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Deficit at ruin with threshold proportional reinsurance

Abstract

In this paper, we focus our analysis on the distribution function and the moments of the deficit at ruin in a model with a threshold proportional reinsurance strategy using the Gerber-Shiu function. This strategy considers a proportional reinsurance, but the retention level is not constant and depends on the surplus. Then a retention level k_1 is applied whenever the surplus is less than a specific threshold b, and a retention level k_2 is applied in the other case.

In a Poisson risk model, we derive the integro-differential equation for the Gerber-Shiu function when the claim amount is exponentially distributed. Then, we obtain the analytical expression for the Gerber-Shiu function for a set of penalty functions. This analytical expression is applicable for several penalty functions and includes, among others, the ruin probability, the time of ruin and the distribution function of the deficit at ruin.

Keywords: Gerber-Shiu function, reinsurance, penalty function.

Introduction

The classical compound Poisson risk model has been studied extensively (Bowers et al., 1997; Gerber, 1979; Dickson, 2005; Asmussen 2000...). In 1998, Gerber and Shiu introduced the so-called Gerber-Shiu function defined as a discounted penalty function payable at ruin. This function allows obtaining, among others, ruin probability, the time to ruin, the level of the surplus both prior and at ruin.

In this paper, using the Gerber-Shiu function, we focus our analysis on the distribution function and the moments of the deficit at ruin in a model with a threshold proportional reinsurance strategy. This strategy considers a proportional reinsurance, but the retention level is not constant and depends on the surplus. Then a retention level k_1 is applied whenever the surplus is less than a specific threshold b, and a retention level k_2 is applied in the other case. Since for the insurer, reinsurance is a tool for controlling the solvency of the portfolio, it seems natural that the retention level should depend on the surplus level at any given moment. The threshold proportional reinsurance strategy is an easy and clear way to include such dependence. The aim of this paper is to analyze the deficit at ruin in a Poisson risk model with a threshold proportional reinsurance strategy.

The paper is organized as follows: In Section 1, the assumptions and some preliminaries are explained. In Section 2 we derive the integro-differential equation for the Gerber-Shiu function in a model with threshold proportional reinsurance strategy and assuming an exponential distribution for the individual claim amount. Then, we obtain the corresponding ordinary differential equation that allows us obtaining the ruin probability, the Laplace transform of the time of ruin, and the distribution function and the moments of the deficit at ruin. In Section 3 we obtain analytical expressions for the distribution function and the moments of the deficit at ruin if ruin occurs for exponential individual claim amount. In Section 4 some numerical results are presented. The final section concludes the paper.

1. Assumptions and preliminaries

In the classical risk theory model, the surplus, $R(t)$, at a given time $t\in[0,\infty)$ is defined as $R(t) = u + ct - S(t)$, with $u = R(0) \geq 0$ being the insurer's initial surplus, $S(t)$ the aggregate claims and c the rate at which the premiums are received.

$S(t)$ is modeled as a compound Poisson process

$$S(t)=\sum_{i=1}^{N(t)} X_i ,$$

where $N(t)$, the number of claims occurring until time t, follows a Poisson process with parameter λ, the amount of individual claims $\{X_i,\ i\geq 1\}$ is a sequence of independent and identically distributed random variables with density function $f(x)$ and $N(t)$ is independent of $\{X_i,\ i\geq 1\}$.

The instantaneous premium rate, c, is proportional to the product of the mean number of claims, λ, and the mean value of the individual claim amount, $E[X]$. In other words, $c = \lambda E[X](1 + \rho)$, where ρ, called the security loading coefficient, is a positive constant, in order to fulfill the net profit condition.

In this model, and in the more general ordinary renewal model, the interclaim-time random variables, $\{T_i\}_{i=1}^{\infty}$, are modeled as a sequence of independent and identically distributed random variables, where T_1 denotes the time until the first claim and T_i, for $i > 1$, denotes the time between the $(i-1)$th and ith claims. Note that in a Poisson process with parameter λ, T_i, $i > 1$ has an exponential distribution with mean $1/\lambda$.

The time to ruin is defined as $T = min\ \{t|R(t) < 0\}$, with $T = \infty$ if $R(t) \geq 0$ for all $t \geq 0$. The ultimate ruin probability is

$$\psi(u) = P\left[T < \infty \mid R(0) = u\right]$$
$$= E\left\{I\left(T < \infty\right) \mid R(0) = u\right\},$$

where $I(A) = 1$ if A occurs and $I(A) = 0$ otherwise.

Let us first consider the effect of a proportional reinsurance. The ceding company (insurer) and the reinsurer agree on a cession percentage, say $(1 - k)$, being k the retention level applied to each claim. Then, in one period, the expected aggregate cost assumed by the insurer is $k\lambda E[X]$ and the expected aggregate cost assumed by the reinsurer is $(1 - k)\lambda E[X]$.

We assume that insurance and reinsurance premiums are calculated following the expected value principle with positive loading factors, being $\rho_R > 0$ the reinsurer loading factor.

The total premium income rate retained by the insurer, c', depends on ρ_R and k, where

$$c' = \lambda E\left[X\right]\left(1+\rho\right) - \left(1-k\right)\left(1+\rho_R\right)\lambda E\left[X\right]. \qquad (1)$$

A new security loading for the insurer, ρ_N, can be defined, considering that $c' = k\lambda\lambda\left[X\right]\left(1+\rho_N\right)$, from (1)

$$\rho_N = \rho_R - \frac{\rho_R - \rho}{k}, \forall k > 0. \qquad (2)$$

If $\rho = p_R$, the total premium paid by the policyholder c is shared between insurer and reinsurance in the same proportion k, so $c' = kc$ and $\rho_N = p$. It is normally assumed that $\rho_R > \rho$, because if $\rho > \rho_R$ the insurer would simply cede his entire portfolio to the reinsurer, a situation which would be senseless.

Let $R^-(T)$ be the surplus immediately before ruin, and $R^+(T)$ the surplus at ruin if ruin occurs. Gerber and Shiu (1998, 2005) define the function

$$\phi(u) = E\left[e^{-\delta T} w\left(R^-(T), \left|R^+(T)\right|\right) I\left(T<\infty\right) \mid R(0)=u\right], \qquad (3)$$

where $\delta \geq 0$ is the discounted factor, and $w(x,j)$, $x \geq 0$, $j > 0$, is the penalty function, so that $\phi(u)$ is the expected discounted penalty payable at ruin. This function is known to satisfy a defective renewal equation (Gerber and Shiu, 1998; Li and Garrido, 2004; Willmot, 2007).

In this paper, we consider a threshold proportional reinsurance strategy, which is defined by a threshold $b \geq 0$. A retention level k_1 is applied whenever the surplus is less than b, and a retention level k_2 is applied in the other case. Then, the premium income retained is c_1 and c_2, respectively. We consider that the retention levels give new positive security loadings for the insurer, i.e., the net profit condition is always fulfilled. From (2), we can define

$$\rho_1 = \rho_R - \frac{\rho_R - \rho}{k_1},$$

$$\rho_2 = \rho_R - \frac{\rho_R - \rho}{k_2}.$$

Graphically,

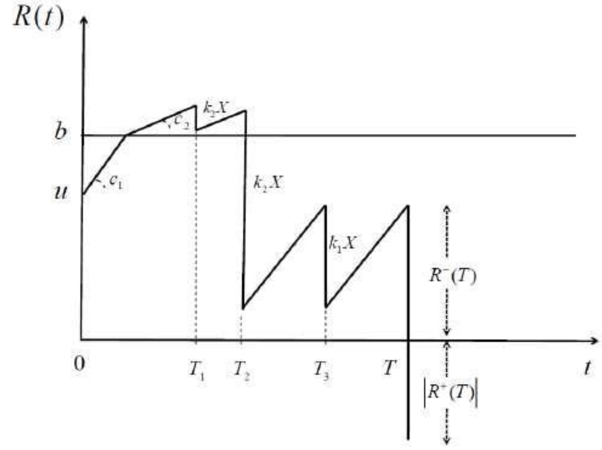

Fig. 1. Threshold reinsurance strategy

Depending on $w(x,j)$, we can obtain different interpretations for the Gerber-Shiu function. In this paper, we will consider the following possibilities, for $w(x,j)$

$$j^m \begin{cases} m=0 \Rightarrow \begin{cases} \phi(u) = E\left[e^{-\delta T} I\left(T<\infty\right) \mid R(0)=u\right] \\ \quad = P\left[e^{-\delta T}\left(T<\infty\right) \mid R(0)=u\right] \\ If\ \delta=0 \Rightarrow \phi(u)=\psi(u) \end{cases} \\ m>0 \Rightarrow \phi(u) = E\left[e^{-\delta T} j^m I\left(T<\infty\right) \mid R(0)=u\right] \end{cases} \qquad (4)$$
$$I\left(j \leq y\right) \Rightarrow \phi(u) = E\left[e^{-\delta T} I\left(j \leq y\right) I\left(T<\infty\right) \mid R(0)=u\right].$$

The interpretations shown in (4) include the discounted factor, δ. For $w(x,j) = 1$, we obtain the defective Laplace transform of the time of ruin being δ the parameter. If we consider $\delta = 0$, in addition to the ultimate ruin probability included in (4), two other interesting functions can be obtained by dividing the Gerber-Shiu function by the probability of ruin: first, for $w(x,j) = j^m$, $m > 0$, the ordinary moments of the deficit at ruin if ruin occurs, and second, for $w(x,j) = I(j \leq y)$, the distribution function of the deficit at ruin if ruin occurs.

2. Integro-differential equation for the Gerber-Shiu function

We derive the integro-differential equations satisfied by the Gerber-Shiu discounted penalty function. The discounted penalty function $\phi(u)$ behaves differently, depending on whether its initial surplus u is below or above the level b. Hence, for notational convenience, we write

$$\phi(u)=\begin{cases}\phi_1(u), & 0\le u<b,\\ \phi_2(u), & u\ge b.\end{cases}$$

The following theorem provides integro-differential equations for $\phi(u)$ in a Poisson process model with unitary exponential individual claim amount.

Theorem 1. The discounted penalty function $\phi(u)$ satisfies the integro-differential equations

$$\phi'(u)=\begin{cases}\phi'_1(u), & 0\le u<b,\\ \phi'_2(u), & u\ge b,\end{cases} \tag{5}$$

where

$$\phi'_1(u)=\frac{\lambda+\delta}{c_1}\phi_1(u)$$

$$-\frac{\lambda}{c_1}\int_0^{\frac{u}{k_1}}\phi_1(u-k_1x)e^{-x}dx-\frac{\lambda}{c_1}\xi_1(u),$$

$$\phi'_2(u)=\frac{\lambda+\delta}{c_2}\phi_2(u)-\frac{\lambda}{c_2}\left[\int_0^{\frac{u-b}{k_2}}\phi_2(u-k_2x)e^{-x}dx\right.$$

$$\left.+\int_{\frac{u-b}{k_2}}^{\frac{u}{k_2}}\phi_1(u-k_2x)e^{-x}dx\right]-\frac{\lambda}{c_2}\xi_2(u),$$

and

$$\xi_1(t)=\int_{\frac{t}{k_1}}^{\infty}w\left(t,k_1x-t\right)e^{-x}dx,$$

$$\xi_2(t)=\int_{\frac{t}{k_2}}^{\infty}w\left(t,k_2x-t\right)e^{-x}dx.$$

Let w($R^-(T|\ R^+(T)|)$) be a nonnegative function of $R^-(T)>0$, the surplus immediately before ruin, and $R^+(T)>0$ the surplus at ruin.

Proof. For $0\le u<b$,

$$\phi_1(u)=\int_0^{\frac{b-u}{c_1}}\lambda e^{-(\lambda+\delta)t}\left[\int_0^{\frac{u+c_1t}{k_1}}\phi(u+c_1t-k_1x)e^{-x}dx\right.$$

$$\left.+\int_{\frac{u+c_1t}{k_1}}^{\infty}w(u+c_1t,k_1x-u-c_1t)e^{-x}dx\right]dt$$

$$+\int_{\frac{b-u}{c_1}}^{\infty}\lambda e^{-(\lambda+\delta)t}\left[\int_0^{\frac{b+c_2\left(t-\frac{b-u}{c_1}\right)}{k_2}}\phi\left(b+c_2\left(t-\frac{b-u}{c_1}\right)-k_2x\right)e^{-x}dx\right. \tag{6}$$

$$\left.+\int_{\frac{b+c_2\left(t-\frac{b-u}{c_1}\right)}{k_2}}^{\infty}w\left(b+c_2\left(t-\frac{b-u}{c_1}\right),k_2x-b-c_2\left(t-\frac{b-u}{c_1}\right)\right)e^{-x}dx\right]dt$$

$$=\lambda\int_0^{\frac{b-u}{c_1}}e^{-(\lambda+\delta)t}\gamma_1(u+c_1t)dt+\lambda\int_{\frac{b-u}{c_1}}^{\infty}e^{-(\lambda+\delta)t}\gamma_2\left(b+c_2\left(t-\frac{b-u}{c_1}\right)\right)dt,$$

where

$$\gamma_1(t)=\int_0^{\frac{t}{k_1}}\phi(t-k_1x)e^{-x}dx+\xi_1(t),$$

$$\gamma_2(t)=\int_0^{\frac{t}{k_2}}\phi(t-k_2x)e^{-x}dx+\xi_2(t).$$

Now, a change of variables in (6) results in

$$\phi_1(u)=\frac{\lambda}{c_1}e^{\frac{(\lambda+\delta)u}{c_1}}\int_u^b e^{-\frac{(\lambda+\delta)t}{c_1}}\gamma_1(t)dt$$

$$+\frac{\lambda}{c_2}e^{\frac{(\lambda+\delta)u}{c_1}}\int_b^{\infty}e^{-\frac{(\lambda+\delta)}{c_2}\left(t-\frac{(c_1-c_2)b}{c_1}\right)}\gamma_2(t)dt. \tag{7}$$

By differentiating (7) with respect to u we obtain

$$\phi'_1(u)=\frac{(\lambda+\delta)}{c_1}\phi_1(u)$$

$$-\frac{\lambda}{c_1}\int_0^{\frac{u}{k_1}}\phi_1(u-k_1x)e^{-x}dx-\frac{\lambda}{c_1}\xi_1(u).$$

Similarly, when $u\ge b$,

$$\phi_2(u)=\int_0^{\infty}\lambda e^{-(\lambda+\delta)t}\left[\int_0^{\frac{u+c_2t}{k_2}}\phi\left(u+c_2t-k_2x\right)e^{-x}dx\right.$$

$$\left.+\int_{\frac{u+c_2t}{k_2}}^{\infty}w\left(u+c_2t,k_2x-u-c_2t\right)e^{-x}dx\right]dt$$

$$=\lambda\int_0^{\infty}e^{-(\lambda+\delta)t}\gamma_2\left(u+c_2t\right)dt.$$

With a change of variable and differentiating with respect to u

$$\phi'_2(u) = \frac{(\lambda+\delta)}{c_2}\phi_2(u) - \frac{\lambda}{c_2}\int_0^{\frac{u-b}{k_2}} \phi_2(u-k_2x)e^{-x}dx - \frac{\lambda}{c_2}\int_{\frac{u-b}{k_2}}^{\frac{u}{k_2}} \phi_1(u-k_2x)e^{-x}dx - \frac{\lambda}{c_2}\xi_2(u),$$

by which the proof is concluded.

From (5), differentiating with respect to u we can obtain the ordinary differential equation

$$\phi''_i(u) - \left(\frac{\lambda+\delta}{c_i} - \frac{1}{k_i}\right)\phi'_i(u) - \frac{\delta}{c_ik_i}\phi_i(u) - \frac{\lambda}{c_i}\left(\frac{1}{k_i}\xi_i(u) + \xi'_i(u)\right) = 0, \quad (8)$$

for i = 1,2 being i = 1 for $0 < u < b$ and i = 2 for $u > b$. If $\frac{1}{k_i}\xi_i(u)+\xi'_i(u)=0$ the ordinary differential equations (8) are independent of *w(x,j)*. The general solution of this new differential equation $\frac{1}{k_i}\xi_i(u)+\xi'_i(u)=0$ is $\xi_i(u) = A_ie^{-\frac{u}{k_i}}$.

For all the specific forms for *w(x,j)* included in (4) it is easy to demonstrate that $\xi_i(u)$ has the form $A_ie^{-\frac{u}{k_i}}$, being

$$A_i = \begin{cases} 1 & \text{If } w(x,j) = j^m, m = 0 \\ m!k_i^m & \text{If } w(x,j) = j^m, m > 0 \\ 1 - e^{-\frac{y}{k_i}} & \text{If } w(x,j) = I(j \le y). \end{cases}$$

Therefore, (8) is

$$\phi''_i(u) - \left(\frac{\lambda+\delta}{c_i} - \frac{1}{k_i}\right)\phi'_i(u) - \frac{\delta}{c_ik_i}\phi_i(u) = 0, \quad (9)$$

with i = 1 for $0 < u < b$ and i = 2 for $u > b$.

The ordinary differential equation (9) is common to all forms of *w(x,j)* included in (4). In the next Section, we obtain the explicit solutions of the Gerber-Shiu function considering (9) and the conditions in which A_i appear.

3. Analytical expressions for the deficit at ruin

In this section, we obtain the analytical expressions for the different particular cases that arise from the Gerber-Shiu function when *w(x,j)* has the forms included in (4).

From (9),

$$\phi(u) = \begin{cases} \phi_1(u) = C_1e^{r_1u} + C_2e^{r_2u}, & 0 \le u < b, \\ \phi_2(u) = D_1e^{s_1u} + D_2e^{s_2u}, & u \ge b, \end{cases} \quad (10)$$

being

$$r_1 = \frac{\delta - \lambda\rho_1 - \sqrt{\delta(2\lambda(2+\rho_1)+\delta) + (\lambda\rho_1)^2}}{2k_1\lambda(1+\rho_1)} < 0,$$

$$r_2 = \frac{\delta - \lambda\rho_1 + \sqrt{\delta(2\lambda(2+\rho_1)+\delta) + (\lambda\rho_1)^2}}{2k_1\lambda(1+\rho_1)} \ge 0,$$

$$s_1 = \frac{\delta - \lambda\rho_2 - \sqrt{\delta(2\lambda(2+\rho_2)+\delta) + (\lambda\rho_2)^2}}{2k_2\lambda(1+\rho_2)} < 0,$$

$$s_2 = \frac{\delta - \lambda\rho_2 + \sqrt{\delta(2\lambda(2+\rho_2)+\delta) + (\lambda\rho_2)^2}}{2k_2\lambda(1+\rho_2)} \ge 0.$$

The coefficients of the solution of the ordinal differential equations (C_i, D_i, i = 1,2) are obtained from a system of four conditions. The first two conditions are common for all *w(x,j)* included in (4) and are $\lim_{u\to\infty}\phi(u) = 0$, and the continuity condition $\phi_1(b) = \phi_2(b)$. The other two are obtained by substituting (10) in (5) and depend on *w(x,j)*,

$$\sum_{i=1}^{2}\frac{C_i}{k_1r_i+1} = A_1,$$

$$\sum_{i=1}^{2}\frac{C_i}{k_2r_i+1}\left(1 - e^{b\left(r_i+\frac{1}{k_2}\right)}\right) + \frac{D_1}{s_1k_2+1}e^{b\left(s_1+\frac{1}{k_2}\right)} = A_2.$$

Once obtained the coefficients C_i, D_i, i = 1,2 the solution of $\phi(u)$ from (10) is,

$$\begin{aligned} \phi_1(u) &= A_1\left[a_{1,2}e^{r_2u} - d_1(\delta)H(u,\delta)\right] + A_2d_2(\delta)H(u,\delta), && 0 \le u < b, \\ \phi_2(u) &= e^{s_1u}\left[A_1\left[E(\delta) - d_1(\delta)J(\delta)\right] + A_2d_2(\delta)J(\delta)\right], && u \ge b, \end{aligned} \quad (11)$$

where $a_{i,j} = (k_ir_j + 1)$, i, j = 1,2.

Being

$$d_1(\delta) = a_{1,2}\left(k_2(r_2 - s_1)e^{\frac{a_{2,2}}{k_2}b} + k_2s_1 + 1\right),$$

$$d_2(\delta)=a_{2,2}(k_2 s_1+1),$$

$$H(u,\delta)=\frac{a_{2,1}a_{1,1}}{dn(\delta)}\left[e^{r_1 u}-\frac{a_{1,2}}{a_{1,1}}e^{r_2 u}\right],$$

$$dn(\delta)=(k_2 s_1+1)(r_1-r_2)(k_1-k_2)$$

$$-k_2\left(a_{1,1}a_{2,2}(s_1-r_1)e^{\frac{ba_{2,1}}{k_2}}-a_{1,2}a_{2,1}(s_1-r_2)e^{\frac{ba_{2,2}}{k_2}}\right),$$

$$J(\delta)=\frac{a_{2,1}a_{1,1}e^{(r_1-s_1)b}-a_{2,1}a_{1,2}e^{(r_2-s_1)b}}{dn(\delta)},$$

$$E(\delta)=a_{1,2}e^{(r_2-s_1)b}.$$

For $A_1 = A_2 = 1$, (11) is the explicit solution of the defective Laplace transform of the time of ruin if claim amount is exponential with unitary mean. This expression can be found, for example, in Castañer (2009).

The ruin probability can be obtained from (11) for $A_1 = A_2 = 1$, $\delta = 0$,

$$\begin{aligned}&\psi_1(u)=1+\left[d_2(0)-d_1(0)\right]H(u,0), && 0\le u<b,\\ &\psi_2(u)=e^{s_1 u}\left(E(0)+J(0)\left[d_2(0)-d_1(0)\right]\right), && u\ge b,\end{aligned}\tag{12}$$

being in this case

$$r_1=-\frac{\rho_1}{k_1(1+\rho_1)},$$

$$s_1=-\frac{\rho_2}{k_2(1+\rho_2)},$$

$$r_2=s_2=0.$$

The distribution of the deficit at ruin if ruin occurs can be obtained from (11) when $A_1=1-e^{-\frac{y}{k_1}}$, $A_2=1-e^{-\frac{y}{k_2}}$ and $\delta=0$, divided by (12). We can observe that it is a mixture of two exponential distributions, with different weights if $0\le u<b$ and $u\ge b$,

$$F_j(u,y)=\begin{cases}F_{1j}(u,y), & 0\le u<b,\\ F_{2j}(u), & u\ge b.\end{cases}$$

$$F_{1j}(u,y)$$
$$=P\left[j\le y\,|\left(R(0)=u,T<\infty\right)\right]$$
$$=1-P(u)e^{-\frac{y}{k_1}}-\left(1-P(u)\right)e^{-\frac{y}{k_2}},$$

$$F_{2j}(y)$$
$$=P\left[j\le y\,|\left(R(0)=u,T<\infty\right)\right]$$
$$=1-(1-Q)e^{-\frac{y}{k_1}}-Qe^{-\frac{y}{k_2}},$$

being

$$P(u)=\frac{1-d_1(0)H(u,0)}{1+\left[d_2(0)-d_1(0)\right]H(u,0)},$$

$$Q=\frac{d_2(0)J(0)}{E(0)+J(0)\left[d_2(0)-d_1(0)\right]},$$

the weights of the distribution. The weight *P(u)* depends on the initial surplus, but in contrast, the weight *Q* is independent of *u*. If the initial surplus is less than *b*, the deficit at ruin if ruin occurs depends on this initial surplus through the weight *P(u)*. But, if the initial surplus is greater than or equal to *b*, the distribution of the deficit at ruin is constant with respect to *u*.

The result that the deficit at ruin if ruin occurs follows a mixed exponential distribution is an expected one and we can give a probabilistic explanation to this.

Let $P[T\in S_1]$ be the probability that the surplus prior to ruin is less than *b* if ruin occurs,

$$P[T\in S_1]=P\left[R^-(T)<b\,|\,T<\infty\right].$$

Let $P[T\in S_1]$ be the probability that the surplus prior to ruin is greater than or equal to *b* if ruin occurs,

$$P[T\in S_2]=P\left[R^-(T)\ge b\,|\,T<\infty\right].$$

By definition $P[T\in S_1]+P[T\in S_2]=1$.

Taking into account the total probability law, we can write

$$F_{1j}(u,y)$$
$$=P\left[j\le y\,|\left(R(0)=u,T\in S_1\right)\right]P[T\in S_1]$$
$$+P\left[j\le y\,|\left(R(0)=u,T\in S_2\right)\right]P[T\in S_2].$$

$P[j\le y\,|\,R(0)=u,T\in S_2]P[T\in S_2]$ is the distribution function of the deficit at ruin if ruin occurs and the surplus prior to ruin is less than *b*, and due to the memory less property of the exponential distribution, it is an exponential distribution with mean k_1. Thus, this gives an interpretation to the weight *P(u)* as the probability that the surplus prior to ruin is less than b if ruin occurs, $P(u) = P[T \in S_1]$.

Let $P[j \le y|(R(0) = u,\ T \in S_2)]$ be the distribution function of the deficit at ruin if ruin occurs and the surplus prior to ruin is greater than b, then this distribution is an exponential distribution with mean k_1. Therefore, $1 - P(u)$ is the probability that the surplus prior to ruin is greater than b if ruin occurs, $1 - P(u) = P[T \in S_2]$. A similar interpretation can be done with respect to $F_{2j}(y)$.

The Gerber-Shiu function is a general one, and includes also the weights $P(u)$ and Q as particular cases. Let us consider first $P(u)$. It can be obtained directly from (5) whereas the penalty function is $w(x,j) = I(x<b)$, and $\delta = 0$. In this case, the Gerber-Shiu function gives $E\left[I(x<b)I(T<\infty)\,|\,R(0)=u\right]$ and its ordinary differential equation is (8) and also (9) because for $w(x,j) = I(x<b)$ it is easy to demonstrate that the corresponding $\xi_i(u)$ has the form $A_i e^{-\frac{u}{k_i}}$, with $A_1 = 1$ and $A_2 = 0$. Then from (11) and (12) for $0 \le u < b$,

$$P(u) = \frac{\phi_1(u)}{\psi_1(u)} = \frac{\left[1-d_1(0)H(u,0)\right]}{1+\left[d_2(0)-d_1(0)\right]H(u,0)}.$$

In order to obtain Q from the Gerber-Shiu function, we have to consider that $w(x,j) = I(x \ge b)$, and $\delta = 0$. Therefore, in this case in (11) and (12), $A_1 = 0$ and $A_2 = 1$ and then

$$Q = \frac{\phi_2(u)}{\psi_2(u)} = \frac{d_2(0)J(0)}{\left(E(0)+J(0)\left[d_2(0)-d_1(0)\right]\right)}.$$

4. Numerical examples

In this Section, some numerical examples are presented with the following values of the parameters: $\lambda = 1$, $\rho = 0.15$, $\rho_R = 0.25$, $b = 2$, $k_1 = 0.8$ and $k_2 = 0.45$.

The distribution function of the deficit at ruin if ruin occurs as a function of the initial surplus is

$$F_{1j}(u,y) = \frac{e^{-0.139u-1.25y}\left(0.644+0.276e^{0.139u}\right)\left(e^{1.25y}-1\right)}{1+0.452\left(e^{-0.139u}-1.125\right)} - \frac{0.191\left(e^{-0.139u}-1.125\right)\left(1-e^{-2.22y}\right)}{1+0.452\left(e^{-0.139u}-1.125\right)}, \quad 0 \le u < 2, \tag{13}$$

$$F_{2j}(y) = 1 - 0.915671e^{-1.25y} - 0.0843291e^{-2.22y}, \quad u \ge 2.$$

In Figure 2, this distribution is displayed

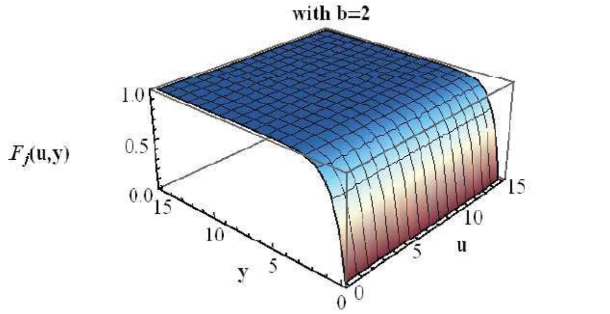

Fig. 2. The distribution function of the deficit at ruin if ruin occurs

The discounted distribution of the deficit at ruin if ruin occurs (with $\delta = 0.03$) is

$$E\left[e^{-\delta T} I(j \le y)\,|\,T<\infty, R(0)=u\right] =$$

$$\begin{cases} \dfrac{e^{-0.264u-1.25y}\left(0.877-0.125e^{0.422u}\right)\left(e^{1.25y}-1\right)}{1+0.452\left(e^{-0.139u}-1.125\right)} \\ -\dfrac{0.024\left(e^{-0.264u}-1.427e^{0.158u}\right)\left(1-e^{-2.22y}\right)}{1+0.452\left(e^{-0.139u}-1.125\right)}, & 0 \le u < 2, \\ \left(0.856394-0.78246e^{-1.25y}\right)e^{-0.317u} \\ -0.0739334e^{-2.22y}e^{-0.317u}, & u \ge 2. \end{cases}$$

In Table 1, we present the values of ruin probability (in column (1)), and the expected deficit at ruin if ruin occurs (in column (3)) for different values of the initial surplus u. The expected deficit at ruin if ruin occurs is the expectation of a mixture of two exponential distributions, the first with mean 0.8, and the second with mean 0.45. The weights of the mixture are indicated in (13). Then, this expectation decreases with respect to u, for $0 \le u < 2$, and is constant for $u \ge 2$.

Table 1. Ruin probability (1) and expected deficit at ruin if ruin occurs (3)

u	$\psi(u)$ (1)	$E\left[yI(T<\infty)\,|\,R(0)=u\right]$ (2)	$E\left[y\,|\left(R(0)=u, T<\infty\right)\right]$ (3) = (2) / (1)
0	0.943442	0.746385	0.791129
0.5	0.913087	0.717609	0.785915
1	0.884768	0.690764	0.780728
1.5	0.858349	0.66572	0.775581
2	0.833703	0.642356	0.770485
2.5	0.809039	0.623352	0.770485
3	0.785105	0.604911	0.770485
3.5	0.761879	0.587016	0.770485
4	0.73934	0.56965	0.770485
4.5	0.717467	0.552798	0.770485
5	0.696242	0.536444	0.770485
5.5	0.675645	0.520574	0.770485
6	0.655657	0.505173	0.770485
8	0.581445	0.447995	0.770485

In order to obtain the total losses if ruin occurs, we have to consider the present value of the deficit at ruin if ruin occurs. In Table 2, we present the values of the expected present value of the deficit at ruin if ruin occurs (column (5)) and the expected total losses if ruin occurs (column (6)), considering that the total losses include the deficit at ruin and the initial surplus u.

Table 2. $E\left[ye^{-\delta T} I(T<\infty) | R(0)=u\right]$ (4), $E\left[ye^{-\delta T} | \left(R(0)=u, T<\infty\right)\right]$ (5) and $u+E\left[ye^{-\delta T} | \left(R(0)=u, T<\infty\right)\right]$ (6)

u	(4)	(5) = (4) / (1)	(6) = u + (5)
0	0.605917	0.642241	0.642241
0.5	0.513665	0.562559	1.06256
1	0.431371	0.487553	1.48755
1.5	0.357689	0.416717	1.916717
2	0.291427	0.349558	2.34956
2.5	0.241328	0.298289	2.79829
3	0.199841	0.254541	3.25454
3.5	0.165486	0.217208	3.71721
4	0.137037	0.185351	4.18535
4.5	0.113479	0.158166	4.65817
5	0.0939709	0.134969	5.13497
5.5	0.0778163	0.115173	5.61517
6	0.0644389	0.0982814	6.09828
8	0.030301	0.0521132	8.05211

In Figure 3, we plot the values included in Table 2. As it can be seen, the expected present value of the deficit at ruin if ruin occurs decreases with u and tends towards zero, whereas the expected total losses if ruin occurs increase with the initial surplus and tend to this initial surplus.

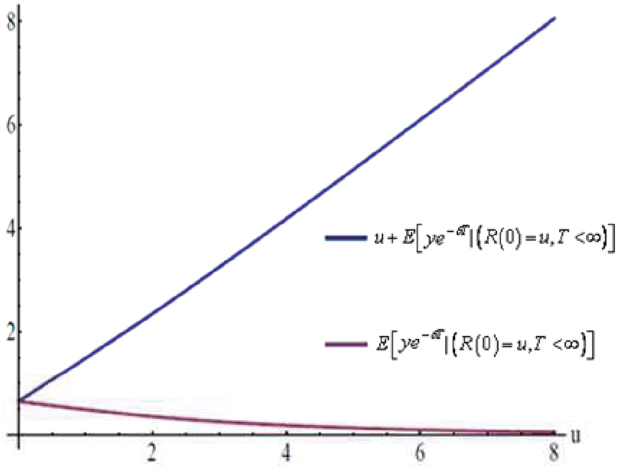

Fig. 3. Expected present value of the deficit at ruin if ruin occurs (5) and expected total losses if ruin occurs (6) for different values of u

Conclusions

In this paper we have analyzed the classical model of risk theory with a dynamic proportional reinsurance strategy. The Gerber-Shiu function allows us to obtain ruin probability and the deficit at ruin. The study of these two measures of solvency is very important for the manager of the portfolio in order to make decisions about the retention levels and the financial requirements (initial surplus) to achieve a desired level of solvency.

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