Classical String Dynamics in Curved Backgrounds

Author: Xavier Ramos Olivé. Advisor: Jorge G. Russo.
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.
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Abstract: In this bachelor thesis, we study a rotating classical string embedded in the 5-dimensional Anti-de-Sitter spacetime (AdS_5), through the analysis of the dynamics of the Polyakov action. We compute its energy E and spin S and discuss the dependence $E(S)$. The function $E(S)$ is explicitly derived in two limiting cases: the short string case (in which the length of the string is considerably smaller than the characteristic length of AdS_5) and the long string case (in which it is considerably greater). In the former case, the dependence $E(S)$ previously derived for a flat space is recovered.

I. INTRODUCTION

The study of string dynamics in curved backgrounds is an active field of research in theoretical physics nowadays. Its motivation, and specially the interest of studying string dynamics in maximally symmetric spaces, such as Anti-de-Sitter spacetime (see [3], [9] and [2]), comes from the AdS/CFT correspondence, also called gauge/gravity duality or Maldacena duality (see [6]).

AdS/CFT stablishes a duality between Quantum Gravity (via String Theory or M-theory), defined in an Anti-de-Sitter space (AdS), and Conformal Field Theories defined on the boundary of AdS. Every magnitude that can be computed in the string-theoretical framework (such as the energy E or the spin S) has a specific corresponding object in the dual Conformal Field Theory.

In this paper we are going to see that when considering a rotating classical string, the spin determines its energy $E(S)$. Moreover, we are going to see how this expression changes in curved backgrounds from the one obtained in a flat spacetime.

II. CLASSICAL STRING DYNAMICS

A. Equation of motion

A *string* is a 2-dimensional manifold (with coordinates (σ, τ) in the *coordinate space*) embedded in spacetime, also known as the *target space* (with coordinates $x^\mu(\sigma, \tau)$).

The dynamics of a classical relativistic string can be described by the *Polyakov action*:

$$\mathcal{S} = -\frac{T}{2} \int d\sigma d\tau \sqrt{-h}\; h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} \tag{1}$$

where $h_{\alpha\beta}$ is a metric defined in the coordinate space (i.e., on the string), $h \equiv det(h_{\alpha\beta})$, and $\eta_{\mu\nu} \equiv diag(-1, +1, \ldots, +1)$. This is a generalization of the *Nambu-Goto action* (see [5] and [10]).

We will focus our attention on closed strings, so the target space functions must be periodic in σ; we choose arbitrarily period 2π, namely:

$$x^\mu(\sigma + 2\pi, \tau) = x^\mu(\sigma, \tau)$$

Varying the action with respect to x^μ, after integrating by parts (boundary contributions are null, as closed strings don't have boundary), we obtain the string's equation of motion:

$$\partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} \partial_\beta x^\nu \right) = 0 \tag{2}$$

The expression (1) has several symmetries (see [8]), namely, it is invariant under:

1. Poincaré group transformations (because of the symmetries in the target space):
$$x^\mu \to \Lambda^\mu_\nu x^\nu + \alpha^\mu$$
2. Diffeomorphisms (string reparameterizations):
$$(\sigma, \tau) \to (\tilde{\sigma}(\sigma, \tau), \tilde{\tau}(\sigma, \tau))$$
3. Weyl transformations (conformal mappings):
$$h_{\alpha\beta} \to G(\sigma, \tau) h_{\alpha\beta}$$

Reparameterization and Weyl invariance provides us with 3 gauge parameters, the same number than the total number of independent variables of $h_{\alpha\beta}$. This allows us to fix the conformal (or orthogonal) gauge (see [5]), where $h_{\alpha\beta} = \eta_{\alpha\beta} \equiv diag(-1, +1)$; equation (2) then becomes linear:

$$(\partial_\sigma^2 - \partial_\tau^2) x^\nu = 0 \tag{3}$$

Introducing light-cone coordinates $\sigma_\pm = \tau \pm \sigma$, it is written as:

$$\partial_+ \partial_- x^\mu = 0$$

So the solution may be written as:

$$x^\mu(\sigma, \tau) = x^\mu_L(\sigma_+) + x^\mu_R(\sigma_-)$$

B. Virasoro Constraints

Constraints come from the metric equation of motion. This one can be expressed in terms of the stress-energy tensor as follows:

$$T_{\alpha\beta} = \frac{-2}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}}$$

Using that $\frac{\partial h}{\partial h^{\alpha\beta}} = -hh_{\alpha\beta}$ (a straightforward computation in dimension 2) we obtain:

$$T_{\alpha\beta} = T\left[\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \frac{h_{\alpha\beta}}{2} h^{\gamma\delta}\partial_\gamma x^\mu \partial_\delta x^\nu \eta_{\mu\nu}\right]$$

In the conformal gauge:

$$\begin{aligned} T_{00} =& T_{11} = \frac{T}{2}\left[\partial_\tau x^\mu \partial_\tau x^\nu \eta_{\mu\nu} + \partial_\sigma x^\mu \partial_\sigma x^\nu \eta_{\mu\nu}\right] \\ T_{01} =& T_{10} = T\partial_\tau x^\mu \partial_\sigma x^\nu \eta_{\mu\nu} \end{aligned}$$

Or using light-cone coordinates, with the corresponding transformed metric ($\eta_{++} = \eta_{--} = 0$ and $\eta_{+-} = \eta_{-+} = -2$):

$$\begin{aligned} T_{\pm\pm} =& \partial_\pm x^\mu \partial_\pm x^\nu \eta_{\mu\nu} \\ T_{+-} =& T_{-+} = 0 \end{aligned}$$

In a vacuum situation $T_{\alpha\beta} = 0$, and using the notation $\dot{x}^\mu = \partial_\tau x^\mu$ and $x'^\mu = \partial_\sigma x^\mu$ they can be written:

$$\begin{aligned} \dot{x}^\mu x'_\mu &= 0 \\ \dot{x}^2 + x'^2 &= 0 \end{aligned}$$

These equations are known as the *Virasoro constraints*.

C. Conserved quantities

Noether's theorem provides us with conserved quantities coming from the symmetries of the lagrangian; the densities associated with Poincaré invariance (see [8]) are:

$$\begin{aligned} P^{\mu\alpha} &= \eta^{\mu\nu}\frac{\partial\mathcal{L}}{\partial[\partial_\alpha x^\nu]} = -T\sqrt{-h}h^{\alpha\beta}\partial_\beta x^\mu \\ J^{\mu\nu\alpha} &= -T\sqrt{-h}h^{\alpha\beta}(x^\mu\partial_\beta x^\nu - x^\nu \partial_\beta x^\mu) \end{aligned}$$

In particular, we can define the *linear* and *angular momenta* in the orthogonal gauge as:

$$\begin{aligned} P^\mu &= \int_0^{2\pi} P^{\mu\tau}\mathrm{d}\sigma = \mathrm{T}\int_0^{2\pi} \partial_\tau \mathrm{x}^\mu \mathrm{d}\sigma \\ J^{\mu\nu} &= \int_0^{2\pi} J^{\mu\nu\tau}\mathrm{d}\sigma = \mathrm{T}\int_0^{2\pi}(\mathrm{x}^\mu\partial_\tau\mathrm{x}^\nu - \mathrm{x}^\nu\partial_\tau\mathrm{x}^\mu)\mathrm{d}\sigma \end{aligned}$$

D. Example: the rotating string solution

Consider a rotating string embedded in a plane:

$$\begin{aligned} t &= p_0\tau \\ x &= \rho\cos(\phi) \\ y &= \rho\sin(\phi) \end{aligned}$$

where $\rho = R\cos(\omega\sigma)$ and $\phi = \tilde{\omega}\tau$, and $R, \omega, \tilde{\omega}, p_0 \in \mathbb{R}$ are constants.

It is a straightforward computation to see that these relations satisfy equation (3) if and only if $\omega^2 = \tilde{\omega}^2$. In order to obtain a 2π-periodic solution, we must require $\omega \in \mathbb{Z}$. The Virasoro constraints are satisfied if and only if $p_0^2 = (R\omega)^2$.

Instead of following the procedure described above to obtain the conserved quantities, we will use polar coordinates ρ and ϕ. In polar coordinates, the metric of the target space becomes:

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2$$

So, before integrating by parts and fixing the orthogonal gauge, the lagrangian reads:

$$\mathcal{L} = -\frac{T}{2}\left[-(t'^2 - \dot{t}^2) + (\rho'^2 - \dot{\rho}^2) + \rho^2(\phi'^2 - \dot{\phi}^2)\right]$$

Hence, the Energy can be obtained as:

$$E = -P_0 = T\int_0^{2\pi}\dot{t}\mathrm{d}\sigma = 2\pi\mathrm{Tp_0} = \pm R\omega$$

where we have used the expression $T = \frac{1}{2\pi\alpha'}$, using units in which $\alpha' = 1$ (see [8]). Similarly, the spin in the plane, $S \equiv J^{xy}$, can be obtained from the symmetry of $\mathcal{L}$ under ϕ-rotation:

$$S = -\frac{T}{2}\int_0^{2\pi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\mathrm{d}\sigma = -\frac{\mathrm{T}}{2}\int_0^{2\pi} -2\rho^2\dot{\phi}\mathrm{d}\sigma = \frac{R^2\tilde{\omega}}{2}$$

So, using that $\omega^2 = \tilde{\omega}^2$, we derive the relation:

$$E^2 = 2\tilde{\omega}S$$

Remark: We said that $\omega, \tilde{\omega} \in \mathbb{Z}$ to obtain a 2π-perdiodic solution. The lowest energy solution for a given spin S has $\omega = \tilde{\omega} = 1$, in which case the parameterized string is a folded string with length $4R$ (in the plane).

III. DYNAMICS IN CURVED BACKGROUNDS

Consider a general target space, a curved manifold with metric:

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu$$

Action (1) can be easily generalized to:

$$\mathcal{S} = -\frac{T}{2}\int d\sigma d\tau\sqrt{-h}\; h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}(x)$$

and fixing the orthogonal gauge:

$$\mathcal{S} = -\frac{T}{2}\int d\sigma d\tau(x'^\mu x'^\nu - \dot{x}^\mu\dot{x}^\nu)G_{\mu\nu}(x) \tag{4}$$

We can in general derive the same relations as above by substituting $\eta_{\mu\nu}$ by $G_{\mu\nu}(x)$; for instance:

$$\begin{aligned} T_{\pm\pm} =& \partial_\pm x^\mu \partial_\pm x^\nu G_{\mu\nu}(x) \\ T_{+-} =& T_{-+} = 0 \end{aligned} \tag{5}$$

A. Example: the rotating string embedded in AdS$_5$

Consider the 5-dimensional Anti-de-Sitter space, AdS_5, whose metric in global coordinates is (see [4]):

$$ds^2 = L^2\left(-\cosh^2 r dt^2 + dr^2 + \sinh^2 r d\Omega_3^2\right)$$
$$d\Omega_3^2 = d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2$$

From (4), the lagrangian reads:

$$\mathcal{L} = -\frac{L^2 T}{2}\Big(-\cosh^2 r(t'^2 - \dot{t}^2) + (r'^2 - \dot{r}^2) + \\ + \sinh^2 r\big[(\theta'^2 - \dot{\theta}^2) + \cos^2\theta(\psi'^2 - \dot{\psi}^2) + \\ + \sin^2\theta(\phi'^2 - \dot{\phi}^2)\big]\Big)$$

Consider a rotating string such that $\theta = \pi/2$, $t = \kappa\tau$, $r = r(\sigma)$, $\phi = \omega\tau$ and $\psi = 0$. From this definitions, we can derive:

$$\partial_\pm t = \frac{\kappa}{2} \qquad \partial_\pm \phi = \frac{\omega}{2}$$
$$\partial_\pm \theta = 0 \qquad \partial_\pm r(\sigma) = \pm\frac{\partial_\sigma r}{2}$$

Using (5), the Virasoro constraints are:

$$T_{\pm\pm} = \frac{L^2}{4}\left[-\kappa^2\cosh^2 r + (\partial_\sigma r)^2 + \omega^2\sinh^2 r\right] = 0$$

From this, we can deduce the maximum value of r:

$$(\partial_\sigma r)^2 = \kappa^2\cosh^2 r - \omega^2\sinh^2 r \geq 0 \implies \\ \implies r \leq \tanh^{-1}\left(\frac{\kappa}{\omega}\right) \equiv R$$

Additionally, for the Virasoro constraints to hold, it has to be the case that:

$$d\sigma = \frac{dr}{\sqrt{\kappa^2\cosh^2 r - \omega^2\sinh^2 r}}$$

And, considering that the string is composed of 4 segments of length R (it is folded as in the case $\omega = 1$ of the previous example), integrating along a period of σ:

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^R \frac{dr}{\sqrt{\kappa^2\cosh^2 r - \omega^2\sinh^2 r}}$$

This last expression is an incomplete elliptic integral of the first kind (see [1]); it can be rewritten as:

$$2\pi = \int_0^{2\pi} d\sigma = \frac{4}{\kappa}\int_0^R \frac{dr}{\sqrt{1 - \left(\frac{\omega^2}{\kappa^2} - 1\right)\sinh^2 r}} \tag{6}$$

Now we can compute the momentum densities necessary to find E and S:

$$P_0^\tau = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -TL^2\dot{t}\cosh^2 r = -TL^2\kappa\cosh^2 r$$
$$P_4^\tau = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = TL^2\dot{\phi}\sinh^2 r\sin^2\theta = TL^2\omega\sinh^2 r$$

Hence, the corresponding conserved quantities, using that $T = 1/2\pi$, $\Lambda \equiv L^2/\alpha' = L^2$, and the change of variables that we used in (6), are:

$$\begin{aligned} E &= -\int_0^{2\pi} P_0^\tau d\sigma = \frac{2\Lambda}{\pi}\int_0^R \frac{\cosh^2 r\ dr}{\sqrt{1 - \left(\frac{\omega^2}{\kappa^2} - 1\right)\sinh^2 r}} \\ S &= \int_0^{2\pi} P_4^\tau d\sigma = \frac{2\Lambda\omega}{\pi\kappa}\int_0^R \frac{\sinh^2 r\ dr}{\sqrt{1 - \left(\frac{\omega^2}{\kappa^2} - 1\right)\sinh^2 r}} \end{aligned} \tag{7}$$

Equation (6) defines implicitly a function $\kappa(\omega)$, from which we theoretically could find $E = f_1(\omega)$ and $S = f_2(\omega)$, that would parametrically define $E = E(S)$. As expressions (7) involve elliptic integrals, it is hard to find out analytically this relation.

In stead, we can try to find $E = E(S)$ in some special limiting cases: when $\frac{\omega}{\kappa} \gg 1$ and when $\frac{\omega}{\kappa} \ll 1$. As we are considering a classical string, we will always deal with $S, \Lambda \gg 1$.

1. *Short string case*

Let us first consider the case in which $\frac{\omega}{\kappa} \gg 1$. In this case, $R = \tanh^{-1}\frac{\kappa}{\omega} \simeq 0 \ll \sqrt{\Lambda} = L$, so the physical interpretation of this limit is that we are considering short strings (with respect to L, the characteristic length of AdS_5).

Let us define $b \equiv \frac{\omega^2}{\kappa^2} - 1$. Using the notation:

$$I_1(b) = \int_0^R \frac{dr}{\sqrt{1 - b\sinh^2 r}}$$
$$I_2(b) = \int_0^R \frac{\sinh^2 r\ dr}{\sqrt{1 - b\sinh^2 r}}$$

We can rewrite (7) as:

$$\begin{aligned} E &= \frac{2\Lambda}{\pi}(I_1(b) + I_2(b)) \\ S &= \frac{2\Lambda\omega}{\pi\kappa} I_2(b) \end{aligned} \tag{8}$$

When $b \gg 1$, we can approximate (see [7] and [1]):

$$I_1(b) \simeq \frac{\pi}{2\sqrt{b}}\left(1 - \frac{1}{4b}\right)$$
$$I_2(b) \simeq \frac{\pi}{4b^{3/2}}$$

Thus, substituting these expressions in (8), and using that $b \gg 1 \implies b^{-1/2} \gg b^{-3/2}$:

$$E \simeq 2\Lambda\left(\frac{1}{2b^{1/2}} + \frac{1}{8b^{3/2}}\right) \simeq \frac{\Lambda}{b^{1/2}}$$

And using that $\sqrt{b+1} = \frac{\omega}{\kappa}$ and that $b \gg 1 \implies b^{-2} \gg b^{-3}$:

$$S \simeq \frac{\Lambda\omega}{2\kappa b^{3/2}} = \frac{\Lambda\sqrt{b^{-2} + b^{-3}}}{2} \simeq \frac{\Lambda}{2b}$$

From the expressions above, and remembering that $\Lambda = L^2$, it is immediate to see that:

$$E^2 = 2L^2 S$$

Hence, we have recovered the result of the flat spacetime, as we expected (because we are studying the short string case, and we expect that if the string is short enough, it will not notice the curvature of spacetime).

2. Long string case

Now let's study the case in which $\frac{\omega}{\kappa} \ll 1$. This case corresponds to long strings, as $b \ll 1 \implies \frac{\omega^2}{\kappa^2} \simeq 1$, hence $R = \tanh^{-1}\frac{\kappa}{\omega} \to \infty \gg \sqrt{\Lambda} = L$ when $\frac{\kappa}{\omega} \simeq 1$.

When $b \ll 1$, we can approximate (see [7] and [1]):

$$I_1(b) \simeq -\frac{1}{2}\ln b$$
$$I_2(b) \simeq \frac{1}{b} + \frac{1}{4}\ln b$$

So, substituting this expressions in (8) we get:

$$E \simeq \frac{2\Lambda}{\pi}\left(\frac{1}{b} - \frac{1}{4}\ln b\right)$$
$$S \simeq \frac{2\Lambda}{\pi}\left(\frac{1}{b} + \frac{1}{4}\ln b\right)$$

where we have used again that $\sqrt{b+1} = \omega/\kappa$ and that if $b \ll 1$, $\sqrt{b+1} \simeq 1$. From these expressions, we deduce that:

$$E - S = \frac{2\Lambda}{\pi}\left(-\frac{1}{2}\ln b\right) = \frac{\Lambda}{\pi}|\ln b|$$

and also, as $b \ll 1$ and $\lim_{b\to 0} b\ln b = 0$:

$$b = \frac{2\Lambda}{\pi S} + \frac{b\ln b}{4S} \simeq \frac{2\Lambda}{\pi S}$$

As a consequence, we can finally deduce that:

$$E - S = \frac{L^2}{\pi}\left|\ln\left(\frac{\pi S}{2L^2}\right)\right| \simeq \frac{L^2}{\pi}|\ln S|$$

IV. CONCLUSIONS

Using the symmetries of action (1), that is a generalization of the Nambu-Goto action (the area action of the world-sheet), we were able to fix the orthogonal gauge, in which the equations of motion and the Virasoro constraints are written in a simple way.

We studied a rotating string in a flat spacetime, and we did it in polar coordinates to gain some insight, in order to be able to do similar computations in curved backgrounds.

We saw that a specific classical rotating string whose movement can be described by a spin S, has a determined energy $E(S)$. We have been able to compute it in a flat space, where the dependence found was $E^2 \propto S$, after fixing the value of ω.

We have seen that to derive a similar relation within a curved background, particularly in AdS_5, it was necessary to deal with some elliptic integrals. We found the energy as a function of the spin in parametric form $E = E(\omega/\kappa)$, $S = S(\omega/\kappa)$ in terms of elliptic functions, and the explicit dependence $E(S)$ in two limiting cases: when considering a long string, we found that $E - S = \frac{L^2}{\pi}|\ln S|$, whereas when we considered a short string, we recovered the flat-spacetime result $E^2 \propto S$.

A physical interpretation of the coincidence of the flat-spacetime result and the limit case for short strings in AdS_5, is that if the string is short (i.e., its characteristic length R is much shorter than the characteristic length of the metric of AdS_5, L) it will not notice the curvature of the space, as any pseudoriemannian manifold is locally flat. So we already expected to recover the flat-spacetime result in this limit.

An open question arises after following this procedure, that we leave as an exercise for the future: is it possible to obtain the result found in the long string limiting case, starting from another (simpler) metric, as we did in the case of the short string, that could be derived from a flat spacetime?

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