_i : i \in I\}$ is a family of objects indexed by I .
2. For every pair $i \leq j$ there is a morphism $f_{ij} : A_i \rightarrow A_j$ with the following properties:

- (a) $f_{ii} = \text{id}_{A_i}$
- (b) $f_{ik} = f_{jk} \circ f_{ij}$ for every $i \leq j \leq k$

Definition 1.17. With the previous notation we define the **direct limit** of the directed system (A_i, f_{ij}) to be:

$$\varinjlim A_i := \bigsqcup_i A_i / \sim$$

With $\sim$ being the following equivalence relation: for $x_i \in A_i$ and $x_j \in A_j$ then $x_i \sim x_j$ if, and only if, $\exists k \in I$ with $i, j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. And a family of canonical functions $\phi_i : A_i \rightarrow \varinjlim A_i$ that maps each element to its class.

The direct limit has a universal property. If one has $(A_i, f_{i,j})$ a direct system of objects of an algebraic category $\mathcal{C}$, $A \in \text{Ob}(\mathcal{C})$ and morphisms $\{\phi_i : A_i \rightarrow A\}$ such that $\phi_j \circ f_{ij} = \phi_i$. For each $Y \in \text{Ob}(\mathcal{C})$ and $\{\psi_i : A_i \rightarrow Y\}$ such that $\psi_j \circ f_{ij} = \psi_i$, there exists a unique $\varphi : X \rightarrow Y$ morphism such that the following commutes:

$$\begin{array}{ccc}
 X_i & \xrightarrow{f_{ij}} & X_j \\
 \phi_i \searrow & & \swarrow \phi_j \\
 & X & \\
 \psi_i \searrow & \downarrow \exists! \varphi & \swarrow \psi_j \\
 & Y &
 \end{array}$$

Definition 1.18. Let X, Y be topological spaces and $f : X \rightarrow Y$ a continuous map. The **inverse image functor** $f_* : \text{Sh}(Y) \rightarrow \text{Sh}(X)$ is defined for $\mathcal{G}$ an abelian sheaf on Y as the sheafification of the following presheaf:

1. On open subset $U \subseteq X$, $(f^{-1}\mathcal{G})(U) = \varinjlim_{f(U) \subseteq V} \mathcal{G}(V)$
2. The restriction maps are defined by the universal property of the direct limit.

We denote $f^{-1}\mathcal{G}$ as **inverse image sheaf** or **pullback sheaf** of $\mathcal{G}$ along f .

Definition 1.19. Let $\mathcal{F}$ be a (pre-)sheaf (of abelian groups) on X and $x \in X$. We define the **stalk** of X at x to be:

$$\mathcal{F}_x = \bigsqcup_{U \ni x} \mathcal{F}(U) / \sim$$

where we consider $(s_1, U_1) \sim (s_2, U_2)$, with $s_1 \in U_1$, $s_2 \in U_2$, if $\exists V \subseteq U_1 \cap U_2$ open subset such that $x \in V$ and $s_1|_V = s_2|_V$.

Remark 1.20. Indeed we are taking $\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U)$ indexed by the open neighborhoods of x with $U \leq V$ if $V \subseteq U$.

The previous definition can be understood as follows: to each $x \in X$, we associate to it an abelian group $\mathcal{F}_x$. To do this, we take a kind of limit over all of those $\mathcal{F}(U)$ associated with each open neighborhood U of x , gradually making them smaller. Thus, the stalks encode information about how the sheaf $\mathcal{F}$ behaves locally.

Example 1.21. Take $X = \mathbb{C}$ and $\mathcal{O}_X$ the sheaf of holomorphic functions defined similarly to the example 1.4. Take $x \in X$, if f, g are holomorphic functions such that are equal on a neighbourhood of x , since both of them can be written as a convergent series (Taylor's expression on x) then $f = g$ there. Hence $\mathcal{O}_{X,x}$ can be identified with the ring of power series convergent in a neighborhood of x .

This is a particular behaviour of holomorphic functions. In general it is not easy to give such an easy intuition of the stalks. Given a sheaf on a topological space X and $x \in X$, one has, for each neighborhood U of x , a natural map sending a section $s \in U$ to the class on the stalk where it belongs. These classes are called **germs** and are denoted as s_x .

Definition 1.22. Given a map of presheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ for any $x \in X$ there is an induced map between the stalks $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ defined by sending the class of (s, U) to the class of $(\phi_U(s), U)$. This is named the **map of stalks**.

Remark 1.23. As ϕ is compatible with the restriction maps, since it is a map of presheaves, the following diagram commutes for any pair $V \subseteq U$ open subsets. Hence, ϕ_x is well defined.

$$\begin{array}{ccccc}
\mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) & & \\
\downarrow \rho_{UV} & \searrow & \downarrow \rho_{UV} & \searrow & \\
& \mathcal{F}_x & \xrightarrow{\quad} & \mathcal{G}_x & \\
\uparrow \rho_{UV} & \swarrow & \uparrow \rho_{UV} & \swarrow & \\
\mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) & &
\end{array}$$

Definition 1.24. Let $\mathcal{F}$ be a (pre)sheaf (of abelian groups) on X , the **support** of $\mathcal{F}$ is defined to be:

$$\text{Supp}(\mathcal{F}) = \{x \in X : \mathcal{F}_x \neq 0\}$$

1.4 Kernel, Image, Cokernel and exact sequences

In this section we will work on abelian sheaves (ie: sheaves of abelian groups).

Definition 1.25. Given a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves the **kernel** of ϕ , denoted by $\text{Ker } \phi$ is the subsheaf of $\mathcal{F}$ defined as follows:

$$\text{Ker } \phi(U) = \{s \in \mathcal{F}(U) : \phi_U(s) = 0\}.$$

Proposition 1.26. Given a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves, $\text{Ker } \phi$ is a sheaf.

Proof. For any $V \subseteq U$ open subsets and $s \in \mathcal{F}(U)$, the definition above is compatible with the restriction maps since $\phi_V(s|_V) = \phi_U(s)|_V$. Hence $\text{Ker } \phi$ is a presheaf.

As $\mathcal{F}$ is a sheaf, the locality axiom holds for $\text{Ker } \phi$ because it is a subsheaf of $\mathcal{F}$. Take $\{U_i\}_i$ an open covering of U an open subset and $\{s_i\}_i$ a family of sections such that $s_i \in \text{Ker } \phi(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for each i, j . As $\mathcal{F}$ is a sheaf the gluing axiom holds, so $\exists s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all i . Now we may see that $\phi(s) = 0$. One have that $\phi(s)|_{U_i} = \phi(s_i)$, as the locality axiom holds for $\mathcal{G}$, we have that $\phi(s) = 0$. $\square$

Proposition 1.27. Given a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves for any $x \in X$ forming stalks commutes with the kernel, $(\text{Ker } \phi)_x = \text{Ker } \phi_x$.

Proof. Take s a section of $\mathcal{F}$ over an open neighbourhood of x such that locally lies in $\text{Ker } \phi$, that is that exists $V \subseteq U$ containing x such that $\phi(s|_V) = 0$ that is that s_x lies in $\text{Ker } \phi_x$. That proves $(\text{Ker } \phi)_x \subseteq \text{Ker } \phi_x$.

For the other inclusion, take $s_x \in \text{Ker } \phi_x \subseteq \mathcal{F}_x$. By definition, $\phi_x(s_x) = 0$ in $\mathcal{G}_x$, so $\exists U$ an open neighbourhood of x and $s \in \mathcal{F}(U)$ such that $s_x = s|_U$, so $(\phi(s))_x = \phi_x(s_x) = 0$. Therefore, there exists V an open subset containing x such that $\phi(s)|_V = 0$, and for ϕ being a morphism of sheaves $0 = \phi(s)|_V = \phi(s|_V)$. Hence, $s|_V \in \text{Ker } \phi|_V$, so $s_x = (s|_V)_x \in (\text{Ker } \phi)_x$. $\square$

Definition 1.28. Given a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves the **image** of ϕ , denoted by $\text{Im } \phi$ is the subsheaf of $\mathcal{G}$ defined as the sheafification of the subpresheaf of $\mathcal{G}$ defined as follows:

$$\text{Im } \phi(U) = \{s \in \mathcal{G}(U) : s \in \text{Im } \phi_U\}$$

Proposition 1.29. Given a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves for any $x \in X$ forming stalks commutes with the image, $(\text{Im } \phi)_x = \text{Im } \phi_x$.

Proof. Take t a section of $\mathcal{G}$ over an open neighbourhood of x such that locally lies in $\text{Im } \phi$ as a presheaf, that is that there exists $V \subseteq U$ containing x such that $t|_V \in \text{Im } \phi|_V$, hence, $t_x \in \text{Im } \phi_x$ as it is known that the stalks are preserved by sheafification.

Take $t_x \in \text{Im } \phi_x$, by definition there exists $s_x \in \mathcal{F}_x$ such that $\phi_x(s_x) = t_x$. So we can take U an open neighbourhood of x such that $\phi|_U(s) = t$. Using the commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\phi_x} & \mathcal{G}_x \end{array}$$

$\phi(s)_x = \phi_x(s_x) = t_x \in (\text{Im } \phi)_x$, using again that the stalks are preserved by sheafifications we get the remaining inclusion. $\square$

Definition 1.30. A morphism of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is said to be **injective** if $\text{Ker } \phi = 0$ and is said to be **surjective** if $\text{Im } \phi = \mathcal{G}$.

Remark 1.31. From 1.27 and 1.29 one has that checking the injectivity and surjectivity of a map ϕ of sheaves is equivalent to checking it for ϕ_x for every $x \in X$.

It is important to notice that the injectivity of ϕ is also equivalent to ϕ_U being injective for all U open sets, this is a direct consequence of the definition of $\text{Ker } \phi$. However it is not true that the surjectivity of ϕ implies all of those ϕ_U to be surjective. Let us give an example.

Take $X = \mathbb{C}^*$ and the sheaves $\mathcal{F}$ and $\mathcal{G}$ defined on U open subset to be

$$\mathcal{F}(U) = \{\text{holomorphic functions on } U\} \text{ (as an additive group)}$$

$$\mathcal{G}(U) = \{\text{non-zero holomorphic functions on } U\} \text{ (as a multiplicative group)}$$

Take the morphism of sheaves ϕ that maps $f \in \mathcal{F}(U)$ to $\phi(f) = \exp 2\pi i f \in \mathcal{G}(U)$. Notice that $\phi(\mathbb{C}^*)$ is not surjective since the function $z \in \mathcal{G}(\mathbb{C}^*)$ is not image of any function, is well-known that $\log(z)$ is not an holomorphic function in $\mathbb{C}^*$. However, ϕ is surjective, indeed locally there exists a branch of logarithm so ϕ_x is surjective for all $x \in \mathbb{C}^*$ and this is equivalent to ϕ being surjective.

Definition 1.32. Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{G} \subseteq \mathcal{F}$ we can define the quotient presheaf as follows:

1. For U open set $(\mathcal{F}/\mathcal{G})(U) = \mathcal{F}(U)/\mathcal{G}(U)$.
2. The restriction maps are those induced by the passing to the quotients $\mathcal{F}(U)/\mathcal{G}(U)$ since the restriction maps of both $\mathcal{F}$ and $\mathcal{G}$ preserves the inclusions $\mathcal{G}(U) \subseteq \mathcal{F}(U)$.

We define the **quotient sheaf** $\mathcal{F}/\mathcal{G}$ to be the sheafification of the previous presheaf.

Definition 1.33. Given a morphism $\phi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves the **cokernel** of ϕ , denoted by $\text{Coker } \phi$, is the quotient sheaf $\mathcal{G}/\text{Im } \phi$

We have the following exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G} \longrightarrow 0 \\ 0 &\longrightarrow \text{Ker } \phi \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \text{Coker } \phi \longrightarrow 0 \end{aligned}$$

Chapter 2

Schemes

2.1 Main definitions

In classical algebraic geometry, for k an algebraically closed field, we have the following equivalence of categories:

$$\{\text{Reduced affine varieties over } k\} \leftrightarrow \left\{ \begin{array}{l} \text{commutative rings with 1, nilpotent-free,} \\ \text{finitely generated over } k \end{array} \right\}$$

Let us now examine some examples of reasons why the right-hand side may, in certain cases, be considered too restrictive.

- (a) In number theory, it is reasonable to desire geometric tools to work with. For this reason, it is not unreasonable to want to associate a "geometric object" to rings such as $\mathbb{Z}$ or $\mathbb{Z}(i)$. However, these are not algebras over a closed field.
- (b) If one wants to study, for example, the affine line (its coordinate ring is $k[X]$ given by the relation above), it is natural to look for the rational functions over it, this is $k(X)$ (quotients of polynomials). The problem is that $k(x)$, while being a finitely generated field extension of k , is not finitely generated over k as a k -algebra.
- (c) One common study in algebraic geometry is the study of intersections of varieties. Take following varieties $V(Y)$, $V(X - Y)$, $V(X^2 - Y)$, and we want to study the intersection of the first with the other two. For the first intersection, we take $k[X, Y]/(Y, X - Y) \cong k[X]/(X) \cong k$. In the second one would take $k[X, Y]/(Y, X^2 - Y) \cong k[X]/(X^2)$, but our relation does not allow nilpotent elements which is why we actually take not (X^2) but $\sqrt{(X^2)}$, yielding $k[X]/(X) \cong k$ again. If our goal is the study of intersections, as we

have done, the two are indistinguishable, yet we would like this not to be the case.

Hence is reasonable to want to have commutative rings with unity to the right of the bijection. Modern algebraic geometry gives an answer:

$$\begin{array}{ccc}
 \{\text{Affine varieties}\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{commutative rings with 1, nilpotent-free,} \\ \text{finitely generated over an algebraically closed field} \end{array} \right\} \\
 \downarrow & & \downarrow \\
 \{\text{Affine schemes}\} & \longleftrightarrow & \{\text{Commutative rings with 1}\}
 \end{array}$$

From now on let R be a commutative ring with unity. Our main goal is to establish a geometric structure associated with R , under the consideration that if R is finitely generated nilpotent-free over an algebraic closed field k , then this structure should resemble an affine variety as closely as possible. Indeed we would like to extend the relation between an affine variety and its coordinate ring. To achieve this, we introduce the notion of an affine scheme. It will be a set of points, known as the spectrum of R , equipped with a topology and a sheaf of rings on it called the sheaf of regular functions.

Definition 2.1. Let R be a commutative ring with unity. The *spectrum* of R , is:

$$\text{Spec}(R) = \{\mathfrak{p} \in R \mid \mathfrak{p} \text{ is a prime ideal}\}$$

As a convention R itself won't be an element of $\text{Spec}(R)$.

Example 2.2. $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[x])$

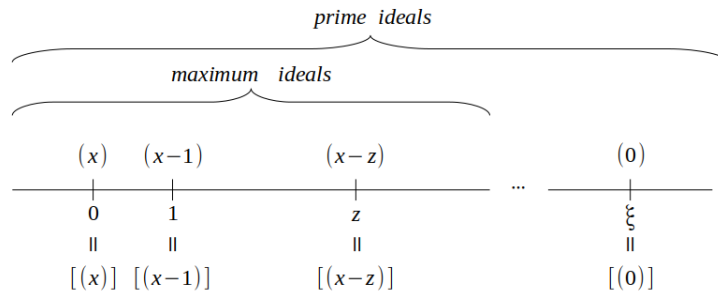


Figure 2.1: A naive representation of $\text{Spec } \mathbb{C}[x]$

The prime ideals of $\mathbb{C}[x]$ are, the maximum ideals $(x - z)$ with $z \in \mathbb{C}$, and the zero ideal (0) , since $\mathbb{C}[x]$ is a domain. In the picture above we have represented the "classical points", one for each maximum ideal (this is one for each complex

number), and one extra point corresponding to (0) , this will be named the **generic point**. Since the zero ideal contains all the others that is why it is pictured somehow anywhere but nowhere at the same time.

Example 2.3. If we try to picture $\text{Spec}(\mathbb{R}[x])$ as we did in the previous example we will get the following points: one for each real number, the generic point, and new points. Those new points correspond to those polynomials of the form $ax^2 + bx + c$ with $b^2 - 4ac < 0$. Thinking of the spectrum of $\mathbb{R}[x]$ we have won one point for each Galois conjugates, this are much more points that in classical algebraic geometry and therefore much more information.

Example 2.4. $\mathbb{A}_{\mathbb{C}}^2 = \text{Spec}(\mathbb{C}[x, y])$

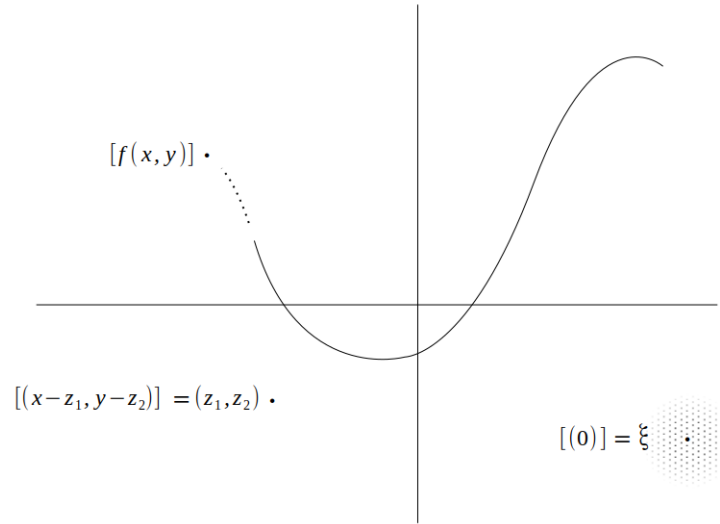


Figure 2.2: A naive representation of $\text{Spec } \mathbb{C}[x, y]$

The maximum ideals correspond to points $[(x - z_1, y - z_2)]$ with $z_1, z_2 \in \mathbb{C}$, those are again the "classical points". We have the generic point corresponding to the zero ideal, as we had before. But now, we have one point for each irreducible curve, since it generates a prime ideal. This new points, $[f(x, y)]$ with f irreducible, are pictured anyway on the curve but nowhere at the same time; for example, the point $[(y - x^2)]$ is not on the parabola $y = x^2$, but, somehow it is everywhere on it. This behaviour gives some intuition that there are bigger points than others. A nice intuition in this point is thinking that the bigger the ideal, the smaller the "size" of the point it represents.

We will now endow our set $\text{Spec}(R)$ with a topology. To do this, we need to introduce notation to refer to specific subsets of our space.

Definition 2.5. For each $\mathfrak{a} \subset R$ an ideal, we define the following subsets

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \text{ contain } \mathfrak{a}\}$$

Lemma 2.6. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R , then $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{ab})$.

Proof. If $\mathfrak{ab} \subseteq \mathfrak{p}$ and suppose $\mathfrak{b} \not\subseteq \mathfrak{p}$. Let us take $b \in \mathfrak{b}$ with $b \notin \mathfrak{p}$ and $a \in \mathfrak{a}$. Now, $ab \in \mathfrak{p}$, so it must be $a \in \mathfrak{p}$, therefore $\mathfrak{a} \subseteq \mathfrak{p}$. On the other direction, is clear that if $\mathfrak{a} \subseteq \mathfrak{p}$ (or $\mathfrak{b} \subseteq \mathfrak{p}$) then $\mathfrak{ab} \subseteq \mathfrak{p}$. $\square$

Lemma 2.7. Let $\{\mathfrak{a}_i\}_{i \in I}$ be a family of ideals of R , then $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$.

Proof. By definition $\sum_i \mathfrak{a}_i$ is the smallest ideal containing all of the ideals $\mathfrak{a}_i$. Hence, $\mathfrak{p} \in V(\sum_i \mathfrak{a}_i)$ if, and only if, $\mathfrak{p} \in \bigcap_i V(\mathfrak{a}_i)$. $\square$

By lemma 2.6 any finite union of subsets $V(\mathfrak{a})$ is again one of those. Moreover, we have proved in lemma 2.7 that any intersection of arbitrary many subsets of the form $V(\mathfrak{a})$ is again of this type. Hence, they define a topology taking them as the closed subsets of the topology. The topology we have defined on $\text{Spec}(R)$ is named **Zariski's topology**.

Definition 2.8. The topology on $\text{Spec}(R)$ is defined by taking the subsets $V(\mathfrak{a})$ to be the closed sets. An open set in **Zariski's topology** is defined as the complement of one of one of those $V(\mathfrak{a})$.

Remark 2.9. It is important to notice that a point $\mathfrak{p} \in \text{Spec}(R)$ is closed if, and only if, it is a maximal ideal. So $\text{Spec}(R)$ is not Hausdorff in general.

Let us focus now on a concrete type of open sets, those that are the complement of a closed set defined by the ideal generated by a single element. Those basic open sets will be of great relevance, not only because they form a base of the topology, but also because they will be associated with localizations by a single element of the ring. As we progress in the text, the importance of these open sets will become evident.

Definition 2.10. For each $f \in R$ a **basic open subset** of $\text{Spec}(R)$ associated to f is $D(f) = \text{Spec}(R) \setminus V((f))$.

Lemma 2.11. For $f \in R$ we have $D(f) = \text{Spec}(R_f)$.

Proof. The prime ideals of R_f are the prime ideals of R that do not contain f , under the correspondence $\mathfrak{p} \mapsto \mathfrak{p}R_f$. $\square$

Proposition 2.12. *Basic open sets of $\text{Spec}(R)$ form a base of the topology.*

Proof. Let U be an open subset of $\text{Spec}(R)$, then there is an ideal $\mathfrak{a}$ of R such that, $U = \text{Spec}(R) \setminus V(\mathfrak{a}) = \text{Spec}(R) \setminus \bigcap_{f \in \mathfrak{a}} V((f)) = \bigcup_{f \in \mathfrak{a}} \text{Spec}(R) \setminus D(f)$. $\square$

To complete the construction of an affine scheme we must define a sheaf structure on $\text{Spec}(R)$. A more or less natural intuition give us the following definition:

1. For any open set $U \subseteq \text{Spec}(R)$, $\mathcal{O}(U)$ is defined to be the set of functions $s : U \rightarrow \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$. The 1 function which maps 1 to each $R_{\mathfrak{p}}$ is an identity, also, the sums and products of those functions are again one of such. Hence $\mathcal{O}(U)$ is a commutative ring with identity.
2. If $V \subseteq U$ are open sets $\rho_{UV} : \mathcal{O}(U) \rightarrow \mathcal{O}(V)$ will be the natural induced restriction morphism of rings.

With this, we have a presheaf. Unfortunately, this is not generally a sheaf. By sheafification (see 1.12) the definition is modified as follows:

1. For any open set $U \subseteq \text{Spec}(R)$, $\mathcal{O}(U)$ is defined to be the set of functions $s : U \rightarrow \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$. And, such that for every $\mathfrak{p} \in U$ there is a neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $a, f \in R$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f \in R_{\mathfrak{q}}$.
2. If $V \subseteq U$ are open sets $\rho_{UV} : \mathcal{O}(U) \rightarrow \mathcal{O}(V)$ will be the natural induced restriction morphism of rings.

Proposition 2.13. *For R a commutative ring with unity, $\mathcal{O}$ defined previously is a sheaf of rings on $\text{Spec}(R)$.*

Remark 2.14. Recall a **local ring** is a ring with a unique maximal ideal.

Proposition 2.15. *Let R be a commutative ring with unity, and $\mathcal{O}$ the sheaf of rings defined before.*

1. For any $\mathfrak{p} \in \text{Spec}(R)$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $R_{\mathfrak{p}}$.
2. For any $f \in R$, $\mathcal{O}(D(f))$ is isomorphic to R_f . In particular $\mathcal{O}(\text{Spec}(R)) = R$.

Proof. 1. We define a morphism $\varphi : \mathcal{O}_{\mathfrak{p}} \rightarrow R_{\mathfrak{p}}$ as follows: for a pair (s, U) with $s \in \mathcal{O}(U)$ and U neighborhood of $\mathfrak{p}$

$$(s, U) \mapsto s(\mathfrak{p})$$

Let us check that φ defines an isomorphism:

- (a) Surjectivity: Any element of $R_{\mathfrak{p}}$ is a/f with $a, f \in R$ and $f \notin \mathfrak{p}$. Hence $D(f)$ is an open neighborhood of $\mathfrak{p}$ such that there exists one of those $s : D(f) \rightarrow \sqcup_{\mathfrak{q} \in D(f)} R_{\mathfrak{q}}$ with $s(\mathfrak{p}) = a/f$.
- (b) Injectivity: Take U an open neighborhood of $\mathfrak{p}$ and $s, t \in \mathcal{O}(U)$ with $s(\mathfrak{p}) = t(\mathfrak{p})$. We can put $s = a/f$ and $t = b/g$ with $a, b, f, g \in R$ and $f, g \notin \mathfrak{p}$. Because of $s(\mathfrak{p}) = t(\mathfrak{p}) \in R_{\mathfrak{p}}$, by definition of localization, $\exists h \notin \mathfrak{p}$ such that $h(ag - bf) = 0 \in R$. Hence $a/f = b/g$ in all $R_{\mathfrak{q}}$ with $f, g, h \notin \mathfrak{q}$. As $D(f) \cap D(g) \cap D(h)$ is an open neighborhood of $\mathfrak{p}$ then $s = t$ in this open subset. We get that $s = t$ in the stalk $\mathcal{O}_{\mathfrak{p}}$.

We get φ is, indeed, an isomorphism.

- 2. Let us proof the first statement. We define a morphism $\psi : R_f \rightarrow \mathcal{O}(D(f))$ as follows: we map a quotient a/f^n to an element $s \in \mathcal{O}(D(f))$ with the property that $s(\mathfrak{p}) = a/f^n$ in every $R_{\mathfrak{p}}$. Let us check that ψ defines an isomorphism:

- (a) Surjectivity: we are not going to prove this statement since it is a bit tedious and the used arguments are not the aim of this work. The proof can be found in [8, Proposition II.2.2.].
- (b) Injectivity: Suppose $\psi(a/f^n) = \psi(b/f^m)$, so by the definition of ψ $a/f^n(\mathfrak{p}) = b/f^m(\mathfrak{p})$ in every $R_{\mathfrak{p}}$ with $\mathfrak{p} \in D(f)$. By definition of localization this means that $\exists h \notin \mathfrak{p}$ such that $h(af^n - bf^m) = 0$ in R . We define now the following ideal of R :

$$\mathfrak{a} = \{h \in R : h(af^n - bf^m) = 0\}$$

So, for all $\mathfrak{p} \in D(f)$ it holds that $\mathfrak{a} \not\subset \mathfrak{p}$. Hence, $V(\mathfrak{a}) \cap D(f) = \emptyset$, so it must be some $l \geq 0$ such that f^l annihilates $af^m - bf^n$. Therefore $a/f^n = b/f^m$ in R_f which give us the injectivity.

For the second part notice that it follows by taking $f = 1$, so $D(f) = \text{Spec}(R)$, and applying what we have just proved.

□

Definition 2.16. A *ringed space* is a pair $(X, \mathcal{O}_X)$ such that X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X .

Definition 2.17. A *morphism of ringed spaces* $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ in which:

- $f : X \rightarrow Y$ is a continuous map.
- $f^\# : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y .

As $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y , for each open set U of Y , we have a ring morphism $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$. Consequently, this induces a natural morphism on the stalks. Indeed, let $p \in X$ be a point, if $s_{f(p)} \in \mathcal{O}_{Y,f(p)}$ is the germ of (s, V) , then, $t = f^\#(s)$ is a section of $\mathcal{O}_X(f^{-1}(V))$, and $p \in f^{-1}(V)$. Therefore the germ t_p of $(t, f^{-1}(V))$ is well defined.

We want $f^\#_{f(p)}(s)$ to vanish at p if s vanishes at $f(p)$. However, this does not happen in general if we do not require any condition on the map $f^\#$. For this to happen we need to ask the maximal ideal of $\mathcal{O}_{Y,f(p)}$ be mapped to the maximal ideal of $\mathcal{O}_{X,p}$. This condition is to require the maps of stalks to be morphisms of local rings.

This is why we will not work on ringed spaces, instead we will work on locally ringed spaces.

Definition 2.18. A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if for all $p \in X$ the stalk $\mathcal{O}_{X,p}$ is a local ring.

Remark 2.19. If R is a commutative ring with unity, the pair $(\text{Spec}(R), \mathcal{O})$ is a locally ringed space. This is a direct consequence of proposition 2.15.

Definition 2.20. For $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ locally ringed spaces, a *morphism of locally ringed spaces* is a pair $(f, f^\#)$, morphism of ringed spaces, such that, for all $p \in X$, the induced morphism $f^\#_p : \mathcal{O}_{Y,f(p)} \rightarrow \mathcal{O}_{X,p}$ satisfies the following condition: if $\mathfrak{m}_Y, \mathfrak{m}_X$ are the maximal ideals of $\mathcal{O}_{Y,f(p)}$ and $\mathcal{O}_{X,p}$ respectively, $(f^\#_p)^{-1}(\mathfrak{m}_X) = \mathfrak{m}_Y$.

Now we are ready to give the definition of affine scheme.

Definition 2.21. An *affine scheme* is a locally ringed space isomorphic to $(\text{Spec}(R), \mathcal{O})$ for some commutative ring with unity R .

Definition 2.22. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ has an open neighborhood U such that $(U, \mathcal{O}_{X|U})$ is an affine scheme. X is called the *underlying topological space* and $\mathcal{O}_X$ the *structural sheaf*. A *morphism of schemes* is a morphism as locally ringed spaces.

If the structural sheaf is understood, and does not bring misunderstands, we will usually denote just by X the scheme $(X, \mathcal{O}_X)$.

2.1.1 Schemes of finite type

We want to generalize the notion of "variety over a field k ". There is a notion of relative schemes, where a base scheme S is chosen.

Definition 2.23. A scheme $(X, \mathcal{O}_X)$ together with a morphism $(X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S)$, named **structural morphism**, is a **scheme over S** .

If the base scheme is affine, $S = \text{Spec}(R)$, we say that X is a **scheme over R** .

Definition 2.24. If X, Y are schemes over S , and $\phi : X \rightarrow Y$ a morphism of schemes, we say it is a **morphism of S -schemes** if the structural morphisms are compatible. This is:

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & S & \end{array}$$

Definition 2.25. If X is a scheme over R , with structural morphism $f : X \rightarrow \text{Spec}(R)$. We say X is of **finite type over R** if $\text{Spec}(R)$ has an open affine covering $\{\text{Spec}(A_i)\}_i$ such that, $f^{-1}(A_i)$ can be covered by affine subsets of X , $\text{Spec}(B_{ij})$, with B_{ij} finitely generated as A_i -algebra.

We will work the most of the time with schemes of finite type over fields. This type of schemes are the closest to varieties, with the difference that we can have non algebraically closed fields, zero divisors...

Example 2.26. 1. $\text{Spec}(k[x_1, \dots, x_n])$ is of finite type over k .

2. More general, $\text{Spec}(k[x_1, \dots, x_n]/\mathfrak{a})$, with $\mathfrak{a}$ an ideal, is of finite type over k .

3. $\mathbb{P}_k^n$ is of finite type over k .

2.2 Subschemes

Definition 2.27. Let $(X, \mathcal{O}_X)$ be a scheme and U an open subset. Then, $(U, \mathcal{O}_{X|U})$ is an scheme, it is called an **open subscheme** of X .

It is not evident that it is an scheme. It suffices for us to see it for the case in which $U = D(f)$ a basic open subset. Then, $(U, \mathcal{O}_{X|U}) \simeq (\text{Spec}(R_f), \mathcal{O}_{\text{Spec}(R_f)})$, see 2.10. As we can cover X with basic open sets we get that for U open subset of X , $(U, \mathcal{O}_{X|U})$ is a scheme.

Definition 2.28. A morphism of schemes $\iota : Y \rightarrow X$ is said to be an **open immersion** if it is an isomorphism with some open subscheme of X .

Example 2.29. Take $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(A_f)$ and $\iota : Y \rightarrow X$ induced by the localization at f . This is an open immersion of $(\operatorname{Spec}(A_f), \mathcal{O}_{\operatorname{Spec}(A_f)})$ in X .

To define what a closed subscheme is, it is not enough to specify a closed subset on X . The construction is more delicate since for a given closed subset $Z \subseteq X$ the sheaf $\mathcal{O}_{X|Z}$ does not behave as a structural sheaf of a scheme. An intuitive approach to the definition is to give $(Z, \mathcal{O}_Z)$ and a morphism $Z \rightarrow X$ that locally looks like $\operatorname{Spec}(A/\mathfrak{a}) \rightarrow \operatorname{Spec}(A)$.

Definition 2.30. A morphism of schemes $\iota : Z \rightarrow X$ is said to be a **closed immersion** if for every affine open subset $\operatorname{Spec}(A) \subset X$, there exists an ideal $\mathfrak{a}$ of A such that $\iota^{-1}(\operatorname{Spec}(A)) \simeq \operatorname{Spec}(A/\mathfrak{a})$.

Definition 2.31. If $Z \subseteq X$ is a closed subset and $\iota : Z \rightarrow X$ a closed immersion, Z is a **closed subscheme** of X .

Example 2.32. Let $X = \operatorname{Spec}(R)$ be an affine scheme and $\mathfrak{a} \subseteq R$ an ideal. The morphism, as rings, $R \rightarrow R/\mathfrak{a}$, gives a closed immersion

$$\operatorname{Spec}(R/\mathfrak{a}) \longrightarrow \operatorname{Spec}(R)$$

Notice that, $\operatorname{Spec}(R/\mathfrak{a}) \simeq V(\mathfrak{a})$ a closed subset of X . Hence, for any ideal $\mathfrak{a} \subseteq R$, $(\operatorname{Spec}(R/\mathfrak{a}), \mathcal{O}_{\operatorname{Spec}(R/\mathfrak{a})})$ is a closed subscheme of X . In Chapter 3 we are going to see that all closed subschemes of an affine scheme are of this type.

Notice that a closed immersion $\iota : Z \rightarrow X$ defines a sheaf of ideals on X , the kernel of:

$$\mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_Z$$

For every open set $U \subseteq X$ we get $\mathcal{I}(U)$ an ideal of the ring $\mathcal{O}_X(U)$. Hence, there is an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z \rightarrow 0$$

Indeed, given $\mathcal{I}$ we can recover Z as follows: $Z = (\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$. Unfortunately, not every sheaf of ideals come about a closed subscheme. We will see in Chapter 3 that a necessary and sufficient condition is ask $\mathcal{I}$ to be quasi-coherent.

2.3 First proprieties

Definition 2.33. A scheme is said to be **irreducible** if the underlying topological space is irreducible (ie: it is not union of two proper closed subsets).

Example 2.34. Take $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ an affine scheme. If A is an integral domain then $(0) \in \text{Spec}(A)$. Suppose that $\text{Spec}(A) = F_1 \cup F_2$ with $F_1 = V(\mathfrak{a}), F_2 = V(\mathfrak{b})$ closed subsets. Then, $\text{Spec}(A) = V(\mathfrak{ab})$. Hence $\mathfrak{ab} \subseteq (0)$, but A is integral, we get a contradiction. We have seen that $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ is an irreducible scheme if A is integral. In particular we have shown that the affine space over k is irreducible.

Definition 2.35. A scheme is said to be **reduced** if $\forall U \subseteq X$ open subset, $\mathcal{O}_X(U)$ do not have nilpotent elements.

Example 2.36. $\text{Spec}(k[x]/(x^2))$ is an example of an affine non reduced scheme, since x is a regular function on $k[x]/(x^2)$ that is nilpotent and not the zero function.

Definition 2.37. A scheme is said to be **integral** if, for all $U \subseteq X$ open subset, $\mathcal{O}_X(U)$ is an integral domain.

Example 2.38. The affine scheme $X = \text{Spec}(k[x, y]/(xy))$ is an example of a non-integral scheme, since x, y are zero divisors of $k[x, y]/(xy)$. Moreover, this is also an example of a scheme with a connected topological space but that is non-irreducible.

We are going to see now a relevant property of affine schemes. Recall that a topological space is said to be quasicompact if for every open covering only a finite number of elements are needed. We say that a scheme is **quasicompact** if the underlying topological space is quasicompact.

Proposition 2.39. An affine scheme $X = \text{Spec}(R)$ is quasicompact.

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of X , we want to see that only a finite number of them are necessary to cover X . As we have proved in 2.12 we can refine the covering and work with $\{D(f_j)\}_{j \in J}$.

Lemma 2.40. A family $\{D(f_j)\}_{j \in J}$ forms a covering of $\text{Spec}(R)$ if, and only if, there exists $\{a_j\}_{j \in J}$, such that, $1 = \sum_j a_j f_j$, and only a finite number of them are non-zero.

Proof. Due to the lemma 2.7 we have $V(\sum_j (f_j))^c = \left(\bigcap_j V(f_j)\right)^c = \bigcup_j D(f_j)$, so they form a covering if, and only if, there is no prime ideal $\mathfrak{p} \in \text{Spec}(R)$ that contains all of those (f_j) , and this happens if, and only if, $\sum_j (f_j) = 1$. Again, this happens if, and only if, $\exists \{a_j\}$ elements of R , only a finite number of them non-zero, such that $1 = \sum_j a_j f_j$. $\square$

As $\{D(f_j)\}_{j \in J}$ is a covering of $X = \text{Spec}(R)$ because of the previous lemma only a finite number of them are necessary. $\square$

Definition 2.41. A scheme $(X, \mathcal{O}_X)$ is said to be **locally noetherian** if it can be covered by open affine subsets, $\{\text{Spec}(R_i)\}_i$, where all of R_i are noetherian. We say it is **noetherian** if the covering can be chosen to be finite.

It is important to notice that we do not require that every open subset of X is the spectrum of a noetherian ring.

Remark 2.42. We have shown that affine schemes are quasicompact. So, it follows from the definition that a scheme is noetherian if, and only if, is both locally noetherian and quasicompact.

Proposition 2.43. An affine scheme $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ is noetherian if, and only if, R is a noetherian ring.

Proof. Let $(X, \mathcal{O}_X) = (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ be an affine scheme. It is clear from the definition that if R is noetherian then the scheme is noetherian, just by taking the trivial covering. We want to see now that if $(X, \mathcal{O}_X)$ is noetherian then R is a noetherian ring. Since the scheme is affine and noetherian it can be covered by a finite number (2.39) of open subsets of the form $D(f) \simeq \text{Spec}(R_f)$ with $f \in R$. Since the localization of a noetherian ring is also noetherian (see [1, Exercise 16.33, solution pg 330]) the problem reduces to a purely algebraic one. We want to prove the following:

If R is a commutative ring with unity, $\{f_1, \dots, f_r\} \in R$, such that, $1 \in (f_1, \dots, f_r)$, and R_{f_i} is noetherian for $i = 1, \dots, r$, then R is noetherian.

Let $\mathfrak{a} \subseteq R$ be an ideal and $\varphi_i : R \rightarrow R_{f_i}$ be the localization map for each i . We claim that,

$$\mathfrak{a} = \bigcap_{i=1}^r \varphi_i^{-1}(\varphi_i(\mathfrak{a})R_{f_i}).$$

To see the proof of that claim see [8, II, Proposition 3.2.]. Take an ascending chain of ideals in R

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$$

Looking in each localization R_{f_i} ,

$$\varphi_i(\mathfrak{a}_1)R_{f_i} \subseteq \varphi_i(\mathfrak{a}_2)R_{f_i} \subseteq \dots$$

As R_{f_i} is noetherian, the chain stabilizes. Now, just a finite number of f_i are necessary to generate the unite ideal, so, because of the claim we get the first chain eventually becomes stationary, and therefore R is noetherian. $\square$

Proposition 2.44. *A scheme $(X, \mathcal{O}_X)$ is locally noetherian if, and only if, for all open affine subset $U = \text{Spec}(R)$, R is a noetherian ring.*

Proof. It follows from the definition that, if for all $U \subseteq X$ open subset, U is of the form $U = \text{Spec}(R)$ with R a noetherian ring, then, X is locally noetherian.

Conversely, we must show that if X is locally noetherian and $U = \text{Spec}(R)$ an affine subset, then R is noetherian. Since the basic open subsets form a base for the topology of $\text{Spec}(A)$ for a ring A , and the localization of a noetherian ring is noetherian, there is a base for the topology on X formed by spectrum of noetherian rings. So U is covered by affine subsets that are the spectrum of noetherian rings. Therefore, it is a direct consequence of the previous result that R is noetherian. $\square$

The noetherian property is not just an algebraic one, it is also a topological one:

Definition 2.45. *A topological space X is said to be **noetherian** if for any descending chain of closed subsets*

$$Y_1 \supseteq Y_2 \supseteq \dots \quad Y_i \text{ closed subset}$$

there exists $m \geq 0$, such that, $Y_m = Y_{m+k}$ for all $k \geq 0$.

Proposition 2.46. *If $(X, \mathcal{O}_X)$ is a noetherian scheme then the underlying topological space is noetherian.*

Proof. X can be covered by a finite number of affine open subsets $\{\text{Spec}(R_i)\}$ with R_i noetherian. So it is sufficient to prove it for $X = \text{Spec}(R)$. Take a descending chain of closed subsets of $\text{Spec}(R)$,

$$V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \dots \quad \text{with } \mathfrak{a}_i \subseteq R \text{ ideal.}$$

Taking ideals,

$$I(V(\mathfrak{a}_1)) \subseteq I(V(\mathfrak{a}_2)) \subseteq \dots$$

This is an ascending chain of ideals in R , which is noetherian, so it becomes stationary. Since $V(I(V(\mathfrak{a}_i))) = V(\mathfrak{a}_i)$ for all i , we get that original chain also becomes stationary. $\square$

2.3.1 A briefly introduction to the dimension of a scheme

We dedicate this section to discussing the concept of dimension. We will provide the initial definitions and some examples. Going further into this topic is beyond the scope of this work, as dimension theory requires complex results and definitions that will be explained in master courses.

Definition 2.47. *The Krull dimension of a ring R is*

$$\dim(R) := \sup\{r \mid \text{exists a chain of primes } \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_r \text{ in } R\}.$$

Definition 2.48. *The Krull dimension of a topological space X is*

$$\dim(X) := \sup\{r \mid \text{exists a chain } Y_0 \subset Y_1 \subset \dots \subset Y_r \text{ irreducible closed subsets of } X\}.$$

Definition 2.49. *The dimension of a scheme is defined to be the dimension of the underlying topological space.*

Remark 2.50. The irreducible closed subsets of $\text{Spec}(R)$ are in correspondence with the prime ideals of R , as we will see in the next chapter. So, for affine schemes $\dim(\text{Spec}(R)) = \dim(R)$.

Example 2.51. If R is a principal ideals domain which is not a field, all the non-zero prime ideals are also maximal, so the longest chains we can get are $\{0\} \subset \mathfrak{p}$. Hence, the dimension of those is 1. In particular, $\dim(K[x]) = \dim(\mathbb{A}_K^1) = 1$.

Although the definition of the dimension of a scheme may seem intuitive, calculating the dimension of simple examples such as the affine space or the projective space, is actually quite complicated. In the case of schemes of finite type, the notion of dimension can be reformulated using Noether's normalization lemma. From now on, unless said otherwise, we will work only with schemes of finite type.

Definition 2.52. *For integral schemes X of finite type over a field k , with function field K the dimension of X is the degree of transcendence of K over k .*

Since every open subset U of X has the same function field than X , $\dim(U) = \dim(X)$.

Example 2.53. 1. $\dim(\mathbb{A}_k^n) = n$.

2. $\dim(\mathbb{P}_k^n) = n$.

3. The dimension of a curve on the plane is 1.

4. Take $Q = \text{Spec}(k[x, y, z]/(x^2 - yz))$, it has function field $k(x, y)$, since we can take $z = y^{-1}x^2$ to eliminate z . This has transcendence degree 2 over k , so $\dim(Q) = 2$.

Remark 2.54. In general all hypersurfaces of a n -dimensional scheme have dimension $n - 1$.

If we take $X = \bigcup_i X_i$ is the decomposition into irreducible components, the dimension of X is just the supreme of all $\dim(X_i)$.

Example 2.55. $X = \text{Spec}(k[x, y, z])$ and $Y = V(\mathfrak{a})$ with $\mathfrak{a} = (x, y, z) \cap (x - 1, y - 1) \cap (x + 1, y + 1)$. Y has three irreducible components, two lines and a point. Hence the dimension of Y is 1.

2.4 Gluing two schemes

One of the main properties of schemes, and one of the reasons that makes the theory much richer than classical algebraic geometry, is the flexibility in gluing schemes together to obtain new schemes, as well as sheaves on them and morphisms between them.

The possibility to obtain new schemes by gluing together old ones along open subschemes is significant, especially compared to the fact that gluing algebraic varieties together easily fails to be a variety. In this chapter, we will focus on a particular case of this fact. We will introduce the technique of gluing two schemes together to obtain an example of great importance: projective spaces and a family of sheaves over them.

Let us consider two schemes $(X_1, \mathcal{O}_{X_1})$ and $(X_2, \mathcal{O}_{X_2})$ and an open subscheme for each one:

$$(X_1, \mathcal{O}_{X_1}) \subseteq (X_{12}, \mathcal{O}_{X_1|_{X_{12}}}) \quad (X_2, \mathcal{O}_{X_2}) \subseteq (X_{21}, \mathcal{O}_{X_2|_{X_{21}}})$$

And let us assume we have an isomorphism of schemes $\tau : X_{21} \rightarrow X_{12}$. Our goal is to define a new scheme X by gluing together X_1, X_2 along X_{12}, X_{21} .

- The topological space:

$$X = X_1 \sqcup X_2 / x \sim \tau(x) \quad \text{for } x \in X_2$$

We equip X with the quotient topology.

Now let us take $\iota_j : X_j \rightarrow X$ for $j = 1, 2$ an open immersion.

- The structural sheaf: the sections of $\mathcal{O}_X$ in an open subset $U \subseteq X$ will be defined by the following exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X(U) & \longrightarrow & \mathcal{O}_X(U \cap X_1) \times \mathcal{O}_X(U \cap X_2) & \longrightarrow & \mathcal{O}_X(U \cap X_1 \cap X_2) \\ & & & & \downarrow \simeq & & \downarrow \simeq \\ & & & & \mathcal{O}_{X_1}(U_1) \times \mathcal{O}_{X_2}(U_2) & \xrightarrow{\alpha} & \mathcal{O}_{X_2}(U_{21}) \end{array} \quad (2.1)$$

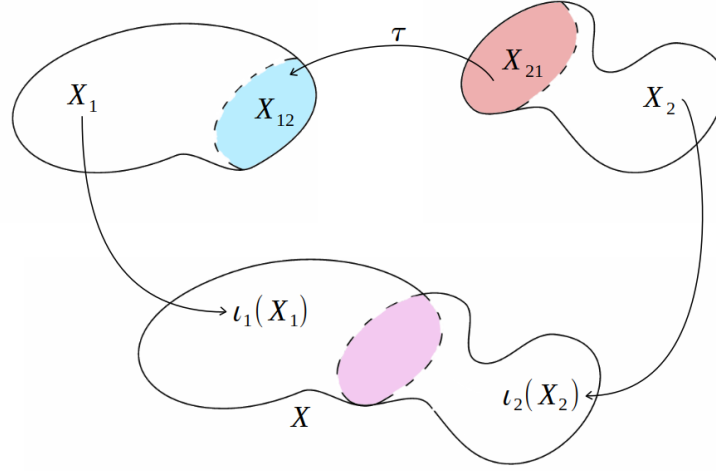


Figure 2.3: Gluing two schemes

Where $U_j = \iota_j^{-1}(U)$ for $j = 1, 2$, $U_{ji} = \iota_j^{-1}(U_1 \cap U_2)$, and α is defined as follows:

$$\begin{aligned} \alpha : \mathcal{O}_{X_1}(U_1) \times \mathcal{O}_{X_2}(U_2) &\longrightarrow \mathcal{O}_{X_2}(U_{21}) \\ (s, t) &\mapsto t|_{U_{21}} - \tau^\#(s|_{U_{12}}) \end{aligned}$$

To recall the definition of $\tau^\#$ see 2.17.

This defines a scheme. To see the construction in general and the proof of this statement see [5, Chapter 4, Proposition 4.3.].

Example 2.56. Take two copies of the affine line $X_1, X_2 = \text{Spec}(k[x])$, and glue them along the common open subset $X_{12} = \text{Spec}(k[x, x^{-1}])$ with the identity map. The resulting scheme consists of two copies of the affine line that overlap everywhere but the origin. Since the gluing process does not affect the origin there are two points that replace the origin in new scheme. This is known as **the affine line with two origin**. Later in this text we will see this is an example of a non-affine scheme.

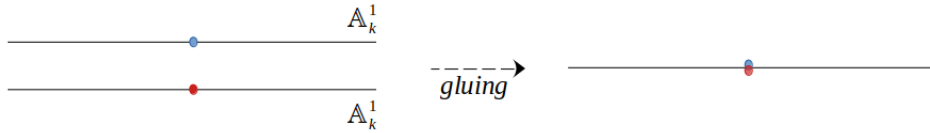


Figure 2.4: Affine line with two origin

Using the gluing technique we are going to see how to construct the affine line and an important family of sheaves on it.

2.4.1 The projective line

Let us now give an example of this construction. Take R a ring and consider $U_0 = \text{Spec}(R[x])$ and $U_1 = \text{Spec}(R[x^{-1}])$ as schemes (both of them are copies of the affine line). Consider the following open subscheme respectively:

$$U_{01} = D(x) \simeq \text{Spec}(R[x, x^{-1}]) \quad U_{10} = D(x^{-1}) \simeq \text{Spec}(R[x^{-1}, x])$$

The result of gluing together U_0 and U_1 along the isomorphism $U_{10} \simeq U_{01}$ is called the **projective line over R** and it is denoted by $\mathbb{P}_R^1$.

Proposition 2.57. $\mathcal{O}_{\mathbb{P}_R^1}(\mathbb{P}_R^1) = R$

Proof. Using the construction of the scheme $\mathbb{P}_R^1$, the sections of the structural sheaf are obtained by the diagram 2.1. So, let us consider the diagram that defines the global section:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_R^1}(\mathbb{P}_R^1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_R^1}(U_0) \times \mathcal{O}_{\mathbb{P}_R^1}(U_1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_R^1}(U_{10}) \\ & & & & \downarrow \simeq & & \downarrow \simeq \\ & & & & R[x] \times R[x^{-1}] & \xrightarrow{\alpha} & R[x, x^{-1}] \end{array}$$

We have used the result 2.15. We want to see that $\text{Ker}(\alpha) = R$. Take $(f(x), g(x^{-1})) \in \text{Ker}(\alpha)$, this is $g(x^{-1}) = f(x)$. Let us assume that

$$f(x) = ax^n + \sum_{i=0}^{n-1} a_i x^i \quad g(x^{-1}) = bx^{-m} + \sum_{i=0}^{m-1} b_i x^{-i}$$

with $a, b \neq 0$, and, without loss of generality $m \geq n$. If $m \geq 1$, then, multiplying by x^m , we will get $b + xh(x) = x^m f(x)$, and evaluating at $x = 0$, $b = 0$ in contradiction to the assumption. We have shown that $\text{Ker}(\alpha) = R$, this ends the prove of $\mathcal{O}_{\mathbb{P}_R^1}(\mathbb{P}_R^1) = R$. $\square$

We can generalise this procedure to construct, the **n -projective scheme**.

Take R a ring and consider now $R[x_0, x_0^{-1}, \dots, x_n, x_n^{-1}]$. For each $i \in \{0, \dots, n\}$, take the ring $R_i = R[x_0 x_i^{-1}, \dots, x_n x_i^{-1}] \simeq R[x_1, \dots, x_n]$. Take the following family of affine schemes $\{\text{Spec}(R_i)\}_{i=0, \dots, n}$. As the example of the projective line we have isomorphisms as "gluing morphisms". Gluing all of them, pair by pair, along the intersections $U_i \cap U_j$, we obtain a new scheme. This is called the **n -projective space over R** , and it is denoted by $\mathbb{P}_R^n$.

2.4.2 A family of sheaves over $\mathbb{P}_R^1$

Once the construction of $\mathbb{P}_R^1$ is given, we can define a family of sheaves over it indexed by $\mathbb{Z}$, $\{\mathcal{O}_{\mathbb{P}}(m)\}_{m \in \mathbb{Z}}$. In order to do it, we will use the same construction that was explained in the first part of this chapter.

The original schemes U_0, U_1 are now an open covering of the projective line over R , using the open immersions $\iota_j : U_j \rightarrow \mathbb{P}_R^1$. Moreover, $U_0 \cap U_1 = \text{Spec}(R[x, x^{-1}])$. Notice that, for each $m \in \mathbb{Z}$ we have an isomorphism

$$R[x, x^{-1}] \xrightarrow[\cdot x^m]{\simeq} R[x, x^{-1}].$$

Those induce a isomorphism of sheaves, one for each m :

$$\tau : \mathcal{O}_{U_1|_{U_0 \cap U_1}} \longrightarrow \mathcal{O}_{U_0|_{U_0 \cap U_1}}.$$

Hence, using the construction of the structural sheaf in the first part of this section we can define the sheaf $\mathcal{O}_{\mathbb{P}_R^1}(m)$ as the sheaf that one obtains while gluing together U_0, U_1 along τ .

Remark 2.58. Using a similar construction, one can define $\{\mathcal{O}_{\mathbb{P}_R^n}(m)\}_{m \in \mathbb{Z}}$, a family of sheaves over $\mathbb{P}_R^n$.

We are going to see now that, for $m \neq 0$, the sheaves $\mathcal{O}_{\mathbb{P}_R^1}(m)$ are not isomorphic to the structural sheaf of $\mathbb{P}_R^1$.

Proposition 2.59. 1. For $m < 0$ we have,

$$\mathcal{O}_{\mathbb{P}_R^1}(m)(\mathbb{P}_R^1) = 0$$

2. For $m > 0$ we have,

$$\mathcal{O}_{\mathbb{P}_R^1}(m)(\mathbb{P}_R^1) = \bigoplus_{k=0}^m R \cdot x^k$$

Proof. Using the notation we introduced in the construction of $\mathcal{O}_{\mathbb{P}_R^1}(m)$, the diagram that defines the global section is:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_R^1(m)}(\mathbb{P}_R^1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_R^1}(m)(U_0) \times \mathcal{O}_{\mathbb{P}_R^1}(m)(U_1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_R^1}(m)(U_{10}) \\ & & & & \downarrow \simeq & & \downarrow \simeq \\ & & & & R[x] \times R[x^{-1}] & \xrightarrow{\alpha} & R[x, x^{-1}] \end{array}$$

The morphism α is:

$$\begin{array}{ccc} \alpha : & R[x] \times R[x^{-1}] & \longrightarrow & R[x, x^{-1}] \\ & (p(x), q(x^{-1})) & \longmapsto & x^m q(x^{-1}) - p(x) \end{array}$$

Let us discuss the two cases separately.

1. For $m < 0$ there are no polynomials $p(x) \in R[x], q(x^{-1}) \in R[x^{-1}]$ such that $x^m q(x^{-1}) = p(x)$. Hence, $\text{Ker } \alpha = 0$. So, $\mathcal{O}_{\mathbb{P}_R^1}(m)(\mathbb{P}_R^1) = 0$.
2. For $m > 0$, take $p(x) \in R[x]$ of degree $d \leq m$. Then $p(x) = \sum_{k=0}^d a_k x^k$. Taking $q(x^{-1})$ to be $\sum_{k=0}^d a_k x^{-m+k}$ one has $x^m q(x^{-1}) = p(x)$. Notice that, given $p(x)$, $q(x^{-1})$ is uniquely determined. Hence,

$$\mathcal{O}_{\mathbb{P}_R^1}(m)(\mathbb{P}_R^1) = \bigoplus_{k=0}^m R \cdot x^k$$

□

Chapter 3

Sheaves of Modules

3.1 Sheaves of modules

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$ -modules** (or a $\mathcal{O}_X$ -module) is a sheaf $\mathcal{F}$ on X such that:

1. For every $U \subseteq X$ open subset, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$ -module.
2. For every pair of open subsets $V \subseteq U \subseteq X$ the restriction maps are compatible with the module structure, this is to require the following diagram to commute:

$$\begin{array}{ccc} \mathcal{F}(U) \times \mathcal{O}_X(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) \times \mathcal{O}_X(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

Definition 3.2. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$ -modules. A **morphism of $\mathcal{O}_X$ -modules** from $\mathcal{F}$ to $\mathcal{G}$ is a sheaves morphism $\mathcal{F} \rightarrow \mathcal{G}$ such that for every $U \subseteq X$ open subset $\mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a morphism of $\mathcal{O}_X(U)$ -modules. We denote by $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the group of morphisms of $\mathcal{O}_X$ -modules from $\mathcal{F}$ to $\mathcal{G}$.

Remark 3.3. We denote by Mod_X the category of $\mathcal{O}_X$ -modules. It is known that is an abelian category.

Definition 3.4. A $\mathcal{O}_X$ -module $\mathcal{F}$ is said to be **free** if it is isomorphic (1.10) to a direct sum of copies of $\mathcal{O}_X$. The number of copies of $\mathcal{O}_X$ (not necessary finite) is the **rank** of $\mathcal{F}$. Furthermore, it is said to be **locally free** if X can be covered by open sets U such that $\mathcal{F}|_U$ is free as a $\mathcal{O}_{X|U}$ -module.

Definition 3.5. The locally free $\mathcal{O}_X$ -modules of rank 1 are called **invertible sheaves**.

3.2 Quasicoherent and coherent sheaves

Let us begin this section by defining a sheaf associated to a R -module on an affine scheme $\text{Spec}(R)$.

Definition 3.6. Let R be a ring and M a R -module. We define the *sheaf* $\tilde{M}$ associated with M on $\text{Spec}(R)$ as follows:

1. For every $U \subseteq \text{Spec}(R)$ open subset, $\tilde{M}(U) = \{s : U \rightarrow \sqcup_{p \in U} M_p\}$ such that for all $p \in U \exists V \subseteq U$ with $p \in V$ and elements $m \in M$ and $f \in R$ such that per each $q \in V$, $f \notin q$ and $s(q) = m/f \in M_q$
2. If $V \subseteq U$ open subsets $\rho_{UV} : \tilde{M}(U) \rightarrow \tilde{M}(V)$ will be the natural restriction map induced by the restriction of R -modules $U \rightarrow V$.

Proposition 3.7. Let R be a ring, M a R -module and $\tilde{M}$ the sheaf defined above on $X = \text{Spec} R$, then:

1. $\tilde{M}$ is an $\mathcal{O}_X$ -module.
2. $\tilde{M}_p \simeq M_p$ for all $p \in \text{Spec}(R)$
3. For all $f \in R$, $\widetilde{M(D(f))} \simeq M_f$, in particular $\tilde{M}(X) \simeq M$.

Proof. For the first statement notice that every step in the construction is compatible with the structure of module so it is clear that $\tilde{M}$ is an $\mathcal{O}_X$ -module. For (2), (3) see 2.15 and replace R by M where needed. $\square$

Example 3.8. Let $X = \text{Spec}(R)$ and $\mathcal{F}$ the sheaf associated to the presheaf defined as:

$$U \mapsto \begin{cases} \mathcal{O}_X(U) & \text{if } 0 \notin U \\ 0 & \text{if } 0 \in U \end{cases}$$

And taking the restriction maps to be those of $\mathcal{O}_X$ if $0 \in V \subseteq U$ and 0 otherwise. Then, $\mathcal{F}$ is an $\mathcal{O}_X$ -module and the stalks are $\mathcal{F}_0 = 0$ and $\mathcal{F}_p = \mathcal{O}_{X,p}$ if $p \neq 0$. Notice that, since $0 \in X$, $\mathcal{F}(X) = 0$. By the previous result $\mathcal{F}$ cannot be of the form $\tilde{M}$ for any R -module M because otherwise $0 = \mathcal{F}(X) = M$ and this would imply that $\mathcal{F}$ is the zero sheaf and it is not.

Example 3.9. Take $R = \mathbb{Z}[x]_{(x)}$ and $X = \text{Spec}(R) = \{(x), (0)\}$. We have only three open sets that are:

$$\emptyset \quad \text{Spec}(R) \quad D((x)) = \{(0)\}.$$

The structural sheaf satisfies

$$\mathcal{O}_X(\emptyset) = 0 \quad \mathcal{O}_X(\text{Spec}(R)) = R \quad \mathcal{O}_X(\{(0)\}) = \mathbb{Z}[x]_{(0)} = \mathbb{Z}(x).$$

An $\mathcal{O}_X$ -module $\mathcal{F}$ can be specified as follows:

$$\mathcal{F}(X) = \mathcal{F}(\emptyset) = 0 \quad \mathcal{F}(\{(0)\}) = \mathbb{Z}(x)$$

Then, $\mathcal{F}$ cannot be of the type $\tilde{M}$ for any R -module M since by the previous proposition we get $M = \mathcal{F}(X) = 0$ and hence $\mathcal{F}$ would be identically 0 and this is not true.

Definition 3.10. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$ -module $\mathcal{F}$ is a **quasicoherent sheaf** if X can be covered by affine subsets $U_i \simeq \text{Spec}(R_i)$, such that, for all i exists M_i a R_i -module that $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$. If all of those M_i can be chosen to be finitely generated as R_i -modules we say that $\mathcal{F}$ is a **coherent sheaf**.

Example 3.11. For any noetherian scheme $(X, \mathcal{O}_X)$, the structural sheaf is quasicoherent.

We are going to see now two important results where the property of the sheaf to be quasicoherent is very relevant. The first one will be related to general lack of surjectivity when taking global sections on a short exact sequence of sheaves, that will be not the case when the first sheaf is quasicoherent. The second one will be the relation between closed subschemes and quasicoherent sheaves of ideals.

Theorem 3.12. Let X be an affine scheme and $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ three $\mathcal{O}_X$ -modules such that

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0. \quad (3.1)$$

If $\mathcal{F}'$ is quasicoherent then:

$$0 \rightarrow \mathcal{F}'(X) \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}''(X) \rightarrow 0 \quad (3.2)$$

is exact.

In order to prove this theorem we need a couple of previous lemmas.

Lemma 3.13. *Let $X = \text{Spec}(R)$ be an affine scheme and $f \in R$. Let $D(f)$ be the basic open subset associated to f and $\mathcal{F}$ a quasicoherent sheaf on X .*

1. *If $s \in \mathcal{F}(X)$ such that $s|_{D(f)} = 0$ then $\exists n > 0$ such that $f^n s = 0$.*
2. *If $t \in \mathcal{F}(D(f))$ then $\exists n > 0$ such that $f^n t$ extends to a global section.*

Proof. 1. As X is an affine scheme, it is quasicompact (see 2.39), and, moreover $\mathcal{F}$ is quasicoherent. Hence $\exists \{D(g_i)\}_{i \in I}$ a finite open covering of X with $\mathcal{F}_{D(g_i)} \simeq M_i$ for some A_{g_i} -modules M_i . We denote by $s_i = s|_{D(g_i)} \in M_i$. Notice that $D(f) \cap D(g_i) = D(fg_i)$ implies $\mathcal{F}_{D(fg_i)} = (\widetilde{M_i})_f$, since $s = 0$ in $\mathcal{F}(D(f))$, this implies that $s = 0$ in $\mathcal{F}(D(fg_i)) = (M_i)_f$ (see 3.7). So the localization by f sends s_i to 0 at $(M_i)_f$, in other words, s_i is annihilated by some power of f . Take $N = \max_{i \in I} \{n_i : f^{n_i} s_i = 0\}$, then $f^N s_i = 0$ for all $i \in I$. Now using that $\mathcal{F}$ is a sheaf, by the locality axiom we get $f^N s = 0$.

2. Take $s \in \mathcal{F}(D(f))$, we want to show that there exists some $N > 0$ such that $f^N s$ can be lifted to a global section of $\mathcal{F}$. Using the same argument that in the previous case, we have $\{D(g_i)\}_{i \in I}$ a finite open covering of X with $\mathcal{F}_{D(g_i)} \simeq M_i$ for some A_{g_i} -modules M_i . Moreover $s|_{D(fg_i)} \in (M_i)_f$ is of the form s_i / f^n for some $s_i \in M_i$. Because of I is finite we can take a common $N > 0$, hence $s_i = f^N s = s_j$ in the intersection $D(f) \cap D(g_i) \cap D(g_j)$. By applying (1) in $\text{Spec}(R_{g_i g_j})$ we have that $f^M (s_i - s_j) = 0$ in $D(g_i) \cap D(g_j)$ for some $M > 0$ sufficiently large. Now, using the gluing axiom, there exists a global section t of $\mathcal{F}$ that extends $f^M s$ (given just by gluing together those $f^M s_i$).

□

Lemma 3.14. *With the hypothesis of theorem 3.12 if $s \in \mathcal{F}''(X)$ and $D(f)$ is an open basic subset such that $s|_{D(f)}$ can be lifted to a section $t \in \mathcal{F}(D(f))$ then there exists $n > 0$ such that $f^n s$ can be lifted to a global section of $\mathcal{F}$.*

Proof. As 3.1 is exact and X is an affine scheme we can take $\{D(g_i)\}_{i \in I}$ a finite open covering of X such that $s|_{D(g_i)}$ can be extended to a section $s_i \in \mathcal{F}(D(g_i))$. In $D(fg_i)$ we have that $s_i|_{D(fg_i)}$ and $t|_{D(fg_i)}$ both lift to $s|_{D(g_i)}$, using the injectivity of the first map of 3.1 we get $(s_i - t)|_{D(fg_i)} \in \mathcal{F}'(D(fg_i))$. Now, using the second statement of the previous lemma, we have that there exists some $n > 0$ and some sections $r_i \in \mathcal{F}'(D(g_i))$ such that $r_i|_{D(fg_i)} = f^n (s_i - t)|_{D(fg_i)}$. So we have that $\sigma_i = r_i + f^n s_i$ are sections of $\mathcal{F}(D(g_i))$ that restricts to $f^n t$ in $D(fg_i)$.

Now we want to glue these σ_i in order to get a global section σ of $\mathcal{F}$. Notice that $\sigma_i|_{D(g_i g_j)} - \sigma_j|_{D(g_i g_j)} \in \mathcal{F}(D(g_i g_j))$ goes to zero in $D(fg_i g_j)$. Using the first

statement of the lemma 3.13, there exists n_{ij} such that $f^{n_{ij}}(\sigma_i|_{D(g_i g_j)} - \sigma_j|_{D(g_i g_j)}) = 0$. We can take a common N for all (i, j) , so all of $f^N t_i \in \mathcal{F}(D(g_i))$ matches on the intersections $D(g_i) \cap D(g_j)$. Hence, using the gluing axiom, we get a global section σ of $\mathcal{F}$ such that $\sigma \mapsto f^{N+n}s$ since each $f^N \sigma_i \mapsto f^{N+n} s|_{D(g_i)}$. This ends the proof. $\square$

Now we can proceed with the proof of the theorem 3.12.

Proof. It only remains to show the surjectivity of the last map, this is a direct consequence of remark 1.31 in the first chapter. Indeed: let s be a global section of $\mathcal{F}''$. Consider $\{D(f_i)\}_{i \in I}$ a finite open covering of X such that $s|_{D(f_i)}$ can be extended to a section $s_i \in \mathcal{F}(D(f_i))$. Using the second lemma there exists $n > 0$ such that $f_i^n s|_{D(f_i)}$ lifts to a global section t_i of $\mathcal{F}$. As $D(f_i^n)$ cover X we have that $1 = \sum_i a_i f_i^n$ (see 2.39 proof) and hence $\sum_i a_i t_i$ lifts s . So we have proved the surjectivity that was remaining. $\square$

In the chapter 2 we have defined closed subscheme of a scheme $(X, \mathcal{O}_X)$ to be: $Z \subseteq X$ a closed subset and $\iota : Z \rightarrow X$ a closed immersion. We have also seen that, the structural sheaf of Z in this case is $\mathcal{O}_X/\mathcal{I}$ for some sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. It is not clear, even in the affine case, that every sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ give rise to a closed subscheme. As we have said in the previous chapter, we are going to see now that we need to ask $\mathcal{I}$ to be quasicohherent. The following results will be quoted since the proofs, although they are not much complicated, are not our main goal.

Proposition 3.15. *Let Z be a closed subscheme of X , with structural sheaf $\mathcal{O}_X/\mathcal{I}$ for $\mathcal{I} \subseteq \mathcal{O}_X$ sheaf of ideals. Then, $\mathcal{I}$ is quasicohherent.*

Proof. See [5, Proposition 10.54] $\square$

Proposition 3.16. *For $(X, \mathcal{O}_X)$ a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ quasicohherent sheaf of ideals. $(\text{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ is a closed subscheme.*

Proof. See [5, Proposition 10.53] $\square$

Now we are able to characterize the closed subschemes of affine schemes.

Proposition 3.17. *Let $X = \text{Spec}(R)$ be an affine scheme. There is the following bijection:*

$$\{\text{Ideals } \mathfrak{a} \subseteq R\} \longleftrightarrow \{\text{closed subschemes of } X\}$$

Proof. We have seen in the example 2.32 that, for $\mathfrak{a} \subseteq R$ ideal, $\text{Spec}(R/\mathfrak{a})$ is a closed subscheme of $\text{Spec}(R)$. Conversely, take Z a closed subscheme given by the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. Since $\mathcal{I}$ is quasicoherent there is some $I \subseteq R$ ideal such that $\mathcal{I} = \tilde{I}$ (see [5, Theorem 10.31]). Then,

$$\mathcal{O}_Z = \text{Coker}(\mathcal{I} \longrightarrow \mathcal{O}_X) = \text{Coker}(\tilde{I} \longrightarrow \tilde{R}) = \widetilde{R/I}$$

□

In particular we have seen the following result.

Corollary 3.18. *All closed subschemes of an affine scheme are affine.*

Chapter 4

Sheaves cohomology

4.1 Introduction

One of the main problems one has when working with sheaves is the fact that a surjective morphism of sheaves may not preserve its surjectivity when taking global sections. That is, when one has a short exact sequence of sheaves

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0 \quad (4.1)$$

when taking the global section functor (ie: $\Gamma(X, -) : Sh(X) \rightarrow Ab$ defined as $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$), one obtains:

$$0 \rightarrow \Gamma(X, \mathcal{F}') \rightarrow \Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, \mathcal{F}''). \quad (4.2)$$

In general we lose the exactness on the right. An example of this behaviour is given in 1.31.

In this chapter we will introduce a first steps in sheaves cohomology. This is defining a family of additive functors

$$\begin{aligned} H^p(X, -) : Sh(X) &\longrightarrow Ab \\ \mathcal{F} &\longmapsto H^p(X, \mathcal{F}) \end{aligned}$$

with the following properties:

1. $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$
2. Functoriality: for any $p \geq 0$, $H^p(X, -)$ is a functor between the category of abelian sheaves and the category of abelian groups.
3. There are connection morphisms: for any short exact sequence of sheaves as 3.1, there is a family of morphisms $\{\delta^p : H^p(X, \mathcal{F}'') \rightarrow H^{p+1}(X, \mathcal{F}')\}_{p \geq 0}$,

named **connection morphisms**, such that we have a long exact sequence that extends 4.2:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(X, \mathcal{F}') & \longrightarrow & \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(X, \mathcal{F}'') \\
 & & & & \searrow \delta^0 & & \\
 & & H^1(X, \mathcal{F}') & \longrightarrow & H^1(X, \mathcal{F}) & \longrightarrow & H^1(X, \mathcal{F}'') \\
 & & & & \searrow \delta^1 & & \\
 & & H^2(X, \mathcal{F}') & \longrightarrow & H^2(X, \mathcal{F}) & \longrightarrow & H^2(X, \mathcal{F}'') \longrightarrow \dots
 \end{array} \tag{4.3}$$

Cohomology groups will "control" the behaviour of $\mathcal{F}$ while being an important family of geometric invariants.

4.2 Čech cohomology

4.2.1 Čech cohomology of a covering

In this section we will take X to be a topological space, $\mathcal{F}$ an abelian sheaf on X , $\mathcal{U} = \{U_i\}_{i \in I}$ a covering of X indexed by I a totally ordered set. In order to make the construction easier (and by the fact that in the following sections we will can lay to this case) we consider I to be finite.

Definition 4.1. For $p \geq 0$ we define the following group:

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0, \dots, i_p}),$$

where $U_{i_0, \dots, i_p}$ denote $U_{i_0} \cap \dots \cap U_{i_p}$.

An element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ is defined by giving $\alpha_{i_0, \dots, i_p} \in \mathcal{F}(U_{i_0, \dots, i_p})$ for all $(p+1)$ -tuple of indexes with $i_0 < \dots < i_p$.

Definition 4.2. For $p \geq 0$ we define the morphism $d^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ given by:

$$(d^p \alpha)_{i_0, \dots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k (\alpha_{i_0, \dots, \widehat{i_k}, \dots, i_{p+1}})|_{U_{i_0, \dots, i_{p+1}}}$$

Proposition 4.3. $d^{p+1} \circ d^p = 0$ for all $p \geq 0$.

Proof. Take $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ and a $(p+1)$ -tuple of indexes with $i_0 < \dots < i_{p+2}$, we want to see the following equality:

$$(d^{p+1}(d^p \alpha_{i_0, \dots, i_{p+1}}))_{i_0, \dots, i_{p+2}} = 0$$

Applying the definition:

$$\begin{aligned} (d^{p+1}(d^p \alpha_{i_0, \dots, i_{p+1}}))_{i_0, \dots, i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k ((d^p \alpha_{i_0, \dots, i_{p+1}})_{i_0, \dots, \widehat{i}_k, \dots, i_{p+2}})_{|U_{i_0, \dots, i_{p+2}}} \\ &= \sum_{k=0}^{p+2} (-1)^k \left(\sum_{s=0}^{p+1} (-1)^s (\alpha_{i_0, \dots, \widehat{i}_s, \dots, i_{p+1}})_{|U_{i_0, \dots, i_{p+1}}} \right)_{i_0, \dots, \widehat{i}_k, \dots, i_{p+2}} |U_{i_0, \dots, i_{p+2}} \\ &= \sum_{0 \leq s < k \leq p+2} ((-1)^{s+k} + (-1)^{s+k-1}) (\alpha_{i_0, \dots, \widehat{i}_s, \dots, \widehat{i}_k, \dots, i_{p+2}})_{|U_{i_0, \dots, i_{p+2}}} = 0 \end{aligned}$$

□

Hence, we have shown that $(C^\bullet(\mathcal{U}, \mathcal{F}), d^\bullet)$ is a finite complex of abelian groups.

Definition 4.4. We denote by $Z^p(\mathcal{U}, \mathcal{F}) = \text{Ker } d^p$, whose elements are named **cocycles**. We denote by $B^p(\mathcal{U}, \mathcal{F}) = \text{Im } d^{p-1}$, whose elements are named **coboundaries**.

Definition 4.5. The p -th Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ is:

$$H^p(\mathcal{U}, \mathcal{F}) := Z^p(\mathcal{U}, \mathcal{F}) / B^p(\mathcal{U}, \mathcal{F}) = \text{Ker } d^p / \text{Im } d^{p-1}$$

Example 4.6. Take $X = \mathbf{S}^1$ with the euclidean topology. Take the following covering $\mathcal{U} = \{U, V\}$ of X and $\mathcal{F}$ the constant sheaf $\mathbb{Z}_x$:

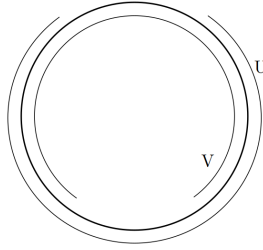


Figure 4.1: A covering of $\mathbf{S}^1$

Then one has:

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{F}) &= \mathbb{Z}_x(U) \times \mathbb{Z}_x(V) \simeq \mathbb{Z} \times \mathbb{Z} \\ C^1(\mathcal{U}, \mathcal{F}) &= \mathbb{Z}_x(U \cap V) \simeq \mathbb{Z} \times \mathbb{Z} \\ d^0 : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{Z} \times \mathbb{Z} \\ (a, b) &\mapsto (b - a, b - a) \end{aligned}$$

Since $\text{Ker } d^0 = \mathbb{Z}(1, 1) \simeq \mathbb{Z}$ and $\text{Im } d^0 \simeq \mathbb{Z}$ the Čech cohomology groups of $\mathbb{Z}_x$ with respect to $\mathcal{U}$ are:

$$H^p(\mathcal{U}, \mathcal{F}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 1 \\ 0 & \text{if } p \geq 2. \end{cases}$$

Example 4.7. Let $\mathbb{P}_k^1$ the projective line over k . We are going to compute the Čech cohomology of $\mathcal{O}_{\mathbb{P}_k^1}(m)$ with respect to the open covering $U_0 = \text{Spec}(k[x])$, $U_1 = \text{Spec}(k[x^{-1}])$, where $U_0 \cap U_1 = \text{Spec}(k[x, x^{-1}])$. The associated Čech complex is the following:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^1}(m)(\mathbb{P}_k^1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^1}(m)(U_0) \times \mathcal{O}_{\mathbb{P}_k^1}(m)(U_1) & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^1}(m)(U_0 \cap U_1) \longrightarrow 0 \\ & & & & \downarrow \simeq & & \downarrow \simeq \\ & & & & k[x] \times k[x^{-1}] & \xrightarrow{\alpha} & k[x, x^{-1}] \end{array}$$

The morphism α is:

$$\begin{aligned} \alpha : \quad k[x] \times k[x^{-1}] &\longrightarrow k[x, x^{-1}] \\ (p(x), q(x^{-1})) &\longmapsto x^m q(x^{-1}) - p(x) \end{aligned}$$

We have seen in 2.59 that:

1. For $m < 0$ we have,

$$\mathcal{O}_{\mathbb{P}_k^1}(m)(\mathbb{P}_k^1) = 0$$

2. For $m \geq 0$ we have,

$$\mathcal{O}_{\mathbb{P}_k^1}(m)(\mathbb{P}_k^1) = \bigoplus_{r=0}^m kx^r$$

Therefore, with the notation of this chapter,

$$H^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}_k^1}(m)) = \text{Ker } \alpha = \begin{cases} \bigoplus_{r=0}^m kx^r & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$$

We are going to compute now $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}_k^1}(m))$. For this let us consider cases depending on m .

1. For $m \geq 0$ any polynomial of $k[x, x^{-1}]$ can be written like $x^m q(x^{-1}) - p(x)$ for suitable q, p . Hence $\text{Im } \alpha = k[x, x^{-1}]$.
2. If $m = -1$ we claim that $\text{Im } \alpha = k[x, x^{-1}]$.

- For $t \neq 1$ one has that, $x^{-t} = x^{-1}(x^{-t+1}) - 0$. And, $x^t = x^{-1}(0) - (-x^t)$.
- For $t = 1$ one has $x^{-1} = x^{-1}(1) - 0$. And, $x = x^{-1}(0) - (-x)$

Hence, every monomial of positive or negative exponent lies in the image.

We have seen:

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}_k^1}(m)) = 0 \quad \text{for } m \geq -1$$

3. For $m \leq -2$ notice that the monomials x^t with $t \in \{-1, -2, \dots, m+1\}$ cannot be written as $x^m q(x^{-1}) - p(x)$. Indeed, suppose $x^t = x^m q(x^{-1}) - p(x)$, with $t \in \{-1, -2, \dots, m+1\}$, for some $q(x^{-1}) \in k[x^{-1}]$ and $p(x) \in k[x]$. It is clear that $p = 0$. Therefore,

$$x^m q(x^{-1}) = x^m \sum_{r=0}^d a_r x^{-r} = \sum_{r=0}^d a_r x^{-r+m} = x^t$$

Then, it would exist some $0 \leq r \leq d$ such that, $a_j = 0$ for all $j \neq r$, $a_r \neq 0$ and $r + m = t$. So, $r = t - m < -1 - m \leq -1 + 2 = 1$. Therefore $r = 0$ and that would imply $a_0 = x^t$.

So we have proved that, as a k -vectorial space, $\text{Im } \alpha$ is the set of finite linear combinations of monomials x^r with $r \notin \{-1, -2, \dots, m+1\}$. Hence, $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}_k^1})(m) = k[x, x^{-1}] / \text{Im } \alpha$ is a k -vectorial space of dimension $1 - m$.

4.2.2 Čech cohomology

Čech cohomology of a covering have the enormous advantage that, given a nice covering, all the calculus involved are completely explicit. However, it presents a series of significant disadvantages. The first of them, and probably the most obvious, is that we want something that does not depend on the covering, and the cohomology we have defined depends in a strong way on that. For example taking the trivial covering gives us trivial cohomology for $p \geq 1$. Moreover, it is not clear that the Čech cohomology of a covering satisfies those desired properties that we asked for in the introduction of this chapter (see 4.1).

To address these problems, we will make the coverings finer. This will enable us to define a cohomology that works well.

Definition 4.8. A *refinement* of an open covering, $\mathcal{U} = \{U_i\}_{i \in I}$ of X , is an open covering $\mathcal{V} = \{V_j\}_{j \in J}$ and a map $\varphi : J \rightarrow I$ such that, $\forall j \in J$, $V_j \subseteq U_{\varphi(j)}$.

Definition 4.9. With the previous notation, the *morphism of refinement* is

$$\begin{aligned} \text{ref}_{\mathcal{U}, \mathcal{V}}^p : C^p(\mathcal{U}, \mathcal{F}) &\longrightarrow C^p(\mathcal{V}, \mathcal{F}) \\ \alpha_{j_0, \dots, j_p} &\longmapsto (\alpha_{\varphi(j_0), \dots, \varphi(j_p)})|_{V_{j_0} \cap \dots \cap V_{j_p}}. \end{aligned}$$

If one takes paper and pencil, with a bit of patience, it is just a matter of calculations to see that the following diagram is commutative:

$$\begin{array}{ccc} C^p(\mathcal{U}, \mathcal{F}) & \xrightarrow{\text{ref}_{\mathcal{U}, \mathcal{V}}^p} & C^p(\mathcal{V}, \mathcal{F}) \\ \downarrow d^p & & \downarrow d^p \\ C^{p+1}(\mathcal{U}, \mathcal{F}) & \xrightarrow{\text{ref}_{\mathcal{U}, \mathcal{V}}^{p+1}} & C^{p+1}(\mathcal{V}, \mathcal{F}). \end{array}$$

Hence, a refinement induces a map between the complexes

$$\text{ref}^\bullet : C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{F}).$$

Therefore, it also induces a map between the cohomology groups:

$$\text{ref}^\bullet : H^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow H^\bullet(\mathcal{V}, \mathcal{F}).$$

The notion of refinement allows us to define a preorder on the set of open coverings of X . We say that $\mathcal{V} \leq \mathcal{U}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. Moreover, this is directed, since $\{U_i \cap V_j\}_{ij} \leq \mathcal{U}$ and $\{U_i \cap V_j\}_{ij} \leq \mathcal{V}$.

Definition 4.10. *The p -th Čech cohomology group of $\mathcal{F}$ is,*

$$H^p(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F})$$

and the direct limit is taken over the set of opening coverings of X ordered by refinement.

Let us see now that, this definition for the cohomology of sheaves is a good answer to those requirements we asked for in the introduction of the chapter.

Proposition 4.11. $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$

Proof. Let us actually check that, for any open covering $\{U_i\}_{i \in I}$ of X , $H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$. We want to see that $\text{Ker } d^0 = \mathcal{F}(X)$. Take $\alpha \in C^0(\mathcal{U}, \mathcal{F})$. It is defined by sections $\alpha_i \in \mathcal{F}(U_i)$. Now, for $i < j$

$$(d^0 \alpha)_{i,j} = \alpha_{j|_{U_i \cap U_j}} - \alpha_{i|_{U_i \cap U_j}}$$

By the locality axiom, this is zero if, and only if, α_i comes from a global section of $\mathcal{F}$. We have seen that $H^0(\mathcal{U}, \mathcal{F}) = \text{Ker } d^0 = \mathcal{F}(X)$, for any open covering of X . Hence, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$. $\square$

Proposition 4.12. *Cohomology groups define a family of functors*

$$\begin{aligned} H^p(X, -) : Sh(X) &\longrightarrow Ab \\ \mathcal{F} &\longmapsto H^p(X, \mathcal{F}) \end{aligned}$$

Proof. Given a morphism of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$, using the definition of it and sheaves axioms, one can show that, $\forall p \geq 0$, it induces a map,

$$\phi^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{G}).$$

With an easy, but tedious computation, one can check that $d^p \circ \phi^p = d^p \circ \phi^{p+1}$. Hence it passes to the cohomology groups. Now, since this happens for any open covering, we have shown that cohomology groups define a family of functors between $Sh(X)$ and Ab . $\square$

To establish the existence of connection morphisms, as well as to prove the Leray's theorem that we will state later, a more complex construction of sheaf cohomology is required. This construction, which involves technical concepts from category theory (that are not the aim of this text), is thereafter shown to coincide with Čech cohomology. This construction can be found in [8, Chapter III, Section 1] and [8, Chapter III, Section 2].

Proposition 4.13. *For any short exact sequence of sheaves $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ there is a family of morphisms $\{\delta^p : H^p(X, \mathcal{F}'') \rightarrow H^{p+1}(X, \mathcal{F}')\}_{p \geq 0}$, such that:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, \mathcal{F}') & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}'') \\ & & & & \searrow \delta^0 & & \\ & & H^1(X, \mathcal{F}') & \longrightarrow & H^1(X, \mathcal{F}) & \longrightarrow & H^1(X, \mathcal{F}'') \\ & & & & \searrow \delta^1 & & \\ & & H^2(X, \mathcal{F}') & \longrightarrow & H^2(X, \mathcal{F}) & \longrightarrow & H^2(X, \mathcal{F}'') \longrightarrow \dots \end{array}$$

Proof. See [5, Appendix B, Proposition B.8.]. $\square$

Theorem 4.14 (Leray's theorem). *Let $\mathcal{U} = \{U_i\}_i$ be an open covering, such that, for all $p \geq 0$ and $i_1 < \dots < i_r$, $H^p(U_{i_1} \cap \dots \cap U_{i_r}, \mathcal{F}) = 0$, then:*

$$H^p(X, \mathcal{F}) = H^p(\mathcal{U}, \mathcal{F}).$$

Proof. See [5, Appendix B, Proposition B.15.]. $\square$

4.3 Grothendieck vanishing theorem

In this section we are going to see the first vanishing theorem of the cohomology of sheaves. Moreover, together with Leray's theorem, we will see that this result allow us to know the cohomology of a separated scheme just by calculations using Čech cohomology with respect any affine covering.

Theorem 4.15 (First vanishing theorem). *Let $X = \text{Spec}(R)$ be an affine scheme and $\mathcal{F}$ a quasicoherent sheaf on X . Then,*

$$H^p(X, \mathcal{F}) = 0 \quad \forall p > 0. \quad (4.4)$$

Proof. First of let us reduce our problem doing some observations:

1. All cohomology groups $H^p(X, \mathcal{F})$ are defined by taking the direct limit of $H^p(\mathcal{U}, \mathcal{F})$ over the directed set of open coverings of X ordered by refinement. Since we have proved that the basic open sets form a base for the topology on X (see 2.12), it is sufficient to prove that, for coverings $\mathcal{U} = \{D(g_i)\}_{i \in I}$, with $g_i \in R$, we have $H^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geq 0$.
2. Since X is affine, then it is quasicompact (see 2.39). Hence, we can take I to be finite. Moreover, we have that all of those g_i , where $i \in I$, generate the unit ideal (see 2.40).
3. As $\mathcal{F}$ is a quasicoherent sheaf $\mathcal{F} = \tilde{M}$ for some R -module M . We have shown in 3.7 that,

$$\begin{aligned} (0) \quad & \Gamma(X, \mathcal{F}) = M \\ (1) \quad & \mathcal{F}(D(g_i)) = M_{g_i} \\ (2) \quad & \mathcal{F}(D(g_i) \cap D(g_j)) = M_{g_i g_j} \\ & \vdots \\ (k) \quad & \mathcal{F}\left(\bigcap_{i=0}^k D(g_i)\right) = M_{g_1 \dots g_k}. \end{aligned}$$

Hence, the Čech complex is formed by products of localizations of M .

Asking for all high dimensional cohomology groups to vanish, is the same as asking for the exactness of the following sequence:

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d_0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d_1} C^2(\mathcal{U}, \mathcal{F}) \longrightarrow \dots \quad (4.5)$$

This is,

$$0 \longrightarrow M \longrightarrow \prod_{i \in I} M_{g_i} \xrightarrow{d_0} \prod_{i < j} M_{g_i g_j} \xrightarrow{d_1} \prod_{i < j < k} M_{g_i g_j g_k} \longrightarrow \dots \quad (4.6)$$

So, for each $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ such that is a cocycle, this is $d^p \sigma = 0$, we want to see it is also a coboundary, in other words, we want to find some $\tau \in C^{p-1}(\mathcal{U}, \mathcal{F})$ such that $d^{p-1} \tau = \sigma$.

We are going to see, in a constructive way, that $\text{Ker } d^1 = \text{Im } d^0$.

Take $\sigma \in \text{Ker } d^1$, $\sigma \in \prod_{i < j} M_{g_i g_j}$, it is defined giving $\sigma_{i,j}$ for $i < j$. We can write $\sigma_{i,j} = \frac{m_{ij}}{(g_i g_j)^{r_{ij}}}$. Notice that, since there are a finite number of g_i , we can take a common r for all $\sigma_{i,j}$. So, we can write $\sigma_{i,j} = \frac{m_{ij}}{(g_i g_j)^r}$. Since σ is a cocycle, for $i < j < k$, we have the following relation in $M_{g_i g_j g_k}$:

$$\frac{m_{jk}}{(g_j g_k)^r} - \frac{m_{ik}}{(g_i g_k)^r} + \frac{m_{ij}}{(g_i g_j)^r} = 0.$$

So, for some $l \geq 0$, multiplying by g_i^{r+l} we have the following relation in $M_{g_j g_k}$:

$$\frac{m_{jk} g_i^{r+l}}{(g_j g_k)^r} = \frac{m_{ik} g_i^l}{g_k^r} - \frac{m_{ij} g_i^l}{g_j^r} = 0. \quad (4.7)$$

Rewriting, it is:

$$\frac{m_{jk} g_i^{r+l}}{(g_j g_k)^r} = \frac{m_{ik} g_i^l g_j^r}{(g_j g_k)^r} - \frac{m_{ij} g_i^l g_k^r}{(g_j g_k)^r} = 0. \quad (4.8)$$

Now, since $D(g_i) = D(g_i^k)$ for any $k \geq 0$, we have that $\{D(g_i^{r+l})\}_{i \in I}$ is also an open covering of X . Therefore, those g_i^{r+l} generate the unit ideal. So, there exist $\{h_i\}_{i \in I}$, where $h_i \in R$, such that, $1 = \sum_{i \in I} h_i g_i^{r+l}$. Let us define $\tau \in \prod_{i \in I} M_{g_i}$ as, for each $j \in I$,

$$\tau_j = \sum_{i \in I} h_i \frac{g_i^l m_{ij}}{g_j^r}.$$

In $\prod_{j < k} M_{g_j g_k}$, this is,

$$\tau_{j,k} = \sum_{i \in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r}.$$

Now, let us show that, indeed, $(d^0\tau)_{j,k} = \sigma_{j,k}$ for each $j < k$.

$$\begin{aligned}
 (d^0\tau)_{j,k} &= \tau_k - \tau_j = \sum_{i \in I} h_i \frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \sum_{i \in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \\
 &= \sum_{i \in I} h_i \left(\frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \right) \\
 &\stackrel{(4.8)}{=} \sum_{i \in I} h_i \frac{g_i^{l+r} m_{jk}}{(g_j g_k)^r} = \frac{m_{jk}}{(g_j g_k)^r} = \sigma_{j,k}.
 \end{aligned}$$

With this we have proved that $H^1(\mathcal{U}, \mathcal{F}) = 0$. The same argument, taking care of the $p + 2$ indices, shows the general case. $\square$

Corollary 4.16. *Let $(X, \mathcal{O}_X)$ be a noetherian scheme such that exist an open affine covering $\mathcal{U} = \{U_i\}_{i \in I}$ with the property that, for all $i_1 < \dots < i_r$, the intersection $U_{i_1} \cap \dots \cap U_{i_r}$ is affine. Then, for any quasicoherent sheaf $\mathcal{F}$ on X ,*

$$H^p(X, \mathcal{F}) = H^p(\mathcal{U}, \mathcal{F}) \quad \forall p > 0. \quad (4.9)$$

Proof. Since X is noetherian, it is quasicompact. Therefore, we can take $\mathcal{U}$ to be finite. By hypothesis, for all $i_1 < \dots < i_r$, the intersection $U_{i_1} \cap \dots \cap U_{i_r}$ is affine, so, by the theorem above, its cohomology vanishes. Hence, the condition of Leray's theorem holds. $\square$

Remark 4.17. The condition we asked for on the open affine covering $\mathcal{U}$, have all its intersections affine, it is satisfied by most usual examples. There is a concept, that is not introduced in this text, that translates the Hausdorff property of topological spaces into a property of schemes. This is known as **separated schemes**. Most of the schemes one work with, for example affine schemes and projective schemes, are examples of such. This type of schemes have the property that the intersection of open affine subschemes is again affine.

With this theorem we can show that the affine line with two origin, defined in 2.56, is not an affine scheme.

Example 4.18. Denote by X the affine line with two origin. X is covered by two copies, U_1, U_2 , of $\text{Spec}(k[x])$, with $U_1 \cap U_2 = \text{Spec}(k[x, x^{-1}])$. The structural sheaf of X is quasicoherent, due to the previous result we have that $H^p(U_1 \cap U_2, \mathcal{O}_X) = 0$ for $p > 0$. Hence, this covering satisfies the condition of Leray's theorem. Therefore, we can compute the cohomology of X using Čech cohomology for this covering.

$$0 \longrightarrow k[x] \times k[x] \xrightarrow{d^1} k[x, x^{-1}] \xrightarrow{d^2} 0$$

So, $H^1(X, \mathcal{O}_X) = \frac{\text{Ker } d^2}{\text{Im } d^1} = \frac{k[x, x^{-1}]}{k[x]} \neq 0$. With this we have shown that the affine line with two origin is not an affine scheme.

Example 4.19. We have seen in 4.7 the computation of the cohomology of $\mathcal{O}_{\mathbb{P}_k^1}(m)$ with respect the covering $U_0 = \text{Spec}(k[x])$, $U_1 = \text{Spec}(k[x^{-1}])$. This covering satisfies the condition of the corollary 4.16, since $U_0 \cap U_1 = \text{Spec}(k[x, x^{-1}])$ is affine. Hence, we have computed all cohomology groups of the family of sheaves $\mathcal{O}_{\mathbb{P}_k^1}(m)$.

We will now present a dimensional vanishing theorem from Grothendieck. The proof of this result is much more technical than this work pretends to. A complete and detailed proof can be found in [8, III, Theorem 2.7].

Theorem 4.20 (Dimensional vanishing theorem). *If $(X, \mathcal{O}_X)$ is a noetherian scheme of dimension n , for any abelian sheaf $\mathcal{F}$ on X ,*

$$H^p(X, \mathcal{F}) = 0 \quad \forall p > n. \quad (4.10)$$

Using the first vanishing theorem and the dimensional vanishing theorem, similar arguments than the ones used in the example 4.7 show the following cohomology groups of the family of sheaves $\mathcal{O}_{\mathbb{P}_R^n}(m)$:

$$\bullet \quad H^0(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}_R^n}(m)) = \begin{cases} R[x_0, \dots, x_n]_m & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$$

Remark 4.21. We denote by $R[x_0, \dots, x_n]_m$ the ring of homogeneous polynomials of degree m in $n + 1$ variables with coefficients in R .

$$\bullet \quad H^p(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}_R^n}(m)) = 0, \text{ for } 0 < p < n.$$

$$\bullet \quad H^n(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}_R^n}(m)) = \begin{cases} R & \text{if } m = -n - 1 \\ 0 & \text{if } m > -n - 1 \end{cases}$$

$$\bullet \quad H^p(\mathbb{P}_R^n, \mathcal{O}_{\mathbb{P}_R^n}(m)) = 0, \text{ for } p > n.$$

Moreover, is a well known result, and follows by the proof of the previous statement, that,

$$\dim_k H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(m)) = \binom{-m-1}{n}. \quad (4.11)$$

4.3.1 Bézout's theorem

As a final part of this work, we are going to present a proof of Bézout's theorem for projective plane curves using almost everything we have seen at this point.

Remark 4.22. To simplify the notation, we will write $V(F)$ to denote $V((F))$.

We wonder how many times do intersect two plane curves without common components $C_1 = V(F_1), C_2 = V(F_2) \subset \mathbb{P}_k^2$. Morally, Bézout's theorem says that, if one count each intersection as it should be counted, the two curves intersects $d_1 d_2$ times, where d_1 is the degree of F_1 , and d_2 is the degree of F_2 . Hence, what we first need is to define what does "to count well intersections" means.

Beforehand, there are many ways to define this concept, but every definition one can give should satisfy a list of requirements or axioms that ensure that the definition is consistent with the properties that the intersection multiplicity should intuitively satisfy. These axioms can be found in [6, Chapter 3, Section 3]. Later is shown that, if a definition that satisfies all of this conditions exists, then is unique. We are going to use a definition based on the relation between points, maximal ideals and coordinate rings at a point.

Definition 4.23. Let k be a closed algebraically field. Let $C_1 = V(f_1), C_2 = V(f_2) \subset \mathbb{A}_k^2$ be two plane curves. We define the **intersection multiplicity** of C_1, C_2 at a point p to be,

$$I_p(C_1, C_2) := \dim_k \left(k[x, y] / (f_1, f_2) \right)_{\mathfrak{m}_p}. \quad (4.12)$$

Where $\mathfrak{m}_p$ denotes the maximal ideal associated to p by the Nullstellensatz theorem.

Lemma 4.24. Let k be a closed algebraically field. Let $C_1 = V(F_1), C_2 = V(F_2)$ two curves in $\mathbb{P}_k^2$. If C_1, C_2 do not have common components, they intersect in a finite number of points.

Proof. Without loss of generality we can suppose they intersect in an affine plane. Indeed, if they intersect at the infinite line they do it in a finite number of points, since it is clear a line and a curve intersect in a finite number of points. Now, take $f_1(x, y) = F_1(1, x, y)$ and $f_2(x, y) = F_2(1, x, y)$, consider them as elements of $k(x)[y]$, in order to work in an Euclidean domain. Then, by Bézout's identity there are some $\frac{a(x)}{d(x)}, \frac{b(x)}{d'(x)}, r(x) \in k(x)$ such that,

$$\frac{a(x)}{d(x)} f_1(x, y) + \frac{b(x)}{d'(x)} f_2(x, y) = r(x).$$

Hence, $a(x)f_1(x, y) + b(x)f_2(x, y) = r(x)d(x)d'(x)$. If $(x_0, y_0) \in V(f_1) \cap V(f_2)$, then $f_1(x_0, y_0) = f_2(x_0, y_0) = 0$ and this implies that $r(x_0)d(x_0)d'(x_0) = 0$. This

is a polynomial in one variable with coefficients in k , that we are supposing to be algebraically closed, hence, there are just a finite number of x_0 that are solution of $r(x)d(x)d'(x) = 0$. By symmetry, repeating the argument changing the variables x and y we get that there are just a finite number of points in the intersection. $\square$

Now we can announce Bézout's theorem for projective plane curves.

Theorem 4.25 (Bézout's theorem). *Let k be an algebraically closed field. If $C_1 = V(F_1), C_2 = V(F_2) \subset \mathbb{P}_k^2$ are two plane projective curves with no common components, $d_1 = \deg(F_1)$ and $d_2 = \deg(F_2)$, the following equality holds,*

$$\sum_{p \in C_1 \cap C_2} I_p(C_1, C_2) = d_1 d_2. \quad (4.13)$$

Before starting with the proof of the theorem, we need to make a brief parenthesis and explain another way to construct projective spaces as schemes.

Similarly to how we constructed affine schemes in Chapter 2, it is possible to construct projective schemes directly, without going through the process of gluing affine schemes as we saw in 2.3. We will only see an outline of this construction. Nevertheless, one should remember that all results can be proved in the similar way as those already seen for affine schemes, adapting the arguments when necessary.

Definition 4.26. *Let R be a commutative ring with unity. We say R is a **graded ring** if it can be written as $R = \bigoplus_{m \in \mathbb{Z}} R_m$, with the property that $R_m R_n \subseteq R_{m+n}$. The elements of R_m are called **homogeneous of degree m** .*

Example 4.27. For any commutative ring with unity R , the ring of polynomials $R[x_0, \dots, x_n]$ is a graded ring. Indeed, if we denote by R_m the set of homogeneous polynomial, in $x_0, \dots, x_n$ variables, of degree m , then, $R[x_0, \dots, x_n] = \bigoplus_{m \geq 0} R_m$. Moreover, it is clear that $R_m R_{m'} \subseteq R_{m+m'}$.

We give now the main steps in the construction.

- Step 1: Take R a graded ring. Define the following set,

$$\text{Proj}(R) = \left\{ \mathfrak{p} \in \text{Spec}(R) \text{ that do not contain } R_+ = \bigoplus_{m > 0} R_m \right\}.$$

- Step 2: We define Zariski's topology over this set by defining the closed subsets to be, for each $\mathfrak{a} \subseteq R$ ideal, $V(\mathfrak{a}) = \{ \mathfrak{p} \in \text{Proj}(R) \mid \mathfrak{a} \subseteq \mathfrak{p} \}$. A base of the topology is given by $\{ D_+(f) := D(f) \cap \text{Proj}(R) \mid f \in R \}$.

- Step 3: a structural sheaf, denoted by $\mathcal{O}_{\text{Proj}(R)}$, on $\text{Proj}(R)$ is constructed. In a way that, $\mathcal{O}_{\text{Proj}(R)}(D_+(f)) = (R_f)_0$.

Definition 4.28. Given R a commutative ring with unity we define the n -projective scheme to be $\mathbb{P}_R^n = \text{Proj}(R[x_0, \dots, x_n])$.

Remark 4.29. This definition coincides with the definition that we gave in 2.4.1 using the gluing technique.

We can start now with the proof of Bézout's theorem. Let us use from now on the notation introduced in the statement of the theorem.

As C_1, C_2 do not have common components they intersect in a finite number of points $\{p_1, \dots, p_s\}$. Using a change of coordinates, if needed, we can suppose that $\{p_1, \dots, p_s\} \subseteq D(x) \simeq k[x, y]$. Hence we can think we are in the affine plane. Taking $Z = V(F_1) \cap V(F_2)$ we can suppose it is a closed subscheme of an affine scheme, by taking the closed immersion to be the inclusion. By the corollary 3.18 Z is also an affine scheme. Recall that for $X = \text{Spec}(R)$ one has that $\mathcal{O}_X(X) = R$, and $\mathcal{O}_X(D(f)) = R_f$. We have that, $\mathcal{O}_Z(Z) = k[x, y]/(f_1, f_2)$, where $f_1 = F_1(1, y, z)$ and $f_2 = F_2(1, y, z)$. This is an artinian ring, hence, it can be written as a direct sum of localization at its maximal ideals. By the Nullstellensatz we know that its maximal ideals are in fact those ideals associated to the points $p_1, \dots, p_s$. Hence,

$$\mathcal{O}_Z(Z) = \bigoplus_{i=1}^s \left(k[x, y]/(f_1, f_2) \right)_{\mathfrak{m}_{p_i}}.$$

Recall that $\mathcal{O}_Z(Z) = \Gamma(Z, \mathcal{O}_Z)$. Now, if we take dimensions we get,

$$\dim_k H^0(Z, \mathcal{O}_Z) = \dim_k \Gamma(Z, \mathcal{O}_Z) = \sum_{i=1}^s \dim_k \left(k[x, y]/(f_1, f_2) \right)_{\mathfrak{m}_{p_i}} = \sum_{i=1}^s I_{p_i}(C_1, C_2). \quad (4.14)$$

Notice now that Z is a closed subscheme, so the structural sheaf on it is given by the short exact sequence of sheaves,

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_Z \rightarrow 0.$$

Taking the long exact sequence of sheaves cohomology, using the computations we have made of the cohomology groups of the structural sheaf on $\mathbb{P}_k^2$ and Grothendieck's vanishing theorem, we can deduce the following:

1. We have an exact sequence,

$$0 \rightarrow H^0(\mathbb{P}_k^2, \mathcal{I}_Z) \rightarrow k \rightarrow H^0(\mathbb{P}_k^2, \iota_* \mathcal{O}_Z) \rightarrow H^1(\mathbb{P}_k^2, \mathcal{I}_Z) \rightarrow 0.$$

By definition of ι_* we have $H^p(\mathbb{P}_k^2, \iota_* \mathcal{O}_Z) = H^p(Z, \mathcal{O}_Z)$. It is clear that the constant functions live in $H^0(Z, \mathcal{O}_Z)$, hence we deduce that $\dim_k H^0(\mathbb{P}_k^2, \mathcal{I}_Z) = 0$. So, we conclude that,

$$\dim_k H^0(Z, \mathcal{O}_Z) = \dim_k H^1(\mathbb{P}_k^2, \mathcal{I}_Z) + 1. \quad (4.15)$$

2. Since Z is a 0 dimensional scheme $H^p(Z, \mathcal{O}_Z) = 0$ for $p \geq 1$. We have the following exact sequence

$$0 \rightarrow H^1(Z, \mathcal{O}_Z) \rightarrow H^2(\mathbb{P}_k^2, \mathcal{I}_Z) \rightarrow H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}) \rightarrow H^2(Z, \mathcal{O}_Z) \rightarrow 0.$$

Substituting the known groups, it is,

$$0 \rightarrow 0 \rightarrow H^2(\mathbb{P}_k^2, \mathcal{I}_Z) \rightarrow 0 \rightarrow 0 \rightarrow 0.$$

We get that, $H^2(\mathbb{P}_k^2, \mathcal{I}_Z) = 0$.

We keep these results for later. Now, let us note that, by taking the intersection of plane curves, we naturally obtain a short exact sequence of graded rings. We are going to use the following notation, for $d \in \mathbb{Z}$,

$$(R[x, y, z])(d) = \bigoplus_{m \geq d} R[x, y, z]_m.$$

We claim that the following is a short exact sequence,

$$0 \rightarrow (R[x, y, z])(-d_1 - d_2) \xrightarrow{\alpha} (R[x, y, z])(-d_1) \oplus (R[x, y, z])(-d_2) \xrightarrow{\beta} (F_1, F_2) \rightarrow 0$$

With the maps α, β defined as, $\alpha(h) = (-hF_2, hF_1)$ and $\beta(h_1, h_2) = h_1F_1 + h_2F_2$.

Since F_1, F_2 do not have common terms, we obtain that β is surjective. Moreover, using that our graded rings are integral domains, we get that α is injective. Finally, we have that $\text{Im } \alpha = \text{Ker } \beta$. Indeed, $(h_1, h_2) \in \text{Ker } \beta \leftrightarrow h_1F_1 = -h_2F_2$, since F_1, F_2 do not have common terms it may exist $h \in (R[x, y, z])(-d_1 - d_2)$ such that $hF_1 = h_2$ and $-hF_2 = h_1$, so, $(h_1, h_2) = (-hF_2, hF_1)$.

We take now the $\sim$ sheaves that were introduced in Chapter 3, adapting the construction to graded rings. Roughly speaking, we associate a sheaf to every graded ring in a similar way we have done with modules. This $\sim$ functor preserves the exactness. Later we check that this associated sheaves are in fact of the form $\mathcal{O}_{\mathbb{P}_k^2}(m)$. With this we will get a short exact sequence of sheaves.

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_k^2}(-d_1 - d_2) \rightarrow \mathcal{O}_{\mathbb{P}_k^2}(-d_1) \oplus \mathcal{O}_{\mathbb{P}_k^2}(-d_2) \rightarrow \mathcal{I}_Z \rightarrow 0. \quad (4.16)$$

To see this step in detail one can see [5, Chapter 12, Section 1], all the theory involved in this part is out of the scope of this work.

Returning to the problem at hand, 4.15 has reduced us to calculating the dimension of $H^1(\mathbb{P}_k^2, \mathcal{I}_Z)$. Taking the long exact sequence of sheaves cohomology in 4.16, and using again the known cohomology groups and the vanishing theorems we get the following exact sequence.

$$\begin{aligned} 0 \rightarrow H^1(\mathbb{P}_k^2, \mathcal{I}_Z) &\rightarrow H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^n}(-d_1 - d_2)) \rightarrow \\ &\rightarrow H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^n}(-d_1)) \oplus H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^n}(-d_2)) \rightarrow H^2(\mathbb{P}_k^2, \mathcal{I}_Z) \rightarrow 0 \end{aligned}$$

Recall that $\dim_k H^n(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(m)) = \binom{-m-1}{n}$ for $m < 0$. Using what we have deduced in (2) we conclude that,

$$\dim_k H^1(\mathbb{P}_k^2, \mathcal{I}_Z) = \binom{d_1 + d_2 - 1}{2} - \binom{d_1 - 1}{2} - \binom{d_2 - 1}{2}. \quad (4.17)$$

We have all the ingredients we need to finish the proof just by computation.

$$\begin{aligned} \sum_{p \in C_1 \cap C_2} I_p(C_1, C_2) &\stackrel{4.14}{=} \dim_k H^0(Z, \mathcal{O}_Z) \\ &\stackrel{4.15}{=} \dim_k H^1(\mathbb{P}_k^2, \mathcal{I}_Z) + 1 \\ &\stackrel{4.17}{=} \binom{d_1 + d_2 - 1}{2} - \binom{d_1 - 1}{2} - \binom{d_2 - 1}{2} + 1 \\ &= d_1 d_2 \end{aligned}$$

The proof of Bézout's theorem is finished.

Bibliography

- [1] A. Altman and S. Kleiman, *A Term of Commutative Algebra*, Worldwide Center of Mathematics (2013).
- [2] J. Dieudonné, *History of algebraic geometry : an outline of the history and development of algebraic geometry ; translated by Judith Sally*, Wadsworth Advanced Books & Software (1985).
- [3] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Text in Mathematics (2004).
- [4] D. Eisenbud and J. Harris, *The geometry of schemes*, Springer, Graduate Text in Mathematics (2000).
- [5] G. Ellingsrud and J. Christian Ottem, *Introduction to schemes*, University of Oslo.
- [6] W. Fulton, *Algebraic Curves*, W.A. Benjamin INC., Brandeis University (1969).
- [7] S. Germain, *Œuvres Philosophiques de Sophie Germain (nov. éd.)*, Librairie de Firmin-Didot et Cie, Paris (1896)
- [8] R. Hartshorne, *Algebraic geometry*, Springer, Graduate Text in Mathematics (1977).
- [9] R. Vakil, *The rising sea: Foundations of Algebraic Geometry*, math216.wordpress.com. **33** (1976).