

THE ELEMENTARY THEORY OF FREE STEINER TRIPLE SYSTEMS

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Abstract. Free Steiner triple systems (STSs) are infinite structures that are naturally characterised by a universal property. We consider the class of free STSs from a model-theoretic viewpoint. We show that free STSs on any number of generators are elementarily equivalent. We axiomatise their theory and show that it is stable.

§1. Introduction. The subject of this article is the first-order theory of a class of combinatorial structures called free Steiner triple systems (STSs). An STS of order λ is a pair $(M, \mathcal{B})$, where M is a set with λ elements, and $\mathcal{B}$ is a collection of 3-element subsets of M (the *blocks*) such that any two $x, y \in M$ are contained in exactly one block. A set M with a collection of 3-element subsets is a *partial STS* (PSTS) if any two elements of M belong to at most one block.

Varying the parameters 2 and 3 gives the more general notion of a Steiner (t, k) -system, where blocks have size k and every t -subset is contained in exactly one block.

Steiner systems are well-known structures, widely studied in the finite case. Recently, certain infinite incarnations have appeared in the model-theoretic literature, e.g., as examples of strongly minimal structures [4], or as specific constructions such as Fraïssé amalgamations [6]. These constructions exhibit a range of stability behaviours: the structure in [6] is TP_2 and NSOP_1 , and thus it lies at the opposite end of the stability spectrum from its counterparts in [4].

An interest has emerged, more broadly, in the model theory of incidence structures—see, for example, [11] for *generic* incidence structures, [21] for projective spaces, [4] and [2] for Steiner systems, and [1] for generalised n -gons. In [1, 2, 11, 21], the structures involved are described as either two-sorted or bipartite, with one sort (resp. part) for points and one for lines. In [4], the vocabulary is relational and one-sorted. As we shall see, here (and in [6]) STSs are one-sorted structures, which means our methods are generally different.

A notable construction method that applies to incidence structures and, more generally, to algebras, generates *free* structures. Universal algebra gives a standard characterization of free constructions, which, in the case of STSs, captures the intuition that an infinite STS can be obtained from an initial set A containing λ vertices with no blocks defined among them by adding, for each pair of vertices $a_1, a_2 \in A$, a new point that gives the unique block to which a_1 and a_2 belong.

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Iterating this process ω times gives an STS. The template example in [9] is, essentially, the result of this construction for $\lambda = 3$.

Standard universal-algebraic tools tell us that such free constructions are uniquely determined up to isomorphism by the cardinality of the generating set A . A series of natural questions then arises, such as which first-order properties, if any, are shared by two free constructions where the sizes of the initial generating sets are different.

Questions of this kind have been addressed, for example, for projective planes [21] and for generalised polygons [1]. Towards the completion of the write-up of our results, we became aware of [26]—see the end of this section for the connection with our work.

In these papers, free structures are characterised via the existence of specific constructions in stages, rather than through the universal algebraic definition. Our choice of a one-sorted, functional language (see below) for STSs makes the connection with universal algebra rather natural, and distinguishes our methods.

A PSTS can be regarded as a first-order structure in the language containing a ternary relation symbol R and satisfying the axioms:

- for all permutations $\sigma \in S_3$, $\forall x_1 x_2 x_3 (R(x_1, x_2, x_3) \rightarrow R(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}))$;
- $\forall x y z (R(x, y, z) \rightarrow y \neq z)$;
- $\forall x y w z (R(x, y, w) \wedge R(x, y, z) \rightarrow w = z)$.

The relation R is interpreted as the set of blocks: three elements x , y , and z are in a block if and only if $R(x, y, z)$. A PSTS that satisfies the additional axiom

- $\forall x y (x \neq y \rightarrow \exists z R(x, y, z))$

is an STS. With this choice of language, a substructure of an STS is a PSTS.

An STS can also be described as a *Steiner quasigroup*, that is, a structure with a single binary operation symbol $\cdot$, interpreted as an operation that satisfies the following axioms:

1. $\forall x y (x \cdot y = y \cdot x)$;
2. $\forall x (x \cdot x = x)$;
3. $\forall x y (x \cdot (x \cdot y) = y)$.

Thus, Steiner quasigroups are an equational class (see [15, 16]). This is what makes it possible to employ universal algebraic tools.

A quasigroup M determines an STS whose blocks are the elements of the graph of the operation $\cdot$ without the diagonal, that is,

$$\mathcal{B} = \{\{x, y, z\} \subseteq M : x \cdot y = z\} \setminus \{\{x\} : x \in M\}.$$

It is easy to see that each STS determines a quasigroup and each quasigroup determines an STS, where

$$x \cdot y = z \text{ iff } R(x, y, z) \text{ or } x = y = z.$$

In fact, as noted in [15, 16], this shows that each STS is bi-interpretable with a Steiner quasigroup, and justifies our choice to use the term *STS* to cover both the functional and relational structures from the end of this section onwards.

A detailed account of the relation between Steiner quasigroups and Steiner systems can be found in [3], where the author shows that each strongly minimal Steiner $(2, k)$ -system constructed in [4] can be *coordinatised* by a quasigroup when

k is a prime power. In particular, in Section 3 STSs are shown to be bi-interpretable with, rather than simply coordinatised by, Steiner quasigroups. Instead, when $k \neq 3$ the strongly minimal Steiner $(2, k)$ -systems constructed in [4] can be interpreted in appropriate quasigroups, but no interpretation can be found in the other direction. This confirms that treating STSs as quasigroups makes algebraic tools available that are not applicable when they are described as incidence structures.

Since the quasigroup axioms are universal, it is clear that a substructure of a Steiner quasigroup is a Steiner quasigroup. Moreover, if N is a Steiner quasigroup and $A \subseteq N$, there is a natural notion of substructure of N generated by A , namely,

$$\langle A \rangle_N = \bigcap \{M \subseteq N : M \text{ is a substructure that contains } A\}.$$

The subscript N may be omitted when clear from the context.

REMARK 1.1. Let M be a Steiner quasigroup generated by $A \subseteq M$. Then

- for every $b \in M$ there is a term $t(\bar{x})$ such that $b = t(\bar{a})$, where $\bar{a}$ enumerates A ;
- $M = \bigcup_{n < \omega} S_n$ for a chain of sets S_n such that $S_0 = A$ and $S_{n+1} = \{a \cdot b \mid a, b \in S_n\}$. Note that $S_n \subseteq S_{n+1}$ since $a \cdot a = a$.

NOTATION. Below we write $S_{<n}$ for $\bigcup_{i < n} S_i$.

DEFINITION 1.2. Let M be a Steiner quasigroup generated by $A \subseteq M$. The sets S_n in Remark 1.1 are called the *levels* of M over A . An element $a \in M$ has *level* n over A if $a \in S_n \setminus S_{<n}$.

As remarked above, in what follows the term STS is used for Steiner quasigroups, too; we rely on the context to clarify the usage.

In Section 2 we characterise free STSs as those that are *freely generated* by a subset in the sense of universal algebra, and we recall some standard universal-algebraic results. Section 3 introduces certain linear orderings for STSs and PSTSs that are an essential tool in our treatment of the theory of free STSs. Moreover, we show that the existence of an *HF-ordering* on an STS is equivalent to its *unconfinedness*, a first-order condition that is satisfied by all infinite free STSs and will essentially axiomatise their theory. In Section 4 we show that the amalgamation property fails for the class $\mathcal{C}_{\text{un}}$ of finite unconfined PSTSs. We also show that the obvious choice of a predimension function does not yield a fruitful definition of *strong substructure* for the construction of a Hrushovski amalgam. An appropriate notion of strong substructure is formulated in Section 5 and it involves HF-orderings. We show that the countable free STS with ω generators is the generic object for $\mathcal{C}_{\text{un}}$ with respect to this notion. We also give appropriate definitions of *n-strong substructure* and *n-amalgamation* and use the setup in [29] to show that the generic is not saturated. In Section 6 we show that infinite unconfined STSs always contain certain binary trees that are an essential ingredient in the proof of our main result in Section 7, where we prove that the theory of infinite unconfined STSs is complete and it is the theory of infinite free STSs—so, in particular, any two free infinite STSs are elementarily equivalent. We also show that the theory is stable and that it has a model which is not free.

Our elementary equivalence result appears to be a special case of the main theorem in [26], developed independently, where the completeness and strict stability are

asserted for the theories of a range of open incidence structures that includes free Steiner systems (hence free STSs). The results in [26] generalise those in [1, 21]. As previously noted, a two-sorted language is chosen in [26]. This allows for a uniform treatment of the range of incidence structures considered in the article. Since here we restrict our attention to STSs, we can adopt the more natural choice of a one-sorted language, which leads to different methods and considerable simplifications.

§2. Free Steiner triple systems: Universal algebra. We give a universal algebraic characterisation of free STSs as those that satisfy a *universal mapping property* (UMP).

DEFINITION 2.1. Let $\mathcal{C}$ be a class of STSs. An STS M generated by A has the UMP over A for $\mathcal{C}$ if for every $N \in \mathcal{C}$, every map

$$\varphi : A \rightarrow N$$

extends to a homomorphism $\hat{\varphi} : M \rightarrow N$.

A standard argument (see, e.g., [17, Section 24, Corollary 1]) ensures that the extension $\hat{\varphi}$ is unique.

DEFINITION 2.2. If M is an STS, a subset $A \subseteq M$ is *independent* if $\langle A \rangle_M$ has UMP over A for the class of all STSs.

M is said to be *freely generated* by A if:

- (i) A is independent;
- (ii) $\langle A \rangle_M = M$.

If M is freely generated by $A \subseteq M$ then M is a *free STS*, and A is a *free base* of M .

The next proposition formalises the intuition that an STS freely generated by a subset A can be constructed from A in stages by adding, at each stage, a new vertex as the product of two existing vertices whose product is still undefined. This construction was first described in [28] as the result of a *free extension process*, and later generalised in [14] to various classes of incidence geometries that include, among others, projective and affine planes, generalised n -gons, k -nets, and Steiner (t, k) -systems. For convenience, we state a version for STSs that lists explicitly the properties we require in the rest of the article.

PROPOSITION 2.3. Let M be an STS and let $A \subseteq M$. The following are equivalent:

1. A is a free base of M .
2. M is generated by A with levels $(S_n \mid n < \omega)$, where $S_0 = A$ and
 - (a) if $a, b \in S_n \setminus S_{<n}$ and $a \neq b$, then $a \cdot b \notin S_n$.
 - (b) if $a \in S_{n+1} \setminus S_n$, there is a unique pair $\{b, c\} \subseteq S_n$ such that $b \neq c$ and $a = b \cdot c$.

Moreover, property (b) implies that if $a \in S_{n+1} \setminus S_n$, and $b, a \cdot b \in S_n$, then $\{b, a \cdot b\}$ is the unique pair given by (b) for a .

PROOF. See [7, Proposition 3.1]. ⊢

Notice that each set $S_{n+1} \setminus S_n$ in the construction above generates a subsystem of M that satisfies condition 2, so $S_{n+1} \setminus S_n$ is a free base for $\langle S_{n+1} \setminus S_n \rangle$, and in

particular an independent set. It follows that M contains independent sets of finite unbounded cardinality. In fact, M also contains an independent set of cardinality ω —a proof for Steiner loops in [24] is easily adapted to STSs.

DEFINITION 2.4. Let M be an STS with a free base $A \subseteq M$. The sequence $(S_n \mid n < \omega)$ of levels described in Proposition 2.3 will be called a *standard free construction of M over A* .

Standard techniques for any variety in universal algebra show that the cardinality of a free base A determines the STS freely generated by A up to isomorphism.

PROPOSITION 2.5. Let M, N be STSs freely generated by $A \subseteq M$ and $B \subseteq N$, respectively. Then $M \cong N$ if and only if $|A| = |B|$. In fact, every bijection between A and B can be extended to an isomorphism between M and N .

PROOF. This follows from Theorem I in [13], and Lemma 2.6 and Theorems 1 and 2 in [22]. See [7, Corollary 2.11] for a self-contained version of these arguments in the case of STSs. $\dashv$

The elementary equivalence of any two infinitely generated free STSs follows from a standard argument for infinitely generated free algebras.

LEMMA 2.6. Let M, N be free STSs and assume A is an infinite free base of M that can be extended to a free base B of N . Then $M \preceq N$.

PROOF. See [17, Chapter 6, Section 38, Theorem 4]. $\dashv$

THEOREM 2.7. If M, N are free STSs with infinite free bases, then $M \equiv N$.

PROOF. See [17, Chapter 6, Section 38, Corollary to Theorem 5]. $\dashv$

§3. HF-orderings and confined configurations. The idea of *free* and *hyperfree extension processes* appears in the context of free projective planes in [28] and has subsequently been generalised to a wider range of incidence structures that includes STSs in [14]. Such processes yield well-founded constructions of incidence structures from existing ones by adding new elements one at a time, with suitable restrictions on the number of incidences. In [21], the related notions of *HF-* (hyperfree) and *F-* (free) *orderings* are formulated for projective planes to dispense with the well-foundedness assumption. These orderings have natural parallels for partial and complete STSs, respectively. Their properties are proved below by adapting arguments that appear in [14, 21] for two-sorted incidence structures.

DEFINITION 3.1. A *free ordering* (or *F-ordering*) of an STS M over $A \subseteq M$ is a linear ordering $<$ of M such that for every $a \in M$ there is a unique pair $\{a_1, a_2\} \subseteq M$ with $a = a_1 \cdot a_2$, and, for each i , either $a_i \in A$ or $a_i < a$. Notice that when $a \in A$, the only possibility is $a_1 = a_2 = a$.

In our discussion of linear orderings we use standard operations as defined, for instance, in [27]. In particular, if $(A, <_A)$ and $(B, <_B)$ are linear orderings and $A \cap B = \emptyset$, then their sum is the linear order

$$(A \cup B, <) = (A, <_A) + (B, <_B),$$

where $<$ is defined as $<_A \cup <_B \cup (A \times B)$.

REMARK 3.2. Let M be an STS with standard free construction $(S_n \mid n < \omega)$ over A . Let $<_n$ be a linear ordering of $S_n \setminus S_{<_n}$ and for $a, b \in M$ define

$a < b$ iff for some n , $b \in S_n \setminus S_{<_n}$ and either $a \in S_{<_n}$ or $a \in S_n \setminus S_{<_n}$ and $a <_n b$.

Then $<$ is an F-ordering of M over $A = S_0$, which is an initial segment, and $<$ is in fact the generalized sum $\sum_{n < \omega} (S_n \setminus S_{<_n}, <_n)$. If every $<_n$ is a well-order, $<$ is a well-order.

REMARK 3.3. Let M be an STS. If there is an F-ordering of M over A , there is another one where A is an initial segment. If the original order is a well-order, so is the new order; moreover, $M = \langle A \rangle$.

PROOF. If $<$ is an F-ordering of M over A , then the sum $(A, < \upharpoonright A) + (M \setminus A, < \upharpoonright (M \setminus A))$ defines an F-ordering of M over A with initial segment A . $\dashv$

DEFINITION 3.4. A *hyperfree ordering* (or *HF-ordering*) of a PSTS A is a linear ordering $<$ of A such that for every $a \in A$ there is at most one pair of smaller elements $\{a_1, a_2\} \subseteq A$ with $a = a_1 \cdot a_2$, that is, there is at most one block $\{a, a_1, a_2\}$ with $a_1, a_2 < a$.

The next proposition explains the connection between free bases and F-/HF-orderings.

PROPOSITION 3.5. Assume M is an STS and $A \subseteq M$. The following are equivalent:

1. A is a free base of M (as a Steiner quasigroup).
2. There is a well-founded F-ordering of M over A .
3. There is a well-founded HF-ordering of M and A is the set of points that are not a product of smaller elements.

PROOF. See [7, Proposition 3.7]. $\dashv$

Free STSs satisfy a property which can be expressed through infinitely many first-order universal sentences.

DEFINITION 3.6. A finite PSTS A is *confined* if every element of A belongs to at least two blocks.

A PSTS $A \neq \emptyset$ is *unconfined*, or *contains no confined configuration*, if no finite $A' \subseteq A$ is confined.

Equivalently, $A \neq \emptyset$ is unconfined if for all finite $A' \subseteq A$ there is $a \in A'$ that belongs to at most one block in A' , that is, a is either not a product of any elements in A' , or the product of exactly one pair of elements of A' .

Note that being unconfined is stronger than not being confined: an unconfined PSTS is hereditarily not confined.

In [18], *confined* partial projective planes are those where every line contains three or more points, and there are three or more lines through every point. If a finite partial plane is not confined, then it is *open*. These notions are later used in [28] and they are central in [21]. Analogues are defined in [14] for the range of incidence structures treated in the article, including Steiner systems. Our definition of unconfined PSTS is equivalent to that of *open* PSTS in [14].

PROPOSITION 3.7.

1. Free STSs are unconfined.
2. Finitely generated unconfined STSs are free.
3. A PSTS is unconfined if and only if it has an HF-ordering.

PROOF. See [7, Propositions 3.10, 3.11, 3.13] for (1), (2), and (3), respectively. $\dashv$

DEFINITION 3.8. Let $A \subseteq B$ be PSTS. An HF-ordering $<$ of B is *over* A if for every $a \in B \setminus A$ there is at most one pair $\{a_1, a_2\}$ of distinct elements of B such that $\{a, a_1, a_2\}$ is a block and for every i , either $a_i < a$ or $a_i \in A$.

Note that if B is an STS, any F-ordering of B over $A \subseteq B$ is an HF-ordering over A , but even if an HF-ordering over A is an F-ordering (over some subset of B), it is not necessarily an F-ordering over A , since elements of A can be a product of smaller elements of A .

LEMMA 3.9. Let $A \subseteq B$ be PSTS. There is an HF-ordering of B over A if and only if there is an HF-ordering of B , where A is an initial segment. In fact, given any HF-ordering $<$ of B over A , the sum $(A, < \upharpoonright A) + (B \setminus A, < \upharpoonright (B \setminus A))$ is an HF-ordering of B with initial segment A .

PROOF. On the one hand, if $<$ is an HF-ordering of B with initial segment A , it is easy to see that it is an HF-ordering over A . On the other hand, assume $<$ is an HF-ordering of B over A . Then the only way in which the sum can fail to be an HF-ordering is for some $a \in B \setminus A$ to be a product of two different pairs of smaller elements, and each element is either smaller than a in $<$ or belongs to A , contradicting that $<$ is an HF-ordering over A . $\dashv$

DEFINITION 3.10. Given an HF-ordering $<$ of the PSTS B and a subset $A \subseteq B$, we define the $<$ -closure of A (in B) as the smallest subset C of B containing A and such that whenever $b \in C$ and $b_1, b_2 < b$ and $\{b, b_1, b_2\}$ is a block, then $b_1, b_2 \in C$. We denote C by $\text{cl}_<(A)$.

It is clear that the $<$ -closure is a finitary closure operator on subsets of B . Moreover,

$$\text{cl}_<(A_1 \cup A_2) = \text{cl}_<(A_1) \cup \text{cl}_<(A_2).$$

LEMMA 3.11. If $<$ is an HF-ordering of B and $A = \text{cl}_<(A) \subseteq B$, then $<$ is an HF-ordering of B over A .

PROOF. If not, there is some $b \in B \setminus A$ and there are distinct b_1, b_2, b_3, b_4 , such that $\{b, b_1, b_2\}$ and $\{b, b_3, b_4\}$ are blocks and, for each i , either $b_i < b$ or $b_i \in A$. Without loss of generality, $b_2 < b_1$. We claim that $b_1 < b$. If not, then $b_1 \in A$ and $b < b_1$, and since $A = \text{cl}_<(A)$, we get $b \in A$, a contradiction. Hence, $b > b_1 > b_2$. By a similar argument, $b > b_3, b_4$. But then $<$ is not an HF-ordering. $\dashv$

LEMMA 3.12. Assume $<$ is an HF-ordering of B , $A = \text{cl}_<(A) \subseteq B$ and $C \subseteq B$ is an initial segment such that $c < a$ whenever $c \in C$ and $a \in A$. Then we can interpolate A between C and the rest of the order of B in such a way that we obtain an HF-ordering. In other words,

$$(C, < \upharpoonright C) + (A, < \upharpoonright A) + (B \setminus (A \cup C), < \upharpoonright B \setminus (A \cup C))$$

is an HF-ordering of B . Moreover, we can change the ordering of A by any other HF-ordering of A and still have an HF-ordering of B .

PROOF. Notice that $(C, < \upharpoonright C) + (A, < \upharpoonright A) = (A \cup C, < \upharpoonright (A \cup C))$. Since $\text{cl}_<(A \cup C) = A \cup C$, Lemma 3.11 ensures that $<$ is an HF-ordering of B over $A \cup C$. For the moreover clause, note that no element of A is a product of elements of C nor a product of an element of C by an element of A . $\dashv$

LEMMA 3.13. If $<$ is an HF-ordering of M , $A \subseteq M$ and $\text{cl}_<(A) = A$, then $\text{cl}_<(\langle A \rangle) = \langle A \rangle$.

PROOF. Let $\langle A \rangle = \bigcup_{n < \omega} A_n$, where $A_0 = A$ and A_{n+1} is the set of elements of $\langle A \rangle \setminus (A_0 \cup \dots \cup A_n)$ which are a product of two elements of $A_0 \cup \dots \cup A_n$. It is enough to prove by induction that, for every n , $\text{cl}_<(A_0 \cup \dots \cup A_n) = A_0 \cup \dots \cup A_n$. This is clear for $n = 0$. Assume $a \in A_{n+1} \setminus (A_0 \cup \dots \cup A_n)$. There are $b, c \in A_0 \cup \dots \cup A_n$ such that $a = b \cdot c$. If $b > a$ or $c > a$, then $a \in \text{cl}_<(A_0 \cup \dots \cup A_n) = A_0 \cup \dots \cup A_n$, a contradiction. Otherwise, $a > b, c$ and $\{b, c\} \subseteq A_0 \cup \dots \cup A_n$ is the unique pair of elements such that $a > b, c$ and $a = b \cdot c$. $\dashv$

§4. Some remarks on predimension. In [18], a notion of *rank* for projective planes is defined whose analogue for STSs coincides with a natural choice for a predimension function. As we shall see, though, this predimension function is not suitable for defining a notion of strong substructure that would lead to the construction of a Hrushovski-style amalgam—in fact, this is what motivates the different choice of predimension to build the uncountably many strongly minimal Steiner systems in [4].

We show that the amalgamation property fails in the class of finite partial unconfined STSs with respect to both the substructure relation, and the obvious definition of strong substructure via predimension. This is why, in Section 5, our construction of a generic object requires a rather different, and possibly not as natural, notion of strong substructure.

We also include a comparison with a class of finite PSTSs, defined in [12] in a wider context, that properly contains the unconfined ones and where predimension works well instead. While these results have no direct application in subsequent sections, they clarify the context and may have independent interest.

DEFINITION 4.1. Let A be a finite PSTS and let $\mathbf{b}(A)$ be the set of blocks in A . Then the *rank*, or *predimension*, of A is

$$\delta(A) = |A| - |\mathbf{b}(A)|.$$

REMARK 4.2. It is well known (see, e.g., Subsection 1.1.1 in [10]) that in a finite STS M of cardinality n :

- any point lies in $\frac{n-1}{2}$ blocks;
- there are $\frac{n(n-1)}{6}$ blocks.

It follows that:

1. an unconfined STS M is infinite, or $|M| = 3$;
2. if M is an STS of cardinality ≥ 3 then $\delta(S) \leq 0$.

For 1, let M be a finite STS such that $|M| > 3$. Then M is a PSTS contained in M such that any $p \in M$ lies in $\frac{n-1}{2}$ blocks. Since $n > 3$, $\frac{n-1}{2} > 1$.

For 2, let M be a finite SMS of cardinality $n \geq 3$. Then

$$\delta(M) = n - \frac{n(n-1)}{6}$$

and $\delta(M) \geq 0$ if and only if $0 \leq n \leq 7$. In fact, the predimension of the Fano plane (the unique STS of order 7) is 0, and the predimension of all other finite, non-trivial STS is negative.

Nevertheless, suitable subclasses of the class of finite PSTSs characterised in terms of non-negative dimension. The next lemma is an adaptation of the corollary to Lemma 1 in [28].

LEMMA 4.3. *The predimension of a finite PSTS A that contains no confined configuration is non-negative.*

PROOF. Suppose $|A| = n$. Since A is unconfined, we can find an HF-ordering

$$a_1 < a_2 < \dots < a_n.$$

Let $A_i := \{a_1, \dots, a_i\}$ for $1 \leq i \leq n$. Then, if $|b(A_i)|$ is the number of blocks in A_i ,

$$\delta(A_i) = \begin{cases} |A_{i-1}| - |b(A_{i-1})| + 1 - 1 = \delta(A_{i-1}) & \text{if } a_i \text{ belongs to one block in } A_i \\ |A_{i-1}| - |b(A_{i-1})| + 1 & \text{if } a_i \text{ belongs to no blocks in } A_i. \end{cases}$$

Therefore $\delta(A_i) \geq \delta(A_{i-1})$ for all i . ◻

REMARK 4.4. The function δ is a *predimension* in the following sense:

1. $\delta(\emptyset) = 0$;
2. $\delta(\{a\}) \leq 1$ (in fact, in our case $= 1$);
3. $\delta(A \cup B) + \delta(A \cap B) \leq \delta(A) + \delta(B)$ if A, B are subsystems of a common PSTS P (*Submodularity* of δ).

This predimension function can be used to define a notion of well-embedded substructure in the class of PSTS.

DEFINITION 4.5. If A is a finite PSTS and $B \supseteq A$ is a PSTS (possibly infinite), we define

$$A \leq^* B \Leftrightarrow \delta(A) \leq \delta(B_0) \text{ for every finite } B_0 \text{ such that } A \subseteq B_0 \subseteq B.$$

If $A \leq^* B$, we say that A is *well embedded* in B .

We compare the behaviour of the predimension δ in the class of finite unconfined PSTS and the class of *oriented* PSTS defined in [12] in the more general case of finite oriented colored k -hypergraphs that have a $\mathcal{G}$ -orientation. The class defined below is the case $k = 3$, $t = 2$ and a single colour in Definition 1.3 in [12].

DEFINITION 4.6. An *orientation* of a PSTS P is a one-to-one mapping $c : b(P) \rightarrow P$ such that $c(e) \in e$ for every block e , where $b(P)$ is the set of blocks of P as in Definition 4.1.

We say that $c(e)$ is an *apex* of e . Given an orientation c of P , we say that $A \subseteq P$ is *c-closed* in P or *closed with respect to c* if for every block e of P with apex in A , $e \subseteq A$.

Let $\mathcal{C}$ be the class of all finite PSTSs. Then

- $\mathcal{C}_{\text{or}} = \{A \in \mathcal{C} \mid A \text{ can be oriented and for every } X \subseteq A \text{ with } |X| \leq 2, X \leq^* A\}$;
- $\mathcal{C}_{\text{un}} = \{A \in \mathcal{C} \mid A \text{ is unconfined}\}$.

We show that the finite unconfined PSTSs are a proper subclass of $\mathcal{C}_{\text{or}}$.

PROPOSITION 4.7. $\mathcal{C}_{\text{un}} \subseteq \mathcal{C}_{\text{or}}$.

PROOF. We check by induction on $|A|$ that if A is unconfined, then it has an orientation. We may assume $A \neq \emptyset$. Let $a \in A$ belong to at most one block of A . By induction hypothesis, $A \setminus \{a\}$ has an orientation c . If a belongs to the block e , we make a the apex of e . This gives an orientation of A extending c .

Now we prove by induction on A that if A is unconfined and $X \subseteq A$, and $|X| \leq 2$, then $X \leq^* A$. It is enough to check that if $X \subseteq A$, then $\delta(X) \geq \min\{|X|, 2\}$. We may assume $A \neq \emptyset$. Choose $a \in A$ belonging to at most one block of A and let $A' = A \setminus \{a\}$. We can apply the induction hypothesis to A' . If $a \notin X$, then $X \subseteq A'$ and the induction hypothesis gives the result. If $a \in X$, let $X' = X \setminus \{a\}$. Then $\delta(X') \geq \min\{|X'|, 2\}$ and, moreover, $\delta(X') = \delta(X)$ or $\delta(X') = \delta(X) + 1$. If $|X| \leq 2$, then $\delta(X) = |X|$ and clearly $\delta(X) \geq \min\{|X|, 2\}$. If $|X| > 2$, then $\min\{|X|, 2\} = 2$ and $\delta(X) \geq \delta(X') + 1 \geq 2$. $\dashv$

The next example shows that $\mathcal{C}_{\text{un}} \neq \mathcal{C}_{\text{or}}$.

EXAMPLE 4.8. Let A be the PSTS consisting of the vertices $\{a_1, \dots, a_9\}$ with horizontal blocks

$$\{a_1, a_2, a_3\}, \{a_4, a_5, a_6\}, \{a_7, a_8, a_9\}$$

and vertical blocks

$$\{a_1, a_4, a_7\}, \{a_2, a_5, a_8\}, \{a_3, a_6, a_9\}.$$

Then A is not unconfined, but it can be oriented and it satisfies the additional condition on subsets X with $|X| \leq 2$. Hence, $A \in \mathcal{C}_{\text{or}} \setminus \mathcal{C}_{\text{un}}$.

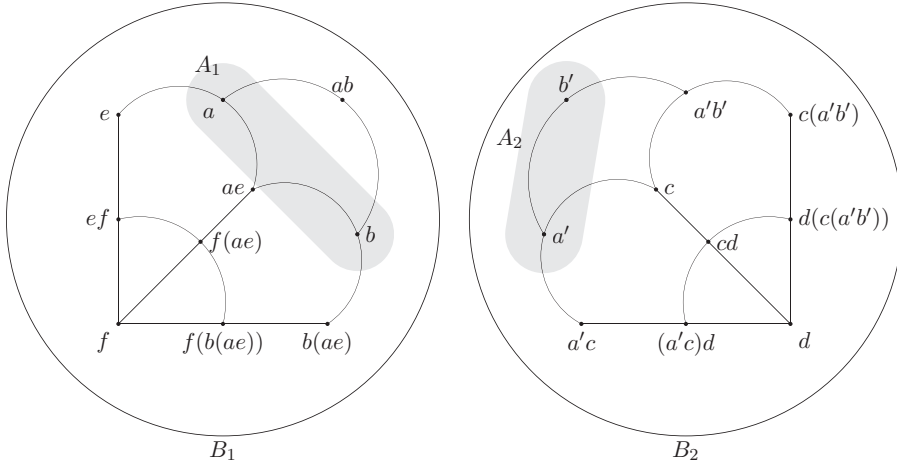
By Corollary 1.1 in [12], the class $(\mathcal{C}_{\text{or}}, \leq^*)$ has the amalgamation property. However, the following example shows that $\leq^*$ does not behave equally well in $\mathcal{C}_{\text{un}}$, which renders the results in [12] not applicable in our case.

EXAMPLE 4.9. Consider two isomorphic PSTSs:

- B_1 on the set $\{a, b, e, f, a \cdot b, a \cdot e, e \cdot f, f \cdot (a \cdot e), b \cdot (a \cdot e), f \cdot (b \cdot (a \cdot e))\}$,
- B_2 on the set $\{a', b', c, d, a' \cdot b', a' \cdot c, c \cdot d, c \cdot (a' \cdot b'), d \cdot (a' \cdot c), d \cdot (c \cdot (a' \cdot b'))\}$,

with blocks as in the figure below.

Let $A_1 = \{a, b\} \subseteq B_1$ and $A_2 = \{a', b'\} \subseteq B_2$. Then the map $\alpha = \{\langle a, a' \rangle, \langle b, b' \rangle\}$ is an isomorphism between A_1 and A_2 .



By inspection, we see that $A_i \leq^* B_i$ for $i = 1, 2$.

Amalgamating B_1 and B_2 over $A_2 = \alpha(A_1)$ forces the identification $a \cdot b \mapsto a' \cdot b'$. The amalgam that results from this identification and no others is in $\mathcal{C}_{\text{or}}$ but not in $\mathcal{C}_{\text{un}}$, because all its vertices are in at least two blocks. It follows that $(\mathcal{C}_{\text{un}}, \leq^*)$ does not have the amalgamation property.

Moreover, in any other amalgam, for each vertex the valency (i.e., the number of blocks to which the vertex belongs) cannot be lower than the valency in the original PSTSs. It follows that any amalgam of B_1 and B_2 over $A_1 \cong A_2$ is not unconfined. Therefore this example shows, more generally, that the amalgamation property does not hold for the class $(\mathcal{C}_{\text{un}}, \subseteq)$, where $\subseteq$ is the substructure relation.

§5. Amalgamation. Since, by Example 4.9, the class $(\mathcal{C}_{\text{un}}, \subseteq)$ does not have the amalgamation property, we introduce a notion of strong substructure based on HF-orderings which makes $\mathcal{C}_{\text{un}}$ a generalised amalgamation class. This leads to the construction of a generic object for $\mathcal{C}_{\text{un}}$ (with respect to the notion of strong substructure) which turns out to be the free STS on ω generators.

DEFINITION 5.1. For $A, B \in \mathcal{C}_{\text{un}}$,

$A \leq B \Leftrightarrow A \subseteq B$ and there is an HF-ordering of B , where A is an initial segment.

When $A \leq B$, we say that A is *strong* in B .

Since A and B are finite, the condition $A \leq B$ is first-order.

To see this, we define certain formulas in the relational language of STSs (so blocks interpret a ternary relation R that is symmetric and irreflexive). In what follows, we implicitly assume that the elements a_i are pairwise distinct. There is a formula $\delta_1(x_1, \dots, x_n)$, in the relational language of PSTSs, that says that $x_1 < x_2 < \dots < x_n$

is an HF-ordering, namely,

$$\delta_1(x_1, \dots, x_n) \equiv \bigwedge_{i=1}^n \left[\bigwedge_{1 \leq j < k < i} \left(R(x_i, x_j, x_k) \rightarrow \bigwedge_{\substack{1 \leq j' < k' < i \\ \{j, k\} \neq \{j', k'\}}} \neg R(x_i, x_{j'}, x_{k'}) \right) \right].$$

Now let S_n denote the symmetric group on n elements, and let

$$\delta_2(x_1, \dots, x_n) \equiv \bigvee_{\pi \in S_n} \delta_1(x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Then $\delta_2(a_1, \dots, a_n)$ holds if and only if there is an HF-ordering of $a_1, \dots, a_n$.

Consider the following modification of δ_2 :

$$\delta_3(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}) \equiv \bigvee_{\substack{\pi \in S_n \\ \rho \in S_m}} \delta_1(x_{\pi(1)}, \dots, x_{\pi(n)}, x_{n+\rho(1)}, \dots, x_{n+\rho(m)}).$$

Then for $A = \{a_1, \dots, a_n\} \subseteq \{a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}\} = B$ we have

$$B \models \delta_3(a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}) \text{ if and only if } A \leq B.$$

So for each m, n , $A \subseteq B$ with $|A| = m$ and $|B| = n + m$, there is a formula that asserts that $A \leq B$.

REMARK 5.2. $\mathcal{C}_{\text{un}}$ is closed under isomorphism and $\leq$ is a partial order on $\mathcal{C}_{\text{un}}$ refining $\subseteq$ and satisfying the following conditions:

1. If $A \leq B$ and $f : B \cong B'$ is an isomorphism such that $f(A) = A'$, then $A' \leq B'$.
2. If $A \subseteq B \subseteq C$, $B \in \mathcal{C}_{\text{un}}$ and $A \leq C$, then $A \leq B$.

This means that $(\mathcal{C}_{\text{un}}, \leq)$ is a *smooth class* as defined in [23].

The following are conditions C3 and C4 in [29].

LEMMA 5.3. *Let $A, A_1, A_2 \in \mathcal{C}_{\text{un}}$. Then*

1. $\emptyset \leq A$;
2. if $B \in \mathcal{C}_{\text{un}}$ is such that $A_1 \subseteq B$ and $A_2 \leq B$, then $A_1 \cap A_2 \leq A_1$.

PROOF. 1. By Proposition 3.7, since every $A \in \mathcal{C}_{\text{un}}$ has an HF-ordering.

2. Given an HF-ordering of B with initial segment A_2 , the elements of A_1 inherit an HF-ordering with initial segment $A_1 \cap A_2$. $\dashv$

If B is a partial unconfined STS, possibly infinite, and $A \subseteq B$ is finite, we define

$$A \leq B \Leftrightarrow A \leq A' \text{ for every } A' \in \mathcal{C}_{\text{un}} \text{ such that } A \subseteq A' \subseteq B.$$

The condition $A \leq B$ is first-order in this case, too: for $m \in \omega$, let $\delta_{4,m}(x_1, \dots, x_n)$ be the formula that says

$$\{x_1, \dots, x_n\} \text{ is strong in any } m\text{-element extension } \{x_1, \dots, x_n, y_1, \dots, y_m\},$$

that is,

$$\forall y_1 \dots y_m \left[\left(\bigwedge_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \neg x_i = y_j \wedge \bigwedge_{1 \leq i < j \leq m} \neg y_i = y_j \right) \rightarrow \delta_3(x_1, \dots, x_n, y_1, \dots, y_m) \right].$$

Now let

$$\text{strong}_m(x_1, \dots, x_n) \equiv \bigwedge_{k \leq m} \delta_{4,k}(x_1, \dots, x_n),$$

and suppose $A = \{a_1, \dots, a_n\} \subseteq B$, where the a_i are pairwise distinct. Then

$$B \models \{\text{strong}_m(a_1, \dots, a_n) : m \in \omega\} \text{ if and only if } A \leq B.$$

We shall see that $(\mathcal{C}_{\text{un}}, \leq)$ has the amalgamation property AP. Now that we have shown that $(\mathcal{C}_{\text{un}}, \leq)$ is smooth, by [23] this will imply that there is a countable unconfined STS F , the *generic model*, such that:

- $F = \bigcup_{n < \omega} C_n$, where $C_i \in \mathcal{C}_{\text{un}}$ and $C_i \leq C_j$ if $i \leq j$;
- if $A \in \mathcal{C}_{\text{un}}$, then there is an embedding $f : A \rightarrow F$ such that $f(A) \leq F$;
- for all $A \leq B$ such that $A \leq F$, there is an embedding $f : B \rightarrow F$ such that $f(B) \leq F$ and $f|_A = \text{id}_A$.

Moreover, up to isomorphism there is a unique countable model satisfying these three properties.

The class $\mathcal{C}_{\text{un}}$ turns out to be an interesting example of a variant of $\leq$ -amalgamation appearing in [29] which we define next in our context and which allows us to establish that our generic is not saturated. We prove that this variant, called n -amalgamation, holds in $\mathcal{C}_{\text{un}}$. We subsequently use it to prove the amalgamation property.

DEFINITION 5.4. Let $A, B \in \mathcal{C}_{\text{un}}$ and $n < \omega$. We write $A \leq^n B$ if $A \subseteq B$ and for every $X \subseteq B \setminus A$ such that $|X| \leq n$, we have $A \leq A \cup X$.

Given a partial unconfined STS B , possibly infinite, and $A \in \mathcal{C}_{\text{un}}$ such that $A \subseteq B$, we write $A \leq^n B$ if $A \leq^n B_0$ for every finite $B_0 \subseteq B$.

In either case we say that A is n -strong in B .

Note that $A \leq B$ if and only if $A \leq^n B$ for every n .

REMARK 5.5. The condition of being an n -strong substructure of a partial unconfined STS B is first-order: for $A = \{a_1, \dots, a_k\} \subseteq B$, where the a_i are pairwise distinct,

$$B \models \text{strong}_n(a_1, \dots, a_k) \text{ if and only if } \{a_1, \dots, a_k\} \leq^n B.$$

LEMMA 5.6. If $A \leq^n B$ and $B \leq^n C$, then $A \leq^n C$.

PROOF. Assume $X \subseteq C \setminus A$ and $|X| \leq n$. We check that $A \leq A \cup X$. Since $B \leq^n C$, we have $B \leq B \cup X$. By Lemma 5.3, $A \cup (X \cap B) \leq A \cup X$. Since $A \leq^n B$, we know that $A \leq A \cup (X \cap B)$. Hence $A \leq A \cup X$. $\dashv$

LEMMA 5.7.

1. Assume $A \leq B$, $b \in B \setminus A$ and there is a block $\{a_1, a_2, b\}$ with $a_1, a_2 \in A$. Then $Ab \leq B$.
2. Assume $A \leq^n B$, $b \in B \setminus A$ and there is a block $\{a_1, a_2, b\}$ with $a_1, a_2 \in A$. Then $Ab \leq^{n-1} B$.

PROOF. 1. Start with an HF-ordering of B with initial segment A . Then $\{a_1, a_2, b\}$ is the only block containing b and elements smaller than b in the order. We can make

b the least element in $B \setminus A$ and obtain, again, an HF-ordering of B , now with Ab as initial segment.

2. Similar proof. $\dashv$

DEFINITION 5.8. We say that $(\mathcal{C}_{\text{un}}, \leq)$ has the n -amalgamation property if for every $A \leq B$ in $\mathcal{C}_{\text{un}}$ there is some $m < \omega$ such that whenever $C \in \mathcal{C}_{\text{un}}$ and $A \leq^m C$, there are $D \in \mathcal{C}_{\text{un}}$ and embeddings $f : B \rightarrow D$ and $g : C \rightarrow D$ such that

$$f \upharpoonright A = g \upharpoonright A, \quad f(B) \leq^n D \text{ and } g(C) \leq D.$$

In [29] Wagner uses the term $\leq^*$ -amalgamation to mean that n -amalgamation holds for every n . The notion also appears in [5] with the name *uniform amalgamation*. In [19], Herwig proves that saturation of the generic is equivalent to $\leq^*$ -amalgamation, but under the additional assumption that there is no infinite ascending chain of minimal pairs (condition C2 in [29]). By Lemma 2.18 in [5], this means the generic does not have *finite closures* according to Definition 2.10 in [5]. When the generic does not contain infinite ascending chains of minimal pairs, the saturation of the generic is in fact equivalent to the conjunction of $\leq^*$ -amalgamation and C2. We show that C2 does not hold in our class $(\mathcal{C}_{\text{un}}, \leq)$, and so the generic is not saturated in our case.

DEFINITION 5.9. A *minimal pair* is a pair (A, B) of elements of $\mathcal{C}_{\text{un}}$ such that $A \subsetneq B$, $A \not\leq B$, and $A \leq A'$ for every A' such that $A \subseteq A' \subsetneq B$.

PROPOSITION 5.10. $\mathcal{C}_{\text{un}}$ contains an infinite ascending chain of minimal pairs, that is, there is a sequence $((A_i, A_{i+1}) \mid i < \omega)$ in $\mathcal{C}_{\text{un}}$ such that each (A_i, A_{i+1}) is a minimal pair. Hence condition C2 of [29] does not hold.

PROOF. Let F_3 be the free STS with three generators a_1, a_2, a_3 , and let $b = (a_1 \cdot a_2) \cdot (a_1 \cdot a_3)$. As shown in [7], the homomorphism $f : F_3 \rightarrow F_3$ induced by $f(a_1) = b$, $f(a_2) = a_2$, and $f(a_3) = a_3$ is an embedding and it is not surjective. It is easy to see that if $A = \{b, a_2, a_3\}$ and $B = \{b, a_1, a_2, a_3, a_1 \cdot a_2, a_1 \cdot a_3\}$, then (A, B) is a minimal pair. This construction can be iterated as follows. Start with N_0 freely generated by $A_0 = \{a_1, a_2, a_3\}$ and let $b_0 = a_1$. Assume N_n is freely generated by $\{b_n, a_2, a_3\}$ and find a proper extension N_{n+1} freely generated by $\{b_{n+1}, a_2, a_3\}$, where $b_n = (b_{n+1} \cdot a_2) \cdot (b_{n+1} \cdot a_3)$. Let $A_{n+1} = A_n \cup \{b_{n+1}, a_2 \cdot b_{n+1}, a_3 \cdot b_{n+1}\}$. Then (A_n, A_{n+1}) is a minimal pair. $\dashv$

PROPOSITION 5.11. $(\mathcal{C}_{\text{un}}, \leq)$ has the n -amalgamation property for every n . In fact, if $A \leq B$ and $n < \omega$ are given, we can always take $m = n + |B \setminus A|$.

PROOF. Assume $A \leq B$ and $A \leq^m C$, where $m = n + |B \setminus A|$. We want to find $B^* \cong_A B$ and $D \in \mathcal{C}_{\text{un}}$ such that $C \leq D$ and $B^* \leq^n D$. We may assume $C \cap B = A$. Note that the blocks containing only elements of A are the same in B and in C . Assume there are $b \in B \setminus A$, $c \in C \setminus A$ and $a_1, a_2 \in A$ such that $\{a_1, a_2, b\}$ is a block of B and $\{a_1, a_2, c\}$ is a block of C . Redefine $A' = A \cup \{c\}$, $B' = (B \setminus \{b\}) \cup \{c\}$ and modify the blocks of B by exchanging b with c to obtain the blocks of B' . Then $B' \cong_A B$ and $B' \cap C = A'$. By Lemma 5.7, we have $A' \leq B'$ and $A' \leq^{m-1} C$. Iterating this process gives $B^* \cong_A B$ such that if $A^* = B^* \cap C$, then $A^* \leq B^*$, $A^* \leq^{n+i} C$ for some $i \geq 0$ and there are no $b \in B^* \setminus A^*$, $c \in C \setminus A^*$ and $a_1, a_2 \in A^*$ such that $\{a_1, a_2, b\}$ is a block of B^* and $\{a_1, a_2, c\}$ is a block of C . Now consider

$D = B^* \cup C$ as a free amalgam of B^* and C over A^* , that is, with the blocks in B^* and in C and no others. Then D is a PSTS. We claim that $D \in \mathcal{C}_{\text{un}}$, $C \leq D$, and $B^* \leq^n D$.

First we show that D has an HF-ordering where C is an initial segment. This proves that $D \in \mathcal{C}_{\text{un}}$ and $C \leq D$. Take an HF-ordering $<_C$ of C and an HF-ordering $<_{B^*}$ of $B^* = A^* \cup (B^* \setminus C)$, where A^* is an initial segment. Then the sum

$$(C, <_C) + (B^* \setminus C, <_{B^*} \upharpoonright (B^* \setminus C))$$

is an HF-ordering of D . If not, there is an element of $B^* \setminus C$ that belongs to different blocks containing smaller elements, but these elements then are in $A^* \cup (B^* \setminus C)$, contradicting the choice of the ordering $<_{B^*}$ of B^* as an HF-ordering over A^* .

Finally, we check that $B^* \leq^n D$. Let $X \subseteq D \setminus B^* = C \setminus A^*$ with $|X| \leq n$; we show that $B^* \leq B^* \cup X$. Since $A^* \leq B^*$, we have an HF-ordering $<_{B^*}$ of B^* with initial segment A^* . Since $A^* \leq^n C$, we know that $A^* \leq A^* \cup X$, and hence there is an HF-ordering $<_{A^* \cup X}$ of $A^* \cup X$ with initial segment A^* . We combine these orderings by taking the sum $(B^*, <_{B^*}) + (X, <_{A^* \cup X} \upharpoonright X)$. Since an element of X cannot have a block containing elements of B^* unless the elements belong to A^* , we obtain an HF-ordering of $B^* \cup X$ with initial segment B^* . $\dashv$

PROPOSITION 5.12. *$(\mathcal{C}_{\text{un}}, \leq)$ has the amalgamation property: if $A, B, C \in \mathcal{C}_{\text{un}}$, $A \leq B$ and $A \leq C$, there are $D \in \mathcal{C}_{\text{un}}$ and embeddings $f : B \rightarrow D$, $g : C \rightarrow D$ such that*

$$f \upharpoonright A = g \upharpoonright A, \quad f(B) \leq D \text{ and } g(C) \leq D.$$

PROOF. The proof of Proposition 5.11 can be used with the same construction of the PSTS $B^* \cong_A B$ and the PSTS D . $\dashv$

PROPOSITION 5.13. *The generic is isomorphic to the free STS with ω generators.*

PROOF. First note that the generic F is an STS, that is, the product of two distinct elements a_1, a_2 is defined: let b be a new vertex and let $A = \{a_1, a_2, b\}$ be an STS (so $\{a_1, a_2, b\}$ is a block). Then $A \in \mathcal{C}_{\text{un}}$ and $\{a_1, a_2\} \leq A$, and, therefore, there is an embedding $f : A \rightarrow F$ over $\{a_1, a_2\}$. Moreover, F must be a free STS, since it has a well-founded HF-ordering. On the other hand, the generic can be constructed by including at each stage an element which does not form a block with any previous elements. Then the corresponding HF-ordering has infinitely many elements which do not form a block with any preceding elements. By Proposition 3.5, these elements belong to a free base of F . $\dashv$

We extend the notion of strong subset so that it applies to infinite PSTSs.

DEFINITION 5.14. Let A, B be arbitrary PSTSs. We write $A \leq B$ if there is an HF-ordering $<$ of B over A , or, equivalently, if there is an HF-ordering of B that has A as an initial segment.

REMARK 5.15. When A is finite and B is unconfined, this definition of $A \leq B$ is equivalent to the definition given after Lemma 5.3, that is, $A \leq B$ if and only if $A \subseteq B$ and $A \leq C$ for every finite C such that $A \subseteq C \subseteq B$. More generally, if A, B are arbitrary PSTSs and B is unconfined, then

$$A \leq B \text{ if and only if } A \subseteq B \text{ and } A \cap C \leq C \text{ for every finite } C \subseteq B.$$

PROOF. Assume $A \subseteq B$ and $A \cap C \leq C$ for every finite $C \subseteq B$. We show that there is an HF-ordering of B over A . The argument is an application of the compactness theorem for propositional logic. For every pair $(a, b) \in B^2$ we introduce a propositional variable $p_{(a,b)}$ meaning $a < b$. The set of sentences

$$\begin{aligned} & \{\neg p_{(a,a)} \mid a \in B\} \cup \{p_{(a,b)} \wedge p_{(b,c)} \rightarrow p_{(a,c)} \mid a, b, c \in B\} \\ & \cup \{p_{(a,b)} \vee p_{(b,a)} \mid a, b \in B, a \neq b\} \end{aligned}$$

defines a linear ordering in B and the set

$$\{\neg(p_{(b,a)} \wedge p_{(c,a)} \wedge p_{(b',a)} \wedge p_{(c',a)}) \mid \{a, b, c\} \neq \{a, b', c'\} \text{ blocks of } B\}$$

expresses that the order is an HF-ordering of B . Finally,

$$\{p_{(a,b)} \mid a \in A, b \in B \setminus A\}$$

expresses that A is an initial segment. By assumption, every finite subset of this set of sentences is satisfiable. By compactness, the whole set is satisfiable. $\dashv$

PROPOSITION 5.16. *If M is an unconfined STS and $A = \text{acl}(A) \subseteq M$, then $A \leq M$.*

PROOF. We can expand the monster model $\mathbb{M}$ by adding an HF-ordering $<$ of $\mathbb{M}$, but we work in $\mathbb{M}$. By P. M. Neumann's Lemma (2.3 in [25]—also [20, Corollary 4.2.2]), given $a_1, \dots, a_n \notin \text{acl}(A)$, for any set B in $\mathbb{M}$ there are $b_1 \dots b_n$ with $b_1, \dots, b_n \equiv_A a_1 \dots a_n$ such that $b_1, \dots, b_n \notin B$. Hence if $M \setminus A = \{a_i \mid i < \kappa\}$ and $p(x_i \mid i < \kappa) = \text{tp}((a_i \mid i < \kappa)/A)$, then, for any set B , the type p has a realization $(b_i \mid i < \kappa)$ such that $\{b_i \mid i < \kappa\} \cap B = \emptyset$. Using this, for any λ we can construct a sequence of models $(M_i \mid i < \lambda)$ containing A such that $M = M_0$, $M_i \cong_A M_j$ and $M_i \cap M_j = A$ for every $i \neq j$. Let $B = \text{cl}_{<}(A) \subseteq \mathbb{M}$ and let $\mu = |B| = |A| + \omega$. We may assume $\mu < \lambda$, which implies $M_i \cap B = A$ for some i (if $A \subsetneq M_i \cap B$ for every $i < \lambda$, we can choose $a_i \in (M_i \cap B) \setminus A$ and the a_i are all distinct, which contradicts $|B| < \lambda$). By Lemma 3.11, $<$ is an HF-ordering over B . Therefore $<_{\upharpoonright A} = <_{\upharpoonright A} \upharpoonright M_i$ is an HF-ordering of M_i over $A = B \cap M_i$. Since $M_i \cong_A M$, there is an HF-ordering of M over A . $\dashv$

COROLLARY 5.17. *If $M_1 \preceq M_2$ are unconfined STSs, then $M_1 \leq M_2$.*

PROOF. By Proposition 5.16 since $\text{acl}(M_1) = M_1$. $\dashv$

§6. Binary trees. This section introduces certain tree constructions in unconfined STSs. These trees are an essential tool for our proof of the completeness of the theory of infinite unconfined STSs in Section 7.

Let M be an unconfined STS. A *binary tree of height* $\alpha \leq \omega$ is a subset $\{a_s \mid s \in 2^{<\alpha}\}$ of M such that:

1. if $s \neq t$, then $a_s \neq a_t$;
2. $a_s = a_{s0} \cdot a_{s1}$;
3. these are all the blocks of the tree: the only block containing $a_\emptyset$ is $\{a_\emptyset, a_0, a_1\}$ and if $s = ti$, the only blocks containing a_s are $\{a_t, a_{t0}, a_{t1}\}$ and $\{a_s, a_{s0}, a_{s1}\}$.

A *leaf* is an element a_s of the tree such that there are no a_{s0}, a_{s1} in the tree with $a_s = a_{s0} \cdot a_{s1}$.

REMARK 6.1.

1. A binary tree of height $n + 1$ can be seen as a construction of $a_\emptyset$ from $\{a_s \mid s \in 2^n\}$ in the form of a binary tree, with the root $a_\emptyset$ at the top. Every free STS embeds many copies of this tree: it suffices to choose a level with at least 2^n elements and take the products of these elements that belong to higher levels, thus obtaining the root $a_\emptyset$ after n steps. If $\{a_s \mid s \in 2^n\}$ are the chosen elements in the given level, the tree can be described as $\{a_s \mid s \in 2^{\leq n}\}$, where $a_s = a_{s_0} \cdot a_{s_1}$ if $s \in 2^{<n}$. A binary tree of height ω can also be seen as a (non well-founded) construction of $a_\emptyset$ over $\emptyset$.
2. The tree $\{a_s \mid s \in 2^{<\alpha}\}$ admits a natural HF-ordering that in the well-founded case reflects the order of construction of the root described in part 1: we order every level arbitrarily and we make the elements of level i precede the elements of level $i + 1$. In particular, $a_s > a_t$ if $s \subsetneq t$.

An alternative HF-ordering can be defined on a binary tree, which will be used in the proof of Proposition 6.6 below.

LEMMA 6.2. *The binary tree of height α has an HF-ordering where the root $a_\emptyset$ is the smallest element instead of being the largest.*

PROOF. If $<$ is a natural HF-ordering of the binary tree, with $a_\emptyset$ at the top, then the reverse order $>$ is also an HF-ordering and now $a_\emptyset$ is the smallest element. $\dashv$

Given an element a of an infinite unconfined STS, we can build a binary tree of arbitrary finite length having a as its root. If the STS is ω -saturated, we can find such a tree of height ω .

LEMMA 6.3.

1. *An HF-ordering of an infinite STS does not have a maximum.*
2. *If $<$ is an HF-ordering of an infinite STS M , then for every $a \in M$, for every $b \geq a$, there are $b_1, b_2 > b$ such that $a = b_1 \cdot b_2$.*

PROOF. 1. Assume $a \in M$ is a maximum. Since $|M| > 3$, the element a belongs to two different blocks, $\{a, a_1, a \cdot a_1\}$ and $\{a, a_2, a \cdot a_2\}$. By maximality, $a > a_1, a_1 \cdot a, a_2, a_2 \cdot a$, in contradiction with $<$ being an HF-ordering.

2. Assume not, and take $b \geq a$ such that if $a = a_1 \cdot a_2$, then $a_1 \leq b$ or $a_2 \leq b$. Since M is infinite, by part 1 we can choose $b_1 > b_2 > b$ such that $b_1 \cdot b_2 < b_1$. Note that $\{b_1, a, a \cdot b_1\} \neq \{b_1, b_2, b_1 \cdot b_2\}$, since otherwise $a = b_1 \cdot b_2$ and $b_1, b_2 > b$. If $b_1 > a \cdot b_1$, then the previous blocks contradict that $<$ is an HF-ordering. Hence, $b_1 < a \cdot b_1$. In this case, $a = b_1 \cdot (a \cdot b_1)$ with $b_1, a \cdot b_1 > b$, a contradiction with the choice of b . $\dashv$

PROPOSITION 6.4. *Let $<$ be an HF-ordering of the infinite STS M and let $n < \omega$. For every $a \in M$ there is a binary tree $T = \{a_s \mid s \in 2^{<n}\}$ in M such that:*

1. $a_s > a$ for every s ;
2. $a_s > a_t$ if $s \subsetneq t$.

PROOF. By induction on n . The initial cases $n = 0, 1$ are trivial. Consider the case $n = 2$. Let $b > a$. We use Lemma 6.3 to get some $b_1 > b_2 > b$ in M such that $b = b_1 \cdot b_2$. We put $a_\emptyset = b_1, a_0 = b_2$, and $a_1 = b$.

Now we assume inductively the result holds for $n \geq 2$. Given $a \in M$, the inductive hypothesis gives some tree $T_0 = \{a_s \mid s \in 2^{<n}\}$ as required for a . Again by the inductive hypothesis we can find in M a tree $T_1 = \{a'_s \mid s \in 2^{<n}\}$ with $a'_s > a_\emptyset$ and condition 2 holds. Now we combine T_0 and T_1 to get a binary tree $T = \{b_s \mid s \in 2^{<n+1}\}$. We set $b_{0s} = a_s$, $b_{1s} = a'_s$, and $b_\emptyset = a_\emptyset \cdot a'_\emptyset$. It is enough to check that $a_\emptyset \cdot a'_\emptyset > a'_\emptyset$. Since $n \geq 2$ this is clear, since otherwise the blocks $\{a'_\emptyset, a'_\emptyset, a'_1\}$ and $\{a'_\emptyset, a_\emptyset, a_\emptyset \cdot a'_\emptyset\}$ are different and contradict that $<$ is an HF-ordering. $\dashv$

COROLLARY 6.5. *Let $<$ be an HF-ordering of the infinite STS M and assume M is ω -saturated as an $(R, <)$ -structure. For every $a \in M$ there is a binary tree $T = \{a_s \mid s \in 2^{<\omega}\}$ in M such that:*

1. $a_s > a$ for every s ;
2. $a_s > a_t$ if $s \subsetneq t$.

PROOF. The tree is a realization of a type over a in M in the extended language with the order relation. The consistency of the type is granted by Proposition 6.4. $\dashv$

Binary trees allow us to give conditions under which a certain element in an infinite unconfined STS M can be placed between a subsystem A of M and the complement $M \setminus A$. In the presence of a suitable amount of saturation, for a $<$ -closed set A we can always find an HF-ordering and a least element in $M \setminus A$ that is above A . This result is essential in the back-and-forth argument that proves the completeness of the theory of infinite unconfined STSs in Section 7.

In the proposition below, the symbol $<$ denotes an HF-ordering, while $\leq$ is used for the relation of being a strong substructure.

PROPOSITION 6.6. *Assume M is an infinite unconfined STS, $A = \langle A \rangle \subseteq M$ and $a \in M \setminus A$. The following are equivalent:*

1. $\text{cl}_<(a) = \{a\}$ for some HF-ordering of M with initial segment A ;
2. $A \leq Aa \leq M$;
3. *there are $M' \succeq M$, an HF-ordering $<'$ of M' and a binary tree $T = \{a_s \mid s \in 2^{<\omega}\}$ in M' such that $\text{cl}_{<'}(A) = A$, $a = a_\emptyset$, $T \succ' A$, and $a_s \succ' a_{s_0}, a_{s_1}$ for every $s \in 2^\omega$.*

Moreover, these conditions imply $a \leq M$.

PROOF. $1 \Rightarrow (2 \text{ and } a \leq M)$. Fix an HF-ordering $<$ of M with initial segment A and such that $\text{cl}_<(a) = \{a\}$. Then we can move a to the bottom (showing $a \leq M$) and we can move a to the top of A , making it the first element of $M \setminus A$ (showing $Aa \leq M$).

$2 \Rightarrow 1$. Since $Aa \leq M$ and $A \leq Aa$, there is an HF-ordering of M having an initial segment Aa , where a is the last element of the segment. Since $a \notin A$, $\text{cl}_<(a) = \{a\}$.

$1 \Rightarrow 3$. Let $<$ be an HF-ordering of M with initial segment A such that $\text{cl}_<(a) = \{a\}$. Let $(M', <') \succeq (M, <)$ be ω -saturated. Then $<'$ is an HF-ordering of M' , $\text{cl}_{<'}(A) = A$, $a \succ' A$, and $\text{cl}_{<'}(a) = \{a\}$. By Corollary 6.5, choose a tree $T_0 = \{a_s \mid s \in 2^{<\omega}\}$ in M' such that $a_s \succ' a$ for every s and such that $a_s \succ' a_t$ if $s \subsetneq t$. Then $a \cdot a_\emptyset \succ' a_\emptyset$, since otherwise the blocks $\{a_\emptyset, a_\emptyset, a_1\} \neq \{a_\emptyset, a, a \cdot a_\emptyset\}$ contradict that $<'$ is an HF-ordering. Let $b_\emptyset = a \cdot a_\emptyset$ and $B_0 = \{b_\emptyset\} \cup T_0$. Note that $\text{cl}_{<'}(B_0) = B_0 \cup \{a\}$. Now we modify the order $<'$, thus obtaining a new HF-ordering $<_0$ such that $<_0 \upharpoonright B_0 = <' \upharpoonright B_0$ and $<_0 \upharpoonright (M \setminus B_0) = <' \upharpoonright (M \setminus B_0)$. This ordering $<_0$ is

obtained by moving B_0 below a but above A , in particular by declaring that $a >_0 b_0$ and $B_0 >_0 A$ and, hence, $a >_0 a_s$ for every s . This can be done by Lemma 3.12. We have obtained an incomplete tree with root a followed by b_0 on the left and the complete tree T_0 on the right. Now b_0 is a leaf and we should complete the tree below b_0 . The construction must be iterated ω times in order to obtain a full binary tree.

$3 \Rightarrow 1$. Let M' , $<'$, and T be as in the assumption. Notice that $\text{cl}_{<'}(T) = T$. By Lemma 6.2 T has an HF-ordering $<''$, where a_0 is the smallest element. By Lemma 3.12 we can assume that $<'' = <' \upharpoonright T$ and A is an initial segment. Let $< = <' \upharpoonright M$. Then $<$ is an HF-ordering of M , A is an initial segment, and $\text{cl}_{<}(a) = \{a\}$. $\dashv$

COROLLARY 6.7. *Let M be an infinite unconfined STS and $A \subseteq M$. If $<$ is an HF-ordering of M such that $(M, <)$ is κ -saturated for some $\kappa > |A|$ and $\text{cl}_{<}(A) = A$, then there is some $a \in M \setminus A$ such that $Aa \leq M$.*

PROOF. By κ -saturation there is some $b \in M$ such that $b > A$. By Corollary 6.5 we find a binary tree $T > b$ of height ω in M and by Proposition 6.6 the root a of the tree verifies $Aa \leq M$. $\dashv$

§7. Completeness and stability. In this section we show that the theory of infinite unconfined STSs is complete and stable. Since all free STSs are models of the theory, it follows that any two free STSs are elementarily equivalent. Moreover, there is a countable model of the theory which is not free.

LEMMA 7.1. *Let M be an unconfined STS. If $A \leq M$ (as in Definition 5.1), then $\langle A \rangle \leq M$ and there is an HF-ordering of $\langle A \rangle$ over A whose final segment $\langle A \rangle \setminus A$ is well-ordered. If N is an unconfined STS and $B \leq N$ and $f : A \cong B$ is an isomorphism of PSTSs, then f can be extended to an isomorphism $f' : \langle A \rangle \cong \langle B \rangle$.*

PROOF. We could use Lemma 3.13 to verify the first assertion, but the following proof also gives the isomorphism. Decompose $\langle A \rangle \setminus A$ into a disjoint union $\bigcup_{n < \omega} A_n$, where A_0 is the set of elements of $\langle A \rangle \setminus A$ which are a product of two elements of A , and A_{n+1} is the set of elements of $\langle A \rangle \setminus (A \cup A_0 \cup \dots \cup A_n)$ which are a product of two elements of $A \cup A_0 \cup \dots \cup A_n$. Fix an HF-ordering of M with initial segment A . There are no blocks between elements of A_0 , and the only blocks of elements of A_0 with smaller elements are blocks with elements of A that justify their inclusion in A_0 . Hence we can move A_0 down, just above A , and we can give A_0 a well-founded HF-ordering over A . This shows that $A \cup A_0 \leq M$. By a similar argument we can move down each A_{n+1} above $A \cup A_0 \cup \dots \cup A_n$ and give it a well-founded HF-ordering over this set. The sequence of these well-founded orders is a well-founded HF-order of $\langle A \rangle \setminus A$ which makes $\langle A \rangle$ an initial segment of the order of M . If we perform a similar decomposition $\bigcup_{n < \omega} B_n$ of $\langle B \rangle \setminus B$, then we can extend f inductively to an isomorphism of partial STSs $f_n : A \cup A_0 \cup \dots \cup A_n \cong B \cup B_0 \cup \dots \cup B_n$. The union of this ascending chain of isomorphisms is the required isomorphism f' . $\dashv$

The following is a stronger form of Corollary 6.7.

LEMMA 7.2. *Let M be an infinite unconfined STS and $A \subseteq M$. If $<$ is an HF-ordering of M , $\text{cl}_{<}(A) = A$, and M is κ -saturated as an $\{R, <\}$ -structure for some $\kappa > |A|$, then $Aa \leq M$ for some $a \in M \setminus \langle A \rangle$.*

PROOF. Since $\text{cl}_{<}(A) = A$, we have $A \leq M$ by Lemma 3.11. Then $\langle A \rangle \leq M$ by Lemma 7.1. By Corollary 6.7 we know that there is $a \in M \setminus \langle A \rangle$ such that $\langle A \rangle a \leq M$. It is easy to see that $Aa \leq \langle A \rangle a$ and hence $Aa \leq M$. $\dashv$

REMARK 7.3. If M is a saturated model of cardinality κ in the language L , then for every extension $L' \supseteq L$ with $|L'| \leq \kappa$, for every set of sentences Σ in L' consistent with $\text{Th}(M)$, there is an expansion M' of M such that $M' \models \Sigma$. If $A \subseteq M$ has cardinality $< \kappa$ we may allow Σ to have parameters in A , since the expansion $M_A = (M, a)_{a \in A}$ is still saturated. Moreover, we can let Σ contain free variables, since we can replace them by constants. The same happens if M is a special model. For details see [8].

For simplicity, the next two results are formulated in terms of saturated models, but we could alternatively use special models (κ -saturated in the lemma and ω_1 -saturated in the theorem), whose existence can always be assumed without extra set-theoretical hypotheses.

LEMMA 7.4. *Let M be a saturated unconfined STS, let $A \leq M$ and assume $|A| < \kappa = |M|$. Then there is an HF-ordering $<$ of M such that M is saturated as an $\{R, <\}$ -structure and $\text{cl}_{<}(A) = A$.*

PROOF. Let $<$ be an HF-ordering of M having A as an initial segment. Notice that the expansion $M_A = (M, a)_{a \in A}$ is saturated. Let $(M'_A, <')$ be a saturated model that is elementarily equivalent to $(M_A, <)$ and has cardinality κ . For each $a \in A$ that is not a product in M of two elements $a_1, a_2 < a$, the sentence $\forall x, y (x, y < a \rightarrow x \cdot y \neq a)$ holds in $(M_A, <)$. Hence, $\text{cl}_{<' }(A) = A$. Since $M_A \cong M'_A$, there is some ordering $<''$ of M such that $(M_A, <'') \cong (M'_A, <')$. Then $<''$ satisfies the requirements. $\dashv$

THEOREM 7.5. *Let M, N be uncountable saturated unconfined STSs. The set of isomorphisms between countable substructures $A \subseteq M$ and $B \subseteq N$ (in particular, A and B are STSs) such that $A \leq M$ and $B \leq N$ is a back-and-forth system between M and N .*

PROOF. The case $A = B = \emptyset$ is allowed and shows that there is at least one such isomorphism. Now assume $f : A \cong B$, where $A \leq M$ and $B \leq N$ are countable substructures, and let $a \in M \setminus A$. We want to find countable substructures $A' \leq M$ and $B' \leq N$ extending A and B with $a \in A'$ and $f' : A' \cong B'$ for some $f' \supseteq f$. The back step (i.e., extending f by adding some $b \in N \setminus B$) is similar. We fix an arbitrary HF-ordering $<_M$ of M , where A is an initial segment and we use Lemma 7.4 to choose an HF-ordering $<_N$ of N such that $(N, <_N)$ is saturated and $\text{cl}_{<_N}(B) = B$.

Case 1. $\text{cl}_{<_M}(a) = \{a\}$, or, equivalently, $Aa \leq M$. Corollary 6.7 gives $b \in N \setminus B$ such that $Bb \leq N$, in which case, by Proposition 6.6 and by changing the ordering $<_N$ if necessary, we may assume that $\text{cl}_{<_N}(b) = \{b\}$ and $b >_N B$. Then f extends to an isomorphism of PSTSs $f' : Aa \cong Bb$. By Lemma 7.1, we have $\langle Aa \rangle \leq M$, $\langle Bb \rangle \leq N$, and f' extends to an isomorphism $f'' : \langle Aa \rangle \cong \langle Bb \rangle$.

Case 2. $\text{cl}_{<_M}(a) \setminus A = \{a_0, \dots, a_n\}$ with $a = a_0 >_M a_1 >_M \dots >_M a_n$. Note that $\text{cl}_{<_M}(a_n) = \{a_n\}$ and $a_n \notin A$. As in Case 1, we can find $b_n \in N \setminus B$ such

that $Bb_n \leq N$, and f extends to an isomorphism $f_n : \langle Aa_n \rangle \cong \langle Bb_n \rangle$ such that $f_n(a_n) = b_n$. Assume inductively that we have found $b_{i+1}, \dots, b_n \in N$ and an isomorphism

$$f_{i+1} : \langle Aa_{i+1} \dots a_n \rangle \cong \langle Bb_{i+1} \dots b_n \rangle$$

with $f_{i+1} \supseteq f$ and such that $f_{i+1}(a_j) = b_j$ for every j . If $a_i \in \langle Aa_{i+1} \dots a_n \rangle$, we let $f_i = f_{i+1}$ and $b_i = f_i(a_i)$. If $a_i \notin \langle Aa_{i+1} \dots a_n \rangle$, we see that $\text{cl}_{<_M}(a_i) = \{a_i\}$ and, as for a_n , we get $b_i \in N \setminus \langle Bb_{i+1} \dots b_n \rangle$ such that $\langle Bb_{i+1} \dots b_n \rangle b_i \leq N$ and f_{i+1} extends to an isomorphism $f_i : \langle Aa_i \dots a_n \rangle \cong \langle Bb_i \dots b_n \rangle$ with $f_i(a_i) = b_i$. Finally we obtain $b_0, \dots, b_n$ and an isomorphism $f_0 : \langle Aa_0 \dots a_n \rangle \cong \langle Bb_0 \dots b_n \rangle$ extending f with $f_0(a_i) = b_i$ for every i . Moreover, $\langle Aa_0 \dots a_n \rangle \leq M$ and $\langle Bb_0 \dots b_n \rangle \leq N$. Since $\langle Bb_0 \dots b_n \rangle \leq N$, if necessary we can change the order on $\langle Bb_0 \dots b_n \rangle$ by that induced by f_0 so that $b_0 >_N b_1 > \dots >_N b_n$.

Case 3. $\text{cl}_{<_M}(a) \setminus A = \{a_i \mid i < \omega\}$ with $a = a_0$ and $a_i >_M a_{i+1}$. We claim that for every $n < \omega$ there are $b_0, b_1, \dots, b_n \in N$ such that:

1. $B \leq Bb_n \leq Bb_{n-1}b_n \leq \dots \leq Bb_0 \dots b_n \leq N$.
2. $Aa_0 \dots a_n \cong Bb_0 \dots b_n$ as PSTSs by some isomorphism f_0 extending f such that $f_0(a_i) = b_i$ for every $i \leq n$. We cannot ensure that $\langle Aa_0 \dots a_n \rangle \cong \langle Bb_0 \dots b_n \rangle$, since $Aa_0 \dots a_n \leq M$ may not hold.

The b_i are obtained inductively, starting with b_n . We choose $b_n \in N \setminus B$ so that $Bb_n \leq N$. It is easy to see that $Aa_n \cong Bb_n$ by an isomorphism extending f . Assume inductively $f_{i+1} : Aa_{i+1} \dots a_n \cong Bb_{i+1} \dots b_n$ with $f \subseteq f_{i+1}$ and $f_{i+1}(a_j) = b_j$, and assume

$$B \leq Bb_n \leq \dots \leq Bb_{i+1} \dots b_n \leq N.$$

There are several cases for a_i . If $a_i = a_j \cdot a_k$ with $i < j$ and $k \leq n$, we take $b_i = b_j \cdot b_k$. If $a_i = a_j \cdot c$ with $i < j \leq n$ and $c \in A$, we put $b_i = b_j \cdot f_{i+1}(c)$. In other cases, we use Lemma 7.2 to get $b_i \in N \setminus Bb_{i+1} \dots b_n$ such that $Bb_i b_{i+1} \dots b_n \leq N$ and which is not a product of elements of $Bb_{i+1} \dots b_n$. In all cases, $Bb_{i+1} \dots b_n \leq Bb_i b_{i+1} \dots b_n \leq N$ and f_{i+1} extends to some $f_i : Aa_i \dots a_n \cong Bb_i \dots b_n$ with $f_i(a_i) = b_i$.

Now we define a set Σ of formulas over B in the language extended by the symbol $<$, with free variables $\{x_i \mid i < \omega\}$ and expressing the following conditions:

1. $(x_i \mid i < \omega)$ realizes the diagram of $(a_i \mid i < \omega)$ over A (as a PSTS) conjugated by the isomorphism f ;
2. $<$ is an HF-ordering and coincides with $<_N$ on B ;
3. $x_i > x_{i+1}$;
4. if $b \in B$ is not a product of smaller elements of B in the order $<_N$, then $\forall xy (x, y < b \rightarrow x \cdot y \neq b)$;
5. if a_i is not the product of smaller elements a_j, a_k nor the product of a smaller a_j with an element of A , then $\forall xy (x, y < x_i \rightarrow x \cdot y \neq x_i)$.

Every finite subset of Σ is satisfiable in N . More precisely, for every $n < \omega$ the order $<_N$ satisfies all five conditions if we restrict Σ to the variables $x_0, \dots, x_n$ and we assign to these variables the sequence $b_0, \dots, b_n$ described above. Note that we can interpret the order as an HF-ordering of N that coincides with $<_N$ on B and has B as an initial segment, followed by $b_n, \dots, b_0$ in ascending order. Now we use Remark 7.3 to obtain an interpretation of Σ in N over B , namely, an HF-ordering $<'$ of N

and a sequence $(b_i \mid i < \omega)$ of elements of N that satisfy all conditions on Σ over B . Then $f \cup \{(a_i, b_i) \mid i < \omega\}$ is an isomorphism of PSTSs between $A(a_i \mid i < \omega)$ and $B(b_i \mid i < \omega)$. Note that $\text{cl}_{<'}(B \cup \{b_i \mid i < \omega\}) = B \cup \{b_i \mid i < \omega\}$ and, therefore, $B(b_i \mid i < \omega) \leq N$. Hence $\langle A(a_i \mid i < \omega) \rangle \leq M$ and $\langle B(b_i \mid i < \omega) \rangle \leq N$, and so, by Lemma 7.1, we can extend f to an isomorphism $f' : \langle A(a_i \mid i < \omega) \rangle \cong \langle B(b_i \mid i < \omega) \rangle$. $\dashv$

COROLLARY 7.6. *Let M, N be infinite unconfined STS. If $A \leq M$, $B \leq N$, and $f : A \rightarrow B$ is an isomorphism of PSTS, then f is a partial elementary map from M to N .*

PROOF. First consider the case where A and B are countable. By Lemma 7.1, $\langle A \rangle \leq M$, $\langle B \rangle \leq N$, and f extends to some isomorphism $f' : \langle A \rangle \cong \langle B \rangle$. We can find uncountable saturated (or special ω_1 -saturated) elementary extensions $M' \succeq M$ and $N' \succeq N$. Notice that $A \leq M'$ and $B \leq N'$. By Proposition 7.5, f' belongs to a back-and-forth system between M' and N' and so it is a partial elementary map from M' to N' and also from M to N .

Consider now the general case. Fix an HF-ordering of M having A as an initial segment. Consider a finite set $A_0 \subseteq A$ and let $A'_0 = \text{cl}_{<}(A_0)$. Since A is an initial segment, $A'_0 \subseteq A$. By Lemma 3.11, we may assume that A'_0 is an initial segment of A . Then $A'_0 \leq A$ and $A'_0 \leq M$. Now order B in such a way that f is an order isomorphism. This gives an HF-ordering of B . Moreover, if $B_0 = f(A_0)$ and $B'_0 = f(A'_0)$, then $B_0 \subseteq B'_0$ and B'_0 is an initial segment of B . Hence $B'_0 \leq B$ and $B'_0 \leq N$. Since A'_0 and B'_0 are countable, by the initial case we know that $f \upharpoonright A'_0$ is elementary. Hence, f is elementary. $\dashv$

The completeness of the theory of infinite unconfined STS is implicit in the proof of Corollary 7.6.

COROLLARY 7.7. *The theory of all infinite unconfined STS is complete; it is the complete theory of every free STS with ≥ 3 generators.*

PROOF. Corollary 7.6 implies that the empty mapping between unconfined STS is elementary. The claim follows directly. $\dashv$

REMARK 7.8. Assume M is an infinite unconfined STS and $A = \langle A \rangle \leq M$. By Corollary 7.6, for any $a \notin A$ the condition $Aa \leq M$ from Proposition 6.6 defines a complete type $\text{tp}(a/A)$: if $b \notin A$ and $Ab \leq M$, the mapping which is the identity on A and sends a to b is elementary.

LEMMA 7.9. *Let M be an infinite unconfined STS and let $\bar{a}, \bar{b}$ be tuples from M . If for some HF-orderings $<$ and $<'$ of M , $\text{cl}_{<}(\bar{a})$ and $\text{cl}_{<' }(\bar{b})$ have the same diagram as PSTS, then $\text{tp}_M(\bar{a}) = \text{tp}_M(\bar{b})$.*

PROOF. By Corollary 7.6, since $\text{cl}_{<}(\bar{a}) \leq M$ and $\text{cl}_{<' }(\bar{b}) \leq M$. $\dashv$

PROPOSITION 7.10. *The theory of free STSs is stable.*

PROOF. We count 1-types over a model M and show that their number is $\leq |M|^\omega$. We assume that M is an elementary substructure of the monster model $\mathbb{M}$. Then $M \leq \mathbb{M}$ by Lemma 5.17, and we can fix an HF-ordering $<$ of $\mathbb{M}$, where M is an initial segment. If $a, b \in \mathbb{M}$ and $\text{cl}_{<}(a), \text{cl}_{<}(b)$ have the same diagram over M as PSTS, then, by Lemma 7.9, they have same type over M . Hence, the number of

1-types over M is bounded by the number of PSTS isomorphism types of countable tuples over M , which is clearly $\leq |M|^\omega$. $\dashv$

PROPOSITION 7.11. *Let M, N be models of the theory of free STSs. If $M \leq N$, then $M \preccurlyeq N$.*

PROOF. By Corollary 5.17 we may assume that N is ω_1 -saturated. Let $\bar{a} \in M$ be a finite tuple. We want to show that $\text{tp}_N(\bar{a}) = \text{tp}_M(\bar{a})$. Fix an HF-ordering $<$ of N with initial segment M and let $\bar{a}'$ be a countable tuple enumerating $\text{cl}_<(\bar{a})$. It follows from Definition 3.10 that elements in $\bar{a}' \setminus \bar{a}$ precede elements in $\bar{a}$ in the ordering. It suffices to show that $\text{tp}_N(\bar{a}') = \text{tp}_M(\bar{a}')$. Let $p(\bar{x}) = \text{tp}_M(\bar{a}')$. By ω_1 -saturation, there is some tuple $\bar{b} \in N$ realizing $p(\bar{x})$. This implies that $\bar{b} \leq N$ (the formulas expressing this belong to $p(\bar{x})$), and moreover $\bar{a}'$ and $\bar{b}$ have the same quantifier-free type as PSTSs. They have the same diagram as STSs. Moreover $\bar{a}' \leq N$. By Corollary 7.6, $\bar{a}' \equiv \bar{b}$ in N . Hence $\text{tp}_N(\bar{a}') = \text{tp}_N(\bar{b}) = \text{tp}_M(\bar{a}')$. $\dashv$

PROPOSITION 7.12. *Let M be an infinite unconfined STS and let $A = \langle A \rangle \subseteq M$ be infinite. Then $A \equiv M$. Moreover,*

1. $\text{acl}(A) \preceq M$;
2. $A \leq M$ if and only if $\text{acl}(A) = A$.

PROOF. Since A is an infinite unconfined STS, by completeness of the theory $A \equiv M$. Claim 1 follows from Proposition 7.11, since, by Proposition 5.16, $\text{acl}(A) \leq M$. Claim 2 follows from 1 and Proposition 7.11. $\dashv$

REMARK 7.13. Let M be an infinite unconfined STS. If $A \subseteq M$, then $\text{acl}(A)$ is the intersection of all substructures $B \supseteq A$ of M such that $B \leq M$.

PROOF. The trivial case where A is contained in a block is easy. Otherwise, $\langle A \rangle$ is infinite and, by Proposition 7.12, $\text{acl}(A)$ contains the intersection of all substructures B of M such that $A \subseteq B \leq M$. On the other hand, if B is one of these structures, again by Proposition 7.12, $\text{acl}(A) \subseteq \text{acl}(B) = B$. $\dashv$

REMARK 7.14. Let F_n be the free STS with $n \geq 3$ generators. Every surjective homomorphism $f : F_n \rightarrow F_n$ is an automorphism of F_n .

PROOF. See [7, Corollary 2.11]. $\dashv$

As remarked by the anonymous referee, a group where every surjective endomorphism is an automorphism is called *Hopfian*. Proofs of this property for groups often use ad-hoc techniques, whereas our argument applies to any free algebras arising from a class that satisfies the hypotheses of Theorems 1 and 2 in [22]—that is, the class contains a finite algebra with more than one element, and is such that any algebra that is free for the finite elements in the class is free for the whole class (for the second condition in the case of STSs see [7, Lemma 2.6]).

PROPOSITION 7.15. *Let F_n be the free STS with $n \geq 3$ generators. Every elementary embedding $f : F_n \rightarrow F_n$ is surjective and, therefore, is an automorphism of F_n .*

PROOF. Let $F_n = \langle \bar{a} \rangle$, where $\bar{a} = a_1, \dots, a_n$ is a free base, and let $a'_i = f(a_i) = t_i(\bar{a})$ for every $i = 1, \dots, n$, where t_i is a corresponding term. Then F_n satisfies

$$\exists x_1 \dots x_n \bigwedge_{i=1}^n a'_i = t_i(x_1, \dots, x_n).$$

Since $f(F_n) \preceq F_n$, the same sentence holds in $f(F_n)$ and, by isomorphism, F_n satisfies

$$\exists x_1 \dots x_n \bigwedge_{i=1}^n a_i = t_i(x_1, \dots, x_n).$$

Choose $\bar{b} = b_1, \dots, b_n \in F_n$ such that $a_i = t_i(\bar{b})$ for every $i = 1, \dots, n$. Then $F_n = \langle b_1, \dots, b_n \rangle$ and $b_1, \dots, b_n$ are independent by Remark 7.14. Now consider the automorphism $g : F_n \rightarrow F_n$ defined by $g(b_i) = a_i$ for $i = 1, \dots, n$. Observe that for every i

$$g(a_i) = g(t_i(\bar{b})) = t_i(g(\bar{b})) = t_i(\bar{a}) = a'_i.$$

This implies that $f = g$ and, therefore, that f is an automorphism. $\dashv$

PROPOSITION 7.16. *If $3 \leq n < m$, there is no elementary embedding of F_m , the free STS with m generators, into F_n , the free STS of n generators, despite the elementary equivalence of F_m and F_n .*

PROOF. Assume $n < m$ and $f : F_m \rightarrow F_n$ is an elementary embedding with $f(F_m) = F'_m$. There is a copy F''_m of F_m such that $F_n \leq F''_m$, so $F_n \preceq F''_m$ by Proposition 7.11. Without loss of generality, $F_m = F''_m$. Let g be the inclusion map $F_n \hookrightarrow F_m$ restricted to F'_m . Then $g \circ f : F_m \rightarrow F_m$ is an elementary injection with $g \circ f(F_m) \neq F_m$, contradicting Proposition 7.15. $\dashv$

PROPOSITION 7.17. *There is a (countable) unconfined STS which is not free.*

PROOF. Let $N = F_3$ be the free STS generated by three elements a_1, a_2, a_3 , and let $b = (a_1 \cdot a_2) \cdot (a_1 \cdot a_3)$. As shown in [7, Example 6.9], the homomorphism $f : N \rightarrow N$ determined by $f(a_1) = b$, $f(a_2) = a_2$, and $f(a_3) = a_3$ is an embedding but it is not surjective. Hence, there is some proper substructure of N isomorphic to N . So we can define an ascending chain $(N_n \mid n < \omega)$ of STSs such that $N_n \cong N$ and $N_n \neq N_{n+1}$. Let $N_\omega = \bigcup_{n < \omega} N_n$. Since N_ω is a union of a countable chain of unconfined STSs and unconfinedness fails on a finite subset, we have that N_ω is unconfined. Moreover, it is not finitely generated. Since it is countable, if it is free then it has a free base B of ω elements and it is (isomorphic to) the generic. There is an embedding $f : F_4 \rightarrow N_\omega$ such that $f(F_4) \leq N_\omega$. Choose n such that the image in f of the base of F_4 is contained in N_n . Then $f(F_4) \leq N_n$ and by Proposition 7.11, $f : F_4 \rightarrow N_n$ is elementary. Since $N_n \cong F_3$, this contradicts Proposition 7.16. Hence, N_ω is not free. $\dashv$

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REFERENCES

- [1] A. AMMER and K. TENT, *On the model theory of open generalised polygons*. *Model Theory*, vol. 3 (2024), no. 2, pp. 433–448.
- [2] J. T. BALDWIN, *Strongly minimal Steiner systems III: Path graphs and sparse configurations*, preprint, 2022, [arXiv:2201.11566](https://arxiv.org/abs/2201.11566).
- [3] ———, *Strongly minimal Steiner systems II: Coordinatization and strongly minimal quasigroups*. *Algebra Universalis*, vol. 84 (2023), no. 20.
- [4] J. T. BALDWIN and G. PAOLINI, *Strongly minimal Steiner systems I: Existence*, 2021. *The Journal of Symbolic Logic*, vol. 86 (2021), no. 4, pp. 1486–1507.
- [5] J. T. BALDWIN and N. SHI, *Stable generic structures*. *Annals of Pure and Applied Logic*, vol. 79 (1996), pp. 1–35.
- [6] S. BARBINA and E. CASANOVAS, *Model theory of Steiner triple systems*. *Journal of Mathematical Logic*, vol. 20 (2020), no. 2, 2050010.
- [7] ———, *Endomorphisms of free Steiner quasigroups*, preprint, 2024, [arXiv:2412.06055](https://arxiv.org/abs/2412.06055).
- [8] E. CASANOVAS, *Compactly expandable models and stability*. *The Journal of Symbolic Logic*, vol. 60 (1995), pp. 673–683.
- [9] K. M. CHICOT, M. J. GRANNELL, T. S. GRIGGS, and B. S. WEBB, *On sparse countably infinite Steiner triple systems*. *Journal of Combinatorial Designs*, vol. 18 (2010), no. 2, pp. 115–122.
- [10] C. J. COLBOURN and A. ROSA, *Triple Systems*, Oxford University Press, Oxford, 1999.
- [11] G. CONANT and A. KRUCKMAN, *Independence in generic incidence structures*. *The Journal of Symbolic Logic*, vol. 84 (2019), pp. 750–789.
- [12] D. M. EVANS, *Block transitive Steiner systems with more than one point orbit*. *Journal of Combinatorial Designs*, vol. 12 (2004), no. 6, pp. 459–465.
- [13] T. FUJIWARA, *Note on the isomorphism problem for free algebraic systems*. *Proceedings. Japan Academy*, vol. 32 (1956), pp. 662–664.
- [14] M. FUNK and K. STRAMBACH, *Free constructions*, *Handbook of Incidence Geometry* (F. Buekenhout, editor), North Holland, Amsterdam, 1995, pp. 739–780.
- [15] B. GANTER and H. WERNER, *Equational classes of Steiner systems*. *Algebra Universalis*, vol. 5 (1975), pp. 125–140.
- [16] ———, *Co-ordinatizing Steiner systems*, *Topics on Steiner Systems* (C. C. Linder and A. Rosa, editors), North Holland, Amsterdam, 1980, pp. 3–24.
- [17] G. GRÄTZER, *Universal Algebra*, second ed., Springer-Verlag, New York, Heidelberg, and Berlin, 1979.
- [18] M. HALL, *Projective planes*. *Transactions of the American Mathematical Society*, vol. 54 (1943), pp. 229–277.
- [19] B. HERWIG, *Weight ω in stable theories with few types*. *The Journal of Symbolic Logic*, (1995), pp. 353–373.
- [20] W. HODGES, *Model Theory*, Cambridge University Press, Cambridge, 1993.
- [21] T. HYTTINEN and G. PAOLINI, *First-order model theory of free projective planes*. *Annals of Pure and Applied Logic*, vol. 172 (2021), 102888.
- [22] B. JÓNSSON and A. TARSKI, *On two properties of free algebras*. *Mathematica Scandinavica*, vol. 9 (1961), no. 1a, pp. 95–101.
- [23] D. W. KUEKER and M. C. LASKOWSKI, *On generic structures*. *Notre Dame Journal of Formal Logic*, vol. 33 (1992), pp. 175–183.
- [24] S. MARKOVSKI and A. SOKOLOVA, *Free Steiner loops*. *Glasnik Matematički*, vol. 36 (2001), no. 56, pp. 85–93.
- [25] P. M. NEUMANN, *The structure of finitary permutation groups*. *Archiv der Mathematik*, vol. 27 (1976), pp. 3–17.
- [26] G. PAOLINI and D. E. QUADRELLARO, *On the model theory of open incidence structures: The rank 2 case*, preprint, 2024, [arXiv:2411.10792](https://arxiv.org/abs/2411.10792).
- [27] J. G. ROSENSTEIN, *Linear Orderings*, Academic Press, New York, 1982.
- [28] L. C. SIEBENMANN, *A characterization of free projective planes*. *Pacific Journal of Mathematics*, vol. 15 (1965), no. 1, pp. 293–298.
- [29] F. O. WAGNER, *Relational structures and dimensions*, *Automorphisms of First-Order Structures* (R. Kaye and D. Macpherson, editors), Clarendon Press, Oxford, 1994, pp. 153–180.

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