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2D Euler system for Hölder continuous vorticities

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Summary

This thesis investigates the two-dimensional incompressible Euler equations under smooth initial data, focusing on three formulations:

1. the classical Euler system,

2. the vorticity-stream forumlation,

3. the integrodifferential (particle-trajectory) formulation.

In Chapter 1, we first show that these three viewpoints are mathematically equivalent. We also develop core background material on vector fields, flows, and singular integral operators. These tools, particularly the singular integral operator theory, lay the groundwork for the vorticity-stream and integrodifferential formulations used in subsequent chapters.

Chapter 2 establishes the local-in-time existence and uniqueness of solutions when the initial vorticity lies in a Hölder space C^γ (with $\gamma \in (0, 1)$). By formulating the problem in terms of the integrodifferential formulation, we show that the relevant operator is locally Lipschitz in an appropriate Banach space. Applying a Picard–Lindelöf-type argument then yields the desired local well-posedness result. Key technical ingredients include singular integral estimates and careful composition bounds in Hölder spaces.

Finally, Chapter 3 addresses the global nature of these solutions. In two dimensions, the vorticity remains constant along particle trajectories and thus stays bounded for all time, preventing finite-time blow-up. Using a method inspired by the analysis of Beale, Kato, and Majda, we show that bounding the vorticity's supremum norm also bounds the velocity gradient, which in turn guarantees global existence. Thus, for sufficiently smooth initial vorticity in 2D, the solutions to the Euler equations extend indefinitely in time and remain regular.

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Contents

Chapter 1
Introduction

In this chapter, we will present the formulation of the problem we are facing. Also, we will prove the equivalence between different formulations that will be presented, which will be needed for the rest of the thesis.

The Navier-Stokes equations are the equations that describe the motion of viscous fluid substances. In our case, we will be studying the case of 2D perfect incompressible flows, with viscosity $\nu = 0$. This special case is described by the Euler equations.

1.1 Navier-Stokes and Euler equations

Incompressible flows of homogeneous fluids in $\mathbb{R}^N$ are solutions to the system of equations

$$\begin{cases} \partial_t v + (v \cdot \nabla)\, v = -\,\nabla p + \nu\, \Delta v, & \qquad (1.1) \\ \operatorname{div} v = 0, \qquad \forall (x,t) \in \mathbb{R}^N \times [0,\infty), & \qquad (1.2) \\ v\big|_{t=0} = v_0, \qquad \forall x \in \mathbb{R}^N, & \qquad (1.3) \end{cases}$$

where $v(\cdot,t) : x \in \mathbb{R}^N \to v(x,t) \in \mathbb{R}^N$ maps each point x to the velocity of the particle in position x at a time t, $p(\cdot,t) : x \in \mathbb{R}^N \to p(x,t) \in \mathbb{R}$ maps each point x to the scalar pressure at a time t and ν is the viscosity of the fluid.

The first equation arises from the conservation of momentum, applying Newton's second law $F = m \cdot a$ to an infinitesimal volume of the fluid. On each infinitesimal volume, the forces acting on it are the pressure (adding the term ∇p to the equation) and the friction caused by the viscosity (which adds the term of the laplacian). Left hand side of the equation encapsulates the acceleration, with the term $(v\cdot\nabla)v$ being the convective acceleration (change in velocity given due to the element of the fluid entering a region where the velocity field is different).

In the system of equations (1.1)-(1.3), div is the divergence of a vector field

$$\operatorname{div}\, v = \nabla \cdot v = \sum_{i=1}^{N} \frac{\partial v^i}{\partial x_i}, \tag{1.4}$$

where the operator ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N} \right), \tag{1.5}$$

when acting against scalar valued maps, and when it acts against a vector value map u, its gradient is the following

$$\nabla u = \left(\frac{\partial u^i}{\partial x_j} \right)_{i,j=1}^{N}. \tag{1.6}$$

Also, the curl of a vector value map $u = (u^1, u^2, u^3)$ is defined as

$$\operatorname{curl} u = \nabla \times u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ u^1 & u^2 & u^3 \end{vmatrix} = \left(\frac{\partial u^3}{\partial x^2} - \frac{\partial u^2}{\partial x^3}, \frac{\partial u^1}{\partial x^3} - \frac{\partial u^3}{\partial x^1}, \frac{\partial u^2}{\partial x^1} - \frac{\partial u^1}{\partial x^2} \right)^t.$$

Finally, the operator Δ is the Laplacian operator

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}. \tag{1.7}$$

With this notation, we define $\frac{D}{Dt}$, the convective or material derivative (the partial along the trajectory of the particle), which will be used later on to simplify the notation

$$\frac{D}{Dt} = \partial_t + v \cdot \nabla = \partial_t + \sum_{i=1}^{N} v^i \frac{\partial}{\partial x_i}. \tag{1.8}$$

The solution of the system of equations (1.1)(1.3) is a flow velocity. It is a vector field, which maps any point x to its vector velocity at a time t, $v(x,t)$.

As initial properties of the solution, we have the different transformations of solutions which also yield solutions (symmetry groups). We state it as the first result of the thesis.

Proposition 1.1.1. *Let v, p be a solution to the Euler or the Navier-Stokes system of equations* (1.1)-(1.3)*. The following transformations are also solutions to the same system:*

i) Galilean invariance: For any $c \in \mathbb{R}^N$, this new velocity and pressure yields also a solution.

$$v_c(x,t) = v(x - ct, t) + c, \qquad p_c(x,t) = p(x - ct, t) \tag{1.9}$$

ii) Rotation symmetry: For any Q rotation matrix (such that $Q^t = Q^{-1}$), this new velocity and pressure yields also a solution.

$$v_Q(x,t) = Q^t v(Qx, t), \qquad p_Q(x,t) = p(Qx, t) \tag{1.10}$$

iii) Scale invariance: For any $\lambda, \tau \in \mathbb{R}$, this new velocity and pressure yields also a solution to the Euler equation.

$$v_{\lambda,\tau}(x,t) = \frac{\lambda}{\tau} v\left(\frac{x}{\lambda}, \frac{t}{\tau}\right), \qquad p_{\lambda,\tau}(x,t) = \frac{\lambda^2}{\tau^2} p\left(\frac{x}{\lambda}, \frac{t}{\tau}\right) \tag{1.11}$$

Also, for any $\tau > 0$, this new velocity and pressure yields also a solution to the Navier-Stokes equation.

$$v_\tau(x,t) = \tau^{-1/2} v\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right), \qquad p_\tau(x,t) = \tau^{-1} p\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right) \tag{1.12}$$

Proof. Each transformation is verified by straightforward substitution back into (1.1)–(1.3) (the Euler or Navier–Stokes system). We give a small explanation on each one:

i) Shifting the spatial variable by ct and adding a constant velocity (same c) does not change the form of the convective derivative.

ii) Incompressibility $\nabla \cdot v = 0$ and the form of $\partial_t v + (v \cdot \nabla)v = -\nabla p + \nu \Delta v$ are preserved under rotations because ∇ and Δ transform covariantly with Q, and Q^t precisely undoes that rotation in the velocity variable.

iii) For $\nu = 0$, the transformation simply rescales space and time. We check that the term $(v \cdot \nabla)v$ and the pressure gradient $-\nabla p$ acquire the correct factors under these dilations so that the Euler system is invariant.

For the Navier–Stokes case (with viscosity $\nu > 0$), the viscous term $\nu\,\Delta v$ picks up a factor $\tau^{-3/2}$ from $x \mapsto x/\tau^{1/2}$ and $t \mapsto t/\tau$, matching the factors on the other terms. Substituting back into the Navier–Stokes equations confirms invariance.

■

1.2 Vector fields and flows

In all this section, $v : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ will be a continuous Lipschitz vector field depending both on x and t. By continuous Lipschitz we mean that $v(x,t)$ is continuous in t for $0 \leq t \leq T$, and it's Lipschitz on x, satisfying

$$\|v(x_1,t) - f(x_2,t)\| \leq L\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^N,\ \forall t \in [0,T].$$

We introduce the following general theorem for ODEs (it is a more restricted result derived from the Picard-Lindelöf Theorem, which will be used later).

Theorem 1.2.1. *Let $v : \mathbb{R}^N \times [a,b] \to \mathbb{R}^N$ a Lipschitz vector field. Given any $(x_0,t_0) \in \mathbb{R}^N \times [a,b]$, there exists a unique continuously differentiable map $x : [a,b] \to \mathbb{R}^N$ such that*

$$\begin{cases} x' = v(x,t) \\ x(t_0) = x_0 \end{cases}$$

The function from the previous theorem 1.2.1, $x(t)$ will be known as the trajectory of the vector field v. Notice that these trajectories do not depend only on t, but also on the initial condition x_0. When thinking about them as functions of x_0, we can define the following concept

Definition 1.2.1. *Given the vector field v, we define $X(\alpha,t)$ as the solution of the following initial value problem*

$$\frac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) = v(X(\alpha,t),t), \qquad X(\alpha,0) = \alpha. \tag{1.13}$$

*This solution is called **flow of v**.*

Physically speaking, $X(\alpha,t)$ represents the position at time t of a particle that, evolving according to the velocity field v, was located at the point α at time 0.

The well-posedness of (1.13) is addressed by the theorem 1.2.1. It turns out that if v is Lipschitz then $X(\cdot,t)$ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ and therefore has a well defined inverse map $X(\cdot,t)^{-1}$.

In a linear transport equation, the solution is simply the translation of the initial position x_0 through the trajectories of the vector field, as stated in the following proposition

Proposition 1.2.1. *Given v a smooth vector field, and X its associated flow, any linear transport equation*

$$\frac{\partial u}{\partial t} + v \cdot \nabla u = 0$$

with initial datum u_0, has as its unique solution $u(X(x,t),t) = u_0(x)$.

Proof. Using the chain rule, we get that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(u(X(x,t),t)\right) &= \nabla u(X(x,t),t) \cdot \frac{\mathrm{d}X}{\mathrm{d}t}(x,t) + \frac{\partial u}{\partial t}(X(x,t),t) \\ &= \nabla u(X(x,t),t) \cdot v(X(x,t),t) + \frac{\partial u}{\partial t}(X(x,t),t) = 0 \end{aligned} \tag{1.14}$$

Now, as its time derivative is zero, we have

$$u(X(x,t),t) = u(X(x,0),0) = u_0(X(x,0)) = u_0(x), \tag{1.15}$$

as claimed. ■

We can now introduce the concept of incompressibility, and relate it with the velocity vector field.

Definition 1.2.2. *A flow of v, $X(\cdot,t)$, is incompressible if it satisfies*

$$\text{vol } X(\Omega,t) = \text{vol } \Omega$$

for all Ω subdomains, and for all t. Here $X(\Omega,t)$ denotes the image set of Ω under the flow map $X(\cdot,t)$.

We will now prove a result which explains why div $v = 0$ and the flow associated to it being incompressible are equivalent. This means the second equation in the original formulation (1.2) implies incompressibility.

Proposition 1.2.2. *For smooth vector fields v, these conditions are equivalent:*

i) *its associated flow is incompressible.*

ii) div $v = 0$.

iii) $J(\alpha,t) := (\det \nabla_\alpha X(\alpha,t)) = 1$ *(Jacobian of the transformation to particle-trajectory mapping)*

Proof. We first prove that

$$\frac{\partial J}{\partial t} = \text{div } v\big|_{(X(\alpha,t),t)} \cdot J(\alpha,t). \tag{1.16}$$

Writing by components, we have that $\nabla_\alpha X(\alpha,t) = \left(\frac{\partial X^i}{\partial \alpha_j}\right)$, so using Jacobi's rule for derivation of matrices, which states

$$\frac{\partial}{\partial t}\det A(t) = \det A(t)\cdot \operatorname{tr}\left(A(t)^{-1}\cdot \frac{\mathrm{d}A(t)}{\mathrm{d}t}\right), \tag{1.17}$$

we have

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial t}\det\left[\frac{\partial X^i}{\partial \alpha_j}(\alpha,t)\right] = \sum_{i,j} A_i^j \frac{\partial}{\partial t}\frac{\partial X^i}{\partial \alpha_j}(\alpha,t), \tag{1.18}$$

with A_i^j the minor of the matrix $\nabla_\alpha X$ associated to the element $\frac{\partial X^i}{\partial \alpha_j}$ (note that these minors come from the inverse).

Now, using the ODE that defines the flow, we have that

$$\sum_{i,j} A_i^j \frac{\partial}{\partial \alpha_j}\frac{\partial X^i}{\partial t}(\alpha,t) = \sum_{i,j} A_i^j \frac{\partial}{\partial \alpha_j} v^i(X(\alpha,t),t) = \sum_{i,j} A_i^j \sum_k \frac{\partial v^i}{\partial x_k}(X(\alpha,t),t)\frac{\partial X^k}{\partial \alpha_j}(\alpha,t). \tag{1.19}$$

Finally, using the linear algebra identity, $A\cdot \operatorname{Adj}(A) = \det(A)\cdot \mathrm{Id}$, by components, we get

$$\sum_j \frac{\partial X^k}{\partial \alpha_j} A_j^i = \delta_i^k J, \tag{1.20}$$

and applying this to the equality we previously got, we get the desired equality.

$$\sum_{i,j} A_i^j \sum_k \frac{\partial v^i}{\partial x_k}(X(\alpha,t),t)\frac{\partial X^k}{\partial \alpha_j}(\alpha,t) = \sum_{i,k}\frac{\partial v^i}{\partial x_k}(X(\alpha,t),t)\delta_i^k J(\alpha,t) \tag{1.21}$$

$$= \sum_i \frac{\partial v^i}{\partial x_i}(X(\alpha,t),t)J(\alpha,t) = \operatorname{div} v\big|_{(X(\alpha,t),t)}\cdot J(\alpha,t). \tag{1.22}$$

Now using the previous computation, we will complete the proof of the result. We will first prove the equivalence between $i)$ and $ii)$. To do so, we will compute the derivative of the volume of any image subdomain $X(\Omega,t)$ in terms of t.

Applying the change of variables $\alpha \to X(\alpha,t)$

$$\int_{X(\Omega,t)} dx = \int_\Omega J(\alpha,t)\,d\alpha, \tag{1.23}$$

and, taking its derivative, we get the following equation

$$\frac{\partial}{\partial t}\int_{X(\Omega,t)} dx = \int_\Omega \frac{\partial}{\partial t} J(\alpha,t)\,d\alpha = \int_\Omega \operatorname{div} v\big|_{(X(\alpha,t),t)}\cdot J(\alpha,t)\,d\alpha = \int_{X(\Omega,t)} \operatorname{div} v\,dx. \tag{1.24}$$

Here we used the equation proved before and we applied the inverse change of variables.

This equation implies both $i) \implies ii)$ and $ii) \implies i)$. This is because if we suppose $i)$ to be true, the derivative must vanish for all subdomains and for all t, so the divergence must be zero. Supposing $ii)$ to be true directly implies $i)$.

Finally, we want to prove equivalence between $ii)$ and $iii)$. First we suppose $ii)$ to be true. It is clear that $J(\alpha, 0) = 1$, as for $t = 0$, the transformation is exactly the identity. Also, using the equation proven before (1.16), we have that $\frac{\partial J}{\partial t} = 0$, so $J(\alpha, t) = 1$ for all α and for all t, so $ii) \implies iii)$.

For the reverse, if we suppose $J(\alpha, t) = 1$ for all t, its time derivative vanishes for all t, as it is a constant. But $\frac{\partial J}{\partial t} = \text{div } v$, so $\text{div } v = 0$ for all t, finishing the proof.

■

Given this result, we will be talking about incompressible flows, but also about incompressible vector fields indistinctly, when $\text{div } v = 0$.

We will now prove a general result about vector fields that will be important later on. To do so, we first need a lemma about the existence of a stream function for any vector field.

Lemma 1.2.1. *Given an incompressible (*$\text{div } v = 0$*) vector field in 2 dimensions, there exists a stream function* $\psi(x, t)$ *such that* $v = \nabla^{\perp}\psi = (-\psi_{x_2}, \psi_{x_1})$

Proof. Any smooth vector field $v(x)$ in two dimensions satisfies the following equation:

$$\text{div } (v) = -\text{curl } (Rv), \text{ with } R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

As $\text{div } v = 0$, we get $\text{curl } (Rv) = 0$, and by Green's Theorem, the integral of Rv over any closed loop is zero. This means that Rv is path-independent, and by Poincaré Lemma, Rv is the gradient of a scalar-valued function. So, $Rv = \nabla\psi$ (and $v = R^t\nabla\psi = \nabla^{\perp}\psi$). ■

1.2.1 Singular Integral Operators

We introduce this subchapter presenting definitions and general results about integral operators which will be really useful during this whole thesis.

Let K be the integral operator defined by

$$Kf(x) = \int_{\mathbb{R}^N} K(x-y)f(y)\, dy \tag{1.25}$$

where the kernel $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is smooth, and homogeneous of degree $1 - N$,

$$K(\lambda x) = \lambda^{1-N}K(x), \qquad \forall\lambda > 0,\, x \neq 0. \tag{1.26}$$

To address the well-posedness of this operator, we present the following result

Proposition 1.2.3. *Given* $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ *smooth and homogeneous of degree* $1 - N$*, the convolution with* K *induces a distribution.*

Proof. We have to prove local integrability around zero, as the function K is smooth away from zero.

Any point $x \in \mathbb{R}^N$ can be written as $x = rw$, with $|r| = w$ and $w \in S^{N-1}$. So

$$|K(x)| = r^{1-N}|K(w)|.$$

Taking a neighborhood $B_\delta(0)$ centered around zero, we can integrate taking the change of variables $x \to rw$, with $dx = r^{N-1}drds$, and $ds = ds(w)$ is the surface measure on S^{N-1}.

We obtain

$$\int_{B_\delta(0)} |K(x)|\,dx = \int_0^\delta \int_{S^{N-1}} r^{1-N}|K(w)|r^{N-1}\,ds\,dr = \int_{S^{N-1}} |K(w)|\,ds \int_0^\delta dr \tag{1.27}$$

This is clearly bounded as K is smooth on any sphere centered at 0, so $K \in L^1_{\text{loc}}$. This shows that $Kf(x)$ is a well defined function if f is smooth and compactly supported. So, the convolution with K induces a distribution, as we claimed. ■

Now that we know Kf is well defined, we can think about its differentiability. For a compactly supported f, $Kf(x)$ is smooth for any point x outside of the support of f, as we can differentiate under the sign of the integral and we never find $x - y$ being zero (being y the integration variable), so we don't find the singularity of K. But, at points $x \in \operatorname{supp} f$, we could find $x - y = 0$, and so we would have problems with the singularity of K. This is why we introduce the following definition

Definition 1.2.3. *The **distributional derivative** ∂_{x_j} of $f \in L^1_{\text{loc}}$ is the linear functional $\partial_{x_j} f$ defined operationally by the formula*

$$\left(\partial_{x_j} f, \varphi\right)_0 = -\left(f, \partial_{x_j}\varphi\right)_0, \qquad \textit{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^N), \tag{1.28}$$

where $(\cdot,\cdot)_0$ is simply the euclidean scalar product.

If we try to find the distributional derivative of K we find an obstacle when doing integration by parts, since, in contrast with K, ∇K is not locally integrable. For this reason, before doing this we are forced to remove a neighborhood of the singularity. In this way we obtain (notice the negative sign of the boundary term due to the normal vector pointing inwards)

$$(K, \partial_{x_j}\varphi)_0 = \int_{\mathbb{R}^N} K(x)\,\partial_{x_j}\varphi(x)\,dx = \lim_{\varepsilon \searrow 0} \int_{|x|\geq\varepsilon} K\partial_{x_j}\varphi\,dx \tag{1.29}$$

$$= \lim_{\varepsilon\searrow 0}\left\{-\int_{|x|\geq\varepsilon} \partial_{x_j}K\varphi\,dx - \int_{|x|=\varepsilon} K\varphi\frac{x_j}{|x|}\,ds\right\}, \tag{1.30}$$

where we have used Green's Theorem for the second equality. Here we have two terms. The second one can be evaluated applying a change of variables $x \to \varepsilon x'$ ($dx = \varepsilon^{N-1}dx'$), and we get:

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\int_{|x|=\varepsilon} K\varphi\frac{x_j}{|x|}\,ds &= \lim_{\varepsilon\searrow 0}\int_{|x'|=1} \varepsilon^{1-N}K(x')\varphi(\varepsilon x')x_j\varepsilon^{N-1}\,ds \\ &= \varphi(0)\int_{|x|=1} K(x)x_j\,ds.\end{aligned} \tag{1.31}$$

Next, concerning the first term, we note it has the following form

Definition 1.2.4. *Let h be locally integrable in $\mathbb{R}^N \setminus \{0\}$. The **principal value** of h is the distribution* $\mathrm{PV}(h)$ *defined by the rule*

$$(PV(h), \varphi)_0 = \mathrm{PV}\int_{\mathbb{R}^N} h\varphi\,dx = \lim_{\varepsilon\searrow 0}\int_{|x|\geq\varepsilon} h\varphi\,dx, \qquad \textit{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^N), \tag{1.32}$$

Note that the existence of the principal value of h is not automatic and strongly relies on the symmetry properties of h.

Example 1.2.1. *An example of a principal value integral is the one-dimensional principal value integral of $h(x) = \frac{1}{x}$, which is a distribution $T = \mathrm{PV}(h)$ defined by its action on a test function $f \in C_0^\infty(\mathbb{R})$ by*

$$(T, f)_0 = (\mathrm{PV}(h), f)_0 = \lim_{\varepsilon \searrow 0} \int_{|x| \geq \varepsilon} \frac{f(t)}{t}\, dt.$$

To see that it's well defined, we recall that

$$\int_{\varepsilon \leq |t| \leq R} h(t)\, dt = 0 \tag{1.33}$$

due to the symmetry of the function h. This property is the key on the well-posedness of the operator, as we will see now.

Given any $f \in C_0^\infty(\mathbb{R})$ test function, the action of the distribution is

$$Tf(x) = \mathrm{PV} \int_{\mathbb{R}} \frac{f(t)}{t}\, dt, = \lim_{\varepsilon \searrow 0} \left\{ \int_{\varepsilon \leq |t| \leq R} \frac{f(t)}{t}\, dt + \int_{|t| > R} \frac{f(t)}{t}\, dt \right\}.$$

The right term presents no problem, as there is not any singularity. We just have to prove the integrability of the first term.

Using the symmetry property (1.33)*, we have that*

$$\int_{\varepsilon \leq |t| \leq R} \frac{f(t)}{t}\, dt = \int_{\varepsilon \leq |t| \leq R} \frac{f(t) - f(0)}{t}\, dt.$$

Now, notice that near $t = \pm\varepsilon$, for small ε, the term $(f(t) - f(0))$ behaves as $t \cdot f'(0)$, so we get rid of the singularity. Away from the origin, the function $\dfrac{1}{t}$ is bounded, and f is smooth, so everything is integrable.

This is how the symmetry property (1.33) *is used to prove the existence of the principal value of h.*

From this example, we can extract the classical example presenting the Hilbert transform.

Example 1.2.2. *A classical example of a principal value value integral is the one-dimensional **Hilbert transform**, defined by*

$$\mathcal{H}f(x) := \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{f(t)}{x - t} = \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{f(x - t)}{t},$$

which is the convolution of $h(x) = \frac{1}{\pi x}$ with f. Notice that this operator can be rewritten as the action of the principal value of the kernel $h(x)$ over a modification of the function f. This modification is given by $\tilde{f} \circ \tau_x$, being $\tilde{f}(x) = f(-x)$, and $\tau_x(t) = t - x$. This transformation of f remains smooth and compactly supported, so the principal value of the kernel $h(x)$ is well defined on this function, as its seen on the previous example 1.2.1, and thus its Hilbert transform is well defined too.

Now, going back to the first term on (1.30), it is the action of the principal-value distribution

$$(\mathrm{PV}(\partial_{x_j}K), \varphi)_0.$$

This principal value is well defined because of the following fact.

Proposition 1.2.4. *Let K be a homogeneous function of degree $1-N$, and smooth outside of $x=0$. Then $\partial_{x_j}K$ is an homogeneous function of degree $-N$ and always has mean-value zero on the unit sphere:*

$$\int_{|x|=1} \partial_j K \, ds = 0. \tag{1.34}$$

Proof. We first prove that $\partial_{x_j}K$ is an homogeneous function of degree $-N$. Notice that by the definition of an homogeneous function of degree $1-N$ (1.26), we have

$$K(\lambda x) = \lambda^{1-N} K(x).$$

By differentiating on each side of the equality, we get

$$\lambda \partial_{x_j} K(\lambda x) = \lambda^{1-N} \partial_{x_j} K(x),$$

and dividing by λ we get the homogeneity of $\partial_{x_j}K$ of degree $-N$.

About the mean-value zero part, let $0 < a < b$, and by Stokes' theorem we can write

$$\int_{a \le |x| \le b} \partial_{x_j} K \, dx = \int_{|x|=b} K \frac{x_j}{|x|} \, ds - \int_{|x|=a} K \frac{x_j}{|x|} \, ds \tag{1.35}$$

Both of them are equal, as making a change of variables by $x \to xb$, one has

$$\int_{|x|=b} K \frac{x_j}{|x|} \, ds = \int_{|x|=1} b^{1-N} K(x) x_j b^{N-1} \, ds = \int_{|x|=1} K(x) x_j \, ds, \tag{1.36}$$

so the right hand side at (1.35) equals to zero.

About the left hand side of (1.35), it can be transformed with the change of variables $x \to rw$, with $r = |x|$ (and so $dx = r^{N-1} dr ds$, with ds the surface measure):

$$\int_{a \le |x| \le b} \partial_j K \, dx = \int_a^b \int_{|w|=1} \partial_{x_j} K(rw) r^{N-1} \, dr \, ds \tag{1.37}$$

Using also the homogeneity, one gets finally the desired integral:

$$\int_a^b \int_{|w|=1} \partial_{x_j} K(rw) r^{N-1} \, dr \, ds = \int_a^b r^{-1} \, dr \int_{|x|=1} \partial_{x_j} K(x) \, ds \tag{1.38}$$

$$= (\log b - \log a) \int_{|x|=1} \partial_{x_j} K(x) \, ds, \tag{1.39}$$

,so we get the claimed equality. ∎

Notice that inside the proof, we also proved that integrating zero on the unit sphere implies integrating zero over rings (1.37) - (1.39). If we write the first term on (1.30) as

$$\mathrm{PV}\int_{\mathbb{R}^N} \partial_{x_j} K\varphi\, dx, = \lim_{\varepsilon\searrow 0}\int_{|x|\geq\varepsilon} \partial_{x_j} K\varphi\, dx = \lim_{\varepsilon\searrow 0}\left\{\int_{\varepsilon\leq|x|\leq R} \partial_{x_j} K\varphi\, dx + \int_{|x|\geq R} \partial_{x_j} K\varphi\, dx\right\} \tag{1.40}$$

The second term will be well defined for a smooth compactly supported φ, so we have to focus on the first term. Notice that (because it integrates zero on rings)

$$\int_{\varepsilon\leq|x|\leq R} \partial_{x_j} K(x)\varphi(x)\, dx = \int_{\varepsilon\leq|x|\leq R} \partial_{x_j} K(x)(\varphi(x)-\varphi(0))\, dx. \tag{1.41}$$

Using the mean value theorem, $|\varphi(x)-\varphi(0)| \leq \|\nabla\varphi\|_\infty |x|$, and this behavior as $|x|$ reduces the size of the singularity to $1-N$, which we know is locally integrable.

With both these facts, we have the following corollary.

Corollary 1.2.1. *Let K be a homogeneous function of degree $1-N$, and smooth outside of $x=0$. Then the distributional derivative of K is the linear functional $\partial_{x_j}K$ defined operationally by:*

$$\partial_{x_j} K = \mathrm{PV}(\partial_{x_j} K) + c_j\delta_0, \text{ with } c_j = \int_{|x|=1} K(x)x_j\, ds. \tag{1.42}$$

In other words, its action over a test function $\varphi \in C_0^\infty(\mathbb{R}^N)$ is defined by

$$(K, \partial_{x_j}\varphi)_0 = (\mathrm{PV}(\partial_{x_j} K), \varphi)_0 + c_j\varphi(0). \tag{1.43}$$

Proof. This result follows from 1.2.4 and from the argument explained with (1.41). ■

Note in particular that the distributional gradient and the pointwise gradient of K do not agree, because of the presence of a Dirac delta.

As what we are using through the thesis is the convolution of the kernel over a function, and not only its action, we must notice that

$$\begin{aligned} K * f(x) &= \int_{\mathbb{R}^N} K(x-y)f(y)\, dy = \int_{\mathbb{R}^N} K(y)f(x-y)\, dy \\ &= \int_{\mathbb{R}^N} K(y)\tilde{f}(\tau_x(y))\, dy = (K, \tilde{f}(\tau_x))_0, \end{aligned} \tag{1.44}$$

where $\tilde{f}(x) = f(-x)$ and $\tau_x(z) = z - x$. This was already used on the example of the Hilbert transform (1.2.2). Now, the distributional derivative of the kernel convoluted against a function will simply be the action of the derivative of the kernel against this modified function. So

$$\begin{aligned} \partial_{x_j} K * f = (K, \partial_{x_j}(\tilde{f}\circ\tau_x))_0 &= (PV(\partial_{x_j} K), \tilde{f}\circ\tau_x) + c_j\tilde{f}\circ\tau_x(0) \\ &= (PV(\partial_{x_j} K), \tilde{f}\circ\tau_x) + c_j f(x) \end{aligned} \tag{1.45}$$

We present now an important general result about vector fields and integral operators.

Proposition 1.2.5. *Any smooth vector field v in 2 dimensions satisfies*

$$\begin{cases} \text{div } v = 0 \\ \text{curl } v = \omega \end{cases} \iff v = K_2 * \omega, \ \textit{with } K_2(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)^t. \tag{1.46}$$

Proof. We start assuming div $v = 0$ and curl $v = \omega$. Using the previous lemma 1.2.1, we know there exists some function $\psi(x,t)$ such that $v = \nabla^\perp \psi$. Taking the curl on this equation, we get

$$\text{curl } v = \omega = \nabla \times \nabla^\perp \psi = \Delta \psi. \tag{1.47}$$

Knowing the fundamental solution of the Laplacian problem, $G(x) = \frac{1}{2\pi} \ln |x|$, then the solution of the problem above is $\psi = G * \omega$, so

$$\psi(x) = \int_{\mathbb{R}^2} G(x-y)\omega(y,t)\, dy = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\omega(y)\, dy. \tag{1.48}$$

Now we just have to compute $\nabla^\perp \psi$ to recover v. We do it by components. We will be differentiating under the sign of the integral, which can be done as ∇G is locally integrable ($\nabla G \in L^1_{\text{loc}}$), as

$$\frac{\partial}{\partial x_i} \ln|x-y| = \frac{x_i - y_i}{|x-y|^2}, \tag{1.49}$$

and so we get

$$\frac{\partial \psi}{\partial x_i} = \int_{\mathbb{R}^2} \frac{1}{2\pi}\frac{x_i - y_i}{|x-y|^2}\omega(y)\, dy. \tag{1.50}$$

As we know that $v = \nabla^\perp \psi = (-\psi_{x_2}, \psi_{x_1})$, we finally obtain the desired conclusion that

$$v = K_2 * \omega = \int_{\mathbb{R}^2} K_2(x-y)\omega(y)\, dy \tag{1.51}$$

For the reverse result, we can explicitly compute the divergence and the curl of v, to verify it. To do so, we use the previous formula (1.45). So, as we have that $v = (K_2, \tilde{\omega}(\tau_x))_0$, its derivative is computed as (by the result mentioned 1.2.1)

$$\partial_{x_j} v = (K_2, \partial_{x_j}(\tilde{\omega} \circ \tau_x))_0 = (\text{PV}(\partial_{x_j} K_2), \tilde{\omega} \circ \tau_x)_0 + c_j \omega(x), \tag{1.52}$$

with $c_j = \int_{|x|=1} K(x) x_j \, ds$. We compute each partial derivative of the kernel, which are as follows:

$$\frac{\partial K_2^1}{\partial x_1} = \frac{1}{\pi}\frac{x_1 x_2}{(x_1^2 + x_2^2)^2} \tag{1.53}$$

$$\frac{\partial K_2^1}{\partial x_2} = \frac{1}{2\pi}\frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2} = \frac{\partial K_2^2}{\partial x_1} \tag{1.54}$$

$$\frac{\partial K_2^2}{\partial x_2} = -\frac{1}{\pi}\frac{x_1 x_2}{(x_1^2 + x_2^2)^2} = -\frac{\partial K_2^1}{\partial x_1}. \tag{1.55}$$

Using these, we can compute each term on the divergence as:

$$\frac{\partial v^1}{\partial x_1}(x) = (\mathrm{PV}(\partial_{x_1} K_2^1), \tilde{\omega} \circ \tau_x)_0 - \frac{1}{2\pi}\omega(x) \int_{|x|=1} x_1 x_2 \, ds, \tag{1.56}$$

$$\frac{\partial v^2}{\partial x_2}(x) = (\mathrm{PV}(\partial_{x_2} K_2^2), \tilde{\omega} \circ \tau_x)_0 + \frac{1}{2\pi}\omega(x) \int_{|x|=1} x_1 x_2 \, ds. \tag{1.57}$$

As $\dfrac{\partial K_2^1}{\partial x_1} + \dfrac{\partial K_2^2}{\partial x_2} = 0$, we get div $v = 0$.

Notice that curl $v = \dfrac{\partial v^2}{\partial x_1} - \dfrac{\partial v^1}{\partial x_2}$. These derivatives are

$$\frac{\partial v^2}{\partial x_1}(x) = (\mathrm{PV}(\partial_{x_1} K_2^2), \tilde{\omega} \circ \tau_x)_0 + \frac{1}{2\pi}\omega(x) \int_{|x|=1} x_1^2 \, ds, \tag{1.58}$$

$$\frac{\partial v^1}{\partial x_2}(x) = (\mathrm{PV}\, \partial_{x_2} K_2^1), \tilde{\omega} \circ \tau_x)_0 - \frac{1}{2\pi}\omega(x) \int_{|x|=1} x_2^2 \, ds. \tag{1.59}$$

Again, as $\dfrac{\partial K_2^2}{\partial x_1} - \dfrac{\partial K_2^1}{\partial x_2} = 0$, we get that

$$\text{curl } v(x) = \frac{1}{2\pi}\omega(x) \int_{|x|=1} x_1^2 + x_2^2 = \omega(x),$$

which completes the whole proof. ■

1.2.2 Decomposition in translation, vorticity and deformation

Let v be a smooth, non-autonomous vector field. Then a first degree Taylor expansion gives us that

$$v(x,t) = v(x_0,t) + \nabla v(x_0,t) \cdot (x - x_0) + \mathcal{O}((x - x_0)^2) \tag{1.60}$$

The matrix ∇v can be decomposed as $\nabla v = \mathcal{D}v + \Omega v$, where

$$\mathcal{D}v = \frac{1}{2}\left(\nabla v + \nabla v^t\right)$$

$$\Omega v = \frac{1}{2}\left(\nabla v - \nabla v^t\right).$$

Note that $\mathcal{D}v$ is always symmetric, and Ωv is always skew symmetric.

It is clear that tr $\mathcal{D} = \sum_{i=1}^N \dfrac{\partial v^i}{\partial x_i} =$ div $v = 0$.

Recall that the curl of any 3D vector field v is defined as

$$\omega = \nabla \times v = \left(\frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3}, \frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1}, \frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2}\right)^t$$

Notice that in the 2 dimensional case, as we can consider

$$v(x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2), 0) \implies \text{curl } v = \left(0, 0, \frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2}\right)^t,$$

the curl is a scalar value.

We will see now that $\Omega v \cdot u = \frac{1}{2}\omega \times u$ for any vector u. Notice that

$$\Omega v = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial v^1}{\partial x_2} - \frac{\partial v^2}{\partial x_1} & \frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1} \\ \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} & 0 & \frac{\partial v^2}{\partial x_3} - \frac{\partial v^3}{\partial x_2} \\ \frac{\partial v^3}{\partial x_1} - \frac{\partial v^1}{\partial x_3} & \frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}, \tag{1.61}$$

and, computing the cross product

$$\omega \times u = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega^1 & \omega^2 & \omega^3 \\ u^1 & u^2 & u^3 \end{vmatrix} = \begin{pmatrix} \omega^2 u^3 - \omega^3 u^2 \\ \omega^3 u^1 - \omega^1 u^3 \\ \omega^1 u^2 - \omega^2 u^1 \end{pmatrix} = 2\Omega v \cdot u, \tag{1.62}$$

as claimed.

Now with this, we can rewrite the previous Taylor expansion as follows:

$$v(x, t) = v(x_0, t) + \frac{1}{2}\omega(x_0) \times (x - x_0) + \mathcal{D}v(x_0)(x - x_0) + \mathcal{O}((x - x_0)^2)$$

The skew-symmetric part, Ωv, represents the local spinning of the fluid element. Since it can be expressed as a cross product, it does not contribute to changes in distances between fluid particles.

The symmetric part, $\mathcal{D}v$, characterizes the deformation of fluid elements, including stretching and shearing. This deformation is inherently volume-preserving in incompressible flows, as $\mathcal{D}v$ is a symmetric matrix with zero trace. A symmetric matrix is diagonalizable with an orthonormal basis and real eigenvalues, which correspond to the principal directions of stretching or compression. These eigenvalues describe the magnitude of deformation along each principal direction. In incompressible flows, the zero-trace property ensures that the sum of the eigenvalues is zero, meaning that any positive stretching in one direction is exactly balanced by compression or stretching in the other directions, preserving the fluid element's volume.

This decomposition is useful as it describes the motion of the fluid. It describes the physical content of "incompressible flow = translation + rotation + shear/stretch with no net expansion."

1.3 Vorticity equation

Our goal in this section will be to prove the following result about equivalence between formulations. This will allow us to work with both of them indistinctly.

Proposition 1.3.1. *For 2D sufficiently smooth flows, the Navier-Stokes equations (1.1)-(1.3) are equivalent to the following system of equations:*

$$\begin{cases} \dfrac{D\omega}{Dt} = \nu\Delta\omega, & (x,t) \in \mathbb{R}^2 \times [0,\infty), \qquad (1.63) \\ \omega\Big|_{t=0} = \omega_0 \;\coloneqq\; \text{curl } v_0, & \forall x \in \mathbb{R}^2, \qquad (1.64) \\ v = K_2 * \omega, & (x,t) \in \mathbb{R}^2 \times [0,\infty), \qquad (1.65) \end{cases}$$

From now on, the above system will be referred to as the vorticity-stream formulation.

Proof. Let's first assume v is a solution for the Navier-Stokes equations (1.1)-(1.3). We define $\omega \coloneqq \text{curl } v$, and we will prove that this ω satisfies the vorticity-stream formulation.

By formally taking the curl on the first equation (1.1), we get

$$\text{curl } (v_t + (v \cdot \nabla)v) = \text{curl } (-\nabla p + \nu\Delta v). \tag{1.66}$$

The first term of the left side becomes

$$\text{curl } v_t = \nabla \times \frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(\nabla \times v) = \frac{\partial \omega}{\partial t} = \omega_t, \tag{1.67}$$

in the same way as the last term of the right side, which becomes

$$\text{curl } (\nu\Delta v) = \nabla \times (\nu\Delta v) = \nu\Delta(\nabla \times v) = \nu\Delta\omega. \tag{1.68}$$

The term involving the pressure vanishes, as the curl of the gradient of any scalar field. Finally, for the second term on the left, we need two identities:

$$(v \cdot \nabla)v = \nabla(\frac{1}{2}v \cdot v) - v \times \text{curl } v, \tag{1.69}$$

$$\nabla \times (a \times b) = a(\nabla \cdot b) - b(\nabla \cdot a) + (b \cdot \nabla)a - (a \cdot \nabla)b. \tag{1.70}$$

Using both of them, one gets

$$\nabla \times ((v \cdot \nabla)v) = -v(\nabla \cdot \nabla \times v) + \nabla \times v(\nabla \cdot v) - (\nabla \times v \cdot \nabla)v + (v \cdot \nabla)\nabla \times v, \tag{1.71}$$

where the first term is zero as the divergence of a curl is zero, and the second one is also zero because of the incompressibility. So, defining curl $v = \omega$, one gets the following equation:

$$\omega_t + (v \cdot \nabla)\omega = (\omega \cdot \nabla)v + \nu\Delta\omega. \tag{1.72}$$

When $N = 2$, we write $v(x,t) = (v_1, v_2, 0)$ and then recall that $\omega = (0, 0, \omega_3)$, so that

$$(\omega \cdot \nabla) = \omega_1 \frac{\partial}{\partial x_1} + \omega_2 \frac{\partial}{\partial x_2} + \omega_3 \frac{\partial}{\partial x_3} = \omega_3 \frac{\partial}{\partial x_3},$$

and therefore

$$(\omega \cdot \nabla)v = \omega_3 \frac{\partial}{\partial x_3} \cdot (v_1, v_2, 0) = (\omega_3 \frac{\partial v_1}{\partial x_3}, \omega_3 \frac{\partial v_2}{\partial x_3}, 0) = 0,$$

as $v(x,t) = v((x_1, x_2), t)$ in 2 dimensions, with no dependence on x_3. So, given a v in 2 dimensions satisfying the Navier-Stokes equation, its vorticity satisfies this final equation

$$\omega_t + (v \cdot \nabla)\omega = \nu\Delta\omega. \tag{1.73}$$

Finally, using a previous result 1.2.5, we know that div $v = 0$ and curl $v = \omega$, so $v = K_2 * \omega$. Also, as curl $v = \omega$, it is obvious that curl $v_0 = \omega_0$.

We need to prove now the other implication. Assume ω satisfies (1.63) and (1.64). As we define $v = K_2 * \omega$ (1.65), by the same result used before (1.2.5), we know div $v = 0$ and curl $v = \omega$. So, we have to prove that v satisfies the first Navier-Stokes equation (1.1). To do so, we define

$$F := \frac{Dv}{Dt} - \nu\Delta v. \tag{1.74}$$

We have already proven above that curl $(F) = \dfrac{Dw}{Dt} - \nu\Delta w = 0$. This means that, by the same argument used in the proof of Lemma (1.2.1), as we know the curl of F is zero (and F is smooth enough as v is), then F is the gradient of a scalar valued function. This way, we can conclude $F = -\nabla p$, so v satisfies the Navier-Stokes equation (1.1). Finally, as curl $v = \omega$, then $v_0 =$ curl ω_0, finishing the proof. ■

It is worth noting that, for sufficiently smooth fluids in two dimensions and with zero viscosity, the vorticity-stream formulation takes the form

$$\frac{Dw}{Dt} = 0, \qquad (x,t) \in \mathbb{R}^2 \times [0,\infty), \tag{1.75}$$

$$\omega\big|_{t=0} = \omega_0 := \text{curl } v_0, \qquad \forall x \in \mathbb{R}^2, \tag{1.76}$$

$$v = K_2 * \omega \qquad (x,t) \in \mathbb{R}^2 \times [0,\infty). \tag{1.77}$$

From this statement, we can derive the following corollary, which is particularly relevant in the context of this thesis.

Corollary 1.3.1. *For sufficiently smooth, two-dimensional, inviscid fluids, its vorticity remains constant along particle trajectories. In other words, it satisfies*

$$\omega(X(\alpha,t),t) = \omega_0(\alpha). \tag{1.78}$$

Proof. This is a consequence of the previous proposition, Proposition 1.3.1. ■

1.4 Particle Trajectory equation

In the previous section, we proved the equivalence between the original formulation of the Euler system of equations, and the vorticity-stream formulation. The objective of this section is to prove the equivalence between those formulations and the formulation of the particle trajectory equation, in order to be able to work indistinctly with any of the formulations, as we will prove the existence of solutions on this last particle trajectory formulation.

So, we aim to prove the following result

Proposition 1.4.1. *Let $v_0(x)$ be a smooth velocity field satisfying* div $v_0 = 0$, *and* $\omega_0 =$ curl v_0. *Let $X(\alpha,t)$ solve:*

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) = \displaystyle\int_{\mathbb{R}^2} K_2\big[X(\alpha,t) - X(\alpha',t)\big]\,\omega_0(\alpha')\,d\alpha', & (1.79) \\[2ex] X(\alpha,0) = \alpha, & (1.80) \end{cases}$$

Define the velocity field v by

$$v(x,t) = \int_{\mathbb{R}^2} K_2[x - X(\alpha', t)]\omega_0(\alpha')\ d\alpha' \tag{1.81}$$

Then, the integrodifferential formulation (1.79)-(1.81) is equivalent to the 2D Euler equation for sufficiently smooth solutions.

Proof. We will see first that a solution of the Euler equation solves this formulation. Using 1.3.1, we know that the Euler and the vorticity formulations are equivalent, so we have ω satisfying

$$\begin{aligned}
&\frac{Dw}{Dt} = 0, && (x,t) \in \mathbb{R}^2 \times [0,\infty), && (1.82)\\
&\omega\big|_{t=0} = \omega_0 \coloneqq \operatorname{curl} v_0, && \forall x \in \mathbb{R}^2, && (1.83)\\
&v = K_2 * \omega && (x,t) \in \mathbb{R}^2 \times [0,\infty). && (1.84)
\end{aligned}$$

Also, by 1.3.1, we know that for the Euler system in 2D, the vorticity is conserved along the trajectories, so

$$\omega(X(\alpha,t),t) = \omega_0(\alpha), \tag{1.85}$$

where X is the flow of the vector field v defined by (1.84).

As X is 1-1 and onto, we can take the change of variables $x' \to X(\alpha', t)$ (with the Jacobian of the transformation being equal to 1 due to incompressibility), and $v(x,t)$ becomes

$$\begin{aligned}
v(x,t) = \int_{\mathbb{R}^2} K_2[x - x']\omega(x',t)\ dx' &= \int_{\mathbb{R}^2} K_2[x - X(\alpha',t)]\omega(X(\alpha',t),t)\ dx' \\
&= \int_{\mathbb{R}^2} K_2[x - X(\alpha',t)]\omega_0(\alpha')\ d\alpha',
\end{aligned} \tag{1.86}$$

thus from (1.84) we get (1.81).

Now, by the ODE defining the flow (1.13), we know that

$$\begin{aligned}
&\frac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) = v(X(\alpha,t),t) = \int_{\mathbb{R}^2} K_2[X(\alpha,t) - X(\alpha',t)]\omega_0(\alpha')\ d\alpha', \\
&X(\alpha,0) = \alpha,
\end{aligned} \tag{1.87}$$

thus getting (1.79) and (1.80), as claimed.

To prove the converse, we assume $X(\alpha,t)$ solves (1.79), (1.80) and (1.81). It is clear that $v = K_2 * (\omega_0 \circ X^{-1})$, and using proposition 1.2.5, we know that $\operatorname{div} v = 0$ and $\operatorname{curl} v = \omega_0 \circ X^{-1}$.

With this, defining $\omega = \omega_0 \circ X^{-1}$, one clearly has (1.84). About (1.83), as $X^{-1}(\cdot, 0) = \mathrm{Id}$, the identity map, this equation is also satisfied.

Regarding (1.82), it is evident that this equation holds true. It asserts that vorticity is conserved along particle trajectories, which aligns with the definition of ω. Specifically, for each point x, the vorticity defined corresponds to the initial vorticity of the particle's initial position. ∎

From now on, when discussing the Euler system of equations, we will use any of the formulations interchangeably, choosing the one that best suits the specific problem at hand.

Chapter 2
Existence of solutions

In this chapter, we will be proving the existence of solutions, using particle-trajectory methods derived from the vorticity equation. This will be done for a smooth initial vorticity $\omega_0 \in C^\gamma$ for some $\gamma \in (0,1)$.

Also, first we will prove the local-in-time existence of solutions, which will be extended to a global-in-time solution in the next chapter.

Recall from Chapter 1 (Proposition 1.4.1) that the integrodifferential formulation is equivalent to the Euler formulation. Hence, it suffices to establish the existence of a solution X to the system

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) = \displaystyle\int_{\mathbb{R}^2} K_2\big[X(\alpha,t) - X(\alpha',t)\big]\,\omega_0(\alpha')\,d\alpha', & (2.1) \\ X(\alpha,0) = \alpha, & (2.2) \end{cases}$$

which is essentially an ordinary differential equation in time. We will apply a version of the Picard–Lindelöf theorem to show that such an X exists. It is a standard result in existence of solutions for ODE.

Theorem 2.0.1. *Let $O \subseteq B$ be an open subset of a Banach space B, and let $F : O \to B$ be a nonlinear operator satisfying the locally Lipschitz condition, that is, for any $X \in O$ there exists $L > 0$ and an open neighborhood U_X such that*

$$\|F(X') - F(X'')\|_B \le L\|X' - X''\|_B \qquad \forall X', X'' \in U_X.$$

Then, for any $X_0 \in O$, there exists a time T such that the ODE

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X) \qquad X\big|_{t=0} = X_0$$

has a unique (local) solution $X \in C^1[(-T,T);O]$.

To apply this result, we have to define the Banach space B, the open subset O and prove the Lipschitz condition on F.

2.1 Definition of the set

First of all, we need to start with some definitions

Definition 2.1.1. *Given $X : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}^M$, the Hölder seminorm with exponent γ, $|\cdot|_\gamma$, is defined by:*

$$|X|_\gamma = \sup_{\substack{\alpha,\alpha' \in \mathbb{R}^N \\ \alpha \neq \alpha'}} \frac{|X(\alpha) - X(\alpha')|}{|\alpha - \alpha'|^\gamma}, \qquad 0 < \gamma \le 1.$$

Functions with finite Hölder seminorm with exponent γ are called $\gamma-$Hölder continuous functions. These functions are uniformly continuous, but not necessarily differentiable.

From it, we can define the seminormed space $C^\gamma(\Omega)$ space as follows

Definition 2.1.2. *The space $C^\gamma(\Omega)$, for $0 < \gamma \leq 1$, is the space of functions $X : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}^M$ that are Hölder continuous with exponent γ, and it is equipped with the Hölder seminorm:*

$$C^\gamma(\Omega) = \left\{X : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}^M \ \middle| |X|_\gamma < \infty\right\}.$$

To define a norm out of the Hölder seminorm, we add the supremum norm as follows

Definition 2.1.3. *Given $X : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}^M$, the Hölder$-\gamma$ norm $\|\cdot\|_\gamma$ is defined by:*

$$\|X\|_\gamma = \|X\|_\infty + |X|_\gamma,$$

being $\|\cdot\|_\infty$ the supremum norm.

With all these definitions, we can define a norm as $|X|_{1,\gamma} = |X(0)| + \|\nabla X\|_\gamma$, and based on this norm we can define the space we will be using in this section as

Definition 2.1.4. $C^{1,\gamma} = \left\{X : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}^M \ \middle| \ |X|_{1,\gamma} < \infty\right\}$

So, basically it consists of functions with bounded and Hölder continuous derivatives.
This $C^{1,\gamma}$ space will be the B used on the theorem.
In advance, we define the open subsets

$$O_M = \left\{X \in B / \inf_{\alpha \in \mathbb{R}^2} \det \nabla_\alpha X > 1/2 \text{ and } |X|_{1,\gamma} < M\right\}.$$

Later in the proof, we will see why it is necessary to introduce them in this way. Furthermore, it can be shown that each element of O_M is a global homeomorphism. For any $M > 1$, these sets are also open and nonempty.

2.2 Additional Results

We will need additional results, presented as lemmas in this section. We divide this section in two subsections, properties on the norm which are straightforward, and properties on the kernel, for which we have to see its action over different functions.

2.2.1 Properties on the norms

Lemma 2.2.1. *Let $X, Y : \mathbb{R}^N \to \mathbb{R}^N$ be smooth, bounded functions. Then, for $\gamma \in (0,1)$ one has*

$$|XY|_\gamma \leq |X|_0|Y|_\gamma + |X|_\gamma|Y|_0 \tag{2.3}$$

$$\|XY\|_\gamma \leq \|X\|_\gamma\|Y\|_\gamma \tag{2.4}$$

Proof. Recall the definition of the Hölder seminorm as the supremum of a quotient. One has:

$$\begin{aligned} |X(\alpha)Y(\alpha) - X(\alpha')Y(\alpha')| &= |X(\alpha)Y(\alpha) - X(\alpha')Y(\alpha) + X(\alpha')Y(\alpha) - X(\alpha')Y(\alpha')| \\ &\leq |X(\alpha) - X(\alpha')||Y(\alpha)| + |Y(\alpha) - Y(\alpha')||X(\alpha')| \\ &\leq |\alpha - \alpha'|^\gamma\{|X|_\gamma|Y|_0 + |Y|_\gamma|X|_0\}. \end{aligned} \tag{2.5}$$

As this follows for all α, α', it follows for the supremum too, so we get the first inequality. Using that $|XY|_0 \leq |X|_0|Y|_0$, the second one follows directly from the definition of the Hölder norm.

$$\begin{aligned} \|XY\|_\gamma &= |XY|_0 + |XY|_\gamma \\ &\leq |X|_0|Y|_0 + |X|_0|Y|_\gamma + |X|_\gamma|Y|_0 \\ &\leq |X|_0|Y|_0 + |X|_0|Y|_\gamma + |X|_\gamma|Y|_0 + |X|_\gamma|Y| \\ &= \|X\|_\gamma\|Y\|_\gamma. \end{aligned} \tag{2.6}$$

■

Lemma 2.2.2. *Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be smooth and invertible, with*

$$|\det(\nabla_\alpha X(\alpha))| \geq c_1 > 0.$$

Then for each $\gamma \in (0, 1)$, there exists $c > 0$ such that

$$\|(\nabla_\alpha X)^{-1}\|_\gamma \leq c\|\nabla_\alpha X\|_\gamma^{2N-1} \tag{2.7}$$

$$|X^{-1}|_{1,\gamma} \leq c|X|_{1,\gamma}^{2N-1}. \tag{2.8}$$

Proof. As X is globally invertible, we begin by differentiating the identity $X^{-1}(X(\alpha)) = \alpha$. Applying the chain rule yields

$$\nabla_x\big(X^{-1}(X(\alpha))\big)\,\nabla_\alpha X(\alpha) = I, \tag{2.9}$$

where I denotes the identity matrix. Using this equation, and by the well known formula for inverse matrices, we have

$$\nabla_x X^{-1}(X(\alpha)) = [\nabla_\alpha X(\alpha)]^{-1} = \frac{\mathrm{Co}(\nabla_\alpha X)}{\det(\nabla_\alpha X)}. \tag{2.10}$$

For the first inequality, applying the norm and using (2.4), we get

$$\|(\nabla_\alpha X)^{-1}\|_\gamma \leq \|\,\mathrm{Co}(\nabla_\alpha X)\|_\gamma\|(\det(\nabla_\alpha X))^{-1}\|_\gamma. \tag{2.11}$$

Using that the cofactor matrix entries are determinants of $(N-1)\times(N-1)$ submatrices of $\nabla_\alpha X$, each of them being a polynomial of degree $N-1$ in the entries of $\nabla_\alpha X$, we can bound it by

$$\|\,\mathrm{Co}(\nabla_\alpha X)\|_\gamma \leq c_2\|\nabla_\alpha X\|_\gamma^{N-1}. \tag{2.12}$$

Using a similar argument on the determinant being a polynomial of degree N in the entries of $\nabla_\alpha X$, the determinant is bounded by

$$\|\det\nabla_\alpha X\|_\gamma \leq c_3\|\nabla_\alpha X\|_\gamma^N. \tag{2.13}$$

As we have a bound on the determinant by hypothesis, we can bound the inverse determinant in terms of the actual determinant by a constant

$$|\det(\nabla_\alpha X(\alpha))| \geq c_1 > 0 \implies \|(\det(\nabla_\alpha X))^{-1}\|_\gamma \leq c_4\|\det(\nabla_\alpha X)\|_\gamma. \tag{2.14}$$

Merging everything together, we finally get the desired bound on the inverse of the gradient of X.

With this and the first equality in (2.10), we have bounded $\|\nabla_x X^{-1}\|_\gamma$. We just need to bound the value $|X^{-1}(0)|$ to have the whole $|X^{-1}|_{1,\gamma}|$ bounded.

Since $\nabla_\alpha X(\alpha)$ remains uniformly invertible (2.14), the inverse function theorem ensures that X^{-1} is locally Lipschitz. In particular, there exists a constant C such that

$$|X^{-1}(0)| = |X^{-1}(0) - X^{-1}(X(0))| \leq C|X(0)| \leq C|X|_{1,\gamma}.$$

Combining this estimate with the bound obtained for $\|\nabla_x X^{-1}\|_\gamma$, we have

$$|X^{-1}|_{1,\gamma} = |X^{-1}(0)| + \|\nabla_x X^{-1}\|_\gamma \leq c|X|_{1,\gamma}^{2N-1}.$$

This concludes the proof. ■

Finally, we present the last lemma on properties of the norm and seminorm that we will need

Lemma 2.2.3. *Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be an invertible transformation with $|\det(\nabla_\alpha X(\alpha))| \geq c_1 > 0$, and let $f : \mathbb{R}^N \to \mathbb{R}^M$ be a smooth function. Then $f \circ X$ and $f \circ X^{-1}$ satisfy:*

$$|f \circ X|_\gamma \leq |f|_\gamma |\nabla_\alpha X|_0^\gamma, \tag{2.15}$$
$$\|f \circ X\|_\gamma \leq \|f\|_\gamma (1 + |X|_{1,\gamma}^\gamma), \tag{2.16}$$
$$\|f \circ X^{-1}\|_\gamma \leq \|f\|_\gamma \left(1 + c|X|_{1,\gamma}^{(2N-1)\gamma}\right). \tag{2.17}$$

Proof. For all $\alpha \neq \alpha'$ (Using the mean value theorem)

$$\begin{aligned}
|f \circ X(\alpha) - f \circ X(\alpha')| &= \frac{|f \circ X(\alpha) - f \circ X(\alpha')|}{|X(\alpha) - X(\alpha')|^\gamma} |X(\alpha) - X(\alpha')|^\gamma \\
&\leq \sup_{\substack{X, X' \in \mathbb{R}^N \\ X \neq X'}} \frac{|f(X) - f(X')|}{|X - X'|^\gamma} \left\{ \sup_{s \in [0,1]} |\nabla_\alpha X[\alpha + s(\alpha' - \alpha)]|^\gamma \right\} |\alpha - \alpha'|^\gamma \\
&\leq |f|_\gamma |\nabla_\alpha X|_0^\gamma |\alpha - \alpha'|^\gamma \implies |f \circ X|_\gamma \leq |f|_\gamma |\nabla_\alpha X|_0^\gamma.
\end{aligned}$$

For the next inequality, we use that $|f \circ X|_0 = |f|_0$:

$$\begin{aligned}
\|f \circ X\|_\gamma &= |f \circ X|_0 + |f \circ X|_\gamma \leq |f|_0 + |f|_\gamma |\nabla_\alpha X|_0^\gamma & (2.18)\\
&\leq |f|_0 + |f|_\gamma + |f|_0 |\nabla_\alpha X|_0^\gamma + |f|_\gamma |\nabla_\alpha X|_0^\gamma & (2.19)\\
&\leq \|f\|_\gamma + \|f\|_\gamma (|X|_0 + |\nabla_\alpha X|_0 + |\nabla_\alpha X|_\gamma)^\gamma = \|f\|_\gamma (1 + |X|_{1,\gamma}^\gamma) & (2.20)
\end{aligned}$$

Finally, using (2.8) and the previous inequality, we get

$$\|f \circ X^{-1}\|_\gamma \leq \|f\|_\gamma (1 + |X^{-1}|_{1,\gamma}^\gamma) \leq \|f\|_\gamma \left(1 + c|X|_{1,\gamma}^{(2N-1)\gamma}\right) \tag{2.21}$$

■

2.2.2 Bounds for integral operators

We now present lemmas to bound the needed integral operators in order to prove the result of the chapter.

Lemma 2.2.4. *Let K_N satisfy equations (1.25) and (1.26). Consider $f \in C^\gamma(\mathbb{R}^N;\mathbb{R}^N)$, $\gamma \in (0,1)$ bounded and with compact support. Define a length scale R such that $R^N = m_f$. Then there exists a constant c independent of f such that*

$$|K_N f|_0 \leq cR|f|_0 \tag{2.22}$$

Proof. First of all, we can bound $K_N(x)$ by writing $x = rw$ with $r = |x|$ and w in the unit sphere, so that $|w| = 1$. Thus,

$$|K_N(x)| = |r^{1-N}K(w)| = |x|^{1-N}|K(w)| \implies |K_N(x)| \leq c|x|^{1-N}. \tag{2.23}$$

With this in mind, we can now divide the integral in two:

$$\begin{aligned} |K_N f(x)| &\leq \left|\int_{|x-y|<R} K_N(x-y)f(y)\,dy\right| + \left|\int_{|x-y|\geq R} K_N(x-y)f(y)\,dy\right| && (2.24)\\ &\leq c_1|f|_0 \int_{|x'|<R} \frac{1}{|x'|^{1-N}}\,dx' + c_2|f|_0 R^{1-N} m_f \leq cR|f|_0 && (2.25)\end{aligned}$$

as claimed. ■

This result implies that the operator K_N is linear and bounded in the space C_0 of continuous functions vanishing at infinity (using a density argument of C_c^γ on C_0, as $C_c^\infty \subset C_c^\gamma \subset C_0$, and C_c^∞ is dense on C_0).

Finally, we will prove the results on the kernel $P_N = \nabla K_N$. It is a vector valued, homogeneous function of degree $-N$:

$$P_N(\lambda x) = \lambda^{-N} P_N(x), \qquad \forall \lambda > 0,\ x \neq 0. \tag{2.26}$$

Moreover, it has mean-value zero on the unit sphere:

$$\int_{|x|=1} P_N\,ds = 0, \tag{2.27}$$

As a consequence, its principal value $\mathrm{PV}(P)$ is a well defined distribution (see the argument on (1.41)) P_N:

$$P_N f(x) = \mathrm{PV}\int_{\mathbb{R}^N} P_N(x-y)f(y)\,dy, \tag{2.28}$$

which, from now on, we will refer to as a Singular Integral Operator (SIO).

Lemma 2.2.5. *Let K_N, f and γ satisfy the same conditions that on lemma 2.2.4. Let the SIO P_N satisfy the equations (2.26), (2.27) and (2.28). Then there exists a constant c,* *independent of R and f such that:*

$$|P_N f|_0 \leq c\left\{|f|_\gamma \varepsilon^\gamma + \max\left(1, log\frac{R}{\varepsilon}\right)|f|_0\right\}, \qquad \forall\varepsilon > 0, \tag{2.29}$$

$$|P_N f|_\gamma \leq c|f|_\gamma. \tag{2.30}$$

Proof. We first split the integral:

$$P_N f(x) = \mathrm{PV}\int_{|x-y|\leq\varepsilon} P_N(x-y)f(y)\,dy + \int_{|x-y|>\varepsilon} P_N(x-y)f(y)\,dy = A + B. \tag{2.31}$$

Notice that (2.27) can be extended to rings, as proved in (1.37) - (1.39), so one has

$$\int_{\delta\leq|x|\leq\varepsilon} P_N\,ds = 0. \tag{2.32}$$

Now, we define A_δ, and as its value will not depend on the delta, the limit will be well defined, bounding A. We are also applying a change of variables $x-y \to z$ and this previous cancellation property.

$$A_\delta = \int_{\delta\leq|z|\leq\varepsilon} P_N(z)f(x-z)\,dz = \int_{\delta\leq|z|\leq\varepsilon} P_N(z)(f(x-z)-f(x))\,dz. \tag{2.33}$$

Using the definition of $|f|_\gamma$, one gets that $|f(x-z)-f(x)| \leq |f|_\gamma|z|^\gamma$. Also, as P_N is homogeneous of degree $-N$, it can be bounded by $|P_N(x)| \leq c|x|^{-N}$ (using the same argument as in (2.23)). We can estimate $|A_\delta|$ by:

$$|A_\delta| \leq \int_{\delta\leq|z|\leq\varepsilon} |P_N(z)||f(x-z)-f(x)|\,dz \leq |f|_\gamma c\int_{\delta\leq|z|\leq\varepsilon} |z|^{-N}|z|^\gamma\,dz. \tag{2.34}$$

Finally, with a change of variables $z \to rw$, with $r = |z|$ and $|w| = 1$ ($dz = r^{N-1}drdw$), we get the desired result for the first part of the integral (where w_N is the surface area of the unit sphere):

$$|A_\delta| \leq c|f|_\gamma \int_{|z|\leq\varepsilon} |z|^{-N}|z|^\gamma\,dz \leq c|f|_\gamma w_N \int_0^\varepsilon r^{\gamma-1}\,dr = c_1|f|_\gamma\varepsilon^\gamma. \tag{2.35}$$

As the bound for A_δ does not depend on delta, we can take the limit $\delta \to 0$, noticing $A = \lim_{\delta\searrow 0} A_\delta$, and the bound on $|A_\delta|$ is valid for $|A|$.

About the second part of the integral, we split it again in two terms

$$|B| \leq \int_{\varepsilon<|x-y|<R} |P_N(x-y)f(y)|\,dy + \int_{|x-y|\geq R} |P_N(x-y)f(y)|\,dy = B_1 + B_2 \tag{2.36}$$

Applying again the bound $|P_N(x)| \leq c|x|^{-N}$, and using the same change of variables, we get

$$B_1 \leq c|f|_0 w_N \int_\varepsilon^R r^{-1}\,dr = cw_N|f|_0 \log\frac{R}{\varepsilon}, \tag{2.37}$$

and with a similar argument, we can bound the second term as follows:

$$B_2 \leq cR^{-N} \int_{\mathbb{R}^N} |f(x)|\, dx \leq cR^{-N} |f|_0 m_f = c|f|_0. \tag{2.38}$$

Finally, working on the constants, we get the final result:

Defining $c' = \max\{c, w_N c\}$, one gets

$$|B| \leq c'|f|_0 (\log \frac{R}{\varepsilon} + 1), \tag{2.39}$$

and, if we let $c_2 = \max\{c_1, c'\}$, we get that

$$|P_N f(x)| \leq c_2 \left(|f|_\gamma \varepsilon^\gamma + |f|_0 (\log \frac{R}{\varepsilon} + 1) \right), \tag{2.40}$$

and so

$$|P_N f|_0 \leq c_3 \left\{ |f|_\gamma \varepsilon^\gamma + \max\left(1, log \frac{R}{\varepsilon}\right) |f|_0 \right\}, \qquad \forall \varepsilon > 0, \tag{2.41}$$

On the other hand, for the second inequality:

$$|P_N f(x) - P_N f(x+h)| \leq \left| \mathrm{PV} \int_{\mathbb{R}^N} P_N(x-y) f(y)\, dy - \mathrm{PV} \int_{\mathbb{R}^N} P_N(x+h-y) f(y)\, dy \right|. \tag{2.42}$$

Again using the property on (2.27), we get:

$$\begin{aligned}
P_N f(x) \;&-\; P_N f(x+h) \\
&= \mathrm{PV} \int_{\mathbb{R}^N} P_N(x-y) \big(f(y) - f(x)\big)\, dy \;-\; \mathrm{PV} \int_{\mathbb{R}^N} P_N(x+h-y) \big(f(y) - f(x+h)\big)\, dy \\
&= \mathrm{PV} \int_{|x-y|<2h} P_N(x-y) \big(f(y) - f(x)\big)\, dy \\
&\quad - \mathrm{PV} \int_{|x-y|<2h} P_N(x+h-y) \big(f(y) - f(x+h)\big)\, dy \\
&\quad + \mathrm{PV} \int_{|x-y|\geq 2h} P_N(x-y) \big(f(x+h) - f(x)\big)\, dy \\
&\quad + \mathrm{PV} \int_{|x-y|\geq 2h} \Big[P_N(x-y) \;-\; P_N(x+h-y)\Big] \big(f(y) - f(x+h)\big)\, dy \\
&= A \;+\; B \;+\; C \;+\; D.
\end{aligned}$$

A and B are both bounded by $ch^\gamma |f|_\gamma$ using the exact same argument as in (2.35). C is zero directly because of the cancellation property (2.27).

The last term will be bounded using the mean value theorem:

$$|P_N(x-y) - P_N(x+h-y)| \leq \sup_{0\leq \xi \leq 1} |\nabla P_N(x-y+\xi h)||h|. \tag{2.43}$$

As $|x-y| \geq 2h \implies |x+h-y| \geq h$, and so one has

$$\frac{1}{2}|x-y| \leq -h + |x-y| \leq |x-y+\xi h| \leq |x-y| + h \leq \frac{3}{2}|x-y|. \tag{2.44}$$

Using the homogeneity of the derivative of P_N (degree $-N-1$), one can write $|\nabla P_N(x)| \leq c|x|^{-N-1}$, and using the last equation, what's inside of ∇P_N is controlled in norm by $c|x-y|$, so following (2.43)

$$\sup_{0\leq\xi\leq 1} |\nabla P_N(x-y+\xi h)||h| \leq ch|x-y|^{-N-1}. \tag{2.45}$$

Finally, applying the bound to the integral:

$$|D| \leq \int_{|x-y|\geq 2h} ch|x-y|^{-N-1}|f|_\gamma |x+h-y|^\gamma \, dy \leq c'h|f|_\gamma \int_{|x-y|\geq 2h} |x-y|^{-N-1+\gamma} \, dy \tag{2.46}$$

$$= c'h|f|_\gamma w_N \int_{2h}^{\infty} r^{-2+\gamma} \, dr \leq c_\gamma h^\gamma |f|_\gamma. \tag{2.47}$$

Merging everything together, we got that

$$|P_N f(x) - P_N f(x+h)| \leq ch^\gamma |f|_\gamma, \tag{2.48}$$

so in the end one gets the desired inequality:

$$|P_N f|_\gamma \leq c|f|_\gamma. \tag{2.49}$$

■

As a comment, we have proved that $P_N : C^\gamma \to C^\gamma$, but it is not true when acting on C^0, as the bound is not controlled only by the 0-norm of the function.

Finally, we need to prove the last lemma of this section:

Lemma 2.2.6. *Let the vorticity $\omega_0 \in C^\gamma$ with compact support, for some $\gamma \in (0,1)$. Given $X, Y \in O_M$, define*

$$G(X)Y = \int_{\mathbb{R}^2} \nabla K_2[X(\alpha) - X(\alpha')][Y(\alpha) - Y(\alpha')]\omega_0(\alpha') \, d\alpha',$$

Then there exists $c > 0$, independent of the particular choice of X and Y, such that:

$$|G(X)Y|_{1,\gamma} \leq c(M)\|\omega_0\|_\gamma |Y|_{1,\gamma}.$$

Proof. We need to bound the supremum norm $|G(X)Y|_0$ and the Hölder norm of the derivative $\|\partial_k G(X)Y\|_\gamma$

First, as the set O_M consists of 1-1 and onto functions, we can take the change of variables $X^{-1}(x) = \alpha$, and so

$$G(X)Y \circ X^{-1}(x) = \int_{\mathbb{R}^2} \nabla K_2(x-x')(Y(X^{-1}(x)) - Y(X^{-1}(x')))f(x') \, dx', \tag{2.50}$$

with f defined as

$$f(x') = \omega_0(\alpha')\big|_{\alpha'=X^{-1}(x')} \det \nabla_x X^{-1}(x'). \tag{2.51}$$

We will divide this integral in two, for $|x-x'|$ bigger and smaller than 1. So, we will have

$$|G(X)Y|_0 \le |I_1| + |I_2|. \tag{2.52}$$

Both terms will be bounded using $|\nabla K_2(x)| \le c|x|^{-2}$, and also using the mean-value theorem, to get:

$$|Y(X^{-1}(x)) - Y(X^{-1}(x'))| \le |(\nabla_\alpha Y \circ X^{-1})\nabla_x X^{-1}|_0 |x-x'|. \tag{2.53}$$

Using this, we can bound the first term (the integral is equal to $\frac{w_N}{N-1}$)

$$\begin{aligned}|I_1| &\le c|(\nabla_\alpha Y \circ X^{-1})\nabla_x X^{-1}|_0 |f|_0 \int_{|x-x'|\le 1} |x-x'|^{-2+1}\, dx' && (2.54)\\ &\le c\|(\nabla_\alpha Y \circ X^{-1})\nabla_x X^{-1}\|_\gamma \|f\|_\gamma. && (2.55)\end{aligned}$$

About the second term, we use the compact support of ω_0 which bounds the support of f, and also using $|x-x'| \ge 1 \implies |x-x'|^{-1} \le 1$, we have:

$$\begin{aligned}|I_1| &\le c|(\nabla_\alpha Y \circ X^{-1})\nabla_x X^{-1}|_0 |f|_0 \int_{\substack{|x-x'|\ge 1\\ x'\in \operatorname{supp}\omega_0}} |x-x'|^{-2+1}\, dx' && (2.56)\\ &\le c\|(\nabla_\alpha Y \circ X^{-1})\nabla_x X^{-1}\|_\gamma \|f\|_\gamma. && (2.57)\end{aligned}$$

By (2.65) found in the proof of the existence 2.3.1, we have $\|f\|_\gamma \le c(M)\|\omega_0\|_\gamma$, an by the lemmas in subsection 2.2.1 about inequalities with norms, one finally gets

$$\begin{aligned}|G(X)Y|_0 &\le c\|(\nabla_\alpha Y \circ X^{-1})\nabla_x X^{-1}\|_\gamma \|f\|_\gamma\\ &\le c(M)\|\omega_0\|_\gamma \|\nabla_\alpha Y\|_\gamma (1+|X|_{1,\gamma}^{3\gamma})\|\nabla_\alpha X\|^3 \le c(M)\|\omega_0\|_\gamma |Y|_{1,\gamma}.\end{aligned} \tag{2.58}$$

For the Hölder norm of $\nabla G(X)Y$, we will compute a distribution derivative of $H(x) = G(X)Y \circ X^{-1}(x)$. Note that $H(x)$ can be written as

$$\begin{aligned}H(x) &= \int_{\mathbb{R}^2} R(x,x')f(x'), \text{ with}\\ R(x,x') &= \nabla K_2(x-x')\left\{Y[X^{-1}(x)] - Y[X^{-1}(x')]\right\}.\end{aligned}$$

We will compute its distributional derivative using the definition 1.2.3

$$
\begin{aligned}
(-\partial_{x_i}\varphi, R(x+x',x')) &= -\int \partial_{x_i}\varphi(x)R(x+x',x')\,dx \\
&= -\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \partial_{x_i}\varphi(x)R(x+x',x')\,dx \\
&= \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \varphi(x)\partial_{x_i}R(x+x',x')\,dx \\
&\quad + \lim_{\epsilon\to 0}\int_{|x|=\epsilon} \varphi(x)R(x+x',x')\frac{x_i}{|x|}\,ds \\
&= \mathrm{PV}\int \varphi(x)\partial_{x_i}R(x+x',x')\,dx \\
&\quad + \nabla_\alpha Y(X^{-1}(x))\nabla_x X^{-1}(x)c_i\varphi(0).
\end{aligned}
$$

Using this formula, we get the distributional derivative of $H(x)$

$$
\begin{aligned}
\partial_{x_k}H(x) = \mathrm{PV}\int_{\mathbb{R}^3} &\partial_{x_k}\nabla K_3(x-x')\big[Y(X^{-1}(x)) - Y(X^{-1}(x'))\big]f(x')\,dx' \\
+ \mathrm{PV}\int_{\mathbb{R}^3} &\nabla K_3(x-x')\nabla_\alpha Y(X^{-1}(x))\partial_{x_k}X^{-1}(x)f(x')\,dx' \\
+ \nabla_\alpha &Y(X^{-1}(x))\nabla_x X^{-1}(x)c_k f(x).
\end{aligned}
$$

Finally, with a really similar argument to the one used on the previous lemma 2.30, we get

$$
\|\partial_{x_k}G(X)Y\|_\gamma \leq c\|\big(\nabla_\alpha Y\circ X\big)\nabla_x X^{-1}\|_\gamma\|f\|_\gamma.
$$

Using the lemmas proved in the previous section 2.2.1, combined with $\|f\|_\gamma \leq c(M)\|\omega_0\|_\gamma$ and (2.58), we get the claimed inequality

$$
|G(X)Y|_{1,\gamma} \leq c(M)\|\omega_0\|_\gamma|Y|_{1,\gamma}.
$$

■

2.3 Proof of the existence

We now have defined all the elements needed for the proof of the theorem:

Theorem 2.3.1. *Let $\omega_0 \in C^\gamma$ with compact support, $\gamma \in (0,1)$, $\omega_0 = \operatorname{curl} v_0$ and $\operatorname{div} v_0 = 0$. Then, for any $M > 0$, there exists a time $T(M) > 0$ and a unique solution $X \in C^1((-T(M),T(M));O_M)$*

Let us define $F(X(\alpha,t)) = \int_{\mathbb{R}^2} K_2[X(\alpha,t) - X(\alpha',t)]\omega_0(\alpha')d\alpha'$. Then, the theorem is shown by proving that $F : O_M \to B$ satisfies the conditions on the Picard theorem. For it, we have to prove that F is bounded and locally Lipschitz.

As functions in O_M are homeomorphisms, we can take the change of variables $x = X(\alpha) \implies d\alpha = \det\nabla_x X^{-1}(x)dx$.

So,

$$F(x) = \int_{\mathbb{R}^2} K_2[x - x']\omega_0(\alpha)\big|_{\alpha=X^{-1}(x')} \det \nabla_x X^{-1}(x')dx', \tag{2.59}$$

and defining

$$f(x') = \omega_0(\alpha)\big|_{\alpha=X^{-1}(x')} \det \nabla_x X^{-1}(x'), \tag{2.60}$$

one can write $F(X) = K_2 f \circ X^{-1}$.

Then, using (2.8), we can bound the norm of F in terms of the norm of $|K_2 f|_{1,\gamma}$, as the norm of $|X|_{1,\gamma}$ is bounded in O_M:

$$|F|_{1,\gamma} \leq |K_2 f|_{1,\gamma}|X^{-1}|_{1,\gamma} \leq c|K_2 f|_{1,\gamma}|X|^3_{1,\gamma} \tag{2.61}$$

We have to study the value of $|K_2 f|_{1,\gamma} = |K_2 f(0)| + \|\nabla K_2 f\|_\gamma$. The bound on the first term is immediate using (2.22):

$$|K_2 f(0)| \leq |K_2 f|_0 \leq c|f|_0 \tag{2.62}$$

It can be seen that $\nabla [K_N f(x)] = -\mathrm{PV}\int_{R_N} P_N(x - x')f(x')\,dx' + cf(x)$ (by 1.2.1, but noticing that as we are applying a convolution, we must take into account the translation). Using this, one gets (also using (2.30))

$$\|\nabla K_2 f\|_\gamma \leq c\|f\|_\gamma + \|P_2 f\|_\gamma \leq c\|f\|_\gamma \tag{2.63}$$

Finally, if we find a bound for the gamma norm of f, we will have bounded both terms on the norm of $K_2 f$

$$\|f\|_\gamma \leq \|\omega_0 \circ X^{-1}\|_\gamma \|\det \nabla_x X^{-1}\|_\gamma \leq c\|w_0\|_\gamma (1 + |X|^{\gamma(2N-1)}_{1,\gamma})c'\|\nabla_\alpha X\|^N_\gamma \tag{2.64}$$

$$\leq c(M)\|\omega_0\|_\gamma, \tag{2.65}$$

where the first step uses (2.4), the second one uses the bound on the inverse (2.17) and the bound on the determinant given by (2.10), (2.14) and (2.13). Finally, using the fact that $|X|_{1,\gamma}$ is bounded by M on O_M, we get the inequality.

So, getting everything together:

$$|K_2 f|_{1,\gamma} \leq c(M)\|\omega_0\|_\gamma, \tag{2.66}$$

and thus F is a bounded operator. What we've really proven, is that $|F(X)|_{1,\gamma} < \infty\ \forall X \in O_M$, so that F maps functions from O_M to B, being well defined.

About the Lipschitz continuity, it is enough to bound the derivative, as if $\|F'(X)\| < \infty\ \forall X \in O_M$, the Lipschitz continuity follows from the following calculation:

$$|F(X') - F(X'')|_{1,\gamma} = \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(X'' + \varepsilon(X' - X''))\ d\varepsilon\right|_{1,\gamma} \tag{2.67}$$

$$\leq \int_0^1 \|F'(X'' + \varepsilon(X' - X''))\|\ d\varepsilon |X' - X''|_{1,\gamma} \tag{2.68}$$

One has

$$F'(X)Y = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(X+\varepsilon Y)\big|_{\varepsilon=0} \tag{2.69}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_{\mathbb{R}^2} K_2[X(\alpha) - X(\alpha') + \varepsilon(Y(\alpha) - Y(\alpha'))]\omega_0(\alpha')\ d\alpha' \Big|_{\varepsilon=0} \tag{2.70}$$

$$= \int_{\mathbb{R}^2} \nabla K_2[X(\alpha) - X(\alpha')][Y(\alpha) - Y(\alpha')]\omega_0(\alpha')\ d\alpha' = G(X)Y. \tag{2.71}$$

It is proved in lemma 2.2.6 that this equation is bounded by

$$|G(X)Y|_{1,\gamma} \leq c(M)\|\omega_0\|_\gamma |Y|_{1,\gamma}. \tag{2.72}$$

So, the derivative is a bounded operator and thus the operator F is locally Lipschitz continuous so the result is proved.

With this all, we have proven the local-in-time existence of solution for the Euler equation in 2D.

Chapter 3
Global nature of the solutions

In chapter 2 we proved the existence of local-in-time solutions when the initial condition is smooth enough ($\omega_0 \in C^\gamma$). In our case (2D Euler equations), the global nature of the solution follows always, and it will be proved in this chapter.

3.1 Principal result

Basically, this chapter will consist on proving the next theorem, which states

Theorem 3.1.1. *Beale-Kato-Majda. Consider a compactly supported initial vorticity $\omega_0 =$ curl v_0, div $v_0 = 0$, with Hölder norm $\|\omega_0\|_\gamma < \infty$ for some $\gamma \in (0,1)$. Let $|\omega(\cdot,s)|_0$ be the supremum norm at fixed time s of a solution to the Euler equation, with initial data ω_0.*

(i) Suppose that for any $T > 0$ there exists $M_1 > 0$ such that the vorticity $\omega(x,t)$ satisfies

$$\int_0^T |\omega(\cdot,s)|_0 \, ds \leq M_1. \tag{3.1}$$

Then, for any T there exists $M > 0$ such that $X \in C^1([0,T);O_M)$ (the solution X exists globally in time).

(ii) Suppose that for any $M > 0$ there is a finite maximal time $T(M)$ of existence of solutions $X \in C^1([0,T(M));O_M)$ and that $\lim_{M\to\infty} T(M) = T^* < \infty$; then necessarily the vorticity accumulates so rapidly that

$$\lim_{t\to T^*} \int_0^t |\omega(\cdot,s)|_0 \, ds = \infty. \tag{3.2}$$

This result is general in 3D. In the case of 2D, the global nature in time follows directly given the property $\omega(X(\alpha,t),t) = \omega_0(\alpha)$, as the vorticity is conserved along particle trajectories, meaning that $|\omega(\cdot,t)|_0 = C$ and therefore having the initial vorticity bounded implies the globalness-in-time of the solution.

We will be proving this result by parts, with the following sketch of the proof:

- Prove that an a priori bound on $|X|_{1,\gamma}$ is enough to have the global nature in time, with a continuation result of an ODE with the function being independent in time.
- Prove that an a priori bound on $\int_0^t |\nabla v(\cdot,s)|_0 \, ds$ implies an a priori bound on $|X|_{1,\gamma}$
- Prove that an a priori bound on $\int_0^t |\omega(\cdot,s)|_0 \, ds$ implies an a priori bound on $\int_0^t |\nabla v(\cdot,s)|_0 \, ds$ and the whole result will be proved.

3.2 Continuation of an Autonomous ODE on a Banach space

In this section we present this standard result of ODE, when the function defining the ODE is independent in time.

Theorem 3.2.1. *Continuation of an Autonomous ODE on a Banach Space. Let $O \subset B$ be an open subset of a Banach space B, and let $F : O \to B$ be a locally Lipschitz continuous operator. Then the unique solution $X \in C^1([0,T);O)$ to the autonomous ODE,*

$$\frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0 \in O,$$

either exists globally in time or $T < \infty$ and $X(t)$ leaves the open set O as $t \nearrow T$.

The case where the solution wouldn't be global in time, is when the solution leaves the open subset where it is defined. Recall that our solutions were defined on

$$O_M = \left\{ X \in B / \inf_{\alpha \in \mathbb{R}^2} \det \nabla_\alpha X > 1/2 \text{ and } |X|_{1,\gamma} < M \right\}. \tag{3.3}$$

So, as for incompressible flows we have $\det \nabla_\alpha X = 1$ (by proposition 1.2.2), the only way for X to leaving the set would be if it grows unbounded. So, this is why the result will be proved the moment we find an a priori bound on $|X|_{1,\gamma}$.

3.3 A priori bound on the solution

Proposition 3.3.1. $|X(\cdot,t)|_{1,\gamma}$ *is a priori bounded by an a priori bound on* $\int_0^t |\nabla v(\cdot,s)|_0 \, ds$.

Proof. Recall that $|X(\cdot,t)|_{1,\gamma} = |X(0,t)| + |\nabla_\alpha X(\cdot,t)|_0 + |\nabla_\alpha X(\cdot,t)|_\gamma$. Also, $X(\alpha,t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} X(\alpha,t) = v(X(\alpha,t),t), \tag{3.4}$$

$$\text{with } \; v(x,t) = \int_{\mathbb{R}^2} K_2(x-y)\omega(y,t)\, dy = (K_2\omega) \tag{3.5}$$

From equation (3.4), by integrating both sides we get

$$|X(0,t)| \leq \int_0^t |v(\cdot,s)| \, ds, \tag{3.6}$$

and by using (2.22) and (3.5), and also using that X preserves volume, so the size of the support of ω is constant in time we finally get that

$$|X(0,t)| \leq \int_0^t |v(\cdot,s)| \, ds \leq C \int_0^t |\omega(\cdot,s)|_0 \, ds, \tag{3.7}$$

which is bounded.

By (3.4), one gets that

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\alpha X(\alpha,t) = \nabla v(X(\alpha,t),t)\nabla_\alpha X(\alpha,t), \tag{3.8}$$

and applying the supremum norm, we finally get:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla_\alpha X(\cdot,t)|_0 \le |\nabla v(\cdot,t)|_0 |\nabla_\alpha X(\cdot,t)|_0 \implies |\nabla_\alpha X(\cdot,t)|_0 \le e^{\int_0^t |\nabla v(\cdot,s)|_0\,ds}, \tag{3.9}$$

obtained by the Grönwall inequality.

A generalization of this inequality is the following Grönwall lemma:

Lemma 3.3.1. *Grönwall Lemma. If u, q, and $c \ge 0$ are continuous on $[0,t]$, c is differentiable, and*

$$q(t) \le c(t) + \int_0^t u(s)q(s)\,ds, \tag{3.10}$$

then

$$q(t) \le c(0)\exp\left(\int_0^t u(s)\,ds\right) + \int_0^t c'(s)\left[\exp\left(\int_s^t u(\tau)\,d\tau\right)\right] ds. \tag{3.11}$$

The Grönwall inequality used above follows from applying the lemma with $c = 0$.

Finally, we need to bound $|\nabla_\alpha X(\cdot,t)|_\gamma$.

$$\begin{aligned}
\frac{d}{dt}|\nabla_\alpha X(\cdot,t)|_\gamma &\le |\nabla v(X(\cdot,t),t)\nabla_\alpha X(\cdot,t)|_\gamma, && (3.12)\\
&\le |\nabla v(X(\cdot,t),t)|_\gamma |\nabla_\alpha X(\cdot,t)|_0 + |\nabla v(\cdot,t)|_0 |\nabla_\alpha X(\cdot,t)|_\gamma, && (3.13)\\
&\le |\nabla v(\cdot,t)|_\gamma |\nabla_\alpha X(\cdot,t)|_0^{\gamma+1} + |\nabla v(\cdot,t)|_0 |\nabla_\alpha X(\cdot,t)|_\gamma, && (3.14)\\
&\le C|\omega(\cdot,t)|_\gamma e^{(\gamma+1)\int_0^t |\nabla v(\cdot,s)|_0\,ds} + |\nabla v(\cdot,t)|_0 |\nabla_\alpha X(\cdot,t)|_\gamma. && (3.15)
\end{aligned}$$

The reasoning behind each step is as follows:

Inequality (3.12) follows from (3.8). Inequality (3.13) uses (2.3) to handle the product in the γ-norm. Inequality (3.14) applies (2.15) for composition in the γ-norm. Lastly, (3.15) is using the bound (3.9) and (2.30) (as $\nabla v = \nabla(K_2\omega)$, so we use the bound on the derivative $|P_N f|_\gamma$).

To finish the proof of the proposition, we have to bound $|\omega(\cdot,t)|_\gamma$ in terms of $|\nabla v(\cdot,t)|_0$.

We will use the following lemma:

Lemma 3.3.2. *Assume that $\omega_0(x) = \omega_0 \in C^\gamma(\mathbb{R}^2)$ and for $|t| \le T$ and $\omega(x,t)$ satisfies the vorticity-stream form*

$$\frac{D\omega}{Dt} = 0$$

and that $X(\alpha,t)$ are the particle trajectories associated with the velocity v that satisfy Eqs. (3.4), (3.5) and (3.8). Then

$$|\omega(\cdot,t)|_\gamma \le |\omega_0|_\gamma \exp\left[\gamma\int_0^t |\nabla v(\cdot,s)|_0\,ds\right],$$

Proof. Using the inverse of the particle trajectory mapping, the transport equation can be written in 2D as

$$\omega(x,t) = \omega_0(X^{-1}(x,t)). \tag{3.16}$$

Also, by definition of the particle-trajectory mapping, it's clear that its inverse satisfies

$$\frac{d}{dt}X^{-1}(x,t) = -v(X^{-1}(x,t),t). \tag{3.17}$$

Using a same argument as before (3.9), applying the gradient and the Grönwall inequality, one gets

$$|\nabla_x X^{-1}(\cdot,t)|_0 \le e^{\int_0^t |\nabla v(\cdot,s)|_0\,ds}. \tag{3.18}$$

Comining all the previous, it is easy to see

$$|\omega(x,t) - \omega(x',t)| = |\omega_0(X^{-1}(x,t)) - \omega_0(X^{-1}(x',t))| \le |\omega_0|_\gamma |\nabla_x X^{-1}(\cdot,t)|_0^\gamma |x-x'|^\gamma, \tag{3.19}$$

and finally dividing by $|x-x'|$, we get the desired bound:

$$|\omega(\cdot,t)|_\gamma \le |\omega_0|_\gamma \exp\left[\gamma\int_0^t |\nabla v(\cdot,s)|_0\,ds\right]. \tag{3.20}$$

■

Combining the bound (3.15) with the lemma, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla_\alpha X(\cdot,t)|_\gamma \le C|\omega_0|_\gamma e^{C_1\int_0^t |\nabla v(\cdot,s)|_0\,ds} + |\nabla v(\cdot,t)|_0 |\nabla_\alpha X(\cdot,t)|_\gamma. \tag{3.21}$$

Integrating both sides of the inequality (and using $X(\alpha,0)=\alpha$), we get

$$|\nabla_\alpha X(\cdot,t)|_\gamma \le C|\omega_0|_\gamma \int_0^t \exp\left[C_1\int_0^s |\nabla v(\cdot,\tau)|_0\,d\tau\right] ds + \int_0^t |\nabla v(\cdot,s)|_0 |\nabla_\alpha X(\cdot,s)|_\gamma\,ds. \tag{3.22}$$

We can use the Grönwall lemma, and we finally get

$$|\nabla_\alpha X(\cdot,t)|_\gamma \le C|\omega_0|_\gamma \int_0^t e^{C_1\int_0^s |\nabla v(\cdot,\tau)|_0\,d\tau} e^{\int_s^t |\nabla v(\cdot,\tau)|_0\,d\tau}\,ds \tag{3.23}$$

$$\le C|\omega_0|_\gamma t\, e^{(C_1+1)\int_0^t |\nabla v(\cdot,s)|_0\,ds}. \tag{3.24}$$

This completes the proof of the proposition. ■

3.4 A priori bound on the velocity

To complete the proof of the initial theorem, we need to see that $\int_0^t |\nabla v(\cdot,s)|_0\,ds$ is controlled by $\int_0^t |\omega(\cdot,s)|_0\,ds$. Recalling the definition of the gradient of $K_2\omega$ used in the previous chapter, we have that

$$\nabla v(x,t) = \mathrm{PV}\int_{\mathbb{R}^2} P_2(x-x')\omega(x',t)\,dx' + c\omega(x,t), \tag{3.25}$$

and also it can be bounded using the relation (2.29):

$$|\nabla v(\cdot,t)|_0 \leq c|\omega(\cdot,t)|_\gamma \epsilon^\gamma + |\omega(\cdot,t)|_0 \left[1+\log\left(R/\epsilon\right)\right] \qquad \forall \epsilon > 0, \tag{3.26}$$

with R a constant independent of time as the measure of the support of $\omega(\cdot,t)$ is independent of the time (X is volume-preserving).

We can fix $\epsilon = |\omega(\cdot,t)|_\gamma^{-1/\gamma}$ (for a fixed time t), to get the following inequality

$$|\nabla v(\cdot,t)|_0 \leq |\omega(\cdot,t)|_0 \left[|\log\left(R|\omega(\cdot,t)|_\gamma\right)| + C\right] \qquad \text{¿+c?} \tag{3.27}$$

Recalling the bound of $|\omega(\cdot,t)|_\gamma$ from the lemma 3.3.2, and plugging it in the previous inequality, one gets:

$$|\nabla v(\cdot,t)|_0 \leq C(\omega_0)|\omega(\cdot,t)|_0 \left[1+\int_0^t |\nabla v(\cdot,s)|_0\,ds\right] \tag{3.28}$$

From this, by basic calculus, we obtain the last inequality we need:

$$\begin{aligned}
\frac{|\nabla v(\cdot,t)|_0}{\left[1+\int_0^t |\nabla v(\cdot,s)|_0\,ds\right]} &\leq C(\omega_0)|\omega(\cdot,t)|_0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\ln\left[1+\int_0^t |\nabla v(\cdot,s)|_0\,ds\right] &\leq C(\omega_0)|\omega(\cdot,t)|_0 \\
\ln\left[1+\int_0^t |\nabla v(\cdot,s)|_0\,ds\right] &\leq C(\omega_0)\int_0^t |\omega(\cdot,s)|_0\,ds \\
1+\int_0^t |\nabla v(\cdot,s)|_0\,ds &\leq \exp\left(C(\omega_0)\int_0^t |\omega(\cdot,s)|_0\,ds\right)
\end{aligned}$$

And with this, we have proven that $\int_0^t |\omega(\cdot,s)|_0\,ds$ controls the term $\int_0^t |\nabla v(\cdot,s)|_0\,ds$, and putting all the previous work together, we have proven that if $\int_0^t |\omega(\cdot,s)|_0\,ds$ remains bounded, then X stays in O_M and the solution is global in time, which happens always in 2D.

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