Universitat de Barcelona

Facultat d'Economia i Empresa

# PRE-MATHEMATICS

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**GEI (Grau d'Empresa Internacional)**

**PROLOGUE**

The aim of this short manual is to provide the student of GEI (Grau d'Empresa Internacional) a thorough summary of the contents of an introductory course for the subject *Mathematics* of GEI.

This course, which does not form part of the curriculum of this degree, intends to homogenize the mathematical skills of students to the level required for the specific subjects in GEI in which the potential use of mathematics is needed. To this end, two of the most important aspects of this potential, such as *derivatives of functions of one real variable* and *systems of linear equations*, will be covered.

The manual includes two sections. The first, "Calculus", studies functions of one real variable putting the emphasis on the applications to the calculus of extreme points (maxima and minima) called optima points in Economics. The second section, "Algebra", does the same with systems of linear equations with the aid of the elementary matrix theory. It is worth noting that each of the sections contains a short list of exercises which tend to ground all the concepts treated there.

Also, at the end of the manual there are some bibliographic references of interest and a glossary of terms to help the student to find the most important concepts quoted here.

Finally, it should be mentioned that this document has been filed in the Digital Repository of the UB (OMADO Collection):

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and that its contents, as well as the mistakes that may be found, are the sole and exclusive responsibility of the author.

Gonzalo Rodríguez Pérez

## INDEX

**SECTION I: Calculus**

# 1. FUNCTIONS OF ONE REAL VARIABLE

## 1.1. Function of One Real Variable and Domain

Definition: A **function of one real variable** (**function** for short) is an application that assigns to any real number of a set $A \subset \mathbb{R}$ one and only one real number. Formally:

$$\begin{array}{ccc} f: A \subset \mathbb{R} & \longrightarrow & \mathbb{R} \\ x \in A & \longrightarrow & y = f(x) \end{array}$$

The set $Domf = A \subset \mathbb{R}$ formed by those real numbers supporting image by $y = f(x)$ will be called the **domain** of the function.

Example: Find the domain of the following functions:

$$(1)\ y = \sqrt[3]{x}.\ (2)\ y = \frac{x-2}{x-5}.\ (3)\ y = \frac{x}{\ln x}.\ (4)\ y = \sqrt{\frac{x-1}{x+1}}.$$

SOLUTION: (1) In this case the domain is the whole real straight line since every real number has a unique cubic root. Thus:

$$Domf = \mathbb{R}\blacksquare$$

(2) Now the domain of definition is all the real numbers except number 5 since we cannot divide by zero. Therefore:

$$Domf = \{x \in \mathbb{R}: x \neq 5\} = ]-\infty, 5[\ \cup\ ]5, +\infty[\blacksquare$$ [1]

(3) The domain will be formed by the real numbers $x \in \mathbb{R}$ that support natural logarithm, i.e., $x > 0$, and $x \neq 1$ since we cannot divide by 0:[2]

$$Domf = \{x \in \mathbb{R}: x > 0 \text{ and } x \neq 1\} = ]0,1[\ \cup\ ]1, +\infty[\blacksquare$$

(4) The existence of the square root implies that:

$$\frac{x-1}{x+1} \geq 0 \text{ equivalent to: } \left\{\begin{array}{c} x-1 \geq 0 \text{ and } x+1 \geq 0 \\ \text{or} \\ x-1 < 0 \text{ and } x+1 < 0 \end{array}\right\} \text{ equivalent to: } \left\{\begin{array}{c} x \geq 1 \text{ and } x \geq -1 \\ \text{or} \\ x < 1 \text{ and } x < -1 \end{array}\right\}.$$

Therefore:

$$Domf = \{x \in \mathbb{R}: x \geq 1 \text{ or } x < -1\} = ]-\infty, -1[\ \cup\ [1, +\infty[\blacksquare$$

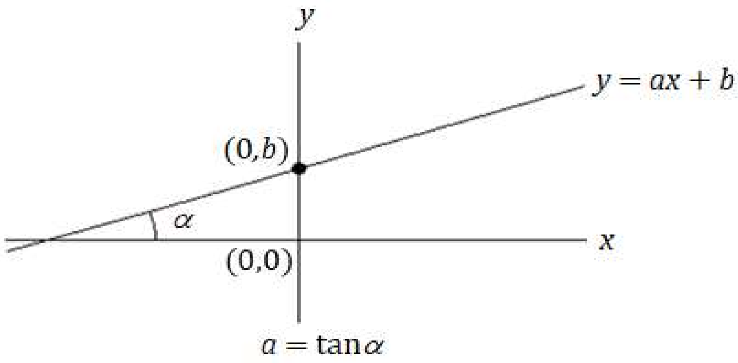
[1] The **open interval** $]a, b[$ contains all the real points that lie between $a$ and $b$.

[2] $\ln 1 = 0$.

1.1.1. *Typical Functions*

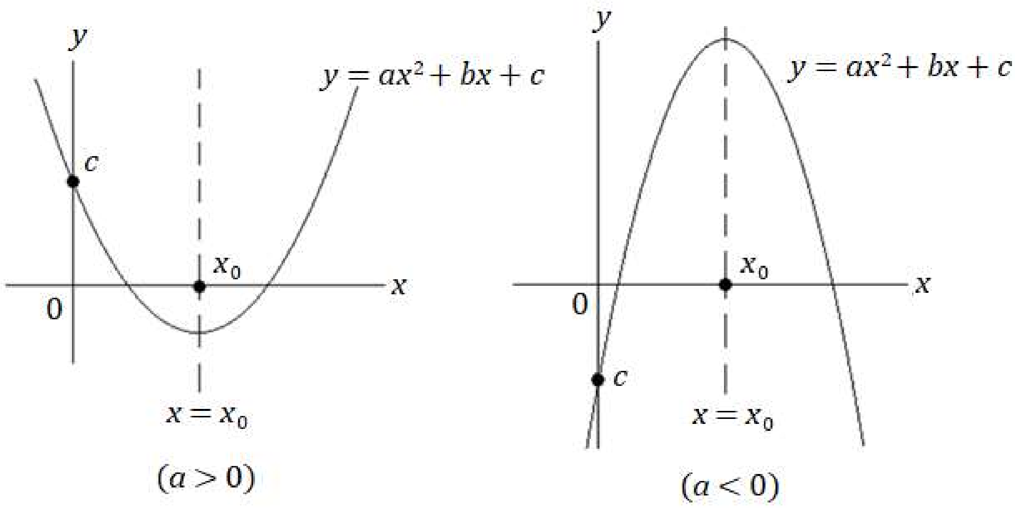
In the sequel, we present the graphics of four of the most usual functions:

a. **Straight Line:** $y = f(x) = ax + b$.[3]




Note that the slope of this straight line is $\tan\alpha = a$.

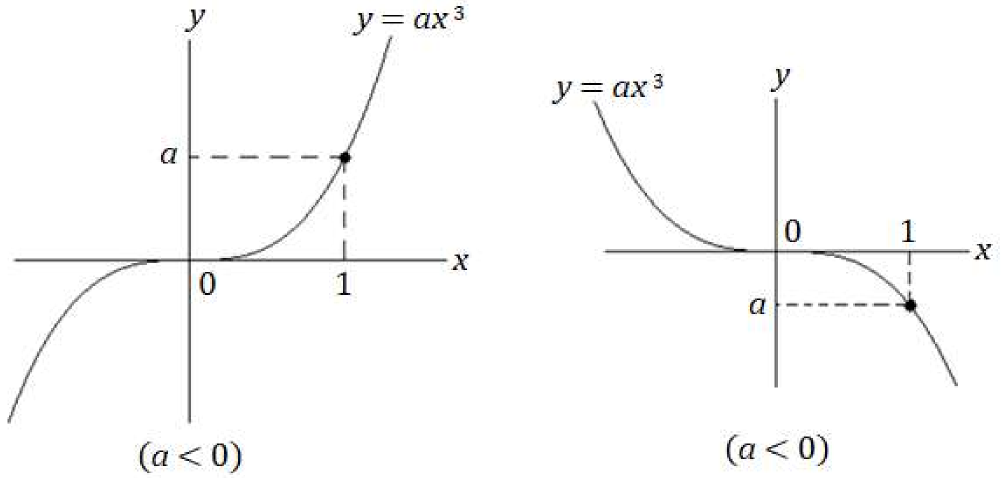
b. **Parabola:** $y = f(x) = ax^2 + bx + c$, where $a \neq 0$.




The point $x_0 = -\frac{b}{2a}$ is either the minimum $(a > 0)$ or the maximum $(a < 0)$of the parabola. Note that the straight line $x = x_0$ is the axis of symmetry. In the case $b = 0$ this axis matches up the $y$-axis.

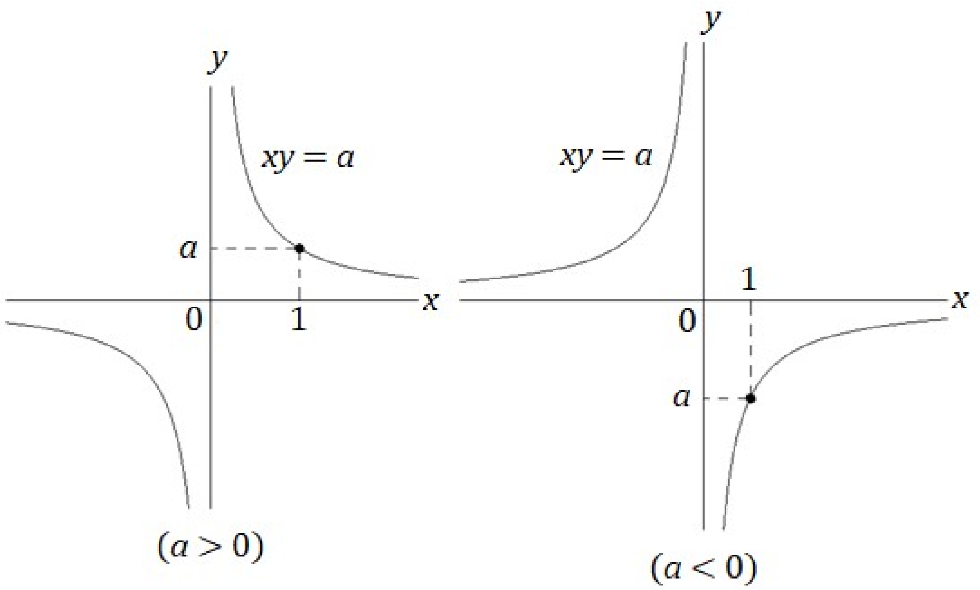
[3] This is the equation of a straight line being not perpendicular to the $x$-axis.

c. **Cubic Parabola:** $y = f(x) = ax^3$, with $a \neq 0$.




In any case the point $x = 0$ is an inflection point.[4]

d. **Hyperbola:** $y = f(x) = \frac{a}{x}$, with $a \neq 0$.[5]




The coordinate axes are its asymptotes.[6]

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[4] An **inflection point** of a function is a point where the "curvature" changes.
[5] Any hyperbola of this type is the locus of points satisfying the equation $x \cdot y =$ constant.
[6] An **asymptote** is a straight line touching the graphic of the function at "infinity".

## 1.2. Exponentials and Logarithms

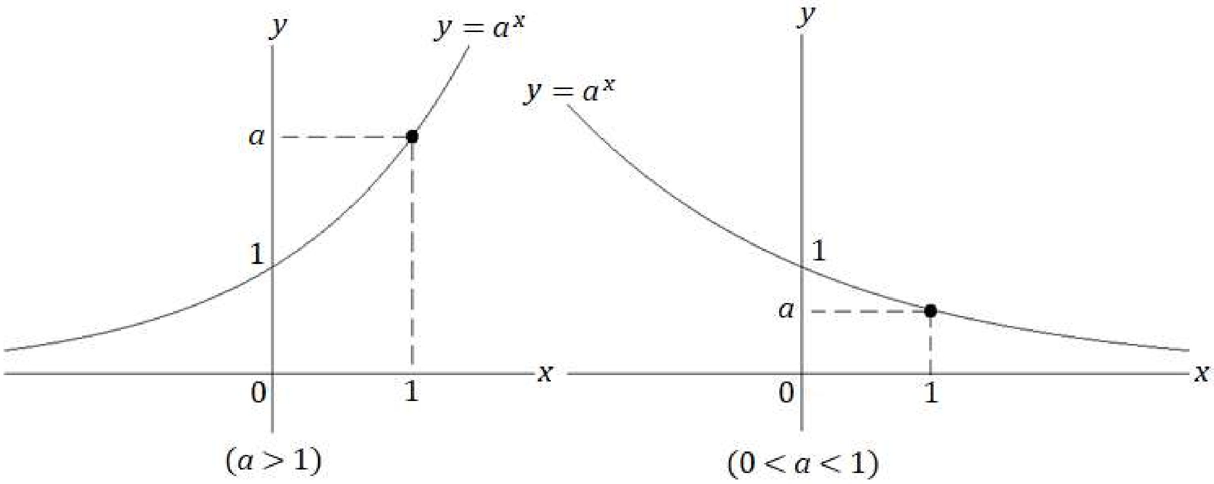
### 1.2.1. *Exponential functions*

These functions are essential in Mathematical Economics. By definition:

<u>Definition</u>: An **exponential function** is of the type:

$$y = f(x) = a^x, \text{ with } a > 0.$$

Graphically:




In either case the $x$-axis is an asymptote. The most important properties of exponential functions are the following:[7]

<u>Properties</u>:

1. $a^{x+y} = a^x \cdot a^y$
2. $a^{-x} = \frac{1}{a^x}$
3. $a^{x-y} = \frac{a^x}{a^y}$
4. $(a^x)^y = (a^y)^x = a^{xy}$
5. $a^0 = 1$

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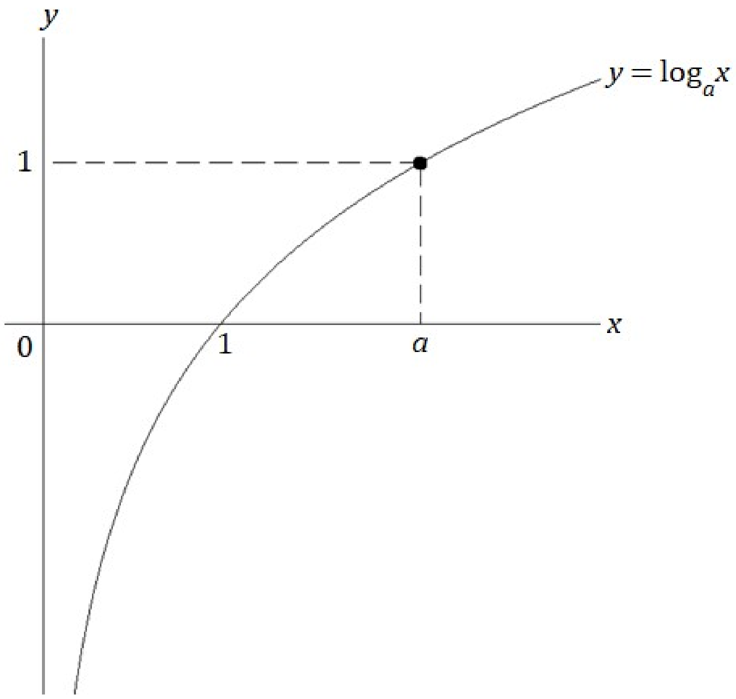
[7] It is advisable to bear in mind these properties from now onwards.

1.2.2. *Logarithmic functions*

Definition: A **logarithmic function** is of the type:

$$y = f(x) = \log_a x, \text{ where } a > 0.$$ [8]

Graphically:




As we can see the $y$-axis is an asymptote. The algebraic properties of logarithms to remember are:

Properties:

1. $\log_a(x \cdot y) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^y) = y \cdot \log_a x$
4. $\log_a(a^x) = a^{\log_a x} = x$
5. $\log_a a = 1$
6. $\log_a 1 = 0$

[8] We have the **natural logarithm** in the case of $a = e$, being $e \cong 2.718282$. This number, called **Euler number**, plays an important role in Mathematics.

### 1.2.3. *Applications of Exponentials and Logarithms*

#### 1.2.3.1 Compounding Interest

Suppose we have an amount of money to invest in order to gain later some revenues. *Compound interest* appears when the monetary value of a principal €$P_0$, some years $t \geq 0$ later, is transformed according to the formula:

$$P(t) = P_0 \cdot (1+i)^t$$

in which $0 < i < 1$ is the "annual interest rate".

Example (Dowling E. T., p. 166, 170): (1) Determine the interest rate needed to have money double in 10 years under annual compounding. (2) A developing country wishes to increase savings from a present level of 5.6 million to 12 million. How long will it take if it can increase savings by 15% a year?

SOLUTION: (1) In this case we have to calculate the interest rate $0 < i < 1$ such that:

$$P(10) = 2 \cdot P_0, \text{ where } t = 10.$$

Indeed:

$$2 \cdot P_0 = P(10) = P_0 \cdot (1+i)^{10} \text{ implies: } (1+i)^{10} = 2.$$

Thus, applying natural logarithms and their properties we deduce that:

$$\ln 2 = \ln(1+i)^{10} = 10 \cdot \ln(1+i) \text{ implies: } \ln(1+i) = \frac{\ln 2}{10} \cong 0.07.$$

Finally doing the same with exponentials:

$$1 + i = e^{\ln(1+i)} = e^{0.07} \text{ implies: } i = e^{0.07} - 1 \cong 0.0725 = 7.25\% \blacksquare$$

(2) In this case we have to find the time $t > 0$ such that:

$$12 = P(t) = P_0 \cdot (1+i)^t = 5.6 \cdot (1 + 15\%)^t = 5.6 \cdot 1.15^t.$$

So:

$$12 = 5.6 \cdot 1.15^t \text{ implies: } 1.15^t = \frac{12}{5.6}$$

and:

$$t \cdot \ln 1.15 = \ln(1.15^t) = \ln\left(\frac{12}{5.6}\right) \text{ implies: } t = \frac{\ln\left(\frac{12}{5.6}\right)}{\ln 1.15} \cong 5.45 \text{ years} \blacksquare$$

1.2.3.2 Exponential Growth

If a variable $X_0$ is steadily increasing (decreasing) by an annually rate of $r$ % then $t \geq 0$ years later we will have a value given by the formula:

$$X(t) = X_0 \cdot e^{r \cdot t} \; (X(t) = X_0 \cdot e^{-r \cdot t}).$$

Example: (1; Dowling E. T., p170) If arable land in the Sahel is eroding by 3.5% a year because of climatic conditions how much of the present land will be left in 12 years?
(2; Sydsaeter, p. 275) The number $N(t)$ of persons who developed influenza $t > 0$ days after those 1,000 individuals has been in contact with the carrier of infection is:

$$N(t) = \frac{1000}{1+999 \cdot e^{-0.39t}}.$$

How many people develop influenza after 20 days? How many days does it take until 800 persons are sick? Will everyone eventually get influenza?
SOLUTION:
(1) In this scenario we have to apply the latest formula:[9]

$$X(t) = X_0 \cdot e^{-r \cdot t} \text{ being: } r = 3.5\% \text{ and } t = 12.$$ [10]

So, the initial level of arable land $X_0$ decreases to 65.7% respect to initial level since:

$$X(12) = X_0 \cdot e^{-0.035 \cdot 12} \cong X_0 \cdot 0.657 = X_0 \cdot 65.7\% \blacksquare$$

(2) After 20 days ($t = 20$) the number of individuals developing influenza is:

$$N(20) = \frac{1000}{1+999 \cdot e^{-0.39 \cdot 20}} = 710 \text{ individuals} \blacksquare$$

The days $t > 0$ does it take until 800 persons are sick satisfies:

$$800 = N(t) = \frac{1{,}000}{1+9 \quad \cdot e^{-0.39t}} \text{ implies: } 1 + 999 \cdot e^{-0.39t} = \frac{1{,}000}{800} = 1.25$$

that implies that it does take 21 days until 800 persons are sick:

$$e^{-0.39t} = \frac{0.25}{999} \text{ implies: } -0.39t = \ln\left(\frac{0.25}{999}\right) \text{ implies: } t = \frac{\ln\left(\frac{0.25}{999}\right)}{-0.39} \cong 21 \blacksquare$$

Taking limit on the formula of $N(t)$ we see everyone eventually get influenza since:

$$\lim_{t \to +\infty} N(t) = \lim_{t \to +\infty} \frac{1000}{1+999 \cdot e^{-0.39t}} = \frac{1000}{1+999 \cdot e^{-\infty}} = \{e^{-\infty} = 0\} = \frac{1000}{1} = 1000 \blacksquare$$

[9] We are dealing with a variable (the arable land) that is steadily decreasing.
[10] The amount of the initial land $X_0$ does not matter as we are going to see.

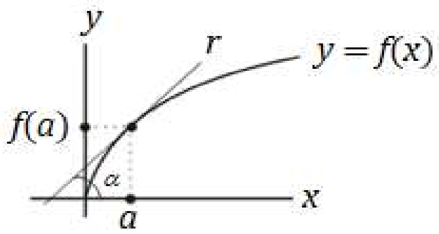
### 1.3. Derivative of a Function

Definition: A function $y = f(x)$ is **differentiable** at point $a \in Domf$ provided that the limit of the difference quotient exists:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) \in \mathbb{R}$$

being the number $\frac{df(a)}{dx} = f'(a) \in \mathbb{R}$ the **derivative** of $y = f(x)$ at point $a \in Domf$.

Geometrically speaking the derivative of a function $y = f(x)$ at point $x = a$ is the slope of the **tangent line** $r$ to the function at point $\left(a, f(a)\right)$. Graphically:




Thus, $f'(a) = \tan \alpha$ and the equation of the tangent line $r$ is:

$$y = f(a) + f'(a) \cdot (x - a).$$

Example: Find the tangent line to the function $y = f(x) = \frac{x-3}{x}$ at point $a = 1$

SOLUTION: First, the value of the function at point $a = 1$ is:

$$f(1) = \frac{1-3}{1} = -2.$$

As the derivative of this function at point $a = 1$ is: [11]

$$f'(x) = \frac{df(x)}{dx} = \frac{3}{x^3} \text{ and } f'(1) = \frac{df(1)}{dx} = \frac{3}{1^2} = 3$$

this equation will be:

$$y = f(1) + f'(1) \cdot (x - 1) = -2 + 3 \cdot (x - 1) = 3x - 5 \blacksquare$$ [12]

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[11] This derivative has been obtained using the typical derivatives and the differentiation rules that we are going to see.

[12] Note we have found the tangent line with no need to draw it.

1.3.1. *Calculating Derivatives*

Given a function $y = f(x)$, in the process of calculating the **derivative function**:

$$y' = f'(x) = \frac{df(x)}{dx}$$

we must bear in mind the following basic derivatives:

1. $f(x) = c$, with $c \in \mathbb{R}$ constant implies: $f'(x) = \frac{df(x)}{dx} = 0$.
2. $f(x) = x^a$ implies: $f'(x) = \frac{df(x)}{dx} = a \cdot x^{a-1}$.
3. $f(x) = \ln x$ implies: $f'(x) = \frac{df(x)}{dx} = \frac{1}{x}$.
4. $f(x) = a^x$, with $a > 0$, implies: $f'(x) = \frac{df(x)}{dx} = a^x \cdot \ln a$.
5. $f(x) = \sin x$ implies: $f'(x) = \frac{df(x)}{dx} = \cos x$.
6. $f(x) = \cos x$ implies: $f'(x) = \frac{df(x)}{dx} = -\sin x$.
7. $f(x) = \tan x$ implies: either $f'(x) = \frac{df(x)}{dx} = \frac{1}{\cos^2 x}$.
8. $f(x) = \arctan x$ implies: $f'(x) = \frac{df(x)}{dx} = \frac{1}{1+x^2}$.

and the basic differentiation rules:

1. $\big(f(x) + g(x)\big)' = f'(x) + g'(x)$
2. $\big(\lambda \cdot f(x)\big)' = \lambda \cdot f'(x)$, being $\lambda \in \mathbb{R}$ constant.
3. $\big(f(x) \cdot g(x)\big)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$, when $g(x) \neq 0$
5. $\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big) \cdot g'(x)$.[13]

With all of this in mind we can find the derivatives of all elemental functions.[14] See the next example

[13] This rule is known as the **chain rule**.

[14] An **elemental function** is a function built up from a finite quantity of exponentials, logarithms, powers, trigonometric functions and constants through the composition of functions and the four fundamental arithmetic operations.

1.3.1.1. Example of Calculus of Derivatives

Example: Calculate the derivatives of the following functions:

$$(1)\ y=\frac{x^2+3x}{e^{x^2}}.\ (2)\ y=\ln\left(\ln\sqrt{1-x^2}\right).\ (3)\ y=+\sqrt{\frac{1-x}{1+x}}.\ (4)\ y=(x^2-1)^{\sin x}.$$

SOLUTION: (1) In this case and taking into account that the derivative of the denominator is applying the chain rule:

$$z=x^2 \text{ implies: } \frac{d\left(e^{x^2}\right)}{dx}=\frac{d(e^z)}{dz}\cdot\frac{dz}{dx}=e^z\cdot\frac{dz}{dx}=e^{x^2}\cdot 2x$$

we obtain thanks to the derivative formula of the quotient:

$$y'=\frac{dy}{dx}=\frac{(2x+3)\cdot e^{x^2}-(x^2+3x)\cdot\left(e^{x^2}\cdot 2x\right)}{\left(e^{x^2}\right)^2}=\frac{e^{x^2}\cdot\left((2x+3)-2x\cdot(x^2+3x)\right)}{\left(e^{x^2}\right)^2}=$$

$$=\frac{-2x^3-6x^2+2x+3}{e^{x^2}}\ \blacksquare$$

(2) Applying the chain rule and the derivative of the logarithm formula we can write:

$$y'=\frac{dy}{dx}=\frac{1}{\ln\left(\sqrt{1-x^2}\right)}\cdot\frac{1}{\sqrt{1-x^2}}\cdot\left(\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\right)=\frac{-x}{(1-x^2)\ln\left(\sqrt{1-x^2}\right)}\ \blacksquare$$

(3) Since $y=+\sqrt{\frac{1-x}{1+x}}=\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$ we get applying the chain rule and the above derivatives:

$$y'=\frac{dy}{dx}=\frac{1}{2}\cdot\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}-1}\cdot\left(\frac{(-1)\cdot(1+x)-(1-x)\cdot 1}{(1+x)^2}\right)=$$

$$=\frac{1}{2}\cdot\left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}\cdot\left(\frac{-2}{(1+x)^2}\right)=\frac{-1}{\sqrt{\frac{1-x}{1+x}}\cdot(1+x)^2}\ \blacksquare$$

(4) With the aid of both natural logarithm:

$$\ln y=\ln\left((x^2-1)^{\sin x}\right)=\sin x\cdot\ln(x^2-1)$$

and the chain rule we can deduce that:

$$\frac{y'}{y}=\frac{d(\ln y)}{dx}=\cos x\cdot\ln(x^2-1)+\sin x\cdot\left(\frac{1}{x^2-1}\cdot 2x\right)=\cos x\cdot\ln(x^2-1)+\frac{2x\cdot\sin x}{x^2-1}.$$

Thus:

$$y'=y\cdot\left(\ln(x^2-1)+\frac{2x\cdot\sin x}{x^2-1}\right)=(x^2-1)^{\sin x}\cdot\left(\cos x\cdot\ln(x^2-1)+\frac{2x\cdot\sin x}{x^2-1}\right)\ \blacksquare$$ [15]

[15] This is a particular case of “logarithm differentiation”.

## 1.4. Applications of Derivatives

### 1.4.1. *L'Hôpital's Rules*[16]

L'Hôpital's rules deal with indeterminate limits when a quotient of differential functions is at stake. These rules affirm basically that:

$$\text{If } \lim_{x\to a}\frac{f(x)}{g(x)}=\left\{\frac{0}{0}\text{ or }\frac{\infty}{\infty}\right\}\text{ and }\lim_{x\to a}\frac{f'(x)}{g'(x)}=L\text{ then }\lim_{x\to a}\frac{f(x)}{g(x)}=L.$$

<u>Example</u>: Calculate the following limits:

$$(1)\lim_{x\to1}\frac{x^2-3x+2}{x^3-1}.\ (2)\lim_{x\to0}\frac{1-\cos x}{x^2}.\ (3)\lim_{x\to+\infty}\frac{\ln}{x}.\ (4)\lim_{x\to2}\left(\frac{x^2+1}{2x+1}\right)^{\frac{1}{x^2-4}}.$$

SOLUTION:

(1)

$$\lim_{x\to1}\frac{x^2-3x+2}{x^3-1}=\left\{\frac{0}{0}\right\}=\{\text{L'Hôpital}\}=\lim_{x\to1}\frac{2x-3}{3x^2}=-\frac{1}{3}\,\blacksquare$$

(2)

$$\lim_{x\to0}\frac{1-\cos x}{x^2}=\left\{\frac{0}{0}\right\}=\{\text{L'Hôpital}\}=\lim_{x\to0}\frac{\sin x}{2x}=\left\{\frac{0}{0}\right\}=\{\text{L'Hôpital}\}=\lim_{x\to0}\frac{\cos x}{2}=\frac{1}{2}\,\blacksquare$$

(3)

$$\lim_{x\to+\infty}\frac{\ln x}{x}=\left\{\frac{\infty}{\infty}\right\}=\{\text{L'Hôpital}\}=\lim_{x\to+\infty}\frac{\left(\frac{1}{x}\right)}{1}=\lim_{x\to+\infty}\frac{1}{x}=\frac{1}{\infty}=\left\{\frac{1}{\infty}=0\right\}=0\,\blacksquare$$

(4) Being $1^\infty$ an indetermination, we can evaluate this limit using the natural logarithm of the limit $L$ we are looking for.[17] Indeed:

$$\ln L=\ln\left(\lim_{x\to2}\left(\frac{x^2+1}{2x+1}\right)^{\frac{1}{x^2-4}}\right)=\lim_{x\to2}\left(\ln\left(\left(\frac{x^2+1}{2x+1}\right)^{\frac{1}{x^2-4}}\right)\right)=\lim_{x\to2}\left(\frac{1}{x^2-4}\right)\ln\left(\frac{x^2+1}{2x+1}\right)$$

$$=\lim_{x\to2}\frac{\ln\left(\frac{x^2+1}{2x+1}\right)}{x^2-4}=\left\{\frac{0}{0}\right\}=\{\text{L'Hôpital}\}=\lim_{x\to2}\frac{\left(\frac{2x+1}{x^2+1}\right)\cdot\left(\frac{2x^2+2x-2}{(2x+1)^2}\right)}{2x}$$

$$=\lim_{x\to2}\frac{x^2+x-1}{x(x^2+1)(2x+1)}=\frac{1}{10}\text{ implies: }L=e^{\ln L}=e^{\frac{1}{10}}\,\blacksquare$$

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[16] More precisely *Bernouilli-L'Hôpital*'s rules.

[17] Thus, the limit $L=e^{\ln L}$ due to properties of logarithms.

1.4.2. *Increase and Decrease of a Function*

In general, the sign of the derivative of a differentiable function at a point will allow us to decide whether this function is increasing or decreasing at this point.

Definition: The function $y = f(x)$ at point $a \in Domf$ is:

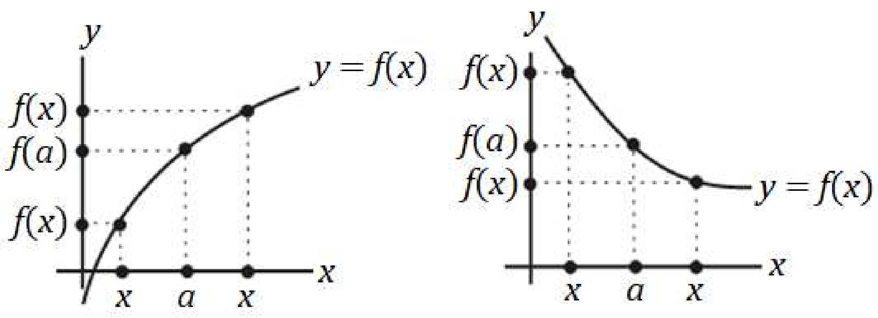
1. **Increasing** if there is an open interval $]a - r, a + r[ \subset Domf$ such that, for any point of this interval $x \in ]a - r, a + r[$ we have: [18]

$$\left.\begin{matrix} x \leq a \\ x \geq a \end{matrix}\right\} \text{implies: } \begin{cases} f(x) \leq f(a) \\ f(x) \geq f(a) \end{cases}$$

2. **Decreasing** if under the same conditions as above:

$$\left.\begin{matrix} x \leq a \\ x \geq a \end{matrix}\right\} \text{implies: } \begin{cases} f(x) \geq f(a) \\ f(x) \leq f(a) \end{cases}$$

Graphically:




As we can see in the first case the function $y = f(x)$ is increasing at point $x = a$ while in the second is decreasing.

The "local" concept of increasing or decreasing function at a point leads, naturally, to the "global" concept of increasing or decreasing of a function over a set: the function is labeled so if this happens at each of the points in the set. The "global" concept of increasing or decreasing is going to appear in the following theorem.

[18] The open interval $]a - r, a + r[$ is the **environment** of the point $a \in Domf$ of radius $r > 0$.

1.4.2.1. Relationship between the Increase or Decrease of a Function and Derivatives

Theorem: Let $y = f(x)$ be differentiable at $A \subset Domf$. In general:

1. If for any $x \in A$ we have $\begin{Bmatrix} f'(x) > 0 \\ f'(x) < 0 \end{Bmatrix}$ then $f(x)$ is $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ in $A$.
2. The function $f(x)$ is $\begin{Bmatrix} \text{increasing} \\ \text{decreasing} \end{Bmatrix}$ then $\begin{Bmatrix} f'(x) \geq 0 \\ f'(x) \leq 0 \end{Bmatrix}$ at $x \in A$.

Note that (2) is not exactly equal to the reciprocal of (1).[19] The problem of the increase or decrease of a differentiable function is resolved if its derivative is different from zero.[20] Let us look at an example:

Example: Determine the intervals of increase and decrease of:

$$(1)\ f(x) = x^2.\ (2)\ f(x) = x^3.\ (3)\ f(x) = \frac{1}{1+x^2}.$$

SOLUTION:

(1) In this case, as $f'(x) = 2x$, we conclude that the function is increasing at any point $x > 0$ and decreasing when $x < 0$; at the point $x = 0$, the previous theorem does not decide in principle■[21]

(2) Now, as $f'(x) = 3x^2$, the function is increasing at any point $x \neq 0$; as before the above theorem does not decide at $x = 0$ either■[22]

(3) Since the associated derivative is:

$$\frac{df}{dx} = f'(x) = -\frac{2x}{(1+x^2)^2}$$

the function is increasing in case that $x < 0$ and decreasing if $x > 0$■[23]

[19] The reciprocal of a property such as "A implies B" is "B implies A".

[20] The points making zero the derivative of a function are called **stationary** or **critical** points. It is worth noting that a critical point can be a minimum, a maximum or an inflection point of the function.

[21] $x = 0$ is a minimum of the function.

[22] $x = 0$ is an inflection point.

[23] At point $x = 0$ the function has a maximum.

### 1.4.3. *Extreme Points of a Function*

In Economics extreme points of a differentiable function (maxima and minima) are very important.[24]

<u>Definition</u>: The function $y = f(x)$ has at point $a \in Domf$:

1. A **global maximum** if:

$$f(x) \leq f(a), \text{ for any point } x \in Domf.$$

2. A **global minimum** if:

$$f(a) \leq f(x), \text{ for any point } x \in Domf.$$

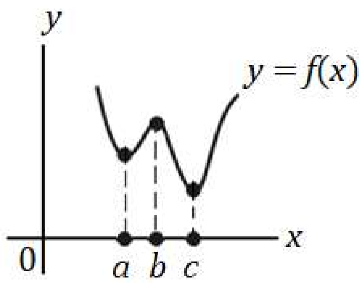
3. A **local maximum** if there exists an environment of $a \in Domf$ such that:

$$f(x) \leq f(a), \text{ for any point } x \in Domf \text{ of this environment.}$$

4. A **local maximum** if there exists an environment of $a \in Domf$ such that:

$$f(a) \leq f(x), \text{ for any point } x \in Domf \text{ of this environment.}$$

Graphically:




In this case the point $x = a$ is a local minimum whereas the point $x = b$ is a local maximum and the point $x = c$ is a global minimum.

In Economics we call **optimum** to any point that satisfies one of the above definitions.

[24] In fact there is a branch in economics called Economic Optimization whose aim is to determine the extreme points of economic functions of either one or more variables with or without additional constraints.

1.4.3.1. Relationship between Extreme Points and Derivatives

As we shall see below, the knowledge of the successive or high-order derivatives of a function will allow us to find their local extreme points.[25] By definition the **successive derivative of order $\boldsymbol{n} \in \mathbb{N}$** of the function $y = f(x)$ is given by the recurrent equality:

$$f^{(n)}(x) = \frac{d^n f(x)}{dx^n} = \{\text{Definition}\} = \frac{d\left(f^{(n-1)}\right)(x)}{dx}.$$

Considering this we can affirm:

Theorem: If at a critical point $a \in Domf$ of the differentiable function $y = f(x)$, the first non-zero derivative is an even derivative, i.e.:

$$f'(x) = \cdots = f^{(n-1)}(x) = 0 \text{ and } f^{(n)}(x) \neq 0, \text{ where } n > 1 \text{ is an even number}$$

then:

$$\left.\begin{matrix} f^{(n)}(x) < 0 \\ f^{(n)}(x) > 0 \end{matrix}\right\} \text{ implies that } a \in Domf \text{ is a local } \begin{Bmatrix} \text{maximum} \\ \text{minimum} \end{Bmatrix} \text{ of } y = f(x).$$ [26]

Example: Examine whether $y = f(x) = \frac{x^2}{1-x}$ has local extreme points.

SOLUTION:

The critical points of this function are $x = 0$ and $x = 2$. Indeed:

$$0 = f'(x) = \frac{df(x)}{dx} = \frac{x(2-x)}{(1-x)^2} \text{ implies: } x = 0 \text{ or } x = 2.$$

Since:

$$f^{(2)}(x) = \frac{d\left(f'(x)\right)}{dx} = \frac{2}{(1-x)^3} \text{ implies: } f^{(2)}(0) = 2 > 0 \text{ and } f^{(2)}(2) = -2 < 0$$

we conclude that $n = 2$ and, consequently:

$x = 0$ is a local minimum and $x = 2$ is a local maximum of $y = f(x)$ ■[27]

---

[25] Determining global extreme points can sometimes be quite complicated.
[26] If $n > 1$ is an odd number, we have not extreme points but inflection points.
[27] Sometimes the second derivative is not enough, and we have to resort to higher-order derivatives to be sure if we have an optimal point or not.

1.4.3.2. Application in Economics: Maximizing Profits

Example: A company dedicated to the commercial exploitation of a certain motorway charges a toll of €7 per vehicle. Considering that the staff costs are €6,000 per day and that maintenance and depreciation per car is given by the function:

$$€\left(2+\frac{x}{5{,}584}\right)$$

where the variable $x > 0$ denotes the number of vehicles circulating per day determine:

1. The cost function as well as the income and profit functions of the company per day.
2. The number of vehicles that maximize these profits as well as their value.
3. The range of profitability per day of the company.

SOLUTION:

(1) Since the income and cost functions depending on $x > 0$ are:

$$I(x) = 7 \cdot x \text{ and } C(x) = \left(2+\frac{x}{5{,}584}\right) \cdot x + 6{,}000$$

we deduce that the daily profits will be the function:

$$B(x) = I(x) - C(x) = -\frac{x^2}{5{,}584} + 5x - 6{,}000 \blacksquare$$

(2) Since:

$$0 = B'(x) = \frac{dB(x)}{dx} = -\frac{x}{2{,}792} + 5 \text{ implies: } x = 13{,}960 \text{ and } B''(x) = \frac{d^2B(x)}{dx^2} = -\frac{1}{2{,}792} < 0$$

13,960 vehicles must go through the motorway in order to maximize the daily profits. Note that the maximum value of this profits will be of $B(13{,}960) = €28{,}900 \blacksquare$[28]

(3) Obviously the range of daily profitability is given by the roots of the equation:

$$0 = B(x) = -\frac{x^2}{5{,}584} + 5x - 6{,}000 \text{ implies: } x_1 = 1{,}256.52 \text{ and } x_2 = 26{,}663.45.$$

Thus, the company will make a profit when a number of vehicles between 1,257 and 26,663 use the motorway■

[28] We define the **extreme value** (**maximum** or **minimum value**) of a function as the value that it gets at an extreme point.

1.4.4. *Elasticity of a Function*

In general, economists usually work with "relative" increments everywhere.[29] Thus, if we change the two "absolute" increments that appear in the difference quotient of the derivative:

$$\Delta a = x - a \text{ and } \Delta f(a) = f(x) - f(a) = f(a + \Delta a) - f(a)$$

by the corresponding relative increments:

$$\frac{\Delta a}{a} \text{ and } \frac{\Delta f(a)}{f(a)} = \frac{f(a + \Delta a) - f(a)}{f(a)}$$

we can define:

<u>Definition</u>: The **elasticity** of a differentiable function $y = f(x)$ at point $a \in Domf$, providing it exists, is the limit:

$$\epsilon_x f(a) = \lim_{\Delta a \to 0} \frac{\left(\frac{\Delta f(a)}{f(a)}\right)}{\left(\frac{\Delta a}{a}\right)} \in \mathbb{R}.$$

<u>Property</u>: In general:

$$\epsilon_x f(a) = \frac{a}{f(a)} \cdot \frac{df(a)}{dx}.$$

In Economics we have three types of elasticity:

- *Rigid elasticity* when $|\epsilon_x f(a)| < 1$
- *Elastic elasticity* when $|\epsilon_x f(a)| > 1$
- *Unitary elasticity* when $|\epsilon_x f(a)| = 1$.[30]

[29] Inflation is said to have risen by 0.5% or the interest rate has dropped by 1%.
[30] For instance, let $Q = f(P)$ be a demand function associated to an economic good. If small variations in its price $P$ cause large variations in the quantity demanded $Q$, we say that this product has an *elastic demand*. If the opposite, we have *rigid demand*. Finally, when the variations of $P$ and $Q$ are similar, we have an *unitary demand*.

1.4.4.1. Application in Economics: Price Elasticity of Demand

As a question of application, the elasticity of a function measures approximately the percentage in change when the independent variable increases by 1%.[31] All of this has important applications in Economics. Let us look at this example of application:

Example: If:

$$Q = f(P) = 650 - 5P - P^2$$

is the demand function of a certain commodity, calculate:

1. The elasticity at the price level of $P = €10$.
2. Approximately he rate of change on the demand if $P = €10$ increases by 2%.

SOLUTION:

(1) Since:

$$\frac{df(P)}{dP} = -5 - 2P$$

the elasticity $\epsilon_P f(10)$ will be of:

$$\epsilon_P f(10) = \frac{10}{f(10)} \cdot \frac{df(10)}{dP} = \frac{10}{650 - 5 \cdot 10 - 10^2} \cdot (-5 - 2 \cdot 10) = -0.5 \blacksquare$$ [32]

Thus, the demand function has rigid elasticity at price of $P = €10$.[33]

(2) Assuming the economic interpretation of the elasticity just exposed above, we deduce that the demand decreases by 0.5% when the commodity price $P = €10$ raises up to 1%.[34] Consequently, if this price raises up to 2%, i.e.:

$$2\% = 2 \cdot (1\%)$$

the demand decreases approximately by 1% since:

$$2 \cdot \left(\epsilon_P f(10)\right) = 2 \cdot (-0.5\%) = -1\% \blacksquare$$

---

[31] Obviously the more the percentage of change of the variable increases the poorer is the approximation.

[32] This number is a percentage.

[33] It is worth noting that goods related to basic needs usually have rigid demand.

[34] Observe that we have a negative elasticity at this price level.

## 1.5. Exercises

1. Calculate the tangent lines to the following functions at the points mentioned:

   (a) $(x) = \frac{e^x}{x-2}$, at $x = 0$. (b) $f(x) = \sin^2 x$, at $x = \pi$.

2. Study the increase as well as the existence of extreme points of the following functions:

   (a) $f(x) = 5 + x^3 - \left(\frac{1}{4}\right)x^4$. (b) $f(x) = x \cdot e^{-x^2}$. (c). $f(x) = \frac{x}{\ln x}$.

3. Find two positive numbers adding up to 21 and such that the product of one of them by the square of the other is maximum.
4. A bookstore receives a book from a publisher at a unit cost of €7 and sets the selling price of €15. At this price the bookstore has sold 1,000 books per month. In order to stimulate sales, the bookstore is going to reduce the selling price knowing that, for every euro of reduction in the price, it might sell 200 more books per month. Under these conditions, determine the number of books to sell each month, and its selling price, in order to maximize profits.
5. A firm has 20,000 m$^2$ of land to build industrial plants. In order to level off and prepare the land, it has to rent some machines with the following costs and restrictions:
   - The rental of each machine is €10,000 per hour.
   - Every machine level off and prepares 25 m$^2$ of land per hour.
   - To operate the machines 20 workers are required at a cost of €1,000 per hour and worker.
   - The cost of transporting each machine is €62,500.

   Under these conditions, calculate the number of machines to hire in order to minimize the total costs.[35]

[35] Hint: the number of hours that $x > 0$ machines are working is $h(x) = \frac{800}{x}$, and all workers are paid whether they are working or not.

SOLUTIONS:

1.

   a. $y = -0.75x - 0.5$

   b. $y = 0$

2.

   a. The function is increasing at $x < 3$ and decreasing at $x > 3$. Hence, $x = 3$ is a local maximum.[36]

   b. The function is increasing at $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and decreasing at $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. Hence, $x = -\frac{1}{\sqrt{2}}$ is a minimum and $x = \frac{1}{\sqrt{2}}$ is a maximum.

   c. The function is increasing at $x > e$ and decreasing at $x < e$. Hence, $x = e$ is a local minimum.

3. 14 and 7

4. The bookstore must sell 1,300 books in order to maximize benefits under a selling price of €13.5

5. 16 machines must be hired if we want to minimize costs.

[36] $x = 0$ is an inflection point.

**SECTION II: Algebra**

# 2. MATRICES AND SYSTEMS OF LINEAR EQUATIONS

## 2.1. Matrices and Determinant of a Square Matrix

### 2.1.1. *Matrix of Real Coefficients*

Numerical matrices (matrices for short) are the fundamental tool that allows us to solve systems of linear equations from a general point of view. We can say that:

Definition: A **matrix** of order $m \times n$ is an array of numbers, called **coefficients**, arranged in $m > 0$ rows and $n > 0$ columns:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

The coefficients $a_{ii}$ form the so-called **main diagonal** of $A$. If the number of rows equals that of the columns, i.e., $m = n$, the matrix $A$ is a **square matrix**.

Consider the following examples:

Example:

1. An example of matrix of order $2 \times 3$ would be:

$$A = \begin{pmatrix} 0 & -1 & 7 \\ 4 & 3 & -11 \end{pmatrix}.$$

2. A remarkable case of square matrix will be:

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \text{ where } \alpha \in \mathbb{R}.[37]$$

[37] From a geometric point of view, this matrix is associated with a rotation in the plane of angle $\alpha$ counterclockwise.

2.1.2. *Determinant of a Square Matrix*

A fundamental characteristic of any square matrix $A$ is its **determinant** denoted by:

$$\det(A) = |A|.$$ [38]

As for applications, we have interest here in determinants of order 2 and 3.[39] We can calculate all these determinants through the following rules:[40]

a. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}.$

b. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{21} \cdot a_{32} \cdot a_{13} - a_{13} \cdot a_{22} \cdot a_{31} -$

$- a_{21} \cdot a_{12} \cdot a_{33} - a_{32} \cdot a_{23} \cdot a_{11}.$

<u>Example</u>: Calculate the determinant of the following square matrices:

1. $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$, for any $\alpha \in \mathbb{R}$.
2. $A = \begin{pmatrix} 0 & -1 & 2 \\ 5 & 3 & 0 \\ -4 & 1 & -2 \end{pmatrix}$.

SOLUTION:

(1) Applying Sarrus's rule we have:

$$|A| = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} = \cos\alpha \cdot \cos\alpha - (-\sin\alpha) \cdot \sin\alpha = \cos^2\alpha + \sin^2\alpha = 1 \blacksquare$$ [41]

(2) In this case:

$$|A| = \begin{vmatrix} 0 & -1 & 2 \\ 5 & 3 & 0 \\ -4 & 1 & -2 \end{vmatrix} =$$

$$= 0 \cdot 3 \cdot (-2) + (-1) \cdot 0 \cdot (-4) + 5 \cdot 1 \cdot 2 -$$

$$-2 \cdot 3 \cdot (-4) - 5 \cdot (-1) \cdot (-2) - 1 \cdot 0 \cdot 0 = 24 \blacksquare$$

[38] The definition of determinant is a complex matter and we do not provide it here. In fact, you can find it in most of the linear algebra textbooks.
[39] It is worth noting that the calculation at hand of high-order determinants comes from general properties of the calculation of determinants of order 3.
[40] Called **Sarrus's Rules**.
[41] Note that the value of the determinant does not depend on the angle of rotation $\alpha$.

## 2.2. Rank of a Matrix

2.2.1. *Submatrix and Minor of a Matrix*

Another basic key of matrix theory that we have to bear in mind is the rank of a matrix. To deal with this, we must previously introduce the concepts of submatrix and minor of a matrix. By definition:

Definition: Let $A$ be a matrix of order $m \times n$.

1. A **submatrix** of $A$ is any matrix obtained from $A$ by deleting $0 \leq m' \leq m$ rows and $0 \leq n' \leq n$ columns.
2. A **minor** of $A$ is the determinant of any square submatrix of $A$.

Example:

1. Prove that $\begin{pmatrix} 0 & -1 \\ -4 & 1 \end{pmatrix}$ is a submatrix of $A = \begin{pmatrix} 0 & -1 & 2 \\ 5 & 3 & 0 \\ -4 & 1 & -2 \end{pmatrix}$.
2. Calculate the **principal minors** associated to the main diagonal of:

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 5 & 3 & 0 \\ -4 & 1 & -2 \end{pmatrix}.$$ [42]

SOLUTION: (1) It is easy to be aware of that since the second row and the third column from $A$ have been deleted■

(2) Obviously the principal minors of order 1 are the coefficients of the main diagonal:

$$0, 3 \text{ and } -2.$$

Those of order 2 will be:

$$\begin{vmatrix} 0 & -1 \\ 5 & 3 \end{vmatrix} = 5, \begin{vmatrix} 3 & 0 \\ 1 & -2 \end{vmatrix} = -6 \text{ and } \begin{vmatrix} 0 & 2 \\ -4 & -2 \end{vmatrix} = 8$$

and the sole minor of order 3 is the determinant of the matrix:

$$|A| = 24$$■[43]

[42] A **principal minor** of a matrix $A$ is a minor associated to a square submatrix whose diagonal coefficients match some of the diagonal coefficients of $A$.

[43] In particular, the whole matrix $A$ is an "improper" submatrix of itself obtained when deleting 0 rows and 0 columns.

2.2.2. *Rank of a Matrix*

In general:

Definition: The **rank** of a matrix $A$, denoted by rank$A$, is the greatest order of their non-zero associated minors.

Let us look at the following example of application:

Example: Calculate the rank of the following matrices:

$$(1)\ A=\begin{pmatrix}0 & -1 & 2\\ 5 & 3 & 0\\ -4 & 1 & -2\end{pmatrix}.\ (2)\ A=\begin{pmatrix}2 & 1 & -1\\ 2 & 0 & 5\\ 0 & -1 & 6\end{pmatrix}.\ (3)\ A=\begin{pmatrix}-1 & 0 & 3 & 2\\ 1 & 4 & 2 & -2\\ 0 & 4 & 5 & 1\end{pmatrix}.$$

SOLUTION:

(1) Since $|A|=24\neq 0$, the rank of the matrix is 3∎

(2) In this case the rank is 2 since:

$$|A|=\begin{vmatrix}2 & 1 & -1\\ 2 & 0 & 5\\ 0 & -1 & 6\end{vmatrix}=0 \text{ and } \begin{vmatrix}2 & 1\\ 2 & 0\end{vmatrix}=-2\neq 0∎$$

(3) Now the rank of $A$ may be at most 3 since it has $m=3$ rows. However, it is at least 2 because we have a non-zero minor of order 2:

$$\begin{vmatrix}-1 & 0\\ 1 & 4\end{vmatrix}=-4\neq 0.$$

Due to the fact that the determinant of the square submatrix formed by the first, the second and the fourth rows is different from zero:

$$\begin{vmatrix}-1 & 0 & 2\\ 1 & 4 & -2\\ 0 & 4 & 1\end{vmatrix}=-4\neq 0$$

we conclude that the rank of the matrix is equal to 3∎[44]

[44] Note that the determinant formed by the first three columns is 0.

### 2.3. Systems of Linear Equations

Definition: A **system of linear equations** (**system** from here onwards) is a set of equations of the form:

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n = b_2 \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n = b_m \end{cases}$$

where the numbers $a_{ij} \in \mathbb{R}$ are the **coefficients**, the variables $x_j$ are the **unknowns** and the numbers $b_i$ are the **constant terms**. In general:

1. Any set of $n$ numbers $x_1^0, x_2^0, \ldots, x_n^0$ satisfying the $m$ equations is said to be a **solution** of the system.
2. If a system has a solution it is said to be a **consistent**, and **inconsistent** otherwise.
3. If the solution of a consistent system is unique, we say the system is **independent**, and **dependent** otherwise.[45]

In order to determine whether a system has solutions or not, as well as their values in the case they exist, matrix theory plays an important role. Here we will only deal with systems of 2 or 3 equations.

Example: Check that $x_0 = 1,\ y_0 = 2$ and $z_0 = 3$ is a solution of the system:

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - z = 1 \\ x + y - 3z = -6 \end{cases}$$

SOLUTION: [46] It is clear since:

$$\left.\begin{array}{r} x_0 - 2y_0 + z_0 = 1 - 2 \cdot 2 + 3 = 0 \\ 2x_0 + y_0 - z_0 = 2 \cdot 1 + 2 - 3 = 1 \\ x_0 + y_0 - 3z_0 = 1 + 2 - 3 \cdot 3 = -6 \end{array}\right\} \blacksquare$$

[45] A consistent system has either a unique solution (independent system) or infinite solutions (dependent system).
[46] We will prove later that this system is independent which means that this solution is unique.

### 2.3.1. *Matrix and Augmented Matrix associated to a System*

The basic tool in the search of the solution of a system is the theorem of Rouché.[47] Before presenting it we need to consider the following concepts:

<u>Definition</u>: The **matrix** and the **augmented matrix** of a system:

$$\begin{cases} a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n = b_1 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 + \cdots + a_{2n} \cdot x_n = b_2 \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \cdots + a_{mn} \cdot x_n = b_m \end{cases}$$

are respectively the matrix of coefficients and this matrix "augmented" with the column of constant terms:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \text{ and } (A;B) = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}.$$ [48]

<u>Example</u>: Find the matrix and the augmented matrix of:

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - z = 1 \\ x + y - 3z = -6 \end{cases}$$

SOLUTION:

It is clear that the matrix of this system is:

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & -3 \end{pmatrix}$$

and thus, the augmented matrix will be:

$$(A;B) = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -3 & -6 \end{pmatrix} \blacksquare$$

[47] Also known as theorem of Rouché-Fröbenius.

[48] Note that the matrix of a system is always a submatrix of the corresponding augmented matrix. So, the rank of the augmented matrix is always greater than or equal to the associated matrix of the system.

2.3.2. *Discussing a System: Theorem of Rouché*

This theorem put in relation the concepts of rank of a matrix and consistent system.

<u>Theorem of Rouché</u>: A system of linear equations with associated matrix $A$ and augmented matrix $(A; B)$ with $n \in \mathbb{N}$ unknowns is:

1. Consistent if and only if $\mathrm{rank}A = \mathrm{rank}(A; B)$.[49]
2. Consistent and independent if and only if $\mathrm{rank}A = \mathrm{rank}(A; B) = n$.
3. Consistent and dependent if and only if $\mathrm{rank}A = \mathrm{rank}(A; B) < n$.[50]

Consider the following example:

<u>Example</u>: Prove that the system:

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - z = 1 \\ x + y - 3z = -6 \end{cases}$$

is consistent and independent.

SOLUTION: [51] Since the determinant of the matrix $A$ of this system is different to zero:

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & -3 \end{vmatrix} = -11 \neq 0$$

we deduce that:

$$\mathrm{rank}A = \mathrm{rank}(A; B) = 3.$$ [52]

Now applying the theorem of Rouché we can affirm that this system is a consistent and independent system because its rank equals the number of unknowns ■ [53]

---

[49] This value of coincidence is called the **rank** of the system. Thus, consistent systems are the only systems having rank.

[50] We call **degree of freedom** of a consistent system to the difference between the number of unknowns and the rank of the system. Note that the rank of a consistent system is always less than or equal to the number of its unknowns and that any consistent and independent system has a degree of freedom equal to 0.

[51] We have already seen that this systema is a consistent system because it has one solution, namely, $x_0 = 1,\ y_0 = 2$ and $z_0 = 3$.

[52] The rank of the augmented matrix is 3 since it has 3 rows, and the matrix of the system is one of their submatrices as we have just mentioned on a footnote of the last page.

[53] i.e., the solution $x_0 = 1,\ y_0 = 2$ and $z_0 = 3$ is unique.

2.3.3. *Solving a Consistent System*

The theorem of Rouché resolves the question of the existence of a solution (or solutions) of any system but does not tell us what it is (they are). Now we are going to study a systematic method associated to Cramer's systems that enable us to find the solutions of every consistent system.

Definition: A system is called a **Cramer's system** if it is a consistent and independent system with an associated square matrix.

Theorem: The sole solution of a Cramer's system with $n \in \mathbb{N}$ unknowns $x_1^0, x_2^0, \dots, x_n^0$ is equal to:

$$x_1^0 = \frac{|A_1|}{|A|}, x_2^0 = \frac{|A_1|}{|A|}, \dots, x_n^0 = \frac{|A_1|}{|A|}$$

where $A_i$ is the square matrix obtained from the matrix of the system $A$ by replacing its "i" column with the column of the constant terms.

Example: Calculate the solution of the Cramer's system:

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - z = 1 \\ x + y - 3z = -6 \end{cases}$$

SOLUTION: [54]

The sole solution of this system is:

$$x_0 = \frac{1}{|A|} \cdot |A_1| = \frac{1}{-11} \cdot \begin{vmatrix} 0 & -2 & 1 \\ 1 & 1 & -1 \\ -6 & 1 & -3 \end{vmatrix} = \frac{1}{-11} \cdot (-11) = 1$$

$$y_0 = \frac{1}{|A|} \cdot |A_2| = \frac{1}{-11} \cdot \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & -6 & -3 \end{vmatrix} = \frac{1}{-11} \cdot (-22) = 2$$

$$z_0 = \frac{1}{|A|} \cdot |A_3| = \frac{1}{-11} \cdot \begin{vmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & -6 \end{vmatrix} = \frac{1}{-11} \cdot (-33) = 3 \blacksquare$$

[54] From previous results this system is obviously a Cramer's system.

2.3.3.1. Cramer's Rule and Equivalent Systems

The basic idea that underlies the Cramer's rule consists in transforming any consistent and dependent system into an "equivalent" Cramer's system.[55] To this end we have first to consider the degree of freedom $k > 0$ of a consistent system with associated matrix $A$ and $n$ unknowns:

$$k = n - \text{rank}A \text{ implies: } \text{rank}A = n - k.$$

This means that a non-zero minor of degree $n - k > 0$ of the matrix $A$ must exist. The next step is to eliminate the equations of the system that do not form part of this minor and to consider further the $k > 0$ unknowns left outside as parameters.[56] The system thus formed is a Cramer's system with $n - k \geq 0$ unknowns and $k \geq 0$ parameters equivalent to the initial one.[57]

Example: Resolve if it is possible the system $\begin{cases} x - 2y + z = 0 \\ 2x + y - z = 1 \\ x + 3y - 2z = 1 \end{cases}$

SOLUTION: Since the ranks of the matrix and the augmented matrix are equal to 2 and it exists a minor of order 2 different to zero:[58]

$$\text{rank}A = \text{rank}\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{pmatrix} = \text{rank}\begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -1 & 1 \\ 1 & 3 & -2 & 1 \end{pmatrix} = 2 \text{ and } \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5 \neq 0$$

we deduce that the system obtained deleting the third equation and considering the variable $z = \alpha \in \mathbb{R}$ as a parameter:

$$\left.\begin{array}{l} 1 \cdot x + (-2) \cdot y = -\alpha \\ 2 \cdot x + 1 \cdot y = 1 + \alpha \end{array}\right\}$$

is a Cramer's system depending on a parameter $\alpha \in \mathbb{R}$. Thus, the solution (infinity of solutions rather) of this system and the initial one is:

$$x_0 = \frac{\begin{vmatrix} -\alpha & -2 \\ 1 + \alpha & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}} = \frac{\alpha + 2}{5}, y_0 = \frac{\begin{vmatrix} 1 & -\alpha \\ 2 & 1 + \alpha \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}} = \frac{3\alpha + 1}{5} \text{ and } z_0 = \alpha, \text{ for any } \alpha \in \mathbb{R} \blacksquare$$

[55] Two systems are called **equivalent** if they have the same solutions.
[56] Roughly speaking a *parameter* is a variable that acts as a constant.
[57] So the solutions of the system will depend on these parameters.
[58] Note that the third equation is equal to the difference between the first and the second.

2.3.3.2. Example of resolution of a System of Linear Equations with Parameters

Example: Resolve for any value of $a \in \mathbb{R}$ the system:

$$\left.\begin{array}{l} x+y=1 \\ ay+z=0 \\ x+(1+a)y+az=1+a \end{array}\right\}.$$

SOLUTION: First we calculate the determinant of the associated matrix:

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & a & 1 \\ 1 & 1+a & a \end{vmatrix} = a^2 - a = a(a-1).$$

Accordingly, we have a Cramer's system for any $a \neq 0$ and $a \neq 1$ with solution:[59]

$$x_0 = \frac{1}{a(a-1)} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & a & 1 \\ 1+a & 1+a & a \end{vmatrix} = \frac{1}{a(a-1)} \cdot a^2 = \frac{a}{a-1}$$

$$y_0 = \frac{1}{a(a-1)} \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1+a & a \end{vmatrix} = \frac{1}{a(a-1)} \cdot (-a) = \frac{-1}{a-1}$$

and:

$$z_0 = \frac{1}{a(a-1)} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 1 & 1+a & 1+a \end{vmatrix} = \frac{1}{a(a-1)} \cdot a^2 = \frac{a}{a-1} \blacksquare$$ [60]

On the other hand, if $a = 0$ we have the consistent system:

$$\left.\begin{array}{l} x+y=1 \\ z=0 \end{array}\right\}$$

with solutions:

$x_0 = \alpha, y_0 = 1 - \alpha$ and $z_0 = 0$, for any value of the parameter $\alpha \in \mathbb{R}\blacksquare$[61]

Finally in the case $a = 1$ it appears to be an inconsistent system since:

$$\text{rank}A = \text{rank}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 2 < 3 = \text{rank}(A;B) = \text{rank}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix} \blacksquare$$

[59] Roughly speaking we have an infinity of Cramer's systems for any value of $a \neq 0$ and $a \neq 1$ with a unique solution.

[60] Note that the solution depends on the value of the parameter $a \in \mathbb{R}$.

[61] Note that we have taken the variable $x$ as the parameter of solutions.

2.3.3.3. The Gauss-Jordan's Alternative Method of resolution of a System: an Example

Finally, we are going to explain through an example how this method works:

Example: Applying the Gauss-Jordan's method solve the system $\begin{cases} x+y=1 \\ 3y+z=0 \\ x+5y+3z=4 \end{cases}$

SOLUTION: This method begins considering the augmented matrix of the system:

$$(A;B)=\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 1 & 5 & 3 & 4 \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$$

together with a set of algebraic operations with the aim of obtaining a matrix in row-echelon form "equivalent" to $(A;B)$.[62] Since $a_{21}=0$, the first step to make consists in "changing" the coefficient $a_{31}=1$ by 0; to this end we can subtract the first row from the second obtaining the equivalent matrix to $(A;B)$:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 4 & 3 & 3 \end{pmatrix}.$$ [63]

Finally, we must put a zero at $a_{32}=4$ without "destroying" the zero $a_{21}=0$. To do this, we can subtract the second row multiplied by $\frac{4}{3}$ from the third row obtaining the equivalent matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 5/3 & 3 \end{pmatrix}.$$

This is the matrix in row-echelon form we are looking for. Observe that this matrix is the augmented matrix of the consistent and independent system equivalent to initial one:

$$\left.\begin{array}{l} x+y=1 \\ 3y+z=0 \\ (5/3)\cdot z=3 \end{array}\right\} \text{with solution: } x_0=\tfrac{8}{5}, y_0=-\tfrac{3}{5} \text{ and } z_0=\tfrac{9}{5}.$$

Accordingly, the initial system is a consistent and independent system with solution:

$$x_0=\frac{8}{5}, y_0=-\frac{3}{5} \text{ and } z_0=\frac{9}{5} \blacksquare$$

[62] Here a matrix in *row-echelon* form has all their coefficients $a_{ij}$ with $i>j$ equal to 0. We say a matrix in row-echelon equivalent to the augmented matrix $(A;B)$ in the sense that it is the augmented matrix of another system with the same solutions as the initial one.

[63] You can always divide a row by a number or subtract to any row another row multiplied by a number. These arithmetic operations are allowed.

2.3.3.4. Application in Economics: Exhaustion of Resources

Example: A craft maker uses three machines to make three toys A, B and C that sells for the unitary prices of €30, €40 and €70. If the time each machine needs to produce each toy as well as the global time available is shown in the table:

| Time | A | B | C | Total |
|---|---|---|---|---|
| Machine 1 | 1h | 2h | 3h | 80h |
| Machine 2 | 1h | 3h | 5h | 120h |
| Machine 3 | 2h | 5h | 8h | 200h |

find the production of A, B and C which, using up the global time, provides an income of €1,900.

SOLUTION: Note that if $x, y, z \geq 0$ are the quantities of A, B and C that uses up the global time it is necessary these quantities satisfy the system of linear equations:

$$\left.\begin{array}{l} 1 \cdot x + 2 \cdot y + 3 \cdot z = 80 \\ 1 \cdot x + 3 \cdot y + 5 \cdot z = 120 \\ 2 \cdot x + 5 \cdot y + 8 \cdot z = 200 \end{array}\right\}.$$

Since the third equation is the sum of the other two, it is a consistent and dependent system equivalent to Cramer's system:

$$\left.\begin{array}{l} x + 2y = 80 - 3\alpha \\ x + 3y = 120 - 5\alpha \end{array}\right\} \text{with } z = \alpha \in \mathbb{R} \text{ as a parameter.}$$

It is easy to see that the solutions in terms of economics are:

$$x_0 = \alpha, y_0 = 40 - 2\alpha \text{ and } z_0 = \alpha, \text{ where } 0 \leq \alpha \leq 40.$$

Finally due to the fact that the income of €1,900 must satisfy:

$$€1{,}900 = €30 \cdot x_0 + €40 \cdot y_0 + €70 \cdot z_0 = 20\alpha + 1{,}600$$

that implies:

$$\alpha = 15$$

we deduce that the craft maker must produce and sell 15 toys of type A and C, and 10 toys of type B in order to obtain an income of €1,900 exhausting the global time■

## 2.4. Exercises

1. Prove that:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)\cdot(c-a)\cdot(c-b) \text{ for any } a,b,c \in \mathbb{R}.$$[64]

2. Calculate the rank of the following matrices:

(a) $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$. (b) $A = \begin{pmatrix} -2 & 4 & 2 & -2 \\ -1 & -1 & 1 & 0 \\ -2 & 1 & 2 & -1 \end{pmatrix}$. (c) $A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 3 & 2 \\ 5 & 7 & 9 \\ 2 & 3 & 3 \end{pmatrix}$.

3. Prove that any homogeneous system is always consistent.[65] Is it always independent? Reason the answer.

4. Resolve, if possible, the following systems:

(a) $\begin{cases} x+y+z=1 \\ 3x-z=0 \\ x+2y+3z=1 \end{cases}$ (b) $\begin{cases} -x+y+z=0 \\ x-2y+2z=1 \\ 8x-7y+z=2 \end{cases}$ (c) $\begin{cases} x+y-mz=1 \\ -x+my+z=1 \\ x-mz=0 \end{cases}$, with $m \in \mathbb{R}$.

5. Consider an economy divided in three sectors: farming F, industry I and services S. To produce one unit of F, $\frac{1}{12}$ units of F, $\frac{1}{4}$ of I and $\frac{1}{6}$ of S is needed; for one unit of I we need $\frac{1}{4}$ units of F, $\frac{1}{4}$ of I and $\frac{1}{4}$ of S, and for one unit of S, $\frac{1}{3}$ units of F, $\frac{1}{4}$ of I and $\frac{1}{3}$ of S is required. Under these assumptions find the amount of units of the three sectors needed to cover exactly the internal and external demands of production of 165 units in each sector.[66]

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[64] This is an example of Vandermonde determinant.

[65] A system is **homogeneous** in the case all their constant terms are 0.

[66] This exercise is an example of Leontieff's *input-output* models. Hint: if $x > 0$ denotes the units of the first sector F, then $x = 165 +$ Internal demand. The internal demand of F comes from the requirements of $x > 0$ units of F, $y > 0$ units of I and $z > 0$ units of S. And so with the other two variables.

SOLUTIONS:

1.

2.

   a. $\text{rank}A = 3$.

   b. $\text{rank}A = 2$.

   c. $\text{rank}A = 3$.

3.

4.

   a. $(x_0, y_0, z_0) = \left(-\frac{1}{2}, 3, \frac{2}{3}\right)$.

   b. $(x_0, y_0, z_0) = \left(\frac{4\alpha-1}{3}, \frac{5\alpha-2}{3}, \alpha\right)$, being $\alpha \in \mathbb{R}$ a parameter.

   c. $(x_0, y_0, z_0) = \begin{cases} (m, 1,1), \text{ for } m \neq 1 \\ (\alpha, 1, \alpha), \text{ for } m = 1 \text{ being } \alpha \in \mathbb{R} \text{ a parameter.} \end{cases}$

5. 576 units of F, 620 units of I and 624 units of S are required.

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## GLOSSARY

Augmented matrix of a system, 29
Consistent system, 28
Cramer's system, 31
Cramer's rule, 32
Critical point of a function, 16
Cubic parabola, 6
Degree of freedom of a system, 30
Derivative of a function, 11
Determinant of a square matrix, 25
Differentiable function, 11
Domain of a function, 4
Elasticity of a function, 20
Equivalent systems, 32
Exponential function, 7
Extreme point of a function, 17
Extreme value of a function, 19
Function of one real variable, 4
Gauss's method, 34
Increase (decrease) of a function, 15
Independent system, 28
Hyperbola, 6
L'Hôpital's rules, 14
Logarithmic function, 8
Matrix of a system, 29
Matrix of real coefficients, 24
Maximum (minimum) of a function, 17
Minor of a matrix, 26
Parabola, 5
Principal minor of a matrix, 26
Rank of a matrix, 27
Sarrus's rules, 26
Solution of a system, 28
Square matrix, 24
Stationary point of a function, 16
Straight line, 5
Submatrix of a matrix, 26
System of linear equations, 28
Tangent of a function, 11
Theorem of Rouché, 30