

Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.

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Abstract

The purpose of this paper is to study phenomena in chaotic zones of area preserving maps using simpler models which are easier to analyse theoretically and numerically. First of all the study of the dynamics in a neighbourhood of the separatrices of a resonant zone is carried out. The well-known separatrix map, defined on a figure eight when needed, is used to determine the location of rotational invariant curves (r.i.c.) inside and outside the resonance. The interest in this part is on a quantitative description of the dynamics in a neighbourhood of the separatrices: to produce theoretical estimates of the width of the stochastic zone, distance to the r.i.c., existence of tiny islands close to the separatrices, ... In every one of the studied items one has tried to complement the limit analytic study with realistic numerical simulations, describing the analogy when possible. After this study, we focus on the formation of larger domains without r.i.c. (e.g. Birkhoff domains). To this end we introduce the biseparatrix map model. Although this is a qualitative model, the mechanism of destruction of the “last” r.i.c., and hence the process of creation of zones without r.i.c., is clarified by means of this simple model. Several numerical examples illustrate the results obtained and are used as a test of the theoretical quantitative predictions.

1 Introduction

It is well-known that in the phase space of an area preserving map (APM), and also in the case of a weakly dissipative perturbation of an APM, the most remarkable fact is the coexistence of chaotic and regular dynamics. It is also known that chaotic regions are related to the interaction of resonances. The goal of this work is to contribute to the description, by means of simple return map models, of the dynamics of a nearly integrable map (or a not so close to integrable map but in selected domains) within chaotic regions.

For concreteness all the maps we shall consider are assumed to be real analytic.

Let F denote an APM (in many applications obtained maybe numerically as, for example, a Poincaré map related to a Hamiltonian flow with $n \geq 1 + \frac{1}{2}$ d.o.f.) and assume that, in the corresponding phase space defined by F , we are able to detect chaotic regions and/or resonant islands (for example, by carrying out a numerical simulation of the map). The dynamics close to the separatrices of the hyperbolic points of a resonant chain of islands, or even in large chaotic regions without rotational invariant curves (r.i.c.), can be well described using suitable return

maps that reflect the main dynamical features of F . Some advantages of this approach using these return maps are:

1. They play the role of paradigms from which dynamical information can be obtained.
2. They depend on some parameters that should be determined from properties of F (maybe numerically, see comments below).
3. They allow, if the assumptions to derive the model hold, to obtain not only qualitative but also *quantitative* information on the dynamics of the original map F .

In this work we obtain several dynamical properties of F using return models. We note that:

- Most of the results discussed through the paper are analytic, based on a theoretical study of return maps (at least the ones concerning the separatrix and double separatrix maps). The formulas predicting width of stochastic regions, number of islands within these regions, ... depend on the parameters of the model which in turn depend on F .
- We systematically confront the analytical results with the numerical ones obtained from direct simulations. In some cases, when no theoretical results are available (for example, to estimate measures of transport between different domains of the phase space), we have tried to obtain some conclusions from the numerical results.

The first difficulty to use a return map model to describe the dynamics of a given APM F in a concrete chaotic region is to estimate the parameters of the model. We are aware of the fact that, for a general system F , these parameters may be only numerically estimated. Some remarks to clarify this point are required:

- All the parameters involved in the models have a clear geometrical meaning and can be estimated, at least numerically, with a reasonable computational effort in most systems.
- To estimate these parameters requires “simple” computations to obtain some periodic orbits and its eigenvalues, invariant manifolds or splittings of separatrices. The cost of these computations (may be expensive for a given system) is lower than the cost of massive simulations to study the qualitative evolution of the system with respect to parameters or to get quantitative information like the width of the chaotic zones or transport properties.
- For simple systems, like the Hénon map, even these higher expensive computations can be carried out (a fact that we use to check the accuracy of the model predictions), but this is not the case one expects for a general system. This motivates the study of these return models to analyse the dynamics. Furthermore these models provide a deep insight into the dynamics of F .

The paper is organised as follows. After a preliminary section whose goal is to set our research into the proper framework, the next two sections are devoted to study the chaotic zone surrounding the separatrices around a resonant island. Hence, in section 3 we consider the *open* case while section 4 deals with the *figure eight* case. Section 5 is devoted to a preliminary study of the dynamics within larger global regions of instability without r.i.c. To end with, some conclusions are given in section 6. Let us describe briefly what these situations refer to and which are the models to study them.

Open case. Consider an *open* map (according to the terminology in [32]), that is, a map having a hyperbolic fixed point such that just one of the branches of the stable manifold intersects transversally only one of the branches of the unstable one. We assume that the angle between the manifolds at the intersection (or the distance between the manifolds or the area bounded by

them) has a small or moderate size. Topologically, the resonance structure looks like the phase space of the flow generated by the Hamiltonian $\mathcal{H}(X, Y) = -Y^2/2 + X^2/2 + X^3/3$ (which is informally known as the “fish” Hamiltonian) although, for the map, the separatrices generically no longer coincide and there is a chaotic region containing the homoclinic tangle. The topological picture is shown in figure 1. Notice that a neighbourhood of the hyperbolic point P_h contains always points inside homoclinic lobes whose iterates do not return close to P_h .

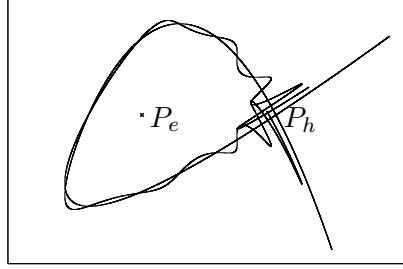


Figure 1: Topology of the open case.

To describe the dynamics inside the chaotic region generated by transversal intersections of the invariant manifolds the so-called separatrix map [49, 5] turns out to be a universal model. It provides not only qualitative but quantitative information concerning the distance to which we can expect islands (of secondary resonances) near the separatrices, the distance to which we can expect r.i.c. inside the resonant domain, the distance to integrable in any relatively close-to-the-separatrix region, the number of the “largest” secondary islands of stability, ... In section 3 we review the separatrix model and illustrate with concrete examples how this information can be obtained from it.

Figure eight case. We consider a “pendulum-like” resonance or, in the [26] terminology, *resonant* (q, m) -islands, which have rotation number q/m . It is defined by a pair of hyperbolic-elliptic periodic orbits such that the invariant manifolds of the hyperbolic points surround the elliptic points. Topologically, the phase space around an island of a resonant chain is like a classical pendulum but, as before, generically the manifolds do not coincide for the map. If the dynamics of F^m is slow, say, it differs from the identity by $\mathcal{O}(\epsilon)$, then a chaotic region exponentially small with respect to ϵ surrounds the separatrices of the island containing the homoclinic tangle generated by all the branches of the invariant manifolds.

The difference with respect to the *open* case (“fish” type of islands) is that in the present one there is a re-injection of *all* the dynamics: both branches of the stable manifold intersect the two branches of the unstable one. In other words, by iteration of F all the points close to P_h inside homoclinic lobes follow the loops of the figure eight and return to a neighbourhood of P_h (the exception are those points on the stable manifold of P_h because their iterates remain close to P_h from some iterate on). The topology of this case is represented in the figure 2 right.

In figure 2 we also illustrate the transition from the “pendulum” shape (left) to a Poincaré figure eight (right). We remark that our interest is on the dynamics in a “fattened separatrix” region containing the separatrices and, in this sense, one can assume for this topological construction the existence of invariant curves inside and outside the pendulum island (corresponding to rotational and librational motion of the pendulum) that bound the region of interest (these curves, if they exist and are close enough to the separatrices, can be determined by the double separatrix model, see section 4 and compare, for instance, with the central island in figure 19 top right). Let us briefly describe the transition shown in figure 2.

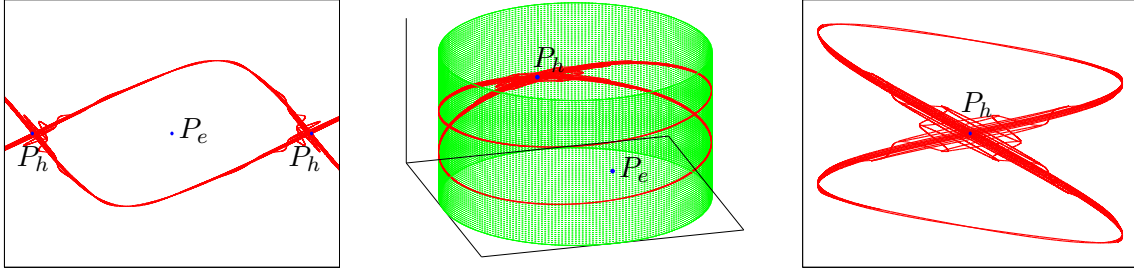


Figure 2: Topology of the figure eight case. Left: A “pendulum-like” structure. Centre: The same structure is represented on a cylinder. Right: The figure eight.

Having a “pendulum” resonant structure like the one on the left, we identify the segment joining the two hyperbolic points with $\mathbb{S}^1$. That is, assume that the segment joining both hyperbolic points P_h of the left plot is parametrised in such a way that they correspond to $x = 0$ and $x = 1$ resp. Then $x \mapsto (\cos(2\pi x), \sin(2\pi x))$ identifies the segment with $\mathbb{S}^1$. In this way the resonance can be seen on a cylinder as it is shown in the centre plot. We consider the sphere obtained as a compactification of the cylinder by collapsing two circles, above and below the resonance, to north and south poles. Then, a Riemann stereographic projection from the elliptic point P_e of the centre plot reduces it to the figure eight shape shown on the right. Note that this transformation maps the elliptic point to ∞ while points outside the “pendulum” island and above the upper separatrix on the left plot are mapped to points inside the upper loop of the figure eight. Similarly, points below the lower separatrix of the “pendulum” structure are mapped to points inside the lower loop of the figure eight. In particular, if y is the vertical coordinate used in the “pendulum” representation then $y = +\infty$ ($y = -\infty$) can be thought as the fixed point located at the centre of the upper (inner) loop of the figure eight. From now on we will refer either to the figure eight case or to the “pendulum” case indistinctly; we hope this will not be confusing. This transformation allows to switch between these two equivalent representations of the same topological structure.

The model considered to analyse the chaotic regions around the separatrices of “pendulum-like” islands is again the separatrix map but now defined on a figure eight (see [31]). For simplicity, we will refer to it as the double separatrix map. In section 4 we obtain a quantitative description of the chaotic zone in a neighbourhood of the separatrices that form the figure eight by means of this model. In particular, we show that the width of the chaotic zone along the two loops of the figure eight depends not only on the splitting of separatrices at the primary transversal homoclinic points but on the shape of the level lines of a suitable Hamiltonian.

Large regions of instability. When studying globally the dynamics of a map it should be considered not only the effect of an isolated resonance but the interaction of infinitely many of them. Resonance overlap can give rise to large domains of chaoticity. These regions, when confined between two r.i.c., are known as Birkhoff zones of instability.

An example of such a zone is shown in figure 3. It has been constructed in an artificial way to display some properties. Consider, for d fixed and e small enough, the Hamiltonian

$$\mathcal{H}(\varphi, J) = J^4 - (2 + ed \cos(\varphi))J^2 - e(1 - d) \cos(\varphi). \quad (1)$$

To define a symplectic map M , close to the time- δ flow of $\mathcal{H}$ for δ small, the generating function $S(\varphi, \bar{J}) = \varphi \bar{J} + \delta \mathcal{H}(\varphi, \bar{J})$ has been used and $\bar{\varphi} = \frac{\partial S}{\partial J}$, $J = \frac{\partial S}{\partial \varphi}$ give M implicitly. This precludes the existence of the inverse in some domains. Elliptic fixed points are located at $P_e^\pm = (\varphi, J_e^\pm) =$

$(0, \pm\sqrt{1+ed/2})$ and at $P_e^\pi = (\pi, 0)$, while $P_h^0 = (0, 0)$ and $P_h^\pm = (\varphi, J_h^\pm) = (\pi, \pm\sqrt{1-ed/2})$ are hyperbolic. The symmetry $(\varphi, J) \leftrightarrow (2\pi - \varphi, -J)$ is apparent.

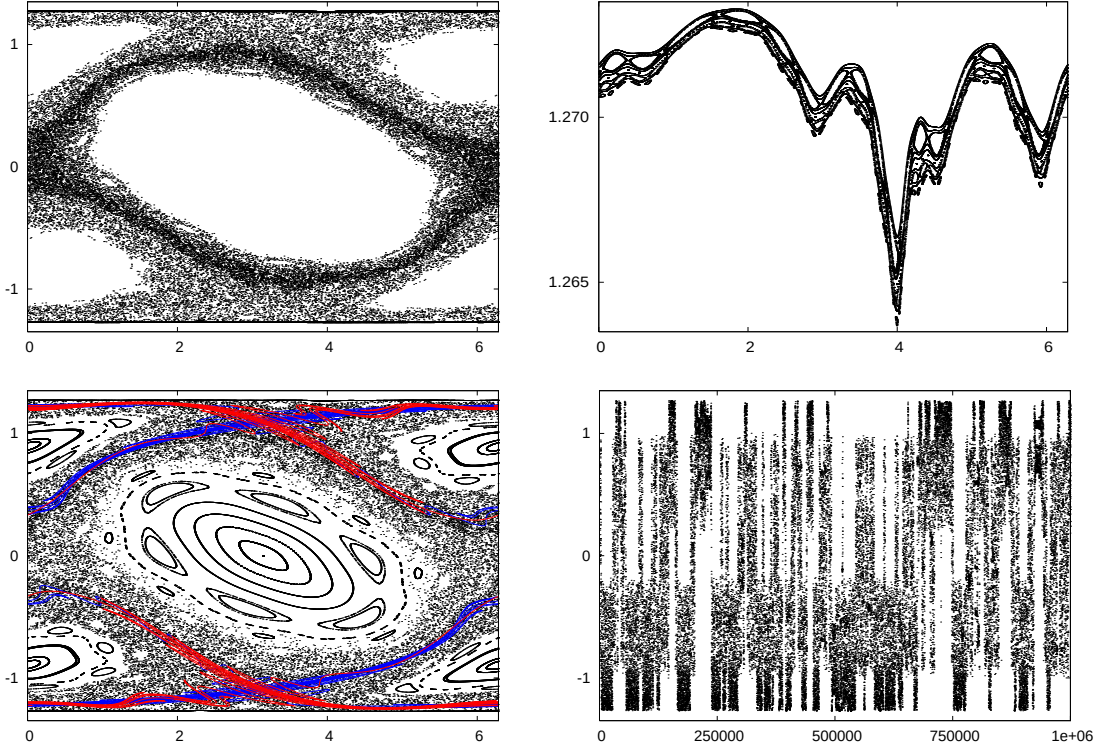


Figure 3: Birkhoff zone. Top left: Chaotic zone and two confining r.i.c. Right: Detail on the upper r.i.c. and some small islands. Bottom left: dynamics in the phase space, manifolds of $P_h^\pm$ and confining r.i.c. Right: J as a function of the number of iterates, starting at $(0.01, 0)$, displaying 1 point every 100 iterates.

The integrable flow related to the Hamiltonian (1), with $e, d \neq 0$, has a foliation of r.i.c. for values $|J| \geq \sqrt{1-ed/2}$ on $\varphi = \pi$, and has three islands. Using small values of e in (1) the three islands are narrow. As we want the map M to be “relatively far” from integrable to have a large chaotic zone we take not too small values of e . The values $e = 0.25, d = -1.6$ have been chosen. Similarly, small δ gives M “too close” to integrable and we took $\delta = 0.8$. Figure 3 displays some results. The top left plot shows a chaotic zone (computed from a single initial point) and two confining r.i.c. The top confining r.i.c. (and other r.i.c. close to it) and some small islands can be seen on the right. The bottom left plot displays some orbits, together with confining r.i.c. and the manifolds of $P_h^\pm$. The dynamics on top of the upper branches of $W^{u,s}(P_h^+)$ can be studied using the separatrix map (see section 3). There are r.i.c. close to the upper branches of $W^{u,s}(P_h^+)$. A key point for this existence is the fact that there is a strong twist in this domain, because the map is no longer invertible for nearby J . On the other hand the lower branches of $W^{u,s}(P_h^+)$ have heteroclinic intersections with the manifolds of P_h^0 (not shown). This precludes the existence of the r.i.c. and allows for the presence of the chaotic zone. As one can see on the top left plot, there is a higher density of points in some places. This is due to the existence of a couple of symmetric Aubry-Mather (A-M) sets, with narrow “holes”, as a remnant of r.i.c. which separate the islands around P_e^π and around P_0^0 for smaller δ . The existence/destruction of these r.i.c. mainly depends on changes in the rotation number when moving δ (e.g., coming too close to $1/9$) and on the weak twist property or even the absence of it. There is numerical evidence that these r.i.c. exist for $0 < \delta \leq 0.6$, are absent for $\delta \geq 0.8$ and are present for δ in some subintervals of $[0.6, 0.8]$ (for, roughly, 59% of its length). The effect of these A-M sets can

be seen on the bottom right plot: J is confined around some of the big islands for a large number of iterates. For some $\delta \in [0.6, 0.8]$ with similar phase portrait and initial point, the number of iterates before the first “crossing” of one of the A-M sets with narrow holes exceeds 5×10^8 .

To see the relative measures of chaotic zone and islands between the confining r.i.c. we have taken a grid with $(\Delta\varphi, \Delta J) = (\pi/3600, 5 \times 10^{-4})$ with a total of ≈ 36.59 Mpixels, the fraction of points in the chaotic zone being 0.6743. To get this figure 10^{11} iterates of the initial point $(0.01, 0)$ have been computed and the related pixels have been marked. The reader is referred to [27] where some numerical simulations were performed to explain the way the chaotic zone is densely covered by a single orbit starting inside. To analyse the points in islands we have studied the connected components of the points outside the chaotic zone. A total of 120 islands, surrounded by points in the large chaotic zone, have been found with a size exceeding 1000 pixels and thousands of smaller islands have been detected. Among the largest islands we have: the central one around P_π^e , the two islands around $P_\pm^e$ and, then, a chain of 8 satellite islands around the central one and two chains of 6 satellite islands around the ones centred on $\varphi = 0$. The relative areas, with respect to the total islands area are: 0.6234, 0.2207, 0.0450 and 0.0495, respectively. All these data have very small changes during last 10^{10} iterates. Checks using different computers, compilers and programs have been done. Even going to 10^{12} iterates the fraction of points in the chaotic zone moves only to 0.6744. A few iterates are able to “penetrate” into the narrow channels which separate some island from one of its satellite islands and some points very close to the boundary of an island are finally detected to have chaotic dynamics.

On the other hand, sometimes the interest is in large regions of instability “located” between the separatrices of two resonant chains of islands. Imagine that, in a situation like the one displayed in figure 3, the r.i.c. on the top are absent. The dynamics is not confined but it can be “effectively confined” due to the existence of A-M sets: the rate of diffusion through them can be extremely small. These zones without r.i.c. located between two chains of resonant islands can be either included in a larger Birkhoff zone or in an open chaotic sea. However, the interest lies not in the whole instability region but in the region delimited by the separatrices of the resonant chains of islands considered. A key point is to decide whether or not there exist r.i.c. between these resonant chains of islands.

A situation where the interest is not in a Birkhoff zone of instability but in the region between two resonant chains appears when studying the evolution of the stability domain (SD) around an elliptic point or in a more general context. Let us clarify what we understand by stability domain.

Definition 1.1 *For a given diffeomorphism $F : U \rightarrow \mathbb{R}^n$, $U \subset \mathbb{R}^n$, and fixed compact sets $K_1 \subset K_2 \subset U$, the stability domain, $\text{SD}(K_1, K_2)$ or simply SD, of F relative to K_1, K_2 is the set of points in K_1 whose iterates remain in K_2 .*

In the definition above it should be understood that we refer to forward and backwards iterates. In some cases it can be interesting to consider only the ones or the other. The following examples could help to clarify the definition:

- In general the choice of the compacts K_1, K_2 must be done according to the dynamical properties we want to study. Consider, for instance, the standard map, STM: $(X, Y) \rightarrow (X', Y') = (X + Y', Y + k \sin(2\pi X))$, with $k = 0.12$ (then the STM has r.i.c.), defined on the cylinder $\mathcal{C} = [0, 1] \times \mathbb{R}$ (see also comments on the STM in section 3). The following could be reasonable definitions of SD:

1. Let $K_1 = [0, 1] \times [-0.5, 0.5]$ and $K_2 = [0, 1] \times [-1, 1]$.
2. Let $K_1 = K_2 = [0.1, 0.9] \times [-0.4, 0.4]$.

Clearly, $SD = K_1$ using definition 1. while it is a set contained in K_1 using definition 2. If we select, instead, $k = 0.2$ (then the STM has no r.i.c) and $K_1 = K_2 = [0, 1] \times [-0.5, 0.5]$, that is, a fundamental domain, then SD consists of a large island around P_e and many other chains of small and tiny islands.

- Let F be an “open” map in the sense described above (an example of an “open” map is the Hénon map). Then, the stability domain can be defined “globally” by taking $K_1 = K_2$ large compact sets for which it is easy to prove analytically that points outside K_1 escape to infinity under iteration (forward or backwards).

Let F_λ , $\lambda \in \Lambda \subset \mathbb{R}$, be a family of APMs depending on the parameter λ . We focus on the stability domain around an elliptic fixed/periodic point P_e (i.e. the sets K_1, K_2 above are chosen large enough). Let $SD = SD(\lambda)$ be the corresponding domain of stability. Moser’s twist theorem [29] implies that if P_e is not at a strong resonance (that is, the eigenvalues are not roots of the unity of order 3 or 4) and some coefficient of the Birkhoff normal form is different from zero, there are r.i.c. around P_e . Let $C(\lambda)$ be the connected component of $SD(\lambda)$ containing P_e , to be denoted also as main component of $SD(\lambda)$. Assume that for $\lambda = \lambda_1$ a chain of islands around P_e is contained in $C(\lambda_1)$ but for $\lambda = \lambda_2 > \lambda_1$ that chain is outside $C(\lambda_2)$. We can refer to this fact in a loose way by saying that the chain of islands is “thrown away” from the main component or that it “moves out” of that component. Of course, changing again λ several chains can be thrown away and there is also the converse possibility: outside chains are “captured” by the main component.

If an expelled chain of islands still has a “pendulum-like” structure the outer separatrices of the islands play a role in the nearby dynamics. The other separatrices playing a main role in the instability region in which we are interested are the ones related to the outermost hyperbolic fixed/periodic point related to the P_e point of the stability domain [44]. As an example consider the Hénon map, depending on a parameter c , which can be written as

$$HM_c : \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} c(1 - X^2) + 2X + Y \\ -X \end{pmatrix}. \quad (2)$$

For $0 < c < 2$ the elliptic E_0 (resp. hyperbolic H_0) fixed point is located at $(1, -1)$ (resp. at $(-1, 1)$) independently of the value of c . When $c > 2$ the elliptic point located at $(-1, 1)$ becomes reflexive hyperbolic. On the other hand, for $c = 0$ the line $Y = -X$ is made of fixed points while for $c < 0$ the elliptic and the hyperbolic points interchange their role via the symmetry $(X, Y, c) \mapsto (-X, -Y, -c)$ (see [40]).

For the map (2) the separatrices of H_0 surround the $SD(c)$ around E_0 . For $c = 1.015$ the order four islands have been thrown away of $C(c)$. This is shown in figure 4 left. For this value of c , there are no r.i.c. between the 1:4 resonance and the separatrices of H_0 : they have been destroyed by the effect of the outer splitting of the separatrices of the order four resonance and by the separatrices of H_0 . In figure 4 right we depict the invariant manifolds of H_0 and of the order four resonant islands. If we continue the separatrices of H_0 long enough they must intersect the order four ones.

Remark. We have used the 1:4 resonance of HM_c (which is known to be a non-generic strong resonance, see [40]) but we shall focus along this work on generic features which are more clearly observed in this 1:4 resonance: destruction of r.i.c, difference of width of the stochastic zones in different regions, existence of tiny islands, global bifurcations of the invariant manifolds, ...

Inside the large instability zones, either in Birkhoff zones or between two chains of resonant islands, there are no r.i.c. but different islands, at least for moderate perturbation size (or,

in other words, for moderate perturbative values of the distance-to-integrable parameter, see section 2.1 for discussion), remain. The destroyed r.i.c. give rise to “cantori” or A-M sets with Cantor structure. To analyse the dynamics in these large instability zones we introduce a new qualitative model: the biseparatrix map. Section 5 contains a full description of the model with several examples.

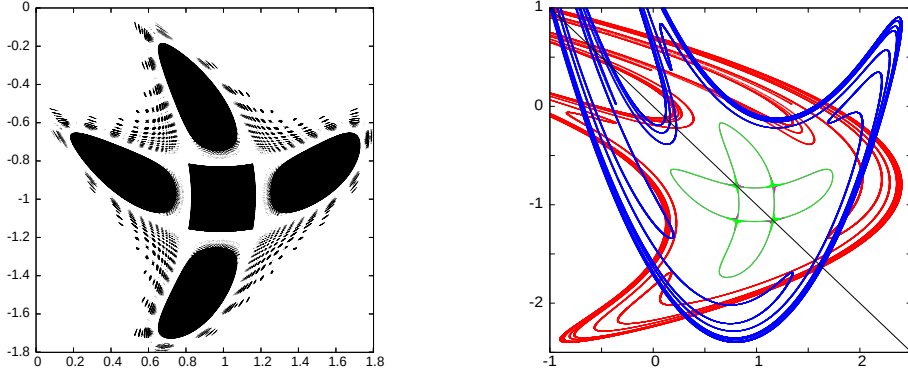


Figure 4: Hénon map (2) with $c = 1.015$. Left: Stability domain. We observe that the 1:4 resonance has been “thrown away” from the connected component of E_0 . Right: Invariant manifolds of H_0 , which have been continued for a relatively large value of the arc length, and the ones of the order four resonance.

The main interest to introduce the biseparatrix map is to be able to analyse whether the large zone of instability exists, that is, to study the destruction of the “outermost” r.i.c. in the considered domain. The biseparatrix model provides just a qualitative description of the breakdown of the “outermost” r.i.c. and of the creation of the instability zone. A further development is necessary in this direction to provide quantitative data for the application to concrete problems. Nevertheless, the model we will introduce allows, for example, to explain qualitatively the evolution of the SD of an APM or the effect of the twistless property in the domain considered (see section 5). In this sense, the biseparatrix map constitutes a first step towards the global description of these zones by means of global return maps.

The three models, that is, the separatrix, double separatrix and biseparatrix maps, are examples of return map models. These types of models are useful to analyse not only qualitatively but in some cases also quantitatively the global dynamics even in non-perturbative regimes.

To conclude this Introduction we note that our study fits within the general aim of Chirikov’s work (see [5]). For the systematic study of chaotic zones we glue several models that capture the relevant aspects of the dynamics in each zone. The modelling approach has to be done at different scales to analyse the dynamics. For every studied property one has tried to complement the limit analytic study of the models with realistic numerical simulations of the system itself. Both approaches are systematically compared.

Along this work $F : U \rightarrow \mathbb{R}^2, U \subset \mathbb{R}^2$ will denote an area preserving analytic nearly integrable map expressed in Cartesian coordinates $Z = (X, Y) \in U$. Our interest, as detailed before, is to analyse some of the dynamical aspects of the original map F . In many cases we use the Hénon map HM_c (or HM_α) as F but also the standard map STM or others (like time-1 maps as used before) are considered. An elliptic (resp. hyperbolic) fixed point of F will be denoted by P_e (resp. P_h), and different superscripts like $P_{h/e}^\pm, P_{h/e}^0, \dots$ will be used to distinguish different fixed/periodic points of F . Eventually we will denote by E_0 an elliptic fixed point of F when the interest is not close to it but in the $(q:m)$ -resonant chain surrounding it, and we keep the notation P_e for the m -periodic elliptic points of the resonant chain related to E_0 . We will refer

by $W^s(P_h)$ ($W^u(P_h)$) to the stable (unstable) manifold of a hyperbolic fixed point P_h and P_{hom} will refer to a homoclinic point of the map F . The distance-to-integrable parameter related to F (see 2 for a discussion on this parameter) will be denoted by ϵ .

On the other hand, to analyse the dynamics of F we use different models. The models are expressed in coordinates (x, y) . These are adapted coordinates defined either in a fundamental domain $\mathcal{U}$ (or $\mathcal{W} = \mathcal{U} \cup \mathcal{D}$ for the double separatrix map DSM) or in a specific domain $\mathcal{D}$ between two resonant chains of islands for the biseparatrix map BSM. In each of the models the relation between the coordinates (x, y) and the Cartesian coordinates (X, Y) in which F is expressed will be detailed.

Remark. Note that sometimes a concrete map can play the role of the original map F or the role of a model depending on the context. This is the case, for instance, of the STM which is a model in sections 3.2 and 4.1.1 that helps to study the existence of r.i.c. of any other original map F (and we consider F to be the Hénon map HM in the examples) while it becomes the original map F in section 5.2.1 (and we use the biseparatrix map BSM as a model to studied it) as it was also the case in this Introduction for the SD illustrations.

2 Some preliminary considerations

In this section we attempt to set this work into the proper research context. We discuss on what we understand by distance-to integrable parameter ϵ and we distinguish between the *a priori stable/unstable* cases in terms of the dependence of the hyperbolicity of the system on ϵ . Moreover, we overview some properties of the separatrix map and we discuss if they can be applied in the *a priori stable case* which will be considered in the following sections.

2.1 The distance-to-integrable parameter

A parameter which is relevant for our purpose is the *distance-to-integrable parameter*. Which is the suitable parameter and its effects depend strongly on the problem that we consider, even if the topological situation is similar. We list some simple ways to do it in the framework of APM's.

1. Assume that the integrable map is given by the time-1 flow of a Hamiltonian $\mathcal{H}_0$ having a homoclinic loop (like the “fish” Hamiltonian) related to a hyperbolic point P_h^0 , and that the map F is obtained as the time-1 flow of a Hamiltonian $\mathcal{H}_0(x, y) + \epsilon \mathcal{H}_1(x, y, t)$, being $\mathcal{H}_1$ 1-periodic in t . The hyperbolic fixed point becomes $P_h(\epsilon) = P_h^0 + \mathcal{O}(\epsilon)$. Then, the usual Melnikov methods allow to predict a splitting of separatrices which is, in general, $\mathcal{O}(\epsilon)$. In this case the dominant eigenvalue at $P_h(\epsilon)$ is bounded away from 1 and we refer to that situation as *a priori unstable*. The hyperbolicity is already present when $\epsilon = 0$. One can take ϵ as the distance-to-integrable and its effect is of the same order of magnitude.
2. Assume that the integrable map is given by the time- ϵ flow of a Hamiltonian $\mathcal{H}_0$ similar to the previous case and that the map F is obtained as the time- ϵ flow of $\mathcal{H}_0(x, y) + \epsilon^r \mathcal{H}_1(x, y, t)$, with $\mathcal{H}_1$ as before, and where the exponent r can be positive or zero. This would be the case for the Hénon map (2) for small values of c and a suitable scaling (see [40]). In the limit $\epsilon \rightarrow 0$ there is no dynamics. The usual suspension and averaging techniques (see, e.g., [2] based on [30]) or, alternatively, the arguments in [10], give an

upper bound for the splitting $s(\epsilon)$ which is exponentially small in ϵ . One can choose as distance-to-integrable either ϵ or $s(\epsilon)$, depending on the purpose.

3. In a neighbourhood of an elliptic fixed/periodic point E_0 of the map F we can be interested in a q/m resonance located at a distance η from E_0 . Then there exists an autonomous Hamiltonian flow interpolating the power m of the resonant Birkhoff normal form of F around E_0 (below $\text{BNF}(F)$ for short), up to some accuracy in a fixed domain [40]. In this context we can define ϵ as a bound of the difference between the Hamiltonian flow approximation and the map in the domain. Bounds for this difference as a function of η are also given in [40]. Note that ϵ depends on the elliptic point E_0 that we consider in the phase space, on the distance η and on a coefficient related to the amplitude of the resonance, coming from the $\text{BNF}(F)$. If we ignore the avoidable resonant terms in $\text{BNF}(F)$, then the dynamics is defined by an integrable twist and there is no hyperbolicity. In other words, the integrable system is *a priori stable*. The hyperbolicity is gone if $\epsilon = 0$.
4. Another possibility to introduce the distance-to-integrable parameter is through an observable of non-integrability. In the case of APM a possibility is the splitting of separatrices, as said before. Then, in a given domain we can choose as ϵ the maximum of the splitting (for instance, using the Lazutkin homoclinic invariant [14, 13] to measure the splitting) of the manifolds of the saddles associated to the islands in the domain. That domain can be close to an elliptic fixed point or in an annulus away from it.

In what follows and through the paper we will refer to ϵ as “the” distance-to-integrable parameter whatever method was used to define it. In general ϵ depends on the domain of interest and varies along the phase space.

2.2 Separatrix map in the *a priori stable/unstable* case

In sections 3 and 4 we deal with the so-called separatrix map SM slightly adapted to different topological structures of the phase space. In particular, in section 3 we derive the model from the APM F we are considering. However, before giving the corresponding details we want to comment on some properties of the SM and to state which of them apply in our setting.

The Chirikov separatrix map describes the dynamics in a close neighbourhood of the separatrices emanating from a hyperbolic fixed point P_h of F . It is given by

$$\text{SM} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b \log |y'| \\ y + \sin(2\pi x) \end{pmatrix}, \quad (3)$$

where (see details in section 3):

- (x, y) are adapted coordinates defined in a fundamental domain $\mathcal{U}$ where the dynamics is analysed. The variable x moves along a fundamental interval of $W^s(P_h)$. The variable y can be seen as a suitably scaled action in the orthogonal to $W^s(P_h)$ direction.
- a can be seen as a shift needed to have the image in the fundamental domain.
- $b = -1/\log(\lambda)$ where λ is the eigenvalue of modulus greater than one of $DF(P_h)$.

Our goal is to analyse the chaotic regions around the separatrices of resonant islands in the phase space of a generic APM F . Assume that a chain of “pendulum like” (q, m) -islands is located at an average distance η from an elliptic point E_0 of F . By P_h we denote the hyperbolic point of the resonant “pendulum like” islands. The distance-to-integrable parameter ϵ and the dominant

eigenvalue λ are related and depend on η . Both parameters ϵ and $\log(\lambda)$ vanish simultaneously at $\eta = 0$ (see section 4.1.1 and [40] for details). In other words, the dynamics in resonant islands fits in the *a priori stable* case according to our discussion in section 2.1. This fact has some consequences on the parameters of SM (although the final expression of SM is the same as if it is derived in the *a priori unstable* case, see also [31]). The following remarks might help the reader familiarised with different derivations of the separatrix map, and can also provide a guide, to the non-familiarised reader, on several approaches to that model.

- 1) Our original system is an APM F , but there are analogous derivations of the separatrix map taking a $1 + \frac{1}{2}$ d.o.f. Hamiltonian system as original system [5, 19, 32, 33, 20, 31].
- 2) Two relevant parameters for the map are the dominant eigenvalue λ at P_h and the size of the splitting A . Typically, in the *a priori unstable* case $\log(\lambda) = \mathcal{O}(1)$ and $A = \mathcal{O}(\epsilon)$. In the *a priori stable* case $\log(\lambda) = \mathcal{O}(\epsilon)$ and $A = \mathcal{O}(\exp(-\text{ctant}/\epsilon^r))$. Furthermore one has to take into account that y has been scaled by A .
- 3) In general, the constant a in (3) depends on λ, A and on the scaling of y . If we *do not scale* y then in the *a priori unstable* case it is of the form $a(\epsilon) = a_0 + o(1)$, $a_0 \neq 0$ and in the *a priori stable* case it is $a(\epsilon) = \mathcal{O}(1/\epsilon)$. These values have to be multiplied in both cases by $|\log(A)|$ if we *do scale* y . And they must be always taken modulus 1.
- 4) The so-called R_ϵ renormalisation usually refers to *a priori unstable* systems of the form $\mathcal{H} = \mathcal{H}_0(x, y) + \epsilon \mathcal{H}_1(x, y, t)$. It is determined by the scalings $\hat{\epsilon} = \lambda\epsilon$, $\hat{h} = \lambda h$, where h is the relative energy measured from the separatrix orbit of the unperturbed system and where λ should verify a suitable condition of the form $\lambda = \exp(Kw_s)$ that implies invariance of the first equation of the separatrix map (3) which is defined (mod 1). Here, K is a suitable constant related to the variational flow along the separatrix of the unperturbed system and w_s is the imaginary frequency of the separatrix orbit (essentially the “flight” time close to the saddle). Details can be found in [19, 20]. In the present SM model the role of ϵ is played by the size of the splitting A , the role of h by the action y and the role of w_s by the constant a .
- 5) The R_ϵ does not, generically, apply to the separatrix map if derived from an APM in the *a priori stable* situation: one needs w_s and ϵ to be independent, at least at first order in ϵ (otherwise the renormalised map has a different constant a). In [19] it was already observed that it is not possible to apply the R_ϵ renormalisation to the resonant islands of the standard map.
- 6) In other words, the imaginary frequency w_s of the separatrix orbit and the perturbation ϵ are related for the model (3) while in a system like $\mathcal{H} = \mathcal{H}_0(x, y) + \epsilon \mathcal{H}_1(x, y, t)$, with $\mathcal{H}_1$ periodic in t , they are not (at least in a first order analysis in ϵ). Hence, in the last situation, the R_ϵ renormalisation applies [19], as it does in the “general” case (*a priori unstable* case) for the separatrix map derived from an APM [31, 46].
- 7) A different renormalisation idea is considered in the works [5, 8]. It refers to the fact that the SM can be approximated by a standard map (see below) which in turn can be considered as the time-1 map of a Hamiltonian pendulum with a perturbation having an infinite number of harmonics. Then, selecting the main harmonic and deriving the SM related to the main island associated to this perturbation, one ends up with a SM different from the initial one but with suitably scaled parameters.

3 Open case: The separatrix map

In this section we assume that the map F has an open structure in the sense described in the Introduction. Hence, the dynamics of F within the stochastic layer generated by the transversal intersection of the separatrix branches can be modelled by the so-called separatrix map (introduced in [49]) and also known as whisker map. In particular, we deal with the Chirikov separatrix map which was introduced in [5] to describe the motion in a vicinity of the separatrix of a perturbed pendulum Hamiltonian obtained from the study of a nonlinear resonance.

It is to be seen as a faithful model: it provides not only qualitative but also quantitative information (at least in a limit sense, that is, in a small enough tubular neighbourhood of the separatrices, exponentially small in ϵ if the splitting is also of this kind). See [31] for a recent overview of different applications of the separatrix map. For instance, estimates on the size of the chaotic zone depend on the distance from the separatrix to the r.i.c. On the other hand, regular motions can exist inside the chaotic zone. These two particular dynamical considerations can be analysed by means of the separatrix map as is going to be explained below.

Although in the present work we focus on the conservative case we should mention that there are also dissipative versions of the separatrix map (see [3, 41]).

3.1 The separatrix model.

The Chirikov SM (3) is a leading explicit approximation to the Zaslavsky separatrix map. The general construction of the Zaslavsky separatrix map relies on two steps: the inner map and the gluing map. The reader is referred to [3] (appendix A) or [31, 39] for a geometrical description of these two steps of the construction. The inner map describes the passage close to the hyperbolic fixed point (see figure 1). It can be explicitly obtained from the normal form analysis around the hyperbolic fixed point P_h of F . The gluing map matches the local coordinates defined for the inner map, which can be analytically extended along the separatrices. It involves more global properties and hence it is not so easily determined.

The relative position of the separatrices determines the gluing map. To fix ideas assume that the local unstable manifold of the hyperbolic point $W^u(P_h)$ extended up to a given fundamental domain (defined between two dynamically consecutive primary homoclinic points, for example) can be seen as a graph of a function $\mathcal{G}$ over the stable manifold in this domain, a situation which holds in the near-integrable case. A first order expression of $\mathcal{G}$ can be obtained from a Poincaré-Melnikov integral (or sum) measuring the distance between the separatrices. Instead, we shall consider a different approach. It is enough to study a fundamental domain going from one homoclinic point to its image. These end points can be identified and this suggests to consider the Fourier expansion of $\mathcal{G}$. Moreover, below we consider that the gluing map is given by the first order harmonic of this function. Indeed, the effect of the higher harmonics is of order two in the amplitude size A of the splitting which should be considered as the natural perturbative parameter (see [5, 6] and also [10] for near-the-identity maps). The corresponding separatrix map, if just the first harmonic is taken, is known as the Chirikov separatrix map. Along this paper, we shall always deal with the Chirikov separatrix map and we will refer to it as “the” separatrix map. The argument in [3] shows that it is generically the suitable one at least in close enough to integrable systems. Furthermore, as we shall see later, the variable y , in a transversal direction to the manifolds, is suitably scaled by the amplitude A of the splitting.

We recall, for reader's convenience, that the separatrix map (3) is given by

$$\text{SM} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b \log |y'| \\ y + \sin(2\pi x) \end{pmatrix}.$$

It is defined on a fundamental domain $\mathcal{U}$, where the dynamics is analysed. It describes the dynamics in a close neighbourhood of the separatrices emanating from a hyperbolic fixed point P_h of F . Parameter a is related to a shift needed to have the image in the fundamental domain (see [3]). On the other hand, $b = -1/\log(\lambda)$ where λ is the eigenvalue of modulus greater than one of $DF(P_h)$.

Remark. The separatrix map as given by (3) is not defined on $y = 0$ neither on its preimages. The dynamics can be considered only for the set Ω of points all whose iterates have $y \neq 0$. We note that Ω^c , the set of points where SM is not defined, is the part of $W^s(P_h)$ in $\mathcal{U}$. In particular, Ω^c is a set of zero Lebesgue measure. In fact, as we deal here with the open case, we should even prevent to have $y < 0$, see figure 5. Points with $y < 0$ approach P_h and move down, losing control on them. Anyway we keep the form (3) because it will be useful for the fully symmetric double separatrix map.

The fundamental domain $\mathcal{U}$ comes from a tubular domain $\hat{\mathcal{U}}$ containing $W^s(P_h)$ between a homoclinic point P_{hom} and $F(P_{\text{hom}})$, after introducing suitable coordinates on it. The domains $\mathcal{U}$ and $\hat{\mathcal{U}}$ are sketched in figure 5. As we are mainly interested in the a priori stable case, we can assume that F is near-the-identity in a neighbourhood of the separatrix, that is, $F(Z) = Z + \mathcal{O}(\epsilon)$ for some small ϵ and $Z \in \hat{\mathcal{U}}$. This maybe implies a reduction of the width of $\hat{\mathcal{U}}$ in the orthogonal direction to the separatrix.

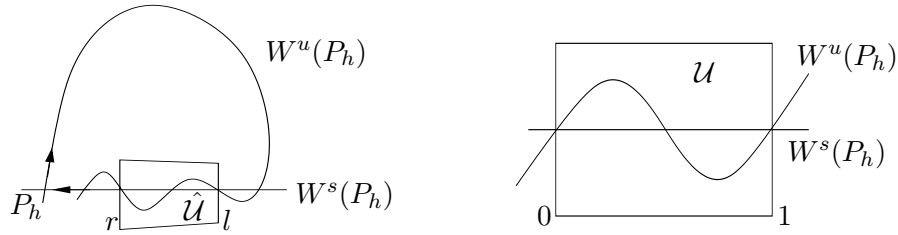


Figure 5: Sketch of the domains $\hat{\mathcal{U}}$ and $\mathcal{U}$.

In particular, the map can be considered to be close to the time- ϵ map of an autonomous Hamiltonian flow $\varphi_t^{\mathcal{H}}$, that is, $F(Z) = \varphi_{t=\epsilon}(Z) + o(\epsilon)$. For instance, assume that $W^s(P_h)$ is close to the stable manifold of the hyperbolic fixed point of the “fish” Hamiltonian $\mathcal{H}(X, Y) = -Y^2/2 + X^3/3 + X^2/2$. Then, a possible choice is to take P_{hom} a homoclinic point of F close to $X = -3/2, Y = 0$. Take a line l through P_{hom} , orthogonal to $W^s(P_h)$. The line l defines the left hand side of the domain $\hat{\mathcal{U}}$ in the original variables (X, Y) . The right hand side r is defined by the relation $r = \varphi_{t=\epsilon}^{\mathcal{H}}(l)$.

The flow box theorem allows to introduce new coordinates x and $\hat{y}$ defined on $[0, 1] \times [\hat{y}_{\min}, \hat{y}_{\max}]$, for suitable $\hat{y}_{\min} < 0, \hat{y}_{\max} > 0$. The x variable is related to the time variable of the interpolating flow scaled by ϵ . Points in l (resp. in r) correspond to $x = 0$ (resp. $x = 1$). The $\hat{y}$ variable is normal to the separatrix of the interpolating flow. In the $(x, \hat{y})$ coordinates the set $\hat{\mathcal{U}}$ is a parallelepiped (see figure 5 right). The values $\hat{y}_{\min, \max}$ should be chosen properly according to the properties of the map we want to study. They must be scaled by the amplitude size of the splitting A , typically exponentially small with respect to the distance-to-integrable parameter ϵ . Hence, the scaling $y = \hat{y}/A$ is carried out to get (3).

The separatrix model is a map from the annulus $\mathcal{U} = [0, 1] \times [y_{\min}, y_{\max}]$ to itself. Coming back

to the original variables, the separatrix map is defined in a fundamental domain $\hat{\mathcal{U}}$ containing the stable manifold $W^s(P_h)$ (which becomes $\{y = 0\}$ in the adapted coordinates in $\mathcal{U}$). The return map to $\hat{\mathcal{U}}$ is defined for all points in $\hat{\mathcal{U}}$ except for those points on $W^s(P_h) \cap \hat{\mathcal{U}}$.

Remark. We can now give some hints about point 3) in the list of remarks of section 2.2. For concreteness we assume we are in the a priori stable case, that one has scaled y by A and that A is exponentially small in ϵ . Other cases are similar. Consider sections Σ_u and Σ_s close to P_h , at a distance d small but fixed, and transversal to $W^u(P_h)$ and $W^s(P_h)$, respectively. Under the flow, the time to go from $\hat{\mathcal{U}}$ to either Σ_u or Σ_s is finite. Hence, the number of steps of F is $\mathcal{O}(1/\epsilon)$. On the other hand, the “flight” time near P_h to pass from Σ_s to Σ_u is $\approx (\log(d) + |\log(y)| + |\log(A)|)/\log(\lambda)$ which has a dominant part $\mathcal{O}(|\log(A)|/\epsilon) = \mathcal{O}(\epsilon^{-2})$.

3.2 Location of invariant curves and islands.

We closely follow ideas of [5, 6] to look for the existence of invariant curves close to the separatrix. By letting $y = y_0 + s$ with $|y_0|$ relatively large, $|y_0| \gg 1$, the separatrix map (3) is rewritten as

$$\begin{aligned} x' &= x + a + b \left(\log |y_0| + \log \left(1 + \frac{s'}{|y_0|} \right) \right) \approx x + \alpha + ks', \\ s' &= s + \sin(2\pi x), \end{aligned} \quad (4)$$

where $\alpha = a + b \log |y_0|$, $k = b/|y_0|$ and the errors are $\mathcal{O}(|y_0|^{-2})$. Then, by setting $w = \alpha + ks$ we get the Chirikov standard map (see [5])

$$\begin{pmatrix} x \\ w \end{pmatrix} \mapsto \begin{pmatrix} x' \\ w' \end{pmatrix} = \begin{pmatrix} x + w' \\ w + k \sin(2\pi x) \end{pmatrix}. \quad (5)$$

In this way, the standard map (5), which can be considered defined on the 2-dimensional torus $\mathbb{T}^2$, approximates the separatrix map (3) for large enough values of $|y|$.

The standard map has been widely studied. For $k = 0$ the map is integrable and the phase space is foliated by r.i.c. (horizontal circles). When k increases some islands appear and the r.i.c. are successively destroyed. In particular, it exists a critical value of k , known as the Greene value $k^* \approx 0.971635/2\pi$ (see [36] and references therein), which has the property of having the “last” r.i.c., that is, for $k > k^*$ no r.i.c. persist. As a consequence, there are r.i.c. for the map (3) provided $|y_0| > |b|/k^*$, meaning that it is expected to have r.i.c. up to a distance

$$d_c \sim |b|/k^* \quad (6)$$

from the separatrix. Note that the relative error in this estimate is $\mathcal{O}(|y_0|^{-1})$ according to the approximation in (4) and it decreases when $|y_0|$ increases, that is, for small values of $\log(\lambda)$. To return to the original coordinates one has to undo the scaling by A . If at some homoclinic point P_{hom} the splitting angle is $\sigma \geq 0$, which is considered to be small enough, and the distance to the image $F(P_{\text{hom}})$ is ℓ , the splitting can be modelled as $A \sin(2\pi\tau/\ell)$, using some local variable τ . Then $A \approx \sigma\ell/(2\pi)$ and we find

$$D_c \sim \sigma\ell/(2\pi k^* \log(\lambda)), \quad (7)$$

as the distance to the r.i.c., which turns out to be of the order of the estimate of the width of the stochastic layer divided by $\log(\lambda)$ (see [31] and references therein). We remark that, if

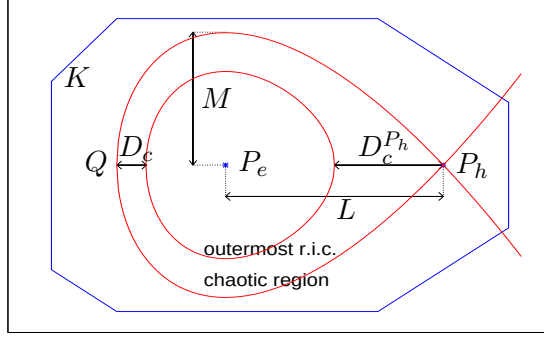


Figure 6: A representation of the “fish”.

the splitting is exponentially small in some relevant parameter, so is then the distance from the separatrix at which r.i.c. exist.

Remark. We can express (7) in terms of the lobe area $\mathcal{A}$. One has $\mathcal{A} \approx \sigma \ell^2 / 2\pi^2$ and then $D_c \sim \mathcal{A}\pi / \ell k^* \log \lambda$. In any case, D_c depends on the homoclinic point considered.

Now we want to see at which distance from the point P_h one can expect r.i.c. First we assume that the map is close to the time- ϵ flow of a “fish” Hamiltonian. The approximation of the map by the “fish” Hamiltonian will be used later in a concrete example (in figure 8 one can observe the fish type topology of the phase space in that situation).

Proposition 3.1 *Consider the Hamiltonian $\mathcal{H} = \frac{1}{2}Y^2 - \alpha X^3 - \beta X^2$, where $\alpha, \beta > 0$ will be made precise later. Let K be a compact whose interior encloses the homoclinic loop formed by the separatrices of $\mathcal{H}$. Assume that, in K , F is close to the time- ϵ flow of $\mathcal{H}$ (for ϵ small enough) and let L be the distance from the elliptic P_e to the hyperbolic P_h fixed points of F . Let D_c be the distance from the invariant manifolds of F to the closest r.i.c. near a point $Q = (-\beta/\alpha, 0)$ on the separatrix opposite to P_h with respect to P_e , as given in (7). Then there exist r.i.c. at a distance from P_h given by*

$$D_c^{P_h} = (3LD_c/2)^{1/2} + \mathcal{O}(D_c^{3/2}). \quad (8)$$

Figure 6 represents the corresponding geometry and the different quantities introduced.

Proof. Let M be the maximum value of Y along the bounded component of the separatrix of $\mathcal{H}$ (see figure 6). It will be apparent later that this value is irrelevant (in a first order approximation of $D_c^{P_h}$). The values of L, M are related to α, β by $\alpha = M^2 L^{-3}, \beta = \frac{3}{2} M^2 L^{-2}$ and P_e, P_h are located at $(-L, 0), (0, 0)$, respectively. The r.i.c. is close to a level curve of $\mathcal{H}$. The level is given, to first order, by $\nabla \mathcal{H}(Q) D_c = -\frac{9M^2}{4L} D_c$, which must be approximately equal to $-\beta (D_c^{P_h})^2$. This gives (8). $\square$

For maps F close to the time- ϵ flow of a more general Hamiltonian or even not close to a flow the estimate of $D_c^{P_h}$ from D_c can be done as follows. First we transport the distance from the $W^s(P_h)$ to the r.i.c. using either the variational of the Hamiltonian flow or the differential DF^{-k} under a suitable number of iterates, k , until we reach a point at a distance r of P_h , close enough to P_h . We are only interested in the transport orthogonal to $W^s(P_h)$. Let Θ be the deformation introduced by this transport. Then, locally near P_h , the map can be approximated by the flow of a Hamiltonian with level lines of the form $XY = \text{ctant}$ in local coordinates. Then we obtain $D_c^{P_h} = (\Theta r D_c)^{1/2}$.

We examine now the existence of islands close to the separatrix (see figure 8). In this discussion we consider islands such that the elliptic point corresponds to a fixed point of the Chirikov standard map (5). Below we provide an approximation of the expected number of these “main” islands. Note that there are other smaller islands of stability (which correspond to periodic orbits of the standard map model) which can be also observed in the figure and that are not considered in the following analysis.

When the r.i.c. are destroyed some islands survive but as $k > k^*$ increases the map (5) becomes more chaotic and the elliptic points become hyperbolic. The fixed points of (5) are $(0, 0)$ (hyperbolic) and $(1/2, 0)$ (elliptic provided $k < 2/\pi =: \tilde{k}$). Hence, the islands around fixed points are destroyed when $y_0 < b/\tilde{k}$. As a consequence, we expect to have islands up to a distance d_i from the separatrix given by a formula like (6) but with $\tilde{k}$ instead of k^*

$$d_i \sim |b|/\tilde{k}. \quad (9)$$

In the same way, formulas like (7) or (8) are obtained for the distances D_i and $D_i^{P_h}$ in original coordinates and to P_h , respectively. If instead of tiny islands close to disappear, with its central elliptic point having trace close to -2, we are interested in islands with its central elliptic point of trace t then the corresponding value of k is $k_t = (2 - t)/(2\pi)$. This allows to obtain the distance $D_{i,t}^{P_h}$ of these islands to P_h . Furthermore, comparing (6) with (9) we see that the ratio of distances to which we can expect to start to have r.i.c. and to have “central” islands is $\approx \tilde{k}/k^*$. From (4) it follows a change by $b \log(\tilde{k}/k^*)$ in the value of x' . Every change by one unit corresponds to an island. Hence, the expected number of “central” islands before the r.i.c. is

$$\#\{\text{islands}\} \approx 1.415 \times b. \quad (10)$$

In figure 7 bottom one can count 15 such islands, in agreement with that estimate for $b = 10$. There are r.i.c. at $y \approx 70$ as can be guessed from the right bottom plot. Hence, the top island on this plot, with $70 < y < 80$, is not of the type we count: it is located beyond the place where r.i.c. are found.

Figure 7 shows the dynamics of SM for $a = 0$ and $b = 1, \sqrt{10}, 10$ in different (x, y) domains. As initial conditions a mesh of points with $x = 0.1, 0.3, 0.5$, and different y values have been taken. The maximal Lyapunov exponent Λ has been also computed for each orbit. If Λ is less than the threshold $\Lambda_c = 2 \times 10^{-5}$ the orbit is considered to be regular and it is plotted in black. Otherwise, it is considered to be chaotic and it is plotted in green.

It is observed in the plots that there are r.i.c. for values of y of the order of 8.2, 21.2 and 68.8 (for $b = 1, \sqrt{10}$ and 10, respectively). Using (6) one gets 6.5, 20.7 and 65.4, quite in agreement with the observations. As said, the relative errors are smaller for large b because then y_0 is larger.

Formulas (6) and (9) give an approximate location of r.i.c. and islands, respectively, of the map (3), in turn an approximation of the dynamics defined by F in a neighbourhood of the separatrix. As a more realistic example we consider the Hénon map in the form

$$HM_\alpha(X, Y) = R_{2\pi\alpha}(X, Y - X^2), \quad (11)$$

where $R_{2\pi\alpha}$ denotes the rotation of angle $2\pi\alpha$, equal to the limit rotation number at the elliptic fixed point $P_e = (0, 0)$ ([18], see also [40] for details and how the “fish” Hamiltonian gives an approximation for α small). The hyperbolic fixed point $P_h = (2 \tan \hat{\alpha}, 2 \tan^2 \hat{\alpha})$, with $\hat{\alpha} = \pi\alpha$ has eigenvalues $\lambda_\pm = 1 + 2 \sin^2 \hat{\alpha} \pm 2 \sin \hat{\alpha} \sqrt{1 + \sin^2 \hat{\alpha}}$. In particular, for $\alpha = 0.1$ one has $P_h \approx (0.64983939, 0.21114562)$ and $\lambda_+ \approx 1.83785279$. In figure 8 left it is shown how the SD is

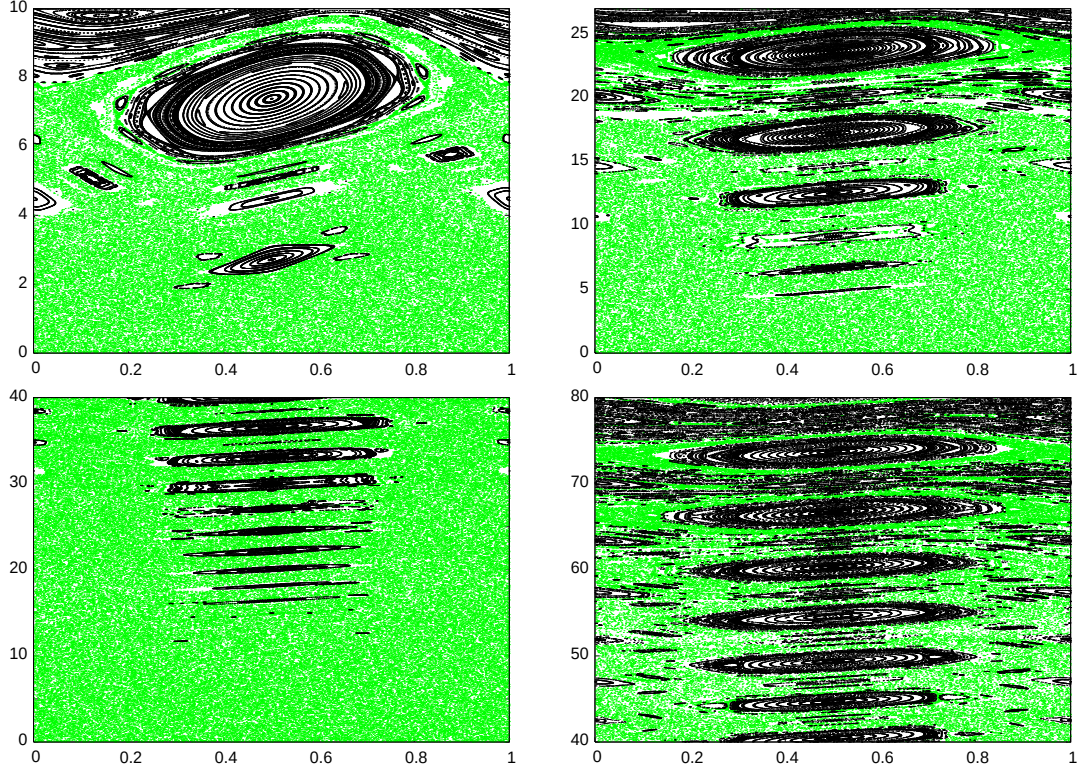


Figure 7: Dynamics of the separatrix map for $a = 0$. First row: left plot for $b = 1$ and right one for $b = \sqrt{10}$. Second row plots done for $b = 10$ in $y < 40$ and $y > 40$, respectively. Black (resp. green) domains correspond to regular (resp. chaotic) orbits. One easily checks that the traces at the centre of the islands decrease from close to 2 to close to -2 from top to bottom.

almost filling up the region delimited by the separatrices of P_h . In figure 8 right, some islands close to P_h are seen, as well as the limit of the stability region. From the computations one obtains the experimental values $(D_c^{P_h})_e \approx 2.94 \times 10^{-3}$ and $(D_i^{P_h})_e \approx 2.08 \times 10^{-3}$.

Numerical computation of the angle between the split separatrices of P_h , measured at the primary homoclinic point $P_{\text{hom}} \approx (-0.3083626, -0.1001931)$ on the symmetry axis $y = \tan(\hat{\alpha})x$, gives $\sigma \sim 1.19 \times 10^{-5}$. Its image is located at $F(P_{\text{hom}}) \approx (-0.1346875, -0.3392363)$ and, hence, the distance between them is $\ell \approx 0.2955$. Formula (7) gives the estimate $D_c \approx 5.95 \times 10^{-6}$. Finally (8) gives $D_c^{P_h} \approx 2.47 \times 10^{-3}$, not too far from the experimental value. Concerning the relatively large islands seen in figure 8 right, it is easy to check that the limit rotation number at the elliptic point is close to $\rho = 0.2279$ and, hence, the trace is given by $t = 2 \cos(2\pi\rho) \approx 0.277$. The corresponding value of k is $k_t \approx 0.274$ and we obtain $D_{i,t}^{P_h} \approx 1.85 \times 10^{-3}$, again not too far from the experimental value, despite α is not very small.

The results can be improved using two modifications in the method. First one is the use of a better Hamiltonian. We begin with the form (2) for the Hénon map, do the change $(u, v) = ((X - Y)/2 + 1, (X + Y)/2)$ so that P_h and P_e are located at $(0, 0)$ and $(2, 0)$, respectively. Let $d = \sqrt{c/2} = \sin \hat{\alpha}$ which has the value ≈ 0.3090167 , not too small. Then

$$\begin{aligned} \mathcal{H}_{\text{impr}}(u, v) = & \left(-1 + \frac{2d^2}{3} - \frac{8d^4}{15}\right)u^2 + \left(1 + \frac{d^2}{3} - \frac{2d^4}{15}\right)v^2 + \left(\frac{1}{3} - \frac{2d^2}{3} + \frac{16d^4}{15}\right)u^3 + \\ & \left(-\frac{d^2}{3} + \frac{2d^4}{3}\right)uv^2 + \left(\frac{d^2}{6} - \frac{2d^4}{3}\right)u^4 + \left(-\frac{d^4}{3}\right)u^2v^2 + \left(\frac{2d^4}{15}\right)u^5 \end{aligned} \quad (12)$$

is a Hamiltonian such that the time- d flow coincides with (2) with error $\mathcal{O}(d^7)$. Expressions like

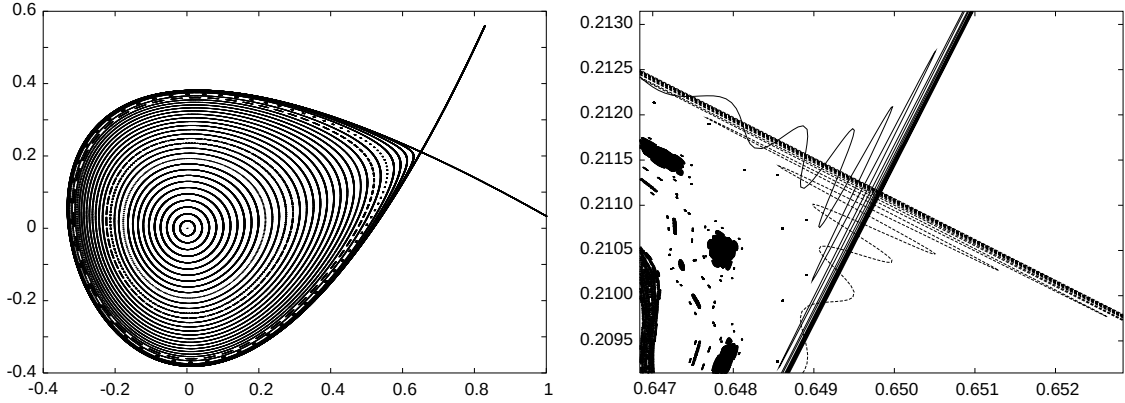


Figure 8: Global phase space (left) and periodic orbits close to the hyperbolic fixed point (right) for the Hénon map ($\alpha = 0.1$). Also the invariant manifolds of the hyperbolic points are shown.

(12) can be extended, formally, to any order in d . If $d = 0$ one recovers the fish Hamiltonian, modulo an affine transformation. In a similar way one can pass from the (u, v) variables in (12) to the variables in (11). Using standard techniques of explicit suspension and averaging (see [2]) it is immediate to obtain bounds for the error between the time- d flow and the map of the form $\exp(-c^*/d)$ for some $c^* > 0$ when the optimal order $N(d)$ is used in expressions generalising (12).

Second improvement deals with the estimate of A . Instead of using as ℓ the distance from P_{hom} to $F(P_{\text{hom}})$ one can locate the homoclinic point which is, roughly, half the way between them and compute its preimage. The arc length along the manifold (it is irrelevant whether we use W^u or W^s) between these homoclinic points gives a better estimate for ℓ , because this arc is “centred” at P_{hom} . One obtains $\ell \approx 0.31728$. With this estimate and working as in proposition 3.1 we obtain $D_c^{P_h} \approx 2.731 \times 10^{-3}$, $D_{i,t}^{P_h} \approx 2.050 \times 10^{-3}$, in better agreement with the experimental results.

4 Figure eight case: The double separatrix map

As observed in the Introduction the homoclinic tangle of a “pendulum-like” resonance is not created just by one splitting of the separatrices but both the inner and outer splittings play a role. Note that this is the case of resonances emanating from an elliptic point even if they are relatively far from it (see [40]). The separatrix map model introduced before can be used to describe the dynamics in tubular domains around the inner and the outer separatrices independently. However the interactions between them are not considered. This leads to the idea of extending the separatrix map to the inner and outer separatrix together.

4.1 The double separatrix model.

Assume that F has a hyperbolic periodic point P_h whose invariant manifolds have a figure eight (“pendulum-like”) structure. To obtain the corresponding model consider a fundamental domain $\mathcal{W} = \mathcal{U} \cup \mathcal{D}$ with $\mathcal{U}$ and $\mathcal{D}$ domains around the outer and inner separatrices of the resonance, respectively. To clarify the meaning of outer and inner separatrices, assume that the map F (or F^m for suitable $m \in \mathbb{Z}$) can be interpolated in a tubular neighbourhood which contains the invariant manifolds of P_h and $F(P_h)$ which bound the resonant island, by a Hamiltonian flow (which is “close” to a pendulum) and express it in action-angle variables (J, ϕ) ; then define the

outer (resp. inner) separatrix to be the one having the first primary homoclinic intersection with larger (resp. smaller) action variable (see figures 12 and 13 for illustrations, and see [40] for more formal definitions).

Domains $\mathcal{U}$ and $\mathcal{D}$ are defined as in the case of the separatrix map. They are related to domains $\hat{\mathcal{U}}$ and $\hat{\mathcal{D}}$ in original variables (see the sketch in figure 9). Then we proceed as in the derivation of the separatrix map, but we add an extra variable s which takes the value $+1$ for points in $\mathcal{U}$ and -1 for points in $\mathcal{D}$. The following generalisation of the separatrix map is obtained,

$$\text{DSM} : \begin{pmatrix} x \\ y \\ s \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_{s,\bar{s}} + b \log |\bar{y}| \pmod{1} \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y) s \end{pmatrix}. \quad (13)$$

The parameters $a_{s,\bar{s}}$ play a minor role in questions like the distance to r.i.c. or islands, but they play a relevant role in the dynamics in the chaotic zone itself. As in the SM case they are related to the shift needed to be in the fundamental domain $\mathcal{W}$. There are four possible ways to pass close to P_h and this gives four possibilities for $a_{s,\bar{s}}$ (see also the remarks below). Parameters similar to $a_{s,\bar{s}}$ were already introduced in [31].

On the other hand, and as in 3.1, $b = -1/\log(\lambda)$, where λ is the dominant eigenvalue of $DF(P_h)$, P_h being at the edge of the “pendulum-like” island. The parameters $\nu_{\bar{s}}$ can be taken such that $\nu_1 = 1$ and $\nu_{-1} = A_{-1}/A_1$, being A_1 and A_{-1} the amplitudes of the outer and inner splittings, respectively. The variable y is such that $y > 0$ means we are outside the manifold W^s either in $\mathcal{U}$ or in $\mathcal{D}$. In particular, if $(x, y) \in \mathcal{U}$ (resp. $\mathcal{D}$), with $y > 0$ it means that $\text{DSM}(x, y) \in \mathcal{U}$ (resp. $\mathcal{D}$). The sketch of figure 9 might help to clarify the definitions above. For simplicity we have plotted the pendulum-like representation using P_h^- and P_h^+ to denote the hyperbolic points that become the point P_h on the figure eight representation (see figure 2 and the transformation details in section 1).

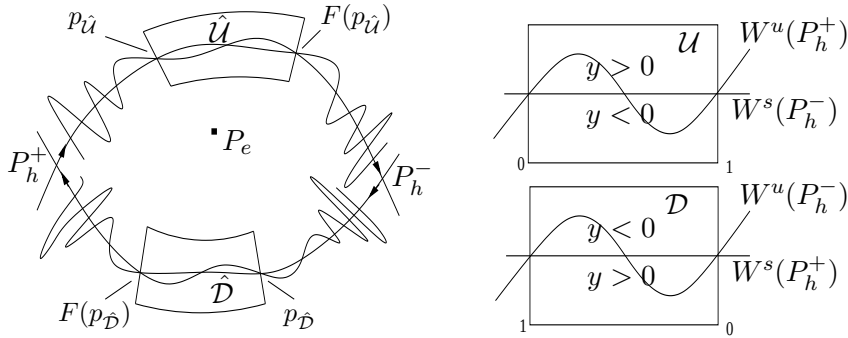


Figure 9: Sketch of the geometry of the fundamental domains $\mathcal{U}$ and $\mathcal{D}$ (right). They are derived from $\hat{\mathcal{U}}$ and $\hat{\mathcal{D}}$ (left), by introducing proper coordinates. The domains $\hat{\mathcal{U}}$ and $\hat{\mathcal{D}}$ are defined with respect to the primary homoclinic points $p_{\hat{\mathcal{U}}}$ and $p_{\hat{\mathcal{D}}}$. In the right plots we show the sign of y (vertical axis) in both domains $\mathcal{U}$ and $\mathcal{D}$, and also the value of x (horizontal axis, from 0 to 1 with the proper sense). Recall that $s = 1$ in $\mathcal{U}$ and $s = -1$ in $\mathcal{D}$ (see text for further details).

Model (13) can be useful for different purposes. For instance, to analyse the transport properties of the map. We note that, in contrast with other topological approaches [32], the DSM reflects the dependency on the size of the splitting. Further investigations should be done in this direction. On the other hand, it can be generalised to a weakly dissipative model, useful to analyse creation/destruction of attractors close to tangencies in the figure eight (see [15]).

Remarks.

1. The parameters $a_{s,\bar{s}}$ are not independent: they satisfy $a_{1,1} + a_{-1,-1} - a_{1,-1} - a_{-1,1} = 0$.

2. The symmetries of F reduce the number of parameters $a_{s,\bar{s}}$ needed in (13). By introducing suitable Cartesian coordinates (X, Y) if necessary we assume that the hyperbolic point of the figure eight is located at $(0, 0)$. Then,
 - (a) If the figure eight is symmetric with respect to the vertical Y -axis, one has $a_{-1,1} = a_{1,-1}$.
 - (b) If the figure eight admits a symmetry with respect to the X -axis, one has $a_{1,1} = a_{-1,-1}$.
 - (c) If both symmetries apply (the fully symmetric case), the four values coincide.
3. Below we use model (13) to estimate the width of the chaotic zones (more precisely, we estimate the distance from r.i.c. and/or islands to the primary intersection of the separatrices in $\mathcal{U}$ and/or $\mathcal{D}$). To this end, the parameters $a_{s,\bar{s}}$ do not play any role (below they will not be considered). They play a relevant role if, for example, one wants to locate islands of stability in the homoclinic lobes between the separatrices. In [42] we study how changes of the parameter a in (3) create/destroy periodic orbits inside these lobes.

4.1.1 Location of invariant curves and islands.

Like in the SM case presented in 3.2, the DSM can be used to predict the distance at which r.i.c. and islands are formed. We focus on r.i.c., the reasoning for islands being similar.

When analysing dynamics outside the resonant island, that is $y > 0$ and s constant equal to 1 (in $\mathcal{U}$ but outside the island) or -1 (in $\mathcal{D}$ but outside the island), the map (13) reduces to a separatrix map (3) after scaling the y variable to normalise $\nu_s = 1$. That is, the scaling depends on the size of the splitting which can be different in $\mathcal{U}$ and $\mathcal{D}$. This means that in the outer or inner part of the resonance we obtain different data concerning distance to r.i.c. and islands (the corresponding distances are obtained applying formula (6) above).

On the other hand, model (13) allows to compute the distance to r.i.c. inside the island. In this case both the inner and outer separatrices play a role that we try now to explain. We use again an energy argument like in 3.2.

Assume that the dynamics of F inside the “pendulum-like” island is modelled by the time- τ flow of an interpolating Hamiltonian $\mathcal{H}(J, \phi)$. This Hamiltonian should also be pendulum-like and we assume that it is expressed in action-angle variables (J, ϕ) . Let h_s be the energy level of the separatrix and $P = (J_s, 0)$ a point in $\mathcal{U}$ (or in $\mathcal{D}$) on the separatrix, that is, $\mathcal{H}(P) = h_s$. Given a point $(J_c, 0)$ on a close enough invariant curve of energy level $h_c = \mathcal{H}(J_c, 0)$ then $h_c \approx h_s + \partial\mathcal{H}/\partial J(J_s, 0)(J_c - J_s)$. Notice that the gradient is not necessarily equal in both domains. The approximation of h_c implies then that J_c is generically different in $\mathcal{U}$ and $\mathcal{D}$ since in the integrable approximation the energy $\mathcal{H}$ is preserved when going from $\mathcal{U}$ to $\mathcal{D}$ and vice versa.

The effect of adding the perturbation coming from the splitting of separatrices is now clear. If the splitting of the perturbed separatrices destroys all the r.i.c. in an “upper strip” in $\mathcal{U}$ between the separatrix $y = 0$ and a curve $\mathcal{C}$, then it defines, by iteration, a “lower strip” in $\mathcal{D}$ between the separatrix $y = 0$ and the image in $\mathcal{D}$ of $\mathcal{C}$. Generically both strips have different width. Obviously, the splitting of the inner curve plays also the same role and implies another strip in $\mathcal{U}$ where we cannot expect to have r.i.c. The size of the stochastic layer in $\mathcal{U}$ (or $\mathcal{D}$) is assumed to be the maximum of the sizes implied by the splitting of the outer (inner) separatrices taking into account the re-injection mechanism explained.

More precisely, each invariant curve of the integrable interpolating Hamiltonian has different energy level, the width of the chaotic region being determined by the energy level of the outermost invariant curve of the resonant island. Of course, the points in an invariant curve of the map have slightly different values of the energy. The outermost invariant curve should be

observed in both $\mathcal{U}$ and $\mathcal{D}$ domains but at different distances. The ratio between the distances, at least in the near-integrable approximation, can be estimated by the ratio of the gradients $\mathcal{H}$ on the inner and outer separatrices. If both gradients and both splittings are equal so are the expected widths of the stochastic layer up and down. The difference between the inner and outer splittings and/or the difference between the gradients can change the expected picture.

We turn our attention to the case of resonant chains of islands emanating from an elliptic fixed point. In [40] it was proved that the inner and outer splittings of the separatrices of the island (or, equivalently, the primary splittings of the separatrices related to both loops of the figure eight representation) are generically of different size. In particular, we expect the width of the chaotic zones to be different around each of the two loops of the figure eight. This difference has been observed in many applications.

Remark. In [8] it was observed the fact that the orbits surrounding P_e are affected by the passage close to the hyperbolic point P_h twice. This results in different size for the chaotic zone inside the island (i.e. in the chaotic region from the separatrices up to an invariant curve surrounding P_e with values of $y < 0$ in $\mathcal{U}$ and $\mathcal{D}$) and outside the island (i.e. in the chaotic region from the separatrices up to a rotational invariant curve with values of $y > 0$ either in $\mathcal{U}$ or $\mathcal{D}$). Our analysis differs from the approach in [8] because

- 1) it shows that the difference of the widths between the inner/outer chaotic zones is a consequence of the difference of the inner/outer splittings which in turn depends on the twist properties of the map [40], and
- 2) it considers the dynamics along the orbit when travelling away from P_h . The dynamics near orbits of an integrable approximation is used to transport distances at the inner and outer parts of the island to any place where the width of the chaotic zone is needed.

Assume F is an APM having an elliptic fixed point E_0 with multipliers $\pm 2\pi i\alpha$, and assume $\alpha \in [0, 1]$ to be close to a resonance, that is, $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$, $(q, m) = 1$, $\delta \in \mathbb{R} \setminus \mathbb{Q}$ relatively small. Let b_1 denote the first Birkhoff coefficient of the Birkhoff normal form of F (BNF(F) for short) around E_0 . If $b_1\delta < 0$ and $m \geq 5$, the BNF(F) has an m order chain of resonances which consists on m islands of width $\mathcal{O}(\delta^{m/4})$ and located at a distance $\eta = \mathcal{O}(\sqrt{\delta})$ from E_0 . Let λ be the dominant eigenvalue of the related hyperbolic point. An integrable approximation, for $m \geq 5$, of the dynamics for such an island is given by the time- $\gamma \approx \log(\lambda) = \mathcal{O}(\delta^{m/4})$ flow of the Hamiltonian vector field generated by

$$\mathcal{H}(J, \psi) = J^2/2 + c_r J^3/3 - (1 + d_r J) \cos(\psi) \quad (14)$$

for suitable $c_r = \mathcal{O}(\delta^{m/4})$ and $d_r = \mathcal{O}(\delta^{m/4-1})$, hence being $c_r \ll d_r$ in a sufficiently small neighbourhood of E_0 (the subscript r refers to “resonance” since the constants c_r and d_r depend on the properties of the resonance considered). Moreover, d_r depends only on b_1 while c_r depends also on the second Birkhoff coefficient, b_2 . After suitable scalings one can consider $d_r > 0$ and $\text{sign}(c_r) = \text{sign}(b_1) \cdot \text{sign}(b_2)$. The variables (J, ψ) are co-rotating coordinates on the m -fold covering space where the original system F around the $(q:m)$ -resonance has been lifted to. See [40] for detailed justification of all these statements.

Let us consider Hamiltonian (14) and, for example, $d_r = 0.5$. The elliptic point and the maximum J_M (resp. minimum J_m) of the outer (resp. inner) separatrix are located on the symmetry line $\psi = 0$. For $c_r = -0.15$, one has $J = J_M \approx 3.199173$ and $J = J_m \approx -1.428648$. Then, the ratio f of the gradients is $f = (\partial\mathcal{H}/\partial J)(J_M)/(\partial\mathcal{H}/\partial J)(J_m) \approx -0.521$. This means that a distance D measured with respect the outer separatrix becomes a distance fD with respect the inner one.

According to the above definition $f \in (-\infty, 0)$. If $c_r > 0$ then $|f| > 1$, while $|f| < 1$ for $c_r < 0$

(recall that $d_r > 0$). For instance, the factor f for $d_r = 0.5$ and $c_r = 0.15$ gives $f \approx -1.422$. For $c_r = 0$ it is clear that $f = -1$ for all values of d_r .

Hamiltonian (14) fits in the general format $\mathcal{H}(J, \psi) = \mathcal{P}(J) - \mathcal{Q}(J) \cos(\psi)$ by taking $\mathcal{P}(J) = J^2/2 - c_r J^3/3$ and $\mathcal{Q}(J) = 1 + d_r J$. A point on $\psi = 0$ at the level of energy h of the separatrix verifies $\mathcal{Q}(J) - (\mathcal{P}(J) - h) = 0$. The corresponding equation related to Hamiltonian (14) has 3 real roots for $c_r \approx 0$. The largest root plays no role since it corresponds to a point located outside the island, the other two are the values of J_M and J_m . However, for large enough values of c_r there exists only one real root meaning that the island has only one of the homoclinic connections (the inner or the outer one, depending on the c_r and d_r coefficients) and the other has been destroyed (the outer part of the island is “open”). In such a case the island is not a “pendulum-like” one.

It is an easy exercise to compute the bifurcation diagram for (14), including changes on the number of fixed points (including escapes to ∞ for $c_r = 0$) and connections between different saddles. The lines in the diagram are given by algebraic functions. The complement consists of 15 open sets in the (d_r, c_r) -plane. Figure 10 displays these sets and the phase portraits of some of them, when a connection of different saddles occur. Note that the island is of “pendulum-like” type only in the domain bounded by $A3B4CD1E$ in the plot. The maximal number of fixed points is 6 and it occurs on the domains containing points marked 2, 4, 5. This value changes by 2 when crossing bifurcation lines through generic points. Note the existence of a codimension two point marked as C . For $d_r = 0$ the Hamiltonian coincides with the popular “reconnection model” used in the study of non-twist APM [34].

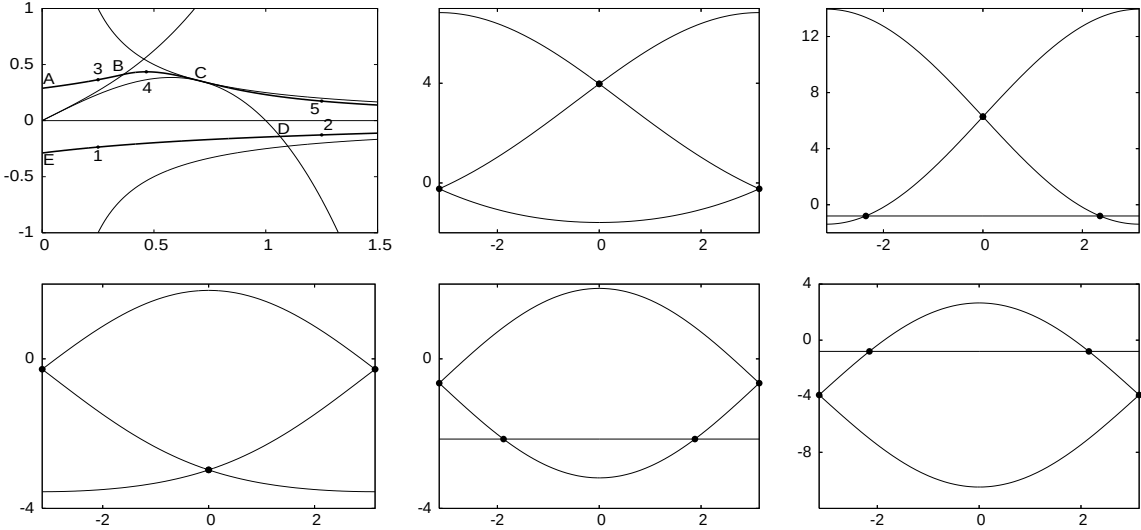


Figure 10: Top left: Bifurcation diagram for (14) as a function of (d_r, c_r) . The thin curves correspond to changes in the number of fixed points or its stability character. The thick ones are the values for which separatrices of different saddles coincide. On these curves the points marked from 1 to 5 denote connections, shown on the top (middle and right for numbers 1,2) and on the bottom (numbers 3 to 5 from left to right). See the text for additional information.

The factor f is not defined for all c_r and d_r values. For a given d_r , to have f well-defined, c_r should be in I_{d_r} , an interval depending on d_r . For instance, for $d_r = 0$ it is easy to check numerically that $c_r \in I_{d_r} \approx (-0.289, 0.289)$, while for $d_r = 0.5$ it is checked that $c_r \in I_{d_r} \approx (-0.197, 0.433)$. The ratio f approaches 0 and $-\infty$ at the ends of this interval (see figure 11).

According to the results in [40] for resonances emanating from an elliptic point with multiplier $2\pi i(q/m + \delta)$ one has $d_r = \mathcal{O}(\delta^{m/4-1})$ and $c_r = \mathcal{O}(\delta^{m/4})$ (for δ small enough or, equivalently, for resonances close to the elliptic point, as detailed above). It turns out that $|f| \approx 1$ if this generic behaviour occurs (see details below). However, in non-generic cases (for instance, for the 1:4 resonance of the Hénon map) or for resonances located far away from the elliptic point (that can be modelled with a Hamiltonian similar to (14) but with, in principle, arbitrary parameters c_r and d_r) it turns out that $|f|$ can be much larger than 1 or closer to 0, as observed in figure 11. Note that in this case the width of the stochastic zones close to the inner and outer separatrices can be of different order of magnitude (see below the estimate of the width of the stochastic regions for the 1:4 resonance of the Hénon map).

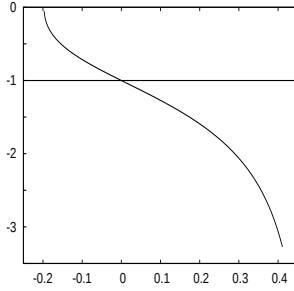


Figure 11: For $d_r = 0.5$ we represent the ratio f (y -axis) as a function of c_r (x -axis).

In order to illustrate the phenomenology for values of the ratio f not too close to -1, we consider $d_r = 0.5$ and two different values of c_r . Figure 12 left shows the island for $c_r = -0.19$ while the right one is for $c_r = 0.4$. The factors are $f \approx -0.23$ and $f \approx -3.064$, respectively. Consequently, one can distinguish clearly in the plots how the distance between the r.i.c. evolves when going from the outer domain $\mathcal{U}$ to the inner one $\mathcal{D}$.

As a summary we note that, assuming both inner and outer splittings to be equal, for $c_r < 0$ it is expected the inner chaotic zone to be narrower than the outer one while for $c_r > 0$ it is expected the inner to be larger. As it is natural the difference between splittings plays a role. We recall that c_r depends on the second Birkhoff coefficient and it turns out that it is necessary $b_2 \neq 0$ to break up the “symmetry” on the r.i.c. in the $\mathcal{U}$ and $\mathcal{D}$ domains (see [40]).

The above considerations give rise to the following general result. We assume that F satisfies

- **A1:** F is an analytic APM around a given fixed elliptic point E_0 , such that the first Birkhoff coefficient at E_0 satisfies $b_1(\delta) = b_1 + o(1)$;
- **A2:** the possible singularities of F remain at a finite distance as $|\delta| \searrow 0$;
- **A3:** if P_h^r is a hyperbolic periodic point on a resonant zone close to E_0 , with dominant eigenvalue λ , assume that the interpolating Hamiltonian flow φ_t has a separatrix $\gamma(t)$ whose closest singularities to $\mathbb{R}$ have $|\text{Im}(t)| = \tau$. One can represent $W_{P_h^r}^u$ and $W_{P_h^r}^s$ also as functions of t , close to $\gamma(t)$ (see [10], [40] and details below for this construction). A suitable periodic function $\mathcal{E}(t)$ measures the distance $W_{P_h^r}^u(t) - W_{P_h^r}^s(t)$. Restrict $\mathcal{E}$ to values of $t \in \mathbb{C}$ of the form $t + i(\tau - \mathcal{O}(\delta^q))$, $t \in \mathbb{R}$ for a suitable $q > 0$. This restriction defines a function G real periodic in t . We require that for any δ , $0 < \delta < \delta_0$, there exist constants $k_1, k_2 > 0$ and $j_2 \leq j_1$, such that $k_1 \delta^{j_1} < |G| < k_2 \delta^{j_2}$ and that the first harmonic c_1 of the Fourier expansion of $G(t)$ verifies $|c_1| > \alpha |G|$, with $\alpha > 0$, a constant independent of δ .

Before stating Theorem 4.1 and for completeness, we make some comments and give details to clarify the role of the assumptions and how to check **A3**. See also [40]. In [10] for general analytic

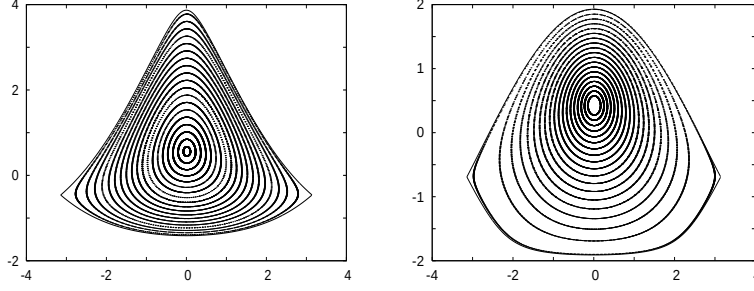


Figure 12: “Pendulum-like” structure of Hamiltonian (14) with $d_r = 0.5$ and $c_r = -0.19$ (left) and $c_r = 0.4$ (right). The different shapes in the phase portrait are translated into a big difference in the distance between r.i.c. in the domains $\mathcal{U}$ and $\mathcal{D}$ (see text for details).

APM of the form $F(z) = z + \epsilon \hat{F}(z, \epsilon)$ a general result was proved concerning upper bounds of the splitting. Only an upper bound was proved because, a priori, F could be integrable (e.g., the time- ϵ flow of a Hamiltonian). Let us comment why the real estimates can be expected to be of the order of the upper bounds:

- In the study of the resonance zone a scaling is applied by some power of δ , see [40]. The possible singularities of F move far away at a distance $\mathcal{O}(\delta^{k_s})$ with $k_s < 0$ and some estimates show that they play no role.
- The invariant manifolds of F at P_h^r can be represented parametrically (see, e.g. [23] for details and examples). W^u can be written as $(x(s), y(s))$ such that $F(x(s), y(s)) = (x(\lambda s), y(\lambda s))$ with $s \in \mathbb{C}$ around $s = 0$. The change $t = \log(s)$ gives a new representation $g_u(t)$ of W^u such that F sends $g_u(t)$ to $g_u(t+h)$, $h = \log(\lambda)$ valid for $\text{Re}(t) < t_u$ for some t_u . F is close to $\varphi_{t=h}$ and, hence, $g_u(t)$ is close to $\gamma(t)$ while $t \in \mathbb{C}$ is in a connected compact set that avoids the singularities of γ . In a similar way we obtain a representation $g_s(t)$ of W^s valid for $\text{Re}(t) > t_s$ and also close to $\gamma(t)$. Both g_u and g_s can be extended on lines of constant imaginary part by $g_u(t+h) = F(g_u(t))$, $g_s(t-h) = F^{-1}(g_s(t))$.
- Around P_h^r the Birkhoff normal form is convergent, uniformly with respect to δ , see [10]. This allows to define an energy $\mathcal{E}$ around P_h^r , $\mathcal{E}(F(x, y)) = \mathcal{E}(x, y)$, and equal to zero on the manifolds. It can be transported in a neighbourhood of $W_{P_h^r}^s$ by iteration. Let us denote simply by $\mathcal{E}(t)$ the function transported up to $g_s(t)$ and then evaluated on $g_u(t)$. By construction it is h -periodic and measures the distance between $W_{P_h^r}^u$ and $W_{P_h^r}^s$.
- As usual, the basic idea is that something very small in $\mathbb{R}$ can be magnified in a complex domain. We restrict $\mathcal{E}$ to a line with imaginary part σ close to (but less than) τ , obtaining a function G . Note that both τ and σ can depend on δ . The fact that γ has singularities with imaginary part τ implies that we should allow for an increase of the norm of G when the line approaches $i\tau$. In a symmetric way we can proceed with $-\tau$. Let $G(t) = \sum_{k \in \mathbb{Z}} c_k \exp(2\pi i t/h)$ be the Fourier expansion of G . Generically c_1, c_{-1} are different from zero. Condition **A3** requires that $|c_1|$ is non-negligible in front of $|G|$. Then, replacing the integral from 0 to h by an integral from $i\sigma$ to $i\sigma + h$, as usual, we obtain the desired exponentially small estimates. Furthermore c_k is $\mathcal{O}(\exp(-2\pi|k|/h))$ showing that only c_1, c_{-1} are relevant. The value of c_0 is bounded due to the existence of homoclinic points in the initial (real) symplectic map.
- It is an easy routine task to obtain g_u, g_s , the normal form around P_h^r and then $\mathcal{E}$ symbolically with numerical coefficients to any desired order for a general APM. Then one can select a suitable value of σ to obtain G and carry out the Fourier analysis numerically to check that the first harmonics are the dominant ones on the real line.

Remarks.

1. If F is given by a closed-form expression then the consequences of the assumption **A3** concerning the behaviour of the splitting may be checked directly. To observe the lack of coincidence of the invariant manifolds one should look into the complex plane, where the exponentially small effects become larger. A systematic way to proceed is to derive an inner (or limit) map describing the dynamics close to τ , and relate the manifolds of the inner map with those of F (matching process), see [21, 22, 17, 45, 12]. In the Appendix A we consider a modified STM (20) as an improved model of the dynamics in resonant islands emanating from an elliptic fixed point E_0 which takes into account the difference between the inner/outer splittings. Starting with this model we derive a suitable inner map and we study the distance between the invariant manifolds of this inner map numerically. This provides an evidence of the exponentially small behaviour, with respect to the small parameter, say ε , of the inner/outer splittings in resonant islands, for all values of $\varepsilon < \varepsilon_0$ for some ε_0 .
2. Recently, V. Gelfreich has announced the following result [11]. Let $\mathcal{F}$ be the set of maps $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that verify: 1) F is of the form $(X, Y) \rightarrow (\bar{X}, \bar{Y}) = (X + \bar{Y}, Y + \epsilon V(X))$, with a 2π -periodic potential $V(X)$, 2) F is analytic in a strip $\mathcal{A}$ of width ζ with respect X, Y , 3) $F|_{\partial\mathcal{A}} \in \mathcal{C}^N$ and $D^{N+1}F|_{\partial\mathcal{A}}$ is piecewise continuous and 4) the origin is a hyperbolic fixed point of F whose invariant stable/unstable manifolds, for $\epsilon > 0$, intersect. Then, generically for $F \in \mathcal{F}$, the splitting of these manifolds asymptotically behaves as $A\epsilon^B \exp(-C/\epsilon)$, for suitable A, B, C which depend on F and N .

Under the conditions **A1**, **A2** and **A3** the outer splitting is larger than the inner one for $|\delta|$ small enough.

Theorem 4.1 *Let F be an APM having an elliptic fixed point E_0 with rotation number $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$, $\delta \in \mathbb{R} \setminus \mathbb{Q}$, assumed to be small, and such that the assumptions **A1**, **A2**, **A3** hold. Assume also that $b_1\delta < 0$. Then, for $|\delta|$ small enough, the width of the outer chaotic zone is larger than the width of the inner one if, and only if, $\text{sign } b_1 \cdot \text{sign } b_2 < 0$. The amplitudes of both stochastic layers are of the order of magnitude of the outer splitting.*

Proof. As $b_1\delta < 0$ the m -resonant chain exists and it is modelled by Hamiltonian (14) being $d_r > 0$, $\text{sign } c_r = \text{sign } b_1 \cdot \text{sign } b_2$, $d_r = \mathcal{O}(\delta^{m/4-1})$ and $c_r = \mathcal{O}(\delta^{m/4})$. For this Hamiltonian $J_M = 2 + d_r + \frac{1}{3}(-4 - 6d_r - 3d_r^2 - d_r^3)c_r + \mathcal{O}(c_r^2)$, $J_m = -2 + d_r + \frac{1}{3}(-4 + 6d_r - 3d_r^2 + d_r^3)c_r + \mathcal{O}(c_r^2)$ and $f = \nabla\mathcal{H}(J_M)/\nabla\mathcal{H}(J_m) = -1 - 8c_r/3 + \mathcal{O}(c_r^2)$. For δ small enough the outer splitting σ_+ is larger than the inner one σ_- (see [40]). Hence the inner chaotic zone has the size of the outer one times f . The effect of σ_- on the outer chaotic zone plays no relevant role since $\sigma_+ \sim \exp(1/\delta) \sigma_-$ for δ small enough (see [40]). If $c_r > 0$ then $|f| > 1$ and the inner chaotic zone is larger than the outer chaotic zone. On the other hand, if $c_r < 0$, $|f| < 1$ and the effect of the σ_+ on the inner zone is smaller than the effect of the inner splitting on the same zone. Anyway, as both c_r and d_r are small, the value of $|f|$ is always close to 1. From the discussion at the beginning of 4.1.1 and the form of (13), the width of the stochastic layers is $\mathcal{O}(\max\{A_1, A_{-1}\})/\log(\lambda)$. As both A_1 and A_{-1} are exponentially small in the current distance-to integrable parameter (a power of δ) the effect of $\log(\lambda)$, which is only a power of δ , is irrelevant. $\square$

Remarks.

1. If $c_r = 0$, that is, if $b_2 = 0$, then one has to consider the effect of the first non-zero term of the form $\tilde{c}_r J^k$, for $k > 2$ in the interpolating Hamiltonian. It turns out that $\tilde{c}_r = \mathcal{O}(\delta^{km/4})$, and then the difference of the splittings plays always a minor role in front the role of the factor f in determining the width of the stochastic layer (for $m \geq 5$).
2. Hamiltonian (14) approximates the dynamics inside a resonant island if it is located close enough to the elliptic point. However, a normal form around a r.i.c. gives rise to the same Hamiltonian up to third order but with different scalings on the coefficients c_r and d_r . Note that the separatrix map approach is perturbative but not local, that is, it does not depend on the proximity to the

elliptic point but to an integrable dynamics. In particular, the same criteria for the existence of r.i.c. can be applied. In this case, however, the inner zone can be larger than the outer and vice versa, depending on the values of c_r and d_r and of the derivatives of the rotation number of an integrable approximation.

4.1.2 Examples

Let us show the effect of the difference of the inner and outer splitting in the location of r.i.c. Figure 13 shows the corresponding period 5 island for the Hénon map (11) with $\alpha = 0.21$. The inner splitting (in terms of the angle measured in the symmetry line) is $\lesssim 3.01 \times 10^{-12}$ while the outer one is approximately 1.412×10^{-3} . The dominant eigenvalue is $\lambda \approx 1.419491$. Observe how this changes the shape of the stochastic layer close to the hyperbolic point shown in the figure. In the inner part (outside the island, left to the hyperbolic point of figure 13 right) the distance to have r.i.c. is expected to be $\mathcal{O}(10^{-12})$ according to (7). Inside the island the distance to have r.i.c. depends on both inner and outer splittings.

For $\alpha = 0.21$ a simple computation shows that $b_1 < 0$ and $b_2 < 0$ (see [40]) hence $c_r > 0$ and the inner zone is expected to be larger than, but of the same order of, the outer one. Figure 14 contains iterates of the point $(0.529402, 0.143704)$ in a neighbourhood of width 8×10^{-5} of the symmetry axis $y = \tan(\pi\alpha)x$, $\alpha = 0.21$. The left one shows a magnification close to the inner zone while the right one is close to the outer zone. The width of both chaotic zones is extremely similar and approximately equal to 6×10^{-4} . This can be explained since the factor f is expected to be $|f| = 1 + \mathcal{O}(\delta^{m/4})$ according to computations in the proof of theorem 4.1.

Concerning figure 14 note that, bounding the chaotic zones close to the points A and B of figure 13, must exist r.i.c. either surrounding the fixed elliptic point at the origin or surrounding the elliptic point of the (1,5)-island. In the left plot we show the r.i.c. obtained by iteration of the points $q_1 = (0, 0.4817)$ and $q_2 = (0, 0.4809)$ in the variables of the plot (which correspond to $\approx (0.380618, 0.295237)$ and $\approx (0.380025, 0.294778)$ respectively, in the variables of (11), compare with figure 13). Iterating q_2 we obtain a r.i.c. (with label (1) in the plot) surrounding the origin. Iterating q_1 we obtain a r.i.c. (with label (5) in the plot) inside the 5-periodic island. This curve is also seen in the domain $\mathcal{U}$, close to B , as shown in the right plot, in which we also show the r.i.c. labelled (1) obtained by iteration of the point $q_3 = (0, 0.82023)$ in the plot variables ($\approx (0.648109, 0.502725)$ in the (11) ones).

The local invariant manifolds related to the hyperbolic 5-periodic points shown in figure 13 extend through the widest horizontal strip of iterated points of each plot, that is, around 0.4811 in the left plot and around 0.82 in the right one. The symmetries of the Hénon map provide the existence of a primary homoclinic point within each of these strips, on the symmetry line $x = 0$ in the figure coordinates (it corresponds to the line ℓ in the original coordinates). Moreover, the white areas in the chaotic zones correspond to islands of stability. For instance, magnifying around the highlighted point, with figure coordinates $\approx (0, 0.481585)$, we find the elliptic island, with period 660, shown in figure 15.

Remark. The rich layer structure shown in figure 13 right causes long time diffusion processes through the stochastic layer. The reader is referred to [47, 48] for a nice numerical exploration of these properties in terms of anomalous transport diffusive processes.

Let us consider the case of the order-4 resonance for the Hénon map (2). We remark that this case is not within the general framework of theorem 4.1 and, moreover, it is expected to have r.i.c. inside and outside the resonant island at a different order of magnitude distance. In figure

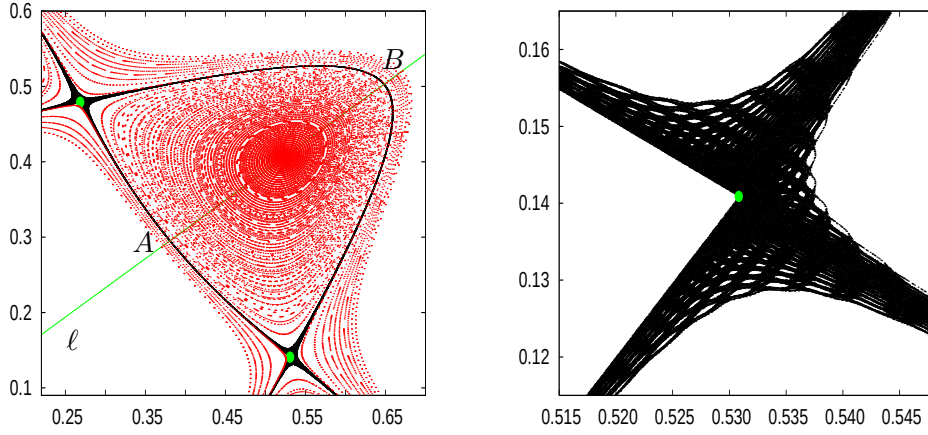


Figure 13: Left: Chaotic layer corresponding to the 5-periodic point of (11) for $\alpha = 0.21$ obtained by iteration of the point $(0.529402, 0.143704)$. Right: A magnification around the hyperbolic point. The position of the hyperbolic points is marked as large green dots in the plots.

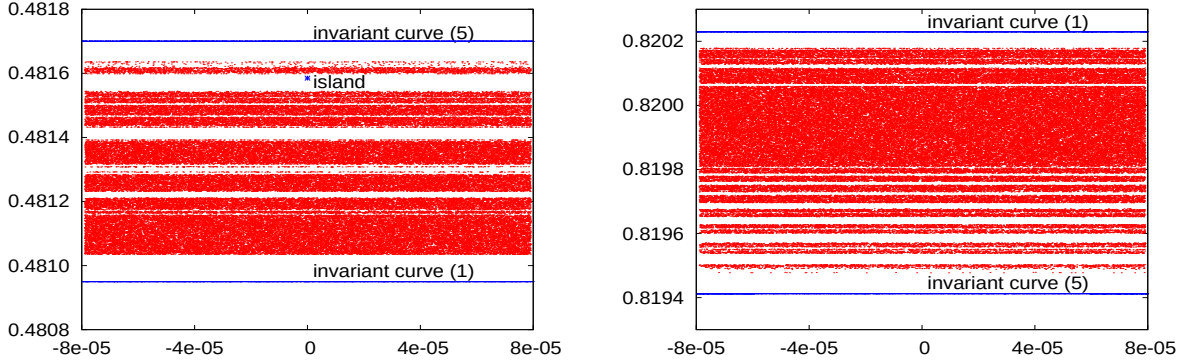


Figure 14: First 10^{10} iterates of point $(0.529402, 0.143704)$, showing only the ones at distance $< 8 \times 10^{-5}$ of the symmetry line ℓ of figure 13. Left: magnification around the inner separatrix (i.e. in $\mathcal{D}$, close to A in fig. 13). Right: around the outer one (i.e. in $\mathcal{U}$, close to B in fig. 13). In both plots the vertical (resp. horizontal) coordinate is taken along the symmetry axis ℓ given by $y = \tan(\pi\alpha)x$ (resp. orthogonal to it). The location of an elliptic point in an island of stability and some r.i.c. bounding the chaotic zones are also shown, see text for details.

16 left, corresponding to the order-4 resonance of the map (2), with $c = 1.015$, we clearly observe different distances in the inner and outer domains. The order-4 resonance of (2) is degenerated and, in particular, it turns out that the inner and the outer splittings are of quite different order of magnitude. Hence, only the transversal intersections of the separatrices corresponding to the outer splitting can be easily observed (figure 16 right).

The inner splitting (in the sense of angle of the splitting measured in the most external-internal homoclinic point) for the value of c shown in figure 16 is of the order 10^{-54} while the outer of order 10^{-1} (see [40]). The dominant eigenvalue of HM_c^4 is $\lambda \approx 1.1284291$. It follows $\nu_{-1} \approx 10^{-53}$ in (13) meaning that the distance we expect to have r.i.c. in the inner part splitting is of the order 10^{-52} according to (7). This explains the sharpness of the central part of the stability domain (it can be observed in the right part of the left figure).

For the distance to r.i.c. inside the island both splittings should be considered. We express the Hénon map HM_c in new coordinates $\hat{X} = 1 + dX$, $\hat{Y} = -1 + dY$, with $d = \sqrt[4]{c-1}$, and denote it

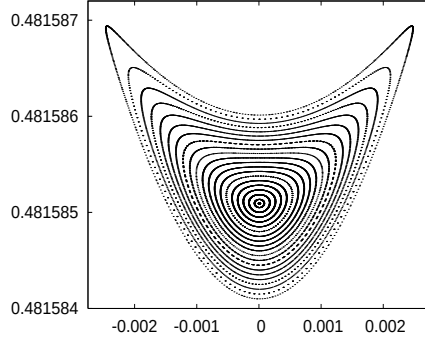


Figure 15: Island of stability around the highlighted point $\approx (0, 0.481585)$ of figure 14 left. Coordinates as in figure 14.

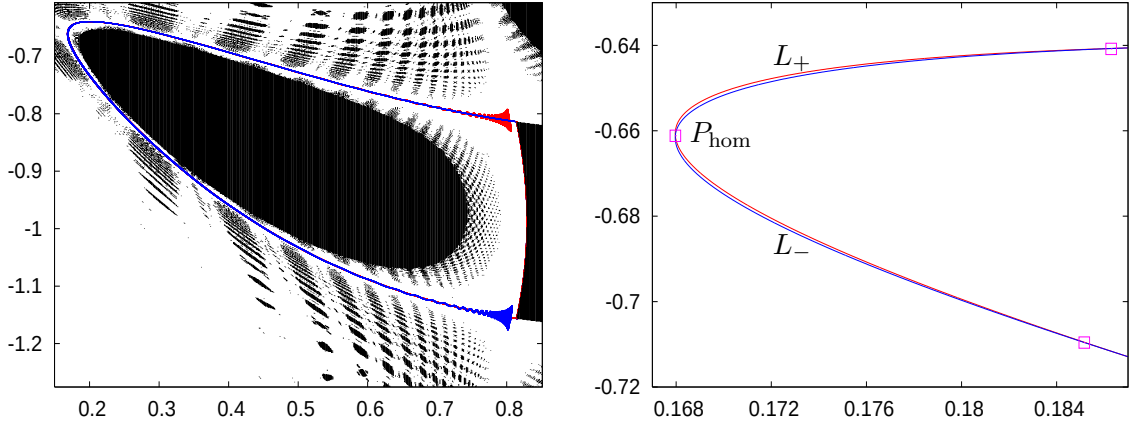


Figure 16: Left: The invariant manifolds of the 4-periodic orbit of (2) $HM_c(X, Y) = ((1 - X^2)c + 2X + Y, -X)$, $c = 1.015$ are depicted. The black regions correspond to points whose orbit does not escape after a large number of iterations of the map (see [44] for details on the algorithm used). Several tiny islands within the stability domain are observed. The r.i.c. limiting the main stability region inside the resonant island can be guessed. Right: Detail of the splitting of the invariant manifolds where the approximated location of homoclinic points is shown.

by HM_d . We use below X, Y instead $\hat{X}, \hat{Y}$, we hope no confusion is produced. The new version of the Hénon map HM_d has the line $r : Y = -X$ and the parabola $QP : Y = X^2 d/2 + X d^4 + X^2 d^5/2$ as reversors. Moreover, HM_d^4 can be interpolated by the time-1 flow of a Hamiltonian $\mathcal{H}(X, Y)$ which can be expressed as a formal power series in d . For our present purposes it is enough to consider the truncated Hamiltonian up to order 3 in d , which is given by

$$\mathcal{H}(X, Y) = -dX^2Y^2 + d^2(X^4Y - XY^4) + d^3 \left(2X^2 + 2Y^2 - \frac{X^6}{3} + 2X^3Y^3 - \frac{Y^6}{3} \right) + \mathcal{O}(d^4).$$

The fixed point (X_4, Y_4) of $\mathcal{H}(X, Y)$ corresponding to the 4-periodic point of HM_d on the line $Y = -X$ and $X < 0$ is such that $X_4 = -\sqrt{2}d + \mathcal{O}(d^4)$. Then, $h = \mathcal{H}(X_4, Y_4) = 4d^5 + \mathcal{O}(d^7)$. There are two intersections, with $X < 0$, of the level $\mathcal{H}(X, Y) = h$ with QP , given by $p_{in} = (-\sqrt{2}d + \mathcal{O}(d^3), d^3 + \mathcal{O}(d^4))$ and $p_{out} = (-\sqrt[4]{24}d + \mathcal{O}(d^2), \sqrt{6}d + \mathcal{O}(d^2))$. The gradient $\nabla \mathcal{H}(X, Y)$ in the tangent to QP direction is essentially given by $\nabla_X \mathcal{H}(X, Y)$. One has $\nabla_X \mathcal{H}(p_{out}) = 8\sqrt[4]{24}d^3 + \mathcal{O}(d^5)$ and $\nabla_X \mathcal{H}(p_{in}) = -4\sqrt{2}d^4 + \mathcal{O}(d^5)$. Taking into account that $d = 2\pi\delta^{1/4} + \mathcal{O}(\delta^{5/4})$ one has $f \approx -2(3/\pi\delta)^{1/4} \approx -1.977\delta^{-1/4}$. If $c = 1 + \epsilon$, $\epsilon > 0$ small, then $\delta \approx \epsilon$. For $\delta = 0.015$ this gives $f \approx -5.64$, while the experimental value observed in figure 16 is close to -5 .

Although the above result concerning the ratio between the width of the chaotic zones seems reasonable enough, the value of δ taken ($\delta = 0.015$) is already too large for the validity of the limit behaviour. In particular, the dynamics in the outside part of the 4-islands in figure 16 is not well-approximated by the dynamics of $\mathcal{H}(X, Y)$. The expected Gevrey character of the series in d of $\mathcal{H}(X, Y)$ is probably the reason why it is not possible to get higher accuracy in the interpolation even when considering approximations of higher order in d . Below we estimate the number of chains of islands surrounding the main 4-order one. This estimation provides an evidence of the fact that the value of δ is too large for the limit behaviour described.

A detailed look at magnifications of figure 16 left (using higher resolution with multiple precision arithmetic) allows to count 15 chains of islands around the islands of period 4. According to (10) one should expect $\approx 1.415b$, where $b \approx 1/\log(1+8\delta) \approx 8.8$. This gives an expected $\#\{\text{islands}\} \approx 12.4$, less than the experimental value, but one has to take into account the following. The chains of islands (and in particular the 15 chains of islands observed) are alternatively organised: there are chains having an elliptic point inside the 4-island displayed in figure 16, close to the central part, on the parabola $QP : Y = c(X^2 - 1)/2 - X$ and chains having a hyperbolic point inside the same zone of the 4-island on QP .

In other words, what plays a role here is the double separatrix map (13) and these islands pass twice, under HM_c^4 , close to the hyperbolic points of period 4. If we neglect the inner splitting, one has to compose (13) with $s = 1$ and with $\bar{s} = -1$ and $\nu_{\bar{s}} = 0$. The effect of the composition is equivalent to use only the map with $s = 1$ and $2b$ instead of b . This would give ≈ 25 islands. Again δ seems to be too large for the validity of the limit behaviour. This is supported by the comparison shown in figure 17 between the theoretical expectations with the number of islands numerically observed where we see a better agreement between the theoretical and numerical expectations for small values of δ .

Finally, we give some comments concerning the existence of islands inside the 4-resonant loop.

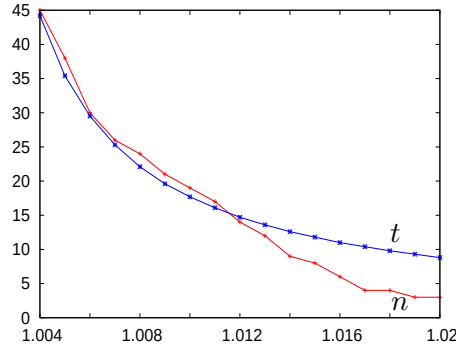


Figure 17: The theoretical prediction (blue line, labelled with t) and the experimental one (red line, with label n) for several values of c . We count the numerically observed islands having an elliptic point on QP in the inner part of the island displayed in figure 16. Hence, the total number of islands considered is, according to the pattern described, half of the total number.

- The islands observed in figure 16 surrounding the main 4-island, are relatively far away from the separatrices related to the 4-hyperbolic points. Recall that the separatrix map for large values of $|y|$ was approximated by a standard map from which it was deduced the existence of these islands.
- These islands “far away” from the separatrices are generically expected to exist for all parameter values of the map.

- But for some range of parameter values can exist islands visiting the homoclinic lobes of the manifolds bounding the 4-loop (see [39, 42]). These islands, if they exist, are located closer to the inner separatrices, very close to the sharp black region at the right central side of figure 16 left. They are not observed in the figure. The size of the islands visiting homoclinic lobes (relative to the area of the lobe) is tiny for most of the parameter values.

The same considerations apply to any “pendulum-like” resonant island.

Remark. For a general diffeomorphism, the fourth order resonance can be studied in terms of Hamiltonian $\mathcal{H}(I, \varphi) = \epsilon I + I^2 + \xi I^2 \cos(4\varphi)$, ξ being a parameter. For the degenerate case $\xi = -1$ it is necessary to consider $\mathcal{H}(I, \varphi) = \pm I + I^2(1 - \cos(\phi)) + \mu I^3(a + b \cos(\phi) + c \sin(\phi))$, $\phi = 4\varphi$, $\mu = |\epsilon|$ and the $\pm$ sign being the sign of ϵ (see [40]). The most relevant case corresponds to $\epsilon < 0$, $\nu = a + b > 0$, which is the case of the fourth order resonance above studied for the Hénon map. The parameter c plays no relevant role. The maximum of the outer separatrix is at $(\varphi, I) \approx (0, (\mu\nu)^{-1/2})$ and the minimum of the inner one at $(\varphi, I) \approx (0, 1/8)$ being $f \approx -1/2 + 3\nu\mu/128 + \mathcal{O}(\mu^2)$ (the gradients have been computed on $\phi = 0$).

5 Dynamics in global zones of instability: The biseparatrix map.

Dynamics in a small neighbourhood of the separatrices bounding a resonant island can be modelled by the separatrix map (3). Relative far from the separatrices the dynamics is also affected by other separatrices related to other resonant structures. To study the effect that the existence of other separatrices has on the r.i.c. and islands we introduce in this section the biseparatrix map. It provides a (very simplified, see comments below) mechanism for the description of the dynamics in Birkhoff zones between r.i.c. in area preserving maps as well as in instability regions between the separatrices of two different resonant chains of islands.

The dynamics of interest concerns the intermediate zone of the instability region. This suggests to define the return model between two chains of resonant islands. On the other hand, we are also interested in instability regions not necessarily confined by r.i.c. Note that the phenomenology in this case, as stated in the Introduction, is the same that in a Birkhoff zone of instability, but points are able to escape to infinity from the region of interest. Usually, Birkhoff zones of instability are defined for twist maps being natural the generalisation to APMs.

Definition 5.1 *Let F be an APM. A Birkhoff zone of instability is a non-contractile annulus between two r.i.c. without r.i.c. inside the annulus.*

Dynamics inside Birkhoff zones is quite complicated. The breakdown of the r.i.c. gives rise to cantori [1, 25, 24, 26]. A cantor is acts as a partial barrier to local diffusion. Certainly the gaps allow trajectories to diffuse but cantori and the tiny islands near them are highly sticky. The stickiness depends on the size of the gaps themselves, on the invariant manifolds of the hyperbolic periodic orbits surrounding the cantor and on the phase space structure of the stability islands close to the cantor [7]. Using renormalisation techniques it was observed that the most robust r.i.c. correspond to noble rotation numbers [16, 24]. As noticed in [38] the renormalisation approach determines the breakdown of a torus for a fixed rotation number but in the generic case the rotation number of the “last” r.i.c. evolves with respect parameters, see [35]. These considerations also hold when dealing with an instability zone between two resonant chains of islands.

Remark. For twist maps it is known that the destruction of a r.i.c. creates a cantor [1, 25]. If, on the other hand, the breakdown of the curve takes place in a non-twist region then one can change variables

to new ones in which the non-twist region of the map expressed in old coordinates becomes a defined twist region in the new coordinates (see [34]).

5.1 The biseparatrix model

Assume we are interested in the dynamics between two concentric chains of islands. More concretely, we look for the dynamics between the upper separatrix of the inner chain and the lower separatrix of the outer chain, and assume that both separatrices are split, as it occurs generically. Assume also that using a suitable, possibly non-linear, radial scaling (and then the preserved measure will be different from the Lebesgue one) the distance between them is $d > 0$. Close to both separatrices we should have a behaviour like the one described by the separatrix map model (3). The simplest *qualitative* model on the domain $0 < v < d$ is given by

$$\text{BSM} : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') - \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}, \quad (15)$$

where $\beta_1 = -1/\log(\lambda)$, $\beta_2 = -1/\log(\mu)$, λ being the eigenvalue of modulus greater than one of the hyperbolic point of the bottom separatrix and μ the corresponding one of the top separatrix. We note that setting $\beta_2 = 0$ we recover (3). The variables (u, v) range on a fundamental domain $\mathcal{D}$ between both chains of islands (as sketched in figure 18 left).

An oversimplification of the general situation has been done: model (15) assumes that the separatrices bounding the interest region are related to hyperbolic orbits of the same period. On the other hand, the splitting of both separatrices has been considered of the same amplitude what means that the vertical coordinate v can be far from uniform with respect the radial variable (transversal to the separatrix) of the original coordinates of F . In this way, the biseparatrix map should be considered as a qualitative model to give an explanation of the breakdown of r.i.c., of the decreasing size or absence of islands when approaching the bottom or top separatrices, etc. It is not intended to extract from it quantitative data (see, however, section 5.2).

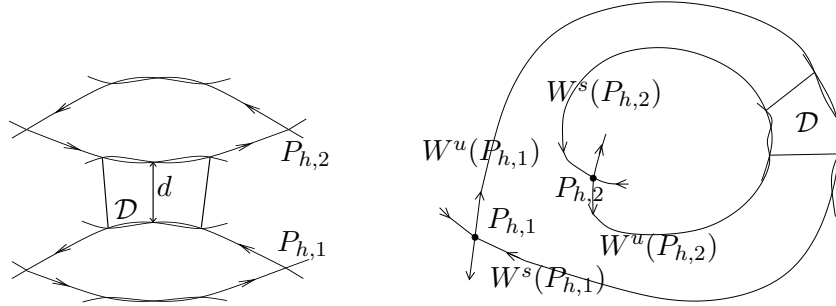


Figure 18: Left: representation of the interest domain $\mathcal{D}$ between two “pendulum” like islands. Right: sketch of the geometry for the biseparatrix map if defined on a figure eight.

Figure 18 shows a scheme of the geometry of the phase space to consider the above model. Consider two chains of resonant m -islands. The geometrical situation for F^m is like the one shown in figure 18 left. Following the steps explained in the Introduction we identify the “two-pendulum” structure with the figure eight. The stereographic projection is done from one of the elliptic points inside one of the islands which is mapped to infinity. The other pendulum structure is then contained inside one of the loops of the figure eight as is shown in figure 18 right. Note that on the left plot can exist r.i.c. at the bottom of the lower separatrix of $P_{h,1}$

and on top of the upper separatrix of $P_{h,2}$, while maybe no r.i.c. appears in $\mathcal{D}$. This would be a model for a Birkhoff zone. Compare with model (1) and figure 3 at the Introduction.

Reasoning as for the separatrix map, by taking $v = v_0 + s$ and assuming both v_0 and $d - v_0$ sufficiently large the map (15) can be approximated by the standard map

$$\text{STM} : \begin{pmatrix} u \\ w \end{pmatrix} \mapsto \begin{pmatrix} u' \\ w' \end{pmatrix} = \begin{pmatrix} u + w' \\ w + k \sin(2\pi u) \end{pmatrix}, \quad (16)$$

where $k = \beta_1/v_0 + \beta_2/(d - v_0)$. Hence, there are r.i.c. provided $k < k^*$. The minimum of k is achieved for $v_0 = d\sqrt{\beta_1}/(\sqrt{\beta_1} + \sqrt{\beta_2})$ and its value is $(\sqrt{\beta_1} + \sqrt{\beta_2})^2/d$. As a consequence, the separatrices have to be far enough in order to have r.i.c. between them, concretely

$$d > (\sqrt{\beta_1} + \sqrt{\beta_2})^2/k^*. \quad (17)$$

In particular, if $\beta_1 \approx \beta_2$ r.i.c. start to appear at distance $v \approx d/2$ when d increases beyond $4\beta_1/k^*$ and $\beta_1 \approx \beta_2$ are kept fixed.

Figure 19 shows the dynamics of the biseparatrix map for $\alpha = 0$ and $(\beta_1, \beta_2, d) = (1, 1, 20)$, $(1, 1, 30)$, $(1, 3, 40)$ and $(10, 12, 250)$, last one exhibiting a lot of islands. The maximal Lyapunov exponent is also computed for the separatrix map, but now $\Lambda_c = 2 \times 10^{-6}$ is taken as threshold to consider the dynamics as chaotic. Obviously, when iterating an initial point all the iterates should satisfy $v \in (0, d)$. Otherwise the point escapes from the annulus between the separatrices and it is removed from the pictures. Figure 20 shows plots of the biseparatrix map for $\alpha = 0$ and $(\beta_1, \beta_2, d) = (10, 12, 250)$ where several islands are observed.

For $\beta_1 = \beta_2 = 1$ formula (17) gives $d > 25.87$. Taking 10^7 initial points on $v = 0$ and performing 10^{11} iterates it is observed d between 26.3 and 26.4, in quite good agreement.

5.2 Global instability zones description by means of the biseparatrix map

We consider an APM $F : [a, b] \times \mathbb{S}^1 \rightarrow [c, d] \times \mathbb{S}^1$ having two resonant chains of islands with rotation numbers q_1/m_1 and q_2/m_2 . Denote by $\hat{F} : [a, b] \times \mathbb{R} \rightarrow [c, d] \times \mathbb{R}$ the lift of F . In a generic situation, $\hat{F}$ is a twist map and the corresponding dynamics in the large instability zone will be analysed in section 5.2.1 using the biseparatrix model. When $\hat{F}$ is not a twist there is a zone in between the two chain of islands with r.i.c. unless there is overlapping of the two chains of resonances. In [3] it was observed that for the non-twist case it is expected to have faster diffusion properties than in the twist case when the last r.i.c. is destroyed. A concrete example of this fact can be seen in [37] for the classical Hill's problem. In section 5.2.2 we adapt the biseparatrix model to describe this situation.

5.2.1 Twist chaotic zone description

Consider the well-known standard map STM (5). As noticed before it is defined on $\mathbb{T}^2$ but, for the purposes of this section, it will be considered defined on the cylinder $\mathbb{S}^1 \times \mathbb{R}$. It has fixed elliptic points $P_{e,1} = (1/2, 0)$ and $P_{e,2} = (1/2, 1)$, while points $P_{h,1} = (0, 0)$ and $P_{h,2} = (0, 1)$ are hyperbolic with separatrices surrounding the elliptic ones giving rise to inner and outer islands.

As mentioned before, for $k \geq k_G/2\pi \approx 0.15438$ the r.i.c. are destroyed and between the two islands a large zone of instability is created. Figure 21 shows the instability zone structure

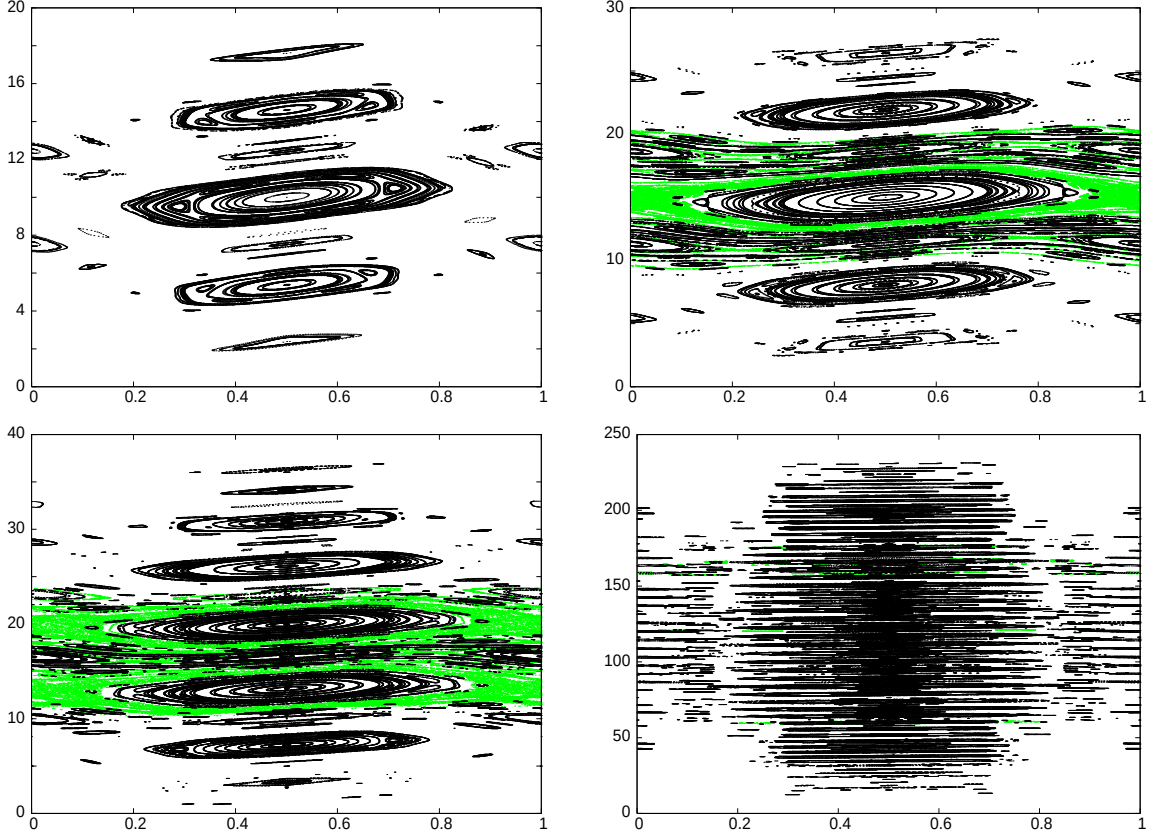


Figure 19: Dynamics of the biseparatrix map for $\alpha = 0$. From left to right and from top to bottom the parameters are $(\beta_1, \beta_2, d) = (1, 1, 20), (1, 1, 30), (1, 3, 40)$ and $(10, 12, 250)$. As in figure 7 black and green denote, respectively, regular and chaotic domains. The white areas correspond to points leaving the domain $0 < v < d$.

for $k = 0.16$ in the central part between the resonant islands of the elliptic fixed points. The manifolds associated to $P_{h,1}$ and $P_{h,2}$ bound the big islands containing $P_{e,1}$ and $P_{e,2}$ which can be partially seen in the bottom right and top left parts of the figure, respectively. Recall that the map has the symmetry $S : (X, Y) \leftrightarrow (1 - X, 1 - Y)$.

As a simple check of coherence of the predictions let us consider the corresponding BSM model. The predictions on the existence of r.i.c. for (15) are based on its approximation by a STM. Now we want to see if we can recover the destruction of r.i.c. for the STM based on the model (15). For $k = 0.16$ we obtain $\beta_1 = \beta_2 \approx 1.0365$ (one has $\lambda = \mu \approx 2.624248$). The amplitude of the splittings (both of them are symmetric by S) depends on where they are measured in figure 21. We have taken the two lobes centred at the homoclinic on $X = 1/2$. The amplitude is ≈ 0.006030 and the distance between the two adjacent homoclinics, measured in the X variable is ≈ 0.338851 , which corresponds to a scaled splitting (to have the u variable ranging in $[0, 1]$ in (15)) $a \approx 0.017796$. Moreover, the distance between the separatrices is $\hat{d} \approx 0.426987$, measured on $X = 1/2$. Then the parameter d of the model should be $d = \hat{d}/a \approx 24$. Finally, we consider $\alpha = 0$.

Figure 22 gives the corresponding picture generated by the BSM model for $d = 27$ and $d = 28$. Initial points are taken on a grid and points escaping from $v \in [0, 1]$ under iteration by (15) are discarded for a more clear representation. The simulations show that there are no r.i.c. for $d = 27$, while they exist for $d = 28$, in good agreement with the theoretical value $d \approx 24$, even taking into account the rough parameter estimates done. Note also the agreement between

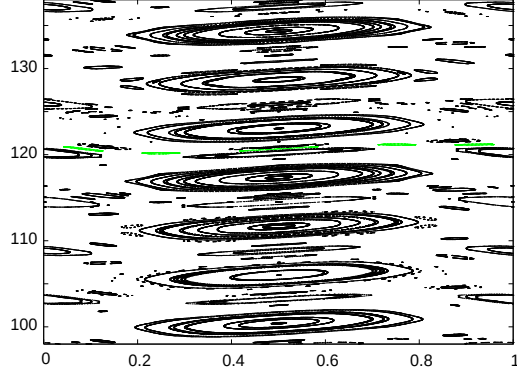


Figure 20: Detail of the islands of the biseparatrix map for $\alpha = 0$ and $(\beta_1, \beta_2, d) = (10, 12, 250)$.

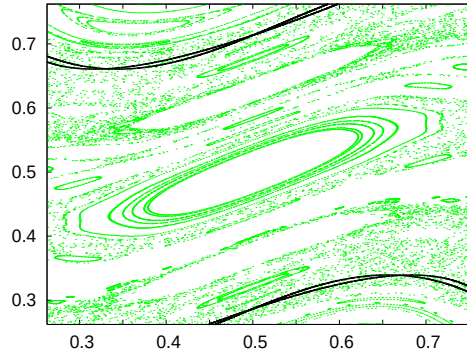


Figure 21: Standard map (5) with $k = 0.16$. Detail of the instability zone between the islands associated to the points $P_{e,1}$ and $P_{e,2}$. The invariant manifolds of $P_{h,1}$ and $P_{h,2}$ are also shown.

figures 22 and 21. For this simple example the model (15) predicts the existence of the different chains of islands giving an idea of its width and also on the parameter value for which the r.i.c. are destroyed, hence predicting the creation of the global chaotic zone. Keep in mind, however, that the STM is periodic in y while the BSM is not.

5.2.2 The vanishing twist case.

As said before in a generic APM there are zones between two chains of islands where $\hat{F}$ is not a twist map. In particular, this can give rise to the existence of meandering curves in the central part of the zone. When the separatrices of the islands responsible of the meanderings destroy them an instability zone appears. However the topological picture is different from the twist case. To describe this situation the biseparatrix map should be modified by changing the minus by a plus in the first equation, that is,

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') + \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}. \quad (18)$$

Different scenarios for different parameters are shown in figure 23. The most relevant fact is the existence of resonant zones which are the largest ones between a strip of central r.i.c. and the separatrices of the chains of islands bounding the instability zone of interest.

As an example take an APM close to the time- ϵ map related to the flow of the Hamiltonian

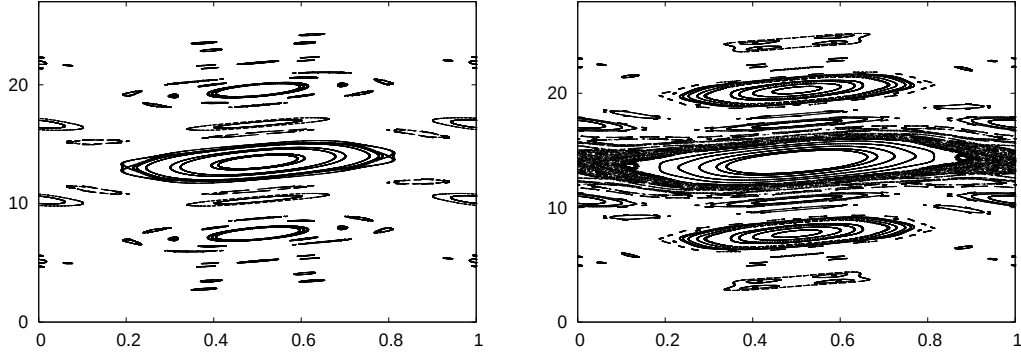


Figure 22: Biseparatrix model with $\alpha = 0$, $\beta_1 = \beta_2 = 1.03649$. Left figure is obtained with $d = 27$ and right one with $d = 28$.

$$\mathcal{H}(X, Y) = -bY + Y^3/3 + \cos(X)$$

$$F_b(X, Y) = (\bar{X}, \bar{Y}) = (X + \epsilon(\bar{Y}^2 - b), Y + \epsilon \sin X). \quad (19)$$

The Hamiltonian mentioned is useful to describe the creation of the so-called meandering curves, see [34]. In figure 24 it is shown the creation of a large non-twist chaotic zone for the map (19). Observe the analogy between figures 23 (top row, left) and 24 (top row, right). We note that the model (18) is able to provide an accurate topological description of the dynamics.

5.2.3 An additional application.

Let us consider the problem of the islands outside the “outermost” r.i.c. around some fixed or periodic elliptic point. Looking at figure 7 one can see that going from the r.i.c. (on top of the plot) down to the chaotic zone, there are islands of decreasing size. A similar thing can be observed in figure 16 left, if we consider the outermost r.i.c. around the period four elliptic point. This is well modelled by the separatrix or double separatrix maps.

Now consider the sketch of the BSM shown in figure 18. One can imagine r.i.c., say $\mathcal{C}$, at the bottom of the lower separatrix of $P_{h,1}$ and on top of the upper separatrix of $P_{h,2}$. Or maybe the separatrices of $P_{h,2}$ have a so large splitting that they prevent the existence of r.i.c. on top of them. We are interested on islands in the region $\mathcal{D}$, on top of the outermost $\mathcal{C}$. Figure 25 left shows an example of such a r.i.c. which has tiny islands around, then they increase size and later on the size decreases again in a quick way, entering in the chaotic zone. In fact, in the example shown, there are no more islands outside the largest one. This phenomenon can not be explained either by (3) or by (13), but as figure 19 shows, can be justified by (15). The discussion after (16) shows that scaling d to 1 and $\beta_j = d\gamma_j, j = 1, 2, v_0 = ds$ then k becomes a function only of s and $\gamma_j, j = 1, 2$: $k = \gamma_1/s + \gamma_2/(1-s)$. A large value of γ_1 and a small one of γ_2 produce that, after several layers of increasing width islands there are few (if any!) outer layers of tiny islands before reaching the chaotic zone, see figure 25 right. We recall that islands with a trace t at the elliptic point are related to a value $k = 2 - 2\pi t$.

6 Conclusions

This work was intended to be a first step towards a systematic description of the dynamics in chaotic zones using simple return maps. In this sense, we have shown how the separatrix and

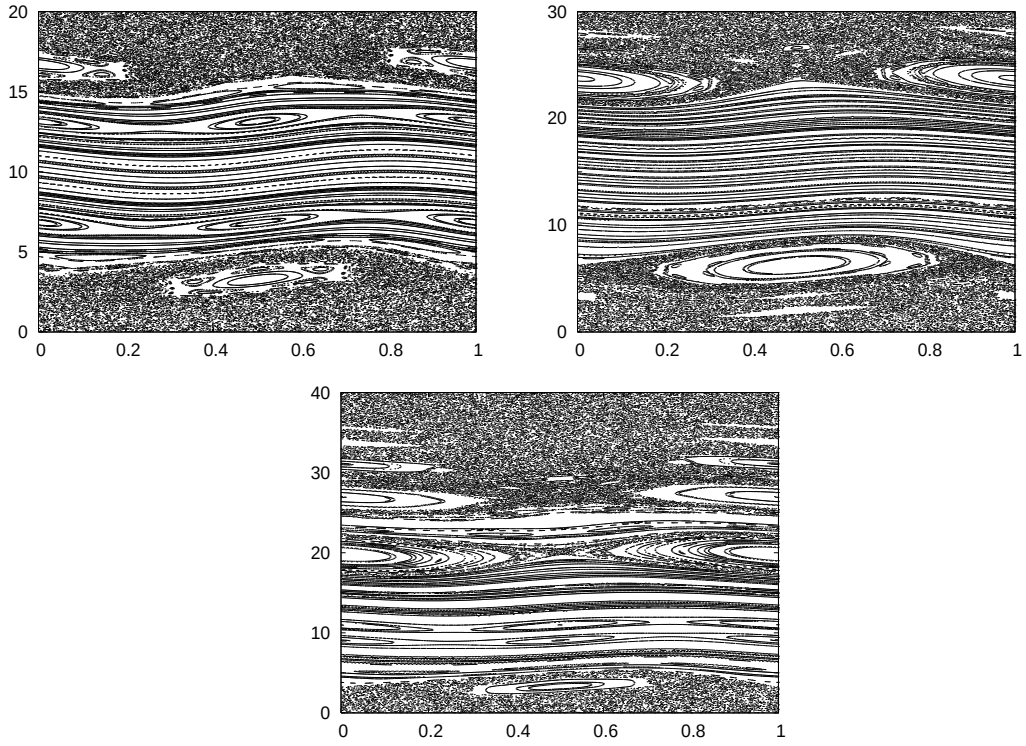


Figure 23: Non-twist Biseparatrix model (18) with $\alpha = 0$. Parameters: $(\beta_1, \beta_2, d) = (1, 1, 20)$, $(\beta_1, \beta_2, d) = (1, 1, 30)$, $(\beta_1, \beta_2, d) = (1, 3, 40)$ (respectively).

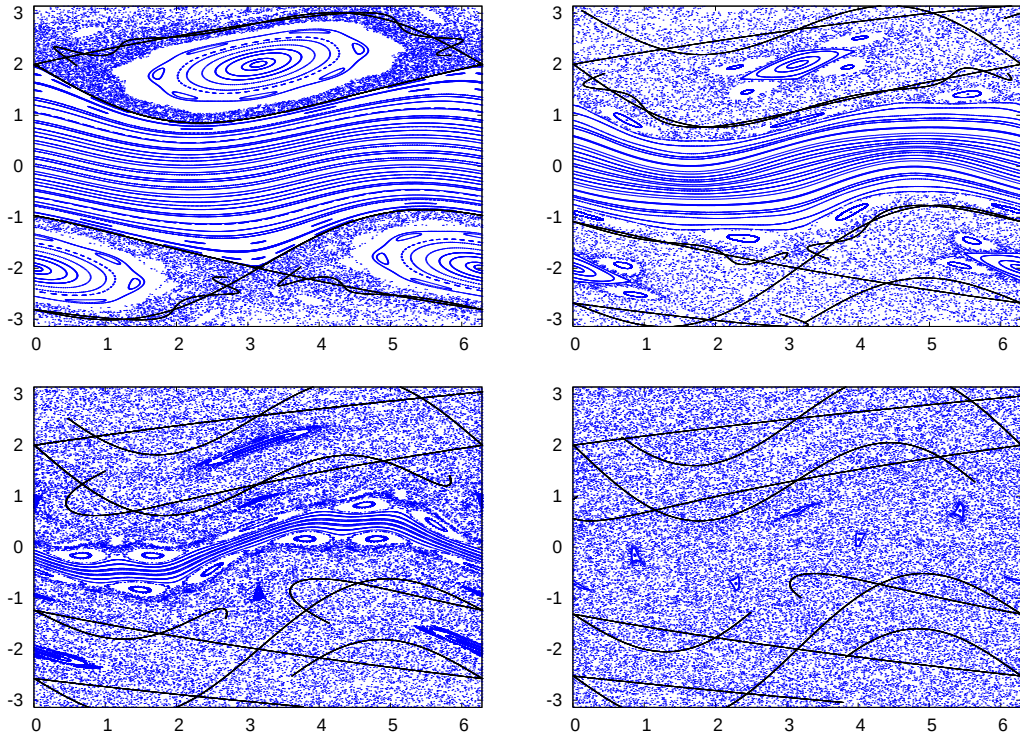


Figure 24: Map (19) with $b = 4$ and $\epsilon = 0.5, 0.75, 1$, and 1.147 respectively. The invariant manifolds of the hyperbolic fixed points are also shown.

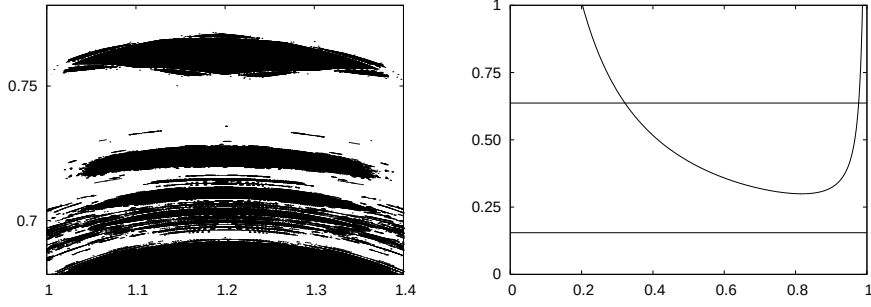


Figure 25: Left: Satellite islands around the outermost r.i.c. of a central island. In this concrete example the Hénon map (2) for $c = 0.531$ has been used. A magnification and an affine transformation have been applied to improve the shape of the plotted islands. Right: Scaled values of k as a function of the scaled value of d for the BSM. Horizontal lines are drawn for $k = k^*, k = \bar{k}$.

the double separatrix models give quantitative information of the stochastic layer in concrete examples. In particular, in a “pendulum-like” island both chaotic zones (close to the inner and outer separatrices) generically are of different size. On the other hand, we have introduced the biseparatrix model to qualitatively describe the dynamics in large twist and non-twist zones created by the interaction of resonances.

Some difficulties on the separatrix model should be addressed in future works. It is not easy for a given family of maps to obtain the parameters of the separatrix model to quantitatively describe the dynamics. Furthermore it has to be studied up to which distance from the separatrix the model gives a good approximation of the real dynamics.

The large instability zones also deserve more detailed studies. We note that nowadays, there is no analytical methodology to obtain quantitative information concerning the destruction of the r.i.c. and the creation of instability zones. Present contribution has to be seen as a first step in this direction. Further investigations should provide suitable quantitative models, maybe numerical ones and/or adapted to the case of study, from which these data will be obtained. A further step towards a quantitative description of global instability should be to relate the different rates of diffusion observed in these zones with the geometry and the remaining objects inside them.

We expect to address some of the questions above in future works.

Acknowledgments

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A The invariant manifolds of the resonant islands: a heuristic justification of why upper bounds are expected to be realistic and how this can be checked

The classical strategy to analyse beyond-all-orders phenomena is to look into the complex plane, close to the singularities of the series involved, where the effects become visible. In particular, for an analytic family F_ϵ of close-to-the-identity maps the splitting of separatrices is a beyond-all-orders phenomenon in terms of the distance-to-integrable parameter (it has an exponentially small upper bound [10]) and its asymptotic behaviour rely on the analysis of the manifolds close to the singularities (the so-called inner problem, see [21, 22, 17, 45, 12]).

In many contexts, one has to deal with maps F_ϵ which are not given in a closed-form (e.g. given as a formal series or obtained numerically as Poincaré maps). However, near a singularity the higher order terms involved in a series expansion of F_ϵ become relevant and, as a general rule, no approximation of F_ϵ will provide full information of such a small phenomena.

This Appendix is devoted to study the splitting of the separatrices surrounding a “pendulum-like” chain of m -resonant emanating from an elliptic fixed point E_0 of a family F_δ of APM’s. It is assumed that E_0 has multiplier $\lambda = \exp(2\pi i\alpha)$ with $\alpha = q/m + \delta$. The dynamics can be described as the time-1 map of a formal Hamiltonian which interpolates a suitable resonant normal form of F_δ [40]. Despite the previous comments one has to approximate the dynamics to get a closed-form model and be able to perform any attempt of analysis.

Following [40] (see also section 4.1.1) we consider an approximation of the time- $\gamma \approx \log(\lambda) = \mathcal{O}(\delta^{m/4})$ map related to the flow of Hamiltonian $\mathcal{H}(q, p) = p^2/2 - (1 + dp) \cos(q)$ as a model of the dynamics in the m -resonant island (for simplicity, in this appendix we denote by d the constant d_r in (14), recall that $d = \mathcal{O}(\delta^{m/4-1})$). The approximation we have considered is described in terms of the generating function $G(q, \bar{p}) = q\bar{p} + \gamma(\bar{p}^2/2 - (1 + d\bar{p}) \cos(q))$ by the relations $\bar{q} = \partial G / \partial \bar{p}$, $p = \partial G / \partial q$. They determine the APM

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} q + \gamma(\bar{p} - d \cos(q)) \\ (p - \gamma \sin(q)) / (1 + \gamma d \sin(q)) \end{pmatrix}, \quad (20)$$

which turns out to be an improved model to the STM (5) describing the dynamics in the m -resonance. The above map is defined for $(q, p) \in [0, 2\pi) \times \mathbb{R}$. The invariant manifolds related to the hyperbolic point $(0, -d)$ show the difference between the inner and outer splittings of the m -resonant island. Properties of this model will be studied in future works [43]. Here we use it to justify why we expect the splittings of the resonant islands emanating from E_0 to generically behave as the exponentially small upper bound, as it implies assumption **A3** of theorem 4.1, and how to check this assumption in that example.

The relevant singularity for the corresponding invariant manifolds are located at $t_*^\pm = i(\pi/2 \mp \arcsin(d))/\sqrt{1-d^2} = i(\pi/2 \mp d + \mathcal{O}(d^2))$, where $+$ refers to the outer separatrices and $-$ to the inner ones (see [40]). As for the STM, the variables q, p behave as $|q| \sim -\log|t - t_*^\pm|$, $|p| \sim |t - t_*^\pm|^{-1}$ close to the singularities $t_*^\pm$. Introducing $q = i(A + u)$, $p = Bv$, with $A = \log(-2i/\gamma d)$ and $B = i/\gamma$, it turns out that a suitable inner map, close to $t_*^\pm$, is

$$\hat{T} : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} u + \bar{v} + e^u \\ \frac{v}{1 + e^u} \end{pmatrix}.$$

The map $\hat{T}$ is a symplectic map with fixed points $v = 0$, $\text{Re } u = -\infty$ ($\text{Im } u$ arbitrary). To

simplify the study of the invariant manifolds of the points at $-\infty$ of $\hat{T}$ we perform the (non-symplectic) change of coordinates $w = e^u$ which gives the map

$$T : \begin{pmatrix} w \\ v \end{pmatrix} \mapsto \begin{pmatrix} \bar{w} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} w \exp\left(\frac{w + \bar{v}}{v}\right) \\ \frac{v}{1 + w} \end{pmatrix}.$$

The map T has a singularity at $w = -1$, while for $w = 0$ all the points are fixed points. We focus on the origin $p_0 = (0, 0)$, which is a parabolic point (the linear part is the identity). An analysis of the quadratic terms shows that the invariant manifolds related to p_0 should have slopes either 0 or -2. The zero slope corresponds to the obvious invariant manifold $v = 0$. The dynamics of T on $v = 0$ reduces to $w \mapsto we^w$ and $w = 0$ becomes a parabolic point (with multiplier 1) of a holomorphic map. Locally around this point the dynamics is conjugated to that of $w \mapsto w + w^2$ inside a cardioid-shape attracting petal. Consequently, the dynamics on $v = 0$ for points in a neighbourhood of p_0 is, for our purposes, trivial: with the exception of a zero measure set either points escape to infinity or return *homoclinically* to p_0 [4, 28]. The global dynamics of T on $v = 0$ is much richer and, in particular, there are other fixed points at $w = 2\pi ik$, $k \in \mathbb{Z} \setminus \{0\}$, which are unstable foci, and also a set of unstable periodic orbits dense in the Julia set, which is a zero measure set (see, for example, [9] and references therein). In any case, the local unstable branch of $v = 0$ coincides with the stable one and we cannot conclude anything concerning the behaviour of the splitting.

A careful analysis of the invariant manifold emanating from p_0 with slope -2 shows that the stable and unstable branches $W_{-2}^{u,s}(p_0)$ do not coincide. Indeed, the distance $\mathcal{D}$ between the stable and the unstable branches measured on the section $Re_w^0 = \{\text{Re}(w) = 0\}$ can be represented as a function of $\beta = \text{Im}(w)$, and it turns out that $\mathcal{D}(\beta) \sim a + b \log \beta + c/\beta$ where $a \approx 7.736$, $b \approx -1$ and $c \approx -2\pi$, with an excellent agreement for values $\beta \leq 0.5$. To compute $\mathcal{D}$ one represents $W_{-2}^{u,s}(p_0)$ as a graph $v = g(w)$ and measures the distance in the v variable. Figure 26 shows the distance $\mathcal{D}(\beta)$ between the stable and unstable branches on $\text{Re}(w) = 0$ as a function of β . Moreover, to provide a strong numerical evidence of the non-existence of homoclinic intersections between $W_{-2}^{u,s}(p_0)$ we considered as a fundamental domain FD the domain between $Re_w^0 \cap W_{-2}^u(p_0)$ and $T(Re_w^0 \cap W_{-2}^u(p_0))$ and computed the distance $\mathcal{D}$ for points in FD. Figure 27 shows the behaviour of the distance between the manifolds in this fundamental domain for $0.15 \leq \text{Im}(w) \leq 0.6$ (measured on Re_w^0). Further details concerning the behaviour of the invariant manifolds of T , the local series representation of them, the Gevrey character of the series, and the numerical computations which provide a rigorous justification of the above statements will be given in [43].

In particular, it follows by continuity that the behaviour of the manifolds showed is inherited by the map (20) since T is derived from it as a limit map. Consequently, the map (20) has an exponentially small splitting, that is, the true behaviour of the splitting is of the order of magnitude of the upper bound (as implies assumption **A3** of theorem 4.1).

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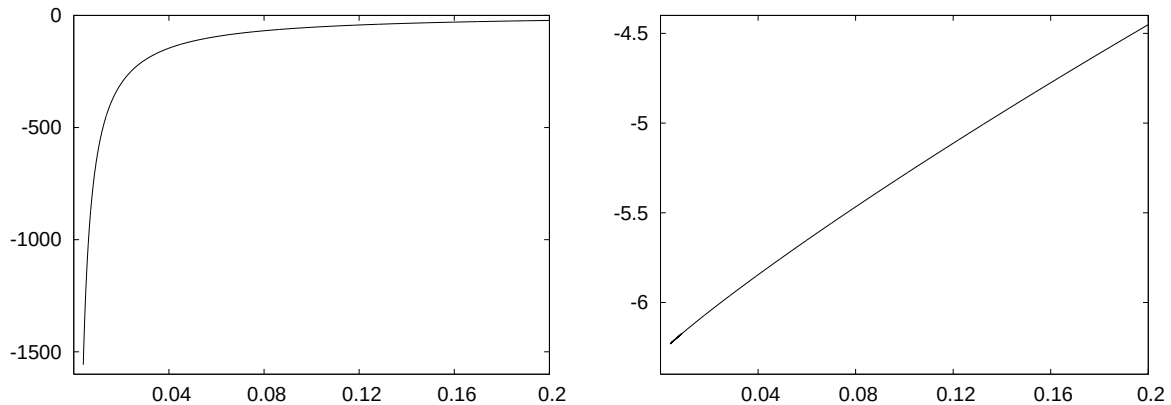


Figure 26: The distance $\mathcal{D}(\beta)$ between unstable and stable branches as a function of $\beta = \text{Im}(w)$ on $\text{Re}(w) = 0$. In the horizontal axis we represent β . Left: $\log(\mathcal{D}(\beta))$ is used as vertical variable. Right: $\beta \log(\mathcal{D}(\beta))$ is used as vertical variable.

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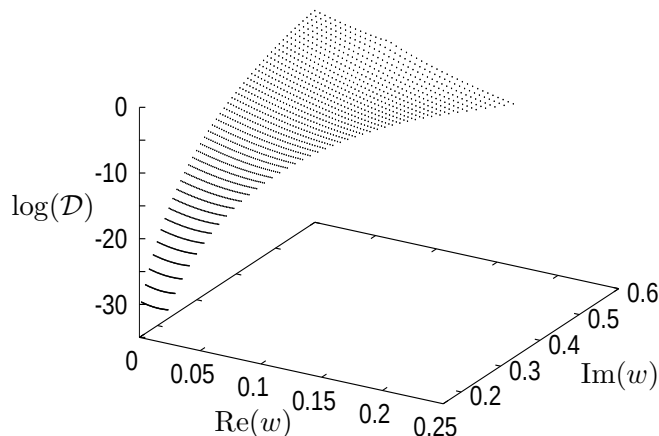


Figure 27: The distance $\mathcal{D}$ between the unstable and stable branches in a fundamental domain. A similar behaviour to the one observed along $\text{Re}(w) = 0$ (see figure 26) is observed in the whole domain, given an evidence of the non-existence of homoclinics between $W_{-2}^{u,s}(p_0)$ of T .

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