

BLACK HOLE SUPERRADIANCE

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Abstract: This study examines two classical mechanisms for extracting energy from rotating black holes. First, the Penrose process is analysed, and the relationship between the energy and angular momentum extracted from the black hole is derived. Second, the phenomenon of black hole superradiance is investigated for massless scalar fields in the Kerr spacetime, leading to the determination of the condition under which incident waves undergo superradiant amplification.

Keywords: General relativity, Black holes, Kerr Metric, Klein-Gordon equation, Penrose Process, Superradiance

SDGs: Quality Education

I. INTRODUCTION

Gravitational energy extraction from a black hole (BH) is expected to happen through quantum-mechanical effects, via the emission of Hawking radiation [1]. In addition, several classical mechanisms for energy extraction exist for rotating (Kerr) BHs. In [2], R. Penrose first proposed a mechanism exploiting the existence of negative-energy states for particles in the ergosphere of a Kerr BH to extract energy and angular momentum from it, a phenomenon known as the Penrose process. A more detailed formulation of it was subsequently presented in [3]. The same year, Ya. B. Zel'dovich suggested the idea that energy can be extracted from a rotating BH via scattering of incident waves, establishing an analogy with the scattering of electromagnetic waves in rotating absorbing cylinders [4]. This phenomenon (termed superradiance) was theoretically confirmed later on by A. A. Starobinsky employing a massless scalar field [5]. This wave amplification led S. Detweiler to study the conditions under which bosonic (scalar field) clouds and instabilities develop around a Kerr BH when the field is massive [6] (see also the more recent work [7]). Importantly, studies of BH superradiance have already been applied to set constraints on the mass of ultra-light bosons –not only scalars, but also vectors–, since the energy extraction produces a decrease of the BH spin producing gaps in the spin–mass plane, and the aforesaid instability of bosonic clouds acts as a source of gravitational waves [8].

In this Bachelor's Thesis (TFG), we first study in detail the spacetime around a rotating BH, described by the Kerr metric, including the properties of a region of the Kerr spacetime known as the ergosphere. The main goal of this work is to delve deeper into the knowledge of general relativity (beyond that covered in the undergraduate course) and high-energy physics, mainly through the study of the Penrose process and the basics of BH superradiance.

Throughout this work, we employ the system of natural units and the metric signature $(-+++)$. We leave some explicit calculations to Appendices A - F.

II. THE KERR METRIC AND ROTATING BLACK HOLES

A. General considerations

The metric components of an axially symmetric spacetime, $g_{\mu\nu}$, are given by Kerr's solution to Einstein's field equations. In Boyer-Lindquist coordinates, the corresponding line element squared $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2 \theta}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 - \frac{4Mar \sin^2 \theta}{\rho^2}d\phi dt + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2}\right)\sin^2 \theta d\phi^2, \quad (1)$$

being $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2$. Using the variables

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \tilde{\omega} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mra}{\Sigma^2}, \quad (2)$$

we can rewrite the line element (1) in an alternative way to showcase its relation with a rotating body:

$$ds^2 = -\frac{\Delta \rho^2}{\Sigma^2}dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2} \sin^2 \theta (d\phi - \tilde{\omega} dt)^2. \quad (3)$$

In these expressions, t represents time, θ and ϕ are respectively the polar and azimuthal angles, with $\theta = 0$ corresponding to the axis of symmetry of the spacetime. The radial-like variable r has a special behaviour, as its origin is at a circumference of radius a centred on the origin of Cartesian coordinates for the equatorial plane, see, e.g., [9]. The parameter $a = J/M$ is the angular momentum of the rotating body, J , over its mass, M .

The determinant of the Kerr metric, g , is

$$g = g_{tt}g_{rr}g_{\theta\theta}g_{\phi\phi} - g_{t\phi}^2 g_{rr}g_{\theta\theta} = -\rho^4 \sin^2 \theta, \quad (4)$$

and the non-zero $g^{\mu\nu}$ components of the inverse metric

read,

$$\begin{aligned} g^{tt} &= -\frac{\Sigma^2}{\Delta\rho^2}, \quad g^{rr} = \frac{\Delta}{\rho^2}, \quad g^{\phi t} = g^{t\phi} = -\frac{2Mra}{\Delta\rho^2}, \\ g^{\theta\theta} &= \frac{1}{\rho^2}, \quad g^{\phi\phi} = \frac{\Delta - a^2 \sin^2 \theta}{\Delta\rho^2 \sin^2 \theta}. \end{aligned} \quad (5)$$

The Kerr metric is a vacuum solution to Einstein's field equations, as it leads to the Ricci tensor being null everywhere. None of the Kerr metric components depends on either ϕ or t . These symmetries imply that $\xi_{(\phi)} = \partial_\phi$ and $\xi_{(t)} = \partial_t$ are two Killing vectors of the metric.

Both the condition $\Delta = 0$ and $\rho^2 = 0$ cause some components of the metric to diverge in Boyer-Lindquist coordinates. It can be seen that the scalar invariants of the curvature tensor do not diverge when the condition $\Delta = 0$ is fulfilled [9], indicating this apparent singularity is just a coordinate artefact. The condition

$$\rho^2 = r^2 + a^2 \cos^2 \theta = 0 \quad (6)$$

does result in a divergence in invariants of the curvature tensor [9], and hence represents a physical singularity. This singularity takes place at the disk defined by $r = 0$ and $\theta = \pi/2$, and indicates the presence of a black hole in the spacetime described solely by the Kerr metric.

B. The event horizon and the ergosurface

The event horizon of a BH is the region of spacetime in which time-like vectors become space-like, in such a way that a particle crossing the event horizon is forced to approach the singularity of the BH. This is manifested in this region being a null-surface. As the components of a vector normal to a $r=\text{const.}$ surface are $\partial_\mu r$, the surface $r=\text{const.}$ defines the event horizon if

$$g^{\mu\nu} \partial_\mu r \partial_\nu r = 0, \quad (7)$$

which imposes that all normal vectors to the surface are null. Hence, the $r=\text{const.}$ null surface satisfies

$$g^{rr} = 0. \quad (8)$$

The condition (8) directly implies $\Delta = 0$; that is,

$$r^2 - 2Mr + a^2 = 0, \quad (9)$$

which has two $r_{h,\pm}$ solutions,

$$r_{h,\pm} = M \pm \sqrt{M^2 - a^2}. \quad (10)$$

The solution $r_{h,+} \equiv r_h$ with the larger distance to the singularity of the black hole defines the relevant radius of the event horizon for an observer outside the black hole. The solution $r_{h,-}$, also defines a null-surface, but it is completely contained within the event horizon. For a non-rotating BH ($a \rightarrow 0$), both solutions for the event horizon in the Schwarzschild metric are recovered.

In the Schwarzschild metric, the surface $g^{rr} = 0$, on which the event horizon also lies in the spherically symmetric spacetime, is also an infinite redshift surface. This is not the case in the spacetime defined by the Kerr metric. To find the infinite redshift surface, which is also called ergosurface in the Kerr metric, let us consider the energy of a photon that a certain observer measures.

The energy E_{obs} of a photon of momentum $\mathbf{K} = K^\mu \partial_\mu$ measured by an observer with 4-velocity $\mathbf{U} = U^\mu \partial_\mu$ is

$$E_{obs} = -U^\mu K_\mu. \quad (11)$$

A static observer has a 4-velocity

$$U^\mu \partial_\mu = \mathcal{R} \xi_{(t)}^\mu \partial_\mu, \quad (12)$$

which is normalised through a parameter $\mathcal{R}$ that satisfies

$$\mathcal{R} = \sqrt{\frac{-1}{g_{tt}}}. \quad (13)$$

Thus, the static observer measures the energy of the photon to be

$$E_{obs} = -\mathcal{R} \xi_{(t)}^\mu K_\mu. \quad (14)$$

The relation between the energy $E_{obs}^{(1)}$ measured by an observer with $\mathcal{R} = \mathcal{R}_1$ and the energy $E_{obs}^{(2)}$ measured by an observer with $\mathcal{R} = \mathcal{R}_2$ is then

$$E_{obs}^{(1)} = \frac{\mathcal{R}_1}{\mathcal{R}_2} E_{obs}^{(2)}. \quad (15)$$

Equation (15) shows that no photon emitted at a point where $\mathcal{R} \rightarrow \infty$ will be observed by any observer. Using the expression (13), it is seen that the condition $g_{tt} = 0$ defines the surface on which this phenomenon takes place. This condition, in turn, leads to

$$2Mr = r^2 + a \cos^2 \theta, \quad (16)$$

whose $r_\pm$ solutions read,

$$r_\pm = \frac{2M \pm \sqrt{4M^2 - 4a^2 \cos^2 \theta}}{2}. \quad (17)$$

The radius

$$r_- = M - \sqrt{M^2 - a^2 \cos^2 \theta} \quad (18)$$

is called inner ergosurface, whereas

$$r_+ = M + \sqrt{M^2 - a^2 \cos^2 \theta} \quad (19)$$

is the outer ergosurface. Both surfaces exhibit the same behaviour, but the relevant region of spacetime for an observer outside the black hole is the exterior one. Both event horizons and the inner ergosurface are contained within the outer ergosurface, except at the axis of symmetry of the BH, where the outer ergosurface and outer event horizon coincide. At the axis of symmetry, the

inner ergosurface and the inner event horizon also coincide, the former being contained by the latter in the other points of the surface.

The ergosphere of a BH is the region between the ergosurface and the event horizon. The restrictions on the trajectories that the region contained within a null-surface imposes are not present in the ergosphere, but limitations do exist on the possible trajectories within the ergosphere, as expected for being $g_{tt} > 0$ in this region.

Consider an observer describing a circular trajectory with angular velocity $\Omega = d\phi/dt$ around the axis of symmetry of the BH. Its 4-velocity $\mathbf{U}$ is

$$\mathbf{U} = \frac{dt}{d\tau} \frac{\partial}{\partial t} + \frac{d\phi}{d\tau} \frac{\partial}{\partial \phi} = \frac{dt}{d\tau} \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right). \quad (20)$$

The norm squared of $\mathbf{U}$ can be written as

$$U^\mu U_\mu = \left(\frac{dt}{d\tau} \right)^2 \left(\xi_{(t)}^\mu + \xi_{(\phi)}^\mu \Omega \right) \left(\xi_{(t)}^\nu + \xi_{(\phi)}^\nu \Omega \right) g_{\mu\nu}. \quad (21)$$

As the 4-velocity is normalized, $U^\mu U_\mu = -1$, we find

$$g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt} < 0. \quad (22)$$

The quadratic coefficient of Ω in the inequality (22) is positive in all spacetime. Thus, the inequality holds when $\Omega_- < \Omega < \Omega_+$, being Ω_- and Ω_+ the solutions to the equation $g_{\phi\phi}\Omega^2 + 2g_{t\phi}\Omega + g_{tt} = 0$,

$$\Omega_{\pm} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}} \right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \quad (23)$$

Both solutions may be re-expressed using the parameter $\tilde{\omega}$ from the expression (2). In particular, we may express Ω_- as

$$\Omega_- = \tilde{\omega} \left(1 - \sqrt{1 - \frac{g_{tt}g_{\phi\phi}}{g_{t\phi}^2}} \right). \quad (24)$$

It is seen from equation (24) that the minimum angular velocity Ω_- is positive for

$$\frac{g_{tt}g_{\phi\phi}}{g_{t\phi}^2} > 0, \quad (25)$$

which is satisfied only for $g_{tt} > 0$. The ergosphere fulfils this condition, and so it can be concluded that static trajectories (which have $\Omega = 0$) cannot exist in the ergosphere, no matter how much energy an observer uses to try to remain static.

This result is a manifestation of the frame dragging in the Kerr metric, of which the component $g_{\phi t}$ of the metric is responsible. The frame dragging is more important the closer to the singularity of the black hole. From equation (22), we see that in the limit where

$$\frac{g_{tt}g_{\phi\phi}}{g_{t\phi}^2} = 1, \quad (26)$$

the values of Ω_- and Ω_+ coincide. Consequently, there is only one permitted angular velocity, which will be denoted as Ω_H , in the limit of approaching this region of spacetime. Equation (26) is equivalent to

$$\Delta\rho^4 = 0. \quad (27)$$

As $\rho^2 = 0$ is the singularity of the BH, the relevant solution to equation (27) is the event horizon. In the limit of approaching the event horizon from outside, the only permitted angular velocity Ω_H is

$$\Omega_H = \tilde{\omega} \Big|_{r_h} = \frac{a}{a^2 + r_h^2} = \frac{a}{2Mr_h}. \quad (28)$$

III. THE PENROSE PROCESS

The geometry of spacetime that is being described in this TFG allows for particles to have negative energy. The energy E of a particle in the Kerr spacetime with mass m can be obtained by contracting the Killing vector $\xi_{(t)}$ with its momentum $\mathbf{P}$:

$$E = -p_t = -p^t g_{tt} - p^\phi g_{\phi t}. \quad (29)$$

The normalisation condition for the 4-momentum ($p_\mu p^\mu = -m^2$) leads to

$$-m^2 = p_t^2 g^{tt} + 2p_t p_\phi g^{t\phi} + p_\theta^2 g^{\theta\theta} + p_\phi^2 g^{\phi\phi} + p_r^2 g^{rr}. \quad (30)$$

The previous expressions may be combined along with the definition (2) to obtain:

$$E = \tilde{\omega} p_\phi + \sqrt{\tilde{\omega}^2 p_\phi^2 - \frac{1}{g^{tt}} \left[p_\phi^2 g^{\phi\phi} + p_\theta^2 g^{\theta\theta} + p_r^2 g^{rr} + m^2 \right]}, \quad (31)$$

where it has been taken into account that at $r \rightarrow \infty$, the energy E of a particle at rest has to be $E = m$, so only the $+$ sign solution is possible out of the two quadratic $\pm$ sign solutions.

The energy of a particle can be negative, if $p_\phi < 0$, and

$$-\tilde{\omega} p_\phi > \sqrt{-\frac{p_\phi^2 (g^{\phi\phi} - \tilde{\omega}^2 g^{tt}) + p_\theta^2 g^{\theta\theta} + p_r^2 g^{rr} + m^2}{g^{tt}}}. \quad (32)$$

The most favorable case in which this inequality can be fulfilled is obtained when $p_\theta = 0$, $p_r = 0$, $m \rightarrow 0$. In this limit, the inequality (32) becomes

$$\tilde{\omega}^2 > -\frac{e^{-2\psi}}{g^{tt}}, \quad (33)$$

where $e^{-2\psi} = g^{\phi\phi} - \tilde{\omega}^2 g^{tt}$. It can be used that

$$g_{tt} = \frac{1}{g^{tt}} + e^{2\psi} \tilde{\omega}^2, \quad (34)$$

to rewrite the inequality (33) as

$$\frac{g_{tt} + e^{2\psi}}{e^{2\psi}} > e^{2\nu - 2\psi}. \quad (35)$$

Thus, we see that, even in the aforesaid favourable scenario, the condition $g_{tt} > 0$ must be fulfilled in order for a particle to have $E < 0$.

This type of solutions are hence only possible inside the ergosphere and are a result of the frame dragging in the Kerr metric. As photons coming from the ergosurface arrive at any observer with a wavelength $\lambda \rightarrow \infty$, no information can be transmitted through the ergosurface, and negative energy particles cannot be measured. Trajectories in which a particle escapes from the ergosurface are however allowed.

Let us consider a particle with energy E_A that falls into the ergosphere of a BH and decays, while being in the ergosphere, into two particles: a particle B with energy E_B that falls into the event horizon, and a particle C with energy E_C that escapes from the ergosurface. If the particle B has $E_B < 0$, by conservation of energy, it is seen that the outgoing particle has a larger energy than that of the incoming particle: $E_C = E_A - E_B > E_A$. The described phenomenon is known as the Penrose process, and involves a loss of mass δM and a loss of angular momentum δJ from the BH. Let us find now the relation between δM and δJ .

The starting point is to consider the only Killing vector that is time-like infinitesimally outside the event horizon of a Kerr black hole, $\xi_{(K)}$, with components

$$\xi_{(K)}^\mu = \xi_{(t)}^\mu + \Omega_H \xi_{(\phi)}^\mu, \quad (36)$$

as has been shown through equation (21) and inequality (22). The contraction of $\xi_{(K)}$ with the 4-momentum P of a particle with energy E_B and angular momentum gives

$$\xi_{(K)}^\mu P_\mu = -E_B + \Omega_H L_B. \quad (37)$$

In the Penrose process, the particle having energy $E_B < 0$ that falls into the event horizon produces an extraction of mass $\delta M = E_B$ and of angular momentum $\delta J = L_B$ from the black hole. As the contraction of two time-like vectors oriented the same way is negative, in the Penrose Process it is satisfied that

$$\delta M > \frac{a}{2Mr_h} \delta J. \quad (38)$$

The previous inequality also reflects a general property of black holes: the area of the event horizon can only increase by means of classical processes. To demonstrate this property for the Penrose process, the area A of the event horizon has been calculated using the line element evaluated at $r = r_h$, when not considering variations in t or r :

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} \Big|_{r_h} = 4\pi (r_h^2 + a^2). \quad (39)$$

In this work, by using the definition for the irreducible mass $M_{irr}^2 = A/16\pi$, it has been found that

$$M_{irr}^2 = \frac{1}{2} \left(M^2 + \sqrt{M^4 - J^2} \right), \quad (40)$$

and the variation δM_{irr} ,

$$\delta M_{irr} = \frac{M_{irr}}{\sqrt{M^2 - a^2}} \left(\delta M - \frac{a}{2Mr_h} \delta J \right), \quad (41)$$

has been obtained. Using the inequality (38), it is seen from equation (41) that $\delta A > 0$, indicating that the area of the event horizon of BHs in the Penrose process can only increase.

IV. SUPERRADIANCE

The massless Klein-Gordon equation,

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \Phi = 0, \quad (42)$$

is the equation that governs the behaviour of a spin-0 massless field Φ on a curved spacetime. The equation may be rewritten in the following convenient way:

$$\frac{1}{\sqrt{-g}} \partial_\nu [\sqrt{-g} g^{\mu\nu} \partial_\mu \Phi] = 0. \quad (43)$$

In the Kerr metric, the massless Klein-Gordon equation is expanded as

$$0 = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \Phi) + \partial_r (\Delta \partial_r \Phi) - \frac{4Mar}{\Delta} \partial_t \partial_\phi \Phi + \left[\frac{-(r^2 + a^2)^2}{\Delta} + a^2 \sin^2 \theta \right] \partial_t \partial_t \Phi + \left(\frac{1}{\sin^2 \theta} + \frac{a^2}{\Delta} \right) \partial_\phi \partial_\phi \Phi. \quad (44)$$

The field Φ must reflect the axial and temporal symmetry of the Kerr spacetime, and so we may propose a solution

$$\Phi(t, r, \theta, \phi) = e^{im\phi} e^{-i\omega t} \tilde{\Phi}(r, \theta), \quad (45)$$

where m and ω are two integration constants. Because of the periodicity in ϕ , the parameter m is a real integer; whereas the parameter ω is complex. In this work, the solution in which ω has no imaginary part has been studied.

It is known, because of the separation of variables of the Hamilton-Jacobi equations in the Kerr metric [9], that Φ can be written as

$$\Phi(t, r, \theta, \phi) = e^{im\phi} e^{-i\omega t} R(r) S(\theta), \quad (46)$$

where the r and θ dependence is separated. The Klein-Gordon equation now implies:

$$\begin{cases} \frac{d}{dr} \left(\Delta \frac{dR}{dr} \right) + R(r) \left[\frac{\omega^2 (r^2 + a^2)^2 - 4Mar\omega m + a^2 m^2}{\Delta} + A \right] = 0 \\ \frac{d^2 S}{d\theta^2} + \cot \theta \frac{dS}{d\theta} - S(\theta) \left[\frac{m^2}{\sin^2 \theta} + \omega^2 a^2 \sin^2 \theta + A \right] = 0, \end{cases} \quad (47)$$

with A a constant (see Appendix G). The equation (47) for $R(r)$ is rewritable using the variables defined by $dr/dr^* = \Delta/(r^2 + a^2)$ and $R = f/\sqrt{r^2 + a^2}$ as

$$\frac{d^2 f}{dr^{*2}} + \frac{f}{(r^2 + a^2)^2} \left[a^2 m^2 + \omega^2 (r^2 + a^2)^2 - \Delta A - 4Mra\omega m - \sqrt{r^2 + a^2} \Delta \frac{d}{dr} \left(\frac{r\Delta}{(r^2 + a^2)^{3/2}} \right) \right] = 0. \quad (48)$$

For $r \gg a$, $r \gg M$, the equation for f becomes

$$\frac{d^2 f}{dr^{*2}} + f\omega^2 + \mathcal{O}(r^{-2}) = 0. \quad (49)$$

Its solution is a linear combination of an outgoing wave $e^{i\omega r^*}$ with coefficient C_0 and an incoming wave $e^{-i\omega r^*}$ with coefficient C_1 ,

$$f_1 \approx C_0 e^{i\omega r^*} + C_1 e^{-i\omega r^*}. \quad (50)$$

In the limit of $r \rightarrow r_h$, in which $\Delta \rightarrow 0$, the equation for f becomes

$$\frac{d^2 f}{dr^{*2}} + f \left(\omega - \frac{am}{2Mr_h} \right)^2 \approx 0, \quad (51)$$

with solution

$$f_2 \approx C_2 \exp[i(\omega - am/2Mr_h)r^*], \quad (52)$$

as the outgoing trajectory is not permitted at $r = r_h$.

The case of a field Φ which initially propagates towards the event horizon, in such a way that $f \approx C_0 \exp[i\omega r^*]$ at large distances to the black hole, has been studied using the Wronskian $W(g_1, g_2)$ of two linearly independent solutions g_1 and g_2 of a differential equation. For a second-order ordinary differential equation $g''(x) + a(x)g'(x) + b(x)g(x) = 0$ with solutions g_1 and g_2 , the Wronskian satisfies that $W'(g_1, g_2) = a(x)W(g_1, g_2)$.

Thus, the Wronskian W of the solutions of f remains constant for all r^* . Considering the solution f_1 from expression (50) and its complex conjugate f_1^* , the Wronskian gives the result

$$W(f_1, f_1^*) = 2i\omega (|C_0|^2 - |C_1|^2). \quad (53)$$

In turn, the Wronskian of the solution f_2 from expression (52) and its complex conjugate f_2^* gives

$$W(f_2, f_2^*) = 2i|C_2|^2 \left(\omega - \frac{a}{2Mr_h} m \right). \quad (54)$$

Hence, it can be concluded that the wave scattered at the event horizon of the black hole has higher energy than the incoming wave (that is, $|C_1|^2 > |C_0|^2$) if it is satisfied that

$$\Omega_H m = \frac{a}{2Mr_h} m > \omega. \quad (55)$$

V. CONCLUSIONS

In this work, classical processes of energy extraction from a rotating black hole have been successfully studied, making use of the undergraduate knowledge of general relativity and mathematical analysis.

- It has been shown that the Kerr metric predicts the existence of axially symmetric black holes, whose event horizon and ergosurface have been determined.
- The condition for the existence of the Penrose process has been found, together with the relation that the extracted mass and angular momentum from the black hole satisfy.
- The condition for black hole superradiance in the Kerr metric has been derived.

The superradiance phenomena may be further studied by considering massive fields, from which black hole instability leading to bosonic clouds may emerge.

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SUPERRADIÀNCIA EN FORATS NEGRES

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Resum: Aquest estudi indaga sobre dos mecanismes clàssics d'extracció d'energia de forats negres en rotació. En primer lloc, s'analitza el procés Penrose i es deriva la relació entre l'energia i el moment angular extrets del forat negre en aquest procés. En segon lloc, s'investiga el fenomen de la superradiància en forats negres per a camps escalars no massius i en l'espai-temps de Kerr, arribant a la determinació de la condició sota la qual les ones incidents experimenten amplificació superradiant.

Paraules clau: Relativitat general, Forats negres, Mètrica de Kerr, Equació de Klein-Gordon, Procés Penrose, Superradiància

ODS: Educació de qualitat

Objectius de Desenvolupament Sostenible (ODS o SDGs)

1. Fi de la desigualtat		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

Appendix A: Useful relations

Throughout this work, the following relations between the variables ρ^2 , Σ^2 and Δ have often been used to do calculations:

$$\Sigma^2 = \Delta^2 + \Delta(4Mr - a^2 \sin^2 \theta) + 4M^2 r^2, \quad (\text{A1})$$

$$\Delta = \rho^2 - 2Mr + a^2 \sin^2 \theta. \quad (\text{A2})$$

Appendix B: Derivation of equation (4)

From the expression (1) along with the definitions on (2), it can be seen that the $g_{\mu\nu}$ components of the Kerr metric are

$$\begin{pmatrix} g_{00} & g_{r0} & g_{\theta 0} & g_{\phi 0} \\ g_{0r} & g_{rr} & g_{\theta r} & g_{\phi r} \\ g_{0\theta} & g_{r\theta} & g_{\theta\theta} & g_{\phi\theta} \\ g_{0\phi} & g_{r\phi} & g_{\theta\phi} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} & 0 & 0 & -\frac{2Mar \sin^2 \theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2Mar \sin^2 \theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2}{\rho^2} \sin^2 \theta \end{pmatrix} \quad (\text{B1})$$

It follows that the determinant of the Kerr metric, g , is

$$g = g_{tt}g_{rr}g_{\theta\theta}g_{\phi\phi} - g_{t\phi}^2 g_{rr}g_{\theta\theta} \quad (\text{B2})$$

$$g = -\frac{(\Delta - a^2 \sin^2 \theta)}{\Delta} \Sigma^2 \sin^2 \theta - \frac{4M^2 a^2 r^2 \sin^4 \theta}{\Delta}. \quad (\text{B3})$$

It is very convenient to use the relations (A1) and (A2) to obtain the determinant, which may be calculated as:

$$\begin{aligned} -g &= \frac{(\Delta - a^2 \sin^2 \theta)}{\Delta} \Sigma^2 \sin^2 \theta + \frac{4M^2 a^2 r^2 \sin^4 \theta}{\Delta} \\ &= \frac{(\Delta - a^2 \sin^2 \theta)}{\Delta} [\Delta^2 + \Delta(4Mr - a^2 \sin^2 \theta) + 4M^2 r^2] \sin^2 \theta \\ &= [\Delta^2 + \Delta(4Mr - 2a^2 \sin^2 \theta) + (\Delta - a^2 \cos^2 \theta - r^2)^2] \sin^2 \theta \\ &= [\Delta^2 - 2a^2 \sin^2 \theta(2Mr + \Delta) + 4Mr(Mr + \Delta) + a^4 \sin^4 \theta] \sin^2 \theta \\ &= [\Delta^2 - 2a^2 \sin^2 \theta(\rho^2 + a^2 \sin^2 \theta) + 4Mr(\rho^2 + a^2 \sin^2 \theta - Mr) + a^4 \sin^4 \theta] \sin^2 \theta \\ &= [\Delta^2 - 2a^2 \sin^2 \theta \rho^2 - a^4 \sin^4 \theta + 4Mr(\rho^2 + a^2 \sin^2 \theta - Mr)] \sin^2 \theta \\ &= [\Delta^2 - a^2 \sin^2 \theta(2\rho^2 - 4Mr) - a^4 \sin^4 \theta + 4Mr\rho^2 - 4M^2 r^2] \sin^2 \theta \end{aligned}$$

As we square Δ , we find that Δ^2 cancels out most of the terms in the previous expression:

$$\begin{aligned} \Delta^2 &= r^4 + 2M^2 r^2 + a^4 - 4Mr^3 - 2Mra^2 + 2r^2 a^2, \\ \Delta^2 &= \rho^4 + a^4 \sin^4 \theta + 4M^2 r^2 + 2\rho^2 a^2 \sin^2 \theta - 4Mr\rho^2 - 4Mra^2 \sin^2 \theta \\ \Delta^2 &= \rho^4 + a^4 \sin^4 \theta + 4M^2 r^2 + a^2 \sin^2 \theta(2\rho^2 - 4Mr) - 4Mr\rho^2. \end{aligned}$$

Hence,

$$g = -\rho^4 \sin^2 \theta.$$

Appendix C: Derivation of equation (27)

From the condition (26), that is, from

$$g_{tt} = \frac{g_{t\phi}^2}{g_{\phi\phi}} , \quad (26)$$

the equation (27) can be derived by making use of the relation (A2):

$$\begin{aligned} 2Mr - \rho^2 &= \frac{4M^2 a^2 r^2 \sin^2 \theta}{\sin^2 \theta [\rho^2(r^2 + a^2 - 2Mr) + 2Mr(a^2 + r^2)]} \\ (2Mr - \rho^2) [\rho^2(r^2 + a^2 - 2Mr) + 2Mr(a^2 + r^2)] &= 4M^2 a^2 r^2 \sin^2 \theta \\ (2Mr - \rho^2) [\rho^2 \Delta + 2Mr(\Delta + 2Mr)] &= 4M^2 a^2 r^2 \sin^2 \theta \\ (2Mr - \rho^2) [\Delta(\rho^2 + 2Mr) + 4M^2 r^2] &= 4M^2 a^2 r^2 \sin^2 \theta \\ \Delta(4M^2 r^2 - \rho^4) + 4M^2 r^2(2Mr - \rho^2) &= 4M^2 a^2 r^2 \sin^2 \theta \\ 4M^2 r^2(\Delta + a^2 \sin^2 \theta - \Delta) - \Delta \rho^4 &= 4M^2 a^2 r^2 \sin^2 \theta \\ 4M^2 r^2 a^2 \sin^2 \theta - \Delta \rho^4 &= 4M^2 a^2 r^2 \sin^2 \theta \\ \Delta \rho^4 &= 0 . \end{aligned}$$

Appendix D: Derivation of equation (39)

The line element squared ds^2 evaluated at $r = r_h$, when not considering variations in t or r is

$$ds^2 = (r_h^2 + a^2 \cos^2 \theta) d\theta^2 + \frac{(r_h^2 + a^2)^2}{r_h^2 + a^2 \cos^2 \theta} \sin^2 \theta d\phi^2 . \quad (D1)$$

From (D1), the area A of the event horizon is obtained as:

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}} \Big|_{r_h} \quad (D2)$$

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{(r_h^2 + a^2)^2 \sin^2 \theta} \quad (D3)$$

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\phi (r_h^2 + a^2) \sin \theta \quad (D4)$$

$$A = 2\pi (r_h^2 + a^2) \int_0^\pi d\theta \sin \theta \quad (D5)$$

$$A = 4\pi (r_h^2 + a^2) . \quad (39)$$

Appendix E: Derivation of expression (41)

From expression (40), the variation $\delta(M_{irr}^2)$ can be obtained

$$\delta(M_{irr}^2) = \frac{1}{2} \left(2M\delta M + \frac{4M^3\delta M - 2J\delta J}{2\sqrt{M^4 - J^2}} \right). \quad (E1)$$

As $\delta(M_{irr}^2) = 2M_{irr}\delta M_{irr}$, we find that δM_{irr} is

$$\delta M_{irr} = \frac{2M\delta M\sqrt{M^4 - J^2} + 2M^3\delta M - J\delta J}{2\sqrt{M^4 - J^2} \cdot 2M_{irr}} \quad (E2)$$

$$\delta M_{irr} = \frac{\delta M(2M\sqrt{M^4 - J^2} + 2M^3) - J\delta J}{4M_{irr}\sqrt{M^4 - J^2}} \quad (E3)$$

$$\delta M_{irr} = \frac{4\delta M M_{irr}^2}{4M_{irr}\sqrt{M^4 - J^2}} - \frac{a\delta J}{4M_{irr}\sqrt{M^2 - a^2}} \quad (E4)$$

$$\delta M_{irr} = \frac{1}{\sqrt{M^2 - a^2}} \left(\frac{\delta M M_{irr}}{M} - \frac{a\delta J}{4M_{irr}} \right) \quad (E5)$$

$$\delta M_{irr} = \frac{M_{irr}}{\sqrt{M^2 - a^2}} \left(\delta M - \frac{a\delta J}{4M_{irr}^2} \right). \quad (41)$$

Appendix F: Derivation of equation (44)

The non-zero derivatives of the auxiliary variables ρ^2 , Δ , and Σ are:

$$\begin{aligned} \partial_r \rho^2 &= 2r, \\ \partial_\theta \rho^2 &= -2a^2 \sin \theta \cos \theta, \\ \partial_r \Delta &= 2r - 2M, \\ \partial_r \Sigma^2 &= 4r(r^2 + a^2) - \partial_r \Delta a^2 \sin^2 \theta, \\ \partial_\theta \Sigma^2 &= -2a^2 \Delta \sin \theta \cos \theta. \end{aligned} \quad (F1)$$

These results, together with (A2), are used for the expansion of the Klein-Gordon equation:

$$\begin{aligned} g^{\mu\nu} \nabla_\mu \nabla_\nu \Phi &= \\ &= \frac{1}{\sqrt{-g}} \partial_\nu \left[\sqrt{-g} g^{\mu\nu} \partial_\mu \Phi \right] = \\ &= \frac{1}{\sqrt{-g}} \left[\partial_\nu (\sqrt{-g}) g^{\mu\nu} \partial_\mu \Phi + \sqrt{-g} \partial_\nu (g^{\mu\nu}) \partial_\mu \Phi + \sqrt{-g} g^{\mu\nu} \partial_\nu \partial_\mu \Phi \right] \\ &= \underbrace{\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}) g^{\mu\nu} \partial_\mu \Phi}_{i)} + \underbrace{\partial_\nu (g^{\mu\nu}) \partial_\mu \Phi}_{ii)} + \underbrace{g^{\mu\nu} \partial_\nu \partial_\mu \Phi}_{iii)} \end{aligned}$$

$$\begin{aligned}
i) \quad & \frac{1}{\sqrt{-g}} \frac{\partial_\nu(-g)}{2\sqrt{-g}} g^{\mu\nu} \partial_\mu \Phi = \\
& = \frac{1}{2(-g)} \partial_\theta(-g) \partial_\theta(\Phi) g^{\theta\theta} + \frac{1}{2(-g)} \partial_r(-g) \partial_r(\Phi) g^{rr} = \\
& = \frac{1}{2(-g)} (2\rho^2 \partial_\theta(\rho^2) \sin^2 \theta + \rho^4 2 \sin \theta \cos \theta) \frac{1}{\rho^2} \partial_\theta \Phi + \\
& + \frac{1}{2(-g)} 2\rho^2 \partial_r(\rho^2) \sin^2 \theta \frac{\Delta}{\rho^2} \partial_r \Phi = \\
& = \frac{1}{\rho^4 \sin^2 \theta} (\partial_\theta(\rho^2) \sin^2 \theta + \rho^2 \sin \theta \cos \theta) \partial_\theta \Phi + \\
& + \frac{1}{\rho^4 \sin^2 \theta} \partial_r(\rho^2) \sin^2 \theta \Delta \partial_r \Phi = \\
& = \left(-2a^2 \frac{1}{\rho^4 \sin \theta} \cos \theta + \frac{1}{\rho^2 \sin \theta} \cos \theta \right) \partial_\theta \Phi + \\
& + \frac{1}{\rho^4} 2r \Delta \partial_r \Phi = \\
& = \frac{\cos \theta}{\rho^4 \sin \theta} (\rho^2 - 2a^2 \sin^2 \theta) \partial_\theta \Phi + \frac{1}{\rho^4} 2r \Delta \partial_r \Phi
\end{aligned}$$

$$\begin{aligned}
ii) \quad & \partial_\nu g^{\mu\nu} \partial_\mu \Phi = \\
& = \partial_r g^{rr} \partial_r \Phi + \partial_\theta g^{\theta\theta} \partial_\theta \Phi + \partial_t g^{\phi t} \partial_\theta \Phi + \partial_t g^{tt} \partial_t \Phi + \\
& + \partial_\phi g^{t\phi} \partial_t \Phi + \partial_\phi g^{\phi\phi} \partial_\phi \Phi = \\
& = \frac{\partial_r \Delta \rho^2 - \partial_r \rho^2 \Delta}{\rho^4} \partial_r \Phi - \frac{2\partial_\theta \rho}{\rho^3} \partial_\theta \Phi = \\
& = \frac{-2r \Delta \partial_r \Phi + 2\rho^2 (r - M) \partial_r \Phi}{\rho^4} + \\
& + \frac{2a^2 \cos \theta \sin \theta}{\rho} \frac{1}{\rho^3} a^2 \partial_\theta \Phi = \\
& = \frac{\rho^2 (2r - 2M) - \Delta 2r}{\rho^4} \partial_r \Phi + \frac{2}{\rho^4} a^2 \cos \theta \sin \theta \partial_\theta \Phi = \\
& = \left[\frac{1}{\rho^2} (2r - 2M) - \frac{\Delta 2r}{\rho^4} \right] \partial_r \Phi + \frac{2}{\rho^4} a^2 \cos \theta \sin \theta \partial_\theta \Phi
\end{aligned}$$

$$\begin{aligned}
iii) \quad g^{\mu\nu} \partial_\nu \partial_\mu \Phi &= \\
&= \frac{\Delta}{\rho^2} \partial_r \partial_r \Phi + \frac{1}{\rho^2} \partial_\theta \partial_\theta \Phi - \frac{\Sigma^2}{\Delta} \frac{1}{\rho^2} \partial_t \partial_t \Phi - \frac{4Mar}{\Delta} \frac{1}{\rho^2} \partial_t \partial_\phi \Phi + \frac{\Delta - a^2 \sin^2 \theta}{\Delta \rho^2 \sin^2 \theta} \partial_\phi \partial_\phi \Phi = \\
&= \frac{\Delta}{\rho^2} \partial_r \partial_r \Phi + \frac{1}{\rho^2} \partial_\theta \partial_\theta \Phi + \left(\frac{-(r^2 + a^2)^2}{\Delta} + a^2 \sin^2 \theta \right) \frac{1}{\rho^2} \partial_t \partial_t \Phi - \frac{4Mar}{\Delta} \frac{1}{\rho^2} \partial_t \partial_\phi \Phi + \left(\frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \frac{1}{\rho^2} \partial_\phi \partial_\phi \Phi .
\end{aligned}$$

Combining *i*), *ii*) and *iii*), we get:

$$i) + ii) + iii)$$

$$\begin{aligned}
&\frac{\cos \theta}{\rho^2} \frac{1}{\sin \theta} \partial_\theta \Phi + \frac{2(r-M)}{\rho^2} \partial_r \Phi + \\
&+ \frac{\Delta}{\rho^2} \partial_r \partial_r \Phi + \frac{1}{\rho^2} \partial_\theta \partial_\theta \Phi + \left(\frac{-(r^2 + a^2)^2}{\Delta} + a^2 \sin^2 \theta \right) \frac{1}{\rho^2} \partial_t \partial_t \Phi - \frac{4Mar}{\Delta} \frac{1}{\rho^2} \partial_t \partial_\phi \Phi + \left(\frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \frac{1}{\rho^2} \partial_\phi \partial_\phi \Phi = 0 .
\end{aligned}$$

We see that $\frac{1}{\rho^2}$ is a common factor

$$\begin{aligned}
&\frac{\cos \theta}{\sin \theta} \partial_\theta \Phi + 2(r-M) \partial_r \Phi + \\
&\Delta \partial_r \partial_r \Phi + \partial_\theta \partial_\theta \Phi + \left(\frac{-(r^2 + a^2)^2}{\Delta} + a^2 \sin^2 \theta \right) \partial_t \partial_t \Phi - \frac{4Mar}{\Delta} \partial_t \partial_\phi \Phi + \left(\frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \partial_\phi \partial_\phi \Phi = 0 \\
&\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \Phi) + \partial_r (\Delta \partial_r \Phi) + \left(\frac{-(r^2 + a^2)^2}{\Delta} + a^2 \sin^2 \theta \right) \partial_t \partial_t \Phi - \frac{4Mar}{\Delta} \partial_t \partial_\phi \Phi + \left(\frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \partial_\phi \partial_\phi \Phi = 0 .
\end{aligned} \tag{44}$$

Appendix G: Further remarks on expression (47)

The constant A is a separation constant that comes from the method of separation of variables applied on the expression (44).

The equation for $S(\theta)$ may be solved using the change of variables $x = \cos \theta$, which reduces the equation to an angular spheroidal wave equation.

The variable r^* allows us to conveniently rewrite the equation for $R(r)$. It may be implicitly defined by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta} . \tag{G1}$$

Thus, it takes the form

$$r^* = r + \frac{2Mr_h}{r_h - r_-} \ln(r - r_h) + 2M \left(1 - \frac{r_h}{r_h - r_-} \right) \ln(r - r_-) . \tag{G2}$$