PERIODIC SOLUTIONS TO INTEGRO-DIFFERENTIAL EQUATIONS: VARIATIONAL FORMULATION, SYMMETRY, AND REGULARITY

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Abstract. We consider nonconstant periodic constrained minimizers of semilinear elliptic equations for integro-differential operators in $\mathbb{R}$. We prove that, after an appropriate translation, each of them is necessarily an even function which is decreasing in half its period. In particular, it has only two critical points in half its period, the absolute maximum and minimum. If these statements hold for all nonconstant periodic solutions, and not only for constrained minimizers, remains as an open problem.

Our results apply to operators with kernels in two different classes: kernels K which are convex and kernels for which $K(\tau^{1/2})$ is a completely monotonic function of τ. This last new class arose in our previous work on nonlocal Delaunay surfaces in $\mathbb{R}^n$. Due to their symmetry of revolution, it gave rise to a 1d problem involving an operator with a nonconvex kernel. Our proofs are based on a not so well-known Riesz rearrangement inequality on the circle $\mathbb{S}^1$ established in 1976.

We also put in evidence a new regularity fact which is a truly nonlocal-semilinear effect and also occurs in the nonperiodic setting. Namely, for nonlinearities in C^β and when $2s+\beta<1$ ($2s$ being the order of the operator), the solution is not always $C^{2s+\beta-\epsilon}$ for all $\epsilon>0$.

1. Introduction

1.1. **A new symmetry result for periodic solutions.**

It is well known that bounded solutions to the semilinear second order ODE $-u''=f(u)$ in $\mathbb{R}$ are —up to a multiplicative factor ± 1 and up to translations— either increasing in $\mathbb{R}$, or even with respect to 0 and decreasing in $(0,+\infty)$, or periodic. This follows immediately from the fact that u must be even with respect to any of its critical points[1] after considering the cases when u' vanishes at none, only one, or at least two points.

It is then natural to ask whether this classification also holds true for the fractional Laplacian, that is, if, for every $0<s<1$, any bounded solution u to

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R} \tag{1.1}$$

belongs to one of the above mentioned three categories. Recall that

$$(-\Delta)^s u(x) := c_s\,\mathrm{P.V.}\int_{\mathbb{R}} dy\,\frac{u(x)-u(y)}{|x-y|^{1+2s}}, \qquad c_s := \frac{s4^s\Gamma(1/2+s)}{\sqrt{\pi}\,\Gamma(1-s)}. \tag{1.2}$$

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[1] This is a consequence of the uniqueness theorem for ODEs: if u' vanishes at $x=0$, say, then $v(x):=u(-x)$ solves the same equation and has same initial position and derivative at $x=0$ as u.

Due to the lack of an analogue uniqueness result for (1.1), the aforementioned argument can not be carried out in the nonlocal case.

The first author learned this question from Solà-Morales when working on our joint paper [19]. The paper dealt with equation (1.1) for $s = 1/2$. To the best of our knowledge the conjecture is still wide open. It is only known to be true for the equation

$$(-\Delta)^{1/2}u = -u + u^2 \quad \text{in } \mathbb{R} \tag{1.3}$$

modelling traveling wave solutions to the Benjamin-Ono equation in hydrodynamics [8, 44] and for the Peierls-Nabarro equation

$$(-\Delta)^{1/2}u = \sin(\pi u) \quad \text{in } \mathbb{R} \tag{1.4}$$

from continuum modeling of dislocations in crystals [43]. These are two really special equations since they are "completely integrable". More precisely, Amick and Toland [2] were able to find, with explicit expressions, all bounded solutions to (1.3). This required $f(u)$ to be exactly $-u+u^2$ and $s = 1/2$.[2] Later, in [49] Toland also found all bounded solutions to the Peierls-Nabarro equation using the fact that a derivative of any solution is the difference of two solutions to the Benjamin-Ono equation.

Both equations (1.3) and (1.4) admit infinitely many periodic solutions of different amplitude. Equation (1.3) admits, on the other hand, a "ground state" solution; that is, an even solution which is decreasing in $(0, +\infty)$. Instead, (1.4) admits a "layer solution", that is, a solution that is increasing in all of $\mathbb{R}$.

In Section 1.5 we will describe other dispersive equations (besides the Benjamin-Ono equation) that lead, when looking for standing or traveling wave solutions, to periodic solutions of semilinear elliptic integro-differential equations fitting in our general framework below.

Back to the fractional Laplacian, for nonlinearities different than the two above there are few works that classify solutions or prove symmetry and monotonicity properties. Let us mention the main ones, including also some of those that establish Sturm-Liouville type results for integro-differential equations —and thus the results are obtained in the absence of an ODE existence and uniqueness theorem. We focus mainly on results on the evenness and monotonicity of certain solutions, since the current paper concerns these properties in the periodic framework.

Chen, Li, and Ou [21] adapted the moving planes method to the fractional Laplacian and proved the even symmetry of positive solutions tending to zero at infinity (sometimes called "ground states").

Still regarding "ground state" solutions, an important 2013 work of Frank and Lenzmann [35] established the uniqueness and nondegeneracy of a constrained minimizer in $H^s(\mathbb{R})$ for subcritical equations of the form

$$(-\Delta)^s u = -u + u^p \quad \text{in } \mathbb{R}.$$

For this, they first prove a nondegeneracy theorem for the linearized equation $(-\Delta)^s v + v - pu^{p-1}v = 0$ at a ground state u. Here, the essential idea relies on a suitable substitute for Sturm-Liouville theory of ODEs to study the sign-changes of the second eigenfunction of an operator $(-\Delta)^s + V(x)$. Then, by means of an implicit function theorem, the nondegeneracy result is used to construct a unique branch of solutions depending on the parameter s. To conclude uniqueness of the solution, they extend the branch up to $s = 1$ and use the uniqueness result for ODEs when $s = 1$.

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[2] Their method consisted of identifying the equation as the boundary relation with respect to a local one in the upper half-plane which can be solved using complex analysis techniques.

For radial solutions to the same equation $(-\Delta)^s u = -u + u^p$ in higher dimensions, Frank, Lenzmann, and Silvestre [36] extended this uniqueness and nondegeneracy result. To do so, they proved Sturmian properties for the radial eigenfunctions of $(-\Delta)^s + V(|x|)$ with V nondecreasing in $(0, +\infty)$.[3]

Their argument uses a Hamiltonian for the fractional Laplacian that had been discovered by the first author, Solà-Morales, and Sire [19, 18].[4]

In the following result we establish the even symmetry and monotonicity of periodic constrained minimizers for a nonlocal Lagrangian whose first variation is a semilinear equation $(-\Delta)^s u = f(u)$. Its proof is based on a not so well-known Riesz rearrangement inequality on $\mathbb{S}^1$ from 1976. It is an open problem to know if evenness holds also for all periodic solutions —not only for constrained minimizers.

Theorem 1.1. *Let $L > 0$, $0 < s < 1$, $G \in C^1(\mathbb{R})$, $\widetilde{G} \in C^1(\mathbb{R})$, $c \in \mathbb{R}$, and set $g := G'$, $\widetilde{g} := \widetilde{G}'$. Assume that $u \in L^\infty(\mathbb{R})$ is $2L$-periodic and minimizes the functional*

$$E(v) := \frac{c_s}{4} \int_{-L}^{L} dx \int_{\mathbb{R}} dy \, \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}} - \int_{-L}^{L} G(v) \tag{1.5}$$

among all $2L$-periodic functions $v \in L^\infty(\mathbb{R})$ satisfying the constraint

$$\int_{-L}^{L} \widetilde{G}(v) = c.$$

We then have:

- *(a) Up to a translation in the variable $x \in \mathbb{R}$, u is an even function in $\mathbb{R}$ which is nonincreasing in $(0, L)$.*[5]
- *(b) Assume that g and $\widetilde{g}$ belong to $C^\delta(\mathbb{R})$ for some $\delta > 0$. Then, $u \in C^{2s+\epsilon}(\mathbb{R})$ for some $\epsilon > 0$ and solves the equation $(-\Delta)^s u = g(u) + \lambda \widetilde{g}(u)$ in $\mathbb{R}$, for some $\lambda \in \mathbb{R}$.*
- *(c) Assume that g and $\widetilde{g}$ belong to $C^{1+\delta}(\mathbb{R})$ for some $\delta > 0$. Then, u' is a continuous function in $\mathbb{R}$ and, up to a translation in the variable $x \in \mathbb{R}$, u is even and $u' < 0$ in $(0, L)$, unless u is constant. In particular, u is an even function in $\mathbb{R}$ which is decreasing in $(0, L)$. Thus, u has only two critical points in $[0, L]$, its maximum at $x = 0$ and its minimum at $x = L$.*

Open problem 1.2. Do the conclusions of Theorem 1.1 hold also for all (smooth) periodic solutions to (1.1)—and not only for constrained minimizers?

It is worth pointing out that some semilinear equations $(-\Delta)^s u = f(u)$ do admit periodic constrained minimizers which are nonconstant. Indeed, in the forthcoming work [17] we will show their existence for the equation $(-\Delta)^s u = -u + u^p$ in $\mathbb{R}$, where $u > 0$ and $p > 1$ is

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[3] Very recently, Fall and Weth [34] improved the nondegeneracy result of [36] by removing the hypothesis which required the "ground state" to be a constrained minimizer. In addition, they also prove its uniqueness but only in dimension $n = 1$.

[4] In a forthcoming article [16], we will make an in depth study of the Hamiltonian for periodic solutions, also in the case of more general integro-differential operators.

[5] In other words, there exists $z \in \mathbb{R}$, such that $y \mapsto u(z + y)$ is an even function in $\mathbb{R}$ which is nonincreasing for $y \in (0, L)$. A comment here is in order. If a similar result, based on a rearrangement, were proved for "ground state" solutions in $\mathbb{R}$, its conclusion should include also the case in which $-u(z + \cdot)$ (with the minus sign) is even and nonincreasing in $(0, L)$. However, this case is not relevant for periodic solutions. Indeed, such a periodic solution (with its minimum at the origin) will be even and nonincreasing in $(0, L)$ after a further translation that places its maximum at the origin.

subcritical, whenever the period $2L$ is large enough.[6] On the other hand, in Appendix C we show that, for f regular enough, every bounded periodic solution to $(-\Delta)^s u = f(u)$ in $\mathbb{R}$ which is a stable solution (here we impose no constraint)[7] must be identically constant. In particular, under no constraint, there are no nonconstant bounded periodic minimizers —not even local minimizers, meaning minimizers among small periodic perturbations.

For more general integro-differential operators, below we will introduce some classes of kernels for which the analogue of Theorem 1.1 also holds; see Corollary 1.5.

1.2. **The Lagrangian for nonlocal periodic problems.** To find solutions to the Dirichlet problem $(-\Delta)^s u = g(u)$ in $(-L, L)$ and $u = \varphi$ in $\mathbb{R} \setminus (-L, L)$ by variational methods, one looks for critical points of the well-known Lagrangian

$$\begin{aligned} &E(v) + \frac{c_s}{4} \int_{\mathbb{R}\setminus(-L,L)} dx \int_{(-L,L)} dy \, \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}} \\ &\quad = \frac{c_s}{4} \iint_{(\mathbb{R}\times\mathbb{R})\setminus((-L,L)^c\times(-L,L)^c)} dxdy \, \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}} - \int_{-L}^{L} G(v) \end{aligned} \tag{1.6}$$

among functions with a prescribed exterior Dirichlet datum φ. Recall that E is the functional in (1.5). However, in the periodic setting, one cannot prescribe any exterior datum, only periodicity. It turns out, as we see from (1.6), that the Lagrangian E of Theorem 1.1 for the periodic framework differs from the Lagrangian for the Dirichlet problem.

To our knowledge, a nonlocal Lagrangian of the same type as E (that is, with $(-L, L) \times \mathbb{R}$ as set of integration), and modeled to find periodic solutions, appeared for first time in 2015 in the arXiv version of [26]. That work concerned the construction, by variational methods, of some nonlocal analogues of Delaunay surfaces. When [26] became available, we communicated to its authors that the first variation of their functional was the well-known nonlocal mean curvature. [26] had not addressed the first variation —that was believed to be a kind of nonlocal mean curvature, but not exactly it. At that time, we were already working on the periodic Lagrangian (1.5) for the fractional Laplacian.[8]

The variational setting for general integro-differential operators that the current paper undertakes serves as foundation for existence results of periodic solutions to integro-differential equations, and not only in dimension one. Indeed, in [15] we established for first time the existence (for any given volume constraint within a period) of nonlocal Delaunay cylinders (i.e., periodic sets with constant nonlocal mean curvature) by variational methods —something that was left as an open problem in [26]. The results in [15] are strongly related to the developments of the present paper.

Another instance is our forthcoming work [17], where we strengthen the existence results of Gui, Zhang, and Du [39] on periodic solutions for the fractional Laplacian with Allen-Cahn

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[6] With the notation of the theorem, here we will have $G(u) = -u^2/2$, $\widetilde{G}(u) = |u|^{p+1}/(p+1)$, and the constant c appropriately chosen so that the Lagrange multiplier λ is 1.

[7] Stability means that the second variation of E at the $2L$-periodic solution is nonnegative definite when acting on $2L$-periodic functions.

[8] That the nonlocal mean curvature is the first variation of the periodic fractional perimeter introduced in [26] already appeared in the Master's Thesis from 2017 of M. Alvinyà [1], directed within our group; see its Proposition 5.0.3. On the other hand, Section 4.2 of the Thesis already introduces the periodic semilinear Lagrangian (1.5), which also appeared, in independent works from ours, in DelaTorre, del Pino, González, and Wei [27] and in Barrios, García-Melián, and Quaas [6].

type nonlinearities, and those of Barrios, García-Melián, and Quaas [6] for Benjamin-Ono type nonlinearities.

1.3. **The symmetry result for general nonlocal operators.**

We also study more general equations, of the form

$$\mathcal{L}_K u = f(u) \quad \text{in } \mathbb{R},$$

where $u : \mathbb{R} \to \mathbb{R}$ is periodic and $\mathcal{L}_K$ is an integro-differential operator defined by

$$\mathcal{L}_K u(x) := \lim_{\epsilon \downarrow 0} \int_{\{y \in \mathbb{R} \colon |x-y| > \epsilon\}} dy \, \big(u(x) - u(y)\big) K(|x-y|) \tag{1.7}$$

and $K : (0, +\infty) \to [0, +\infty)$.

In some results, we will assume one or both of the standard conditions on the growth of K:

$$K(t) \le \frac{\Lambda}{t^{1+2s}} \quad \text{for all } t > 0 \tag{1.8}$$

and

$$K(t) \ge \frac{\lambda}{t^{1+2s}} \quad \text{for all } t > 0, \tag{1.9}$$

for some constants $0 < s < 1$ and $0 < \lambda \le \Lambda$.

For semilinear equations involving the fractional Laplacian, the Lagrangian can be constructed using a suitable extension to one more real variable, converting the problem into a local one. This extension technique has been used in [23, 33, 39] to show the existence of periodic solutions to certain classes of semilinear equations of the type $(-\Delta)^s u = f(u)$ in $\mathbb{R}$. However, for general integro-differential operators $\mathcal{L}_K$ no extension exists a priori, and the Lagrangian must be defined directly “downstairs”. We will easily see that the appropriate Lagrangian for $2L$-periodic solutions to $\mathcal{L}_K u = g(u)$ in $\mathbb{R}$ is, in analogy to the case of the fractional Laplacian,

$$\mathcal{E}_{\mathcal{L}_K}(u) := \frac{1}{2}[u]_K^2 - \int_{-L}^{L} G(u), \tag{1.10}$$

where $g = G'$ and

$$[u]_K := \Big(\frac{1}{2} \int_{-L}^{L} dx \int_{\mathbb{R}} dy \, |u(x) - u(y)|^2 K(|x-y|) \Big)^{1/2}. \tag{1.11}$$

Note that if $\mathcal{L}_K = (-\Delta)^s$ then $\mathcal{E}_{\mathcal{L}_K}$ is the Lagrangian E in (1.5).

To prove the evenness and monotonicity of $2L$-periodic constrained minimizers, we use a periodic rearrangement. For a $2L$-periodic function $u : \mathbb{R} \to \mathbb{R}$, the periodic symmetric decreasing rearrangement of u is defined as the $2L$-periodic function $u^{*\,\mathrm{per}} : \mathbb{R} \to \mathbb{R}$ such that $u^{*\,\mathrm{per}} \chi_{(-L,L)}$ is the Schwarz rearrangement of $|u| \chi_{(-L,L)}$ on $\mathbb{R}$. Recall that the Schwarz rearrangement of $v : (-L, L) \to \mathbb{R}$, denoted by v^*, is the nonnegative function which is equimeasurable with $|v|$, nonincreasing in $(0, L)$, and even with respect to 0 —that is, v^* is obtained by replacing the superlevel-sets of $|v|$ by symmetric intervals.[9]

The classical Pólya-Szegő inequality for functions vanishing on the boundary of $(-L, L)$ states that

$$\int_{-L}^{L} |(v^*)'|^2 \le \int_{-L}^{L} |v'|^2, \quad \text{if } v(-L) = v(L) = 0. \tag{1.12}$$

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[9] Most rearrangements are defined by rearranging the superlevel-sets of $|u|$ and not of u. One might define the rearrangement by considering the superlevel-sets of u, but then one needs an additional assumption on u, such as, for instance, being bounded from below.

This inequality remains true in the local periodic setting, that is, if we assume v to be $2L$-periodic (not necessarily with $v(\pm L) = 0$) and we replace v^* by $v^{*\,\mathrm{per}}$. To see this, simply replace v by $v - \min_{\mathbb{R}} v$, consider a translate $v(\cdot + z)$ of v so that $v(\cdot + z) - \min_{\mathbb{R}} v$ vanishes at $-L$ and L, and then apply (1.12).

Instead, in the nonlocal case, the study of periodic versions of (1.12) becomes nontrivial. Indeed, the nonlocal semi-norm $[u]_K$ feels the changes after rearrangement in all other periods besides $(-L, L)$. This will have a nontrivial impact when localizing the problem to $(-L, L)$. Indeed, in [41] it was shown that the Gagliardo $W^{s,2}(-L, L)$ semi-norm,

$$[u]_{W^{s,2}(-L,L)} := \Big(\int_{-L}^{L} dx \int_{-L}^{L} dy \, |u(x) - u(y)|^2 |x - y|^{-1-2s} \Big)^{1/2}, \tag{1.13}$$

may increase, after symmetric decreasing rearrangement, for some smooth functions u with compact support in $(-L, L)$. Note, however, that our semi-norm $[u]_K$ differs from this last semi-norm in its set of integration $(-L, L) \times \mathbb{R}$.

A possible analogue to the rearrangement inequality (1.12) for the fractional perimeter functional on periodic sets was raised as an open problem by Dávila, del Pino, Dipierro, and Valdinoci in [26]. We solved it affirmatively in [15]. Our proof was based on a Riesz rearrangement inequality on the circle $\mathbb{S}^1$, established independently in 1976 by Baernstein and Taylor [5] and by Friedberg and Luttinger [37]. It is stated below in Theorem 4.1.[10] We then realized that similar ideas would work for equations involving the fractional Laplacian instead of the more involved nonlocal mean curvature operator, giving rise to the current paper.

The following Theorem 1.4 establishes the analogue to the rearrangement inequality (1.12) for the nonlocal semi-norm $[\cdot]_K$ on $2L$-periodic functions and with respect to the periodic symmetric decreasing rearrangement. Our result applies to three different classes of kernels K. The first one is given by those kernels K which are convex. This is an interesting class that contains the kernel of the fractional Laplacian $K(t) = c_s t^{-1-2s}$. However, this class does not allow to conclude symmetry for the nonlocal Delaunay cylinders in [15], where we had to deal with the nonconvex kernel

$$K(t) = (t^2 + a^2)^{-(n+s)/2} \qquad \text{for some } a > 0 \text{ and } n \geq 2. \tag{1.14}$$

Our second class[11] —which includes (1.14) and also the kernel of the fractional Laplacian— are those kernels K for which $\tau > 0 \mapsto K(\tau^{1/2})$ is a completely monotonic function. Let us recall this notion.

Definition 1.3. A function $J : (0, +\infty) \to \mathbb{R}$ is *completely monotonic* if J is infinitely differentiable and satisfies

$$(-1)^k \frac{d^k}{d\tau^k} J(\tau) \geq 0 \qquad \text{for all } k \geq 0 \text{ and all } \tau > 0.$$

Clearly $K(\tau^{1/2})$ is completely monotonic in $\tau > 0$ when K is either the kernel (1.14) or the one of the fractional Laplacian, since we have, respectively, $J(\tau) = (\tau + a^2)^{-(n+s)/2}$ and $J(\tau) = c_s \tau^{-(1+2s)/2}$ in Definition 1.3.

We can now state our main result.

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[10] Theorem 4.1 will lead to a periodic rearrangement inequality for quantities integrated in $(-L, L) \times (-L, L)$, as in (1.13), but with $|x - y|^{-1-2s}$ replaced by some kernels $\overline{K}(|x - y|)$ which are $2L$-periodic in x for all given y.

[11] In Appendix B we will show that none of these two classes contains the other.

Theorem 1.4. *Let* $0 < s < 1$, $K : (0, +\infty) \to [0, +\infty)$ *satisfy the upper bound* (1.8), $L > 0$, *and* $[\cdot]_K$ *be defined by* (1.11).

- (*a*) *Assume that one of the following holds:*
  - (*i*) K *is convex.*
  - (*ii*) *The function* $\tau > 0 \mapsto K(\tau^{1/2})$ *is completely monotonic.*
  - (*iii*) K *is nonincreasing in* $(0, L)$ *and vanishes in* $[L, +\infty)$.

  Then, $[u^{*\,\mathrm{per}}]_K \leq [u]_K$ *for every* $2L$*-periodic function* $u : \mathbb{R} \to \mathbb{R}$ *such that* $u \in L^2(-L, L)$.
- (*b*) *Assume that* u *is* $2L$*-periodic, belongs to* $L^2(-L, L)$*, satisfies* $[u^{*\,\mathrm{per}}]_K = [u]_K < +\infty$, *and that one of the following holds:*
  - (*i*)′ K *is convex in* $(0, +\infty)$ *and strictly convex*[12] *in* $(c, +\infty)$ *for some* $c \geq 0$.
  - (*ii*)′ K *is as in* (*ii*) *and* K *is not identically zero.*
  - (*iii*)′ K *is decreasing in* $(0, L)$ *and vanishes in* $[L, +\infty)$.

  Then, for some $z \in \mathbb{R}$, *we have that either* $u = u^{*\,\mathrm{per}}(\cdot + z)$ *or* $u = -u^{*\,\mathrm{per}}(\cdot + z)$.

Case (*iii*) will apply if the kernel K is nonincreasing and has compact support and, at the same time, the period is large enough.

Regarding the class (*ii*), a characterization of complete monotonicity is given by Bernstein's theorem; see [10], [50, Theorem 12b in Chapter IV], and [40, Section 14.3]. Roughly speaking, it states that a function is completely monotonic if and only if it is the Laplace transform of a nonnegative measure. In the context of Theorem 1.4, this characterization reads as follows: condition (*ii*) is equivalent to the existence of a nondecreasing function $h : [0, +\infty) \to \mathbb{R}$ such that, if μ denotes the Borel measure defined by $\mu((a, b)) := h(b) - h(a)$ for all $0 \leq a \leq b < +\infty$, the kernel K is of the form

$$K(t) = \int_0^{+\infty} d\mu(r)\, e^{-t^2 r} \quad \text{for all } t > 0, \tag{1.15}$$

and the integral in (1.15) converges for all $t > 0$.[13] In Remark 6.2 we will compute the measure μ for the kernel K in (1.14) and for the kernel of the fractional Laplacian.

This representation of K by means of the Laplace transform is the key ingredient to prove Theorem 1.4 for the class (*ii*). For this, we write the Lagrangian $[\cdot]_K$ in (1.11) as a double integral over $(-L, L) \times (-L, L)$ with respect to the $2L$-periodic kernel

$$\overline{K}(t) := \sum_{k \in \mathbb{Z}} K(|t + 2kL|) \qquad \text{for } t \in \mathbb{R}.$$

In order to apply the Riesz rearrangement inequality on the circle, $\overline{K}$ must be nonincreasing in $t \in (0, L)$. This is a nontrivial question, for instance, for the nonlocal mean curvature kernel (1.14) which appeared in [15]. In that paper, we had the idea of expressing it in terms of a Laplace transform, and this reduced the problem to verify the monotonicity of the classical periodic heat kernel. We then realized that this method works for all kernels which arise from the Laplace transform of a nonnegative measure. This led us to a new class for us: kernels such

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[12] By K being strictly convex in an interval I we mean that $K(\lambda t_1 + (1 - \lambda)t_2) < \lambda K(t_1) + (1 - \lambda)K(t_2)$ for all $t_1 \neq t_2$ in I and all $\lambda \in (0, 1)$. Thus, in case K were C^2, this is more general than the condition $K'' > 0$ at all points in I.

[13] In all our results, and also examples in Appendix B, we will only use a special case of (1.15), namely, when the measure is of the form $\mu(A) = \int_A dr\, \kappa(r)$ for some $\kappa : (0, +\infty) \to [0, +\infty)$ measurable (and not identically zero). That is, μ will be absolutely continuous with respect to the Lebesgue measure.

that $K(\sqrt{\cdot})$ is completely monotonic. Then, with the keywords "completely monotonic" and "fractional" at hand, we found that in independent works from ours, Claassen and Johnson [22] and Bruell and Pei [12] had also used the Laplace transform to prove symmetry and monotonicity properties of periodic solutions associated to standing or travelling waves of some nonlocal dispersive equations.[14]

Indeed, for the particular case of the fractional Laplacian, part (a) of Theorem 1.4 is also proved by Claassen and Johnson [22]. That is, they showed that if $K(t) = t^{-1-2s}$ then $[u^{*\,\mathrm{per}}]_K \leq [u]_K$ for $2L$-periodic functions u. Their proof follows the same lines as ours for Theorem 1.4 (ii). However, they do not cover the case of equality described in part (b) of Theorem 1.4, which is the key point to ensure that all constrained minimizers must be symmetric and monotone. Also, their arguments are more involved than ours since they consider the kernel for the fractional heat equation. In addition, for the case of the fractional Laplacian, our proof of the Pólya-Szegő type inequality for convex kernels —i.e., our proof of Theorem 1.4 (i)— applies and is much simpler.

A class of kernels which contains the cases (i), (ii), and (iii) is the one of nonincreasing kernels —note that, in case (i), convexity and the decay estimate (1.8) lead to nonincreasing monotonicity. Thus, it is tempting to think that Theorem 1.4 could hold for all nonincreasing kernels. This is however not true, as we show in Appendix B with a simple example.

Theorem 1.4 leads to the evenness and monotonicity of every constrained minimizer of the Lagrangian functional $\mathcal{E}_{\mathcal{L}_K}$. This is the content of the following result.

Corollary 1.5. *Let $L > 0$, $0 < s < 1$, K satisfy the upper bound* (1.8)*, $G \in C^1(\mathbb{R})$, $\widetilde{G} \in C^1(\mathbb{R})$, $c \in \mathbb{R}$, and set $g := G'$, $\widetilde{g} := \widetilde{G}'$. Let $\mathcal{E}_{\mathcal{L}_K}$ be as in* (1.10)*. Assume that K satisfies one of the three conditions $(i)'$, $(ii)'$, or $(iii)'$ of Theorem 1.4 (b).*

If $u \in L^\infty(\mathbb{R})$ is $2L$-periodic and minimizes $\mathcal{E}_{\mathcal{L}_K}$ among $2L$-periodic functions $v \in L^\infty(\mathbb{R})$ satisfying $\int_{-L}^{L} \widetilde{G}(v) = c$, then, up to a translation in the variable $x \in \mathbb{R}$, u is even and nonincreasing in $(0, L)$.

Assume, in addition, that K satisfies the lower bound (1.9) *and that g and $\widetilde{g}$ belong to $C^{1+\delta}(\mathbb{R})$ for some $\delta > 0$. Then, u' is a continuous function in $\mathbb{R}$ and, up to a translation in the variable $x \in \mathbb{R}$, u is even and $u' < 0$ in $(0, L)$, unless u is constant. In particular, u is an even function in $\mathbb{R}$ which is decreasing in $(0, L)$. Thus, u has only two critical points in $[0, L]$, its maximum at $x = 0$ and its minimum at $x = L$.*

Within the proof of the corollary, we will see that if we further assume $u \geq 0$, then the condition $u \in L^\infty(\mathbb{R})$ is not necessary in the first conclusion of the result, i.e., u is even and nonincreasing in $(0, L)$ after a translation. Thus, we may consider constrained minimizers in classes of functions which are not necessarily bounded, such as the class $L^2(-L, L)$ for instance.

In Remark 6.1 we will point out that Corollary 1.5 also holds for minimizers with other type of constraints. For instance, a constraint on the quasi-norm of u in a Lorentz space.

To prove the last statements of Corollary 1.5 and of Theorem 1.1 on strict monotonicity, we need to develop a strong maximum principle for periodic solutions to integro-differential equations. To our knowledge, this is the first time that such a result is proven, even for the fractional Laplacian.

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[14] In Section 1.5 we will explain the relation between our semilinear elliptic equations and some dispersive equations.

Theorem 1.6. *Let* $0 < s < 1$ *and* $L > 0$. *Assume that* $K : (0,+\infty) \to [0,+\infty)$ *satisfies the upper bound* (1.8) *and that* K *satisfies one of the three conditions* $(i)'$, $(ii)'$, *or* $(iii)'$ *of Theorem 1.4 (b). Suppose that* $c \in L^\infty(\mathbb{R})$ *and that* $v \in C^{2s+\gamma}(\mathbb{R})$ *for some* $\gamma > 0$.

If v *is odd,* $2L$*-periodic, and solves*

$$\begin{cases} \mathcal{L}_K v \le c(x)v & \text{in } (0,L), \\ v \le 0 & \text{in } (0,L), \end{cases}$$

then either $v < 0$ *in* $(0,L)$ *or* v *is identically equal to* 0.

We have a simple direct proof of this lemma, as that of the maximum principle for the fractional Laplacian without a periodicity condition, when the kernel K is convex. Instead, for the class of nonconvex kernels with $K(\tau^{1/2})$ being completely monotonic in τ, the proof uses the Laplace transform as when proving Theorem 1.4.

1.4. **Regularity.** In this paper we also study the Hölder regularity of bounded $2L$-periodic distributional solutions to $\mathcal{L}_K u = f(u)$ in $\mathbb{R}$ when $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$. Our regularity results are employed to establish some parts of the results stated above. By *$2L$-periodic distributional solution* we mean that u is $2L$-periodic, belongs to $L^1(-L,L)$, and satisfies

$$\int_{-L}^{L} u\,\mathcal{L}_K\psi = \int_{-L}^{L} f(u)\psi$$

for every $2L$-periodic and smooth function ψ.

The boundedness of solutions $u \in L^2(-L,L)$ such that $[u]_K$ is finite is guaranteed either if $s > 1/2$ or under standard subcritical-type assumptions on the nonlinearity f; see Proposition 7.1.

Recall that, with $f \in C^\beta(\mathbb{R})$ as above, bounded solutions to $-\Delta u = f(u)$ in $B_1 \subset \mathbb{R}^n$ satisfy $u \in C^{2+\beta}(B_{1/2})$ (if $\beta > 0$ is not an integer), a result which is obtained by simply combining Calderón-Zygmund and Schauder's estimates. If $\beta = k$ is an integer and $n > 1$, then u is $C^{2+k-\epsilon}$ for all $\epsilon > 0$, but to prove or disprove that u is always of class C^{2+k} still seems to be an unsettled problem.[15]

In view of the local result, one would expect that bounded solutions to $\mathcal{L}_K u = f(u)$ with $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$ are of class $C^{2s+\beta-\epsilon}$ for every $\epsilon > 0$ (or even perhaps $C^{2s+\beta}$). In the following result we prove that this claim is indeed true when $s \ge (1-\beta)/2$. However, if $0 < \beta < 1$ and $0 < s < (1-\beta)/2$, the best regularity that one can expect for bounded solutions is to be of class $C^{2s/(1-\beta)}$, as explained in the following theorem and subsequent comments. This exponent $2s/(1-\beta)$ is smaller than $2s+\beta$ when $0 < s < (1-\beta)/2$, and seems to be new on the literature of fractional semilinear elliptic equations.

We emphasize that the next result remains true also in the nonperiodic setting (i.e., for the interior regularity in the Dirichlet problem) with an additional assumption on the Hölder semi-norm of K; see Remark 8.2 and the observations right after it.

Theorem 1.7. *Let* K *satisfy the bounds* (1.8) *and* (1.9) *for some* $0 < s < 1$, f *belong to* $C^\beta(\mathbb{R})$ *for some* $\beta > 0$, *and* $u \in L^\infty(\mathbb{R})$ *be a* $2L$*-periodic distributional solution to* $\mathcal{L}_K u = f(u)$ *in* $\mathbb{R}$.

Then, the following holds:

(*i*) *If* $\beta < 1$ *and* $s < \frac{1-\beta}{2}$, *then* $u \in C^{\frac{2s}{1-\beta}-\epsilon}(\mathbb{R})$ *for all* $\epsilon > 0$. *Moreover,* $\|u\|_{C^{2s/(1-\beta)-\epsilon}(\mathbb{R})}$ *is bounded by a constant which depends only on* s, β, ϵ, L, λ, Λ, $\|f\|_{C^\beta(\mathbb{R})}$, *and* $\|u\|_{L^\infty(\mathbb{R})}$.

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[15] In the case $k = 0$, and thus f is merely continuous, it has been posed as an open question in [47] whether all solutions have continuous second derivatives.

(*ii*) *If* $s \geq \frac{1-\beta}{2}$ *(this holds, in particular, if* $\beta \geq 1$*), then* $u \in C^{\beta+2s-\epsilon}(\mathbb{R})$ *for all* $\epsilon > 0$*. Moreover,* $\|u\|_{C^{\beta+2s-\epsilon}(\mathbb{R})}$ *is bounded by a constant which depends only on* s*,* β*,* ϵ*,* L*,* λ*,* Λ*,* $\|f\|_{C^{\beta}(\mathbb{R})}$*, and* $\|u\|_{L^{\infty}(\mathbb{R})}$*.*

In our forthcoming work [25], it will be shown with an example that in case (*i*) (and also in the nonperiodic setting) one cannot go beyond $C^{2s/(1-\beta)}$ regularity. This a truly semilinear-fractional effect that does not occur for the linear equation $\mathcal{L}_K u = h(x)$, and neither for local semilinear equations.

To see that (*i*) is a natural result, note that a simple heuristic calculation —assuming that $r < 1$ and $(-\Delta)^s(|x|^r) \approx |x|^{r-2s} = (|x|^r)^{(r-2s)/r}$. Here we have $\beta = 1 - 2s/r$ and thus $r = 2s/(1-\beta)$. Besides this heuristic argument, the precise reason why the corresponding proof of (*ii*) (or its analogue in the local case) breaks down in case (i), is that the composition of Hölder functions $f(v)$, for $f \in C^{\beta}$ and $v \in C^{\sigma}$, behaves differently when σ and β are both smaller than 1, and one obtains merely $f(v) \in C^{\beta\sigma}$. In the iteration argument of the proof, one uses in each step this composition argument, never reaching a Hölder exponent greater or equal than 1.

1.5. Fourier multiplier operators and dispersive equations.

In this work we also address the variational structure of equations of the form $\mathcal{L}u = f(u)$, where $\mathcal{L}$ is a Fourier multiplier operator defined by

$$\mathcal{L}u(x) := \sum_{k\in\mathbb{Z}} \ell(\tfrac{\pi k}{L}) u_k e^{\frac{i\pi k}{L}x} \tag{1.16}$$

and $u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\frac{i\pi k}{L}x}$ is the Fourier series expansion of a $2L$-periodic function u. Here, $\ell : \mathbb{R} \to \mathbb{R}$ is called the symbol (or multiplier) of $\mathcal{L}$. As we will see, this class of operators includes all $\mathcal{L}_K$ from (1.7) with K satisfying the upper bound (1.8) —the case $\ell(\xi) = |\xi|^{2s}$ being the fractional Laplacian $(-\Delta)^s$.

Equations of the form $\mathcal{L}u = f(u)$ naturally arise in many physical models. For instance, in the context of wave propagation, the general nonlinear equation

$$v_t - \mathcal{L}(v_x) + g(v)_x = 0 \qquad \text{for } x \in \mathbb{R},\ t \geq 0 \tag{1.17}$$

was considered in [9, 20] to describe long-crested, long-wavelength disturbances of small amplitude propagating in one direction in a dispersive media. Here, $v \equiv v(x,t)$ represents and amplitude or a velocity, x is usually associated with distance to some given point in the spatial domain of propagation, t is proportional to the time variable, and the operator $\mathcal{L}$ acts on the spatial variable. When looking for periodic traveling wave solutions of the form

$$v(x,t) \equiv u(x-ct) = \sum_{k\in\mathbb{Z}} u_k e^{\frac{i\pi k}{L}(x-ct)},$$

where $2L$ is the spatial period and c the velocity of propagation, the dispersive equation (1.17) rewrites in terms of u as $-cu' - \mathcal{L}(u') + g(u)' = 0$. Then, commuting the derivative with $\mathcal{L}$, one arrives to the equation $\mathcal{L}u = f(u)$ with $f(u) = -cu + g(u) + A$, where A is a constant of integration. The Benjamin-Ono equation $(-\Delta)^s u = -u + u^p$ mentioned in Section 1.1 is a concrete example coming from (1.17), as well as the Benjamin equation $\beta(-\Delta)u - \gamma(-\Delta)^{1/2}u = -u + u^2$ with $\beta > 0$ and $\gamma > 0$, which is a further generalization of Benjamin-Ono where short and long range diffusion compete against each other. For the generalized Korteweg-de Vries equation $v_t + v_{xxx} + g(v)_x = 0$, which corresponds to (1.17) taking $\mathcal{L} = -\partial_{xx}$, one gets

$-u'' = -cu+g(u)+A$. An example coming from a model not covered by (1.17) is the one for the Benjamin-Bona-Mahony equation $v_t+v_x+vv_x-v_{xxt} = 0$, which leads to $-cu'' = (c-1)u-u^2+A$.

The integro-differential equation $\mathcal{L}_K u = f(u)$ considered in this work also falls within the above Fourier framework $\mathcal{L}u = f(u)$. Indeed, the operators $\mathcal{L}_K$ defined in (1.7), with K satisfying the upper bound (1.8), are multiplier operators with a symbol $\ell = \ell_K$ which can be explicitly written in terms of K; see Lemma 3.1. This follows from the simple fact that the standard Fourier basis are eigenfunctions of $\mathcal{L}_K$. In turn, the equation $\mathcal{L}_K u = f(u)$ covers some of the examples from wave propagation presented before. In particular, for those examples, Corollary 1.5 leads to symmetry properties for traveling waves found by a constrained minimization of the associated Lagrangian $\mathcal{E}_{\mathcal{L}_K}$ given in (1.10). This applies, for instance, to the nontrivial constrained minimizers for large enough periods $2L$ found by Chen and Bona [20] when the symbol comes from a kernel as the ones described in Theorem 1.4 (b). Moreover, our Corollary 1.5 is, in a sense, the counterpart of the symmetry results found by Bruell and Pei [12] for dispersive equations with weak dispersion to the case of strong dispersion, once we restrict ourselves to constrained minimizers. Recall here that in [12], apart from complete monotonicity and/or homogeneity, the symbol is assumed to have a power-like decay at infinity —this last property is the one which refers to the words "weak dispersion". Hence, their results do not cover integro-differential operators like the fractional Laplacian, for which the symbol has instead power growth at infinity.

The proofs of symmetry and monotonicity presented in this paper, which are based on periodic rearrangement, only work for operators of the form $\mathcal{L}_K$ and some kernels K, but not for general multiplier operators $\mathcal{L}$. However, the description of the variational structure of $\mathcal{L}_K u = f(u)$ —briefly commented in Section 1.3 and fully developed in Section 2— easily extends to $\mathcal{L}u = f(u)$. This is useful since there are models of interest in the literature (such as the Benjamin equation) which cannot be described solely by using an integro-differential operator. That is why we will show in Section 3 that the Lagrangian expression (2.2) generalizes to multiplier operators $\mathcal{L}$ as in (1.16). More precisely, we will show that $\mathcal{L}u = g(u)$ is the Euler-Lagrange equation of the functional

$$\mathcal{E}_{\mathcal{L}}(u) := \int_{-L}^{L} \Big(\frac{1}{2}\, u\, \mathcal{L}u - G(u)\Big); \tag{1.18}$$

see (3.8), (3.9), and Lemma 3.3 for more details. The reader may also look at Section 3 for the precise assumptions required on the symbol ℓ of $\mathcal{L}$ in order to construct the Lagrangian (1.18).

To finish this subsection, let us mention that the Fourier series approach also leads to a nice relation between the integro-differential expressions for the half-Laplacian on $\mathbb{R}$ and on $\mathbb{S}^1$. We give the details of this in Appendix A.

Plan of the paper.

- In Section 2 we describe the variational formulation for linear and semilinear equations. We define here the notions of periodic weak and distributional solutions.
- In Section 3 we generalize the variational formulation of Section 2 to multiplier operators for Fourier series, introduced above in Section 1.5.
- In Section 4 we show a nonlocal Pólya-Szegő inequality for the periodic symmetric decreasing rearrangement, Theorem 1.4.
- In Section 5 we establish a strong maximum principle for periodic solutions of integro-differential equations, Theorem 1.6.
- In Section 6 we provide the proofs of Theorem 1.1 and Corollary 1.5.

- In Section 7 we prove $L^\infty(\mathbb{R})$ estimates for weak periodic solutions.
- In Section 8 we establish the $C^\alpha(\mathbb{R})$ estimates of Theorem 1.7 for bounded periodic distributional solutions.
- In Appendix A we investigate the connection between the half-Laplacian acting on periodic functions in $\mathbb{R}$ and on functions defined in $\mathbb{S}^1$.
- In Appendix B we discuss on the classes of kernels considered in Theorem 1.4.
- In Appendix C we establish that every stable periodic solution is identically constant.

2. Variational formulation for nonlocal periodic problems

In this section we address the variational approach for periodic solutions to the linear equation $\mathcal{L}_K u = h$ in $\mathbb{R}$ and also to the semilinear equation $\mathcal{L}_K u = f(u)$ in $\mathbb{R}$. In the linear case we assume the function $h : \mathbb{R} \to \mathbb{R}$ to be periodic.

As mentioned in Section 1.3 of the Introduction, the well-known Lagrangian (1.6) for the Dirichlet problem for the fractional Laplacian differs from the Lagrangian (1.5) for the periodic problem. The same situation occurs for the general integro-differential operator $\mathcal{L}_K u$ defined in (1.7). In this section we will show that $\mathcal{L}_K u = g(u)$ in $\mathbb{R}$ is the Euler-Lagrange equation for critical points of the functional (recall that $g = G'$)

$$\mathcal{E}_{\mathcal{L}_K}(u) = \frac{1}{4}\int_{-L}^{L} dx \int_{\mathbb{R}} dy\, |u(x) - u(y)|^2 K(|x-y|) - \int_{-L}^{L} G(u) \tag{2.1}$$

acting on $2L$-periodic functions u. To show this, we will express the functional explicitly in terms of $\mathcal{L}_K$. By a basic integration by parts formula for $\mathcal{L}_K$ acting on periodic functions, we will indeed see that (2.1) rewrites as

$$\mathcal{E}_{\mathcal{L}_K}(u) = \int_{-L}^{L} \Big(\frac{1}{2}\, u\, \mathcal{L}_K u - G(u)\Big). \tag{2.2}$$

In view of (2.1) it is now natural to define an associated semi-norm and inner product as follows. Given a $2L$-periodic function $u : \mathbb{R} \to \mathbb{R}$, we set

$$[u]_K := \Big(\frac{1}{2}\int_{-L}^{L} dx \int_{\mathbb{R}} dy\, |u(x) - u(y)|^2 K(|x-y|)\Big)^{1/2}.$$

In addition, given another $2L$-periodic function v, assuming that $[u]_K < +\infty$ and $[v]_K < +\infty$, we define the symmetric bilinear form

$$\langle u, v\rangle_K := \frac{1}{2}\int_{-L}^{L} dx \int_{\mathbb{R}} dy\, \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) K(|x-y|).$$

This motivates the following weak formulation for periodic solutions to the equation $\mathcal{L}_K u = h$ in $\mathbb{R}$. Given a $2L$-periodic function h, with $h \in L^2(-L, L)$, we say that a $2L$-periodic function $u : \mathbb{R} \to \mathbb{R}$ is a *periodic weak solution to $\mathcal{L}_K u = h$ in $\mathbb{R}$* if $u \in L^2(-L, L)$, $[u]_K < +\infty$, and

$$\langle u, v\rangle_K = \int_{-L}^{L} hv \tag{2.3}$$

for every $2L$-periodic function v with $v \in L^2(-L, L)$ and $[v]_K < +\infty$. In Lemma 2.1 we will show that, if u is smooth enough and satisfies (2.3) for all such functions v, then

$$\int_{-L}^{L} (\mathcal{L}_K u - h)\psi = \langle u, \psi\rangle_K - \int_{-L}^{L} h\psi = 0$$

for every $2L$-periodic and smooth function ψ. From this, by density and periodicity we conclude that $\mathcal{L}_K u(x) = h(x)$ for almost every $x \in \mathbb{R}$.

In addition, in Proposition 2.4 we will state a consistency-type result establishing that periodic weak solutions are also weak solutions in the Dirichlet sense in every interval under their own exterior data. This result will be needed later in the paper.

A final comment on another notion of periodic solution to $\mathcal{L}_K u = h$ in $\mathbb{R}$ is in order. Based on our integration by parts formula (Lemma 2.1) we may also introduce the concept of *periodic distributional solution* simply by asking that $u \in L^1(-L, L)$ is $2L$-periodic and that

$$\int_{-L}^{L} u\, \mathcal{L}_K \psi = \int_{-L}^{L} h\psi$$

for every $2L$-periodic and smooth function ψ. This notion of solution is convenient when considering more general right-hand sides h, such as distributions compactly supported in $(-L, L)$, with the understanding that $\int h\psi$ means the action of the distribution h on the test function ψ.

Be aware that in [31, 45, 46], whose regularity results will be used in Section 8, the authors call "weak solution" to what we refer as "distributional solution". The two notions coincide in the usual Dirichlet case if u belongs to $L^2(\mathbb{R})$ and has finite $W^{2,s}$ semi-norm. The same holds true in the periodic setting. More precisely, as a consequence of Lemma 2.1, we will see that every periodic distributional solution u satisfying $u \in L^2(-L, L)$ and $[u]_K < +\infty$, is also a periodic weak solution (and vice-versa).

The following lemma is the basic integration by parts formula associated to $\mathcal{L}_K$ when acting on periodic functions. It will justify the appropriate notion of periodic solution in the weak sense, and it will give the suitable kinetic term of the Lagrangian functional associated to $\mathcal{L}_K$.

Lemma 2.1. *Let K satisfy the upper bound (1.8). If u and ψ are $2L$-periodic functions, $u \in L^2(-L, L)$, $[u]_K < +\infty$, and $\psi \in C^\alpha(\mathbb{R})$ for some $\alpha > 2s$,*[16] *then*

$$\int_{-L}^{L} u\, \mathcal{L}_K \psi = \frac{1}{2} \int_{-L}^{L} dx \int_{\mathbb{R}} dy \, \big(u(x) - u(y)\big)\big(\psi(x) - \psi(y)\big) K(|x-y|). \tag{2.4}$$

If in addition $u \in C^\alpha(\mathbb{R})$ for some $\alpha > 2s$, then

$$\int_{-L}^{L} u\, \mathcal{L}_K \psi = \int_{-L}^{L} (\mathcal{L}_K u)\psi\,. \tag{2.5}$$

Remark 2.2. Before proving the lemma, a few remarks on (2.4) are in order. Recall that by definition

$$\mathcal{L}_K \psi(x) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} dy\, (\psi(x) - \psi(y)) K_\epsilon(|x-y|) =: \lim_{\epsilon \downarrow 0} \mathcal{L}_{K_\epsilon} \psi(x), \quad \text{where } K_\epsilon := K\chi_{(\epsilon,+\infty)}.$$

If $\psi \in C^\alpha(\mathbb{R})$ (ψ not being necessarily periodic) and $\alpha > 2s$, then $\mathcal{L}_{K_\epsilon}\psi(x)$ is uniformly bounded for all $\epsilon > 0$ and all $x \in \mathbb{R}$, and the pointwise limit as $\epsilon \downarrow 0$ is well defined. This can be checked as follows. First, note that we can assume without loss of generality $2s < \alpha < 2$, since s is smaller than 1. Using the upper bound (1.8) we see that (the second line is needed only if

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[16] Here, $\alpha > 1$ is allowed as well.

$2s \geq 1$ and hence $1 < \alpha < 2$)

$$\begin{aligned} |\mathcal{L}_{K_\epsilon}\psi(x)| &= \Big| \int_{\mathbb{R}} dz \, \big(\psi(x) - \psi(x-z)\big) K_\epsilon(|z|) \Big| \\ &= \frac{1}{2} \Big| \int_{\mathbb{R}} dz \, \big(2\psi(x) - \psi(x-z) - \psi(x+z)\big) K_\epsilon(|z|) \Big| \\ &\leq 4\Lambda \|\psi\|_{L^\infty(\mathbb{R})} \int_1^{+\infty} \frac{dt}{t^{1+2s}} + 2\Lambda \|\psi\|_{C^\alpha(\mathbb{R})} \int_0^1 dt \, \frac{t^\alpha}{t^{1+2s}} \leq C \end{aligned} \tag{2.6}$$

for all $\epsilon > 0$ and all $x \in \mathbb{R}$, where C is a constant depending only on Λ, s, α, and $\|\psi\|_{C^\alpha(\mathbb{R})}$. Similar arguments also show that $|\mathcal{L}_{K_\epsilon}\psi(x) - \mathcal{L}_{K_\delta}\psi(x)|$ tends to zero as ϵ and δ tend to 0. This proves the well-definiteness of the left-hand side of (2.4), since we are assuming that $u \in L^2(-L, L) \subset L^1(-L, L)$.

On the other hand, observe that no principal value is required on the right-hand side of (2.4). To see this, simply write $K = K^{1/2}K^{1/2}$ and use the Cauchy-Schwarz inequality to get

$$\frac{1}{2} \int_{-L}^{L} dx \int_{\mathbb{R}} dy \, |u(x) - u(y)||\psi(x) - \psi(y)| K(|x-y|) \leq [u]_K [\psi]_K \, . \tag{2.7}$$

Hence, $|u(x) - u(y)||\psi(x) - \psi(y)|K(|x-y|)$ is integrable in $(-L, L) \times \mathbb{R}$. Here we have also used that $[\psi]_K < +\infty$ if $\alpha > s$.

Proof of Lemma 2.1. Set $K_\epsilon = K\chi_{(\epsilon,+\infty)}$ for $\epsilon > 0$, as before. We claim that it suffices to prove the lemma for K_ϵ instead of K. Indeed, by definition $\mathcal{L}_K\psi(x) = \lim_{\epsilon \downarrow 0} \mathcal{L}_{K_\epsilon}\psi(x)$ and, therefore, from (2.6) and dominated convergence it follows that $\lim_{\epsilon\downarrow 0} \int_{-L}^{L} u \, \mathcal{L}_{K_\epsilon}\psi = \int_{-L}^{L} u \, \mathcal{L}_K \psi$. On the other hand, using the fact that $0 \leq K_\epsilon \leq K$ for all $\epsilon > 0$ and (2.7), by dominated convergence we get that

$$\begin{aligned} \lim_{\epsilon\downarrow 0} \int_{-L}^{L} dx \int_{\mathbb{R}} dy \, \big(u(x) - u(y)\big)\big(\psi(x) - \psi(y)\big) K_\epsilon(|x-y|) \\ = \int_{-L}^{L} dx \int_{\mathbb{R}} dy \, \big(u(x) - u(y)\big)\big(\psi(x) - \psi(y)\big) K(|x-y|). \end{aligned}$$

This shows the claim. Thus, we will work with K_ϵ from now on.

In the next step we check the integrability properties of certain integrals. This is required to justify the calculations proving the lemma. Writing $K_\epsilon = {K_\epsilon}^{1/2}{K_\epsilon}^{1/2}$ and using the Cauchy-Schwarz inequality and the upper bound (1.8) on the growth of K, we see that

$$\begin{aligned} &\int_{-L}^{L} dx \int_{\mathbb{R}} dy \, |\psi(x) - \psi(y)| K_\epsilon(|x-y|) \, |u(x)| \\ &\qquad \leq \sqrt{2}[\psi]_K \Big(\int_{-L}^{L} dx \, |u(x)|^2 \int_{\mathbb{R}\setminus(x-\epsilon,x+\epsilon)} dy \, K(|x-y|) \Big)^{1/2} \leq \frac{C}{\epsilon^s} \, [\psi]_K \|u\|_{L^2(-L,L)}, \end{aligned}$$

where C only depends on Λ and s. Therefore, $(\psi(x) - \psi(y))K_\epsilon(|x-y|)u(x)$ is absolutely integrable in $(-L, L) \times \mathbb{R}$. This also proves that $(\psi(x) - \psi(y))K_\epsilon(|x-y|)u(y)$ is absolutely integrable in $(-L, L) \times \mathbb{R}$, in view of (2.7) and the fact that $|u(y)| \leq |u(y) - u(x)| + |u(x)|$. These integrability properties justify the forthcoming computations.

We now address the proof of (2.4) for K_ϵ. Interchanging the name of the variables x and y, we express the first integral as follows:

$$\begin{aligned}\int_{-L}^{L} u\,\mathcal{L}_{K_\epsilon}\psi &= \int_{-L}^{L} dx \int_{\mathbb{R}} dy \,\big(\psi(x)-\psi(y)\big) K_\epsilon(|x-y|)\,u(x)\\ &= \frac{1}{2}\int_{-L}^{L} dx \int_{\mathbb{R}} dy \,\big(\psi(x)-\psi(y)\big) K_\epsilon(|x-y|)\,u(x)\\ &\quad + \frac{1}{2}\int_{-L}^{L} dy \int_{\mathbb{R}} dx \,\big(\psi(y)-\psi(x)\big) K_\epsilon(|x-y|)\,u(y).\end{aligned} \tag{2.8}$$

Since u and ψ are $2L$-periodic, the change of variables $\overline{x} = x - 2kL$, $\overline{y} = y - 2kL$ yields

$$\begin{aligned}&\int_{-L}^{L} dy \int_{\mathbb{R}} dx \,\big(\psi(y)-\psi(x)\big) K_\epsilon(|x-y|)\,u(y)\\ &\qquad = \sum_{k\in\mathbb{Z}} \int_{-L}^{L} dy \int_{(2k-1)L}^{(2k+1)L} dx \,\big(\psi(y)-\psi(x)\big) K_\epsilon(|x-y|)\,u(y)\\ &\qquad = \sum_{k\in\mathbb{Z}} \int_{-(2k+1)L}^{-(2k-1)L} d\overline{y} \int_{-L}^{L} d\overline{x} \,\big(\psi(\overline{y})-\psi(\overline{x})\big) K_\epsilon(|\overline{x}-\overline{y}|)\,u(\overline{y})\\ &\qquad = -\int_{-L}^{L} d\overline{x} \int_{\mathbb{R}} d\overline{y} \,\big(\psi(\overline{x})-\psi(\overline{y})\big) K_\epsilon(|\overline{x}-\overline{y}|)\,u(\overline{y}).\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we conclude that

$$\int_{-L}^{L} u\,\mathcal{L}_{K_\epsilon}\psi = \frac{1}{2}\int_{-L}^{L} dx \int_{\mathbb{R}} dy \,\big(\psi(x)-\psi(y)\big)\big(u(x)-u(y)\big) K_\epsilon(|x-y|).$$

This shows (2.4) for K_ϵ in place of K, and finishes the proof of the lemma. □

Given $2L$-periodic functions u and v such that $[u]_K < +\infty$ and $[v]_K < +\infty$, we have considered the symmetric bilinear form

$$\langle u, v\rangle_K := \frac{1}{2}\int_{-L}^{L} dx \int_{\mathbb{R}} dy \,\big(u(x)-u(y)\big)\big(v(x)-v(y)\big) K(|x-y|),$$

which is well defined by (2.7). In particular, $[u]_K^2 = \langle u, u\rangle_K$. Moreover, if in addition $u \in C^2(\mathbb{R})$ and $v \in L^2(-L, L)$, Lemma 2.1 yields

$$\langle u, v\rangle_K = \int_{-L}^{L} (\mathcal{L}_K u)\, v.$$

Let us now turn our attention to the Lagrangian associated to the linear and semilinear equations. To compare it with the nonperiodic scenario, we begin by considering the Dirichlet problem associated to the linear equation in a bounded open interval $I \subseteq \mathbb{R}$, namely,

$$\begin{cases} \mathcal{L}_K u = h & \text{in } I,\\ \quad u = g & \text{in } \mathbb{R}\setminus I,\end{cases} \tag{2.10}$$

where h is the given right-hand side and g corresponds to the exterior data. It is well known that (2.10) is the Euler-Lagrange equation for critical points of the Lagrangian

$$\mathcal{E}_D(u) := \frac{1}{4}\Big\{ \int_I dx \int_I dy\, |u(x)-u(y)|^2 K(|x-y|) \\ + 2\int_I dx \int_{I^c} dy\, |u(x)-u(y)|^2 K(|x-y|)\Big\} - \int_I hu$$

among functions satisfying $u = g$ in $\mathbb{R}\setminus I$ —see also (1.6) to write this as a single double integral in $(\mathbb{R}\times\mathbb{R})\setminus(I^c\times I^c)$.

In contrast, consider now the periodic scenario. Let $L>0$ and $I=(-L,L)\subset\mathbb{R}$. We want to study the variational framework of $2L$-periodic solutions to $\mathcal{L}_K u = h$ in $\mathbb{R}$, where $h:\mathbb{R}\to\mathbb{R}$ is a $2L$-periodic function. Here, no boundary data can be imposed (and hence the class of competitors above for the Dirichlet problem makes no sense here), but the solution is asked to be periodic and the equation to hold in the whole real line.

In the following Lemma 2.3 we show that, as announced, $\mathcal{L}_K u = h$ is the Euler-Lagrange equation among $2L$-periodic functions of a different functional than the one in the Dirichlet case. It is given by

$$\begin{aligned} \mathcal{E}_{\mathcal{L}_K}(u) &= \frac{1}{4}\int_I dx \int_{\mathbb{R}} dy\, |u(x)-u(y)|^2 K(|x-y|) - \int_I hu \\ &= \frac{1}{2}\langle u,u\rangle_K - \int_I hu = \frac{1}{2}[u]_K^2 - \int_I hu. \end{aligned} \tag{2.11}$$

Using the periodicity of u, sometimes it will be useful to write $\mathcal{E}_{\mathcal{L}_K}(u) = \frac{1}{2}[u]_K^2 - \int_I hu$ with

$$[u]_K = \Big(\frac{1}{2}\int_I dx \int_I dy\, |u(x)-u(y)|^2 \sum_{k\in\mathbb{Z}} K(|x-y+2kL|)\Big)^{1/2}. \tag{2.12}$$

This will be useful, for instance, in the proof of Theorem 1.4 to get our results on the periodic symmetric decreasing rearrangement.

Lemma 2.3. *Let K satisfy the upper bound* (1.8). *Assume that u and h are $2L$-periodic functions such that $u\in L^2(-L,L)$, $[u]_K<+\infty$, and $h\in L^1(-L,L)$. Then,*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_{\mathcal{L}_K}(u+t\psi) = \langle u,\psi\rangle_K - \int_{-L}^{L} h\psi = \int_{-L}^{L}\big(u\,\mathcal{L}_K\psi - h\psi\big) \tag{2.13}$$

for every $2L$-periodic smooth function ψ. In particular, if $h\in L^2(-L,L)$ and u is a critical point of $\mathcal{E}_{\mathcal{L}_K}$ among $2L$-periodic smooth perturbations, then $\mathcal{L}_K u = h$ in $\mathbb{R}$ in the periodic weak sense.

Proof. Lemma 2.1 directly gives (2.13). Now, if u is a critical point of $\mathcal{E}_{\mathcal{L}_K}$, from (2.13) we deduce that

$$\langle u,\psi\rangle_K = \int_{-L}^{L} h\psi$$

for every $2L$-periodic smooth ψ. By density, this equality extends to every $2L$-periodic function ψ such that $\psi\in L^2(-L,L)$ and $[\psi]_K<+\infty$. Thus, u is a periodic weak solution to $\mathcal{L}_K u = h$ in $\mathbb{R}$. □

Note that, once the variational framework for the linear periodic case is developed, we easily get that the semilinear equation $\mathcal{L}_K u = g(u)$ in $\mathbb{R}$ is the Euler-Lagrange equation of the functional $\mathcal{E}_{\mathcal{L}_K}$ given in (2.1). That is, for the semilinear case we simply replace $\int_I hu$ by $\int_I G(u)$ on the right-hand side of (2.11), where $G' = g$.

To finish this section, we establish a consistency-type result that will be needed within the proof of Theorem 1.7. Recall that Theorem 1.7 establishes the Hölder regularity of bounded periodic distributional solutions to the equation $\mathcal{L}_K u = f(u)$. The next proposition shows that periodic distributional solutions, in the linear case, are also distributional solutions in the usual Dirichlet sense. It follows easily from the proposition that periodic weak solutions are also weak solutions in the usual Dirichlet sense, provided one uses the standard (nonperiodic) integration by parts formula and the corresponding (nonperiodic) semi-norm to define weak solutions in the Dirichlet sense.

Proposition 2.4. *Let K satisfy the upper bound* (1.8). *Assume that u and h belong to $L^1(-L, L)$, u is $2L$-periodic, and*

$$\int_{-L}^{L} u\,\mathcal{L}_K\psi = \int_{-L}^{L} h\psi$$

for every $2L$-periodic smooth function ψ.

Then,

$$\int_{\mathbb{R}} u\,\mathcal{L}_K\varphi = \int_{\mathbb{R}} h\varphi \tag{2.14}$$

for every $\varphi \in C_c^\infty(-L, L)$. In particular, u is a distributional solution of $\mathcal{L}_K u = h$ in the usual Dirichlet sense.

Proof. Let $I = (-L, L)$, $\varphi \in C_c^\infty(I)$, and let $\varphi_e \in C^\infty(\mathbb{R})$ denote its $2L$-periodic extension to $\mathbb{R}$. We claim that

$$\int_{\mathbb{R}} u\,\mathcal{L}_K\varphi = \int_I u\,\mathcal{L}_K\varphi_e\,. \tag{2.15}$$

Recall that $u \in L^1(I)$ is $2L$-periodic. To prove the claim let us write

$$\begin{aligned}
\int_{\mathbb{R}} u\,\mathcal{L}_K\varphi &= \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, u(x)\big(\varphi(x) - \varphi(y)\big)K(|x-y|)\\
&= \int_I dx \int_I dy\, u(x)\big(\varphi_e(x) - \varphi_e(y)\big)K(|x-y|)\\
&\quad + \int_I dx \int_{\mathbb{R}\setminus I} dy\, u(x)\varphi_e(x)K(|x-y|) - \int_{\mathbb{R}\setminus I} dx \int_I dy\, u(x)\varphi_e(y)K(|x-y|).
\end{aligned} \tag{2.16}$$

Observe that the first two integrals on the last expression in (2.16) are absolutely convergent since $u \in L^1(I)$, K satisfies (1.8), $\varphi_e \in C^2(I)$, and $\varphi_e|_I$ has compact support in I. For the third integral, using that φ has support in $(-L+\delta, L-\delta) =: I_\delta$ for some $\delta > 0$, and also that u is

$2L$-periodic, we see that

$$\begin{aligned}\int_{\mathbb{R}\setminus I} dx \int_I dy\, |u(x)||\varphi_e(y)|K(|x-y|) &\le \|\varphi\|_{L^\infty(\mathbb{R})} \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_{I+2kL} dx \int_{I_\delta} dy\, |u(x)|K(|x-y|) \\ &= \|\varphi\|_{L^\infty(\mathbb{R})} \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_I d\bar{x} \int_{I_\delta - 2kL} d\bar{y}\, |u(\bar{x})|K(|\bar{x}-\bar{y}|) \\ &\le \Lambda \|\varphi\|_{L^\infty(\mathbb{R})} \|u\|_{L^1(I)} \int_{\mathbb{R}\setminus(-\delta,\delta)} dt\, |t|^{-1-2s} < +\infty.\end{aligned}$$

This means that $u(x)\varphi_e(y)K(|x-y|)$ is absolutely integrable in $(\mathbb{R}\setminus I)\times I$ and therefore, arguing as in (2.9) and replacing there $k\in\mathbb{Z}$ by $k\in\mathbb{Z}\setminus\{0\}$, we see that

$$\int_{\mathbb{R}\setminus I} dx \int_I dy\, u(x)\varphi_e(y)K(|x-y|) = \int_I dx \int_{\mathbb{R}\setminus I} dy\, u(x)\varphi_e(y)K(|x-y|).$$

Thus, (2.16) leads to

$$\begin{aligned}\int_{\mathbb{R}} u\,\mathcal{L}_K\varphi = \int_I dx \int_I dy\, u(x)\big(\varphi_e(x)-\varphi_e(y)\big)K(|x-y|) \\ + \int_I dx \int_{\mathbb{R}\setminus I} dy\, u(x)\big(\varphi_e(x)-\varphi_e(y)\big)K(|x-y|) = \int_I u\,\mathcal{L}_K\varphi_e.\end{aligned}$$

This proves the claim (2.15).

From (2.15), using that u is a distributional $2L$-periodic solution to $\mathcal{L}_K u = h$ in $\mathbb{R}$, we deduce that

$$\int_{\mathbb{R}} u\,\mathcal{L}_K\varphi = \int_I u\,\mathcal{L}_K\varphi_e = \int_I h\varphi_e = \int_{\mathbb{R}} h\varphi,$$

since φ vanishes outside I. This establishes (2.14). ☐

3. FOURIER MULTIPLIER OPERATORS

In this section we describe the variational structure of periodic solutions to semilinear equations of the form $\mathcal{L}_K u = f(u)$ from the Fourier side point of view, and also for more general multiplier operators. This has applications to travelling waves of some dispersive equations as we have explained in Section 1.5.

The first key observation is that the standard Fourier basis of unitary complex exponentials (and, thus, also the sines and cosines) are eigenfunctions of $\mathcal{L}_K$; see Lemma 3.1 below. All what is needed for this is the kernel to be even. Therefore, our integro-differential operators have a simple representation in the Fourier side by means of multipliers. More precisely, if we set

$$\ell_K(\xi) := \frac{1}{|\xi|}\int_{\mathbb{R}} dz\, \big(1-\cos(z)\big)K\Big(\frac{|z|}{|\xi|}\Big) \quad \text{for } \xi\in\mathbb{R}\setminus\{0\}, \quad \text{and } \ell_K(0) := 0, \tag{3.1}$$

then, for every $2L$-periodic function $u:\mathbb{R}\to\mathbb{C}$ written as

$$u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\frac{i\pi k}{L}x} \quad \text{with} \quad u_k := \frac{1}{2L}\int_{-L}^{L} dx\, u(x)e^{-\frac{i\pi k}{L}x}, \tag{3.2}$$

we will see in Lemma 3.1 that, formally,

$$\mathcal{L}_K u(x) = \sum_{k\in\mathbb{Z}} \ell_K(\tfrac{\pi k}{L}) u_k e^{\frac{i\pi k}{L}x}.$$

This leads us to study the variational structure of periodic solutions to semilinear equations $\mathcal{L}u = f(u)$ also for general multiplier operators of the form

$$\mathcal{L}u(x) = \sum_{k\in\mathbb{Z}} \ell(\tfrac{\pi k}{L}) u_k e^{\frac{i\pi k}{L}x}, \tag{3.3}$$

where the given function $\ell : \mathbb{R} \to \mathbb{R}$ is called the symbol of $\mathcal{L}$. Clearly, $\mathcal{L} = \mathcal{L}_K$ for $\ell = \ell_K$.

In the case of integro-differential operators, (3.1) shows that $\mathcal{L}_K$ has a positive definite symbol ℓ_K. In fact, thanks to the bounds (1.8) and (1.9) on K, in Lemma 3.1 we will check that

$$\frac{\lambda}{c_s} |\xi|^{2s} \leq \ell_K(\xi) \leq \frac{\Lambda}{c_s} |\xi|^{2s} \tag{3.4}$$

for all $\xi \in \mathbb{R}$, where c_s is given by (1.2). However, as we will see below, in the Fourier series approach to periodic solutions to $\mathcal{L}u = f(u)$ we can also allow symbols which change sign and which are bounded by a power-like behavior at infinity not necessarily of order less than 2. More precisely, we consider symbols $\ell : \mathbb{R} \to \mathbb{R}$ such that

$$\ell(\xi) = \ell(-\xi) \quad \text{for all } \xi \in \mathbb{R} \tag{3.5}$$

and

$$\limsup_{\xi\uparrow+\infty} |\xi|^{-p}|\ell(\xi)| < +\infty \quad \text{for some } p \geq 0. \tag{3.6}$$

Note that the later is a weaker assumption than the second bound in (3.4) if $p > 2s$. This class of symbols contains, for example, the case $\ell(\xi) = \beta\xi^2 - \gamma|\xi|$ associated to the Benjamin equation —simply take $p = 2$ in (3.6).

Let us start by proving that the Fourier basis is formed by eigenvectors of the integro-differential operator $\mathcal{L}_K$.

Lemma 3.1. *Let K satisfy the upper bound* (1.8), *and let $L > 0$. Then,*

$$\mathcal{L}_K\big(e^{\frac{i\pi k}{L}\cdot}\big)(x) = \ell_K(\tfrac{\pi k}{L}) e^{\frac{i\pi k}{L}x} \tag{3.7}$$

for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, where ℓ_K is given by (3.1). *Moreover, the second inequality in* (3.4) *holds and, if K also satisfies the lower bound* (1.9), *the first inequality in* (3.4) *also holds. In addition, if $\mathcal{L}_K = (-\Delta)^s$ then $\ell_K(\xi) = |\xi|^{2s}$ for all $\xi \in \mathbb{R}$.*

Proof. Thanks to (1.8), the integral defining $\ell_K(\xi)$ in (3.1) is well defined.

Since the case $k = 0$ is clear, we may assume $k \neq 0$. Now, by a change of variables we easily see that

$$\begin{aligned}
\mathcal{L}_K\big(e^{\frac{i\pi k}{L}\cdot}\big)(x) &= \int_{\mathbb{R}} d\widetilde{z}\, \big(e^{\frac{i\pi k}{L}x} - e^{\frac{i\pi k}{L}(x+\widetilde{z})}\big) K(|\widetilde{z}|) \\
&= e^{\frac{i\pi k}{L}x} \frac{L}{\pi|k|} \int_{\mathbb{R}} dz\, (1 - e^{iz}) K\Big(\frac{L|z|}{\pi|k|}\Big) = \ell_K(\tfrac{\pi k}{L}) e^{\frac{i\pi k}{L}x},
\end{aligned}$$

where we have used that $\sin(z)$ is and odd function. This proves (3.7).

We now address the other statements in the lemma. It is well known that

$$\int_{\mathbb{R}} dz\, \frac{1-\cos(z)}{|z|^{1+2s}} = \frac{\sqrt{\pi}\,\Gamma(1-s)}{s4^s\Gamma(1/2+s)} = \frac{1}{c_s};$$

see [13, Lemma 3.1.3] and (1.2). Hence, in case that $\mathcal{L}_K = (-\Delta)^s$, we have

$$\ell_K(\xi) = |\xi|^{2s} c_s \int_{\mathbb{R}} dz\, \frac{1-\cos(z)}{|z|^{1+2s}} = |\xi|^{2s}.$$

From this, the first and second inequalities in (3.4) for a general kernel K follow by the growth estimates (1.9) and (1.8), respectively. $\square$

Remark 3.2. In analogy to (3.7), if $\mathcal{F}(v)(\xi) := \int_{\mathbb{R}} dx\, v(x)e^{-ix\xi}$ denotes the Fourier transform of $v \in L^1(\mathbb{R})$, then $\mathcal{F}(\mathcal{L}_K v) = \ell_K \mathcal{F}(v)$ for all v in the Schwartz space of rapidly decreasing functions in $\mathbb{R}$. To see this simply note that

$$\begin{aligned}\mathcal{L}_K v(x) &= \lim_{\epsilon\downarrow 0} \int_{\{z\in\mathbb{R}:\,|z|>\epsilon\}} dz\, \big(v(x)-v(x-z)\big)K(|z|)\\ &= \frac{1}{2}\int_{\mathbb{R}} dz\, \big(2v(x)-v(x-z)-v(x+z)\big)K(|z|),\end{aligned}$$

and then

$$\begin{aligned}\mathcal{F}(\mathcal{L}_K v)(\xi) &= \frac{1}{2}\int_{\mathbb{R}} dz\, \big(2\mathcal{F}(v)(\xi)-\mathcal{F}(v(\cdot-z))(\xi)-\mathcal{F}(v(\cdot+z))(\xi)\big)K(|z|)\\ &= \frac{1}{2}\int_{\mathbb{R}} dz\, \mathcal{F}(v)(\xi)\big(2-e^{-iz\xi}-e^{iz\xi}\big)K(|z|)\\ &= \mathcal{F}(v)(\xi)\int_{\mathbb{R}} dz\, (1-\cos(z\xi))K(|z|) = \mathcal{F}(v)(\xi)\ell_K(\xi),\end{aligned}$$

in view of (3.1).

Given a kernel K, let us now look at the Fourier description of the Lagrangian $\mathcal{E}_{\mathcal{L}_K}$. This will allow us to find a Lagrangian $\mathcal{E}_{\mathcal{L}}$ for general multiplier operators $\mathcal{L}$. First of all, note that if u is a $2L$-periodic $\mathbb{R}$-valued function and u_k are given by (3.2), then $u_{-k} = \overline{u_k}$ for all $k \in \mathbb{Z}$, where $\overline{u_k}$ denotes the complex conjugate of $u_k \in \mathbb{C}$. In such case, we can also write

$$u(x) = \frac{a_0}{2} + \sum_{k>0}\Big(a_k \cos\big(\tfrac{\pi k}{L}\,x\big) + b_k \sin\big(\tfrac{\pi k}{L}\,x\big)\Big),$$

where

$$a_k := \frac{1}{L}\int_{-L}^{L} dx\, u(x)\cos\big(\tfrac{\pi k}{L}\,x\big), \qquad b_k := \frac{1}{L}\int_{-L}^{L} dx\, u(x)\sin\big(\tfrac{\pi k}{L}\,x\big)$$

for all $k \geq 0$. In fact, we have that $2u_k = a_k - ib_k$. Hence, if u is $\mathbb{R}$-valued, thanks to Lemma 3.1 we see that the Lagrangian in (2.2) can be expressed as

$$\mathcal{E}_{\mathcal{L}_K}(u) = 2L\sum_{k\in\mathbb{Z}} \ell_K(\tfrac{\pi k}{L})\frac{1}{2}|u_k|^2 - \int_{-L}^{L} G(u) = \frac{L}{2}\sum_{k>0}\ell_K(\tfrac{\pi k}{L})(|a_k|^2+|b_k|^2) - \int_{-L}^{L} G(u). \tag{3.8}$$

With this at hand, it is natural to define the Lagrangian $\mathcal{E}_{\mathcal{L}}$ on the Fourier side, for the operator

$$\mathcal{L}u(x) = \sum_{k\in\mathbb{Z}} \ell(\tfrac{\pi k}{L})u_k e^{\frac{i\pi k}{L}x}$$

with ℓ as in (3.5), and for $2L$-periodic $\mathbb{R}$-valued functions u, by

$$\begin{aligned}\mathcal{E}_{\mathcal{L}}(u) &:= L\Big(\ell(0)|u_0|^2 + 2\sum_{k>0}\ell(\tfrac{\pi k}{L})|u_k|^2\Big) - \int_{-L}^{L} G(u)\\ &= \frac{L}{4}\Big(\ell(0)|a_0|^2 + 2\sum_{k>0}\ell(\tfrac{\pi k}{L})(|a_k|^2+|b_k|^2)\Big) - \int_{-L}^{L} G(u).\end{aligned} \tag{3.9}$$

Here we have taken into account that, despite $\ell_K(0) = 0$, one may have $\ell(0) \neq 0$ for certain multiplier operators $\mathcal{L}$. Using now that $u_{-k} = \overline{u_k}$ and $2u_k = a_k - ib_k$ for $\mathbb{R}$-valued functions, and (3.5), from (3.9) we see that actually

$$\mathcal{E}_{\mathcal{L}}(u) = \int_{-L}^{L} \Big(\frac{1}{2}\, u\, \mathcal{L}u - G(u)\Big),$$

as expected by looking at identity (2.2) for $\mathcal{E}_{\mathcal{L}_K}$. All these computations are formal, but they indeed work if the symbol ℓ satisfies (3.6) and we require enough regularity on u (and, therefore, we get fast enough decay of its Fourier coefficients u_k compared to $\ell(\frac{\pi k}{L})$ as $k \uparrow +\infty$) in order to ensure that all the sums above are absolutely convergent; see Remark 3.4 for more details.

To finish, in the following lemma we verify that $\mathcal{E}_{\mathcal{L}}$ is a functional whose critical points are solutions to the equation $\mathcal{L}u = g(u)$, where G is of class C^1 in $\mathbb{R}$ and $g = G'$. This is, therefore, the semilinear extension of Lemma 2.3 for $\mathcal{E}_{\mathcal{L}_K}$, now for multiplier operators on the Fourier side.

Lemma 3.3. *Let $\mathcal{L}$ be given by* (3.3) *with ℓ satisfying* (3.5) *and* (3.6). *Let G be of class C^1 on $\mathbb{R}$ and $g = G'$. Assume that $u : \mathbb{R} \to \mathbb{R}$ is a $2L$-periodic function such that*

$$\sum_{k\in\mathbb{Z}} \big(1 + |\ell(\tfrac{\pi k}{L})|\big)|u_k|^2 < +\infty$$

and $g(u) \in L^1(-L, L)$. Then,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_{\mathcal{L}}(u + t\psi) = \int_{-L}^{L} \big(\mathcal{L}u - g(u)\big)\psi$$

for every $2L$-periodic smooth function $\psi : \mathbb{R} \to \mathbb{R}$. Therefore, if in addition u is a critical point of $\mathcal{E}_{\mathcal{L}}$ among $2L$-periodic smooth perturbations, then $\mathcal{L}u = g(u)$ in $L^1(-L, L)$.

Proof. Let ψ_k be the complex Fourier coefficients of ψ as in (3.2). By (3.5), we get

$$\int_{-L}^{L} \psi\, \mathcal{L}u = 2L \sum_{k\in\mathbb{Z}} \ell(\tfrac{\pi k}{L}) u_k \psi_{-k} = 2L \sum_{k\in\mathbb{Z}} \ell(\tfrac{\pi k}{L}) u_{-k} \psi_k = \int_{-L}^{L} u\, \mathcal{L}\psi.$$

Therefore,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}_{\mathcal{L}}(u + t\psi) = \int_{-L}^{L} \frac{1}{2}\big(\psi\, \mathcal{L}u + u\, \mathcal{L}\psi\big) - \int_{-L}^{L} g(u)\psi = \int_{-L}^{L} \big(\mathcal{L}u - g(u)\big)\psi,$$

as desired. The last statement in the lemma follows by density.

We point out that the assumption (3.6) on ℓ guarantees that $\sum_{k\in\mathbb{Z}} |\ell(\frac{\pi k}{L})||\psi_k|^2 < +\infty$ whenever ψ is smooth. This in turn is used to ensure, using the Cauchy-Schwarz inequality, that the above computations based on Parseval identity make sense. ☐

Remark 3.4. Here we analyze which Hölder regularity must be imposed on u in order to ensure the assumption

$$\sum_{k\in\mathbb{Z}} |\ell(\tfrac{\pi k}{L})||u_k|^2 < +\infty \tag{3.10}$$

of Lemma 3.3. Clearly, (3.10) holds if, for some $\epsilon > 0$, u satisfies

$$\limsup_{k\to\pm\infty} |k|^{1+\epsilon} |\ell(\tfrac{\pi k}{L})||u_k|^2 < +\infty. \tag{3.11}$$

In particular, if the Fourier multipliers of $\mathcal{L}$ satisfy (3.5) and the growth estimate (3.6), then (3.11) holds whenever

$$\limsup_{k\uparrow+\infty} |k|^{\frac{1+\epsilon+p}{2}}|u_k| < +\infty. \tag{3.12}$$

Finally, by a simple argument using integration by parts, it is easy to show that $|u_k| \le C|k|^{-m}$ whenever $u \in C^m(\mathbb{R})$ for some $m \in \mathbb{N}$. On the other hand, a classical result states that if $u \in C^r(\mathbb{R})$ for some $0 < r \le 1$ then $|u_k| \le C|k|^{-r}$; see [51, estimate (4.1) in page 45]. A combination of these two estimates shows that (3.12) will hold for some $\epsilon > 0$ whenever $u \in C^\alpha(\mathbb{R})$ for some $\alpha > (p+1)/2$. Hence, in such case (3.10) will also hold. In particular, if $\ell = \ell_K$ then by (3.4) it suffices to take $\alpha > s + 1/2$.

4. Periodic nonlocal Pólya-Szegő inequality

The purpose of this section is to prove Theorem 1.4, that is, the periodic symmetric decreasing rearrangement inequality for $[\cdot]_K$, for three different classes of kernels K. Recall that in Appendix B we will show that none of these three classes is contained in the other.

A crucial ingredient of the proof of Theorem 1.4 is the following Riesz rearrangement inequality on the circle found independently in 1976 by Baernstein and Taylor [5] and by Friedberg and Luttinger [37]. Recall that, for a 2π-periodic function f, in Section 1.3 we defined the rearrangements f^* and $f^{*\mathrm{per}}$.

Theorem 4.1. ([5, Theorem 2], [37, Theorem 1], [14, Theorem 2], [4, Theorem 7.3]) *Let $f, h, g : \mathbb{R} \to \mathbb{R}$ be three nonnegative 2π-periodic measurable functions. Assume that g is even, as well as nonincreasing in $(0, \pi)$.*

Then,

$$\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, f(x)g(x-y)h(y) \le \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy\, f^*(x)g(x-y)h^*(y). \tag{4.1}$$

In addition, if g is decreasing in $(0, \pi)$ and the left-hand side of (4.1) is finite, then equality holds in (4.1) if and only if at least one of the following conditions holds:

- (*i*) *Either f or h is constant almost everywhere.*
- (*ii*) *There exists $z \in \mathbb{R}$ such that $f(x) = f^{*\mathrm{per}}(x+z)$ and $h(x) = h^{*\mathrm{per}}(x+z)$ for almost every $x \in \mathbb{R}$.*[17]

Inequality (4.1) was first discovered, independently, in [5] and [37]. Both references contain more general inequalities: [5] deals with the sphere $\mathbb{S}^n$, whereas [37] deals with a product of more than three functions. The result [4, Theorem 7.3] by Baernstein is also more general than (4.1), as it deals with a Riesz rearrangement inequality on the sphere $\mathbb{S}^n$ and not only on the circle. Moreover, [4] treats more general functions of $f(x)$ and $h(y)$ than simply the product $f(x)h(y)$. Inequality (4.1) can be found in [3], another work of Baernstein, in a more general form where g is also rearranged.

Instead, the statement in Theorem 4.1 concerning equality in (4.1) follows from Burchard and Hajaiej [14, Theorem 2], who treated the case of equality in $\mathbb{S}^n$ for the first time, even for $n = 1$. For this result, we also cite [4, Theorem 7.3] since, being less general than [14], fits precisely with our statement in Theorem 4.1.

We find reference [4] to be the simplest one for looking up all assertions of Theorem 4.1.

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[17] This is equivalent to the condition $f(\cdot - z) = f^*$ and $h(\cdot - z) = h^*$ a.e. in $(-\pi, \pi)$. Note also that, in case (*i*), if f is constant then it could happen that h does not agree with any translation of its periodic rearrangement.

All the above mentioned references, except for [3], use the *method of polarization* to prove Theorem 4.1. When finishing this article, we have found that the method of polarization has been also used in the following very recent papers on fractional equations. They do not deal, however, with periodic solutions. First, DelaTorre and Parini [28] and Dieb, Ianni, and Saldaña [29] use the method to prove uniqueness of least energy solutions for fractional equations, while Benedikt, Bobkov, Dhara, and Girg [7] employ it to establish the nonradiality of the second eigenfunction of the fractional Laplacian in a ball.

Proof of Theorem 1.4. Since the periodic Riesz rearrangement inequality (4.1) requires the functions f and h to be nonnegative, we first need to reduce the proof to the case $u \geq 0$.

First, observe that $||u(x)|-|u(y)|| \leq |u(x)-u(y)|$ for all x and y in $\mathbb{R}$, and that $u^{*\,\mathrm{per}} = |u|^{*\,\mathrm{per}}$. Thus, to prove part (a) we can assume without loss of generality that u is nonnegative.

Regarding part (b), assume that $[u^{*\,\mathrm{per}}]_K = [u]_K < +\infty$. We claim that u does not change sign (in the usual "almost everywhere" or "essential" sense, since we only assume $u \in L^2(-L, L)$). Indeed, otherwise there would exist two subsets U and V of $\mathbb{R}$ with positive measure such that $||u(x)| - |u(y)|| < |u(x) - u(y)|$ for all $x \in U$ and $y \in V$. In addition, due to periodicity, we can assume that $|x - y| < L$ for all $x \in U$ and all $y \in V$ making the sets U and V smaller if necessary. Let us now use that $K > 0$ in $(0, L)$ (actually $K > 0$ everywhere in cases $(i)'$ and $(ii)'$) and that $[|u|^{*\,\mathrm{per}}]_K \leq [|u|]_K$, since we will have already proved part (a) for nonnegative functions. We deduce that $[|u|^{*\,\mathrm{per}}]_K \leq [|u|]_K < [u]_K = [u^{*\,\mathrm{per}}]_K$, leading to a contradiction with the fact that $u^{*\,\mathrm{per}} = |u|^{*\,\mathrm{per}}$.

To prove part (b) of the theorem, since we now know that u does not change sign, we will later assume $u \geq 0$ and prove that $u = u^{*\,\mathrm{per}}(\cdot + z)$ for some $z \in \mathbb{R}$. In case that $u \leq 0$ we may replace u by $-u$ (since $(-u)^{*\,\mathrm{per}} = u^{*\,\mathrm{per}}$ and $[-u]_K = [u]_K$) and then conclude $u = -u^{*\,\mathrm{per}}(\cdot + z)$.

Since in one step of the proof —namely, in (4.4) below— we will need the kernel to be integrable around its singularity, we first approximate it by bounded kernels. We will do this in a different way for each of the three classes (i), (ii), and (iii). This first step is not necessary if K is bounded.

We first deal with the case (i) of convex kernels. Let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence of positive numbers tending to zero and such that K is differentiable at t_i. We replace K in $(0, t_i)$ by the affine function which is tangent to the graph of K at the point $(t_i, K(t_i))$. Since K is convex, we obtain in this way a nondecreasing sequence of bounded and convex kernels K_i in $[0, +\infty)$ converging to K as $t_i \downarrow 0$. By monotone convergence, it thus suffices to show the claimed rearrangement inequality for each of the bounded convex kernels K_i.

Now we explain the approximation for K as in (ii). In this case we consider, for $i = 1, 2, 3, \ldots$, the kernels $K_i(t) := K(\sqrt{t^2 + 1/i^2})$ for $t \geq 0$. If K is as in (1.15) for some nonnegative measure μ, then

$$K_i(t) = \int_0^{+\infty} d\mu(r)\, e^{-r/i^2} e^{-t^2 r} \quad \text{for all } t \geq 0. \tag{4.2}$$

By the comments following Theorem 1.4, $\tau > 0 \mapsto K_i(\tau^{1/2})$ is completely monotonic since it is the Laplace transform of the nonnegative measure $e^{-r/i^2} d\mu(r)$. Also, by monotone convergence we see that $K_i(t) \uparrow K(t)$ for all $t > 0$ as $i \uparrow +\infty$. Finally, for any given $i \geq 1$, we have $K_i(t) \leq K_i(0) = K(1/i) < +\infty$, and hence K_i are bounded kernels in $[0, +\infty)$. Once again, by monotone convergence, it thus suffices to show the claimed rearrangement inequality for each of the bounded kernels K_i.

In the case (iii), we define $K_i(t) := K(t)$ if $t \geq 1/i$ and $K_i(t) =: K(1/i)$ if $t \in (0, 1/i)$. We have that K_i is bounded and still belongs to the class (iii).

Now, in whichever of the three ways that K_i has been defined, set

$$\overline{K}_i(t) := \sum_{k\in\mathbb{Z}} K_i(|t + 2kL|) \quad \text{for } t \in \mathbb{R}.$$

Note that $\overline{K}_i$ is bounded uniformly in $t \in \mathbb{R}$ by the upper bound (1.8) and since K_i is bounded. Using that u is $2L$-periodic, we write

$$2[u]^2_{K_i} = \int_{-L}^{L} dx \int_{\mathbb{R}} dy\, |u(x) - u(y)|^2 K_i(|x - y|) = \int_{-L}^{L} dx \int_{-L}^{L} dy\, |u(x) - u(y)|^2 \overline{K}_i(x - y). \quad (4.3)$$

Since $\overline{K}_i$ is integrable around the origin, we can expand $|u(x) - u(y)|^2$ to get

$$\begin{aligned}\int_{-L}^{L} dx \int_{-L}^{L} dy\, |u(x) - u(y)|^2 \overline{K}_i(x - y) = &\int_{-L}^{L} dx\, u(x)^2 \int_{-L}^{L} dy\, \overline{K}_i(x - y) \\ &+ \int_{-L}^{L} dy\, u(y)^2 \int_{-L}^{L} dx\, \overline{K}_i(x - y) \\ &- 2 \int_{-L}^{L} dx \int_{-L}^{L} dy\, u(x)u(y)\overline{K}_i(x - y).\end{aligned} \quad (4.4)$$

Note that $\overline{K}_i$ is even with respect to $x = 0$, $2L$-periodic, and bounded. Hence, we have that

$$\int_{-L}^{L} dy\, \overline{K}_i(x - y) = \int_{-L}^{L} \overline{K}_i < +\infty.$$

In addition, $\int_{-L}^{L} u^2 = \int_{-L}^{L} (u^{*\,\mathrm{per}})^2$. Therefore, the claimed inequality of the theorem will follow if we show that

$$\int_{-L}^{L} dx \int_{-L}^{L} dy\, u(x)u(y)\overline{K}_i(x - y) \leq \int_{-L}^{L} dx \int_{-L}^{L} dy\, u^{*\,\mathrm{per}}(x)u^{*\,\mathrm{per}}(y)\overline{K}_i(x - y). \quad (4.5)$$

To prove (4.5), by scaling we may assume that $L = \pi$. According to Theorem 4.1, to get (4.5) it is enough to show that $\overline{K}_i$ is a nonnegative and nonincreasing function in $(0, \pi)$. That $\overline{K}_i \geq 0$ is clear from its definition in terms of K_i. We now show the monotonicity of $\overline{K}_i$ in two different ways, depending on whether we are in assumption (i) or (ii) of the theorem. Note that in the case (iii) it holds that $\overline{K}_i(t) = K_i(|t|)$ for all $t \in (-\pi, \pi)$, and hence the monotonicity of $\overline{K}_i$ is an immediate consequence of the assumptions in (iii).

Proof of the inequality for K as in (i):

To show the monotonicity of $\overline{K}_i$ in $(0, \pi)$, we rename by $-k - 1$, with $k = 0, 1, 2, \ldots$, those indices which are negative in the definition of $\overline{K}_i$. In this way we obtain

$$\overline{K}_i(t) = \sum_{k\geq 0} \big(K_i(2k\pi + t) + K_i(2(k + 1)\pi - t)\big) \quad \text{for } t \in (0, \pi).$$

Thus, if $0 < t_1 < t_2 < \pi$, we have

$$\begin{aligned}\overline{K}_i(t_1) - \overline{K}_i(t_2) = &\sum_{k\geq 0} \big(K_i(2k\pi + t_1) + K_i(2(k + 1)\pi - t_1) \\ &- K_i(2k\pi + t_2) - K_i(2(k + 1)\pi - t_2)\big).\end{aligned} \quad (4.6)$$

We now observe that if f is a convex function, $a<b<c<d$, and $a+d=b+c$, then

$$f(b)+f(c)\le f(a)+f(d), \qquad \text{with strict inequality if } f \text{ is stictly convex in } [a,d]. \tag{4.7}$$

This follows from convexity, since for $\lambda=(d-b)/(d-a)\in(0,1)$ we have that $b=\lambda a+(1-\lambda)d$ and $c=(1-\lambda)a+\lambda d$.

Using (4.7) with $a=2k\pi+t_1$, $b=2k\pi+t_2$, $c=2(k+1)\pi-t_2$, and $d=2(k+1)\pi-t_1$, we see that each term in the sum over $k\ge 0$ of (4.6) is nonnegative. Thus, $\overline{K}_i$ is nonincreasing in $(0,\pi)$. This leads to (4.5) and finishes the proof of the inequality stated in the theorem for convex kernels K.

Case of equality for K as in $(i)'$:

We now assume that K is convex in $(0,+\infty)$ and strictly convex in $(c,+\infty)$ for some $c>0$. Recall that we may assume $u\ge 0$.

Set $M(t):=K(t)-K_1(t)$, where K_1 is the kernel constructed at the beginning of the proof for $i=1$. Recall that t_1 is a point of differentiability of K (for which we may assume that $0<t_1<c$) and that K_1 is characterized by: $K_1=K$ in $(t_1,+\infty)$, K_1 is affine in $(0,t_1)$, K_1 is convex in $(0,+\infty)$, and $K_1\le K$ in $(0,+\infty)$. Observe that both K_1 and M are nonnegative convex functions in $(0,+\infty)$ that satisfy the upper bound (1.8). We now split the integral

$$\int_{-L}^{L}dx\int_{\mathbb{R}}dy\,|u(x)-u(y)|^2K(|x-y|)=\mathcal{E}_1(u)+\mathcal{M}(u), \tag{4.8}$$

where

$$\begin{aligned}\mathcal{E}_1(u)&:=\int_{-L}^{L}dx\int_{\mathbb{R}}dy\,|u(x)-u(y)|^2K_1(|x-y|),\\ \mathcal{M}(u)&:=\int_{-L}^{L}dx\int_{\mathbb{R}}dy\,|u(x)-u(y)|^2M(|x-y|).\end{aligned} \tag{4.9}$$

By the inequality of Theorem 1.4 (i) (applied now to the kernels K_1 and M) we have $\mathcal{E}_1(u^{*\,\mathrm{per}})\le\mathcal{E}_1(u)$ and $\mathcal{M}(u^{*\,\mathrm{per}})\le\mathcal{M}(u)$. Combining these two inequalities with the assumption that $[u^{*\,\mathrm{per}}]_K=[u]_K<+\infty$, we deduce $\mathcal{E}_1(u^{*\,\mathrm{per}})=\mathcal{E}_1(u)$ (observe here that all these quantities are finite, since $[u^{*\,\mathrm{per}}]_K=[u]_K<+\infty$, and $0\le K_1\le K$ and $0\le M\le K$). It now follows from (4.3) and (4.4) (which required the kernel K_1 to be integrable) that there must be equality in (4.5) for $i=1$, that is, for the kernel K_1.

Now, as before, by scaling we may assume $L=\pi$. From (4.6) and (4.7) we obtain that $\overline{K}_1$ is decreasing in $(0,L)=(0,\pi)$, since K_1 is convex in $(0,+\infty)$ and strictly convex in $(c,+\infty)$ (and, hence, for each $k>c/(2\pi)$ the k-th summand in (4.6) is positive, and all others are nonnegative). As there must be equality in (4.5) for $\overline{K}_i=\overline{K}_1$, we deduce from the strict Riesz rearrangement inequality on the circle (Theorem 4.1 (i) and (ii); which requires the kernel $\overline{K}_1$ to be decreasing in $(0,\pi)$ and also $u\ge 0$) that a translate of u is equal to $u^{*\,\mathrm{per}}$.

Proof of the inequality for K as in (ii):

To check the monotonicity of $\overline{K}_i$, from (4.2) we see that

$$\overline{K}_i(t)=\sum_{k\in\mathbb{Z}}K_i(|t+2kL|)=\sum_{k\in\mathbb{Z}}\int_0^{+\infty}d\mu(r)\,e^{-r/i^2}e^{-(t+2kL)^2r}=\int_0^{+\infty}d\mu(r)\,e^{-r/i^2}\,\Phi(t,r), \tag{4.10}$$

where $\Phi(t,r) := \sum_{k\in\mathbb{Z}} e^{-(t+2kL)^2 r}$. Now we claim that, for every $r>0$, the function $t \mapsto \Phi(t,r)$ is decreasing in $(0,L)$. This follows from the fact that the fundamental solution of the heat equation with $2L$-periodic boundary conditions is decreasing in $(0,L)$; we refer to [15, Appendix B] where one can find several references for this result, but also an elementary self-contained proof. Therefore, by (4.10) and since the measure μ is nonnegative, $\overline{K}_i$ is nonincreasing in $(0,L)$. By the comments right after (4.5), this finishes the proof.

Case of equality for K as in $(ii)'$:

The arguments will be similar to the ones above for the case $(i)'$. Recall that we may assume $u \geq 0$. Set $M(t) := K(t) - K_1(t)$, where $K_1(t) = K(\sqrt{t^2+1})$. Hence

$$M(t) = \int_0^{+\infty} d\mu(r)\,(1-e^{-r})e^{-t^2 r}.$$

Observe that both $K_1(\sqrt{\tau})$ and $M(\sqrt{\tau})$ are completely monotonic functions of $\tau \in (0,+\infty)$ as can be seen by differentiating under the integral sign. Both also satisfy the upper bound (1.8). We now perform the splitting (4.8) but using the current definitions of K_1 and M in (4.9). From part (ii) of the theorem, we know that $\mathcal{E}_1(u^{*\,\mathrm{per}}) \leq \mathcal{E}_1(u)$ and $\mathcal{M}(u^{*\,\mathrm{per}}) \leq \mathcal{M}(u)$. Now, combining these two inequalities with the assumption that $[u^{*\,\mathrm{per}}]_K = [u]_K < +\infty$, we deduce $\mathcal{E}_1(u^{*\,\mathrm{per}}) = \mathcal{E}_1(u)$. Since K_1 is bounded in $[0,+\infty)$, we can argue as in (4.3) and (4.4) to deduce that there must be equality in (4.5) for $i=1$. As before, by scaling we can also assume $L=\pi$. Since we know that $\overline{K}_1$ is decreasing in $(0,\pi)$ by (4.10) (we use here the hypothesis $K \not\equiv 0$ to ensure that $\mu \not\equiv 0$), from the strict Riesz rearrangement inequality on the circle (see Theorem 4.1 (i) and (ii)) we deduce that a translate of u is equal to $u^{*\,\mathrm{per}}$.

Proof of the inequality for K as in (iii):

As pointed out after (4.5), after rescaling to have $L=\pi$, the proof in this case is clear, since $\overline{K}_i = K_i(|\cdot|)$ is nonincreasing in $(0,\pi)$ for all $i \geq 1$.

Case of equality for K as in $(iii)'$:

The argument is very similar to the previous ones. From them we see that it suffices to split K as $K = K_0 + M$, where K_0 and M both satisfy the hypothesis of Theorem 1.4 (iii), and in addition K_0 is bounded and decreasing in $(0,L)$.[18]

We accomplish this as follows. Define $K_0 : [0,+\infty) \to [0,1]$ by

$$K_0(t) := 1 - \frac{1}{K(t)+1} = \frac{K(t)}{K(t)+1}\,.$$

Note that K_0 is bounded, vanishes in $[L,+\infty)$, and is decreasing in $[0,L]$ (since K is decreasing in this interval). It remains to check that $M(t) := K(t) - K_0(t)$ is nonnegative, satisfies the upper bound (1.8), vanishes in $[L,+\infty)$, and is nonincreasing in $(0,L)$. The first three properties are obvious, since $M(t)$ is given by

$$M(t) = \frac{K(t)^2}{K(t)+1} \leq K(t).$$

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[18] Note that using K_1 (i.e., the cutoff of K described in the first step of the proof of the theorem for case (iii)) as a candidate for K_0 will not work, since K_1 is constant near 0.

Concerning the monotonicity of M, let $0 < t_1 < t_2 \leq L$. Since $K(t_1) \geq K(t_2)$, we have

$$\frac{K^2(t_1)}{K^2(t_2)} \geq \frac{K(t_1)}{K(t_2)} \geq \frac{K(t_1)+1}{K(t_2)+1},$$

which shows the monotonicity of M. ☐

5. A STRONG MAXIMUM PRINCIPLE FOR PERIODIC WEAK SOLUTIONS

To prove the strict monotonicity in the statements of Theorem 1.1 and Corollary 1.5, we need the strong maximum principle Theorem 1.6 for periodic solutions. We had never seen such type of result for integro-differential equations in the periodic setting.

Towards the proof, let us try to determine the sign of $\mathcal{L}_K v(x_0)$ at some $x_0 \in (0, L)$ where $v(x_0) = 0$ by splitting the integral over $\mathbb{R}$ in the definition of $\mathcal{L}_K$ into intervals of length L. Recall that, in Theorem 1.6, v is odd, $2L$-periodic, and with $v \leq 0$ in $(0, L)$. For the first two intervals, the integral of $-v(x)K(|x_0 - x|)$ over $(-L, 0) \cup (0, L)$ will give a positive sign, since v is odd and the negative mass of v is closer to x_0 than the positive one. Instead, by the same reason, the integral over $(-2L, -L) \cup (L, 2L)$ will give a negative sign, and so on, leading to an alternating sequence. There are thus competing terms and it is not clear why $\mathcal{L}_K v(x_0)$ should be positive. If $K(\sqrt{\cdot})$ is completely monotonic we bypass this difficulty using the Laplace transform, as in the proof of Theorem 1.4, and obtain a sign for $\mathcal{L}_K v(x_0)$. Whereas if K is convex as in $(i)'$, a simpler direct proof is indeed possible, by comparing groups of four intervals of length L (and not only two). The details go as follows.

Proof of Theorem 1.6. We provide first a detailed proof for the case when $K(\sqrt{\cdot})$ is completely monotonic. At the end of the proof we show why the same proof works for the slightly simpler cases $(i)'$ and $(iii)'$, with the obvious adaptations.

For $K(\sqrt{\cdot})$ completely monotonic, we have split the proof in two cases for the sake of clarity of the ideas, although the second proof is valid for both cases.

Case $K(\sqrt{\cdot})$ completely monotonic and $s < 1/2$:

Recall that if $s < 1/2$, then the principal value in the definition of $\mathcal{L}_K v(x_0)$ is not necessary, since $x \mapsto |v(x_0) - v(x)||x_0 - x|^{-1-2s}$ belongs to $L^1(\mathbb{R})$; recall that $v \in C^{2s+\gamma}(\mathbb{R})$. This justifies the forthcoming calculations.

We prove the result by contradiction. Assume that v is not identically zero and that there exists $x_0 \in (0, L)$ such that $v(x_0) = 0$. Then, using the expression for K in (1.15) based on the Laplace transform, we have

$$\begin{aligned}\mathcal{L}_K v(x_0) &= \int_{\mathbb{R}} dx\, (v(x_0) - v(x))K(|x_0 - x|) = -\int_0^{+\infty} d\mu(r) \int_{\mathbb{R}} dx\, v(x) e^{-|x_0 - x|^2 r} \\ &= -\int_0^{+\infty} d\mu(r) \sum_{k\in\mathbb{Z}} \int_{-L}^{L} dx\, v(x) e^{-|x_0 - x + 2kL|^2 r} \\ &= -\int_0^{+\infty} d\mu(r) \int_{-L}^{L} dx\, v(x) \Phi(x_0 - x, r),\end{aligned} \tag{5.1}$$

where $\Phi(t, r) := \sum_{k\in\mathbb{Z}} e^{-(t+2kL)^2 r}$.

We know that the function $t \mapsto \Phi(t, r)$ is even, $2L$-periodic, and decreasing in $(0, L)$, as explained in the proof of Theorem 1.4 (ii); see the comment right after (4.10). Splitting the integration domain as $(-L,L)=(-L,0)\cup(0,L)$ and using that v is odd, one obtains that

$$\int_{-L}^{L} dx\, v(x)\Phi(x_0-x,r)=\int_0^L dx\, v(x)\left(\Phi(x_0-x,r)-\Phi(x_0+x,r)\right). \tag{5.2}$$

Since $\Phi(\cdot,r)$ is even, $2L$-periodic, and decreasing in $(0,L)$ it holds that

$$\Phi(x_0-x,r)=\Phi(|x_0-x|,r)>\Phi(x_0+x,r)$$

for all $x_0\in(0,L)$ and all $x\in(0,L)$. This follows from the inequalities $|x_0-x|<x_0+x$ and $|x_0-x|<2L-(x_0+x)$, which mean that $|x_0-x|$ is closer to 0 than x_0+x to 0 and to $2L$.

Finally, using that $v\le 0$ in $(0,L)$ and $v\not\equiv 0$ we obtain that (5.2) is negative and thus, by (5.1), that $\mathcal{L}_K v(x_0)>0$. This contradicts the inequality $\mathcal{L}_K v(x_0)-c(x_0)v(x_0)\le 0$, since we are assuming that $v(x_0)=0$.

Case $K(\sqrt{\cdot})$ completely monotonic and $1/2\le s<1$:

The proof follows that of the case $s<1/2$, with the only difference that we must to take into account the principal value in the definition of $\mathcal{L}_K$. We again assume that $v(x_0)=0$ for some $x_0\in(0,L)$. In order to have more symmetric expressions in what follows, we observe, since v is bounded, that

$$\lim_{\epsilon\downarrow 0}\int_{A_\epsilon} dx\,\frac{v(x_0)-v(x)}{|x_0-x|^{1+2s}}=0,\qquad \text{where } A_\epsilon:=\bigcup_{k\in\mathbb{Z}\setminus\{0\}}\left((x_0-\epsilon,x_0+\epsilon)+2kL\right),$$

and

$$\lim_{\epsilon\downarrow 0}\int_{D_\epsilon} dx\,\frac{v(x_0)-v(x)}{|x_0-x|^{1+2s}}=0,\qquad \text{where } D_\epsilon:=\bigcup_{k\in\mathbb{Z}}\left((-x_0-\epsilon,-x_0+\epsilon)+2kL\right).$$

Hence, proceeding as in (5.1) and (5.2) we obtain (using also the upper bound (1.8))

$$\begin{aligned}\mathcal{L}_K v(x_0)&=\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}\setminus(x_0-\epsilon,x_0+\epsilon)} dx\,(v(x_0)-v(x))K(|x_0-x|)\\&=\lim_{\epsilon\downarrow 0}\int_{\mathbb{R}\setminus(A_\epsilon\cup D_\epsilon\cup(x_0-\epsilon,x_0+\epsilon))} dx\,(v(x_0)-v(x))K(|x_0-x|)\\&=-\lim_{\epsilon\downarrow 0}\int_0^{+\infty} d\mu(r)\int_{(-L,L)\setminus((x_0-\epsilon,x_0+\epsilon)\cup(-x_0-\epsilon,-x_0+\epsilon))} dx\, v(x)\Phi(x_0-x,r)\\&=\lim_{\epsilon\downarrow 0}\int_0^{+\infty} d\mu(r)\int_{(0,L)\setminus(x_0-\epsilon,x_0+\epsilon)} dx\,(-v(x))\left(\Phi(x_0-x,r)-\Phi(x_0+x,r)\right).\end{aligned}$$

We assume again, to reach a contradiction, that there exists $z_0\in(0,L)$ such that $v(z_0)<0$. By what we have seen in the previous case $s<1/2$ concerning the function Φ, we know that the integrand satisfies

$$-v(x)\left(\Phi(x_0-x,r)-\Phi(x_0+x,r)\right)\ge 0\qquad \text{for all } x\in(0,L).$$

Hence, $\mathcal{L}_K v(x_0)$ is a limit of a nondecreasing sequence in ϵ, a sequence which, moreover, is positive for ϵ small enough (precisely when $z_0\notin[x_0-\epsilon,x_0+\epsilon]$). This shows that $\mathcal{L}_K v(x_0)>0$, which leads to the contradiction.

Case K convex or K as in $(iii)'$:

In this case (5.1) must be replaced by

$$\mathcal{L}_K v(x_0) = -\int_{-L}^{L} dx\, v(x)\overline{K}(x_0 - x), \qquad \text{where } \overline{K}(t) := \sum_{k\in\mathbb{Z}} K(|t + 2kL|).$$

Note that $\overline{K}$ has the same properties as $\Phi(\cdot, r)$: it is even, $2L$-periodic, and decreasing in $(0, L)$. The third property is obvious for K as in $(iii)'$. Instead, for K as in $(i)'$, it follows exactly arguing as in (4.6) and (4.7) (with K_i and π replaced by K and L), and as in the last paragraph of the proof of Theorem 1.4 (b) for $(i)'$. Hence, the entire argument of the proof in case $(ii)'$ applies to the current cases $(i)'$ and $(iii)'$, with the only modification that Φ is to be replaced by $\overline{K}$ and the integral in $d\mu(r)$ is to be removed. □

6. PROOFS OF THEOREM 1.1 AND COROLLARY 1.5

In this section we prove the symmetry and monotonicity results for constrained minimizers, Corollary 1.5 and Theorem 1.1 —the latter being essentially a special case of Corollary 1.5.

Proof of Corollary 1.5. Recall that $u^{*\,\mathrm{per}}\chi_{(-L,L)}$ is the Schwarz rearrangement of the absolute value $|u|\chi_{(-L,L)}$. Thus it will be useful to first assume that u is nonnegative. With $u \geq 0$, it follows that

$$\int_{-L}^{L} G(u) = \int_{-L}^{L} G(|u|) = \int_{-L}^{L} G(u^{*\,\mathrm{per}}).$$

The last equality follows from the fact that $u^{*\,\mathrm{per}}$ is a rearrangement of $|u|$ and by standard properties of rearrangements; see for instance [42, Section 3.3]. For the same reason the constraint $\int_{-L}^{L} \widetilde{G}(v) = c$ is satisfied by $u^{*\,\mathrm{per}}$ too. Thus, $u^{*\,\mathrm{per}} \in L^\infty(\mathbb{R})$ is an admissible competitor.

Since u is a constrained minimizer of $\mathcal{E}_{\mathcal{L}_K}$, we deduce that $[u]_K \leq [u^{*\,\mathrm{per}}]_K$. But the reversed inequality also holds by Theorem 1.4 (a). We conclude that there is equality, and by Theorem 1.4 (b), that $u = \pm u^{*\,\mathrm{per}}(\cdot + z)$ for some $z \in \mathbb{R}$. Since $u \geq 0$, we must have $u = u^{*\,\mathrm{per}}(\cdot + z)$. From this, the first conclusion of the corollary (i.e., after a translation, u is even and nonincreasing in $(0, L)$) follows.

Note that, when assuming $u \geq 0$, we do not need the condition $u \in L^\infty(\mathbb{R})$ in the previous argument in case we minimize among a class of functions which are not bounded, such as for instance $L^2(-L, L)$.

Let now u be arbitrary (not necessarily nonnegative). In view of the assumption $u \in L^\infty(\mathbb{R})$ there exists a constant d such that $u + d \geq 0$. Define now

$$G^d(t) := G(t - d), \qquad \text{and} \qquad \mathcal{E}^d_{\mathcal{L}_K}(v) := \frac{1}{4}[v]_K^2 - \int_{-L}^{L} G^d(v),$$

and $\widetilde{G}^d(t) := \widetilde{G}(t - d)$. It is easy to check that if u is a constrained minimizer of $\mathcal{E}_{\mathcal{L}_K}$ then $u + d$ is a constrained minimizer of $\mathcal{E}^d_{\mathcal{L}_K}$ under the constraint $\int_{-L}^{L} \widetilde{G}^d(v) = c$. Applying now the corollary to the nonnegative constrained minimizer $u + d$ we obtain that

$$u = (u + d)^{*\,\mathrm{per}}(\cdot + z) - d \qquad \text{for some } z \in \mathbb{R}.$$

Thus u has the claimed form, i.e., u is even and nonincreasing in $(0, L)$.

Let us now prove the second statement of the corollary. Note that, since $u \in L^\infty(\mathbb{R})$ and G and $\widetilde{G}$ belong to $C^{2+\delta}(\mathbb{R})$ for some $\delta > 0$, both $G(u)$ and $g(u)$ belong to $L^\infty(\mathbb{R})$. This justifies the forthcoming arguments. Also observe that for a constrained minimizer u the semi-norm $[u]_K$ must be finite, since $G(u)$ is bounded.

First, since $\mathcal{E}_{\mathcal{L}_K}(u) \le \mathcal{E}_{\mathcal{L}_K}(v)$ for all $2L$-periodic functions $v \in L^\infty(\mathbb{R})$, u is a critical point of $\mathcal{E}_{\mathcal{L}_K}$ among $2L$-periodic smooth perturbations satisfying the constraint and, thus, by Lemma 2.3 (and the paragraph following it), $\mathcal{L}_K u = g(u) + \lambda \widetilde{g}(u)$ in $\mathbb{R}$ in the periodic weak sense for some $\lambda \in \mathbb{R}$. Thanks to (2.4) in Lemma 2.1, this last equation also holds in the periodic distributional sense.

We can now apply the regularity Theorem 1.7 (*ii*) (with $\beta = 1 + \delta$) to the equation

$$\mathcal{L}_K u = h(u) \quad \text{in } \mathbb{R}, \qquad \text{where} \qquad h := g + \lambda \widetilde{g} \in C^{1+\delta}(\mathbb{R}).$$

Therefore we obtain that $u \in C^{1+\nu}(\mathbb{R})$ for some $\nu > 2s$. Hence $u' \in C^\nu(\mathbb{R})$ and we can differentiate the equation to obtain

$$\mathcal{L}_K u' = h'(u)u' \quad \text{in } \mathbb{R}.$$

In view of the evenness and monotonicity result of the corollary we know that u' is odd and $u' \le 0$ in $(0, L)$. Applying now Theorem 1.6 to $v = u'$, we obtain that $u' < 0$ in $(0, L)$, unless u is constant. □

Remark 6.1. The statement of Corollary 1.5 dealing with the monotonicity (but not the strict monotonicity) remains true for more general constraints than those of the form $\int_{-L}^{L} \widetilde{G}(u) = c$. Indeed, any constraint which is preserved by the symmetric decreasing rearrangement in $(-L, L)$ can be assumed, as long as G is even (or equivalently, depends only on the absolute value of its argument). Under such assumptions the proof remains identical, with the only difference that the first step —assuming u to be nonnegative— becomes superfluous.

An example for such a constraint is the quasi-norm in the Lorentz space $L^{p,q}(-L, L)$, which is given for $1 \le p < +\infty$ and $1 \le q \le +\infty$ by

$$\|u\|_{L^{p,q}(-L,L)} := \begin{cases} \left(p \displaystyle\int_0^{+\infty} dt\, t^{q-1} \big|\{x \in (-L, L) : |u(x)| > t\}\big|^{q/p}\right)^{1/q}, & 1 \le q < +\infty, \\ \sup_{t>0} t \big|\{x \in (-L, L) : |u(x)| > t\}\big|^{1/p}, & q = +\infty. \end{cases}$$

Note that $\|u\|_{L^{p,q}(-L,L)}$ cannot be expressed as $\int_{-L}^{L} \widetilde{G}(u)$ when $p \ne q$, but it is invariant under the symmetric decreasing rearrangement since it is defined in terms of level sets of $|u|$.

We will now prove Theorem 1.1. It follows from Corollary 1.5 and the results of Sections 2 and 8. Since it deals with the fractional Laplacian, let us make the following observation.

Remark 6.2. For both the kernel K of the fractional Laplacian and for K as in (1.14), the function $K(\tau^{1/2})$ is completely monotonic. This has been easily checked right after Definition 1.3. Alternatively, let us check that, for both kernels, the equivalent condition (1.15), with $\mu \ge 0$, regarding the Laplace transform, also holds. For this purpose, we apply the Laplace transform to the function $r \mapsto r^{\gamma-1}$, which amounts to the equality

$$w^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} dr\, r^{\gamma-1} e^{-wr}, \tag{6.1}$$

where $\Gamma(\gamma) := \int_0^{+\infty} dr\, r^{\gamma-1} e^{-r}$ is the Gamma function. The identity (6.1) follows simply by a change of variables $r \mapsto wr$ in the definition of the function Γ. To prove the completely monotonic property for both kernels, we use the equivalent condition (1.15), which reads as

$$K(t) = \int_0^{+\infty} d\mu(r)\, e^{-t^2 r} \quad \text{for all } t > 0,$$

for some nonnegative measure μ. We conclude the claim from (6.1) applied to, respectively, $w = t^2$ and $\gamma = (1+2s)/2$, and $w = t^2 + a^2$ and $\gamma = (n+s)/2$. In particular we obtain that (1.15) is satisfied for the measures, respectively,

$$d\mu(r) = \frac{1}{\Gamma(\frac{1+2s}{2})} r^{s-1/2} dr, \qquad \text{and} \qquad d\mu(r) = \frac{1}{\Gamma(\frac{n+s}{2})} r^{(n+s)/2-1} e^{-a^2 r} dr.$$

Proof of Theorem 1.1. The statements (a) and (c) are a special case of Corollary 1.5, applied with $K(t) = c_s t^{-1-2s}$. This kernel satisfies both $(i)'$ and $(ii)'$ in the assumptions of Corollary 1.5, as pointed out in Remark 6.2.

The proof of (b) follows the same line as the one in Corollary 1.5. The difference is that g and $\widetilde{g}$ belong now to $C^{\delta}(\mathbb{R})$ only (instead of $C^{1+\delta}(\mathbb{R})$), and hence $(-\Delta)^s u = h(u)$ in $\mathbb{R}$ in the periodic distributional sense for some $h \in C^{\delta}(\mathbb{R})$. We can now apply the regularity Theorem 1.7 (i) or (ii), depending on the value of δ (called β there). Observe that in both cases (i) and (ii) we deduce that $u \in C^{\alpha}(\mathbb{R})$ for some $\alpha > 2s$. Then, using (2.5), we obtain that $(-\Delta)^s u = h(u)$ in $\mathbb{R}$ in the classical sense. □

7. $L^{\infty}(\mathbb{R})$ ESTIMATES FOR PERIODIC WEAK SOLUTIONS

In this section we give an L^{∞} estimate for periodic weak solutions to subcritical semilinear equations. More precisely, under standard subcritical-type assumptions on the nonlinearity f, we show that every periodic weak solution to the semilinear equation $\mathcal{L}_K u = f(x,u)$ in $\mathbb{R}$ is bounded. Here, the kernel K is only assumed to be even and satisfy the lower bound (1.9). To state the result, for $s < 1/2$, let us denote by

$$2_s^* := \frac{2}{1-2s} = \frac{1+2s}{1-2s} + 1 \tag{7.1}$$

the fractional Sobolev exponent.

Proposition 7.1. *Let K satisfy the lower bound (1.9) for some $0 < s < 1$ and some $\lambda > 0$. Let $u : \mathbb{R} \to \mathbb{R}$ be a $2L$-periodic function with $u \in L^2(-L,L)$ and $[u]_K < +\infty$.*

- (i) *Assume that $1/2 < s < 1$. Then, $\|u\|_{L^{\infty}(\mathbb{R})} \le C(\|u\|_{L^2(-L,L)} + [u]_K) < +\infty$ for some constant C depending only on L, s, and λ.*
- (ii) *Assume that $0 < s \le 1/2$ and that u is a $2L$-periodic weak solution to*

$$\mathcal{L}_K u = f(x,u) \quad \text{in } \mathbb{R}, \tag{7.2}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is $2L$-periodic in the first variable and satisfies

$$|f(x,t)| \le C_0(1+|t|^p) \quad \text{for all } (x,t) \in (-L,L) \times \mathbb{R}, \tag{7.3}$$

for some constant C_0, some exponent $1 \le p < 2_s^* - 1$ if $0 < s < 1/2$, and some $1 \le p < +\infty$ if $s = 1/2$. Then, $\|u\|_{L^{\infty}(\mathbb{R})} \le C$ for some constant C depending only on L, s, λ, C_0, p, $\|u\|_{L^2(-L,L)}$, and $[u]_K$.

The proof of Proposition 7.1 (ii) will follow the classical argument of Brezis and Kato [11]; see also [48, B.3 Lemma]. This argument has been already adapted to the fractional framework; see, possibly among others, [30, Proposition 5.1.1] for a nonperiodic semilinear problem and [6, Lemma 2.3] for a periodic linear problem, both treating only the operator $(-\Delta)^s$. We carry this out here for the operator $\mathcal{L}_K$ putting, in addition, special care on tracking what the constant C in Proposition 7.1 depends on.

To carry out the proof, we will use the auxiliary Lemma 7.2, which plays the role of the chain rule in the local case. More precisely, in the case of the Laplacian, one starts by testing the equation against u^{2r+1}. Then, from the chain rule, one uses

$$\frac{2r+1}{(r+1)^2}|\nabla(u^{r+1})|^2 = (2r+1)u^{2r}|\nabla u|^2 = \nabla u \cdot \nabla(u^{2r+1}).$$

It is precisely to adapt this step to the nonlocal operator $\mathcal{L}_K$ where we make use of the following simple lemma. In its statement, the left hand side of the inequality plays the role of $|\nabla(u^{r+1})|^2$ in the local framework, while the right hand side plays the role of $\nabla u \cdot \nabla(u^{2r+1})$ —this later is the one which links (7.2) to its weak formulation.

Lemma 7.2. *For every a and b in $\mathbb{R}$, and every $M \geq 0$ and $r \geq 0$, it holds*

$$\big(a\min\{|a|,M\}^r - b\min\{|b|,M\}^r\big)^2 \leq 2(r+2)(a-b)\big(a\min\{|a|,M\}^{2r} - b\min\{|b|,M\}^{2r}\big).$$

Proof. Assume first that $ab \leq 0$. Then,

$$\begin{aligned}&\big(a\min\{|a|,M\}^r - b\min\{|b|,M\}^r\big)^2\\ &\quad= a^2\min\{|a|,M\}^{2r} + b^2\min\{|b|,M\}^{2r} - 2ab\min\{|a|,M\}^r\min\{|b|,M\}^r\\ &\quad\leq a^2\min\{|a|,M\}^{2r} + b^2\min\{|b|,M\}^{2r} - ab\big(\min\{|a|,M\}^{2r} + \min\{|b|,M\}^{2r}\big)\\ &\quad= (a-b)\big(a\min\{|a|,M\}^{2r} - b\min\{|b|,M\}^{2r}\big),\end{aligned}$$

and the lemma follows in this case.

Therefore, from now on we can assume that $ab > 0$. Note also that the case where both a and b are negative numbers follows easily once we know the inequality for all positive a and b. In addition, if $a = b$ the statement is clear, and hence we can assume $a \neq b$ and, by symmetry, we can further assume that

$$0 < b < a.$$

At this point, we split the problem in three possible situations, as follows.

Case $M < b < a$: Then,

$$\begin{aligned}\big(a\min\{|a|,M\}^r - b\min\{|b|,M\}^r\big)^2 &= (M^r a - M^r b)^2 = (a-b)(M^{2r}a - M^{2r}b)\\ &= (a-b)\big(a\min\{|a|,M\}^{2r} - b\min\{|b|,M\}^{2r}\big).\end{aligned}$$

Case $0 < b < a \leq M$: By the Cauchy-Schwarz inequality we see that

$$\begin{aligned}\big(a\min\{|a|,M\}^r - b\min\{|b|,M\}^r\big)^2 &= (a^{r+1} - b^{r+1})^2 = \Big(\int_b^a dt\,(r+1)t^r\Big)^2\\ &\leq (r+1)^2(a-b)\int_b^a dt\,t^{2r} = \frac{(r+1)^2}{2r+1}(a-b)(a^{2r+1} - b^{2r+1})\\ &\leq (r+1)(a-b)\big(a\min\{|a|,M\}^{2r} - b\min\{|b|,M\}^{2r}\big).\end{aligned}$$

Case $0 < b \leq M < a$: Then,

$$\begin{aligned}\big(a\min\{|a|,M\}^r - b\min\{|b|,M\}^r\big)^2 &= (M^r a - b^{r+1})^2\\ &\leq 2\Big(M^r a - (M^{2r}a)^{\frac{r+1}{2r+1}}\Big)^2 + 2\Big((M^{2r}a)^{\frac{r+1}{2r+1}} - b^{r+1}\Big)^2.\end{aligned} \tag{7.4}$$

Now, on the one hand, note that $M^r a \geq (M^{2r}a)^{\frac{r+1}{2r+1}} \geq M^{r+1}$. Thus

$$\begin{aligned}\Big(M^r a - (M^{2r}a)^{\frac{r+1}{2r+1}}\Big)^2 &\leq (M^r a - M^{r+1})^2 = (a-M)(M^{2r}a - M^{2r+1})\\ &\leq (a-b)\big(a\min\{|a|,M\}^{2r} - b\min\{|b|,M\}^{2r}\big).\end{aligned} \tag{7.5}$$

On the other hand, by the Cauchy-Schwarz inequality and using that $M < a$, we get

$$\begin{aligned}&\Big((M^{2r}a)^{\frac{r+1}{2r+1}} - b^{r+1}\Big)^2 = \Big(\int_b^{(M^{2r}a)^{\frac{1}{2r+1}}} dt\,(r+1)t^r\Big)^2\\ &\leq (r+1)^2\big((M^{2r}a)^{\frac{1}{2r+1}} - b\big)\int_b^{(M^{2r}a)^{\frac{1}{2r+1}}} dt\,t^{2r} = \frac{(r+1)^2}{2r+1}\big((M^{2r}a)^{\frac{1}{2r+1}} - b\big)(M^{2r}a - b^{2r+1})\\ &\leq (r+1)(a-b)\big(a\min\{|a|,M\}^{2r} - b\min\{|b|,M\}^{2r}\big).\end{aligned}$$

Combining this with (7.4) and (7.5) we see that the lemma also holds for $0 < b \leq M < a$. □

We now prove the L^∞ estimate for periodic weak solutions.

Proof of Proposition 7.1. In the following, we denote

$$\|u\|_{W^{s,2}(-L,L)} := \Big(\int_{-L}^{L} |u|^2 + \int_{-L}^{L} dx \int_{-L}^{L} dy\, \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\Big)^{1/2}.$$

Clearly, the lower bound (1.9) gives

$$\|u\|^2_{W^{s,2}(-L,L)} \leq \|u\|^2_{L^2(-L,L)} + \frac{2}{\lambda}[u]_K^2 < +\infty. \tag{7.6}$$

The proof of (i) is straightforward. Since $1/2 < s < 1$, [32, Theorem 8.2] and (7.6) yield

$$\|u\|_{L^\infty(-L,L)} \leq C\|u\|_{W^{s,2}(-L,L)} \leq C\big(\|u\|_{L^2(-L,L)} + \sqrt{2\lambda^{-1}}[u]_K\big)$$

for some constant C depending only on L and s.

We now address the proof of (ii) for $0 < s < 1/2$. Given nonnegative constants M and r, and $t \in \mathbb{R}$, set

$$\varphi_{r,M}(t) := t\min\{|t|,M\}^r.$$

It is not hard to show that $\varphi_{r,M}(u)$ is a $2L$-periodic function with $\varphi_{r,M}(u) \in L^2(-L,L)$ and $[\varphi_{r,M}(u)]_K < +\infty$ for all $r \geq 0$ and $M \geq 0$. Using Lemma 7.2, taking $\varphi_{2r,M}(u)$ as a test function in the weak formulation of (7.2) given by (2.3) with $h = f(x,u)$, and using the subcritical growth (7.3), we see that

$$\begin{aligned}[\varphi_{r,M}(u)]_K^2 &\leq 2(r+2)\langle u, \varphi_{2r,M}(u)\rangle_K = 2(r+2)\int_{-L}^{L} f(x,u)\,\varphi_{2r,M}(u)\\ &\leq C_1(r+1)\int_{-L}^{L}(1+|u|^p)|\varphi_{2r,M}(u)|,\end{aligned} \tag{7.7}$$

where $C_1 = 4C_0$. Adding $\|\varphi_{r,M}(u)\|^2_{L^2(-L,L)}$ on both sides of (7.7), using (7.6), applying the fractional Sobolev inequality [32, Theorem 6.7] to the left-hand side, and using that

$$|u|^p|\varphi_{2r,M}(u)| = |u|^{p-1}\varphi_{r,M}(u)^2 \tag{7.8}$$

on the right-hand side, we deduce

$$\Big(\int_{-L}^{L}|\varphi_{r,M}(u)|^{2_s^*}\Big)^{2/2_s^*}\leq C_2(r+1)\Big(\int_{-L}^{L}|\varphi_{2r,M}(u)|+\int_{-L}^{L}|u|^{p-1}\varphi_{r,M}(u)^2+\int_{-L}^{L}\varphi_{r,M}(u)^2\Big) \tag{7.9}$$

for some constant C_2 depending only on L, λ, s, and C_0.

Since we are assuming that $p<2_s^*-1$, for every $\epsilon>0$ (to be chosen later on) there exists a constant $C_\epsilon>0$ (depending only on ϵ, s, and p) such that

$$t^{p-1}+1\leq \epsilon t^{2_s^*-2}+\frac{C_\epsilon}{t}\quad\text{for all } t>0.$$

In particular, from (7.9), the definition of $\varphi_{r,M}$, and (7.8), we deduce that

$$\Big(\int_{-L}^{L}|\varphi_{r,M}(u)|^{2_s^*}\Big)^{2/2_s^*}\leq C_2(r+1)\Big((1+C_\epsilon)\int_{-L}^{L}|\varphi_{2r,M}(u)|+\epsilon\int_{-L}^{L}|u|^{2_s^*-2}\varphi_{r,M}(u)^2\Big). \tag{7.10}$$

Now, by Hölder's inequality,

$$\int_{-L}^{L}|u|^{2_s^*-2}\varphi_{r,M}(u)^2\leq\Big(\int_{-L}^{L}|u|^{2_s^*}\Big)^{\frac{2_s^*-2}{2_s^*}}\Big(\int_{-L}^{L}|\varphi_{r,M}(u)|^{2_s^*}\Big)^{2/2_s^*},$$

and thus (7.10) finally yields

$$\begin{aligned}\Big(\int_{-L}^{L}|\varphi_{r,M}(u)|^{2_s^*}\Big)^{2/2_s^*}&\leq C_2(r+1)(1+C_\epsilon)\int_{-L}^{L}|\varphi_{2r,M}(u)|\\&\quad+C_2(r+1)\epsilon\|u\|_{L^{2_s^*}(-L,L)}^{2_s^*-2}\Big(\int_{-L}^{L}|\varphi_{r,M}(u)|^{2_s^*}\Big)^{2/2_s^*}.\end{aligned} \tag{7.11}$$

Recall that $\|u\|_{L^2(-L,L)}$ and $[u]_K$ are finite. Therefore, the fractional Sobolev inequality and (7.6) show that $\|u\|_{L^{2_s^*}(-L,L)}\leq C\|u\|_{W^{s,2}(-L,L)}<+\infty$.

Let us now choose $r_1>0$ such that $2r_1+1=2_s^*$, and also choose $\epsilon>0$ small enough depending only on L, λ, C_0, s, and $\|u\|_{W^{s,2}(-L,L)}$, such that

$$C_2(r_1+1)\epsilon\|u\|_{L^{2_s^*}(-L,L)}^{2_s^*-2}\leq\frac{1}{2}. \tag{7.12}$$

Then, from (7.11) we get that

$$\Big(\int_{-L}^{L}|\varphi_{r_1,M}(u)|^{2_s^*}\Big)^{2/2_s^*}\leq C_3\int_{-L}^{L}|\varphi_{2r_1,M}(u)| \tag{7.13}$$

for some constant C_3 depending only on L, λ, C_0, s, p, and $\|u\|_{W^{s,2}(-L,L)}$. Now, letting $M\uparrow+\infty$ and using monotone convergence in (7.13), in view of the definition of r_1 we conclude that

$$\||u|^{r_1+1}\|_{L^{2_s^*}(-L,L)}^2=\Big(\int_{-L}^{L}|u|^{2_s^*(r_1+1)}\Big)^{2/2_s^*}\leq C_3\int_{-L}^{L}|u|^{2r_1+1}=C_3\|u\|_{L^{2_s^*}(-L,L)}^{2_s^*}\leq C_4 \tag{7.14}$$

for some constant C_4 depending only on L, λ, C_0, s, p, and $\|u\|_{W^{s,2}(-L,L)}$.

We now come back to (7.10). We stress that ϵ has already been chosen according to (7.12). By letting $M\uparrow+\infty$ in (7.10) and using monotone convergence, we deduce that

$$\Big(\int_{-L}^{L}|u|^{2_s^*(r+1)}\Big)^{2/2_s^*}\leq C(r+1)\Big(\int_{-L}^{L}|u|^{2r+1}+\int_{-L}^{L}|u|^{2r+2_s^*}\Big) \tag{7.15}$$

for all $r \geq 0$, where the constant C depends only on L, λ, C_0, s, p, and $\|u\|_{W^{s,2}(-L,L)}$. Since $2_s^* > 1$, we clearly have

$$t^{2r+1} \leq 1 + t^{2r+2_s^*} \quad \text{for all } t \geq 0. \tag{7.16}$$

From (7.15) and (7.16) we get that

$$\Big(1 + \int_{-L}^{L} |u|^{2_s^*(r+1)}\Big)^{\frac{1}{2_s^* r}} \leq \big(C(r+1)\big)^{\frac{1}{2r}}\Big(1 + \int_{-L}^{L} |u|^{2r+2_s^*}\Big)^{\frac{1}{2r}}, \tag{7.17}$$

where C depends only on L, λ, C_0, s, p, and $\|u\|_{W^{s,2}(-L,L)}$.

We now define

$$r_{m+1} := \frac{2_s^*}{2} r_m = \Big(\frac{2_s^*}{2}\Big)^m r_1 \quad \text{for all } m \geq 1$$

(r_1 has already been defined by the relation $2r_1 + 1 = 2_s^*$). Clearly, $2r_{m+1} + 2_s^* = 2_s^*(r_m + 1)$, and $r_m \uparrow +\infty$ as $m \uparrow +\infty$. We also set

$$A_m := \Big(1 + \int_{-L}^{L} |u|^{2_s^*(r_m+1)}\Big)^{\frac{1}{2_s^* r_m}} \quad \text{for all } m \geq 1.$$

Combining (7.17) for $r = r_{m+1}$ with (7.14) we deduce that

$$\begin{aligned} A_{m+1} &\leq \Big(C\Big(\frac{2_s^*}{2}\Big)^m r_1 + C\Big)^{(\frac{2}{2_s^*})^m \frac{1}{2r_1}} A_m \leq \cdots \leq A_1 \prod_{j=1}^{m} \Big(C\Big(\frac{2_s^*}{2}\Big)^j r_1 + C\Big)^{(\frac{2}{2_s^*})^j \frac{1}{2r_1}} \\ &= A_1 \exp\Big\{\frac{1}{2r_1} \sum_{j=1}^{m} \Big(\frac{2}{2_s^*}\Big)^j \log\Big(C\Big(\frac{2_s^*}{2}\Big)^j r_1 + C\Big)\Big\} \leq C_5 \end{aligned}$$

for some constant $C_5 \geq 1$ depending only on L, λ, C_0, s, p, and $\|u\|_{W^{s,2}(-L,L)}$. Here we also used that $2_s^* > 2$ to see that the sum above is uniformly bounded in $m \geq 1$. Therefore,

$$\|u\|_{L^{2_s^*(r_m+1)}(-L,L)} \leq C_5^{\frac{r_m}{r_m+1}} \leq C_5$$

for all $m \geq 1$. Since this bound is uniform in m, letting $m \uparrow +\infty$ (and, thus, $r_m \uparrow +\infty$) we get $\|u\|_{L^\infty(-L,L)} \leq C_5$. By periodicity, $\|u\|_{L^\infty(\mathbb{R})} \leq C_5 < +\infty$, as desired.

Let us finally address the proof of *(ii)* for $s = 1/2$. In this case we are assuming (7.3) for some $1 \leq p < +\infty$. Since p is finite, we can take $0 < s' < 1/2$ such that $p < \frac{1+2s'}{1-2s'}$, which yields $p < 2_{s'}^* - 1$ by (7.1). The idea now is to carry out the proof made for the case $0 < s < 1/2$ but replacing s by s' everywhere. To do so, we only need to bound $\|u\|_{W^{s',2}(-L,L)}$ by $\|u\|_{L^2(-L,L)}$ and $[u]_K$. Note that (1.9) for $s = 1/2$ gives $K(t) \geq \lambda(2L)^{2s'-1}t^{-1-2s'}$ for all $0 < t \leq 2L$. Hence,

$$\|u\|_{W^{s',2}(-L,L)}^2 \leq \|u\|_{L^2(-L,L)}^2 + \frac{1}{\lambda}(2L)^{1-2s'}[u]_K^2, \tag{7.18}$$

as desired. Now, going back to the beginning of the proof, from the weak formulation of (7.2) we obtain (7.7). This is the only point where we used that u is a periodic weak solution. This means that, from (7.1) up to the end of the proof, we simply need to replace s by s', and (7.6) by (7.18), everywhere to conclude $\|u\|_{L^\infty(\mathbb{R})} < +\infty$. □

Remark 7.3. Let $0 < s < 1/2$. Consider a sequence of semilinear equations of the form

$$\mathcal{L}_{K_j} u_j = f_j(x, u_j) \quad \text{in } \mathbb{R},$$

where f_j is $2L$-periodic in the first variable. From Proposition 7.1 we see that, in the subcritical case $1 \leq p < 2_s^* - 1$, if

- $|f_j(x,t)| \leq C_0(1+|t|^p)$ for all j and all $(x,t) \in (-L,L) \times [0,+\infty)$,
- K_j satisfies (1.9) uniformly in j, and
- u_j is a periodic weak solution to $\mathcal{L}_{K_j} u_j = f_j(x,u_j)$ in $\mathbb{R}$ such that $\|u_j\|_{L^2(-L,L)}$ and $[u_j]_{K_j}$ are uniformly bounded in j (hence, $\|u_j\|_{W^{s,2}(-L,L)}$ too),

then $\|u_j\|_{L^\infty(\mathbb{R})}$ are uniformly bounded in j. We took a special care to track the dependence of the constants appearing in the proof of Proposition 7.1 (*ii*) precisely to get this type of uniform estimates. It can be useful, for instance, when one uses a monotone iteration argument to construct solutions to semilinear integro-differential equations.

8. $C^\alpha(\mathbb{R})$ ESTIMATES FOR BOUNDED PERIODIC DISTRIBUTIONAL SOLUTIONS

The aim of this section is to prove Theorem 1.7.

Before providing the proof, let us first comment on the optimality of the Hölder exponents appearing in Theorem 1.7. The ϵ appearing in the statement of Theorem 1.7 (*ii*) is due to the fact that $\beta+2s$ may be an integer. In particular, if $\beta+2s$ is not an integer, and if in the iteration argument of the proof no Hölder exponent ever is an integer, then we indeed get $u \in C^{\beta+2s}(\mathbb{R})$. On the contrary, the ϵ appearing in the statement of Theorem 1.7 (*i*) has a different nature. It comes from the fact that the iteration method of the proof yields $u \in C^{2s\beta_k}(\mathbb{R})$ for all $k \geq 1$, where β_k is a sequence of exponents such that $\beta_k \uparrow 1/(1-\beta)$ but $\beta_k < 1/(1-\beta)$ for all k.

Currently we do not know whether Theorem 1.7 (*i*) is sharp or if it holds true with $\epsilon = 0$ as well. But what we know for sure is that the exponent $2s/(1-\beta)$ cannot be improved, as shown in our forthcoming work [25].

We have seen that Proposition 2.4 allows us to apply the known regularity results for distributional Dirichlet solutions to the class of periodic distributional (and in particular also to weak) solutions. Thus, let us recall the following regularity result of [31]. Be aware that what we call here distributional solution is called weak solution in [31].

Theorem 8.1. ([31, Theorem 3.8]) *Let K satisfy* (1.8) *and* (1.9), *and let $u \in L^\infty(\mathbb{R})$ be a distributional (Dirichlet) solution to $\mathcal{L}_K u = h$ in $I = (-L,L)$, i.e., a solution satisfying* (2.14) *for all $\varphi \in C_c^\infty(I)$.*

Then, the following holds:

(*i*) *Assume that $h \in L^\infty(I)$. If $s \neq 1/2$, then $u \in C^{2s}(\overline{I}/2)$ and*

$$\|u\|_{C^{2s}(\overline{I}/2)} \leq C\big(\|u\|_{L^\infty(\mathbb{R})} + \|h\|_{L^\infty(I)}\big).$$

If $s = 1/2$, then for every $\epsilon > 0$, $u \in C^{2s-\epsilon}(\overline{I}/2)$ and

$$\|u\|_{C^{2s-\epsilon}(\overline{I}/2)} \leq C\big(\|u\|_{L^\infty(\mathbb{R})} + \|h\|_{L^\infty(I)}\big).$$

The constants C depend only on L, s, λ, and Λ (and also on ϵ in the case $s = 1/2$).

(*ii*) *Assume that $h \in C^\alpha(\overline{I})$ and $u \in C^\alpha(\mathbb{R})$ for some $\alpha > 0$. If $\alpha + 2s$ is not an integer, then $u \in C^{\alpha+2s}(\overline{I}/2)$ and*

$$\|u\|_{C^{\alpha+2s}(\overline{I}/2)} \leq C\big(\|u\|_{C^\alpha(\mathbb{R})} + \|h\|_{C^\alpha(\overline{I})}\big).$$

If $\alpha + 2s$ is an integer, then for every $\epsilon > 0$, $u \in C^{\alpha+2s-\epsilon}(\overline{I}/2)$ and

$$\|u\|_{C^{\alpha+2s-\epsilon}(\overline{I}/2)} \leq C\big(\|u\|_{C^\alpha(\mathbb{R})} + \|h\|_{C^\alpha(\overline{I})}\big).$$

The constants C depend only on L, s, α, λ, and Λ (and also on ϵ in the case that $\alpha + 2s$ is an integer).

The last case of (*ii*) in Theorem 8.1 ($\alpha + 2s$ is an integer) is not stated explicitly in [31, Theorem 3.8], but it follows from the first case of (*ii*) by replacing α by $\alpha - \epsilon$.

As we showed in Proposition 2.4, if we assume that h is $2L$-periodic, the conclusions of the Theorem 8.1 also hold for $2L$-periodic distributional solutions to $\mathcal{L}_K u = h$ in $\mathbb{R}$, replacing I (and $I/2$) by $\mathbb{R}$ in (*i*) and (*ii*). More precisely, since $\mathcal{L}_K u = h$ in the whole real line and u is $2L$-periodic, one can apply the theorem to the sets $I/2 + a \subset I + a$ for different values of $a \in \mathbb{R}$ (to cover a neighborhood of $\overline{I}$) and obtain an estimate on the Hölder norm of u in all of $\mathbb{R}$. Thus, we can use Theorem 8.1 to prove Theorem 1.7 on periodic solutions.

Proof of Theorem 1.7. We will prove the theorem by distinguishing several cases, but we will treat all of them essentially using a same bootstrap argument based on Theorem 8.1. We will also make use of the following simple facts regarding the composition of Hölder continuous functions.

Let $f \in C^{\beta}(\mathbb{R})$ and $v \in C^{\sigma}(\mathbb{R})$ be a $2L$-periodic function. Then, there exists some constant $C_{\beta,\sigma} > 0$ depending only on $\|f\|_{C^{\beta}(\mathbb{R})}$ and $\|v\|_{C^{\sigma}(\mathbb{R})}$ such that the following holds:

$$\text{If } 0 \le \beta, \sigma < 1, \text{ then } f(v) \in C^{\beta\sigma}(\mathbb{R}) \text{ and } \|f(v)\|_{C^{\beta\sigma}(\mathbb{R})} \le C_{\beta,\sigma}. \tag{8.1}$$

$$\text{If } \max\{\beta, \sigma\} \ge 1, \text{ then } f(v) \in C^{\min\{\beta,\sigma\}}(\mathbb{R}) \text{ and } \|f(v)\|_{C^{\min\{\beta,\sigma\}}(\mathbb{R})} \le C_{\beta,\sigma}. \tag{8.2}$$

The proof of these statements follows, for instance, from [24, Theorem 16.31] and the periodicity of v; [24, Theorem 16.31] does not cover the case $\beta \ge 1$ and $\sigma < 1$, but this case follows very easily from the fact that f is Lipschitz. As already mentioned right above this proof, the periodicity allows to extend the estimates of the Hölder norm of the composition $f(v)$ from bounded intervals to the whole real line.

Assume $s \neq 1/2$ (the case $s = 1/2$ will be explained at the end of the proof). Since $f(u) \in L^{\infty}(\mathbb{R})$, an application of Theorem 8.1 (*i*) gives

$$\|u\|_{C^{2s}(\overline{I}/2)} \le C\big(\|u\|_{L^{\infty}(\mathbb{R})} + \|f(u)\|_{L^{\infty}(I)}\big),$$

where $I = (-L, L)$. Since u is $2L$-periodic, this argument applies to all translates of u (since they are also periodic solutions), and thus we actually deduce that

$$\|u\|_{C^{2s}(\mathbb{R})} \le C\big(\|u\|_{L^{\infty}(\mathbb{R})} + \|f(u)\|_{L^{\infty}(\mathbb{R})}\big). \tag{8.3}$$

To simplify the exposition, we first assume in the second and fourth case below that β is not an integer, and that all the Hölder exponents appearing in the forthcoming arguments never coincide with an integer when using Theorem 8.1. Then, at the end of the proof we will explain what should be modified in case that some Hölder exponent eventually hits an integer when using Theorem 8.1.

<ins>*Case* $2s \le 1 - \beta$:</ins>

Since s and β are positive, we infer from $2s \le 1 - \beta$ that $0 < \beta < 1$ and $0 < s < 1/2$. Set

$$\beta_k := \sum_{j=0}^{k} \beta^j \quad \text{for } k = 1, 2, 3, \dots$$

Note that $\beta_k \uparrow 1/(1-\beta)$ as $k \uparrow +\infty$. Since we are assuming $2s/(1-\beta) \le 1$, we have $2s\beta_k < 1$ for all $k \ge 1$.

From (8.3) and the fact that $\beta < 1$ we have $u \in C^{2s}(\mathbb{R}) \subset C^{2s\beta}(\mathbb{R})$. Since $2s < 1$ and $\beta < 1$, (8.1) yields $f(u) \in C^{2s\beta}(\mathbb{R})$. Therefore, Theorem 8.1 (*ii*) gives $u \in C^{2s(\beta+1)}(\mathbb{R}) = C^{2s\beta_1}(\mathbb{R})$. Since $2s\beta_1 < 1$, from (8.1) once again we obtain $f(u) \in C^{2s\beta_1\beta}(\mathbb{R})$. Also, since $\beta < 1$, we have

$u \in C^{2s\beta_1\beta}(\mathbb{R})$. Applying Theorem 8.1 (*ii*) we get $u \in C^{2s\beta_2}(\mathbb{R})$ with $\beta_2 = \beta_1\beta + 1 = \beta^2 + \beta + 1$. Iterating this argument, and recalling that $2s\beta_k < 1$ for all $k \geq 1$, we see that $u \in C^{2s\beta_k}(\mathbb{R})$ for all $k \geq 1$. Using that $\beta_k \uparrow 1/(1-\beta)$, this yields $u \in C^{\frac{2s}{1-\beta}-\epsilon}(\mathbb{R})$ for all $\epsilon > 0$, as desired.

Case $2s > 1 - \beta$*, with* $0 < \beta < 1$*,* $0 < s < 1/2$*:*

Arguing as in the case $2s \leq 1 - \beta$, we see that $u \in C^{2s\beta_k}(\mathbb{R})$ whenever $2s\beta_{k-1} < 1$, with the understanding that $\beta_0 := \beta$. Note that $2s\beta_0 < 1$ and hence we can start the iteration. Recall that $\beta_k \uparrow 1/(1-\beta)$ and, since we are now assuming that $2s > 1-\beta$, we have $2s/(1-\beta) > 1$. Hence, there exists a unique $k_0 \geq 1$ such that $2s\beta_{k_0} \geq 1$ and $2s\beta_{k_0-1} < 1$. Now, since $u \in C^{2s\beta_{k_0}}(\mathbb{R})$ and $2s\beta_{k_0} \geq 1$, we have $u \in C^1(\mathbb{R})$. Thus, $f(u) \in C^\beta$ by (8.2). Applying Theorem 8.1 (*ii*) we deduce that $u \in C^{\beta+2s-\epsilon}(\mathbb{R})$ for all $\epsilon > 0$, as desired.

Case $2s > 1 - \beta$*, with* $0 < \beta < 1$*,* $1/2 < s < 1$*:*

Since $\beta < 1 < 2s$, from (8.3) we get $u \in C^1(\mathbb{R})$. Thus, $f(u) \in C^\beta(\mathbb{R})$ by (8.2). Applying Theorem 8.1 (*ii*) we end up with $u \in C^{\beta+2s-\epsilon}(\mathbb{R})$ for all $\epsilon > 0$.

Case $\beta > 1$ *(and hence* $2s > 1-\beta$*) and* $s \neq 1/2$*:*

Let $j_0 \geq 1$ be an integer such that $2s(j_0 - 1) < \beta \leq 2sj_0$. From Theorem 8.1 with $\alpha = 2sj$ and the periodicity of u we see that

$$\|u\|_{C^{2sj}(\mathbb{R})} \leq C\big(\|u\|_{C^{2s(j-1)}(\mathbb{R})} + \|f(u)\|_{C^{2s(j-1)}(\mathbb{R})}\big) \tag{8.4}$$

whenever the right-hand side is finite —recall that we are assuming that $2sj \notin \mathbb{Z}$. Here, (8.4) for $j = 1$ must be interpreted as (8.3). Observe that if $u \in C^{2sj}(\mathbb{R})$ and $\beta \geq 1$, then $f(u) \in C^{\min\{\beta,2sj\}}(\mathbb{R})$ by (8.2). Therefore, (8.4) can be iterated for $j = 1, \dots, j_0$ to get $u \in C^{2sj_0}(\mathbb{R}) \subset C^\beta(\mathbb{R})$, which yields $f(u) \in C^\beta(\mathbb{R})$ by (8.2). Now, Theorem 8.1 leads to $u \in C^{\beta+2s-\epsilon}(\mathbb{R})$ for all $\epsilon > 0$, as desired.

Case $s = 1/2$*, or* β *is an integer, or if we hit an integer (in the 2nd and 4th case above):*

Assume first that $s \neq 1/2$. During the iteration processes explained in the previous cases, if we know that u and $f(u)$ belong to $C^\alpha(\mathbb{R})$ for some $\alpha < \beta$ and it turns out that $\alpha + 2s$ hits an integer (clearly, this situation cannot happen in the first and third case), we simply diminish a bit the exponent α in order to avoid that integer and we continue the iteration.

Finally, the case $s = 1/2$ is treated analogously but, instead of (8.3), using that

$$\|u\|_{C^{2s-\epsilon}(\mathbb{R})} \leq C\big(\|u\|_{L^\infty(\mathbb{R})} + \|f(u)\|_{L^\infty(\mathbb{R})}\big)$$

for any $\epsilon > 0$ arbitrarily small, which follows from Theorem 8.1 (*i*) and periodicity. □

Remark 8.2. With the same iteration method used here one can show that the analogue of Theorem 1.7 in the nonperiodic scenario also holds for regular enough kernels K satisfying (1.8) and (1.9). Note that from the first iteration one would only get a bound on the Hölder regularity of u in $\overline{I}/2$, but not in the whole real line. Hence, one cannot use Theorem 8.1 (*ii*) for the next iteration since its right-hand side involves $\|u\|_{C^\alpha(\mathbb{R})}$. However, for more regular kernels there is an improvement of Theorem 8.1 (*ii*). More precisely, if K additionally satisfies

$$[K]_{C^\alpha(\mathbb{R}^n\setminus B_r)} \leq \Lambda r^{-n-2s-\alpha} \quad \text{for all } r > 0, \tag{8.5}$$

then

$$\|u\|_{C^{\alpha+2s}(\overline{I}/2)} \leq C\big(\|u\|_{L^\infty(\mathbb{R})} + \|h\|_{C^\alpha(\overline{I})}\big) \quad \text{if } \alpha + 2s \text{ is not an integer;}$$

see [31, Theorem 3.8 (*c*)].

Therefore, for kernels K satisfying (8.5) one can obtain the following result: *given $f \in C^{\beta}(\mathbb{R})$ for some $\beta > 0$, every $u \in L^{\infty}(\mathbb{R})$ distributional solution to $\mathcal{L}_K u = f(u)$ in $I = (-L, L)$ satisfies*

(i) $u \in C^{\frac{2s}{1-\beta}-\epsilon}(\overline{I}/2)$ *for all* $\epsilon > 0$ *if* $2s < 1-\beta$,

(ii) $u \in C^{\beta+2s-\epsilon}(\overline{I}/2)$ *for all* $\epsilon > 0$ *if* $2s \geq 1-\beta$.

In the periodic scenario this difficulty does not appear thanks to the periodicity of u, even for nonregular kernels.

APPENDIX A. THE HALF-LAPLACIAN ON $\mathbb{S}^1$

We shall now explain the relation between the integro-differential expressions for the half-Laplacian on $\mathbb{R}$ for periodic functions and the half-Laplacian on $\mathbb{S}^1$. Any 2π-periodic function u on the real line can be identified with a function $u_{\mathbb{S}^1}$ in the unit circle by $u_{\mathbb{S}^1}(e^{ix}) = u(x)$ for all $x \in \mathbb{R}$. With this at hand, we will see below that $(-\Delta)^{1/2}u$ corresponds to the Dirichlet-to-Neumann map for the harmonic extension of $u_{\mathbb{S}^1}$ to the unit disk. This map also has an integro-differential expression on $\mathbb{S}^1$ for $u_{\mathbb{S}^1}$. It is noticeable that the expression that one gets for $u_{\mathbb{S}^1}$ on $\mathbb{S}^1$ is the same as the one for u on $\mathbb{R}$, namely (1.2) with $s = 1/2$, except that $\mathbb{R}$ must be replaced by $\mathbb{S}^1$ and u by $u_{\mathbb{S}^1}$, but the kernel remains the same. The purpose of this section is to give the details of all these observations.

By definition,

$$(-\Delta)^{1/2}u(x) := \frac{1}{\pi}\,\mathrm{P.V.}\int_{\mathbb{R}} dy\, \frac{u(x)-u(y)}{|x-y|^2} \tag{A.1}$$

for all $u : \mathbb{R} \to \mathbb{R}$ whenever the integral and the limit make sense. Assume that u is 2π-periodic. Then, using the Fourier series expansion $u(x) = \sum_{k\in\mathbb{Z}} u_k e^{ikx}$, where $u_k := \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, u(x)e^{-ikx}$, we know form Lemma 3.1 that

$$(-\Delta)^{1/2}u(x) = \sum_{k\in\mathbb{Z}} |k| u_k e^{ikx}.$$

To the 2π-periodic function u, let us associate the function $u_{\mathbb{S}^1} : \mathbb{S}^1 \subset \mathbb{C} \to \mathbb{R}$ given, for $p = e^{ix}$, by

$$u_{\mathbb{S}^1}(p) = u_{\mathbb{S}^1}(e^{ix}) := u(x) = \sum_{k\in\mathbb{Z}} u_k e^{ikx} = \sum_{k\in\mathbb{Z}} u_k (e^{ix})^k = \sum_{k\in\mathbb{Z}} u_k p^k.$$

The function $u_{\mathbb{S}^1}$ can be naturally extended to a harmonic function in the unit disk $u_{\mathbb{D}} : \mathbb{D} \subset \mathbb{C} \to \mathbb{R}$ by writing $z = rp = re^{ix} \in \mathbb{C}$, where $r \geq 0$ and $p = e^{ix} \in \mathbb{S}^1$, and

$$\begin{aligned} u_{\mathbb{D}}(z) = u_{\mathbb{D}}(rp) &:= \sum_{k\in\mathbb{Z}} u_k r^{|k|} p^k = \sum_{k\in\mathbb{Z}} u_k r^{|k|} (e^{ix})^k \\ &= \sum_{k\geq 0} u_k (re^{ix})^k + \sum_{k>0} u_{-k} (re^{-ix})^k = \sum_{k\geq 0} u_k z^k + \sum_{k>0} u_{-k} \overline{z}^k. \end{aligned} \tag{A.2}$$

The fact that $\Delta u_{\mathbb{D}} = 0$ in $\mathbb{D}$ follows directly from the right-hand side of (A.2) and the fact that $\Delta = 4\partial_z \partial_{\overline{z}} = 4\partial_{\overline{z}}\partial_z$. It is also clear that $u_{\mathbb{D}} = u_{\mathbb{S}^1}$ on $\mathbb{S}^1$. Now, one can use this extension into the unit disk to interpret $(-\Delta)^{1/2}u$ as a Dirichlet-to-Neumann map on $\mathbb{S}^1$, that is,

$$(\partial_r u_{\mathbb{D}})(p) = \frac{\partial}{\partial r}\Big|_{r=1} u_{\mathbb{D}}(z) = \frac{\partial}{\partial r}\Big|_{r=1} \sum_{k\in\mathbb{Z}} u_k r^{|k|} p^k = \sum_{k\in\mathbb{Z}} |k| u_k e^{ikx} = (-\Delta)^{1/2}u(x). \tag{A.3}$$

This Dirichlet-to-Neumann map $u_{\mathbb{S}^1}(p) \mapsto (\partial_r u_{\mathbb{D}})(p)$ on $\mathbb{S}^1$ can now be used to get an integro-differential expression on $\mathbb{S}^1$ for $(-\Delta)^{1/2}u_{\mathbb{S}^1}$. The following lemma shows that, as we claimed at the beginning of this section, the formula that one gets for $(-\Delta)^{1/2}u_{\mathbb{S}^1}$ on $\mathbb{S}^1$ is the same as (A.1) for $(-\Delta)^{1/2}u$ on $\mathbb{R}$ replacing $\mathbb{R}$ by $\mathbb{S}^1$ and u by $u_{\mathbb{S}^1}$, but leaving the same kernel.

Lemma A.1. *Given $u_{\mathbb{S}^1} \in C^\alpha(\mathbb{S}^1)$ for some $\alpha > 1$, let $u_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}$ be the harmonic extension of $u_{\mathbb{S}^1}$ to the unit disk. Then,*

$$(\partial_r u_{\mathbb{D}})(p) = \frac{1}{\pi} \,\mathrm{P.V.}\!\int_{\mathbb{S}^1} dq\, \frac{u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q)}{|p-q|^2} =: (-\Delta)^{1/2} u_{\mathbb{S}^1}(p)$$

for all $p \in \mathbb{S}^1$, where the second equality must be understood as a definition of an operator $(-\Delta)^{1/2}$ on $\mathbb{S}^1$.

Proof. It is well known that one can write $u_{\mathbb{D}}(z) = \int_{\mathbb{S}^1} dq\, P(z,q) u_{\mathbb{S}^1}(q)$ for all $z \in \mathbb{D}$, where P denotes the Poisson kernel on the unit disk. It is given by

$$P(z,q) := \frac{1-|z|^2}{2\pi|z-q|^2} \quad \text{for all } z \in \mathbb{D} \text{ and } q \in \mathbb{S}^1.$$

The fact that a harmonic function which is constant on $\mathbb{S}^1$ must be constant in $\mathbb{D}$ yields that $\int_{\mathbb{S}^1} dq\, P(z,q) = 1$ for all $z \in \mathbb{D}$. Therefore, for every $p \in \mathbb{S}^1$ and every $\epsilon > 0$ we have

$$\begin{aligned}(\partial_r u_{\mathbb{D}})(p) &= \lim_{\delta \downarrow 0} \frac{u_{\mathbb{D}}(p) - u_{\mathbb{D}}((1-\delta)p)}{\delta} = \lim_{\delta\downarrow 0} \int_{\mathbb{S}^1} dq\, (u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q)) \frac{1}{\delta} P((1-\delta)p, q) \\ &= \lim_{\delta\downarrow 0} \Big(\int_{\mathbb{S}^1 \cap \{|p-q|>\epsilon\}} + \int_{\mathbb{S}^1 \cap \{|p-q|\le\epsilon\}} \Big) dq\, (u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q)) \frac{1}{\delta} P((1-\delta)p, q) \\ &=: A_\epsilon + B_\epsilon,\end{aligned} \tag{A.4}$$

where A_ϵ denotes the limit as $\delta \downarrow 0$ of the integral over $\mathbb{S}^1 \cap \{|p-q| > \epsilon\}$, and B_ϵ the limit of the other integral.

To compute A_ϵ we use that $p \neq q$. Then, since $|p| = 1$, we see that

$$\lim_{\delta\downarrow 0} \frac{1}{\delta} P((1-\delta)p,q) = \lim_{\delta\downarrow 0} \frac{1-|(1-\delta)p|^2}{2\pi\delta|(1-\delta)p-q|^2} = \lim_{\delta\downarrow 0} \Big(1-\frac{\delta}{2}\Big) \frac{1}{\pi|(1-\delta)p-q|^2} = \frac{1}{\pi|p-q|^2}.$$

By dominated convergence, we conclude that

$$\begin{aligned}\lim_{\epsilon\downarrow 0} A_\epsilon &= \lim_{\epsilon\downarrow 0} \int_{\mathbb{S}^1 \cap \{|p-q|>\epsilon\}} dq\, (u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q)) \lim_{\delta\downarrow 0} \frac{1}{\delta} P((1-\delta)p,q) \\ &= \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \int_{\mathbb{S}^1 \cap \{|p-q|>\epsilon\}} dq\, \frac{u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q)}{|p-q|^2}.\end{aligned} \tag{A.5}$$

Let us now compute B_ϵ. To this end, we can assume without loss of generality that $1 < \alpha \le 2$. By the symmetry of $\mathbb{S}^1 \cap \{|p-q| \le \epsilon\}$ with respect to the line $\mathbb{R}p$,

$$\int_{\mathbb{S}^1 \cap \{|p-q|\le\epsilon\}} dq\, \frac{u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q)}{|(1-\delta)p-q|^2} = \frac{1}{2} \int_{\mathbb{S}^1 \cap \{|p-q|\le\epsilon\}} dq\, \frac{2u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q) - u_{\mathbb{S}^1}(q_s)}{|(1-\delta)p-q|^2}, \tag{A.6}$$

where $q_s \in \mathbb{S}^1$ denotes the reflected point to q with respect to $\mathbb{R}p$. On the one hand, since $\mathbb{S}^1$ has a tangent line at p, we see that the points q, p, and q_s tend to be aligned as q approaches p. Indeed, $|p - q - (q_s - p)| = O(|p-q|^2)$. It is not hard to show that this, together with the fact that $u_{\mathbb{S}^1} \in C^\alpha(\mathbb{S}^1)$ for some $1 < \alpha \le 2$, yields that

$$|2u_{\mathbb{S}^1}(p) - u_{\mathbb{S}^1}(q) - u_{\mathbb{S}^1}(q_s)| = O(|p-q|^\alpha) \quad \text{as } q \text{ tends to } p. \tag{A.7}$$

On the other hand, for every $0 \le \delta \le 1$ and $q \in \mathbb{S}^1$ it holds that $\delta \le |(1-\delta)p - q|$, and we therefore obtain that

$$|p-q| \le |p-(1-\delta)p| + |(1-\delta)p-q| = \delta + |(1-\delta)p-q| \le 2|(1-\delta)p-q|. \tag{A.8}$$

Finally, since $\frac{1}{\delta}P((1-\delta)p,q) = \frac{1-\delta/2}{\pi|(1-\delta)p-q|^2}$, combining (A.6), (A.7), and (A.8) we get

$$|B_\epsilon| \le C \int_{\mathbb{S}^1 \cap \{|p-q|\le\epsilon\}} dq\, |p-q|^{\alpha-2} \le C\epsilon^{\alpha-1}$$

for some constant $C > 0$ independent of ϵ. Thus, we conclude that $\lim_{\epsilon\downarrow 0} B_\epsilon = 0$, and the lemma follows from this, (A.5), and (A.4). □

Combining the previous results, we deduce the following identities at the level of energies.

Corollary A.2. *Given $u : \mathbb{R} \to \mathbb{R}$ 2π-periodic, let $u_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{R}$ be defined by $u_{\mathbb{S}^1}(e^{ix}) := u(x)$ for all $x \in \mathbb{R}$, and let $u_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}$ be the harmonic extension of $u_{\mathbb{S}^1}$ to the unit disk.*

Then,

$$\begin{aligned}\mathcal{E}_{(-\Delta)^{1/2}}(u) &= \frac{1}{4\pi}\int_{-\pi}^{\pi} dx \int_{\mathbb{R}} dy\, \frac{|u(x)-u(y)|^2}{|x-y|^2} = \frac{1}{2}\int_{\mathbb{D}} |\nabla u_{\mathbb{D}}|^2 \\ &= \frac{1}{4\pi}\int_{\mathbb{S}^1} dp \int_{\mathbb{S}^1} dq\, \frac{|u_{\mathbb{S}^1}(p)-u_{\mathbb{S}^1}(q)|^2}{|p-q|^2},\end{aligned}$$

where $\mathcal{E}_{(-\Delta)^{1/2}}$ is the energy defined in (1.10) *with $G = 0$.*

Proof. Using Lemma 2.1, that $u(x) = u_{\mathbb{D}}(e^{ix})$, and (A.3) we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \int_{\mathbb{R}} dy\, \frac{|u(x)-u(y)|^2}{|x-y|^2} = \int_{-\pi}^{\pi} u\,(-\Delta)^{1/2}u = \int_{\mathbb{S}^1} u_{\mathbb{D}}\,\partial_r u_{\mathbb{D}}. \tag{A.9}$$

Now, the divergence theorem and the fact that $u_{\mathbb{D}}$ is harmonic shows that the right-hand side of (A.9) equals $\int_{\mathbb{D}} |\nabla u_{\mathbb{D}}|^2$.

Finally, using Lemma A.1 and since $u_{\mathbb{D}} = u_{\mathbb{S}^1}$ on $\mathbb{S}^1$, a standard symmetrization argument (as the one in (2.8) in the proof of Lemma 2.1) shows that

$$\int_{\mathbb{S}^1} u_{\mathbb{D}}\,\partial_r u_{\mathbb{D}} = \frac{1}{2\pi}\int_{\mathbb{S}^1} dp \int_{\mathbb{S}^1} dq\, \frac{|u_{\mathbb{S}^1}(p)-u_{\mathbb{S}^1}(q)|^2}{|p-q|^2},$$

as claimed. □

Remark A.3. We follow the notation above, $u_{\mathbb{S}^1}(e^{ix}) = u(x)$. We finish this section by proving that $(-\Delta)^{1/2}u(x) = (-\Delta)^{1/2}u_{\mathbb{S}^1}(e^{ix})$ for all $x \in \mathbb{R}$ without using the extension argument above via $u_{\mathbb{D}}$, but directly at the level of integro-differential expressions.

Note that, by the periodicity of u, we have

$$(-\Delta)^{1/2}u(x) = \frac{1}{\pi}\int_{\mathbb{R}} dy\, \frac{u(x)-u(y)}{|x-y|^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} dy\,(u(x)-u(y)) \sum_{k\in\mathbb{Z}} \frac{1}{|x-y+2k\pi|^2}. \tag{A.10}$$

Recall that

$$s^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^{+\infty} dt\, t^{\gamma-1}e^{-st} \quad \text{for all } s > 0.$$

Hence, assuming that $0 < x - y < 2\pi$ (the case $-2\pi < x - y < 0$ is analogous), we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{1}{|x-y+2k\pi|^2} &= \sum_{k\in\mathbb{Z}} \int_0^{+\infty} dt\, t e^{-|x-y+2k\pi|t} \\ &= \int_0^{+\infty} dt\, t e^{-(x-y)t} \sum_{k\geq 0} (e^{-2\pi t})^k + \int_0^{+\infty} dt\, t e^{(x-y)t} \sum_{k<0} (e^{2\pi t})^k \\ &= \int_0^{+\infty} dt\, t \frac{e^{-(x-y)t}}{1-e^{-2\pi t}} + \int_0^{+\infty} dt\, t \frac{e^{(x-y)t} e^{-2\pi t}}{1-e^{-2\pi t}} \\ &= \int_0^{+\infty} dt\, t \frac{e^{-(x-y-\pi)t} + e^{(x-y-\pi)t}}{e^{\pi t} - e^{-\pi t}} = \int_0^{+\infty} dt\, t \frac{\cosh((\pi - x + y)t)}{\sinh(\pi t)}.\end{aligned}$$

For every $0 < r < 2\pi$ it holds that (see [38, Number 8 in Section 3.524 on page 377])

$$\int_0^{+\infty} dt\, t \frac{\cosh((\pi - r)t)}{\sinh(\pi t)} = \frac{1}{2 - 2\cos(r)},$$

and since $|e^{ix} - e^{iy}|^2 = 2 - 2\cos(x-y)$, we deduce that

$$\sum_{k\in\mathbb{Z}} \frac{1}{|x-y+2k\pi|^2} = \frac{1}{|e^{ix} - e^{iy}|^2}.$$

Plugging this identity in (A.10), we conclude that

$$(-\Delta)^{1/2} u(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} dy\, \frac{u_{\mathbb{S}^1}(e^{ix}) - u_{\mathbb{S}^1}(e^{iy})}{|e^{ix} - e^{iy}|^2} = \frac{1}{\pi} \int_{\mathbb{S}^1} dq\, \frac{u_{\mathbb{S}^1}(e^{ix}) - u_{\mathbb{S}^1}(q)}{|e^{ix} - q|^2} = (-\Delta)^{1/2} u_{\mathbb{S}^1}(e^{ix}).$$

Appendix B. On the class of kernels considered in Theorem 1.4

In this section we shall discuss the three conditions in the rearrangement Theorem 1.4 imposed on the kernel K by exhibiting several examples. In particular we show that the theorem is false in general for nonincreasing kernels, which is the most natural class which contains all three conditions. We also show that neither of the three classes is contained in one of the others.

Let us start by exhibiting a simple example which shows that the statement in Theorem 1.4, namely, that $[u^{*\,\mathrm{per}}]_K \leq [u]_K$ for every $2L$-periodic function $u : \mathbb{R} \to \mathbb{R}$ such that $u \in L^2(-L, L)$, does not hold for a nonincreasing kernel K. Take, for example, K_ϵ to be the characteristic function of $[0, L+\epsilon]$ for some $0 < \epsilon < L$, and define $\overline{K}_\epsilon(t) := \sum_{k\in\mathbb{Z}} K_\epsilon(|t + 2kL|)$ as in the proof of Theorem 1.4. Since $K_\epsilon(|t|) = \chi_{[-L-\epsilon, L+\epsilon]}(t)$ for all $t \in \mathbb{R}$, we have that $\overline{K}_\epsilon$ is the even $2L$-periodic function given in $[-L, L]$ by

$$\overline{K}_\epsilon(t) = \begin{cases} 1 & \text{if } t \in (-L+\epsilon, L-\epsilon), \\ 2 & \text{if } t \in [-L, -L+\epsilon] \cup [L-\epsilon, L]. \end{cases}$$

Note that in particular $\overline{K}_\epsilon$ is smaller in $(0, L-\epsilon)$ than in $(L-\epsilon, L)$. This last observation shows that our proof of Theorem 1.4 cannot work for such a kernel, since the Riesz rearrangement inequality of Theorem 4.1 does not apply. In fact, we now show that the conclusion given in Theorem 1.4 is indeed false for the kernel $K_\epsilon = \chi_{[0,L+\epsilon]}$.

Indeed, by using (2.12) for $[u]_{K_\epsilon}$, expanding the square $|u(x) - u(y)|^2$ as $u^2(x) + u^2(y) - 2u(x)u(y)$, and using the fact that rearrangement does not modify the double integrals containing $u^2(x)$ and $u^2(y)$ (assuming u to be in $L^2(-L, L)$ for the integrals to be finite) we see that

$[u^{*\,\mathrm{per}}]_{K_\epsilon} \le [u]_{K_\epsilon}$ is equivalent to the inequality

$$\int_{-L}^{L} dx \int_{-L}^{L} dy\, u^{*\,\mathrm{per}}(x) u^{*\,\mathrm{per}}(y) \overline{K}_\epsilon(x-y) \ge \int_{-L}^{L} dx \int_{-L}^{L} dy\, u(x)u(y)\overline{K}_\epsilon(x-y). \tag{B.1}$$

We now show that this inequality does not hold for some functions $u \in L^2(-L,L)$. For instance, take

$$u = a\chi_{(-\frac{L}{2}-\delta, -\frac{L}{2}+\delta)} + b\chi_{(\frac{L}{2}-\delta, \frac{L}{2}+\delta)},$$

where a and b are two positive numbers and $0 < \delta < L/2$. If we chose δ sufficiently small compared to ϵ, then

$$\int_{-L}^{L} dx \int_{-L}^{L} dy\, u(x)u(y)\overline{K}_\epsilon(x-y) = 4\delta^2\big(a^2\overline{K}_\epsilon(0) + b^2\overline{K}_\epsilon(0) + 2ab\overline{K}_\epsilon(L)\big).$$

The function $u^{*\,\mathrm{per}}$ restricted to $(-L,L)$ is equal to

$$u^{*\,\mathrm{per}} = \max\{a,b\}\chi_{(-\delta,\delta)} + \min\{a,b\}\chi_{(-2\delta,-\delta]\cup[\delta,2\delta)}$$

which, for δ small enough, leads to

$$\int_{-L}^{L} dx \int_{-L}^{L} dy\, u(x)^{*\,\mathrm{per}} u^{*\,\mathrm{per}}(y)\overline{K}_\epsilon(x-y) = 4\delta^2\big(a^2\overline{K}_\epsilon(0) + b^2\overline{K}_\epsilon(0) + 2ab\overline{K}_\epsilon(0)\big).$$

Therefore, since $\overline{K}_\epsilon(L) = 2 > 1 = \overline{K}_\epsilon(0)$, (B.1) does not hold for this function u. Indeed, we have shown that for every choice of positive numbers a and b it holds that $[u^{*\,\mathrm{per}}]_{K_\epsilon} > [u]_{K_\epsilon}$.

Let us now comment on the class of kernels K considered in the statement of Theorem 1.4. It is simple to show that the collection of kernels satisfying (*iii*) does not contain and is not contained in the collection of kernels described neither by (*i*) nor by (*ii*).

We now show that the collection of kernels satisfying (*i*) does not contain and is not contained in the collection of kernels described by (*ii*). On the one hand, the kernel $K(t) = (t^2+1)^{-(1+2s)/2}$ satisfies the growth condition (1.8) with $\Lambda = 1$, and belongs to the class (*ii*) as we have seen right after Definition 1.3 or in Remark 6.2. However, K is strictly concave for t close to the origin. Therefore, K is as in (*ii*) but it does not satisfy (*i*), that is, K is not convex.[19]

On the other hand, if a kernel K is as in (*ii*) then it is infinitely differentiable in $(0,+\infty)$. This follows form Definition 1.3 or by differentiating (1.15). At the same time, there exist convex (and even strictly convex) kernels which are not infinitely differentiable in $(0,+\infty)$. These kernels will fulfill (*i*) but not (*ii*). Nevertheless, the lack of regularity is not the only property that prevents a convex kernel to belong to the class (*ii*). In Lemma B.1 we provide an example of a smooth kernel that fulfills (*i*) but not (*ii*).

We finally mention that there exist kernels that satisfy both (*i*) and (*ii*), such as the kernel of the fractional Laplacian (see Remark 6.2).

Lemma B.1. *Given* $0 < s < 1$, *set*

$$K(t) = \int_{t^2}^{+\infty} dr \int_{r}^{+\infty} dx\, \frac{2+\sin x}{x^{s+5/2}} \quad \textit{for all } t > 0.$$

Then, K *is infinitely differentiable in* $(0,+\infty)$*, satisfies* (1.8) *and* (1.9)*, is strictly convex in* $(0,+\infty)$*, but* $\tau > 0 \mapsto K(\tau^{1/2})$ *is not a completely monotonic function.*

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[19] Instead, recall that the function $\tau \mapsto K(\tau^{1/2})$ is convex, as follows from Definition 1.3.

Proof. We first show that K satisfies (1.8) and (1.9). Since $1 \le 2+\sin x \le 3$ for all $x>0$,

$$\frac{t^{-1-2s}}{(s+\frac{3}{2})(s+\frac{1}{2})} = \int_{t^2}^{+\infty} dr \int_r^{+\infty} \frac{dx}{x^{s+5/2}} \le K(t) \le 3\int_{t^2}^{+\infty} dr \int_r^{+\infty} \frac{dx}{x^{s+5/2}} = 3\frac{t^{-1-2s}}{(s+\frac{3}{2})(s+\frac{1}{2})}.$$

Moreover, since $x \mapsto x^{-s-5/2}(2+\sin x)$ is infinitely differentiable in $(0,+\infty)$, so is K.

Let us now check that K is strictly convex in $(0,+\infty)$. Set

$$f(\tau) := \int_\tau^{+\infty} dr \int_r^{+\infty} dx\, \frac{2+\sin x}{x^{s+5/2}} \quad \text{for all } \tau>0.$$

Then $K(t)=f(t^2)$, which yields $K''(t)=4t^2 f''(t^2)+2f'(t^2)$. Note that

$$f'(\tau) = -\int_\tau^{+\infty} dx\, \frac{2+\sin x}{x^{s+5/2}} \quad \text{and} \quad f''(\tau) = \frac{2+\sin\tau}{\tau^{s+5/2}}.$$

Therefore,

$$\begin{aligned} K''(t) &= 4t^2 f''(t^2)+2f'(t^2) = 4\frac{2+\sin(t^2)}{t^{2s+3}} - 2\int_{t^2}^{+\infty} dx\, \frac{2+\sin x}{x^{s+5/2}} \\ &\ge \frac{4}{t^{2s+3}} - 2\int_{t^2}^{+\infty} \frac{3\,dx}{x^{s+5/2}} = \frac{4}{t^{2s+3}} - \frac{6t^{-2s-3}}{s+\frac{3}{2}} = 4t^{-2s-3}\Big(1-\frac{3}{2s+3}\Big) > 0 \end{aligned}$$

for all $t>0$. That is, K is strictly convex in $(0,+\infty)$.

Finally, it remains to check that $\tau \mapsto K(\tau^{1/2}) = f(\tau)$ is not a completely monotonic function. If it was, then we would have $f'''(\tau) \le 0$ for all $\tau>0$. However,

$$f'''(\tau) = \frac{\cos\tau}{\tau^{s+5/2}} - \Big(s+\frac{5}{2}\Big)\frac{2+\sin\tau}{\tau^{s+7/2}} = \frac{1}{\tau^{s+5/2}}\Big(\cos\tau - \Big(s+\frac{5}{2}\Big)\frac{2+\sin\tau}{\tau}\Big),$$

which changes sign infinitely many times for τ big enough. ☐

Appendix C. Stable periodic solutions are constant

We show in this appendix that stable periodic solutions of $\mathcal{L}_K u = f(u)$ in $\mathbb{R}$ must be constant. Here the corresponding variational problem has no constraint. That is, stability means that the second variation of

$$\mathcal{E}_{\mathcal{L}_K} := \frac{1}{2}[u]_K^2 - \int_{-L}^{L} F(u),$$

at the $2L$-periodic solution u, is nonnegative definite when acting on $2L$-periodic functions. This condition is stated in an equivalent way in (C.1). This result applies to all kernels having the standard growth bounds.

Lemma C.1. *Let K satisfy the bounds* (1.8) *and* (1.9)*, $L>0$, and $f=F' \in C^{1+\beta}(\mathbb{R})$ for some $\beta>0$.*

If $u \in L^\infty(\mathbb{R})$ is a $2L$-periodic stable solution of $\mathcal{L}_K u = f(u)$ in $\mathbb{R}$, then u is constant.

Proof. As in the proof of Corollary 1.5 or of Theorem 1.1, one can see that $u' \in C^\nu(\mathbb{R})$ for some $\nu > 2s$ and that u' satisfies the linearized equation $\mathcal{L}_K u' = f'(u)u'$ in $\mathbb{R}$. In view of the stability of u, the second variation of $\mathcal{E}_{\mathcal{L}_K}$ at u is nonnegative, i.e,

$$D^2\mathcal{E}_{\mathcal{L}_K}(u)(\xi,\xi) := \frac{1}{2}\int_{-L}^{L} dx \int_{\mathbb{R}} dy\, |\xi(x)-\xi(y)|^2 K(|x-y|) - \int_{-L}^{L} f'(u)\xi^2 \ge 0 \tag{C.1}$$

for all $2L$-periodic functions $\xi \in C^{2s+\epsilon}(\mathbb{R})$ and for all $\epsilon>0$.

Multiplying by u' the linearized equation satisfied by u' and integrating in $(-L, L)$ we deduce from the integration by parts formula Lemma 2.1 (applied with u and ψ both replaced by u', a $2L$-periodic function of class $C^{\nu}(\mathbb{R})$ with $\nu > 2s$) that $D^2\mathcal{E}_{\mathcal{L}_K}(u)(u', u') = 0$. Since $||u'(x)| - |u'(y)|| \leq |u'(x) - u'(y)|$ it follows that $D^2\mathcal{E}_{\mathcal{L}_K}(u)(|u'|, |u'|) \leq D^2\mathcal{E}_{\mathcal{L}_K}(u)(u', u') = 0$, which in turn shows that $D^2\mathcal{E}_{\mathcal{L}_K}(u)(|u'|, |u'|)$ vanishes too, by (C.1) used with $\xi = |u'|$.

Thus the function

$$w(t) := D^2\mathcal{E}_{\mathcal{L}_K}(u)(|u'| + t\eta, |u'| + t\eta)$$

vanishes at $t = 0$ and, in view of (C.1), is nonnegative for all $t \in \mathbb{R}$ and all $2L$-periodic functions η. It follows that $w'(0) = 0$. Computing $w'(0)$, integrating by parts, and using that η is arbitrary, we obtain that $|u'|$ is a weak solution of

$$\mathcal{L}_K|u'| - f'(u)|u'| = 0 \text{ in } \mathbb{R}.$$

Thus, by our regularity result Theorem 1.7, it follows that $|u'|$ is of classe C^{ω} for some $\omega > 2s$. Therefore $\mathcal{L}_K|u'|$ makes sense pointwise and $|u'|$ is a pointwise solution of the equation.

Now, by the well known strong maximum principle for the operator $\mathcal{L}_K$ in $\mathbb{R}$, and since $|u'| \geq 0$ in $\mathbb{R}$, we must have that either $|u'| \equiv 0$ or $|u'| > 0$ in $\mathbb{R}$ (we are not using here the strong maximum principle of Section 5 for periodic solutions, whose proof is harder). Now $|u'| > 0$ in $\mathbb{R}$ is not possible, since u is periodic. Thus $|u'| \equiv 0$ and hence u is constant in $\mathbb{R}$. $\square$

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