UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques
i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

Lefschetz properties for monomial ideals

Autor: Pau Garcia Comas

Director: Dra. Rosa Maria Miró-Roig
Realitzat a: Departament
de Matemàtiques i Informàtica

Barcelona, 10 de juny de 2025

Contents

Abstract

This work is centred on studying a conjecture affirming the existence of monomial ideals satisfying the weak Lefschetz property for any number of generators. We prove the conjecture for three variables, and show that it also holds in any number of variables when the number of generators is big enough. We also show that for a higher number of variables, the problem can be reduced to a problem of the strong Lefschetz property in one less variable.

Aquest treball se centra en l'estudi d'una conjectura afirmant l'existència d'ideals monomials amb qualsevol nombre de generadors satisfent la *weak Lefschetz property*. Demostrem la conjectura en el cas de tres variables, i mostrem també que és verídica si el nombre de generadors és prou gran. Mostrem també que en més variables, el problema es pot reduïr a un altre sobre la *strong Lefschetz property* en una variable menys.

Notation: In this work K will denote a characteristic zero field, R the polynomial ring $K[x_1, \ldots, x_n]$, and $\mathfrak{m}$ the maximal ideal $(x_1, \cdots, x_n)$.

2020 Mathematics Subject Classification. 13A02, 16W50

Acknowledgements

I would like to begin this work by thanking my advisor *Dra. Rosa Maria Miró* for her continuous support during the making of this project, for introducing me to this area of study and encouraging me to learn more. Her insights and recommendations have been immensely valuable.

I would also like to thank my family and my friends for the immense support I have received throughout all of my studies.

Chapter 0

Introduction

0.1 Historic motivation

In 1924, Solomon Lefschetz formulated what is now known as the Hard Lefschetz Theorem, stating that the multiplication by a power of a hyperplane section induces isomorphisms between the cohomology groups of a non-singular, complex and irreducible projective variety. Richard P. Stanley used then this fact in 1980 to show that the multiplication maps:

$$\times L^{2d-t} : (R/(x_1^{a_1+1}, \ldots, x_n^{a_n+1}))_d \longrightarrow (R/(x_1^{a_1+1}, \ldots, x_n^{a_n+1}))_{t-d}$$

are isomorphisms for any d, $0 \leq d \leq \left[\frac{t}{2}\right]$. Here, $L = x_1 + \cdots + x_n$, $a_1, \ldots, a_n$ are natural numbers and $t = a_1 + \cdots + a_n$. This in now known as Stanley's theorem. Junzo Watanabe independently proved this fact in 1987 using representation theory. This theorem is now considered to be of great interest, and has since then received many other proofs by different authors.

This result inspired the definition of the strong and weak Lefschetz properties. A graded Artinian algebra A over a field K is said to satisfy the strong Lefschetz property (SLP) if there is an element $L \in A_1$ such that all the linear maps

$$\times L^d : A_t \longrightarrow A_{t+d}$$

have maximal rank for any choice of t and d. If we only demand rank maximality when $d = 1$, then A is said to have the weak Lefschetz property (WLP). The Lefschetz properties have become an area of great interest and have been found to have connections to many areas of mathematics including algebraic topology, combinatorics and algebraic geometry.

0.2 Structure of the work and methods

The work is divided in three chapters. In chapter one, we introduce graded rings and prove the most relevant results about graded algebras, including Hilbert's theorem asserting the existence of Hilbert polynomials and Hilbert's syzygy theorem about the existence of minimal free resolutions. In chapter two, we give an introduction to representation theory that will be used to prove Stanley's theorem in the next chapter.

In chapter 3, we define the Lefschetz properties and prove Stanley's theorem, which asserts the SLP for monomial complete intersections. The rest of the work is then focused on the study of this conjecture:

Conjecture 3.1.12. *Let $R = K[x_1, \cdots, x_n]$. For each $d \geq 1$ and μ with $n \leq \mu \leq \binom{n+d-1}{n-1}$ there exists an ideal $I \subseteq R$ generated by μ degree d monomials such that R/I is Artinian and satisfies the WLP.*

The cases in which $n = 2$ or $d = 2$ are solved. Also, when $\mu = n$, the result is a direct consequence of Stanley's theorem. In our work we will also show that the conjecture is true whenever $n = 3$, $\mu = n + 1$ or $\mu \geq \binom{d}{n} - 2$. Furthermore, we will show that if a similar conjecture about the SLP is true for some value of n, then conjecture 3.1.12 is true for $n + 1$. To do that, we first construct algebras in $n + 1$ variables by considering an algebra A in n variables, and extend it using a tensor product; $B = A \otimes_K K[y]/(y^n)$. We show that maps of multiplication by a lineal form in B behave in an analogous way to the maps of multiplication by the power of a linear form in A. This allows us to reduce the problem to algebras in one less variable, where we can computationally found algebras with the desired properties for small degrees. Then we show that if the WLP holds when the number of generators is big enough, it will be conserved when adding more.

Throughout the making of this work, we have used the programming language Macaulay2, both to obtain concrete findings and to seek evidence for statements that are later proven in the work.

The reader is expected to be familiar with the basics of commutative algebra, including Noetherian rings, modules and tensor products.

Chapter 1

Preliminaries

The aim of this chapter is to introduce basic concepts of graded rings and algebras, as well as some of their most important properties. We will also introduce monomial ideals, which will be the most important objects used throughout the rest of the work, and their most relevant properties which we will be continually using.

1.1 Graded rings

Definition 1.1.1. *A graded ring is a ring A such that its additive group can be written as $A = \bigoplus_{n=0}^{\infty} A_i$ such that $A_i A_j \subseteq A_{i+j}$. If A is also a K-algebra and every A_i is a K-vector space, we will say that A is a graded K-algebra.*

Let A be a graded ring. A homogeneous element of degree i is an element of A_i. Every element of A can be written as a sum $a = a_0 + a_1 + \cdots + a_m$, with every a_i homogeneous of degree i. We say that the a_i are the homogeneous components of a.

Example 1.1.2. $R = K[x_1, x_2, \dots, x_n]$ is a graded K-algebra, where $R_0 = K$ and R_d is the vector space of degree d homogeneous polynomials.

Definition 1.1.3. *An element $x \in A$ is a regular element if it is not a zero divisor. A sequence of elements $x_1, \dots, x_n$ is a regular sequence if x_i is a regular element of $A/(x_1, \dots, x_{i-1})$ for all i, and $(x_1, \dots, x_n) \neq A$. The depth of A is the maximal length of a regular sequence. It is denoted* $\operatorname{depth} A$.

Definition 1.1.4. *A graded Noetherian ring is Cohen-Macaulay if its depth equals its Krull dimension.*

Definition 1.1.5. *Let A be a graded ring. A graded A-module is an A-module M such that its additive group can be written as $M = \bigoplus_{j \in \mathbb{Z}} M_j$, and it satisfies $A_i M_j \subseteq M_{i+j}$. If A is a K-algebra we also demand every M_j to be a K-vector space.*

Definition 1.1.6. *Let M, N be graded A-modules. A morphism of graded A-modules is a morphism of A-modules $\varphi : M \longrightarrow N$ such that $\varphi(M_t) \subseteq N_t$ for all t.*

Definition 1.1.7. *Let M be a graded A-module. A submodule $M' \subseteq M$ is a graded submodule if for every $m' \in M'$, all the homogeneous components of m' belong to M'. In this case, M' is a graded module with the grading $M'_t = M' \cap M_t$.*
A homogeneous ideal is an ideal $I \subseteq A$ that is a graded submodule of A.

If M is a graded A-module and M' is a graded submodule, M/M' admits a graded A-module structure by setting $(M/M')_t$ to be the projection of M_t.

Example 1.1.8. Kernels and images of graded morphisms are graded submodules.

Definition 1.1.9. *Given a graded A-module M and an integer d, the graded shifted A-module $M(d)$ is defined as $(M(d))_t = M_{d+t}$.*

Definition 1.1.10. *Let M be a graded A-module. The annihilator of M is the set*

$$\operatorname{Ann}(M) = \{a \in A \mid am = 0 \textit{ for all } m \in M\}.$$

If $m \in M$, $\operatorname{Ann}(m) = \{a \in A \mid am = 0\}$.

Definition 1.1.11. *Let A be a graded K-algebra and let M be a graded A-module. The Hilbert function of M, $H(M, \_) : \mathbb{Z} \longrightarrow \mathbb{Z}$, is defined as $H(M, d) = \dim_K M_d$.*

Example 1.1.12. It is satisfied $H(R, d) = \binom{n+d-1}{n-1}$ for any $d \geq 0$.

Suppose M and N are graded R-modules. Then the tensor product $M \otimes_K N$ admits the grading

$$(M \otimes N)_t = \{\sum_i m_i \otimes n_i \mid m_i \in M_r, n_i \in N_s, r + s = t\}.$$

If A and B are graded K-algebras, the grading of $A \otimes B$ endows it with graded K-algebra structure.

Example 1.1.13. Suppose $I \subseteq R$ is a homogeneous ideal and $d \geq 1$. Then

$$R/I \otimes_K (K[y]/(y^d)) \cong R[y]/(I^e + (y^d)),$$

where I^e denotes the extension of I to $R[y]$. In essence, tensoring with $K[y]/(y^d)$ can be seen as adding y^d to the set of generators of the ideal.

1.2 Artinian algebras

Artinian algebras will be of utmost importance in the third chapter of this work. In this section, we define Artinian graded K-algebras and introduce their most important properties

Definition 1.2.1. *For a homogeneous ideal $I \subseteq R$, the algebraic variety $V(I)$ is the set of all points p of the projective space $\mathbb{P}^{n-1}$ such that, in homogeneous coordinates, $f(p) = 0$ for all $f \in I$.*

Definition 1.2.2. *Let A be a ring. We say that A is Artinian if satisfies the descending chain condition. That is, any descending chain of ideals of A is stationary. If I is an ideal of A, we say that I is Artinian if A/I is Artinian.*

Proposition 1.2.3. *Let $I \subseteq R$ be a homogeneous ideal. Then R/I is Artinian if and only if there is an integer d such that $H(R/I, t) = 0$ for all $t > d$.*

Proof. Write $A = R/I$. Define the ideals $J_k = \bigoplus_{i=k}^{\infty} A_i$. The chain $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ is descending and therefore stationary. Choose d such that $J_{d+1} = J_{d+2}$. Notice that $J_{d+1} = J_{d+2} \oplus A_{d+1}$, so $J_{d+1} = J_{d+2}$ implies $A_{d+1} = 0$. Thus, $A_t = 0$ for all $t > d$.
Conversely, suppose there is a d such that $H(A, t) = 0$ for all $t > d$. Then A is a finitely generated vector space of dimension $\sum_{k=0}^{d} H(A, k)$ and satisfies the descending chain condition for vector subspaces. As every ideal of A is also a vector subspace, A is Artinian. □

Remark 1.2.4. Other conditions equivalent to R/I being Artinian are the Krull dimension of R/I being 0, and if K is algebraically closed, $V(I) = \emptyset$.

Example 1.2.5. Consider the ideal $I = (x^3, y^3, z^3) \subseteq K[x, y, z]$. Every monomial of degree seven or more is multiple of at least one among x^3, y^3 and z^3. Therefore $(R/I)_t = 0$ for all $t \geq 7$ and R/I is Artinian.

Definition 1.2.6. *Let R/I be an Artinian graded K-algebra.*

1. *The maximum integer d such that $H(R/I, d) \neq 0$ is the socle degree of R/I.*
2. *The h-vector of R/I is the sequence $(h_0, h_1, \cdots, h_d)$ where $h_t = H(R/I, t)$ and d is the socle degree of R/I.*

Definition 1.2.7. *We say that an h-vector* $h = (h_0, h_1, \cdots, h_d)$ *of a graded Artinian K-algebra is unimodal if for some k,* $h_0 \leq h_1 \cdots \leq h_k \geq h_{k+1} \geq \cdots \geq h_d$ *(i.e. it does not increase again after decreasing once). We say that that h has a peak at degree i if* $h_{i-1} \leq h_i \geq h_{i+1}$.
We say that h is symmetric if $h_i = h_{d-i}$ *for all* $i \leq d$.

Example 1.2.8. The h-vector of the algebra $K[x,y,z]/(x^3, y^3, z^3)$ is $(1,3,6,7,6,3,1)$, which is symmetric and unimodal finding a unique peak at degree 3.

Example 1.2.9. Let $R = K[x_1, x_2, x_3, \ldots, x_{10}, a, b, c]$ and $S = K[y_1, y_2, y_3, \ldots, y_{10}, u, v, w]$. Then R has S-module structure by partial differentiation. That is, for a $g \in S$ and an F in R, the product gF is calculated by considering $y_i = \partial_{x_i}$, $u = \partial_a$, $v = \partial_b$, $w = \partial_c$.
Let $T = \{m_1, \ldots, m_{10}\}$ be the set of degree 3 monomials in the variables a, b, c and $F = \sum_{i=1}^{10} x_i m_i$. Define the ideal of S, $I = \mathrm{Ann}_S F = \{g \in S \mid gF = 0\}$. Then S/I is Artinian and its h-vector is

$$(1, 13, 12, 13, 1)$$

which is non-unimodal. See [4, Proposition 3.3] for further details.

Let A be a Cohen-Macaulay K-algebra. Let $(L_1, L_2, \cdots, L_d)$ be a regular sequence of maximal length of linear forms. Notice that the Krull dimension of $A/(L_1, L_2, \cdots, L_d)$ is 0, and therefore it is an Artinian ring. By definition,

$$A/(L_1, L_2, \ldots, L_d)$$

is the Artinian reduction of A.

Example 1.2.10. Let $X \subseteq \mathbb{P}^n$ be a finite set of points and $A = R/I_X$. The Artinian reduction of A is $R/(I_X, L)$, where L is a linear form dodging every point. In the particular case where X is $\{[1:0:0], [0:1:0], [0:0:1]\}$, $I_X = (xy, xz, yz)$ and L can be taken to be $x + y + z$.

Definition 1.2.11. *For an Artinian graded algebra* R/I*, the socle of* R/I *is the ideal*

$$\mathrm{Soc(R/I)} = 0 : \mathfrak{m} = \{f \in (R/I) \mid x_i f = 0 \textit{ for all } 1 \leq i \leq n\}.$$

Notice that if the socle degree of R/I is d, then$(R/I)_d \subseteq \mathrm{Soc}(R/I)$. If $(R/I)_d = \mathrm{Soc}(R/I)$ we will say that R/I is level. If R/I is level and $\dim(R/I)_d = 1$, we say that R/I is Gorenstein.

Example 1.2.12. Let $I = (f_1, \ldots, f_n)$ be a homogeneous Artinian complete intersection (that is, an Artinian ideal generated by a regular sequence). Then, R/I is Gorenstein.

If $R = K[x, y]$, and R/I is Gorenstein, then I is generated by a regular sequence. In general, Gorenstein ideals in more than two variables don't have to be complete intersections. The ideal given in example 1.2.9 is an example of a Gorenstein ideal that is not a complete intersection.

1.3 Hilbert polynomials

The aim of this section is to prove one of the most important and well known results in the theory of graded algebras, which is that the Hilbert function of a graded module can be described by a polynomial.

Definition 1.3.1. *A function $F : \mathbb{Z} \longrightarrow \mathbb{Z}$ is of polynomial type of degree d if there is a degree d polynomial $P \in \mathbb{Q}[x]$ and an integer $n_0 \in \mathbb{Z}$ such that for all $n > n_0$, $F(n) = P(n)$.*

Remark 1.3.2. Two polynomials can coincide in at most a finite amount of values. Therefore the polynomial P from the previous definition is unique.

Definition 1.3.3. *For a function $F : \mathbb{Z} \longrightarrow \mathbb{Z}$ we define the function ΔF as $\Delta F(n) = F(n+1) - F(n)$ for all $n \in \mathbb{Z}$.*

Functions of polynomial type can be characterized in terms of successive applications of the operator Δ.

Lemma 1.3.4. *Given a function $F : \mathbb{Z} \longrightarrow \mathbb{Z}$, the following are equivalent:*

1. *F is of polynomial type of degree d.*
2. *$\Delta^d F(n)$ is constant and non-zero for sufficiently large n.*

Proof. See [3, Lemma 4.1.2]. □

Lemma 1.3.5. *Let M be a finitely generated graded R-module. There exists a chain of graded submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ such that for all i, $0 \leq i \leq r-1$, $M_{i+1}/M_i \cong R/\mathfrak{p}_i(-l_i)$, where each $\mathfrak{p}_i$ is a homogeneous prime ideal of R and $l_i \in \mathbb{Z}$.*

Proof. Notice that as R is Noetherian and M is a finitely generated R-module, M is Noetherian. Let Φ be the set of graded submodules of M that admit such a chain of submodules and let M' be a maximal element of Φ. Define $M'' = M/M'$. If $M'' = 0$ we are done. Otherwise, M'' is also Noetherian and therefore there exists a homogeneous element $m \in M''$ such that $\mathfrak{p} := \operatorname{Ann}(m)$ is a maximal element of

$\{\mathrm{Ann}(m'') \mid m'' \in M''$ is a non-zero homogeneous element$\}$. We claim that $\mathfrak{p}$ is a prime ideal.

Let $a, b \in R$ such that $ab \in \mathfrak{p}$. Suppose that $b \notin \mathfrak{p}$. Expressing a, b as the sum of their homogeneous components, we can suppose that a and b are homogeneous. As $b \notin \mathfrak{p}$, $bm \neq 0$. Certainly, $\mathfrak{p} \subseteq \mathrm{Ann}(bm)$. By maximality, $\mathrm{Ann}(bm) = \mathfrak{p}$. As $abm = 0$, $a \in \mathrm{Ann}(bm)$ and therefore $a \in \mathfrak{p}$. This shows that $\mathfrak{p}$ is a prime ideal.

As $\mathfrak{p} = \mathrm{Ann}(m)$, $Rm \cong R/\mathfrak{p}(-l)$, where $l = \deg m$. Let N be the preimage of Rm in M. Then $N \supsetneq M'$ and N also admits a chain of submodules like in the statement of the lemma. If $0 = M'_0 \subseteq M_1 \subseteq \cdots \subseteq M'_r = M'$ is a chain for M', $0 = M'_0 \subseteq M_1 \subseteq \cdots \subseteq M'_r = M' \subseteq N$ is one for N, as $N/M' \cong R/\mathfrak{p}(-l)$. This contradicts the maximality of M' in Φ, and therefore $M/M' = 0$. □

Theorem 1.3.6 (Hilbert). *Suppose K is algebraically closed. Let M be a finitely generated graded R-module and $d = \dim V(\mathrm{Ann}(M))$. The Hilbert function $H(M, \_) : \mathbb{Z} \longrightarrow \mathbb{Z}$ is of polynomial type of degree d.*

Proof. We proceed by induction on d. Consider a chain $0 = M_0 \subseteq M_1 \subseteq \ldots M_r = M$ with $M_{i+1}/M_i \cong R/\mathfrak{p}_i(-l_i)$ given by the previous lemma. Then $H(M, n) = \sum\limits_{i=0}^{r-1} H(R/\mathfrak{p}_i, n - l_i)$. Notice that $\mathrm{Ann}(M) = \mathfrak{p}_0\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1}$, so that $V(\mathrm{Ann}(M)) = \bigcup\limits_{i=0}^{r-1} V(\mathfrak{p}_i)$ and therefore $d = \max\limits_{0 \leq i \leq r-1} (\dim V(\mathfrak{p}_i))$. Thus it is enough to prove it when $M = R/\mathfrak{p}$ with $\mathfrak{p}$ a homogeneous prime ideal. It is important to notice that the leading coefficient of the polynomial will be positive as the Hilbert function will be positive for sufficiently large n.

If $d = -1$, then $\mathfrak{p} = \mathfrak{m}$ and therefore $H(R/\mathfrak{p}, n) = 0$ for all $n > 0$.

Suppose now that $V(\mathfrak{p})$ has dimension $d \geq 0$, and that every graded R-module M' with $\dim V(\mathrm{Ann}(M')) = d - 1$ has Hilbert function of polynomial type of degree $d - 1$. Let $L \in R_1$ be a linear form not in $\mathfrak{p}$. Then $V(\mathfrak{p} + (L))$ is of dimension $d - 1$ and by induction hypothesis $H(R/(\mathfrak{p}, L), n)$ is of polynomial type of degree $d - 1$. The exact sequence

$$0 \longrightarrow (R/\mathfrak{p})_n \xrightarrow{\times L} (R/\mathfrak{p})_{n+1} \longrightarrow ((R/(\mathfrak{p}, L))_{n+1} \longrightarrow 0$$

implies $H(R/\mathfrak{p}, n+1) = H(R/\mathfrak{p}, n) + H(R/(\mathfrak{p}, L), n+1)$, or equivalently, $\Delta H(R/\mathfrak{p}, n) = H(R/(\mathfrak{p}, x), n+1)$. We apply the operator Δ^{d-1} to both sides of the equation to obtain $\Delta^d H(R/\mathfrak{p}, n) = \Delta^{d-1} H(R/(\mathfrak{p}, x), n+1)$. $\Delta^{d-1} H(R/(\mathfrak{p}, x), n+1)$ is constant and non-zero for sufficiently large n, therefore $\Delta^d H(R/\mathfrak{p}, n)$ is too. We can conclude that $H(R/\mathfrak{p}, \_)$ is of polynomial type of degree $d - 1$, as we wanted to see. □

Definition 1.3.7. *The only polynomial P such that $P(n) = H(M,n)$ for large enough n is called the Hilbert polynomial of M. If $X \subseteq \mathbb{P}^n$ is an algebraic variety, the Hilbert polynomial of X is the Hilbert polynomial of R/I_X.*

Remark 1.3.8. The existence of Hilbert polynomials is also true for fields that are not algebraically closed, but it is not true in general that its degree is the dimension of $V(M)$. For example, consider $K = \mathbb{R}$ and $I = (x^2+y^2+z^2) \subseteq R = \mathbb{R}[x,y,z]$. Then $V(I) = \emptyset$, but $H(R/I,t) = 2t+1$ for all $t \geq 0$. A method to prove the existence of Hilbert polynomials in any field will be discussed in the next section.

Example 1.3.9. Let $X \subseteq \mathbb{P}^n$ be a set of three different points. As $\dim X = 0$, we expect the Hilbert polynomial of X to be a constant, which we will show to be 3 regardless of their relative position. However, the first values of the Hilbert function will depend on whether the points are aligned.
Consider first the case where the three points are not aligned. After a change of coordinates we can suppose $X = \{[1:0:0],[0:1:0],[0:0:1]\}$ so that $I_X = (xy,xz,yz)$. Notice that any monomial that is not the power of a variable is 0 in R/I_X, so $\{x^d,y^d,z^d\}$ is a basis of $(R/I_X)_d$ for all $d \geq 1$. In particular $H(R/I_X,0) = 1$ and $H(R/I_X,d) = 3$ for all $d \geq 1$ and the Hilbert polynomial of X is 3.
Suppose now that the three points are aligned. After suitably changing coordinates, $X = \{[1:0:0],[0:1:0],[1:1:0]\}$ and $I_X = (xy(x-y),z)$. The set $\{x,y\}$ is a basis of $(R/I_X)_1$, and $\{x^2,xy,y^2\}$ is a basis of $(R/I)_2$. For degree greater than two, we have the relation $xy^2 = x^2y$, so any monomial of degree $d \geq 3$ divisible by xy is equal to xy^{d-1}. We conclude that $\{x^d,xy^{d-1},y^d\}$ is a basis of $(R/I)_d$ and the Hilbert function of R/I_X is $(1,2,3,3,3,\cdots)$. In particular, the Hilbert polynomial of X is 3.

1.4 Free resolutions

Free resolutions are a very powerful tool for the study of graded modules. In this section we will define minimal free resolutions and show how they can be used to determine the Hilbert function of any finitely generated module. Recall that for a ring A, an exact sequence of A-modules is a chain of A-modules $M_0,\dots M_k$ and morphisms $f_i : M_i \longrightarrow M_{i+1}$

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \longrightarrow \cdots \longrightarrow M_{k-2} \xrightarrow{f_{k-2}} M_{k-1} \xrightarrow{f_{k-1}} M_k$$

such that $\mathrm{Im} f_i = \ker f_{i+1}$ for all $i \in \{0,1,\cdots,k-1\}$. Notice that if $M_0 = 0$, f_1 is injective, while if $M_k = 0$, f_{k-2} is surjective.

Let M be a finitely generated A-module. A free resolution of M is an exact sequence

$$0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where every F_i is a free A-module. If A and M are graded, we will also demand that every F_i and φ_i is graded.

A free resolution of an R-module is minimal if $\varphi_0 \otimes id_{R/\mathfrak{m}}$ is an isomorphism and $\varphi_i \otimes id_{R/\mathfrak{m}} = 0$ for all $i = 1, \dots, s$. This last condition equates to, after fixing bases for F_i and F_{i-1}, all entries of the matrix of φ_i belong to $\mathfrak{m}$.

After finding a free resolution of an R-module, we can determine its entire Hilbert function using the following proposition:

Proposition 1.4.1. *Let $k \geq 2$. For any exact sequence of finite dimensional K-vector spaces,*

$$0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \longrightarrow \cdots \longrightarrow E_{k-1} \xrightarrow{f_{k-1}} E_k \xrightarrow{f_k} 0$$

we have $\sum\limits_{i=1}^{k} (-1)^i \dim E_i = 0$.

Proof. For any $i \in \{1, \dots, k\}$ we have $\dim E_i = \dim \ker f_i + \dim \mathrm{Im} f_i = \dim \mathrm{Im} f_{i-1} + \dim \mathrm{Im} f_i$. Therefore the sum $\sum\limits_{i=1}^{k} (-1)^i \dim E_i = \sum\limits_{i=1}^{k} (-1)^i (\dim \mathrm{Im} f_{i-1} + \dim \mathrm{Im} f_i)$ is telescoping and we are left with just $\dim \mathrm{Im} f_0 + (-1)^k \dim \mathrm{Im} f_k$. Both these dimensions are zero. ☐

Example 1.4.2. Let $R = K[x, y, z]$ and let $I = (x^3, y^3, z^3)$. We want to find a minimal free resolution of R/I. We have the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$, but I is not a free R-module. Let e_1, e_2, e_3 be a basis of R^3 with each e_i homogeneous of degree 0. I is generated by the three elements x^3, y^3, z^3, so the image of the R-module morphism $\varphi_1 : R^3 \longrightarrow R$ determined by $\varphi_1(e_1) = x^3, \varphi_1(e_2) = y^3, \varphi_1(e_3) = z^3$; is I. In order for φ_1 to be a graded morphism, we define it on $R(-3)^3$.

We have then the exact sequence

$$0 \longrightarrow \ker \varphi_1 \longrightarrow R(-3)^3 \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0.$$

The kernel of φ_1 is $\langle y^3 e_1 - x^3 e_2, z^3 e_1 - x^3 e_3, z^3 e_2 - y^3 e_3 \rangle$, therefore we consider the graded morphism $\varphi_2 : R(-6)^3 \longrightarrow R(-3)^3$ determined by $\varphi_2(e_1) = y^3 e_1 - x^3 e_2, \varphi_2(e_2) = z^3 e_1 - x^3 e_3, \varphi_2(e_3) = z^3 e_2 - y^3 e_3$. The kernel of φ_2 is generated by the element $z^3 e_1 - y^3 e_2 + x^3 e_3$, therefore if $\varphi_3 : R(-9) \longrightarrow R(-6)^3$ is the only

R-module morphism such that $\varphi_3(1) = z^3e_1 - y^3e_2 + x^3e_3$, we obtain the exact sequence

$$0 \longrightarrow R(-9)^3 \xrightarrow{\varphi_3} R(-6)^3 \xrightarrow{\varphi_2} R(-3)^3 \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} R/I \longrightarrow 0$$

This is the minimal free resolution of R/I.
Every free resolution completely determines the Hilbert function $H(M,\_)$ as well as the Hilbert polynomial.
We have $H(R(-d),t) = \binom{2+t-d}{2}$, where we have to use the convention $\binom{a}{b} = 0$ if $a < b$. The Hilbert function of R/I is then

$$H(R/I,t) = \binom{2+t}{2} - 3\binom{-1+t}{2} + 3\binom{-4+t}{2} - \binom{-7+t}{2}.$$

Writing the binomials in their polynomial form we obtain $\binom{2+t}{2} - 3\binom{-1+t}{2} + 3\binom{-4+t}{2} - \binom{-7+t}{2} = 0$, therefore $H(R/I,t) = 0$ for all $t \geq 7$. For $t < 7$, the h-vector can be fully calculated from the previous formula and found to be $(1,3,6,7,6,3,1)$.

Free resolutions exist for any graded R-module.

Theorem 1.4.3 (Hilbert syzygy theorem)**.** *For every exact sequence of graded R-modules*

$$0 \longrightarrow E \longrightarrow \bigoplus_{1\leq j\leq a_{n-1}} R(-d_{n-1,j}) \longrightarrow \cdots$$

$$\longrightarrow \bigoplus_{1\leq j\leq a_1} R(-d_{1,j}) \longrightarrow \bigoplus_{1\leq j\leq a_0} R(-d_{0,j}) \longrightarrow M \longrightarrow 0$$

there exist integers a_n and $d_{n,j}$ with $1 \leq j \leq a_n$ such that $E = \bigoplus_{1\leq j\leq a_n} R(-d_{n,j})$.

Proof. We proceed by induction on the number of variables n. If $n = 0$ then M is a finitely generated K-vector space and $E \to M$ is a graded isomorphism. Let $(e_1, \cdots, e_m)$ be a basis of homogeneous elements of M and $d_i = \deg e_i$. Then $E = \bigoplus_{1\leq i\leq m} (Ke_i) \cong \bigoplus_{1\leq i\leq m} K(-d_i)$.
Suppose $n > 0$. Let $R' = R/(x_n)$ and $N = \ker(\bigoplus_{1\leq j\leq a_0} R(-d_{0,j}) \to M)$. Consider the following commutative diagram of exact rows and columns:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E(-1) \arrow[r] \arrow[d] & \bigoplus\limits_{1\leq j\leq a_{n-1}} R(-d_{n-1,j}-1) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus\limits_{1\leq j\leq a_{1}} R(-d_{1,j}-1) \arrow[r] \arrow[d] & N(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & \bigoplus\limits_{1\leq j\leq a_{n-1}} R(-d_{n-1,j}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus\limits_{1\leq j\leq a_{1}} R(-d_{1,j}) \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E/x_nE \arrow[r] \arrow[d] & \bigoplus\limits_{1\leq j\leq a_{n-1}} R'(-d_{n-1,j}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \bigoplus\limits_{1\leq j\leq a_{1}} R'(-d_{1,j}) \arrow[r] \arrow[d] & N/x_nN \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

where the morphisms from the first row to the second are multiplication by x_n. By induction hypothesis, $E/x_nE = \bigoplus\limits_{1\leq j\leq m} R'(-\ell_j)$ for some integers ℓ_j. We want to check that $E = \bigoplus\limits_{1\leq j\leq m} R(-\ell_j)$. Choose homogeneous elements $z_1, \dots, z_m \in E$ with $\deg z_i = \ell_i$ such that their projections to E/x_nE form a basis. Let E' be the submodule of E generated by $z_1, \dots, z_n$. Notice $E = E' + x_nE$. Let us see this implies $E' = E$.

We have $x_n(E/E') = E/E'$. Suppose that $E/E' \neq 0$ and let t be the minimum integer such that $(E/E')_t \neq 0$. Then $t' = t + 1$ is the minimum integer such that $(x_n(E/E'))_{t'} \neq 0$, which is a contradiction, thus $E/E' = 0$. This proves that $E =< z_1, \dots, z_n >$. It remains to see that $z_1, \dots, z_m$ are linearly independent.

Suppose that exist $a_1, \dots, a_m \in R$ not all zero and such that $\sum\limits_{1\leq i\leq m} a_i z_i = 0$. We can suppose, dividing the relation by any common divisor, that $\gcd(a_1, \dots, a_m) = 1$. Projecting to E/x_nE, we obtain $\sum\limits_{1\leq i\leq m} \overline{a_i z_i} = 0$ and therefore a_i must be zero on R'. Equivalently, $x_n | a_i$ for all i. This contradicts $\gcd(a_1, \dots, a_m) = 1$ and therefore $z_1, \dots, z_n$ are linearly independent. ☐

Example 1.4.4. We also calculate the minimal free resolution of the coordinate ring of a set of three points in a projective plane. First suppose that the three points are not aligned so that the coordinate ring is R/I with $I = (xy, xz, yz)$ after a suitable coordinate change. The image of the graded R-module morphism $\varphi_1 : R(-2)^3 \longrightarrow R$ determined by $\varphi_1(e_1) = xy, \varphi_1(e_2) = xz, \varphi_1(e_3) = yz$ is I. The kernel of φ_1 is $< -ze_1 + ye_2, -ye_2 + xe_3 >$. This is a free graded submodule of $R(-2)^3$, so it is isomorphic to $R(-3)^2$. We conclude that the minimal free resolution of R/I takes the form

$$0 \longrightarrow R(-3)^2 \longrightarrow R(-2)^3 \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

This gives the Hilbert function $H(R/I,t) = \binom{t+2}{2} - 3\binom{t}{2} + 2\binom{t-1}{2}$. Rewriting the binomials as polynomials, we obtain that $H(R/I,t) = 3$ for all $t \geq 1$.
If the three points are aligned we can suppose the ideal is $I = (xy(x-y), z)$. Firstly we consider the morphism $\varphi_1 : R(-3) \oplus R(-1) \longrightarrow R$ given by $\varphi_1(p,q) = xy(x-y)p + qz$. Setting the basis $e_1 = (1,0)$ and $e_2 = (0,1)$, the kernel of φ_1 is $< ze_1 - xy(z-y)e_2 >$. Notice that $\deg e_1 = 3$ and $\deg e_2 = 1$, so $\ker \varphi_2$ is a rank 1 graded free R-module generated by a degree 4 element, and therefore isomorphic to $R(-4)$. We conclude the minimal free resolution of R/I is

$$0 \longrightarrow R(-4) \longrightarrow R(-3) \oplus R(-1) \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

This gives the Hilbert function $H(R/I,t) = \binom{t+2}{2} - (\binom{t+1}{2} + \binom{t-1}{2}) + \binom{t-2}{2}$. We can see that it is 3 for all $t \geq 2$ and $H(R/I,1) = 2$, as expected.

Remark 1.4.5. Following a similar procedure as in these examples, we can determine the entire Hilbert function of any finitely generated module using a free resolution, which will always be a linear combination of binomials with integer coefficients. As binomials are polynomials, a free resolution determines the Hilbert polynomial of a module. We can conclude that the existence of Hilbert polynomials is a corollary of Hilbert's syzygy theorem, and does not depend on the field being algebraically closed. However, using this method does not show the equivalence between the degree of the polynomial and the dimension of the algebraic variety $V(\mathrm{Ann}(M))$ in an obvious manner.

Remark 1.4.6. The property of an algebra being Cohen-Macaulay, Gorenstein, or a complete intersection, can be characterized in terms of its minimal free resolution. Suppose the minimal resolution of R/I is

$$0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} R/I \longrightarrow 0,$$

With $F_s \neq 0$. Then

- If R/I is of depth d, it is Cohen-Macaulay if and only if $d = n - s$.
- R/I is Gorenstein if and only if it is Cohen-Macaulay and the rank of F_s is one.

- I is generated by a regular sequence if and only if the rank of F_1 is the codimension of $V(I)$ in $\mathbb{P}^{n-1}$. Then, if $I = (f_1, \dots, f_n)$ with $\deg f_i = d_i$ its minimal free resolution is given by

  $$F_k = \bigoplus_{1 \le k_1 \le \cdots \le k_i} R(-d_{k_1} - \cdots - -d_{k_i})$$

  See [6, Theorem 1.1].

- R/I is level if and only if it is Cohen-Macaulay and all the summands of F_s have the same twist.

From this, we can see that any complete intersection is Gorenstein, and any Gorenstein algebra is level. Also, if $R = K[x,y]$ and I is a homogeneous ideal such that R/I is Gorenstein and artinian, the minimmal free resolution of R/I is given by

$$0 \longrightarrow R(-d) \longrightarrow F_1 \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

For some positive d. Then F_1 must be of rank two, as the alternating sum of the ranks must be zero. Therefore I is generated by a regular sequence.

1.5 Monomial ideals

Definition 1.5.1. *An ideal $I \subseteq R$ is a monomial ideal if it can be generated by monomials.*

Definition 1.5.2. *For a polynomial f, we denote by* Supp$f$ *the set of monomials with non-zero coefficients in f.*

Proposition 1.5.3. *Let $I \subseteq R$ be a monomial ideal. For a polynomial f of R, $f \in I$ if and only if* Supp$f \subseteq I$.

Proof. Suppose $I = (m_1, \cdots, m_k)$ where each m_i is a monomial. For some polynomials $f_1, \dots, f_k \in R$, $f = \sum_{i=1}^{k} m_i f_i$. Writing the polynomials f_i as a sum of monomials, we obtain that all the monomials in the support of f are multiples of some m_i. Conversely, if $f = \sum_{i=1}^{k} a_i m_i$ where every m_i is a monomial in I and $a_i \in K$, $f \in I$. □

Proposition 1.5.4. *Let $I \subseteq R$ be a monomial ideal. The set of non-empty degree d monomials is a basis of $(R/I)_d$.*

Proof. Let $\mathcal{B}$ be the set of degree d monomials that are non-zero on R/I. As every degree d homogeneous polynomial is a linear combination of degree d monomials, $\mathcal{B}$ generates $(R/I)_d$. We also need to see it is a linearly independent set. Suppose it is not. Let f be a linear combination of elements of $\mathcal{B}$ that is zero on $(R/I)_d$. By proposition 5.3, $\mathrm{Supp} f \subseteq I$, but as $\mathrm{Supp} f \subseteq \mathcal{B}$, $\mathrm{Supp} f = \emptyset$ and so $f = 0$. $\square$

The n-torus $(K^*)^n$ acts on R by $(a_1, \cdots, a_n) \cdot f(x_1, \cdots, x_n) = f(a_1 x_1, \cdots, a_n x_n)$. Notice that monomial ideals are invariant under this action. Therefore, if I is a monomial ideal, the action is well defined on R/I.

Chapter 2

Representation theory

The aim of this chapter is to introduce Lie algebras and the basic notions of representation theory. In the next chapter, this theory will be used to prove one of the most central theorems about Lefschetz properties.
The content of this chapter is adapted from [7].
Recall that we are working under the hypothesis that K is of characteristic 0.

2.1 Lie algebras

Definition 2.1.1. *A Lie algebra is a vector space* $\mathfrak{g}$ *with a bilinear operator*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

such that for all $x, y, z \in \mathfrak{g}$*:*

- $[x,y] = -[y,x]$ *(Anticommutativity).*
- $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ *(Jacobi identity).*

Example 2.1.2. Let V be a vector space. The set of K-endomorphism of V, $\mathrm{End}(V)$, is a Lie algebra with the operation $[f,g] = fg - gf$.

Definition 2.1.3. *The Lie algebra* $\mathfrak{sl}_n$ *is formed by the* $n \times n$ *matrices with trace 0. It is a subalgebra of* $\mathrm{End}(K^n)$*.*

We focus now on the case $n = 2$. $\mathfrak{sl}_2$ is a three-dimensional vector space which admits the basis:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This three elements satisfy what we will call the fundamental relations:

$$[h,e] = 2e, [e,f] = h, [h,f] = -2f.$$

Any other relationship between them is deduced from the bilinearity and anti-commutativity of the $[\cdot,\cdot]$ operator.

Definition 2.1.4. *A homomorphism of Lie algebras* $\mathfrak{g}, \mathfrak{g}'$ *is a K-linear map* $\rho : \mathfrak{g} \longrightarrow \mathfrak{g}'$ *such that* $\rho([x,y]) = [\rho(x), \rho(y)]$ *for all* $x, y \in \mathfrak{g}$.

Definition 2.1.5. *Let* $\mathfrak{g}$ *be a Lie algebra and V a vector space. A representation of* $\mathfrak{g}$ *on V is a homomorphism* $\mathfrak{g} \longrightarrow \mathrm{End}(V)$.

Whenever there exists a representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$, $\mathfrak{g}$ acts on V in the following way. Let $x \in \mathfrak{g}$ and $v \in V$. Then $x \cdot v = \rho(x)(v)$. We will then say that V is a $\mathfrak{g}$-module. If $W \subseteq V$ is a linear subspace, we will say that it is a submodule of V if $xw \in W$ for all $w \in W$ and $x \in \mathfrak{g}$. V is irreducible if it has no submodules other than $\{0\}$ and V itself.

Definition 2.1.6. *Let V be a vector space and* $L, H, D \in \mathrm{End}(V)$. L, H, D *are an* $\mathfrak{sl}_2$*-triple if they satisfy the relations:*

$$[L,D] = H, [H,L] = 2L, [H,D] = -2D.$$

Note that if L, H, D are an $\mathfrak{sl}_2$-triple, there exists a unique representation $\rho : \mathfrak{sl}_2 \longrightarrow \mathrm{End}(V)$ such that

$$\rho(e) = L, \rho(h) = H, \rho(f) = D.$$

Definition 2.1.7. *Let V be an* $\mathfrak{sl}_2$*-module. A weight vector is an eigenvector of* $u \mapsto h \cdot u$*, and the corresponding eigenvalue is its weight. If v is a weight vector and* $f \cdot v = 0$*, then v is a lowest weight vector and its weight is called a lowest weight. If* $e \cdot v = 0$ *and v is a weight vector, v is a highest weight vector and its weight is a highest weight.*

Proposition 2.1.8. *Suppose V is an* $\mathfrak{sl}_2$*-module and* $v \in V$ *is a weight vector. There exist* $\lambda \in K$ *and integers N and n such that setting* $v_0 = f^N \cdot v$ *and* $v_k = e^k \cdot v_0$ *for* $0 \leq k$*:*

1. $h \cdot v_k = (\lambda + 2k)v_k$ *when* $0 \leq k \leq n$.
2. $e \cdot v_k = v_{k+1} \neq 0$ *for* $0 \leq k < n$ *and* $e \cdot v_n = 0$.
3. $f \cdot v_0 = 0$ *and* $f \cdot v_k = k(-\lambda - k + 1)v_{k-1}$ *for* $k > 0$.

In particular, v_0 *is a lowest weight vector and* v_n *is a highest weight vector.*

Proof. Let c be the weight of v. Notice that as $[h,f] = -2f$ and $hf = [h,f] + fh = -2f + fh$, $hf \cdot v = -2f \cdot v + fh \cdot v = -2f \cdot v + cf \cdot v = (c-2)f \cdot v$. In particular $f \cdot v$ is a weight vector of weight $(c-2)$ and so applying the same argument, $hf^2 \cdot v = hff \cdot v = (c-4)f^2 \cdot v$. Applying this process over and over shows that $hf^k \cdot v = (c-2k)f^k \cdot v$ for all $k \geq 0$. Therefore, for any N such that $f^N \cdot v \neq 0$, $\{f^k \cdot v\}_{0 \leq k \leq N}$ are eigenvectors of h with different eigenvalues and are thus linearly independent. N can not be chosen to be arbitrarily large since V is finite dimensional.

Let N be the largest integer such that $f^N \cdot v \neq 0$ and define $v_0 = f^N \cdot v$, $\lambda = c - 2N$. Define recursively $v_{k+1} = e \cdot v_k$ for all $k \geq 0$. By construction, v_0 is a weight vector of weight λ. As $[h,e] = 2e$, an argument similar to the first one of the proof shows that $he^k \cdot v_0 = (\lambda + 2k)e^k \cdot v$. Also, there is some n such that $e^{n+1} \cdot v_n = 0$. This shows 1 and 2.

For part 3, $f \cdot v_0 = 0$ is true by construction and the identity for $k = 1$ follows from

$$fe \cdot v_0 = (-[e,f] + ef) \cdot v_0 = (-h + ef) \cdot v_0 = -h \cdot v_0 = -\lambda v_0.$$

The rest follows by induction:

$$\begin{gathered} f \cdot v_{k+1} = fe \cdot v_k = (-[e,f] + ef) \cdot v_k = (-h + ef) \cdot v_k = \\ -(\lambda + 2k)v_k + k(-\lambda - k + 1)e \cdot v_{k-1} = (k+1)(-\lambda - k)v_k. \end{gathered}$$

□

Proposition 2.1.9. *In the previous proposition, $\lambda = -n$.*

Proof. In equation 3 of the proposition we set $k = n + 1$ to obtain:

$$0 = f \cdot v_{n+1} = (n+1)(-\lambda - n) \cdot v_n.$$

As $v_n \neq 0$, $-\lambda - n = 0$. □

Remark 2.1.10. Let V be a vector space of dimension $n + 1$ and choose a basis $v_0, \ldots, v_n$. Then the rules from proposition 2.1.8 with $\lambda = -n$ give V irreducible $\mathfrak{sl}_2$-module structure. Furthermore, this is the only irreducible $\mathfrak{sl}_2$-module up to isomorphism.

2.2 Operations between modules and reducibility

Let $\mathfrak{g}$ be a Lie algebra and suppose V and V' are $\mathfrak{g}$-modules. Then the direct sum $V \oplus V'$ also has $\mathfrak{g}$-module structure by defining

$$x \cdot (v + v') = (x \cdot v) + (x \cdot v')$$

for any $x \in \mathfrak{g}, v \in V, v' \in V'$.

Definition 2.2.1. *Let V be a $\mathfrak{g}$-module and let $W \subseteq V$ be a submodule. A supplementary submodule of W is a submodule $W' \subseteq V$ such that $V = W \oplus W'$. We say that V is completely reducible if any submodule admits a supplementary submodule.*

Proposition 2.2.2. *Any finite dimensional $\mathfrak{sl}_2$-module is completely reducible.*

Proof. See Weyl's theorem from page 28 of [10]. $\square$

This allows us to break down any $\mathfrak{sl}_2$-module into irreducible submodules in the following way:
Choose any weight vector v. Then proposition 2.1.8 gives us a linearly independent set $\{v_0, v_1, \dots v_n\}$ and the linear subspace $W =< v_0, v_1, \cdots, v_n >$ is an irreducible submodule. Choose a supplementary W' and repeat the same process on W'. Notice that this eventually stops as V is finite dimensional. Also, all weights are integers.

Definition 2.2.3. *For an integer n, define $V[n] = \{v \in V \mid h \cdot v = nv\}$. These spaces are called weight spaces.*

If V and V' are $\mathfrak{g}$-modules of some Lie algebra $\mathfrak{g}$, it is possible to define a $\mathfrak{g}$-module structure on $V \otimes V'$ with the action determined by

$$x \cdot (v \otimes v') = (x \cdot v) \otimes v' + v \otimes (x \cdot v')$$

for all $x \in \mathfrak{g}, v \in V, v' \in V$.

Chapter 3

Lefschetz properties

In this chapter, we will first be defining the Lefschetz properties for Artinian graded algebras, and we will then study a conjecture asserting the existence of monomial ideals satisfying the weak Lefschetz property.

3.1 Lefschetz properties

Definition 3.1.1. *Let R/I be a graded Artinian K-algebra. A linear form $L \in (R/I)_1$ is a weak Lefschetz element if for all $t \in \mathbb{N}$, the K-vector space morphism $\times L : (R/I)_t \longrightarrow (R/I)_{t+1}$ of multiplication by L has maximum rank. Equivalently, it is always injective or surjective.*
L is a strong Lefschetz element if $\times L^d : (R/I)_t \longrightarrow (R/I)_{t+d}$ has maximum rank for all t, d.
It is said that R/I has the weak (resp. strong) Lefschetz property if there exists a weak (resp. strong) Lefschetz element.

We use the abbreviations WLP and SLP to refer the the weak and strong Lefschetz properties. Obviously, the strong Lefschetz property implies the weak.

Remark 3.1.2. Even for monomial ideals, the characteristic of the field can have a significant effect on the WLP. For example, Juan C. Migliore, Rosa M. Miró-Roig and Uwe Nagel show in [11] that for the ideal $I = (x^{10}, y^{10}, z^{10}, x^3y^3z^3) \subseteq R = K[x, y, z]$, R/I fails the WLP in characteristics 2, 3 and 11 but has it in every other characteristic. For this reason, we will be working in characteristic zero.

Remark 3.1.3. For an Artinian graded K-algebra R/I, fix a basis of every $(R/I)_t$. For a lineal form $L \in (R/I)_1$, the matrices of the maps $\times L^d : (R/I)_t \longrightarrow (R/I)_{t+d}$ depend on the coefficients of L, and the property of L failing to be a strong or weak Lefschetz element is a set of homogeneous polynomial equations on these coefficients. Therefore, the set of weak or strong Lefschetz elements of R/I is a Zariski open set of the dual space $\mathbb{P}^{n-1^*}$ which may be empty.

Definition 3.1.4. *Let R/I be an Artinian graded K-algebra with socle degree d. We say that R/I has the strong Lefschetz property in the narrow sense if there exists a linear form $L \in (R/I)_1$ such that $\times L^{d-2t} : (R/I)_t \longrightarrow (R/I)_{d-t}$ is bijective for all $d = 0, 1, \ldots, [\frac{d}{2}]$.*

Remark 3.1.5. An Artinian algebra R/I satisfies the SLP in the narrow sense if and only if it satisfies the SLP and its h-vector is symmetric.

Using the fact that $K[x]$ is a principal ideal domain, it is easy to check that every homogeneous Artinian ideal of $K[x]$ satisfies the SLP. Tadahito Harima, Juan Migliore, Uwe Nagel, and Junzo Watanabe also show in [8, Proposition 4.4] that any homogeneous Artinian algebra in two variables and characteristic 0 satisfies the SLP. The best known result in more than two variables is the following. It is of great interest and has been independently proven by many authors. It asserts the SLP in the narrow sense for a certain class of ideals.

Theorem 3.1.6 (Stanley)**.** *All monomial complete intersections of R (these are, ideals of the form $(x_1^{a_1}, \ldots, x_n^{a_n})$ with all $a_i \geq 1$) satisfy the SLP in the narrow sense, with $x_1 + \cdots + x_n$ as a strong Lefschetz element.*

Many proofs of this theorem are known. See for example [20], [21, Corollary 3.5], [14, Remark 4.3], [19, Theorem 5] or [18, Theorem 3.1]. We will explain Watanabe's original proof in detail in the next section.

Algebras in three or more variables may not satisfy the WLP. We will later show that the ideal $I = (x^3, y^3, z^3, xyz) \subseteq R = K[x, y, z]$ fails the WLP.
We now present algorithms to determine whether a linear form is a Lefschetz element of an algebra. Let I be an Artinian homogeneous ideal and let $L \in (R/I)_1$ be a non-zero linear form. First notice that the cokernel of the multiplication by L, $(R/I)_d \longrightarrow (R/I)_{d+1}$, is $(R/(I, L))_{d+1}$. We can then determine whether $\times L : (R/I)_d \longrightarrow (R/I)_{d+1}$ has maximal rank in the following way.

- If $H(R/I, d) \leq H(R/I, d+1)$, $\times L$ has maximal rank if and only if it is injective. This is equivalent to the equation

$$H(R/(I, L), d+1) = H(R/I, d+1) - H(R/I, d).$$

- If $H(R/I, d) \geq H(R/I, d+1)$, $\times L$ has maximal rank if it is surjective, that is, if $H(R/(I, L), d+1) = 0$.

Notice that if $(R/(I,L))_{d+1} = 0$, then $(R/(I,L))_{t+1} = 0$ for all $t \geq d$. This proves the following:

Proposition 3.1.7. *If* $\times L : (R/I)_d \longrightarrow (R/I)_{d+1}$ *is surjective, then*

$$\times L : (R/I)_t \longrightarrow (R/I)_{t+1}$$

is surjective for all $t \geq d$.

We can also conclude the following:

Corollary 3.1.8. *Let* $I \subseteq R$ *be a homogeneous Artinian ideal satisfying the WLP. Then the h-vector of* R/I *is unimodal.*

Proof. Let d be the first natural such that $h_d \geq h_{d+1}$ and let L be a weak Lefschetz element. Then, as $\times L : (R/I)_d \longrightarrow (R/I)_{d+1}$ has maximal rank, it must be surjective. Then, for all $t \geq d$, the maps $\times L : (R/I)_t \longrightarrow (R/I)_{t+1}$ are surjective and therefore $h_t \geq h_{t+1}$. □

Let $h = (1, h_1, \dots, h_d)$ be the h-vector of R/I. We define its derived vector as $\Delta h = (1, \max(h_1 - 1, 0), \max(h_2 - h_1, 0), \cdots, \max(h_d - h_{d-1}, 0))$. We can conclude that R/I satisfies the WLP if and only if the h-vector of $R/(I,L)$ is Δh. Using the language Macaulay2 ([5]), we can do it in the following way. First, we introduce a function that returns the h-vector of a given graded Artinian algebra:

```
Hv= B->(
        i:=0;
        vect:={};
        d:=1;
        while d>0 do (
                vect=append(vect,d);
                i=i+1;
                d=hilbertFunction(i,B);
        );
        return vect;
);
```

Then, we define functions to determine the h-vector of the cokernel, and the derived of the h-vector of R/I:

```
cokern=(B,L)->(
        C:=Hv(B/(L));
        l:=length Hv B;
```

```
        while (length C<l) do C=append(C,0);
        return C;
);
derivative=Ll->(
        i:=0;
        deri:={1};
        while i<(length Ll-1) do (
                deri=append(deri,max{Ll#(i+1)-Ll#i,0});
                i=i+1;
        );
        return deri;
);
```

We are adding zeroes to the end of the h-vector of the cokernel to guarantee it has the same length as the derived h-vector.
Finally we just need to compare the derived h-vector with the h-vector of the cokernel:

```
WLP=(B,L)->(derivative Hv B==cokern (B,L));
```

Reasoning similarly, L is a strong Lefschetz element if and only if

$$H(R/(I,L^s),t) = \max(h_t - h_{t-s}, 0)$$

for all s varying from 1 to the socle degree of R/I.
There is an analog of proposition 3.1.7 for injectivity.

Proposition 3.1.9. *Suppose R/I is level of socle degree s. Suppose $\times L : (R/I)_d \longrightarrow (R/I)_{d+1}$ is injective for some $d < s$, then $\times L : (R/I)_t \longrightarrow (R/I)_{t+1}$ is injective for all $t \leq d$.*

Proof. We will show that if $\times L : (R/I)_t \longrightarrow (R/I)_{t+1}$ fails to be injective for some $t < s$, then $\times L : (R/I)_{t+1} \longrightarrow (R/I)_{t+2}$ also fails to be injective. Let J be the kernel of the graded morphism $\times L : (R/I) \longrightarrow (R/I)(1)$. Notice that $\times L : (R/I)_t \longrightarrow (R/I)_{t+1}$ fails to be injective if and only if $J_t \neq 0$. If $J_t \neq 0$, for any non-zero $f \in J_t$ there is some indeterminate x_i such that $x_i f \neq 0$. Thus $J_{t+1} \neq 0$. □

Example 3.1.10. The hypothesis that the algebra is level is necessary. Define the ideal $I = (x^2, xy, xz, y^4, z^4) \subseteq R = K[x,y,z]$. Notice that R/I is not level as $x \in 0 : \mathfrak{m}$. The multiplication $\times y : (R/I)_2 \longrightarrow (R/I)_3$ is injective, but $\times y : (R/I)_1 \longrightarrow (R/I)_2$ is not, as $xy = 0$.

This work is centred on studying the following conjecture, formulated by Filip Jonsson Kling in [12]:

Conjecture 3.1.11. *Let $R = K[x_1, \cdots, x_n]$. For each $d \geq 1$ and μ with $n \leq \mu \leq \binom{n+d-1}{n-1}$ there exists an ideal $I \subseteq R$ generated by μ degree d monomials such that R/I is Artinian and satisfies the SLP.*

In his work, he proves the case $d = 2$. Notice also that the result is true for $n = 2$ due to [8, Proposition 4.4].
We limit ourselves to considering only the WLP, so we want to study a weaker version of the conjecture:

Conjecture 3.1.12. *Let $R = K[x_1, \cdots, x_n]$. For each $d \geq 1$ and μ with $n \leq \mu \leq \binom{n+d-1}{n-1}$ there exists an ideal $I \subseteq R$ generated by μ degree d monomials such that R/I is Artinian and satisfies the WLP.*

Remark 3.1.13. In [1], Nasrin Altafi and Mats Boij show that for certain numbers of generators, any monomial Artinian graded algebra satisfies the WLP. This conjecture addresses a related problem. That is, for which numbers of generators and degrees is the failure of the WLP guaranteed. The statement that there are none is equivalent to the veracity of the conjecture.

For monomial ideals, the following proposition is very useful for the study of the weak and strong Lefschetz properties.

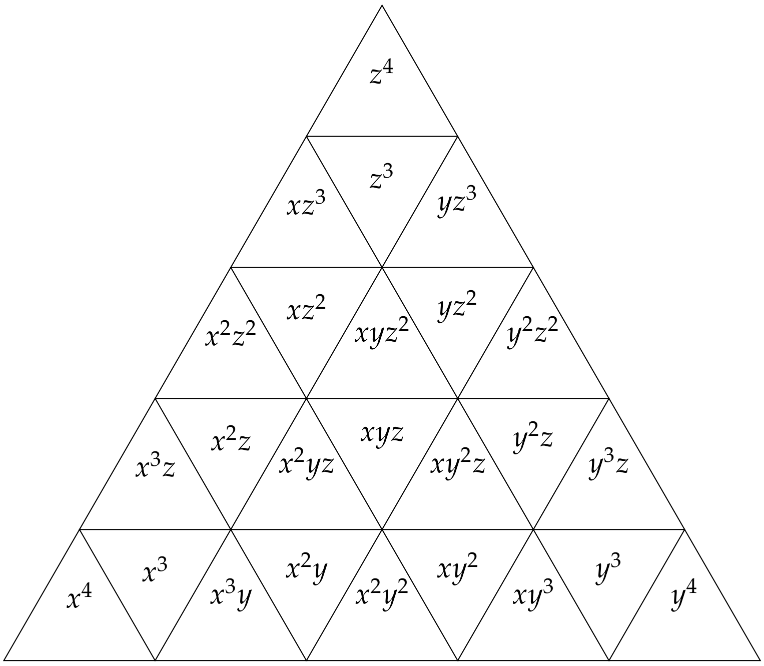
Proposition 3.1.14. *Let $I \subseteq R$ be an Artinian monomial ideal. If R/I satisfies the SLP (resp. WLP), $x_1 + \cdots + x_n$ is a strong (resp. weak) Lefschetz element.*

Proof. Set $A = R/I$ and let $L = a_1x_1 + \cdots + a_nx_n \in (R/I)_1$ be a linear form with every a_i non-zero. The function

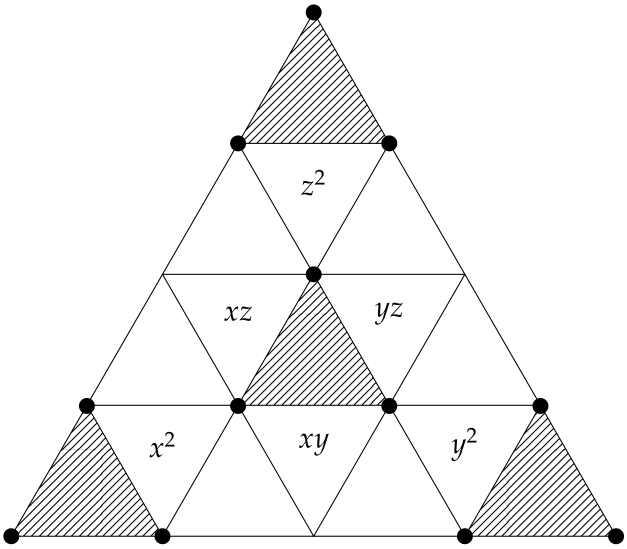
$$\begin{aligned} A &\longrightarrow A \\ f(x_1, \cdots, x_n) &\longmapsto f(a_1x_1, \cdots, a_nx_n) \end{aligned}$$

is a graded K-algebra automorphism of A and sends $x_1 + \cdots + x_n$ to L. This induces isomorphisms of the graded K-algebras $A/((x_1 + \cdots x_n)^k) \cong A/(L^k)$ for all $k \geq 1$ and in particular they have the same Hilbert functions. For a non-zero $L \in A_1$, the property of being a strong or weak Lefschetz only depends on the Hilbert functions of $A/(L^k)$, and as the set of weak or strong Lefschetz elements is a Zariski open set, it will always have elements outside all the hyperplanes $a_i = 0$ as long as it is non-empty, so the result follows. $\square$

From now on, L will denote the lineal form $x_1 + \cdots + x_n$ When $n = 3$, we can grafically represent monomials in the following way: Take an equilateral triangle and divide it in smaller equilateral triangles such that each border has $d + 2$ vertices. Then vertices are identified with degree $d + 1$ monomials , small triangles oriented in the same way as the big triangle are identified with degree d monomials, and the inversely oriented small triangles identify with monomials of degree $d - 1$. See the following example for $d = 4$.



If we want to signal that a monomial belongs to a certain ideal, we will shade its corresponding triangle or highlight its vertex. For example, for $I = (x^3, y^3, z^3, xyz)$ we have the following graph:



If m is a degree d monomial represented by an upwards pointing triangle, Lm is the sum of its vertices, and if m' has degree $d-1$ and is represented by a downwards pointing triangle, Lm' is the sum of the three triangles with which it shares an edge. Looking at the previous graph, we can observe that

$$L(x^2+y^2+z^2-xy-xz-yz)=0.$$

In particular $\times L : (R/I)_2 \longrightarrow (R/I)_3$ is not injective. As $H(R/I,2)=H(R/I,3)=6$, R/I does not satisfy the WLP.

Remark 3.1.15. Despite R/I not having the WLP, its h-vector is (1,3,6,6,3) which is unimodal. So while all Artinian graded algebras enjoying the WLP have a unimodal h-vector, the unimodality of the h-vector does not imply the WLP.

3.2 Proof of Stanley's theorem

In this section we will show a connection between the strong Lefschetz property in the narrow sense for Artinian graded K-algebras and the theory of $\mathfrak{sl}_2$-modules. Using this, we will show that the SLP in the narrow sense is closed under tensor product and Stanley's theorem will follow trivially.

Recall that we use the letters e, h, f to refer to the generators of $\mathfrak{sl}_2$

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix}, h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, f=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

.

Theorem 3.2.1. *Let A be an Artinian graded K-algebra and $g \in A_1$. Then, A satisfies the SLP in the narrow sense with g as a Lefschetz element if and only if there exists a representation $\rho : \mathfrak{sl}_2 \longrightarrow \operatorname{End}(A)$ such that:*

1. *$\rho(e)$ is the multiplication by g.*
2. *The weight space decomposition of A as an $\mathfrak{sl}_2$-module is the same as the decomposition of A as a graded ring.*

Proof. Suppose A satisfies the SLP in the narrow sense with g as a Lefschetz element. Let E be the endomorphism of A of multiplication by g. If s is the socle degree of A, then $E^{s+1} = 0$. As E is nilpotent, it has no non-zero eigenvalues. First we construct a basis of A such that the matrix of $\times g$ is a direct sum of Jordan blocks (of eigenvalue 0). The first part of the basis is $\{1, g, \dots, g^s\}$. This gives a Jordan block of size s. Then, let $\overline{a_1}, \dots, \overline{a_t}$ be a basis of $\ker(\times L : A_{s-1} \longrightarrow A_s)$, and for every i let $a_i \in A_1$ such that $g^{s-2}a_i = \overline{a_i}$. Then we add to the basis the elements:

$$\begin{gathered}\{a_1, ga_1, \dots g^{s-2}a_1\},\\ \{a_2, ga_2, \dots g^{s-2}a_2\},\\ \vdots\\ \{a_t, ga_t, \dots g^{s-2}a_t\}.\end{gathered}$$

Each of this successions gives a Jordan block of size $s-2$. We then repeat this process for the kernel of $\times g : A_{s-i} \longrightarrow A_{s-i+1}$ for all $i \leq \left[\frac{s}{2}\right]$. Each step creates parts of the basis with Jordan blocks of size $s-2i$. After finalizing this process we obtain a desired basis of A. Also, every element of this basis is homogeneous. Notice that if m_k is the amount of Jordan blocks of size k, then $m_{s-2i+1} = H(A,i) - H(A,i-1)$.

We now construct morphisms H and F so that E, H, F is an $\mathfrak{sl}_2$-triple.
Suppose $a, ga, \dots, g^j a$ is one of the parts of the basis and $V =< a, ga, \dots, g^j a >$. Let E_V be the restriction of E to V. Define H_V to be the endomorphism of V with matrix

$$\operatorname{diag}(-j, -j+2, \dots, j)$$

in the given basis. Define also F_V to have the entries $k(j-k+1)$ in the superdiagonal for $k = 1, 2 \dots, j$.

$$H_V = \begin{pmatrix} -j & & & & \\ & -j+2 & & & \\ & & -j+4 & & \\ & & & \ddots & \\ & & & & j \end{pmatrix}, F_V = \begin{pmatrix} 0 & 1j & & & & \\ & 0 & 2(j-1) & & & \\ & & 0 & 3(j-2) & & \\ & & & \ddots & \ddots & \\ & & & & 0 & j1 \\ & & & & & 0 \end{pmatrix}.$$

Notice that H_V and F_V are constructed such that together with E_V, they satisfy the rules of proposition 2.1.8. Therefore there is a representation $\mathfrak{sl}_2 \longrightarrow \mathrm{End}(V)$ that maps (h, e, f) to (H_V, E_V, F_V). Doing this for every part of the basis yields by direct sum a representation ρ on A. Notice that if v is in in the basis, it has degree t if and only if its weight is $2t - s$. Thus, $A_t = A[2t - s]$ for all t.

Conversely, if such a representation exists, decomposing V into h-invariant spaces allows us to build a basis with the same properties as in the first implication and thus A satisfies the SLP in the narrow sense with g as a Lefschetz element. □

Theorem 3.2.2. *Suppose A, A' are Artinian graded K-algebras with the SLP in the narrow sense and g, g' are respective Lefschetz elements. Then $B = A \otimes_K A'$ also satisfies the SLP in the narrow sense with $G = g \otimes 1 + 1 \otimes g'$ as a Lefschetz element.*

Proof. Denote by s and s' the socle degrees of A and A' respectively. The actions of $\mathfrak{sl}_2$ on A and A' given by the previous theorem extend to B as defined in the previous section. Then the action of e is multiplication by G. If $x \in A$ and $x' \in A'$ are homogeneous of degrees d and t respectively, $x \otimes x'$ has degree $d + t$. Also, as x has weight $2d - s$ and y has weight $st - s'$

$$\begin{aligned} h \cdot (x \otimes x') &= (h \cdot x) \otimes x' + x \otimes (h \cdot x') \\ &= (2d - s)x \otimes x' + (2t - s')x \otimes x' \\ &= (2d - (s + s')x \otimes x'). \end{aligned}$$

So $\mathrm{weight}(x \otimes x') = 2(d + t) - (s + s')$. Therefore the decomposition of B as a graded algebra coincides with the weight space decomposition. □

Corollary 3.2.3 (Stanley's theorem)**.** *Every monomial complete intersection satisfies the SLP.*

Proof. $K[x]/(x^a)$ satisfies the SLP in the narrow sense for any $a \geq 1$. The rest is obtained by noticing that

$$K[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n}) \cong K[x_1]/(x^{a_1}) \otimes_K \cdots \otimes_K K[x_n]/(x^{a_n}),$$

and applying the previous theorem. □

3.3 Case of three variables

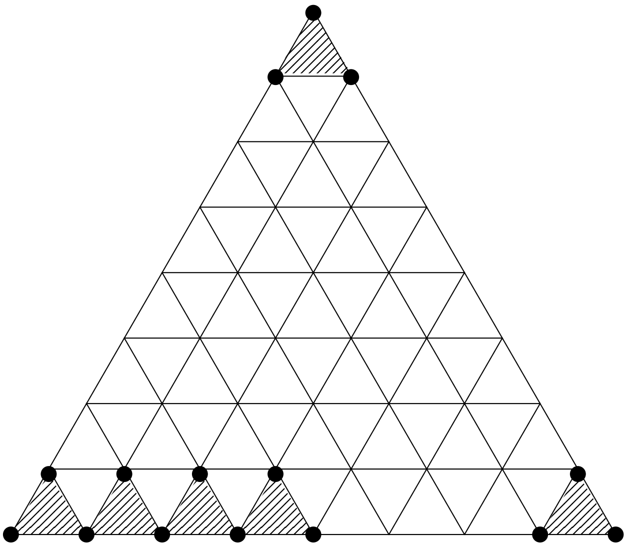
We will begin our study of conjecture 3.1.12 in the case of three variables. For any $k \in \{0, 1, \ldots, d-2\}$, I^d_{3+k} will denote the ideal generated by the first $3+k$ monomials of the succession

$$x^d, y^z, z^d, x^{d-1}y, x^{d-2}y^2, \ldots, x^2y^{d-2}.$$

Example 3.3.1. The ideal I^7_6 is

$$I^7_6 = (x^7, y^7, z^7, x^6y, x^5y^2, x^4y^3).$$

It can be illustrated with the following graph:



Our aim is to show that these ideals satisfy the WLP. To start, we wish to study the h-vectors of the algebras R/I^d_{3+k}. First we study the Hilbert function when $I = (x^d, y^d, z^d)$. The minimal free resolution

$$0 \longrightarrow R(-3d) \longrightarrow R(-2d)^3 \longrightarrow R(-d)^3 \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

determines the Hilbert function $H(R/I, t) = \binom{t+2}{2} - 3\binom{t+2-d}{2} + 3\binom{t+2-2d}{2} - \binom{t+2-3d}{2}$. We make the following observations:

i For $0 \leq t < d$, we simply get $H(R/I, t) = \binom{t+2}{2}$.

ii When $d \leq t < 2d$, $H(R/I, t) = \binom{t+2}{2} - 3\binom{t+2-d}{2} = -t^2 + (3d-3)t - \frac{3}{2}d^2 + \frac{9}{2}d - 2$. As a function of t, this is a quadratic polynomial. If the polynomial is seen as a real function, it reaches its maximum at $t^* = \frac{3d-3}{2}$. If d is odd then t^* is an integer, and the only point where the Hilbert function reaches its maximum. Otherwise the two maximums of the Hilbert function occur at $t_{\pm} = t^* \pm \frac{1}{2}$, being $t_- = \left[\frac{3d-3}{2}\right]$ the first peak. Notice also that this function is symmetrical around t^*.

iii For $2d-2 \leq t < 3d-2$, the Hilbert function is $H(R/I,t) = \binom{t+2}{2} - 3\binom{t+2-d}{2} + 3\binom{t+2-2d}{2} = -\frac{1}{2}t^2 + (-3d+\frac{3}{2})t + \frac{9}{2}d^2 - \frac{9}{2}d + 1$. Writing $s = 3d-3-t$, we see $0 \leq s < d$ and $H(R/I,t) = \binom{s+2}{2}$, so this part of the Hilbert function reflects the part described in i.

iv $H(R/I,t) = 0$ for all $t \geq 3d-2$.

This describes the entirety of the h-vector of R/I. In particular it is symmetric and unimodal. Now we wish to study how it changes when adding some generators to the ideal. Start by considering $I_4^d = (x^d, y^d, z^d, x^{d-1}y)$. We will calculate the Hilbert function of R/I_4^d for degrees smaller than $2d-1$.
The monomials in I_4^d multiples of $x^{d-1}y^1$ differ from all multiples of y^d or z^d for all degrees less than $2d-1$, so we only need to answer how many monomials does $x^{d-1}y$ generate in degree $t \geq d$ that are not multiples of x^d. A simple combinatoric computation shows the answer is $t-d+1$. We show another way to find it using free resolutions.

Consider two monomials $x^{d-k}y^k$ and $x^{d-k-1}y^{k+1}$ (in the previous case, $k=0$). Let M be the quotient of ideals $M = (x^{d-k}y^k, x^{d-k-1}y^{k+1})/(x^{d-k}y^k)$. The minimal free resolution of M is

$$0 \longrightarrow R(-d-1) \xrightarrow{\times x} R(-d) \longrightarrow M \longrightarrow 0$$

where the morphism $R(-d) \longrightarrow M$ sends 1 to the generator $[x^{d-k-1}y^{k+1}]$, and the morphism $R(-d-1) \xrightarrow{\times x} R(-d)$ is due to the torsion $x[x^{d-k-1}y^{k+1}] = 0$. This shows $M \cong (R/(x))(-d)$. From the resolution we obtain $H(M,t) = \binom{t-d+2}{2} - \binom{t-d+1}{2}$ which equals $t-d+1$ for all $t \geq d$. In particular $t-d+1$ is the Hilbert polynomial of M.

Let h be the Hilbert function of $R/(x^d,y^d,z^d)$. Then, for $d \leq t < 2d-1$, $H(R/I_4^d,t) = h(t) - (t-d+1)$. Suppose now that we also want to add the monomial $x^{d-2}y^2$ to the ideal. The first collisions between the contributions of $x^{d-2}y^2$ and y^d will be at degree $2d-2$. At lower degrees, the amount of degree t monomials that are multiples of $x^{d-2}y^2$ but not $x^{d-1}y$ will be $t-d+1$. Taking $I_5^d = (x^d, y^d, z^d, x^{d-1}y, x^{d-2}y^2)$, we obtain $H(R/I_5^d,t) = h(t) - 2(t-d+1)$. Similar considerations show that $H(R/I_6^d,t) = h(t) - 3(t-d+1)$ for $d \leq t < 2d-3$.

We can keep subsequently adding the monomials $x^{d-4}y^4, x^{d-5}y^5, \cdots$ and repeat the same arguments to obtain the following:

Proposition 3.3.2. *Let k be a natural number between 0 and $d-2$. Then*

$$H(R/I^d_{3+k}, t) = h(t) - k(t-d+1)$$

whenever $d \leq t < 2d-k$.

We wish to use this result to study the peak of the Hilbert function. We already know that in the given range for t, $h(t) = -t^2 + (3d-3)t - \frac{3}{2}d^2 + \frac{9}{2}d - 2$, therefore

$$H(R/I^d_{3+k}, t) = -t^2 + (3d-3)t - \frac{3}{2}d^2 + \frac{9}{2}d - 2 - k(t-d+1).$$

Considering this expression as a function of $t \in \mathbb{R}$, this is a quadratic polynomial with negative leading coefficient. The derivative is $-2t + 3d - 3 - k$, which becomes zero at $t^*_k = \frac{3d-3-k}{2}$, which is the point where the maximum value is reached. This value is an integer if an only if $d \not\equiv k \mod 2$. If d and k have the same parity, the Hilbert function has a double peak at degrees $t^*_k \pm \frac{1}{2}$. This allows us to classify the peaks of h-vectors of the algebras R/I^d_{3+k}.

Proposition 3.3.3. *Let h_k be the Hilbert function of R/I^d_{3+k}.*

1. *If $0 \leq k \leq d-4$, h_k has a peak at degree $[\frac{3d-3-k}{2}]$. This peak is doubled if and only if $d \equiv k \mod 2$.*

2. *If $d-4 \leq k \leq d-3$, h_k has a peak at degree d, which is doubled for $k = d-4$.*

3. *For $k = d-2$, h_k has a doubled peak at degrees $d-1$ and d.*

Proof.

1. Part 1 has already been shown.

2. Notice that $H(R,d) - H(R,d-1) = d+1$, while $h_k(d-1) = H(R,d-1)$ and $h_k(d) = H(R,d) - k - 3$. As $k < d-2$, $h_k(d) > h_k(d-1)$. In this range, our previous analysis still shows that the Hilbert function at degrees $t \in \{d, d+1\}$ is the polynomial $-t^2 + (3d-3)t - \frac{3}{2}d^2 + \frac{9}{2}d - 2 - k(t-d+1)$. For $k = d-4$, we obtain that the maximum is at $t^*_k + \frac{1}{2}$, so there is a doubled peak at degrees d and $d+1$. For $k = d-3$, $t^*_k = d$, so $h_k(d) > h_k(d+1)$.

3. We only need to observe that $h_k(d-2) < h_k(d-1)$ as h_k coincides with $H(R,\_)$ for degrees less than d, and $h_k(d) = H(R,d) - k - 3 = H(R,d) - (d+1) = H(R,d-1) = h_k(d-1)$.

□

To study the WLP of these ideals, we need to study the Hilbert functions of the algebras $R/(I^d_{3+k}, L)$. The next proposition is enough to state that $H(R/(I^d_{d+1}, L), d) = 0$.

Proposition 3.3.4. *Let I be a monomial ideal. Suppose that $z^d \in I$ and all but at most one monomials of the form $x^{d-i}y^i$ are in I. Then $\times L : (R/I)_{d-1} \longrightarrow (R/I)_d$ is surjective.*

Proof. Regarding R as $K[x,y,L]$, allows us to consider L a monomial. Then the ideal (I, L) contains L, all but one degree d monomials that are not multiples of L, and $(L-x-y)^d$. Notice that the linear subspace

$$(L)_d + (\text{all but one degree } d \text{ monomials that are not multiples of } L)_d$$

of R_d has codimension 1. The element $(L-x-y)^d$ does not belong to this subspace as it contains all degree d monomials in the support. Therefore $I_d + (L)_d = R_d$, and thus $(R/(I,L))_d = 0$, which is equivalent to $\times L : (R/I)_{d-1} \longrightarrow (R/I)_d$ being surjective. □

Proposition 3.3.5. *Consider the map $\times L : (R/I^d_{3+k})_{d-1} \longrightarrow (R/I^d_{3+k})_d$.*

1. *If $k = d-2$, it is an isomorphism.*
2. *If $k \leq d-2$, it is injective.*
3. *For any homogeneous ideal J generated in degree d containing I^d_{d+1}, $\times L : (R/J)_{d-1} \longrightarrow (R/J)_d$ is surjective.*

Proof.

1. By the previous proposition, $\times L : (R/I^d_{d+1})_{d-1} \longrightarrow (R/I^d_{d+1})_d$ is surjective. As $H(R/I^d_{d+1}, d-1) = H(R/I^d_{d+1}, d)$ it must also be injective.
2. If $3+k \leq d+1$ and $Lf = 0$ for some $f \in (R/I^d_{3+k})_{d-1}$, the projection of f to (R/I^d_{d+1}) multiplied by L is also 0. This forces f to be zero in (R/I^d_{d+1}) and as the projection $(R/I^d_{3+k})_{d-1} \longrightarrow (R/I^d_{d+1})_{d-1}$ is an isomorphism, $f = 0$ in $(R/I^d_{3+k})_{d-1}$. This shows that the kernel of $\times L : (R/I^d_{3+k})_{d-1} \longrightarrow (R/I^d_{3+k})_d$ is $\{0\}$.

3. Consider the commutative diagram

$$\begin{tikzcd}
(R/I^d_{d+1})_{d-1} \arrow[r, "\times L"] \arrow[d, "\pi_{d-1}"] & (R/I^d_{d+1})_d \arrow[d, "\pi_d"] \\
(R/J)_{d-1} \arrow[r, "\times L"] & (R/J)_d
\end{tikzcd}$$

where the downwards arrows are the natural projections. As both

$$(R/I^d_{d+1})_{d-1} \xrightarrow{\times L} (R/I^d_{d+1})_d$$

and π_d are surjective, $\times L : (R/J)_{d-1} \longrightarrow (R/J)_d$ is also surjective.

$\square$

Corollary 3.3.6. *Any monomial ideal generated in degree d that contains I^d_{d+1} satisfies the WLP.*

Proposition 3.3.7. *The ideal I^d_d satisfies the WLP.*

Proof. We already know that $\times L : (R/I^d_d)_{d-1} \longrightarrow (R/I^d_d)_d$ is injective. Let us check that $\times L : (R/I^d_d)_d \longrightarrow (R/I^d_d)_{d+1}$ is surjective. Notice that:

- $x^{d+1}, z^{d+1} \in I^d_d$
- If $i \in \{3, 4, \dots, d\}$, $x^i y^{d-i} \in I^d_d$ and therefore $x^i y^{d-i+1} \in I^d_d$.
- As $y^d \in I^d_d$, $xy^d, y^{d+1} \in I^d_d$.

Thus I^d_d contains z^{d+1} and all degree $d+1$ monomials not multiples of z except $x^{d-1}y^2$, concluding that $\times L : (R/I^d_d)_d \longrightarrow (R/I^d_d)_{d+1}$ is surjective. Then $\times L : (R/I^d_d)_t \longrightarrow (R/I^d_d)_{t+1}$ is also surjective for all $t \geq d$. $\square$

We now generalize proposition 3.3.5 to establish that $H(R/(I^d_{3+k}, L), t) = 0$ whenever it is required.

Proposition 3.3.8. *Set $t = d + \left[\frac{d-k-1}{2}\right]$. The maps $\times L : (R/I^d_{3+k})_{t-1} \longrightarrow (R/I^d_{3+k})_t$ are surjective.*

Proof. Set $I = I^d_{3+k}$. We show that $(I, L)_t = R_t$. I is generated by x^d, y^d, z^d and the monomials $x^{d-i}y^i$ for $i \in \{1, \dots, k\}$, so $(I)_t$ contains all the monomials of the form $x^{t-i}y^i$ for i from 0 to $k + t - d$ and from d to t, and is missing $2d - k - t - 1 = d - k - 1 - \left[\frac{d-k-1}{2}\right]$ monomials not multiples of z.

I also contains all monomials of the form $x^{t-d-i}y^iz^d$, with i ranging from 0 to $t - d$. This includes a total of $t - d + 1 = \left[\frac{d-k-1}{2}\right] + 1$ monomials. Notice that $\left[\frac{d-k-1}{2}\right] + 1 \geq d - k - 1 - \left[\frac{d-k-1}{2}\right]$.

Once again, see R as $K[x, y, L]$. Then $(I, L)_t$ contains all degree t multiples of L. Combined with the monomials of the form $x^{t-i}y^i$ they generate a linear subspace of codimension $d - k - 1 - \left[\frac{d-k-1}{2}\right]$. We will see that adding the $x^{t-d-i}y^i(L - x - y)^d$ as generators completes the space.

Let T be the set of degree t monomials in I_t that are not multiples of z. Define the vector space $V = I_t / < T >$. We show that the projections of $x^{t-d-i}y^i(L - x - y)^d$ generate V. Choosing the basis of V, $\{x^{t-i}y^i\}_{k+t-d<i<d}$ the matrix of coordinates of the $x^{t-d-i}y^i(L - x - y)^d$ is

$$M = \begin{pmatrix} \binom{d}{k+t-d+1} & \binom{d}{k+t-d+2} & \cdots & \binom{d}{d-2} & \binom{d}{d-1} \\ \binom{d}{k+t-d+2} & \binom{d}{k+t-d+3} & \cdots & \binom{d}{d-1} & \binom{d}{d} \\ \binom{d}{k+t-d+1} & \binom{d}{k+t-d+2} & \cdots & \binom{d}{d} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{d}{k+2t-2d+1} & \binom{d}{k+2t-2d+2} & \cdots & 0 & 0 \end{pmatrix}$$

so we want M to have maximal rank. This can be seen as a consequence of [1, Lemma 3.2]. □

Now that we have established surjectivity for a lot of maps of multiplication by L, we need to show that the rest are injective. We will do that by controlling the decrease of the Hilbert function of the cokernel of ×L when more generators are added to the ideal.

Proposition 3.3.9. *Let* $0 \leq k \leq d - 3$ *and let* $t, d \leq t < 2d - k - 1$.

$$H(R/(I^d_{3+k}, L), t) - H(R/(I^d_{4+k}, L), t) \leq 1.$$

Proof. Set $A = R/(L)$, $I = I^d_{3+k}, \tilde{I} = I^d_{4+k}$. Let J and $\tilde{J}$ be the projections of I and $\tilde{J}$ to A respectively. Notice that J_t is generated, as a K-vector space, by the set of degree d monomials in I_t. We know that $\dim(\tilde{I})_t - \dim(I)_t = t - d + 1$, while also $\dim(\tilde{I})_{t-1} - \dim(I)_{t-1} = t - d$. Therefore $(\tilde{J})_t$ has $t + d + 1$ more generators than $(J)_d$ with at least $t - d$ more relations. This forces the difference in dimensions to be at most one. □

Remark 3.3.10. Recall that for $d \leq t < 2d - k$, $H(R/I^d_{3+k}, t) = -t^2 + (3d-3)t - \frac{3}{2}d^2 + \frac{9}{2}d - 2 - k(t - d + 1)$. The difference $H(R/I^d_{3+k}, t) - H(R/I^d_{3+k}, t-1)$ is $3d - 2t - k - 2$, so the ideals I^d_{3+k} satisfy the WLP if and only if

$$H(R/(I^d_{3+k}, L), t) = \max(3d - 2t - k - 2, 0)$$

for all $t \geq d$.

Every Artinian monomial complete intersection (that is, an ideal

$$(x_1^{a_1}, \dots, x_n^{a_n})$$

of $K[x_1, \dots, x_n]$) satisfies the SLP (and in particular, the WLP). Therefore the Hilbert function of $R/(x^d, y^d, z^d, L)$ for $t \geq$ is $\max(3d - 2t - 2, 0)$.

Theorem 3.3.11. *For any $k \in \{0, 1, \dots, d-2\}$, the ideal I^d_{3+k} satisfies the WLP.*

Example 3.3.12. Before proceeding with the proof, we illustrate the idea with an example. Consider the case $d = 10$. We write a table of values of $H(R/I^{10}_{3+k}, t)$. Depending on t from 10 to 14, and k from 0 to 8. We know where zeroes, can be located, and we also know that $R/(x^{10}, y^{10}, z^{10})$ satisfies the WLP, so this allows us to fill part of the table.

k t	0	1	2	3	4	5	6	7	8
10	8								0
11	6						0	0	0
12	4				0	0	0	0	0
13	2		0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0

We also know that, when moving horizontally, the values can only decrease one at a time. This forces the full table to complete in a unique manner.

k t	0	1	2	3	4	5	6	7	8
10	8	7	6	5	4	3	2	1	0
11	6	5	4	3	2	1	0	0	0
12	4	3	2	1	0	0	0	0	0
13	2	1	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0

And from this table we see the ideals satisfy the WLP.

Proof. We want to show that $H(R/(I^d_{3+k}, L), t) = \max(3d - 2t - k - 2, 0)$ for $t \geq d$. This is true for $k = 0$, as $R/(x^d, y^d, z^d)$ satisfies the WLP. We also know that $H(R/(I^d_{3+k}, L), t) = 0$ as long as $3d - 2t - k - 2 \leq 0$. Also, $H((R/I^d_{3+k}, L), t) \leq H((R/I^d_{4+k},, L), t) + 1$. The fact that $H(R/(I^d_3, L), t) = \max(3d - 2t - 2, 0)$ and $H(R/(I^d_{3+k}, L), d + \left[\frac{d-k-1}{2}\right]) = 0$, forces the previous inequality to be sharp when $H(R/(I^d_{3+k}, L), t) > 0$, and thus $H(R/(I^d_{3+k}, L), t) = \max(3d - 2t - k - 2, 0)$, so the ideals I^d_{3+k} possess the WLP. $\square$

Theorem 3.3.13. *For each $d \geq 1$ and μ with $3 \leq \mu \leq \binom{d+2}{2}$ there exists an ideal $I \subseteq R$ generated by μ degree d monomials such that R/I is Artinian and satisfies the WLP.*

Example 3.3.14. Suppose $d = p$ is a prime number and K is a field of characteristic p. Let $I = (x^p, y^p, z^p)$. We know that I satisfies the WLP in characteristic 0. However, notice that while $L^{p-1} \neq 0$ on R/I, $LL^{p-1} = L^p = x^p + y^p + z^p = 0$. Therefore $\times L : (R/I)_{p-1} \longrightarrow (R/I)_p$ is not injective and I fails the WLP. This shows the relevance of the characteristic in the WLP and justifies the choice of characteristic 0. In [13], Jizhou Li and Fabrizio Zanello give a complete characterisation of the primes p for which an ideal $(x^\alpha, y^\beta, z^\gamma)$ satisfies the WLP in characteristic p.

Example 3.3.15. Even in characteristic 0, the ideals I^d_{3+k} do not satisfy the SLP in general. For example, in degree $d = 3$, the ideal $I^3_5 = (x^3, y^3, z^3 x^2 y, xy^2)$ fails the SLP. For that, we use the following Macaulay2 code, which prints the h-vectors of R/I^3_5 and $R/(I^3_5, L^3)$:

```
R=QQ[x,y,z];
L=x+y+z;
A=R/(x^3,y^3,z^3,x^2*y,x*y^2);
C =A/(L^3);
print(apply((0..5),i->hilbertFunction(i,A)));
print(apply((0..5),i->hilbertFunction(i,C)));
```

The result is

```
(1,3,6,5,3,0)
(1,3,6,4,1,0)
```

Notice that $H(R/I^3_5, 1) = H(R/I^3_5, 4) = 3$, but $H(R/(I^3_5, L), 4) = 1 \neq 0$, so $\times L^3 : (R/I^3_5)_1 \longrightarrow (R/I^3_5)_4$ fails to be an isomorphism.

3.4 Generalisation to $n + 1$ variables

We wish to generalise our previous work to the case of more variables. We will work on R and the extension $R \otimes K[y]$, which is isomorphic to $R[y]$. The Hilbert function of $R[y]$ is $H(R[y], t) = \binom{n+t}{n}$.
Firstly, we want to study the minimal free resolution of R/I, where $I = I(x_1^d, \cdots, x_n^d)$. We will use the following

Theorem 3.4.1. *For any $n, d \geq 1$, the minimal free resolution of R/I has the form*

$$\begin{tikzcd}
0 \arrow[r] & R(-nd) \arrow[r] & R(-(n-1)d)^n \arrow[r] & \cdots \arrow[r] & R(-kd)^{\binom{n}{k}} \arrow[r] & \cdots \\
{} \arrow[r] & R(-d)^n \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}$$

Proof. See [6, Theorem 1.1]. □

Example 3.4.2. In the case of four variables, the minimal free resolution of the ideal I is

$$\begin{tikzcd}
0 \arrow[r] & R(-4d) \arrow[r] & R(-3d)^4 \arrow[r] & R(-2d)^6 \\
{} \arrow[r] & R(-d)^4 \arrow[r] & R \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}$$

and considering that $H(R, t) = \binom{t+3}{3}$ for all $t \geq 0$, the Hilbert function of R/I is

$$H(R/I, t) = \binom{t+3}{3} - 4\binom{t-d+3}{3} + 6\binom{t-2d+3}{3} - 4\binom{t-3d+3}{3} + \binom{t-4d+3}{3}.$$

Proposition 3.4.3. *The algebra R/I satisfies the following properties:*

1. *Its socle degree is $nd - n$.*
2. *The h-vector is symmetric.*
3. *It has a peak at degree $[\frac{nd-n}{2}]$.*

Proof. Set $m = x_1^{d-1} \cdots x_n^{d-1}$ and $A = R/I$.

1. The projection of m to A is non-zero and belongs to A_{nd-n}, so $A_{nd-n} \neq 0$. Any monomial of degree at least $nd - n + 1$ is divisible by the d-th power of at least one variable, so $A_{nd-n+1} = 0$.

2. This is a direct consequence of the fact that A satisfies the SLP in the narrow sense.

3. The algebra satisfies the SLP, so the h-vector must be unimodal. As it is also symmetric, a peak is necessarily found in the middle.

□

Our objective is to study conjecture 3.1.12 by generalizing the methods used to prove the case of three variables. Let $I \subseteq R$ be an Artinian monomial ideal generated in degree d and define $A = R/I$, $B = A \otimes_K (K[y]/(y^d))$. If I can be generated by μ monomials, then B can be seen as a quotient of $R[y]$ by an ideal generated by $\mu + 1$ monomials (see example 1.1.13). For this section, let $L = x_1 + \cdots + x_n \in R$, and $\overline{L} = L \otimes 1 + 1 \otimes y$. We want to characterize the WLP of B in terms of some property of A. First, we relate their Hilbert functions in the following way:

Proposition 3.4.4.

$$H(B,t) = \sum_{i=0}^{d-1} H(A, t-i)$$

Proof. The idea is to show we can express B_d as a direct sum

$$B_d \cong \bigoplus_{i=0}^{d-1} A_{d-i} \otimes y_i.$$

For any $i \in \{0, \ldots, d-1\}$ let φ_i be the R-module morphism defined by the composition of the multiplication $\otimes y^i : R(-i) \longrightarrow R \otimes_K K[y]$ and the projection $R \otimes_K K[y] \longrightarrow B$. The kernel of this map is I, so it factors through R/I and defines a monomorphism $\overline{\varphi_i} : A(-i) \longrightarrow B$ making the following diagram commute:

$$\begin{tikzcd}
R(-i) \arrow[r, "\times y^i"] \arrow[d] \arrow[dr, "\varphi_i"] & R \otimes_K K[y] \arrow[d] \\
A(-i) \arrow[r, "\overline{\varphi_i}"] & B.
\end{tikzcd}$$

For $i \neq j$ the images of $\overline{\varphi_i}$ and $\overline{\varphi_j}$ have zero intersection. Also, any element of B can be written as $\sum_{i=0}^{d-1} f_i \otimes y^i$ with $f_i \in R$, so we have an equality of vector spaces

$$< \overline{\varphi_0}(A), \overline{\varphi_1}(A), \ldots, \overline{\varphi_{d-1}}(A) >= B.$$

This implies $H(B,t) = \sum_{i=0}^{d-1} H(A, t-i)$. □

Corollary 3.4.5. *The socle degree of B is equal to the socle degree of A plus $d-1$.*

In order to find the multiplication maps to be surjective when needed, we give the following generalisation of proposition 3.3.5.

Proposition 3.4.6. *For $t \geq 1$*

1. *The map $\times\overline{L} : B_{t-1} \longrightarrow B_t$ is surjective if and only if $\times L^d : A_{t-d} \longrightarrow A_d$ is surjective.*
2. *The map $\times\overline{L} : B_{t-1} \longrightarrow B_t$ is injective if and only if $\times L^d : A_{t-d} \longrightarrow A_d$ is injective.*

Proof. Identify $R \otimes_K K[y]$ as $R[y]$ and regard $R[y]$ as $K[x_1, \dots, x_n, L]$. Notice $y = \overline{L} - x_1 - \cdots - x_n$. The evaluation of y as $-x_1 - \cdots - x_n$ induces an isomorphism $R[y]/(\overline{L}) \cong R$. Using this:

$$\begin{aligned} B/(\overline{L}) \cong R[y]/(I^e + (y^d) + (\overline{L})) \cong ((R[y])/(\overline{L}))/(I^e + (y^d)) \cong \\ \cong R/(I^e + ((-x_1 - \cdots - x_n)^d)) \cong A/(L^d). \end{aligned}$$

Therefore $B/(\overline{L})$ and $A/(L^d)$ are isomorphic as graded K-algebras and thus they have the same Hilbert function. In particular, $(B/(\overline{L}))_t = 0$ if and only if $(A/(L^d))_t = 0$. This shows 1.

For 2, notice that $\times\overline{L} : (B)_{t-1} \longrightarrow (B)_t$ is injective if and only if $H(B/(\overline{L})) = H(B,t) - H(B,t-1)$, while $\times L^d : A_{t-d} \longrightarrow A_t$ is injective if and only if $H(A/(L),t) = H(A,t) - H(A,t-d)$.

As $H(B,t) = \sum\limits_{i=0}^{d-1} H(A,t-i)$ and $H(B,t-1) = \sum\limits_{i=0}^{d-1} H(A,t-i-1)$, $H(B,t) - H(B,t-1) = H(B,t) - H(B,t-d)$, so if either map is injective, both are. $\square$

Remark 3.4.7. In the proof of proposition 3.4.6, we show that

$$H(B,t) - H(B,t-1) = H(A,t) = H(A,t-d).$$

Therefore, the maps

$$\times\overline{L} : B_{t-1} \longrightarrow B_t$$

are expected to be injective (resp. surjective) exactly when the maps

$$\times L^d : A_{t-d} \longrightarrow A_t$$

are expected to be injective (resp. surjective).

Corollary 3.4.8. *The algebra B satisfies the WLP if and only if all the maps*

$$\times L^d : A_{t-d} \longrightarrow A_t$$

have maximal rank. In particular, the SLP of A implies the WLP of B.

Definition 3.4.9. *Let $f \in R/I$ be a non-zero homogeneous element of degree s. We say that f is faithful if all the maps*

$$\times f : (R/I)_{t-s} \longrightarrow (R/I)_t$$

have maximal rank.

We conclude that the problem of finding monomial ideals in $n+1$ variables satisfying the WLP can be reduced to the problem of finding monomial in n variables such that L^d is faithful.

Corollary 3.4.10. *If conjecture 3.1.11 is true for some number of variables n and degree d, conjecture* 3.1.12 *is true for $n+1$ and d.*

Proof. Let μ be an integer with $n+1 \leq \mu \leq \binom{n+d}{n}$. We consider two cases.

- If $\mu \leq \binom{n+d-1}{n}$, let $I \subseteq S$ be a monomial ideal generated by $\mu - 1$ degree d monomials satisfying the SLP. Then $R/I \otimes_K (K[y]/(y^d))$ is quotient of $R[y]$ by an ideal generated by μ degree d monomials and satisfies the WLP.
- Suppose $\mu \geq \binom{n+d-1}{n}$. Let $I \subseteq R$ be a monomial ideal generated by $\binom{n+d-1}{n} - 1$ degree d monomials satisfying the SLP. Then the ideal $J = I^e + (y^d)$ of $R[y]$ satisfies the WLP and is generated by $\binom{n+d-1}{n}$ monomials. Then, $H(R[y]/J, d) = H(R[y], d) - \binom{n+d-1}{n} = H(R[y], d-1) = H(R[y]/J, d-1)$, as $I_{d-1} = 0$. Then, as

  $$\times \overline{L} : (R[y]/J)_{d-1} \longrightarrow (R[y]/J)_d$$

  has maximal rank, it is an isomorphism. Define:

  $$\tilde{J} = J + (\text{any set of } \mu - \binom{n+d-1}{n} \text{ degree } d \text{ monomials not in } J).$$

  For any $t < d$, the maps $\times L : (R[y]/I)_{t-1} \longrightarrow (R[y]/I)_t$ resemble the maps $\times L : (R[y])_{t-1} \longrightarrow (R[y])_t$ and thus are injective. The map $\times \overline{L} : (R[y]/\tilde{J})_{d-1} \longrightarrow (R[y]/\tilde{J})_d$ is surjective because in the commutative diagram

  $$\begin{tikzcd} (R[y]/J)_{d-1} \arrow[r, "\times \overline{L}"] \arrow[d, "\pi_{d-1}"] & (R[y]/J)_d \arrow[d, "\pi_d"] \\ (R[y]/\tilde{J})_{d-1} \arrow[r, "\times \overline{L}"] & (R[y]/\tilde{J})_d, \end{tikzcd}$$

$\times \overline{L} : (R[y]/J)_{d-1} \longrightarrow (R[y]/J)_d$ and $\pi_d : (R/J)_d \longrightarrow (R/\tilde{J})_d$ are surjective. Then
$\times \overline{L} : (R/\tilde{J})_{t-1} \longrightarrow (R/\tilde{J})_t$ is also surjective for all $t \geq d$ and we conclude that $R/\tilde{J}$ satisfies the WLP.

□

With the idea from the previous proof and the following result from Nasrin Altafi and Samuel Lundqvist we can solve the case when the number of generators is sufficiently large:

Theorem 3.4.11. *Let I be an Artinian ideal of R generated by degree $d \geq 3$ monomials. If $H(R/I, d) \leq 2$, R/I satisfies the SLP.*

Proof. See [2, Theorem 2]. □

Corollary 3.4.12. *Conjecture 3.1.12 is true for $d = 2$ and for $d \geq 3$ and $\mu \geq \binom{n+d-1}{n-1} - 2$.*

Proof. This is a direct consequence of corollary 3.4.8, the fact that conjecture 3.1.11 is true for $d = 2$, and theorem 3.4.11. □

3.5 The strong Lefschetz property of a monomial almost complete intersection

Notice that Stanley's theorem covers the case $\mu = n$ in conjecture 3.1.11. In this section we wish to cover the cases $\mu = n + 1$ of conjecture 3.1.11 (and thus also conjecture 3.1.12). For this section, let S denote the ring $K[x, y]$ and $R = K[x, y, z_1, \ldots, z_n]$ for some $n \geq 1$. Define the ideals $J = (x^d, y^d, x^{d-1}y) \subseteq S$ for some $d \geq 3$ and set $A = S/J$. A satisfies the SLP as it is an Artinian graded K-algebra in two variables. Our aim is to show that for any $a_1, \ldots, a_n \geq 1$, the ideal $(x^d, y^d, x^{d-1}y, z^{a_1}, \ldots, z^{a_n}) \subseteq R$ also does. In particular, setting $a_1 = \cdots = a_n = d$ gives us the desired result.
First we study the Hilbert function of A. We will do that from the minimal free resolution.

Proposition 3.5.1. *The minimal free resolution of A takes the form :*

$$0 \longrightarrow S(-d-1) \oplus S(-2d+1) \longrightarrow S(-d)^3 \longrightarrow S \longrightarrow A \longrightarrow 0.$$

Proof. I is generated by three elements $x^d, y^d, x^{d-1}y$, so the image of the morphism

$$\varphi_1 : S(-d)^3 \longrightarrow S$$

that sends a graded basis (e_1, e_2, e_3) of degree d elements to $(x^d, y^d, x^{d-1}y)$ is the kernel of $S \longrightarrow A$. The kernel of φ_1 is generated by the syzygies $ye_1 - xe_3, x^{d-1}e_2 - y^{d-1}e_3$ and $y^d e_1 - x^d e_2$. This last element is redundant because

$$y^d e_1 - x^d e_2 = y^{d-1}(ye_1 - xe_3) - x(x^{d-1}e_2 - y^{d-1}e_3).$$

For a graded basis of $S(-d-1) \oplus S(2d-1)$, (f_1, f_2), with $\deg f_1 = d+1$ and $\deg f_2 = 2d-1$, define the morphism $\varphi_2 : S(-d-1) \oplus S(2d-1) \longrightarrow S(-d)^3$ sending f_1 to $ye_1 - xe_3$ and f_2 to $x^{d-1}e_2 - y^{d-1}e_3$. Then $\mathrm{Im}\varphi_2 = \ker \varphi_1$. By Hilbert's syzygy theorem, φ_2 is injective, so the minimal free resolution takes the given form.

□

Proposition 3.5.2. *The Hilbert function of A is*

$$H(A,i) = \begin{cases} 0 & \text{if } i < 0 \text{ or } i > 2d-3 \\ i+1 & \text{if } 0 \le i \le d-1 \\ 2d-i-2 & \text{if } d \le i \le 2d-3. \end{cases}$$

Proof. For an integer j define $(j)_+ = j$ if j is non-negative and $(j)_+ = 0$ otherwise. Notice that the Hilbert function of S is $H(S,i) = (i+1)_+$. Then from the minimal free resolution of A we obtain.

$$H(A,i) = (i+1)_+ - 3(i-d+1)_+ + (i-d)_+ + (i-2d+2)_+.$$

Notice that this immediately gives $H(A,i) = i+1$ for $0 \le i \le d-1$. For $d \le i \le 2d-3$, we obtain

$$H(A,i) = (i+1) - 3(i-d+1) + (i-d) = 2d-i-2.$$

When $i > 2d-3$, the last therm $(i-2d+2)_+$ cancels out the rest of the summands leaving $H(A,i) = 0$ when $i > 2d-3$. □

Example 3.5.3. The h-vector of $S/(x^{10}, y^{10}, x^9 y)$ is

$$(1,2,3,4,5,6,7,8,9,10,8,7,6,5,4,3,2,1)$$

The pattern is always the same. It increases one at a time until reaching d, decreases 2 once, and decreases one at a time until it reaches 1.

The next definition is from [14].

Definition 3.5.4. *Let $h = (h_0, h_1, \dots, h_s)$ be the h-vector of an Artinian graded algebra of socle degree s. h is in the class $\mathcal{H}$ if it satisfies*

$$h_{i-1} \leq h_{s-i} \leq h_i \text{ for all } 1 \leq i \leq \left[\frac{s}{2}\right] \tag{3.1}$$

or

$$h_{s-i+1} \leq h_i \leq h_{s-i} \text{ for all } 1 \leq i \leq \left[\frac{s}{2}\right]. \tag{3.2}$$

Proposition 3.5.5. *The h-vector of A satisfies equation 3.2 from the previous definition.*

Proof. The socle degree of A is $s = 2d - 3$, so $\left[\frac{s}{2}\right] = d - 2$. Suppose $1 \leq i \leq d - 2$. Then

- $h_{s-i+1} = i$.
- $h_i = i + 1$.
- If $i < d - 2$, $h_{s-i} = i + 1$. If $i = d - 2$, $h_{s-i} = h_{d-1} = d = i + 2$.

In either case, $h_{s-i+1} \leq h_i \leq h_{s-i}$ for all $1 \leq i \leq d - 2$. ☐

Theorem 3.5.6. *For a polynomial ring $R = K[x_1, \dots, x_n]$, suppose M is an Artinian graded R-module with the SLP and $l \in R_1$ is a strong Lefschetz element. Then the h-vector of M is in the class $\mathcal{H}$ if and only if $l + y \in R[y]$ is a strong Lefschetz element of $M \otimes_k K[y]/(y^m)$ for all $m \geq 0$.*

Proof. See [14, Theorem 3.10]. ☐

Corollary 3.5.7. *For all $m \geq 1$, $K[x,y,z]/(x^d, y^d, x^{d-1}y, z^m)$ satisfies the SLP.*

Proof. Follows directly from the fact that $S/(x^d, y^d, x^{d-1}y)$ satisfies the SLP with its h-vector in the class $\mathcal{H}$, and

$$K[x,y,z]/(x^d, y^d, x^{d-1}y, z^m) \cong S/(x^d, y^d, x^{d-1}y) \otimes_K K[z]/(z^m).$$

☐

In fact, using the results from the paper of Melissa Lindsey, we can extend the result to any number of variables.

Corollary 3.5.8. *Given non-negative integers $a_1, \dots, a_n$, the algebra*

$$R/(x^d, y^d, x^{d-1}y, z_1^{a_1}, \dots, z_n^{a_n})$$

satisfies the SLP.

Proof. See [14, Theorem 3.10] and [14, Corollary 3.4]. ☐

3.6 Difficulties in higher variables

Although we have reduced the problem of finding monomial ideals in $n+1$ variables satisfying the WLP to finding monomial ideals in n variables such that L^d is faithful, this problem proves to be quite elusive even in the case $n = 3$. In this section we show that some natural families of ideals of $R = K[x,y,z]$ actually fail this property.
A first intuitive attempt would be considering a succession similar to the one used in section 3.3. When taking the ideal generated by x^d, y^d, z^d and all degree d monomials coprime with z, we obtain the following obstruction.

Proposition 3.6.1. *Suppose $d \geq 3$ and let $I = (x,y)^d + (z^d)$. Then, the multiplication $\times L^d : (R/I)_{d-2} \longrightarrow (R/I)_{2d-2}$ fails required surjectivity.*

Proof. First we show that $H(R/I, d-2) \geq H(R/I, 2d-2)$. Notice that $H(R/I, d-2) = \binom{d}{2}$. Defining $\mathcal{B}_{d-1}$ to be the set of all degree $d-1$ monomials in variables x, y, the set of monomials $z^{d-1}\mathcal{B}_{d-1}$ forms a basis of $(R/I)_{2d-2}$. Therefore $H(R/I, 2d-2) = d$. As for all $d \geq 3$, $\binom{d}{2} \geq d$, $H(R/I, d-2) \geq H(R/I, 2d-2)$.

Let $f = \sum_{i=0}^{d-1} a_i x^{d-1-i} y^i z^{d-1}$ be in the image of $\times L^d : (R/I)_{d-2} \longrightarrow (R/I)_{2d-2}$, and let F be a lift of f to R_{2d-2} that is a multiple of L^d. Then $F(1,-1,z) \in K[z]$ is a multiple of z^d. The coefficient of z^{d-1} in $F(1,-1,z)$ is 0, so $\sum_{i=0}^{d-1} (-1)^i a_i = 0$. In particular, $f \neq x^{d-1}z^{d-1}$ so $\times L^d : (R/I)_{d-2} \longrightarrow (R/I)_{2d-2}$ is not surjective. □

Another possible attempt is considering the succession starting with x^d, y^d, z^d and then taking monomials in the lexicographical order.

$$x^d, y^d, z^d, x^{d-1}y, x^{d-1}z, x^{d-2}y^2, x^{d-2}yz, x^{d-2}z^2, \dots$$

This also fails.

Proposition 3.6.2. *For $d \geq 6$, let I be the ideal generated by the first 17 monomials of this succession. That is*

$$I = (x^d, y^d, z^d, x^{d-1}y, x^{d-1}z, x^{d-2}y^2, x^{d-2}yz, x^{d-2}z^2, x^{d-3}y^3, x^{d-3}y^2z, x^{d-3}yz^2, x^{d-3}z^3,$$
$$x^{d-4}y^4, x^{d-4}y^3z, x^{d-4}y^2z^2, x^{d-4}yz^3, x^{d-4}z^4).$$

Then $\times L^d : (R/I)_{d-4} \longrightarrow (R/I)_{2d-4}$ fails required injectivity, as $x^{d-4}L^d = 0$.

Proof. We should first see that this injectivity is indeed required. $H(R/I, d-4) = \binom{d-2}{2}$, so we must calculate $H(R/I, 2d-4)$ and check it is larger.
A monomial $x^i y^j z^k$ is non-zero in $(R/I)_{2d-4}$ if and only if it satisfies the following conditions

$$\begin{cases} i+j+k = 2d-4 \\ i < d-4 \\ j < d \\ k < d \end{cases}$$

We want to determine the amount of solutions of this equation in $\mathbb{N}^3$. After fixing an $i \in \{0, 1, \ldots, d-5\}$, the system reduces to

$$\begin{cases} j+k = 2d-4-i \\ j < d \\ k < d \end{cases}$$

which has $i+3$ solutions. Varying i, we obtain

$$H(R/I, 2d-4) = \sum_{i=0}^{d-5} (i+3) = 3+4+\cdots+d-2 = \binom{d-1}{2} - 3.$$

For $d \geq 6$, $\binom{d-1}{2} - 3 \geq \binom{d-2}{2}$.
Notice that every monomial in the support of $x^{d-4}L^d$ is zero as it is either a multiple of x^d, or a multiple of some monomial $x^{d-4}y^j z^{4-j}$, so $x^{d-4}L^d = 0$. □

3.7 Random ideals with a faithful power of L

After encountering difficulties to find a family of ideals in three variables such that L^d is faithful, we still aim to show such ideals exist, at least for low degrees and three variables. We do that computationally, and following this process:

1. Fix integers $d \geq 3$ and k, $2 \leq k \leq \binom{d+2}{2} - 3$. We wish to show there is a monomial ideal generated by $3+k$ monomials such that L^d is faithful· We omit the cases $k=0$ and $k=1$ because they are already covered by Stanley's theorem and section 3.5.

2. Take a randomly generated set B of k degree d monomials excluding x^d, y^d, z^d.

3. Define $I = (B) + (x^d, y^d, z^d)$

4. Check whether the maps of multiplication by L^d in R/I have maximal rank. If they do we are done. Otherwise go back to step 2.

5. Repeat this process for all $k \in \{2, 3, \dots, \binom{d+2}{2} - 3\}$.

Where for this section, $R = K[x, y, z]$. To run this algorithm, we implement three Macaulay 2 functions. First, we implement one to generate a random monomial ideal for given k and d.

```
    idealRandom=(d,k)->(
        B:=random(flatten entries basis(d,R));
        B=select(B,m-> not member(m,{x^d,y^d,z^d}));
        B=take(B,k);
        I:=ideal(B)+ideal(x^d,y^d,z^d);
        return I;
);
```

Secondly, we introduce another function to check the property for a random ideal.

```
    randomDLP=(d,k)->(
        I:=idealRandom(d,k);
        return hasMaxRank(promote((x+y+z)^d,R/I));
);
```

Here we are using the *hasMaxRank* function from the package *MaximalRankProperties* ([17]), which checks faithfulness of a given homogeneous element. Lastly, we implement a function that, for a given d, it generates random ideals with k generators until one satisfying the property is found. It then prints the number k and the amount of attempts taken to find the ideal.

```
   checkDegree=(d)->(
        k:=2;
        att:=1;
        while(k<binomial(d+2,2)-3) do (
                if (randomDLP(d,k)===true) then (
                        print (k,att);
                        k=k+1;
                        att=1;
                )
                else (
                        att=att+1;
                );
        );
);
```

If the checkDegree function is unable to find an Artinian ideal generated by $3 + k$ degree d monomials, it will not halt. However, for all d up to 50, we have run the checkDegree function and it has completed its task. This prompts us to postulate the following conjecture.

Conjecture 3.7.1. *For any integers* $d \geq 2$ *and* $0 \leq k \leq \binom{d+2}{2} - 3$ *there is an Artinian ideal of the ring* $R = K[x, y, z]$ *generated by* $3 + k$ *degree* d *monomials such that* L^d *is a faithful element.*

The veracity of this conjecture would imply the case $n = 4$ of conjecture 3.1.12, and this conjecture is implied by the case $n = 3$ of conjecture 3.1.11.

Bibliography

[1] Nasrin Altafi and Mats Boij, *The weak Lefschetz property of equigenerated monomial ideals*, Journal of Algebra **556** (2020), 136–168.

[2] Nasrin Altafi and Samuel Lundqvist, *Monomial ideals and the failure of the strong Lefschetz property*, Collectanea Mathematica **73** (2022), no. 3, 383–390.

[3] Winfried Bruns and H. Jürgen Herzog, *Cohen-Macaulay Rings*, 2 ed., Cambridge Studies in Advanced Mathematics, no. 2, Cambridge University Press, 1998.

[4] Armando Cerminara, Rodrigo Gondim, Giovanna Ilardi, and Giuseppe Zappalà, *On the Hilbert function of Gorenstein algebras of socle degree four*, Journal of Pure and Applied Algebra **224** (2020), no. 12, 106434.

[5] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www2.macaulay2.com`.

[6] Elena Guardo and Adam Tuyl, *Powers of complete intersections: graded Betti numbers and applications*, Illinois journal of mathematics **49** (2005), 265–279.

[7] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe, *Lefschetz properties*, pp. 97–140, Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[8] Tadahito Harima, Juan Migliore, Uwe Nagel, and Junzo Watanabe, *The weak and strong Lefschetz properties for Artinian K-algebras*, Journal of Algebra **262** (2002), 99–126.

[9] Robin Hartshorne, *Algebraic Geometry*, 1 ed., Graduate Texts in Mathematics, vol. 52, Springer Nature, New York, NY, 1977 (eng).

[10] James E. Humphreys, *Semisimple Lie algebras*, pp. 15–41, Springer New York, New York, NY, 1972.

[11] Uwe Nagel Juan C. Migliore, Rosa M. Miró-Roig, *Monomial ideals, almost complete intersections and the weak Lefschetz property*, Transactions of the American Mathematical Society **363** (2009), 229–257.

[12] Filip Jonsson Kling, *The strong Lefschetz property for quadratic reverse lexicographic ideals*, Proceedings of the American Mathematical Society Series B **11** (2024), no. 35, 390–401.

[13] Jizhou Li and Fabrizio Zanello, *Monomial complete intersections, the weak Lefschetz property and plane partitions*, Discrete Mathematics **310** (2010), no. 24, 3558–3570.

[14] Melissa Lindsey, *A class of Hilbert series and the strong Lefschetz property*, Proceedings of the American Mathematical Society **139** (2011), no. 1, 79–92.

[15] Juan C. Migliore and Giuseppe Favacchio, *Lefschetz properties in algebra, geometry and combinatorics: Notes for the preparatory school*, May 2024.

[16] Rosa M. Miró-Roig, *Determinantal ideals*, Progress in mathematics ; 264, Basel, 2008 (eng).

[17] Lisa Nicklasson, *Maximal rank properties*, Available at https://github.com/LisaNicklasson/MaximalRankProperties-Macaulay2-package, 2023.

[18] Ho Phuong and Tran Quang Hoa, *A new proof of Stanley's theorem on the strong Lefschetz property*, Colloquium Mathematicum **173** (2023), 1–8.

[19] Les Reid, Leslie G. Roberts, and Moshe Roitman, *On complete intersections and their hilbert functions*, Canadian Mathematical Bulletin **34** (1991), no. 4, 525–535.

[20] Richard P. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM Journal on Algebraic Discrete Methods **1** (1980), no. 2, 168–184.

[21] Junzo Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math **11** (1987), 303–312.