Treball final de grau

GRAU DE
MATEMÀTIQUES

Facultat de Matemàtiques
Universitat de Barcelona

ASYMPTOTIC SERIES IN QUANTUM MECHANICS

Joan Garcia i Tormo

Directors: Bartomeu Fiol Núñez
Alejandro Haro Provinciale
Realitzat a: Departament de Física
Fonamental.
Departament de Matemàtica
Aplicada i Anàlisi. UB

Barcelona, 24 de juny de 2014

Asymptotic series in quantum mechanics

Full report of Joan Garcia i Tormo's bachelor's degree thesis
to apply for the bachelor's degree in mathematics

Thesis' supervisors
Dr. Bartomeu Fiol Núñez
Dr. Alejandro Haro Provinciale

July 2014
Universitat de Barcelona

UNIVERSITAT DE BARCELONA

I am so hungry.
The Master

Acknowledgments

First of all I want to thank Tomeu Fiol for his guidance in the forthcoming period of my life, and to him and Àlex Haro for the supervision and support of my bachelor's thesis. I also want to thank my friends for the company, with a special mention to Armajac and Guillem for their discussions on the subject. Last, but far from least, I want to thank my brothers Xavier and Albert and my parents for the support over the years.

Contents

List of Figures

Chapter 1

Introduction and objectives

1.1 Preface

In the late 19th century the quantum theory was born. Physicists like Planck and Einstein introduced the idea of quantisation, which was later formalised and justified by other scientists like Heisenberg, Schrödinger and Dirac. This new theory explained the non-intuitive results that were obtained from experiments at the time, like the famous Stern-Gerlach experiment. Arguments based on Newton's classical mechanics were not able to describe them, and quantum mechanics was the theory created to explain them. The application of this theory to fields led to the development of quantum field theories, which are the ones physicists use at present.

Here we present two formulations of quantum mechanics. An axiomatic one, where we will compute an elemental example and motivate this thesis, and the Feynman formulation, which will be useful for our calculations.

1.2 Quantum mechanics

A popular way to introduce quantum mechanics is with the following postulates.

Postulate 1 *(States)*
The state of a quantum mechanical system is given by a vector (which we will call ket) in a complex vector space $\mathcal{H}$ *with hermitian inner product* $< \cdot | \cdot >$. $\mathcal{H}$ *may be finite or infinite dimensional; in the latter, it is required to be a separable Hilbert space. All the information of the physical system is contained in the state vector.*

Postulate 2 *(Observables)*
The observables of a quantum mechanical system correspond to self-adjoint linear operators on $\mathcal{H}$. *Its eigenvectors define a base in the Hilbert space.*

Let us recall here that all eigenvalues of a self-adjoint operator are real, and also that in this case eigenvectors corresponding to different eigenvalues are orthogonal. We denote by $|a_i\rangle$ the eigenvectors with eigenvalue a_i of an operator A.

Postulate 3 *(Measurements)*

- *Given a self-adjoint operator (an observable) A, states which are eigenvectors of A are the ones which the value of the observable is a well-defined number. The result obtained by measuring an observable in such a state can only be a real number a_i, eigenvalue of A. The probability to obtain a certain a_i is*

$$P_{A|\psi\rangle}(a_i) = |\langle a_i|\psi\rangle|^2$$

or, defining the projector $\Pi_{a_i} := |a_i\rangle\langle a_i|$,

$$P_{A|\psi\rangle}(a_i) = \|\Pi_{a_i}|\psi\rangle\|^2.$$

- *Given an observable A and states $|\psi_1\rangle$, $|\psi_2\rangle$ eigenvectors of A with eigenvalues a_1 and a_2 respectively, the result of the measure of a state $|\Phi\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle$ will be a_i, with probability*

$$\frac{|\alpha_i|^2}{|\alpha_1|^2 + |\alpha_2|^2}, \qquad i = 1, 2.$$

So, our initial state, if normalised to 1, will turn to $|\psi_i\rangle$ with a probability $|\langle\psi_i|\Phi\rangle|^2$ after a filtering measure.

Postulate 4 *(Dynamics)*

Between measurements, the time evolution of a state $|\psi(t)\rangle \in \mathcal{H}$ is given by

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle,$$

where H is the Hamiltonian operator.

Now we will solve explicitly a typical problem: the harmonic oscillator.

1.2.1 Harmonic oscillator

The Hamiltonian for the harmonic oscillator is

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2X^2, \tag{1.1}$$

where X and P are the position and momentum operators, which satisfy the commutation relation $[X, P] = i\hbar$. In the position representation, the momentum operator is given by $-i\hbar\frac{\mathrm{d}}{\mathrm{d}X}$. We will use this later on.

To solve the problem, i.e. to find the eigenvectors and eigenvalues of H (that is, the energy for each state) we introduce following operators,

$$a_{\pm} := \sqrt{\frac{m\omega}{2\hbar}} \left(X \mp i \frac{P}{m\omega} \right).$$

which are called creation and annihilation operators, respectively. We will see why in a moment. The following properties can easily be checked,

$$a_{\pm}^{\dagger} = a_{\mp}, \quad [a_-, a_+] = 1, \quad a_+ a_- = \frac{1}{\hbar\omega} H - \frac{1}{2}$$

It is usual to denote these operators by $a^{\dagger}$ (creation) and a (annihilation). For compactness we will stick with the $a_{\pm}$ notation for the moment, though. Using these operators, we can solve the problem without choosing a representation. We can write the Hamiltonian in terms of the new operators,

$$H = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) = \hbar\omega \left(N + \frac{1}{2} \right),$$

where $N := a_+ a_-$ is the number operator. Since H is a linear function of N, we can diagonalize them simultaneously. Let $|n\rangle$ be the eigenvectors of N with eigenvalue n. We have

$$H |n\rangle = \left(n + \frac{1}{2} \right) \hbar\omega |n\rangle \quad \Longrightarrow \quad E_n = \left(n + \frac{1}{2} \right) \hbar\omega.$$

So, we have the energy eigenvalues. Now, it is straightforward to see that

$$[N, a_-] = -a_-, \quad [N, a_+] = a_+ \quad \Longrightarrow \quad N a_{\pm} |n\rangle = (n \pm 1) a_{\pm} |n\rangle,$$

becoming clear the notation for the $a_{\pm}$ operators. From the last equation, we set

$$a_- |n\rangle = c |n-1\rangle$$

where c is a constant, real and positive by convention. Since $N = a_+ a_-$ is the number operator, we have

$$n = \langle n| a_+ a_- |n\rangle = |c|^2 \geq 0. \tag{1.2}$$

Then,

$$a_- |n\rangle = \sqrt{n} |n-1\rangle, \quad a_+ |n\rangle = \sqrt{n+1} |n+1\rangle.$$

Applying repeatedly a_- to a state $|n\rangle$, equation (1.2) leads to $n \in \mathbb{N}$; otherwise we would obtain negative values for n. Therefore, our system has a ground state energy, which is

$$E_0 = \frac{1}{2} \hbar\omega.$$

Applying successively a_+ to the ground state, we obtain the orthonormal eigenvectors of H,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a_+)^n |0\rangle \text{, with eigenvalues} \quad E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \ n \in \mathbb{N}.$$

In this case we were able to find an exact solution. However, in general the interesting problems can not be solved exactly. That is why we usually resort to perturbation theory: if H_0 is an exactly solvable problem, we consider a small dimensionless parameter λ such that the Hamiltonian $H = H_0 + \lambda H_1$ describes our model, and we calculate the results as a power series in λ. A typical method in quantum mechanics to calculate them is the Rayleigh-Schrödinger theory, a time independent perturbation theory. It is presented in many quantum mechanics books. [1], for example. We will now present this method, and apply it to the anharmonic oscillator, that is, to the harmonic oscillator with a quartic perturbation term in its Hamiltonian, which was first studied by Bender and Wu [2].

1.2.2 Rayleigh-Schrödinger theory

Consider a time-independent Hamiltonian H that can be written as

$$H = H_0 + \lambda V,$$

where H_0 (H with λ=0) is a Hamiltonian which can be exactly solved and λV is the small perturbation. Let $|n^{(0)}\rangle$ be the energy eigenvectors of H_0 and $E_n^{(0)}$ the eigenvalues, i.e.

$$H_0 |n^{(0)}\rangle = E_n^{(0)} |n^{(0)}\rangle .$$

We want the eigenvectors and eigenvalues of the full Hamiltonian. So, we have to solve

$$(H_0 + \lambda V) |n\rangle = E_n |n\rangle .$$

We will assume that the energy spectrum is non-degenerate. Let $\Delta_n := E_n - E_n^{(0)}$ be the energy shift. Plugging it in the equation, it reads

$$(E_n^{(0)} - H_0) |n\rangle = (\lambda V - \Delta_n) |n\rangle . \tag{1.3}$$

If we multiply both sides by $\langle n^{(0)}|$ on the left, we see that $(\lambda V - \Delta_n) |n\rangle$ has no component along $|n^{(0)}\rangle$. So, we can define

$$\Phi_n := 1 - |n^{(0)}\rangle \langle n^{(0)}| = \sum_{k \neq n} |k^{(0)}\rangle \langle k^{(0)}|$$

and consider the inverse operator of $(E_n^{(0)} - H_0)$. Let us denote it by $1/(E_n^{(0)} - H_0)$. We have just seen that it is well defined when multiplies Φ_n on the right,

$$\frac{1}{E_n^{(0)} - H_0}\Phi_n = \sum_{k \neq n} \frac{1}{E_n^{(0)} - E_k^{(0)}} |k^{(0)}\rangle \langle k^{(0)}| .$$

Since

$$(\lambda V - \Delta_n) |n\rangle = \Phi_n(\lambda V - \Delta_n) |n\rangle ,$$

and taking into account that for $\lambda \to 0$, $|n\rangle \to |n^{(0)}\rangle$ we write, from (1.3),

$$|n\rangle = c_n(\lambda) |n^{(0)}\rangle + \frac{1}{E_n^{(0)} - H_0} \Phi_n(\lambda V - \Delta_n) |n\rangle , \quad \text{with} \quad \lim_{\lambda\to 0} c_n(\lambda) = 1.$$

We notice that $c_n(\lambda) = \langle n^{(0)}|n\rangle$. We can set $c_n(\lambda) = 1$ to simplify the notation. If we want normalised vectors as a result, we can always normalise them at the end. Then, our ket reads

$$|n\rangle = |n^{(0)}\rangle + \frac{\Phi_n}{E_n^{(0)} - H_0}(\lambda V - \Delta_n) |n\rangle , \tag{1.4}$$

where we have written Φ_n in the numerator for notation compactness. We can write

$$\Delta_n = \lambda \langle n^{(0)}| V |n\rangle , \tag{1.5}$$

because $(\lambda V - \Delta_n) |n\rangle$ has no component along $|n^{(0)}\rangle$, as we mentioned earlier. Now we will expand $|n\rangle$ and Δ_n in powers of λ and match the terms of the same power. So, we write

$$|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + ... \quad ; \quad \Delta_n = \lambda\Delta_n^{(1)} + \lambda^2\Delta_n^{(2)} + ...$$

First we substitute it in equation (1.5) and match the terms;

$$\begin{aligned} \lambda^1 : \quad & \Delta_n^{(1)} = \langle n^{(0)}| V |n^{(0)}\rangle \\ \lambda^2 : \quad & \Delta_n^{(2)} = \langle n^{(0)}| V |n^{(1)}\rangle \\ & \quad \vdots \\ \lambda^N : \quad & \Delta_n^{(N)} = \langle n^{(0)}| V |n^{(N-1)}\rangle \end{aligned}$$

We see that to calculate the energy shift of order N we only need $|n\rangle$ up to order $N-1$. So, we already have the expression for $\Delta_n^{(1)}$. Doing the same with (1.4) to obtain the states,

$$\begin{aligned} & |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + ... = \\ &= |n^{(0)}\rangle + \frac{\Phi_n}{E_n^{(0)} - H_0}(\lambda V - \lambda\Delta_n^{(1)} - ...)(|n^{(0)}\rangle + \lambda |n^{(1)}\rangle + ...). \end{aligned}$$

Matching terms (we just write the first one now),

$$\lambda^1 : \quad |n^{(1)}\rangle = \frac{\Phi_n}{E_n^{(0)} - H_0}(V - \Delta_n^{(1)}) |n^{(0)}\rangle = \frac{\Phi_n}{E_n^{(0)} - H_0} V |n^{(0)}\rangle .$$

With this expression, we can obtain $\Delta_n^{(2)}$,

$$\Delta_n^{(2)} = \langle n^{(0)}| V \frac{\Phi_n}{E_n^{(0)} - H_0} V |n^{(0)}\rangle .$$

Now we can compute $|n^{(2)}\rangle$, and so on. Therefore, with this method we can compute the perturbation series and solve the problem.

Let us now apply this method to the anharmonic oscillator. Before we do that though, we notice that we can use a compact notation for Δ_n and write, denoting by V the perturbation term,

$$\Delta_n = \lambda V_{nn} + \lambda^2 \sum_{k\neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + ... \quad \text{with} \quad V_{nk} := \langle n^{(0)}| V |k^{(0)}\rangle .$$

Example (Anharmonic oscillator)

Let H_0 be the harmonic oscillator Hamiltonian, and let us consider the Hamiltonian of the anharmonic oscillator,

$$H = H_0 + \lambda X^4 = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 + \frac{\omega^3 m^2}{\hbar}\lambda X^4.$$

where λ is dimensionless, as we introduced before. As we have seen in the previous section, $E_n^{(0)} = \hbar\omega(n + 1/2)$. We will compute $E_0^{(1)}$ and $E_0^{(2)}$. To simplify the calculations, we will write X^4 in terms of the creation and annihilation operators. Now we will denote by $a^\dagger$ the creation operator, and thus a will be the annihilation one. Recalling that $N = a^\dagger a$,

$$X^4 = \frac{\hbar^2}{4m^2\omega^2}(a^4 + a^2 a^{\dagger 2} + a^2 2N + a^2 + a^{\dagger 2} a^2 + a^{\dagger 4} + a^{\dagger 2}(2N + 1) + 2Na^2 +$$
$$+2Na^{\dagger 2} + 4N^2 + a^2 + a^{\dagger 2} + 4N + 1).$$

The point of writing all the expression is to notice that to compute $E_0^{(1)}$ we need $\langle 0^{(0)}| X^4 |0^{(0)}\rangle$, and since $a|0^{(0)}\rangle = 0$ and the states are orthogonal, only two terms contribute, $a^2 a^{\dagger 2}$ and $(2N + 1)^2$. If we compute them, since $aa^\dagger = N + 1$ we obtain

$$\langle 0^{(0)}| aaa^\dagger a^\dagger |0^{(0)}\rangle = 2, \quad \langle 0^{(0)}| (2N + 1)^2 |0^{(0)}\rangle = 1.$$

Therefore,

$$\Delta_0^{(1)}(\lambda) = \langle 0^{(0)}| \frac{\omega^3 m^2}{\hbar}\lambda X^4 |0^{(0)}\rangle = \frac{3}{4}\hbar\omega\lambda. \tag{1.6}$$

Analogously,

$$\Delta_0^{(2)}(\lambda) = \sum_{k\neq 0} \frac{|\langle 0^{(0)}| \frac{\omega^3 m^2}{\hbar}\lambda X^4 |k^{(0)}\rangle|^2}{E_0^{(0)} - E_k^{(0)}} =$$

$$= \frac{|\langle 0^{(0)}| \frac{\omega^3 m^2}{\hbar}\lambda X^4 |2^{(0)}\rangle|^2}{E_0^{(0)} - E_2^{(0)}} + \frac{|\langle 0^{(0)}| \frac{\omega^3 m^2}{\hbar}\lambda X^4 |4^{(0)}\rangle|^2}{E_0^{(0)} - E_4^{(0)}}$$

$$\Delta_0^{(2)}(\lambda) = -\frac{21}{8}\hbar\omega\lambda^2 \tag{1.7}$$

Remark In chapter 3 we will study the anharmonic oscillator. In order to avoid cumbersome expressions due to the units, we can consider a dimensionless Hamiltonian with the following change: if we set $X = \eta Y$, with $\eta^2 = \hbar/m\omega$, and divide the Hamiltonian by $m\eta^2/\hbar^2$, we obtain the dimensionless expression we are looking for,

$$H' = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}Y^2} + \frac{1}{2}Y^2 + \lambda Y^4.$$

The new energy is indeed the same, but without units,

$$E' = E\frac{m\eta^2}{\hbar^2} = \frac{E}{\hbar\omega}.$$

If we keep computing terms, we obtain

$$E_0 = \hbar\omega\left(\frac{1}{2} + \frac{3}{4}\lambda - \frac{21}{8}\lambda^2 + \frac{333}{16}\lambda^3 - \frac{30885}{124}\lambda^4 + ...\right). \tag{1.8}$$

We should worry about the behaviour of the coefficients; the series we obtain for the ground state energy of the perturbed oscillator is not convergent! However, we should expect this divergence. As we said before, in quantum mechanics and quantum field theory we obtain results as a power series of a parameter λ. But, as Dyson argued in [3], for λ negative the system is unstable and so the radius of convergence of the series must be 0.

Even if the series do not converge, they behave in a specific way, which we can describe analytically; they are asymptotic series. We will focus our efforts on the study of this kind of series, starting next chapter. Now we will present the path integral formulation of quantum mechanics, which we will need to reproduce the asymptotic behaviour of the series (1.8).

1.3 Path integral formulation

The general idea of this formulation is that, in contrast to classical mechanics, where there exists a unique path which the particles follow (the path given by Hamilton's principle, that is, the one which minimizes the action), in quantum mechanics we have to take into account all possible paths that connect our initial and final states to obtain the evolution of the system. So, our goal now is to obtain from this idea a formulation of quantum mechanics equivalent to the one presented in section 1.2, which is the usual one. We present this method because it is the one that we will use in chapter 3, since it will allow us to use methods which can be easily generalised to quantum field theory. We will just give a brief introduction, an idea of this formulation, just as we did in the previous section. We will follow Sakurai's introduction of this formulation [1].

We will work with real time so the idea underneath is clear, and at the end we will give the expressions for imaginary time, because they are the ones that we will use. We will also comment some results concerning the partition function, which we will need afterwards.

First, let us notice that we can interpret $\langle x_N, t_N | x_1, t_1 \rangle$ as the transformation function that connects the two sets of base kets at different times. Since we can insert the identity operator, written as

$$\int \mathrm{d}^3 x' \, |x', t'\rangle \, \langle x', t'| = 1$$

wherever we need, we can write

$$\langle x_N, t_N | x_1, t_1 \rangle = \int \mathrm{d}x_{N-1} \int \mathrm{d}x_{N-2} ... \int \mathrm{d}x_2 \, \langle x_N, t_N | x_{N-1}, t_{N-1} \rangle \times$$

$$\times \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \, ... \, \langle x_2, t_2 | x_1, t_1 \rangle \, .$$

From this expression, it is clear that we have to take into account the sum over all possible paths that connect our initial and final states in the space-time plane.

Let us use, for now, the following notation for the classical action,

$$S(n, n-1) := \int_{t_{n-1}}^{t_n} \mathrm{d}t L_{classical}(x, \dot{x}).$$

When writing this, we understand that the integration is along a definite path. Consider a definite path connecting (x_1, t_1) and (x_N, t_N). If we focus on a small segment of the path, between (x_{n-1}, t_{n-1}) and (x_n, t_n), let us associate $\exp[iS(n, n-1)/\hbar]$ with that segment. Going along the whole path,

$$\prod_{n=2}^{N} e^{\frac{i}{\hbar} S(n,n-1)} = e^{\frac{i}{\hbar} \sum_{n=2}^{N} S(n,n-1)} = e^{i \frac{S(N,1)}{\hbar}}.$$

We are considering just a specific path, but in some way we should take all possible paths into account. That is, we should integrate over $x_2, ..., x_{N-1}$, and this should correspond to $\langle x_N, t_N | x_1, t_1 \rangle$. Before we do that, though, let us check if the expressions we are dealing with make sense in the classical limit.

If we consider the $\hbar \to 0$ limit, the oscillations of $\exp[iS(N, 1)/\hbar]$ are huge, and so, close paths will cancel with each other and will not contribute. We see that only the path given by Hamilton's principle will survive. Indeed, as it satisfies $\delta S(N, 1) = 0$, paths close to this one won't oscillate that much, and so constructive interference will be possible. Therefore, only paths very close to the classical one will contribute, and in the $\hbar \to 0$ limit only the classical path will contribute, as it should be.

So, let us introduce a weight factor to the expressions,

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \frac{1}{w(\Delta t)} e^{\frac{i}{\hbar} S(n,n-1)},$$

and let us assume that the weight factor is independent of the potential V (which is part of the Lagrangian). To obtain it, we have to calculate $S(n, n-1)$ for $\Delta t \to 0$ (so, we can approximate our path of integration by a straight line). As we are assuming the weight factor to be independent of the potential, we can set $V = 0$ and obtain it using the orthonormality of the position eigenvectors at the same time. Doing this, we obtain

$$\frac{1}{w(\Delta t)} = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}.$$

Therefore,

$$\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} e^{\frac{i}{\hbar}S(n,n-1)},$$

and so

$$\langle x_N, t_N | x_1, t_1\rangle =$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N-1}{2}} \int \mathrm{d}x_{N-1} \int \mathrm{d}x_{N-2} ... \int \mathrm{d}x_2 \prod_{n=2}^{N} e^{\frac{i}{\hbar}S(n,n-1)}$$

with x_N and t_N fixed. If we define

$$\int_{x_1}^{x_N} [\mathrm{d}x(t)] := \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N-1}{2}} \int \mathrm{d}x_{N-1} \int \mathrm{d}x_{N-2} ... \int \mathrm{d}x_2,$$

then we can write

$$\langle x_N, t_N | x_1, t_1\rangle = \int_{x_1}^{x_N} [\mathrm{d}x(t)] e^{i\int_{t_1}^{t_N} \frac{L_{classical}(x,\dot{x})}{\hbar}}. \tag{1.9}$$

This is a path integral, which we can easily identify by the $[\mathrm{d}x(t)]$ notation. To see that this is equivalent to the previous section, and for a more detailed process see, for instance, [1].

Now, as we said, we will use the partition function, and will work with imaginary time. Let us give some ideas about that, without going into too much detail. For a more comprehensive description see, for instance, [4], [5].

1.3.1 Partition function

We know that in statistical mechanics, the partition function can be written as

$$Z = \sum_s e^{-\beta E_s}, \qquad \beta = \frac{1}{K_B T}$$

where the sum is over all the states, and T is the temperature. Now, it can be easily seen (as is shown by Sakurai, [1]) that the time evolution of a wave function can be written as

$$\psi(x,t) = \int \mathrm{d}^3x' K(x,t;x',t_0)\psi(x',t_0),$$

where

$$K(x,t;x',t_0)\psi(x',t_0) = \sum_a \langle x|a\rangle \langle a|x'\rangle\, e^{-i\frac{E_a(t-t_0)}{\hbar}}$$

is known as the propagator, which can also be written as

$$K(x,t;x',t_0) = \langle x|\, e^{-i\frac{H(t-t_0)}{\hbar}}\, |x'\rangle\,. \tag{1.10}$$

In fact, it is the Green function for the time dependent wave equation,

$$\begin{cases} \left(-\left(\frac{\hbar^2}{2m}\right)\nabla^2 + V(x) - i\hbar\frac{\partial}{\partial t}\right) K(x,t;x',t_0) = -i\hbar\delta^3(x-x')\delta(t-t_0) \\ \\ K(x,t;x',t_0) = 0, \quad t < t_0 \end{cases}$$

So, if we are given the wave function at the initial time, multiplying by the propagator and integrating over all space we solve the problem. But let us not discuss this any further, and instead let us go back to (1.10). We see that if we set $x = x'$ and integrate, we obtain the trace of the time-evolution operator in the x-representation. As the trace is independent of the representation we choose, we can calculate it in a basis where the operator is diagonal.

Now, we notice that if we write it,

$$\int \mathrm{d}^3x' K(x',t;x',0) = \sum_a e^{-\frac{iE_at}{\hbar}}$$

what we obtain is just a sum over all states, just like the partition function in statistical mechanics was. It can be seen that an analytic continuation of time t to a purely imaginary time can connect the two, with $\beta \mapsto it/\hbar$.

In chapter 3 we will work with imaginary time and the partition function, as we will understand quantum mechanics as a 0+1 quantum field theory (that is, with fields that only depend on time). Even if the process we just mentioned is beyond our reach, the idea of the partition function is familiar, since we know it from statistical mechanics. Let us give some results on the quantum partition function that we will use afterwards.

As we just said, $Z(\beta) = \mathrm{Tr}e^{-\beta H}$ describes evolution in imaginary time. If H is bounded from below, as in our case, then the ground state energy is given by

$$E_0 = \lim_{\beta\to\infty}\left(-\frac{1}{\beta}\ln \mathrm{Tr}e^{-\beta H}\right).$$

This expression should not surprise us: cooling the system (considering the β-infinite limit) we leave no energy for the excited states, and so the term corresponding to the ground state energy becomes more important in the series.

Now that we work with imaginary time, let us denote by q the positions. Then, the expression for the path integral that we obtained, equation (1.9), looks like

$$\int_{q(t')=q'}^{q(t'')=q''} [\mathrm{d}q(t)] e^{-\frac{S(q)}{\hbar}}.$$

Then, it can be seen, for example as shown in [4] that

$$Z(\beta) = \mathrm{Tr} e^{-\beta H} = \int [\mathrm{d}q(t)] e^{-\frac{S(q)}{\hbar}},$$

We will use the expression for the partition function of the harmonic oscillator, Z_0. The calculation is not difficult, but it is long and does not provide any useful knowledge for the sake of this thesis. So, we give directly its result:

$$Z_0(\beta) = \frac{e^{-\frac{\beta\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}.$$

1.4 Structure of the thesis

We have just given a brief introduction to quantum mechanics. Chapter 2 deals with the study of a 0+0 quantum field theory, as a first example where we can compute everything explicitly. We introduce there the steepest descent method for approximating integrals and the formalism for asymptotic series. We also introduce the Borel sum, as a method to sum some divergent series. In chapter 3 we study the anharmonic oscillator. First we obtain the perturbative series for the ground state energy using Mathematica (the script is included in appendix A), and then we obtain the behaviour at large order of the coefficients using instantons. Chapter 4 summarises the work, points out the limitations and hints at future related lectures.

1.5 Objectives

- Learn some methods used in quantum field theory and use them on an example (the anharmonic oscillator).
- Give a mathematical introduction to asymptotic series, that is, to Gevrey asymptotics.
- Study the steepest descent method for approximating some integrals on the complex plane.
- Reproduce the results for the anharmonic oscillator found in the papers.
- Learn a way to interpret some of the divergent series that are obtained in quantum field theory.

- Get familiar with specific physics papers in an area of personal interest.

Chapter 2

0-dimensional example

In this chapter we introduce a simple model of a quantum field theory, so we get used to the methods and definitions of asymptotic series and approximation of integrals.

2.1 The model

In quantum field theory we deal with fields that depend on spatial and temporal coordinates, $\Phi(\vec{x}, t)$. In this sense, we can understand quantum mechanics as a 0+1 QFT, because the states are fields which depend only on time, $\Phi(t)$. Going one step further, we can consider a theory where the fields are constants of the space time. In this section we consider a model described by an integral where Φ does not depend on any coordinate,

$$Z(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4\right)}. \tag{2.1}$$

2.2 Analyticity

First we prove that (2.1) is analytic on the cut plane, that is, on the complex λ-plane minus the negative real axis. Let us first consider $\lambda \in \mathbb{R}$. The integrals converges for $\lambda \geq 0$ and diverges for $\lambda < 0$. It's easy to see this. Differentiating cases: for $\lambda > 0$,

$$\int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4\right)} < \int_{-\infty}^{\infty} \mathrm{d}\Phi e^{-\frac{\lambda}{4!}\Phi^4} < +\infty \quad \Longrightarrow \quad \text{converges}$$

for $\lambda = 0$,

$$\int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} e^{-\frac{\Phi^2}{2}} = 1 < +\infty \quad \Longrightarrow \quad \text{converges}$$

and for $\lambda < 0$,

$$\exists[-k,k] \subset \mathbb{R}, k \in \mathbb{N} : \forall \Phi \notin [-k,k], \frac{1}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4 < 0 \implies$$

$$\implies \forall \Phi \notin [-k,k], e^{-\left(\frac{1}{2}\Phi^2+\frac{\lambda}{4!}\Phi^4\right)} > 1 \implies \int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}\Phi^2+\frac{\lambda}{4!}\Phi^4\right)} \text{ diverges}$$

Now we want to show that the function is analytic on the cut plane. The proof of this result is not as straightforward as the previous one. In fact, we will follow [4] to obtain this. However, before that we will present the steepest descent method (or saddle-point method) for approximating an integral, which we will use from now on.

2.2.1 Steepest descent method

Our goal is to estimate the value of an integral of the form

$$I(\lambda) = \int_C f(x) e^{\lambda S(x)} \mathrm{d}x,$$

where C is a contour on the real plane, and $f(x)$ and $S(x)$ are analytic functions such that $S(x)$ has a single maximum at $x = x_0$. The idea is that the main contribution to the integral will be given by the neighbourhood of x_0. As [6] shows, under some assumptions we can obtain the following estimate using Watson's lemma,

$$I(\lambda) \sim e^{-\lambda S(x_0)} \sum_{i=0}^{\infty} \Gamma\left(\frac{i+\alpha}{\mu}\right) \frac{c_i}{\lambda^{\frac{i+\alpha}{\mu}}}, \quad \lambda \to +\infty,$$

where μ,α are constants such that $\mu \in \mathbb{R}^+$, $\mathrm{Re}(\alpha) > 0$, and the coefficients c_i are related to the ones that we obtain considering the expansion at first order of $f(x)$ and $S(x)$ around x_0. However, we do not need such a general result, as we are only interested in the behaviour at leading order. The formula we are looking for is known as the Laplace approximation. Let us now give an heuristic argument for this approximation.

Keeping in mind that the major contribution comes from the maximum x_0 of $S(x)$, let us consider the leading terms of the Taylor series for $S(x)$ and $f(x)$ around x_0, and let $a \to -\infty$, $b \to +\infty$. Then,

$$\int_a^b f(x) e^{\lambda S(x)} \mathrm{d}x \approx f(x_0) e^{\lambda S(x_0)} \int_{-\infty}^{\infty} e^{\frac{\lambda}{2} S''(x_0)(x-x_0)^2} \mathrm{d}x \approx$$

$$\approx f(x_0) e^{\lambda S(x_0)} \sqrt{\frac{2\pi}{-\lambda S''(x_0)}}, \quad \lambda \to +\infty. \tag{2.2}$$

This is the approximation that we need. However, as we will encounter integrals on the complex plane, what we will prove is an extension of this result known as the steepest descent method.

Let us consider the previous integral on the complex plane,

$$I(\lambda) = \int_C f(z) e^{\lambda S(z)} \mathrm{d}z.$$

As before, C is a contour on the complex plane, and $f(z)$ and $S(z)$ are analytic functions. The idea of the method is that we apply Cauchy's theorem to deform the contour C so that it passes through the point of major contribution, and follows the path which makes decrease the value of the function S more rapidly (that is why we call it steepest descent method). This way, we can approximate the function by the values around that point. Let us now show that the point of major contribution corresponds to a saddle point of $\mathrm{Re}[S(z)]$.

Let $S(z) = u(x,y) + iv(x,y)$, where $z = x + iy$ and u, v are real, and let $z_0 = x_0 + iy_0$ be a non-degenerate critical point of $S(z)$. Then, we can write

$$0 = S'(z) = \frac{\mathrm{d}u}{\mathrm{d}z} + i\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z}\right).$$

Using the Cauchy-Riemann equations, we obtain

$$0 = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \Longrightarrow \qquad \frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} = \frac{\partial u}{\partial y}\Big|_{(x_0,y_0)} = 0.$$

Therefore, (x_0, y_0) is a critical point of u. Since u is an harmonic function (i.e. $\Delta u = 0$), (x_0, y_0) must be a saddle point of $u(x,y) = \mathrm{Re}(S(z))$, as we said before. Now we are motivated to define the saddle point of $S(z)$.

Definition 2.2.1 *Let $S : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. $z_0 \in \mathbb{C}$ is called a non-degenerate saddle point of $S(z)$ if*

$$\frac{\mathrm{d}S}{\mathrm{d}z}\Big|_{z=z_0} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 S}{\mathrm{d}z^2}\Big|_{z=z_0} \neq 0.$$

We will use Morse's lemma to construct the asymptotic expansion of integrals in the case of non-degenerate saddle points. Before that, we need a previous result.

Lemma 2.2.2 *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function in a neighbourhood of the origin, and $f(0) = 0$. Then, in some neighbourhood there exists a function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = zg(z)$, with g holomorphic and $g(0) = \frac{\mathrm{d}f(z)}{\mathrm{d}z}|_{z=0}$.*

Proof From the identity

$$f(z) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tz)\mathrm{d}t = z\int_0^1 \frac{\mathrm{d}f(y)}{\mathrm{d}y}\Big|_{y=tz}\mathrm{d}t,$$

we obtain

$$g(z) = \int_0^1 \frac{\mathrm{d}f(y)}{\mathrm{d}y}\Big|_{y=tz}\mathrm{d}t \quad \text{and} \quad g(0) = \frac{\mathrm{d}f(z)}{\mathrm{d}z}\Big|_{z=0}.$$

□

Now we are ready to prove Morse's lemma. We will give the proof for the one-dimensional case, but the lemma also holds for higher dimensions. See, for instance, [7].

Theorem 2.2.3 (Morse lemma) *Let S be a holomorphic function on a domain $U \subset \mathbb{C}$, and let $z_0 \in U$ be a non-degenerate saddle point of S. Then, there exists a local biholomorphism $\varphi : W \to V$, where $W \subset \mathbb{C}$ and $V \subset U$ are neighbourhoods of $w = 0$ and z_0 respectively, such that $S(\varphi(w)) - S(z_0)$ is quadratic in terms of w.*

Proof Without loss of generality, we translate the origin to z_0, i.e., $z_0 = 0$, $S(0) = 0$. Using the previous lemma, $S(z) = zg(z)$. We can also apply the lemma to $g(z)$, since the origin is a saddle point for $S(z)$, and so $g(0) = \frac{\mathrm{d}S}{\mathrm{d}z}|_{z=0} = 0$. Therefore, $g(z) = zh(z)$ and $S(z) = z^2 h(z)$. We see that

$$h(0) = \frac{S''(0)}{2}.$$

So, we consider the change of coordinates $z = \varphi(w)$ defined by

$$w = z\sqrt{\frac{h(z)}{h(0)}}.$$

Then,

$$S(\varphi(w)) = w^2 \frac{S''(0)}{2},$$

as we wanted. To see that it is a valid change of variables, we use the inverse function theorem. We have to see that the derivative does not vanish at 0. Since $S'(0) = 0$, and $\frac{\mathrm{d}^2 S(\varphi(w))}{\mathrm{d}w^2}\Big|_{w=0} = S''(0)$ from the previous expression, we obtain, applying the chain rule,

$$\frac{\mathrm{d}^2 S(\varphi(w))}{\mathrm{d}w^2}\Big|_{w=0} = S''(0)(\varphi'(0))^2 \qquad \Longrightarrow \qquad |\varphi'(0)| = 1.$$

□

Now we are ready to give the asymptotic expansion of the integral.

Theorem 2.2.4 (Steepest descent method) *Let $f(z)$ and $S(z)$ be holomorphic functions in an open, bounded and simply connected set $\Omega_z \subset \mathbb{C}$ such that $I_z = \Omega_z \cap \mathbb{R}$ is connected. Assume that $z_0 \in \Omega_z$ is a non-degenerate saddle point of $S(z)$ and that the integral $I(\lambda)$ converges absolutely for $\lambda \in \mathbb{R}$ large enough. Then,*

$$\int_{I_z} f(z)e^{\lambda S(z)}\mathrm{d}z \approx \sqrt{\frac{2\pi}{\lambda}}\frac{e^{\lambda S(z_0)}}{\sqrt{-S''(z_0)}}\left(f(z_0)+O(\lambda^{-1})\right), \quad \lambda \to +\infty. \tag{2.3}$$

Proof Let us recall our integral,

$$I(\lambda) = \int_{I_z} f(z)e^{\lambda S(z)}\mathrm{d}z.$$

First of all, we change the contour of integration. We deform I_z to a new contour $I'_z \subset \Omega_z$ in a way that it passes through the saddle point z_0, shares the endpoints with I_z and satisfies that $\mathrm{Im}[S(z)]$ is constant along the path. By Cauchy's integral theorem, the value of the integral does not change with this new contour because they share the endpoints. The last condition is required so the new path is indeed a path of steepest descent. Let us see it explicitly.

As we did at the beginning of this section, let $S(z) = u(x, y) + iv(x, y)$. Now, let γ be a differentiable curve with parametric equations $x = x(s)$, $y = y(s)$, where s is the arc length parameter of the curve in the xy plane. If we consider u as a third axis, orthogonal to x and y, $u = u(x, y)$ defines a surface Σ in the (x, y, u) space. If we now consider

$$\begin{cases} x = & x(s) \\ y = & y(s) \\ u = & u(x(s), y(s)), \end{cases} \tag{2.4}$$

they form a set of parametric equations of a curve on the surface Σ. Since s is the arc length parameter of the xy curve, we can write

$$\cos\theta = \frac{\mathrm{d}x}{\mathrm{d}s}, \quad \sin\theta = \frac{\mathrm{d}y}{\mathrm{d}s},$$

and so u changes along γ like

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = \partial_x u\cos\theta + \partial_y u\sin\theta.$$

The steepness of 2.4 is given by the angle α it forms with the u-axis. If we calculate the tangent vectors, we see that this angle is

$$\cos\alpha = \frac{\frac{\mathrm{d}u}{\mathrm{d}s}}{\sqrt{1+\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2}}.$$

We want the path of steepest descent, so we have to find a maximum of this function. Applying the chain rule, it is straightforward to see that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\alpha = \frac{-\partial_x u \sin\theta + \partial_y u \cos\theta}{\left(1+\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2\right)^{\frac{3}{2}}}.$$

So the derivative is 0 when the numerator is 0, and applying the Cauchy-Riemann equations we see that this is the condition we required at the beginning of the proof,

$$-\partial_x u \sin\theta + \partial_y u \cos\theta = -(\partial_y v \sin\theta + \partial_x v \cos\theta) = -\frac{\mathrm{d}v}{\mathrm{d}s} = 0.$$

When this condition holds, we see that

$$\frac{\mathrm{d}^2(\cos\alpha)}{\mathrm{d}\theta^2} = -\frac{\mathrm{d}u}{\mathrm{d}s}\frac{1}{\left(1+\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2\right)^{\frac{3}{2}}}.$$

Therefore, the condition allows us to obtain the path of steepest descent (or ascent) that we need.

So, with this new contour I_z' we can assume that $\mathrm{Re}[S(z)]$ has a maximum at z_0. Let $U \subset \Omega_z$ be a neighbourhood of z_0. We can split the integral into two parts, one over a neighbourhood $U \cap I_z'$ of z_0, which we will denote by $I_0(\lambda)$, and the other over the remaining path. We will now show that the second one is exponentially smaller than $I_0(\lambda)$, and so can be ignored.

Let $V = \Omega_z \setminus U$. We are concerned about the integral along $C = I_z' \cap V$. If $M = \sup_{z\in V} \mathrm{Re}[S(z)]$ and $\lambda_0 \in \mathbb{R}^+$ is such that

$$\int_C \left|f(z)e^{\lambda_0 S(z)}\right| \mathrm{d}z < \infty,$$

then, for all $\lambda \geq \lambda_0$, since for $z \in V$ we have that $\left|e^{(\lambda-\lambda_0)(S(z)-M)}\right| \leq 1$, we can write

$$\left|\int_C f(z)e^{\lambda S(z)}\mathrm{d}z\right| \leq \int_C |f(z)|\left|e^{\lambda S(z)}\right| \mathrm{d}z =$$

$$= \int_C |f(z)|\, e^{\lambda M}\left|e^{\lambda_0(S(z)-M)}e^{(\lambda-\lambda_0)(S(z)-M)}\right| \mathrm{d}z \leq$$

$$\leq \int_C |f(z)|\, e^{\lambda M}\left|e^{\lambda_0(S(z)-M)}\right| \mathrm{d}z =$$

$$= \left(e^{-\lambda_0 M}\int_C \left|f(z)e^{\lambda_0 S(z)}\right| \mathrm{d}z\right) e^{\lambda M} =: Ke^{\lambda M}, \qquad K\text{constant}. \tag{2.5}$$

Therefore, we are only concerned about the integral over the neighbourhood of the saddle point, $U \cap I_z'$. Hence we can use Morse's lemma to change the variables of integration (notice that the lemma gives a local coordinate). We have

$$I(\lambda) \approx I_0(\lambda) = \int_{U\cap I'_z} f(z)e^{\lambda S(z)}\mathrm{d}z = e^{\lambda S(z_0)} \int_{U\cap I'_z} f(z)e^{\lambda(S(z)-S(z_0))}\mathrm{d}z.$$

We will use the notation of the lemma. Let I_w be a contour of integration such that $\varphi(I_w) = U \cap I'_z$. Applying the lemma, since we know that $\mathrm{Re}[S(z)]$ has a maximum at z_0,

$$I_0(\lambda) = e^{\lambda S(z_0)} \int_{I_w} f(\varphi(w))e^{-\lambda\frac{(-S''(z_0)w^2)}{2}}|\varphi'(w)|\mathrm{d}w.$$

From the proof of Morse's lemma, we know that $\varphi(U)$ is a neighbourhood containing the origin, $\varphi(0) = z_0$ and also that $|\varphi'(0)| = 1$. If we expand $\varphi(w)$ in its Taylor series, and just keep the zero-order term, we obtain

$$I_0(\lambda) \approx f(z_0)e^{\lambda S(z_0)} \int_{I_w} e^{-\lambda\frac{(-S''(z_0)w^2)}{2}}\mathrm{d}w.$$

Now, we can change the contour of integration of our integral, and consider the integral over all $\mathbb{R}$, using again (2.5). The main contribution is the saddle point (the origin, which is contained by both sets), and at infinity the function vanishes. So, we obtain a well-known integral, which leads to the desired result

$$I(\lambda) = \sqrt{\frac{2\pi}{\lambda}}\frac{e^{\lambda S(z_0)}}{\sqrt{-S''(z_0)}}\left(f(z_0) + O(\lambda^{-1})\right), \quad \lambda \to +\infty. \tag{2.6}$$

□

Remark If the function $S(z)$ has multiple isolated saddle points, the same argument holds if we take a path passes through them all. Then, all we have to do is sum over all saddle points. The contribution of each saddle point will be given by (2.6).

Before we continue, we will give a direct application of this formula on the real plane: Stirling's approximation for the factorial, which we will use later on.

Example From the definition of the gamma function, we can write

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\mathrm{d}t = \int_0^\infty e^{-x\left(\frac{t}{x}-\ln t\right)}\mathrm{d}t = x^{x+1}\int_0^\infty e^{-x(z-\ln z)}\mathrm{d}z,$$

where in the last step we have introduced the change of variables $t = xz$. Now, using Laplace's approximation (2.2),

$$\Gamma(x+1) \sim x^{x+1}e^{-x}\sqrt{\frac{2\pi}{x}} = \sqrt{2\pi x}x^x e^{-x}.$$

Setting $x = n$ and recalling that $\Gamma(n+1) = n!$, we obtain

$$n! \sim \sqrt{2\pi n}n^n e^{-n}, \quad \text{for} \quad n \to +\infty. \tag{2.7}$$

2.2.2 Analyticity on the cut plane

We have seen that for $\lambda \geq 0$ integral (2.1) converges. Clearly, for small and positive values of λ (remember that λ is a small parameter) it is dominated by the saddle point at the origin. Let $\lambda \in \mathbb{C}$. We only have to worry about $\mathrm{Re}[\lambda]$, because the imaginary part is bounded. Also, if $\mathrm{Re}[\lambda] \geq 0$ the integral remains finite, since we are in the same case as before. For $\mathrm{Re}[\lambda] < 0$ the integral diverges, and so we have to perform the analytic continuation of the integral. We need $\mathrm{Re}[\lambda\Phi^4]$ to remain positive, so the integral remains finite. Hence, we have to rotate the contour of integration as we change the argument of λ. We choose a contour C such that $\arg(\Phi) = -\frac{1}{4}\arg(\lambda)$. Then

$$\mathrm{Re}[\lambda\Phi^4] = |\lambda||\Phi^4|\cos(\theta_\lambda + 4\theta_\Phi) > 0 \quad \forall\lambda \in \mathbb{C}.$$

Depending on the direction in which we rotate in the λ-plane, we obtain

- $\lambda = -|\lambda| + \varepsilon i$: $Z(\lambda) = \int_{C_+} e^{-\left(\frac{\Phi^2}{2}+\lambda\frac{\Phi^4}{4!}\right)}\mathrm{d}\Phi$
- $\lambda = -|\lambda| - \varepsilon i$: $Z(\lambda) = \int_{C_-} e^{-\left(\frac{\Phi^2}{2}+\lambda\frac{\Phi^4}{4!}\right)}\mathrm{d}\Phi$

where $\varepsilon \ll 1$, $\varepsilon \in \mathbb{R}$; C_+ corresponds to $\arg\Phi = -\frac{\pi}{4}(\mod \pi)$ and C_- corresponds to $\arg\Phi = \frac{\pi}{4}(\mod \pi)$, as seen in figure 2.1. The two integrals remain finite, and for $\lambda \to 0^-$ we can evaluate them using steepest descent. In order to write the integral (2.1) in the same form as appears in the theorem, we just have to consider the change $\Phi \mapsto \Phi'/\sqrt{\lambda}$. This way we have the whole dependence in λ in front of the exponent, and identifying $\lambda' = \lambda^{-1}$ we have the parameter that tends to infinity. To find the paths of steepest descent we just have to consider the constant-phase contours, as we showed in the proof of the method. A detailed description of the process for similar cases, and also for higher-order saddle points (when the second and third derivative at the saddle point also vanish, for example) can be found in [8].

Therefore, we see that for $\lambda \to 0^-$ the integrals are still dominated by the saddle point at the origin. Indeed, the other saddle points, which correspond to $\Phi_s^2 = -\frac{6}{\lambda}$ (S_1 and S_2 in figure 2.1), are negligible, since

$$\Phi_s^2 = -\frac{6}{\lambda} \Rightarrow e^{-\left(\frac{\Phi_s^2}{2}+\lambda\frac{\Phi_s^4}{4!}\right)} = e^{\frac{3}{2\lambda}} \ll 1 \quad \text{because} \quad \lambda \to 0^-.$$

In the next chapter we will need to calculate the discontinuity of the integral on the cut to obtain the imaginary part of the energy. As a guide, let us see what is obtained here. We have to compute the difference between the two integrals;

$$Z(\lambda + \varepsilon i) - Z(\lambda - \varepsilon i) = \int_{C_+ - C_-} e^{-\left(\frac{\Phi^2}{2}+\lambda\frac{\Phi^4}{4!}\right)}\mathrm{d}\Phi.$$

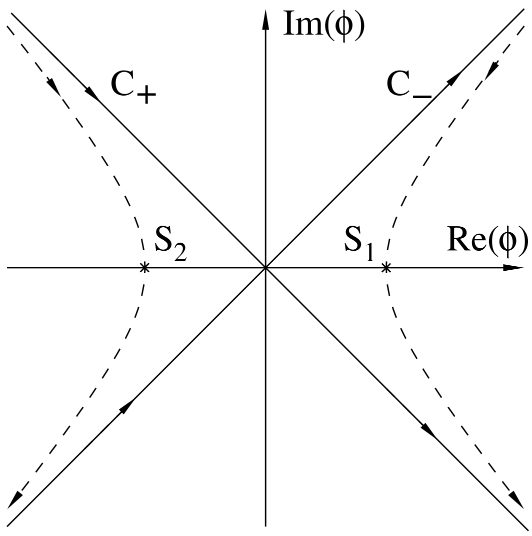

Figure 2.1: Contours of integration for the analytic continuation

The contour $C_+ - C_-$ can be deformed into the sum of the dashed ones in figure 2.1. So, the contributions from the saddle point at the origin cancel, and thus the other saddle points now dominate the integral. Therefore,

$$\mathrm{Im}[Z(\lambda)] \sim e^{\frac{3}{2\lambda}}.$$

2.3 Perturbative expansion

Now we will write $Z(\lambda)$ as a power series. Here we will be able to obtain an exact expression, and see its behaviour. This will be an illustrative example of this kind of series, as for the anharmonic oscillator we will not be able to obtain an exact analytic expression. So, first we set

$$e^{-\frac{\Phi^4}{4!}\lambda} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Phi^4 \lambda}{4!}\right)^n .$$

Then, plugging this expression in (2.1),

$$Z(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\Phi^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left(\frac{\Phi^4}{4!}\right)^n \lambda^n .$$

We have seen that the integral converges for $\lambda \geq 0$, but it can not be expressed in terms of usual functions. In order to associate an asymptotic series to it, which is what we expect as we argued in the introduction, let us we change the order of the sum and the integral.

$$Z(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\lambda^n}{(4!)^n} \int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} \Phi^{4n} e^{-\frac{\Phi^2}{2}} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\lambda^n}{(4!)^n} \frac{1}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{4n+1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{4n+1}{2}}}.$$

Simplifying, we obtain

$$Z(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{(4n-1)!!}{n!(4!)^n} \lambda^n =: \sum_{n=0}^{\infty} c_n (-\lambda)^n, \tag{2.8}$$

where $(2n-1)!! = \frac{(2n)!}{2^n n!}$.

Now we prove that the radius of convergence r of the power series is 0, as expected. We just have to compute it,

$$r = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{(4n-1)!!}{n!(4!)^n}}{\frac{(4n+3)!!}{(n+1)!(4!)^{n+1}}} = \lim_{n\to\infty} \frac{4!(n+1)}{(4n+3)(4n+1)} = 0.$$

As we said, this is an asymptotic series. In section 2.5 we will formalise the concept. However, let us now see how it behaves, to get an idea. An asymptotic series is a divergent series which truncated after a finite number of terms gives an approximation of a function. We can see this here very clearly. In figure 2.2 we present the behaviour of the series up to order 5 and 10, and we compare it with the exact result (integral 2.1 computed with Mathematica). As we were just saying, computing more terms of the series does not assure a better estimate of the function. As [9] discusses, the best possible estimate of the initial function is usually obtained by optimal truncation, keeping terms up to the smallest one in the series. However, it is not a very good approximation. A better approach is Borel resummation, which we discuss in section 2.6.

Before we formalise the concept of asymptotic series, we are going to evaluate the integral that led us to (2.8) using Laplace's method. Even though we were able to compute c_n explicitly, it is worth the effort as a guide for the anharmonic oscillator.

2.4 Semiclassical method

Here, the simplest version of the approximation will be enough. We will use that if $f(x)$ is a function with a minimum at x_0,

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, e^{-f(x)} \approx e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}}. \tag{2.9}$$

We have to apply this formula to the following integral

$$\int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} e^{-\frac{\Phi^2}{2} + \ln \Phi^{4n}}.$$

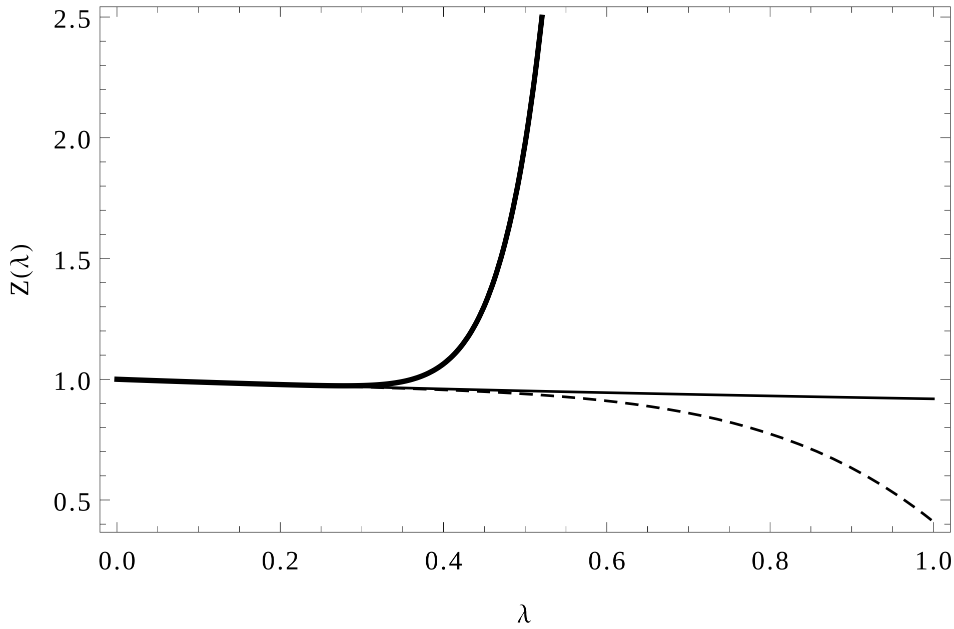

Figure 2.2: Values of $Z(\lambda)$ given by power series (2.8) up to order $n = 5$ (dashed line), $n = 10$ (thick line) and computing exactly the integral (2.1).

The minimum corresponds to $\Phi_0^2 = 4n$. Therefore,

$$\int_{-\infty}^{\infty} \mathrm{d}\Phi \frac{1}{\sqrt{2\pi}} e^{-\frac{\Phi^2}{2}+\ln \Phi^{4n}} \approx \frac{1}{\sqrt{2\pi}} e^{-2n}(4n)^{2n}\sqrt{\frac{2\pi}{2}} = \frac{1}{\sqrt{2}} e^{-2n}(4n)^{2n}.$$

We computed the exact result of this integral a while ago, and it is $(4n-1)!!$. We have to compare both results. We will do it using Stirling's approximation (2.7) (we are concerned about the large n behaviour). Here, we will use the approximation in a simplified form: for n large, $\ln n! \approx n \ln n - n$. Then

$$\ln\left(\frac{e^{-2n}(4n)^2}{\sqrt{2}}\right) = \ln\left(e^{-2n}\right) + \ln\left((4n)^{2n}\right) - \ln\sqrt{2} = -2n + 2n\ln 4n - \frac{1}{2}\ln 2,$$

and so

$$\ln(4n-1)!! = \ln\left(\frac{(4n)!}{4^n(2n)!}\right) \approx -2n + 2n\ln 4n.$$

As we used Stirling's approximation, we are working with large values of n. Hence the contribution of $\ln\sqrt{2}$ is negligible, and so we have obtained the same result, as expected.

Now we will formalise the concept of asymptotic series, and we will prove that the power series (2.8) is indeed asymptotic. To do so, we will introduce some basic concepts of Gevrey asymptotics, following [10] and [11]. One of the objectives of the thesis is to study asymptotic series, and since it is a new concept, we will prove some extra results (than the ones we would strictly need) to set a solid frame.

2.5 Gevrey asymptotics

First we define the regions where we will consider our functions.

Definition 2.5.1 *The set*

$$S(\gamma,\rho)=\{z=re^{i\varphi}\in\mathbb{C}:0<r<\rho,|\varphi|<\gamma/2\}$$

is called a sector of radius $\rho>0$ and opening $\gamma\in(0,2\pi]$. A closed sector is

$$\bar{S}(\gamma,\rho)=\{z=re^{i\varphi}\in\mathbb{C}:0<r\leq\rho,|\varphi|\leq\gamma/2\}.$$

Henceforth, $\alpha\in(0,1]$. Now we define what we mentioned earlier as an asymptotic series.

Definition 2.5.2 *We say that a formal power series $\hat{f}=\sum_{n=0}^{\infty}f_nz^n$ is α-Gevrey if there exist positive constants C,K such that for every non-negative integer n,*

$$|f_n|\leq CK^nn!^{\alpha}.$$

The set of all α-Gevrey power series is denoted by $\mathbb{C}[[z]]_{\alpha}$.

In order to interpret these series, we will regard them as an asymptotic expansion of a function.

Definition 2.5.3 *We say that an analytic function f in a sector S is α-Gevrey if for every closed subsector $\bar{S}_1\subset S$, there exist positive constants C,K such that for every non-negative integer n and every $z\in\bar{S}_1$,*

$$\frac{1}{n!}|f^{(n)}(z)|\leq CK^nn!^{\alpha}.$$

The set of all α-Gevrey functions in a sector S is denoted by $G_{\alpha}(S)$.

Definition 2.5.4 *We say that an analytic function f in a sector S is asymptotic α-Gevrey to a formal power series $\hat{f}$, or that $\hat{f}$ is the α-Gevrey asymptotic expansion of f (in which case we will write $f\cong_{\alpha}\hat{f}$) if for every closed subsector $\bar{S}_1\subset S$ there exist positive constants C,K such that for every non-negative integer n and every $z\in\bar{S}_1$,*

$$|r_f(z,n)|\leq CK^nn!^{\alpha},$$

where r_f is the residue,

$$r_f(z,n)=z^{-n}\left(f(z)-\sum_{k=0}^{n-1}f_kz^k\right).$$

The set of all asymptotic α-Gevrey functions in a sector S is denoted by $A_{\alpha}(S)$.

Now we present some results concerning the definitions we just gave.

Proposition 2.5.5 *Let f be an analytic function in a sector S, asymptotic α-Gevrey to a formal power series $\hat{f}$. Then, $\hat{f}$ is α-Gevrey.*

Proof Let $z \in S$, and let $\bar{S}_1 \subset S$ be a closed subsector such that $z \in \bar{S}_1$. Then,

$$|r_f(z,n) - f_n| = |z r_f(z,n+1)| \leq |z| C K^{n+1} (n+1)!^{\alpha}.$$

Therefore,

$$f_n = \lim_{S \ni z \to 0} r_f(z,n).$$

Since for any closed subsector we have $|r_f(z,n)| \leq C K^n n!^{\alpha}$, the same bound holds when z goes to zero, and so we obtain

$$|f_n| = \left| \lim_{S \ni z \to 0} r_f(z,n) \right| \leq C K^n n!^{\alpha}.$$

□

Proposition 2.5.6 *Let f be an analytic function in a sector S and $\hat{f} = \sum_{n \geq 0} f_n z^n$ be a formal power series. Then, f is asymptotic α-Gevrey to $\hat{f}$ if, and only if, f is α-Gevrey and for every non-negative integer n*

$$\lim_{S \ni z \to 0} f^{(n)}(z) = n! f_n. \tag{2.10}$$

Proof Let us assume first that $f \cong_{\alpha} \hat{f}$. Let $\hat{f}^{(l)}$ be the term wise $l-$derivative formal series of $\hat{f}$, that is,

$$\hat{f}^{(l)}(z) = \sum_{n \geq 0} n(n-1)...(n-l+1) f_n z^{n-l} =$$

$$= \sum_{m \geq 0} (m+1)...(m+l) f_{m+l} z^m = \sum_{m \geq 0} f_m^{(l)} z^m,$$

and let $r_{f^{(l)}}$ be its residue. Let $\bar{S}_1 \subset S$ and $\bar{S}_2 \subset S$ be closed subsectors such that $\bar{S}_1 \subsetneq \bar{S}_2$. Let $\delta > 0$ be small enough so that for every $z \in \bar{S}_1$, $B(z, |z|\delta) \subset \bar{S}_2$. Let C_2, K_2 be the α-Gevrey constants of f in the sector $\bar{S}_2$, $z \in \bar{S}_1$ and $n, l \in \mathbb{Z}^+$. We can write

$$z^n r_{f^{(l)}}(z,n) = \frac{\mathrm{d}^l}{\mathrm{d}z^l} (z^{n+l} r_f(z,n+l)).$$

From Cauchy's integral formula, we know

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\varsigma)}{(\varsigma - z)^{n+1}} \mathrm{d}\varsigma$$

for a circle C. Therefore, we can write

$$\frac{\mathrm{d}^l}{\mathrm{d}z^l}(z^{n+l}r_f(z,n+l)) = \frac{l!}{2\pi i}\int_{|w-z|=\delta|z|}\frac{w^{n+l}r_f(w,n+l)}{(w-z)^{l+1}}\mathrm{d}w.$$

Using this, as well as $(n+l)! \leq 2^{n+l}n!l!$, we obtain

$$|r_{f^{(l)}}(z,n)| \leq C_2\left(2^\alpha K_2\frac{1+\delta}{\delta}\right)^l l!^{1+\alpha}(2^\alpha(1+\delta)K_2)^n n!^\alpha,$$

which in fact proves that $f^{(l)} \cong_\alpha \hat{f}^{(l)}$. Then, since $r_f(z,0) = f(z)$ we have

$$|f^{(l)}(z)| = |r_{f^{(l)}}(z,0)| \leq C_2\left(2^\alpha K_2\frac{1+\delta}{\delta}\right)^l l!^{1+\alpha}.$$

So, f is α-Gevrey. Also,

$$\lim_{S\ni z\to 0} f^{(n)}(z) = \lim_{S\ni z\to 0} r_{f^{(n)}}(z,0) = f_0^{(n)} = n!f_n.$$

Conversely, assume that f is α-Gevrey in the sector S and that the previous limit holds. Let

$$R_f(z,u,n) = z^{-n}\left(f(z) - \sum_{k=0}^{n-1}\frac{f^{(k)}(u)}{k!}(z-u)^k\right)$$

be the remainder of the Taylor series. We can write

$$R_f(z,u,n) = \frac{z^{-n}(z-u)^n}{(n-1)!}\int_0^1 f^{(n)}(u+\lambda(z-u))(1-\lambda)^{n-1}\mathrm{d}\lambda$$

for every $z,u \in S$ and $n \geq 0$. Let $\bar{S}_1 \subset S$ be a closed subsector, and let C, K be the corresponding α-Gevrey constants in $\bar{S}_1$. For $z,u \in \bar{S}_1$, we have $|f^{(n)}(u+\lambda(z-u))| \leq CK^n(n!)^{1+\alpha}$. So,

$$|R_f(z,u,n)| \leq |z|^{-n}|z-u|^n CK^n n!^\alpha.$$

Since

$$r_f(z,n) = \lim_{S\ni u\to 0} R_f(z,u,n),$$

we obtain that

$$|r_f(z,n)| \leq \lim_{\bar{S}\ni u\to 0}|z|^{-n}|z-u|^n CK^n(n!)^\alpha = CK^n n!^\alpha,$$

which completes the proof. □

From the proof of the proposition, we see that the asymptotic α-Gevrey expansion $\hat{f}$ of an asymptotic α-Gevrey function f in a sector S is unique.

The final result we give is the Borel-Ritt Gevrey theorem, which tells us that in a sector of small opening, every α-Gevrey power series is the asymptotic expansion of an analytic function.

Theorem 2.5.7 (Borel-Ritt) *Let* $\hat{f} \in \mathbb{C}[[z]]_\alpha$ *and let* S *be a sector of opening* $\beta < \alpha\pi$. *Then, there exists a function* f *analytic in* S *such that* $f \cong_\alpha \hat{f}$.

Before proving this theorem, it is convenient to introduce the idea of Laplace and Borel transforms. We will not go into much detail though.

Laplace and Borel transforms

Definition 2.5.8 *Let* S *be a sector of infinite radius, and let* f *be an analytic function of exponential size at most* $k > 0$ *in* S, *and continuous at the origin. We say that the function*

$$g(z) = z^{-k} \int_0^\infty f(u) e^{-\left(\frac{u}{z}\right)^k} \mathrm{d}(u^k)$$

is the Laplace transform with index k *of the function* f, *and we denote it by* $\mathfrak{L}_k f$.

It can be seen that $\mathfrak{L}_k f$ is defined and analytic in a region which contains a sector $\hat{S}$ of opening smaller than $\beta + \pi/k$ (where β is the opening of the sector S), but arbitrarily close to it.

We will now prove a result for Laplace transforms that we will later use in the proof of the Borel-Ritt theorem.

Lemma 2.5.9 *For* $f(u) = u^\lambda$, *with* $\lambda \in \mathbb{R}^+$,

$$\mathfrak{L}_k f(z) = \Gamma(1 + \lambda/k) z^\lambda.$$

Proof Let us first recall the definition of the gamma function. For $n > 0$,

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} \mathrm{d}t.$$

Applying the definition of $\mathfrak{L}_k f$ the result is straightforward,

$$\mathfrak{L}_k f(z) = z^{-k} \int_0^\infty f(u) e^{-\left(\frac{u}{z}\right)^k} \mathrm{d}(u^k) = \int_0^\infty z^\lambda z^{-k} \left(\frac{u}{z}\right)^{\frac{k\lambda}{k}} e^{-\left(\frac{u}{z}\right)^k} \mathrm{d}(u^k) =$$

$$= z^\lambda \Gamma(1 + \lambda/k).$$

□

Now, since are working with asymptotic series, we would like to define an operator that transforms them to analytic functions. Roughly speaking, this is what we will call the Borel transform.

Definition 2.5.10 *Let $\hat{f}(z) = \sum_{n\geq 0} f_n z^n$ be a formal power series. We say that the power series*

$$\sum_{n=0}^{\infty} \frac{f_n z^n}{\Gamma(1+n/k)}$$

is the Borel transform of $\hat{f}$ with index k, and we denote it by $\hat{\mathfrak{B}}_k \hat{f}(z)$.

Remark The hat on the operator $\mathfrak{B}$ indicates that we are applying it to a power series. The Borel transform is an integral transform, and by $\hat{\mathfrak{B}}_k$ we mean the term wise application of $\mathfrak{B}_k$ to a power series.

These transforms are usually used to give asymptotic series a physical meaning, to make them finite. In the next section we will see how.

It can be seen that for suitable function spaces, $\mathfrak{B}_k$ is the inverse operator of $\mathfrak{L}_k$. We will discuss this transforms no further though. For a complete description see, for instance, [11].

We are now prepared to prove the Borel-Ritt Gevrey theorem.

Proof (of theorem 2.5.7) Let $\hat{f}(z) = \sum_{n\geq 0} \tilde{f}_n z^n \in \mathbb{C}[[z]]_\alpha$, and let $f(u)$ be its Borel transform. For $|u|$ sufficiently small $f(u)$ is analytic, because $\hat{f} \in \mathbb{C}[[z]]_\alpha$. Let d be the bisecting direction of S, and define for sufficiently small $\rho > 0$

$$g(z) = z^{-\frac{1}{\alpha}} \int_0^{\rho e^{id}} f(u) e^{\left(-\frac{u}{z}\right)^{\frac{1}{\alpha}}} \mathrm{d}(u^{\frac{1}{\alpha}}).$$

This function g is called the finite Laplace transform of f with index $1/\alpha$. We will see that $g(z) \cong_\alpha \hat{f}(z)$.

Let us first argue the analyticity of $g(z)$. Since we are considering the finite Laplace transform, the limits of the integral are finite. Also, we know that $f(u)$ is analytic, and so is the exponential function, which we can write as its power series. Since we have a product of two convergent series, and the exponential is absolutely convergent, the term wise integration of the product series will also be analytic. Therefore, $g(z)$ is analytic.

Now, we have to see that

$$|r_g(z,N)| = \left| z^{-N} \left(z^{-\frac{1}{\alpha}} \int_0^{\rho e^{id}} f(u) e^{-\left(\frac{u}{z}\right)^{\frac{1}{\alpha}}} \mathrm{d}(u^{\frac{1}{\alpha}}) - \sum_{n=0}^{N-1} \tilde{f}_n z^n \right) \right| \leq C K^N N!^\alpha. \tag{2.11}$$

First, let us write $\tilde{f}_n = \Gamma(1+\alpha n) f_n$. Since $f(u)$ is analytic, we can assume that $|f_n| \leq C_n$, for all $n \geq 0$. Now, if we set $u = re^{id}$, with $0 \leq r \leq \rho$ in the lhs of inequality (2.11) we obtain

$$\left| z^{-\frac{1}{\alpha}} \int_0^{\rho} f(re^{id}) e^{-\left(\frac{re^{id}}{z}\right)^{\frac{1}{\alpha}}} e^{idn} \mathrm{d}(r^{\frac{1}{\alpha}}) - \sum_{n=0}^{N-1} f_n z^n \Gamma(1+\alpha n) \right|.$$

For a given $\varepsilon > 0$, take z such that $\cos(\alpha^{-1}[d - \arg z]) \geq \varepsilon$. Then, since we are looking for an upper bound, and we will enter the modulus into the integral,

$$\left| e^{-\left(\frac{re^{id}}{z}\right)^{\frac{1}{\alpha}}} \right| = \left| e^{-\left(\frac{re^{id}}{|z|e^{i\arg(z)}}\right)^{\frac{1}{\alpha}}} \right| \leq e^{-\varepsilon\left(\frac{r}{|z|}\right)^{\frac{1}{\alpha}}}.$$

Using this, and lemma 2.5.9 in the second term, we obtain

$$\left| g(z) - \sum_{n=0}^{N-1} f_n \Gamma(1+\alpha n) z^n \right| \leq$$

$$\leq |z|^{-\frac{1}{\alpha}} \int_0^\rho \left| f(re^{id}) - \sum_{n=0}^{N-1} f_n \left(re^{id}\right)^n \right| e^{-\varepsilon\left(\frac{r}{|z|}\right)^{\frac{1}{\alpha}}} \mathrm{d}\left(r^{\frac{1}{\alpha}}\right) +$$

$$+ \sum_{n=0}^{N-1} |f_n||z|^{-\frac{1}{\alpha}} \int_\rho^\infty r^n e^{-\varepsilon\left(\frac{r}{|z|}\right)^{\frac{1}{\alpha}}} \mathrm{d}\left(r^{\frac{1}{\alpha}}\right) \leq$$

$$\leq |z|^{-\frac{1}{\alpha}} C_N |u|^N \int_0^\rho e^{-\varepsilon\left(\frac{r}{|z|}\right)^{\frac{1}{\alpha}}} \mathrm{d}\left(r^{\frac{1}{\alpha}}\right) +$$

$$+ \sum_{n=0}^{N-1} C_n \left(\frac{1}{\rho}\right)^{N-n} |z|^{-\frac{1}{\alpha}} \int_\rho^\infty r^N e^{-\varepsilon\left(\frac{r}{|z|}\right)^{\frac{1}{\alpha}}} \mathrm{d}\left(r^{\frac{1}{\alpha}}\right),$$

where in the last inequality we have used that, since $f(u)$ is analytic, there exist real numbers C_N such that

$$\left| f(u) - \sum_{n=0}^{N-1} f_n u^n \right| \leq C_N |u|^N,$$

and also that

$$|f_n| \leq C_n \left(\frac{r}{\rho}\right)^{N-n}$$

for $r \geq \rho$, $0 \leq n \leq N-1$. Now, since $r = |u|$, we can join the two terms in a single sum up to N, taking into account that since we are integrating positive functions, the integral over all values of r (that is, from 0 to infinity) will be an upper bound of the integrals. Therefore, our expression will be bounded from above by

$$\sum_{n=0}^{N} C_n \rho^{n-N} |z|^{-\frac{1}{\alpha}} \int_0^\infty r^N e^{-\varepsilon\left(\frac{r}{|z|}\right)^{\frac{1}{\alpha}}} \mathrm{d}\left(r^{\frac{1}{\alpha}}\right).$$

Recalling the definition of the gamma function, this integral is straightforward if we name y the argument of the exponential,

$$\int_0^\infty r^N e^{-\varepsilon\left(\frac{r}{|z|}\right)^{\frac{1}{\alpha}}} \mathrm{d}\left(r^{\frac{1}{\alpha}}\right) = \frac{|z|^{\frac{1}{\alpha}}}{\varepsilon} \left(\frac{|z|^{\frac{1}{\alpha}}}{\varepsilon}\right)^{N\alpha} \int_0^\infty y^{N\alpha} e^{-y} \mathrm{d}y =$$

$$= \frac{|z|^N}{\varepsilon^{N\alpha+1}}\Gamma(1+N\alpha)|z|^{\frac{1}{\alpha}}.$$

Hence, we have obtained that

$$\left|g(z) - \sum_{n=0}^{N-1} f_n \Gamma(1+\alpha n) z^n\right| \leq |z|^N \varepsilon^{-1-N\alpha}\Gamma(1+N\alpha)\sum_{n=0}^{N} C_n \rho^{n-N}.$$

To prove (2.11), it just remains to see that

$$W_N := \varepsilon^{-1-N\alpha}\Gamma(1+N\alpha)\sum_{n=0}^{N} C_n \rho^{n-N} \leq CK^N N!^\alpha \tag{2.12}$$

for suitable C, K. We can assume that

$$C_n \leq \tilde{C}\tilde{K}^n \Gamma(1+n\alpha_1)$$

for $0 \leq \alpha_1 < +\infty$, and so we can take C_N outside of the sum. Let $\alpha_2 := \alpha + \alpha_1$. Clearly, $\Gamma(1+\alpha N)\Gamma(1+\alpha_1 N) \leq \Gamma(1+\alpha_2 N)$. Then, for every ρ we can find suitable constants to satisfy the Gevrey condition, because the sum is finite. We can check it in every case; for $\rho > 1$,

$$\sum_{n=0}^{N} \rho^{n-N} \leq \sum_{n=0}^{\infty} \rho^{-n} = \frac{\rho}{\rho-1}.$$

For $\rho < 1$,

$$\sum_{n=0}^{N} \rho^{n-N} = \frac{1-\frac{1}{\rho^{N+1}}}{1-\frac{1}{\rho}} < \left(\frac{2}{\rho}\right)^N$$

and for $\rho = 1$,

$$\sum_{n=0}^{N} 1 = N < 2^N.$$

Therefore, we have suitable constants $C, K > 0$ such that

$$W_N \leq CK^N \Gamma(1+N\alpha_2).$$

This is equivalent to (2.12). We just need to comment two things to see it: first, the result holds for $\alpha_1 = 0$ (that is, $\alpha_2 = \alpha$) because $f(u)$ is analytic. Second, changing the constants C, K if necessary, we identify $\Gamma(1+N\alpha)$ with $N!^\alpha$, in the light of Stirling's formula (2.7). This completes the proof. □

Remark If we define the homomorphism $J : A_\alpha(S) \to \mathbb{C}[[z]]_\alpha$, Borel-Ritt theorem shows that if the opening of S is at most $\alpha\pi$, then J is surjective. However, it is not injective. As seen in [11], if we consider

$$f(z) = e^{-cz^{\frac{1}{\alpha}}},$$

with c a constant, it is easy to see that $f(z) \cong_\alpha \hat{0}$ in a sector S of opening at most $\pi\alpha$, where $\hat{0}$ stands for the zero power series.

It can be seen that if f is analytic in a sector S of opening $\beta < \pi\alpha$, and $f \cong_\alpha \hat{0}$, then f is exponentially small in S. In this same direction we have Watson's theorem [12], which we will not prove.

Theorem 2.5.11 (Watson) *Let S be a sector of opening $\beta > \pi\alpha$, and let $f \in A_\alpha(S)$. If f decays exponentially $1/\alpha$ in S, then $f \equiv 0$.*

To end this section we will see that $Z(\lambda)$ is 1-Gevrey, and asymptotic 1-Gevrey to its power series (2.8) (and so, using proposition 2.5.5 we will have that power series (2.8) is also 1-Gevrey).

First, recalling the definition of an α-Gevrey function, we have to find constants C, K and α such that

$$\frac{1}{n!}\left|Z^{(n)}(\lambda)\right| \leq CK^n n!^\alpha.$$

We have to compute the derivative of $Z(\lambda)$. Recalling (2.1),

$$\frac{\mathrm{d}^n Z}{\mathrm{d}\lambda^n} = \frac{\mathrm{d^n}}{\mathrm{d}\lambda^n}\int_{-\infty}^{\infty}\mathrm{d}\Phi\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{1}{2}\Phi^2+\frac{\lambda}{4!}\Phi^4\right)}.$$

Since the function inside the integral is well behaved, we can enter the derivative,

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}e^{-\frac{1}{2}\Phi^2}\frac{\mathrm{d^n}}{\mathrm{d}\lambda^n}\left(e^{-\frac{\lambda}{4!}\Phi^4}\right) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}(-1)^n\left(\frac{\Phi^4}{4!}\right)^n e^{-\left(\frac{1}{2}\Phi^2+\frac{\lambda}{4!}\Phi^4\right)}.$$

So, taking $\mathrm{Re}[\lambda] \geq 0$, we recover the integral we computed before,

$$\left|Z^{(n)}(\lambda)\right| \leq \int_{-\infty}^{\infty}\mathrm{d}\Phi\frac{1}{\sqrt{2\pi}}\left(\frac{\Phi^4}{4!}\right)^n e^{-\left(\frac{1}{2}\Phi^2+\frac{\mathrm{Re}[\lambda]}{4!}\Phi^4\right)} \leq$$

$$\leq \int_{-\infty}^{\infty}\frac{\mathrm{d}\Phi}{\sqrt{2\pi}}\left(\frac{\Phi^4}{4!}\right)^n e^{-\frac{1}{2}\Phi^2} = \frac{1}{\sqrt{2\pi}}\frac{1}{(4!)^n}\Gamma\left(2n+\frac{1}{2}\right)2^{2n+\frac{1}{2}} = \frac{(4n-1)!!}{(4!)^n},$$

i.e.

$$\frac{1}{n!}\left|Z^{(n)}(\lambda)\right| \leq \frac{(4n-1)!!}{n!(4!)^n},$$

which, as we can see, are exactly the modulus of the coefficients $|c_n|$ of the power series (2.8). Let us see now from this expression that $Z(\lambda)$ is 1-Gevrey.

Using Stirling's formula (2.7),

$$\frac{1}{n!}\frac{(4n-1)!!}{(4!)^n} = \frac{1}{n!}\frac{(4n)!}{4^n(2n)!}\frac{1}{(4!)^n} \sim \left(\frac{2}{3}\right)^n\frac{\sqrt{n}n^n e^{-n}}{\sqrt{\pi}n}.$$

If we multiply and divide by $\sqrt{2\pi}$ we can apply once again Stirling and recover an $n!$. Then, we will have that the power series is 1-Gevrey, and we can guess that suitable constants C, K (according to (2.5.2)) will be $1/\pi\sqrt{2}$

and 2/3 respectively. However, we will try to find the best bound for C. Therefore, we will prove the inequality by induction, finding a better bound for C than $1/\pi\sqrt{2}$. Our assumption will be that

$$\frac{(4n-1)!!}{n!(4!)^n} \leq C\left(\frac{2}{3}\right)^n n!. \tag{2.13}$$

For $n = 1$ we have

$$\frac{3}{24} \leq C\frac{2}{3} \quad \Rightarrow \quad C \geq \frac{3}{16}$$

setting our constant C to 3/16 (notice that $3/16 < 1/\pi\sqrt{2}$). Now, our induction hypothesis is that (2.13) holds for $n-1$, and we have to prove it for n. Expanding the factorials and using Stirling,

$$\frac{(4n-1)!!}{n!(4!)^n} = \frac{(4n-1)(4n-3)}{24n}\frac{(4n-5)!!}{(n-1)!(4!)^{n-1}} \leq$$

$$\leq \frac{(4n-1)(4n-3)}{24n}C(n-1)!\left(\frac{2}{3}\right)^{n-1} \leq Cn!\left(\frac{2}{3}\right)^n \iff$$

$$\iff \frac{(4n-1)(4n-3)}{24n} \leq n\frac{2}{3} \iff 1 \leq 4n$$

that holds for every $n \geq 1$. Therefore, we have proved that

$$\frac{(4n-1)!!}{n!(4!)^n} \leq \frac{3}{16}\left(\frac{2}{3}\right)^n n!$$

i.e. that $Z(\lambda)$ is 1-Gevrey with $C = 3/16$ and $K = 2/3$.

Now, it is straightforward to see that $Z(\lambda)$ is 1-Gevrey to its power series, using proposition 2.5.6. We have to see that

$$\lim_{\lambda\to 0} Z^{(n)}(\lambda) = n!c_n,$$

and since we can enter the limit to the integral, this is the exact same calculation that we just did. Thus, we have obtained that $Z(\lambda)$ is 1-Gevrey, and asymptotic 1-Gevrey to its power series. Using proposition 2.5.5 we have that the power series (2.8) is also 1-Gevrey, and, in fact, the same proof that we have given for $Z(\lambda)$ holds for its power series (because, as we mentioned earlier, in this example we are able to compute everything exactly and so it is expected that we arrive at the exact same result by both ways), and so (2.8) is 1-Gevrey with constants $C = 3/16$ and $K = 2/3$.

To end this chapter, we are going to give a method to solve the divergence of the power series that we encounter: Borel summation.

2.6 Borel summation

Up to now we have mentioned that asymptotic series appear often in quantum field theory. We have seen their behaviour, and given some results about them. We have also mentioned optimal truncation as a way to understand these divergent series, but now we are going to give a much more powerful way to reproduce the original function.

As we have introduced in the previous section, the Borel transform compensates the divergent behaviour of our series. Additionally, we have introduced the Laplace transform, which as we have commented can be seen as the inverse operator of the Borel one. Keeping this in mind, along with the fact that the Laplace transform of an analytic function is analytic in suitable regions, as we mentioned earlier, we think about the following process: given a divergent power series, we can calculate its Borel transform, sum the resulting series (which we will be able to do for an asymptotic series) and then apply the Laplace transform to the result of the sum. This way, we can sum the series and obtain an analytic function as a result, which is our objective. Therefore, the definition of the Borel sum is now clear.

Definition 2.6.1 *Let*

$$\hat{f}(w) = \sum_{n=0}^{\infty} f_n w^n$$

be a divergent power series, with $f_n \sim n!$. The Borel sum of $\hat{f}(w)$ is defined as

$$s(\hat{f})(z) := \mathfrak{L}_1[\hat{\mathfrak{B}}_1 \hat{f}],$$

understanding that we compute the sum of the Borel transform before applying Laplace.

We will not write the subindices of the transforms when we talk about Borel summation; we will directly understand that they are 1.

So now we have a way to sum our asymptotic power series (2.8) and obtain a finite result. However, there is an obvious setback in this process: we have to be able to sum the Borel transform of the series. In some cases this will be easy, but usually it is going to be a tough job.

In our case, if we denote by $\hat{Z}$ power series (2.8), we have to compute the following sum:

$$\hat{\mathfrak{B}}\hat{Z}(w) = \sum_{n=0}^{\infty} (-1)^n \frac{(4n-1)!!}{(4!)^n (n!)^2} w^n.$$

This is not an easy task, and so we compute it with Mathematica. The result involves a complete elliptic integral of the first kind, so we will not write it. Now we have to calculate its Laplace transform, and compare the result with the initial integral $Z(\lambda)$ (2.1). We present in figure 2.3 both

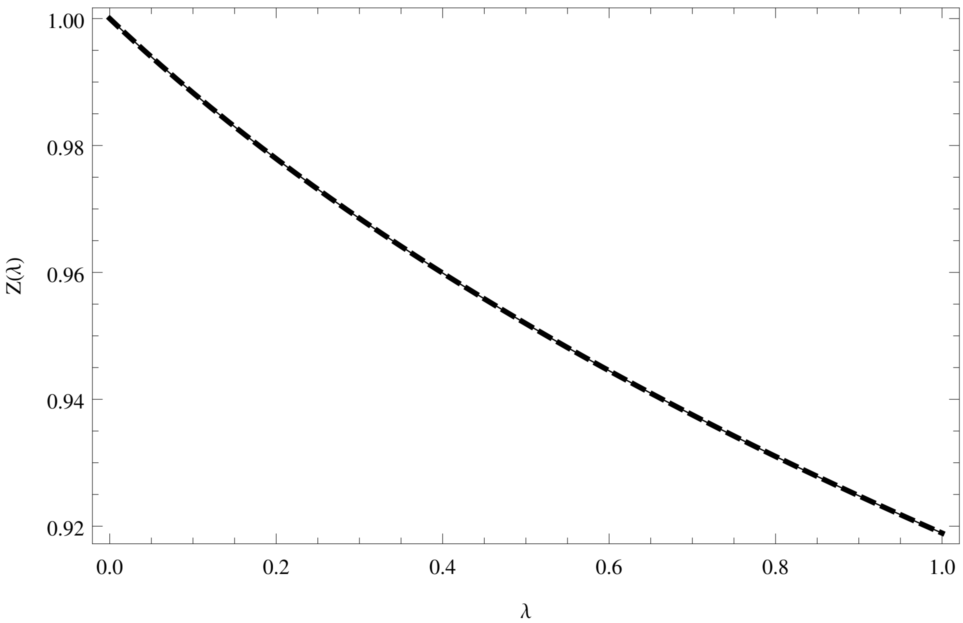

Figure 2.3: Comparison of the Borel sum of power series (2.8) (thick, dashed line) and the integral $Z(\lambda)$ (2.1) computed exactly with Mathematica.

results. As we can see, they coincide very well. In fact, we can check with Mathematica that in this case they are equals. In general, the Borel sum of the series gives a great approximation to the original function.

If we compare this figure with 2.2 (notice that in both cases we have represented the same segment of the function), we clearly see that the Borel summation is a very powerful method to solve the divergence. With this method we take into account all the terms of the series, and so the result is better than any truncation.

Chapter 3

Anharmonic Oscillator

In this chapter we focus on the anharmonic oscillator. We compute the perturbative series, and using semiclassical methods we reproduce the perturbative expansion.

3.1 Perturbative series

We will discuss the anharmonic oscillator, that is, a quantum harmonic oscillator with a quartic perturbation in its Hamiltonian. We will understand it as a field theory in a one-dimensional space time, described by the Hamiltonian

$$H = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\varphi^2 + \lambda\varphi^4.$$

Our eigenvalue problem will be defined by

$$\begin{cases} \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{4}x^2 + \frac{1}{4}\lambda x^4\right)\Phi(x) = E(\lambda)\Phi(x), \\ \\ \lim_{x\to\pm\infty}\Phi(x) = 0. \end{cases} \tag{3.1}$$

This is the coordinate representation of the Hamiltonian, as is rigorously shown in Appendix A of Bender and Wu's original paper [2]. We will not obtain these equations, because we would have to introduce some field theory concepts, and we would be going too far away from our main goal.

Now we reproduce the first terms in the perturbation series that are given in [2], following their steps. To do so, let us write the perturbation series for the ground-state energy as

$$E_0(\lambda) = \frac{1}{2} + \sum_{n=1}^{\infty} A_n\lambda^n. \tag{3.2}$$

Our objective is to calculate the coefficients A_n. Let us write

$$\Phi(x) = \sum_{n=0}^{\infty} \lambda^n e^{-\frac{x^2}{4}} B_n(x),$$

where $B_{-1} = 0$, $B_0 = 1$ and $B_n(x)$ are the polynomials to be determined. If we plug this in (3.1), we obtain

$$\sum_{n=0}^{\infty} \lambda^n \left(x B_n'(x) - B_n''(x) + \frac{1}{4} x^4 B_{n-1}(x) \right) = \left(\sum_{n=1}^{\infty} A_n \lambda^n \right) \left(\sum_{m=0}^{\infty} \lambda^m B_m(x) \right). \tag{3.3}$$

We have written a small script with Mathematica that solves this equation. Since we use Mathematica for some calculations along this thesis, we include the script for this one, as it is the most relevant. It is presented in Appendix A. Let us, however, compute here A_1 and A_2, as it will illustrate what Mathematica does.

First we will compute A_1. Our initial equation is the one that corresponds to the terms with λ^1. So, it is

$$x B_1'(x) - B_1''(x) + \frac{1}{4} x^4 = A_1.$$

Writing the $B_i(x)$ as the following sum

$$B_i(y) = \sum_{j=1}^{2i} y^{2j} B_{i,j} (-1)^i$$

where $x = \sqrt{2} y$, and computing the derivatives of $B_1(x)$ yields

$$-4y^2 B_{1,1} - 16 y^4 B_{1,2} + 2 B_{1,1} + 24 y^2 B_{1,2} + y^4 = A_1.$$

Matching the terms with the same power of y, we obtain $B_{1,2} = \frac{1}{16}$, $B_{1,1} = \frac{3}{8}$ and $A_1 = 2B_{1,1} = \frac{3}{4}$, in accordance with our initial calculation with the Rayleigh-Schrödinger theory, (1.6). Analogously, for the λ^2 terms the equation we get is

$$4y^2 B_{2,1} + 16 y^4 B_{2,2} + 48 y^6 B_{2,3} + 128 y^8 B_{2,4} - 2 B_{2,1} - 24 y^2 B_{2,2} - 120 y^4 B_{2,3} -$$
$$-448 y^6 B_{2,4} - 2y^6 B_{1,1} - 4 y^8 B_{1,2} = -\frac{9}{16} y^2 - \frac{3}{16} y^4$$

which leads to $A_2 = -2B_{2,1} = -\frac{21}{8}$, so we also recover (1.7). Now we realise that $A_n = 2(-1)^{n+1} B_{n,1}$, which in fact can be easily seen by just looking at the equations.

The asymptotic behaviour of the series was guessed correctly in [2], and it is

$$A_n \sim (-1)^{n+1} \left(\frac{6}{\pi^3} \right)^{\frac{1}{2}} \Gamma\left(n + \tfrac{1}{2}\right) 3^n =: \mu(n). \tag{3.4}$$

We present in figure 3.1 the relative difference between the coefficients A_n computed with Mathematica (up to $n = 100$) and the asymptotic behaviour of the series. As expected, the difference goes to 0 for large values of n.

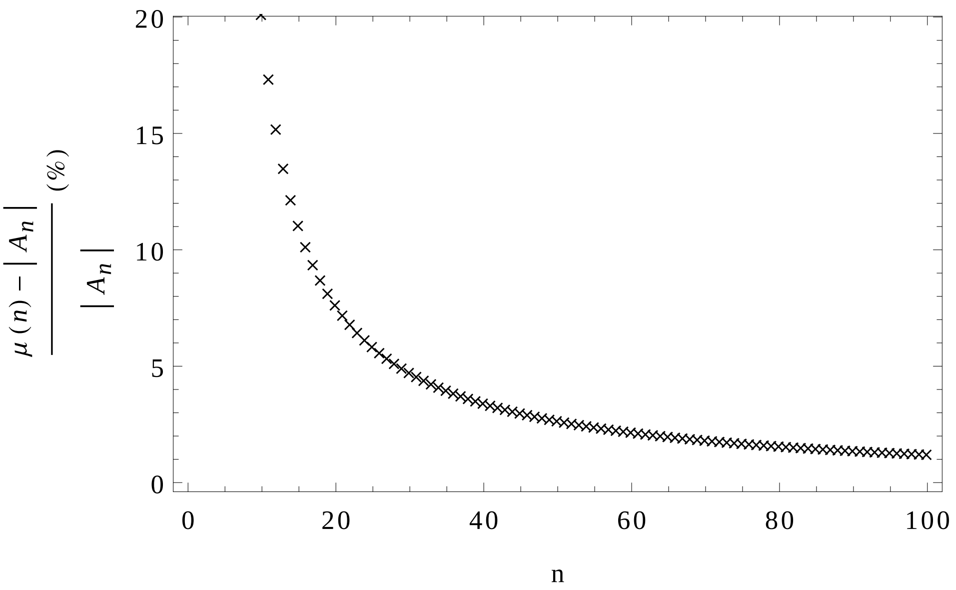

Figure 3.1: Difference between Rayleigh-Schrödinger coefficients and the asymptotic behaviour (3.4).

Our goal now is to obtain the asymptotic behaviour (3.4) analytically. This is proved rigorously in [13]. However, the proof is beyond the reach of this thesis, as it uses a lot of results demonstrated in [14] and this alone could be the subject of another thesis. What we will do instead is obtain the result using semiclassical methods. Before we do that though, we will prove from the asymptotic behaviour that the power series of the ground state energy is 1-Gevrey. The computation of the result is analogous to the one we did on the previous chapter, and so we will skip the details. Remembering that

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi},$$

and using Stirling's formula (2.7), it is straightforward to see that

$$|A_n| \leq \sqrt{\frac{3}{2\pi^2}} 3^n n!,$$

i.e. that the power series of the ground state energy of the anharmonic oscillator is 1-Gevrey with constants $C = \sqrt{3/2\pi^2}$ and $K = 3$. A comment is in order here. This behaviour is all we will be able to give about the series. However, by Borel-Ritt theorem 2.5.7 we know that there exists an analytic function function f such that the power series is its asymptotic expansion. Therefore we can relate the series to an analytic function, giving physical meaning to it.

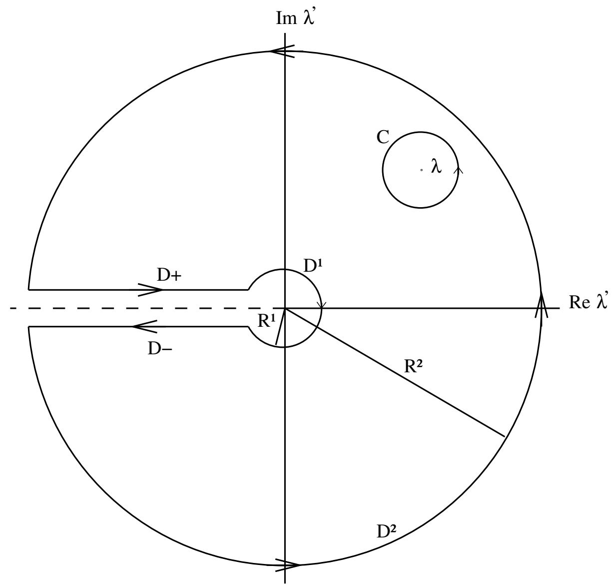

Figure 3.2: Change of the contour of integration on the complex λ' plane to obtain the dispersion relation (3.5).

More results can be obtained using Borel summation. As we have seen in the previous chapter, we can relate the series to an analytic function, but we can not talk about uniqueness. However, as Simon et al. prove in [15], in this case the Borel summation method sums the power series of the ground state energy (the coefficients we compute with Mathematica in appendix A) to the exact eigenvalue. We will not reproduce this result here, though.

3.2 Semiclassical Methods

Our objective now is to evaluate the large order behaviour of the perturbation series. To do so, first we will obtain $\mathrm{Im}[E(\lambda)]$ by semiclassical methods, following once again [4]. Here we are going to use instanton solutions, alternative to the WKB method of quantum mechanics. The calculation of this result using the WKB approximation can be found in [16]. Once we have $\mathrm{Im}[E(\lambda)]$, we will use Cauchy's formula to reproduce the perturbative expansion. Let us see how, following Bender and Wu's argument [16].

We will have an analytic function $E(\lambda)$ on the cut plane. By Cauchy's theorem, we write

$$E(\lambda) = \frac{1}{2\pi i} \oint_C \mathrm{d}\lambda' \frac{E(\lambda')}{\lambda' - \lambda},$$

where C is a circle on the complex λ' plane. We can change the contour C to $D = D_+ + D_- + D^1 + D^2$, as shown in figure 3.2. If we let $R_1 \to 0$,

$R_2 \to \infty$, the contributions from D_1 and D_2 vanish. Since

$$\lim_{\varepsilon \to 0}[E(\lambda' + i\varepsilon) - E(\lambda' - i\varepsilon)] = 2i\mathrm{Im}[E(\lambda')],$$

the integral is reduced to

$$E(\lambda) = \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\lambda' \frac{\mathrm{Im}[E(\lambda')]}{\lambda' - \lambda}.$$

Now, if we expand $E(\lambda)$ in a power series of λ,

$$E(\lambda) = \sum_{n \geq 0} A_n \lambda^n$$

and recall that

$$\frac{1}{\lambda' - \lambda} = \sum_{n \geq 0} \frac{\lambda^n}{(\lambda')^{n+1}},$$

we obtain, comparing both expressions,

$$A_n = \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\lambda' \frac{\mathrm{Im}[E(\lambda')]}{(\lambda')^{n+1}}. \tag{3.5}$$

So, our goal now is to obtain $\mathrm{Im}[E(\lambda)]$ and use (3.5) to reproduce the asymptotic behaviour (3.4).

We work with the Hamiltonian

$$H = -\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 + \frac{1}{2}q^2 + \lambda q^4. \tag{3.6}$$

We want to compute the ground state energy, $E_0(\lambda)$. To do so, we will use the path integral representation of the partition function,

$$\sum_{n \geq 0} e^{-\beta E_n} = \mathrm{Tr} e^{-\beta H} = \int_{q(-\beta/2) = q(\beta/2)} [\mathrm{d}q(t)] e^{-S(q(t))}, \tag{3.7}$$

where $S(q)$ is the Euclidean action,

$$S(q) = \int_{-\beta/2}^{\beta/2} \mathrm{d}t \left(\frac{1}{2}\dot{q}(t)^2 + \frac{1}{2}q(t)^2 + \lambda q(t)^4\right).$$

As we said in the introduction, as the temperature becomes lower the term corresponding to the ground state becomes more important in the series. Therefore, we obtain the ground state energy from the large β limit,

$$E_0(\lambda) = \lim_{\beta \to +\infty} -\frac{1}{\beta} \ln \mathrm{Tr} e^{-\beta H}.$$

Notice that we impose a periodic motion, and so the path integral is over periodic trajectories.

Now, analogous to section 2.2.2, we have to calculate the discontinuity on the cut to obtain the imaginary part of the ground state energy. As before, we have that (3.7) is analytic in the half plane $\mathrm{Re}[\lambda] > 0$, and in fact, using analytic continuation we have analyticity on the cut plane, working now in the $q(t)$ space. This is not surprising, as for $\lambda \in \mathbb{R}^-$ the Hamiltonian is unbounded from below, so we expect a singularity at λ=0. Thus we will now essentially follow the previous chapter, but as the path integral is more complicated, the procedure will be trickier.

We rotate the contour of integration in the $q(t)$ space, $q(t) \to q(t)e^{-i\theta}$, where θ is time independent, in such a way that $\mathrm{Re}[\dot{q}(t)]$ remains positive, because we want paths regular enough, that is, those for which in the $\varepsilon \to 0$ limit $(q(t+\varepsilon) - q(t))^2/\varepsilon$ remains finite. This way we ensure the existence of the continuum limit of the discretised path integral, as [4] argues. Therefore, we need $\mathrm{Re}[\lambda q^4(t)]$, $\mathrm{Re}[\dot{q}^2(t)] > 0$. For λ negative,

$$\mathrm{Re}[\dot{q}^2(t)] > 0 \iff \cos(2\theta_q) > 0 \iff -\frac{\pi}{4} < \theta < \frac{\pi}{4},$$

and, taking $\theta_\lambda \in (\pi/2, \pi)$,

$$\mathrm{Re}(\lambda q^4) > 0 \iff \cos(\theta_\lambda + 4\theta_q) > 0 \iff \frac{\pi}{8} < \theta < \frac{\pi}{4},$$

where $\theta = -\arg q(t)$, and the same if $-\pi < \theta_\lambda < -\pi/2$. For λ small, the two path integrals are dominated by the saddle point at the origin. However, like in the previous chapter, when we compute the difference between the two integrals, the contribution cancels. So, as before, we have to find non-trivial saddle points from the Euclidean classical action. So, we have to solve

$$-\ddot{q}(t) + q(t) + 4\lambda q^3(t) = 0 \quad (\lambda < 0) \tag{3.8}$$
$$q(-\beta/2) = q(\beta/2).$$

A solution of this equation is

$$q^2(t) = \frac{-1}{4\lambda}.$$

However, its contribution to the integral is of order $e^{\beta/16\lambda}$ and so is negligible in the large β limit. We need solutions with an action that remains finite for $\beta \to +\infty$. These solutions are called instantons.

3.2.1 Instantons

The solutions we are seeking correspond to a periodic motion in real time in the potential $-V(q) = -q^2/2 - \lambda q^4$. So, they will exist and be oscillations around the minima of the potential, $q = \pm\sqrt{-1/(4\lambda)}$. We can find the period of these solutions; integrating equation (3.8),

$$\frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2 - \lambda q^4 = \varepsilon, \qquad \varepsilon < 0.$$

Since it is a periodic motion, as we just said, let q_0, q_1 be the two return points (that is, points with $q > 0$ and where $\dot{q}$ vanishes). We have

$$\dot{q} = \sqrt{q^2 + 2\lambda q^4 + 2\varepsilon} \qquad \Longrightarrow \qquad \int_{t_0}^{t_1} \mathrm{d}t = \int_{q_0}^{q_1} \mathrm{d}q \frac{1}{\sqrt{q^2 + 2\lambda q^4 + 2\varepsilon}}.$$

Since it is a periodic motion, the integral over t is half a period. So, we obtain

$$\beta = 2 \int_{q_0}^{q_1} \mathrm{d}q \frac{1}{\sqrt{q^2 + 2\lambda q^4 + 2\varepsilon}}.$$

In the infinite β limit, $\varepsilon, q_0 \to 0$. Then, with a little help from Mathematica to skip long calculations, we can check that the classical solution becomes

$$q_c(t) = \pm\sqrt{-\frac{1}{2\lambda}} \frac{1}{\cosh(t - t_0)}$$

and the corresponding classical action is

$$S(q_c) = -\frac{1}{3\lambda} + O(e^{-\beta}/\lambda), \tag{3.9}$$

which now remains finite in the limit. Since the classical action is invariant under time translations, for β finite the classical solution has a free parameter t_0, $0 \leq t_0 < \beta$. So, we now have two one-parameter families of saddle points. We have to evaluate their contribution at leading order. To do so, we will use the Gaussian approximation.

We expand the action around the saddle point we just obtained. We set $q(t) = q_c(t) + r(t)$. Then, $\dot{q} = \dot{q}_c + \dot{r}$. If we plug this in the action, take into account (3.9), expand in powers of r staying in order 2 and take the $\lambda \to 0^-$, $\beta \to \infty$ limit, we obtain, writing directly the expression of (3.7),

$$\mathrm{Im}[\mathrm{Tr} e^{-\beta H}] = \frac{1}{i} e^{\frac{1}{3\lambda}} \int [\mathrm{d}r(t)] e^{-\frac{1}{2}\int \mathrm{d}t (\dot{r}^2 + r^2 + 12 q_c^2 r^2 \lambda)}.$$

We can express it in terms of an operator M, which is just the second functional derivative of the action,

$$M(t_1, t_2) = \frac{\delta^2 S}{\delta q(t_1) \delta q(t_2)}\Big|_{q=q_c} = \left[-\left(\frac{\mathrm{d}}{\mathrm{d}t_1} \right)^2 + 1 + 12\lambda q_c^2(t_1) \right] \delta(t_1 - t_2).$$

Then, the integrals reads

$$\mathrm{Im}[\mathrm{Tr} e^{-\beta H}] = \frac{1}{i} e^{\frac{1}{3\lambda}} \int [\mathrm{d}r(t)] e^{-\frac{1}{2}\int \mathrm{d}t_1 \mathrm{d}t_2 r(t_1) M(t_1, t_2) r(t_2)}.$$

Now, differentiating equation (3.8) with respect to t we see that, since $\dot{q}_c(t)$ is square integrable, it is an eigenvector of M with eigenvalue 0, $M\dot{q}_c = 0$. This is not surprising, because the action S does not change under infinitesimal variations of $q(t)$, which correspond to a variation of t_0, i.e. proportional to $\dot{q}_c$ since it is a saddle point.

We face a problem here, though. Since we have the eigenvalue 0, the Gaussian integration leads to a solution which is infinite, because it is proportional to $(\det M)^{-1/2}$. To solve this, we will use the method of collective coordinates.

3.2.2 Collective coordinates

We need a collective coordinate, that is, an integration variable associated with time translation. The point is to use a set of integration variables in which the time parameter appears explicitly, so we can integrate exactly over the parameter that describes the saddle points (the time translation).

Let t_0 be the time collective coordinate. We will use the following identity,

$$1 = \frac{1}{\sqrt{2\pi\xi}} \int \mathrm{d}t_0 \left[\int \mathrm{d}t \dot{q}_c(t) \dot{q}(t+t_0) \right] e^{-\frac{1}{2\xi}\left[\int \mathrm{d}t \dot{q}_c(t)(q(t+t_0)-q_c(t))\right]^2}. \quad (3.10)$$

It can be verified using the change of variables $t_0 \to \tau$, with

$$\tau = \int \mathrm{d}t \dot{q}_c(t)(q(t+t_0) - q_c(t)).$$

If we put identity (3.10) to the path integral (3.7), the new action is

$$S(q) + \frac{1}{2\xi} \left[\int \mathrm{d}t \dot{q}_c(t)(q(t+t_0) - q_c(t)) \right]^2.$$

We notice that it no longer is translation invariant. Now, the saddle point equation will be

$$\frac{\partial}{\partial q(t)} \left(S(q) + \frac{1}{2\xi} \left[\int \mathrm{d}t \dot{q}_c(t)(q(t+t_0) - q_c(t)) \right]^2 \right) =$$

$$= \frac{\partial S}{\partial q(t)} + \frac{1}{\xi} \dot{q}_c(t-t_0) \int \mathrm{d}t' \dot{q}_c(t'-t_0)(q(t') - q_c(t'-t_0)) = 0.$$

For $q(t) = q_c(t-t_0)$ the equation is clearly satisfied. Now, instead of the operator M, the determinant generated by Gaussian integration around the saddle point will be

$$\det \left(\frac{\partial^2 S}{\partial q_c(t_1) \partial q_c(t_2)} + \frac{1}{\xi} \dot{q}_c(t_1 - t_0) \dot{q}_c(t_2 - t_0) \right),$$

which we obtain from the new action. As the new term is constant, we see that this new operator has the same eigenvalues as the previous one, except for the eigenvalue that corresponds to the eigenvector $\dot{q}_c$, which now is $\|\dot{q}_c\|^2/\xi$ instead of 0, like we needed.

We have to normalise the path integral now. To do so, we compare it to the partition function $Z(\beta)$ of the harmonic oscillator, that is, to its value at $\lambda = 0$, which we know that reduces to $e^{-\beta/2}$ in the large β limit. Let M_0 be the operator M for $\lambda = 0$,

$$M_0(t_1, t_2) = \left[-\left(\frac{\mathrm{d}}{\mathrm{d}t_1} \right)^2 + 1 \right] \delta(t_1 - t_2).$$

As is shown in [4], what can be calculated is $\det(M+\varepsilon)\det(M_0+\varepsilon)^{-1}$, where ε is an arbitrary constant. Since for $\varepsilon \to 0$ the last expression vanishes like ε, let

$$\det{}' M M_0^{-1} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \det(M + \varepsilon)(M_0 + \varepsilon)^{-1}.$$

As we have just seen, if we denote by $|1\rangle$ the unitary eigenvector proportional to $\dot{q}_c$ and define $\mu := \|\dot{q}_c\|^2/\xi$, what we have to calculate is

$$\det(M + \mu\, |1\rangle\langle 1|) M_0^{-1} = \lim_{\varepsilon \to 0} \det(M + \varepsilon + \mu\, |1\rangle\langle 1|)(M_0 + \varepsilon)^{-1}.$$

We can do it using a simple result on determinants, which we present as a lemma.

Lemma 3.2.1 *Let A be a square, invertible matrix, and let $\vec{u}$, $\vec{v}$ be column vectors. Then,*

$$\det(A + \vec{u}\vec{v}^T) = (1 + \vec{v}^T A^{-1} \vec{u}) \det(A).$$

Proof If $A = Id$, we obtain the result taking the determinant on both sides of the following equality,

$$\begin{pmatrix} I & 0 \\ \vec{v}^T & 1 \end{pmatrix} \cdot \begin{pmatrix} I + \vec{u}\vec{v}^T & \vec{u} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ -\vec{v}^T & 1 \end{pmatrix} = \begin{pmatrix} I & \vec{u} \\ 0 & 1 + \vec{v}^T \vec{u} \end{pmatrix}$$

Now the result clearly holds for any invertible matrix A, because

$$\det(A + \vec{u}\vec{v}^T) = \det(A)\det(I + A^{-1}\vec{u}\vec{v}^T) = \det(A)(1 + \vec{v}^T A^{-1} \vec{u}).$$

$\square$

Then, since

$$(M + \varepsilon)^{-1} = \varepsilon^{-1}(M\varepsilon^{-1} + 1)^{-1} = \varepsilon^{-1}(1 - M\varepsilon^{-1} + M^2\varepsilon^{-2} + ...),$$

$M\,|1\rangle = 0$, as we said before, and $|1\rangle$ is unitary, we obtain

$$\det(M + \varepsilon + \mu\, |1\rangle\langle 1|)(M_0 + \varepsilon)^{-1} = \det(M + \varepsilon)(M_0 + \varepsilon)^{-1} \left(1 + \frac{\mu}{\varepsilon} \right).$$

And thus, letting $\varepsilon \to 0$,

$$\det(M + \mu \left|1\right\rangle \left\langle 1\right|)M_0^{-1} = \det{}' M M_0^{-1} \frac{\|\dot{q}_c\|^2}{\xi}.$$

Now we notice that t_0 can be eliminated from the determinant, because all functions depend only on $t - t_0$. Then, if we go back to the integral (3.10), and we replace $q(t + t_0)$ by $q_c(t)$ (we just keep the leading order), we see that the integral does not depend on t_0, and so its contribution is

$$\frac{1}{\sqrt{2\pi\xi}} \beta \|\dot{q}_c\|^2.$$

If we now plug all expressions to the partition function, recall the expression for the partition function of the harmonic oscillator (which is $e^{-\beta/2}$ in the large β limit), take into account that we have two families of saddle points, that we have to divide by $2i$ to obtain the imaginary part, and we consider the $\beta \to +\infty$ limit, we obtain

$$\mathrm{Im}[\mathrm{Tr} e^{-\beta H}] \sim \frac{1}{i} \frac{\beta}{\sqrt{2\pi}} e^{-\beta/2} \|\dot{q}_c\| (\det{}' M M_0^{-1})^{-1/2} e^{1/3\lambda} \tag{3.11}$$

Now we have to calculate the determinant. However, this is no easy task. Jean Zinn-Justin [4] shows a method to obtain the result of this kind of determinants, but we will not reproduce it here. We just give the result; it is

$$\det{}' M M_0^{-1} = -\frac{1}{12}.$$

We have to take the square root of this expression, and so there is an ambiguity of sign. It can only be resolved by the analytic continuation from λ positive to λ negative, as [4] argues.

If we look at equation (3.11), only $\|\dot{q}_c\|$ remains to be calculated. Once again, with a little help from Mathematica to compute the integral, we obtain

$$\|\dot{q}_c\| = \left(\int \dot{q}_c^2 \mathrm{d}t\right)^{\frac{1}{2}} = \sqrt{\frac{-1}{3\lambda}}.$$

Writing everything together,

$$\mathrm{Im}[\mathrm{Tr} e^{-\beta H}] \sim -\frac{2\beta}{\sqrt{2\pi}} \frac{e^{-\beta/2}}{\sqrt{-\lambda}} e^{1/3\lambda} (1 + O(\lambda)). \tag{3.12}$$

Now, for $\lambda \to 0^-$, $\beta \to +\infty$ we can write

$$\mathrm{Im}[\mathrm{Tr} e^{-\beta H}] \sim \mathrm{Im}[e^{-\beta E_0(\lambda)}] = \mathrm{Im}[e^{-\beta(\mathrm{Re}[E_0(\lambda)] + i\mathrm{Im}[E_0(\lambda)])}].$$

Since the imaginary part of E_0 is exponentially small in the λ limit,

$$e^{-\beta i \mathrm{Im}[E_0(\lambda)]} = \cos(\beta \mathrm{Im}[E_0(\lambda)]) - i \sin(\beta \mathrm{Im}[E_0(\lambda)]) \approx 1 - i\beta \mathrm{Im}[E_0(\lambda)],$$

and so

$$\mathrm{Im}[\mathrm{Tr}e^{-\beta H}] \sim -\beta \mathrm{Im}[E_0(\lambda)]e^{-\beta \mathrm{Re}[E_0(\lambda)]} \sim -\beta e^{-\beta/2}\mathrm{Im}[E_0(\lambda)].$$

Therefore, if we plug this expression on the lhs of (3.12) we obtain the desired result,

$$\mathrm{Im}[E_0(\lambda)] = \frac{2}{\sqrt{2\pi}} \frac{e^{\frac{1}{3\lambda}}}{\sqrt{-\lambda}}(1 + O(\lambda)). \tag{3.13}$$

To reproduce equation (3.4), the asymptotic behaviour given by the perturbative method, we just have to put the result we just obtained in the dispersion relation, equation (3.5), and compute the integral, as we said at the beginning of the section:

$$E_n = \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\lambda' \frac{\mathrm{Im}[E(\lambda')]}{(\lambda')^{n+1}} = \frac{1}{\pi}\int_{-\infty}^{0} \mathrm{d}\lambda' \frac{2}{\sqrt{2\pi}} \frac{e^{\frac{1}{3\lambda'}}}{\sqrt{-\lambda'}} \frac{1}{(\lambda')^{n+1}} =$$

$$= (-1)^{n+1}\left(\frac{6}{\pi^3}\right)^{\frac{1}{2}} \Gamma\left(n + \tfrac{1}{2}\right) 3^n. \tag{3.14}$$

The evaluation of this last integral is straightforward, recalling the definition of the gamma function.

Chapter 4

Conclusions

Many series that appear in quantum field theory are not convergent. However, they are asymptotic series, and so they grow in a controlled way which we can describe and obtain analytically. We have studied their behaviour, and we have seen that we can understand them as an asymptotic expansion of an analytic function. In this direction we have presented the Borel summation method. Using adequate transformations, we can compute the sum and obtain a finite result which, as we have mentioned, converges to the exact result.

In the process we have learned a useful method to approximate integrals on the complex plane: the steepest descent method. From the idea that saddle points dominate the behaviour of the integral, we have been able to obtain a simple formula.

In chapter 3 we have studied the anharmonic oscillator, understanding it as a field theory in a one-dimensional space time. We have seen that classical solutions are related to very large order in perturbation theory: if we can obtain the imaginary part of the energy using semiclassical methods, Cauchy's theorem allows us to obtain the coefficients.

Also, path integral formulation of quantum mechanics has allowed us to use semiclassical methods which can easily be generalised to field theory, such as the instanton solutions. We have also encountered typical procedures, like the collective coordinates method.

There are many ways to extend this work. We have just given a brief description of the partition function. We have skipped some calculations, and we have introduced the Laplace and Borel transforms, but we have not studied them in detail. In addition, we have presented just one summation method, and we have not commented its limitations.

It would also have been interesting to obtain the asymptotic behaviour for the ground state energy of the anharmonic oscillator using perturbation theory, as well as study the Stokes phenomenon or use Feynman diagrams for some computations.

However, I hope to study many of these things in my near future, as they will appear in some of next year's lectures.

References

[1] J.J. Sakurai, “Modern Quantum Mechanics,” Addison-Wesley. 1994.

[2] C. M. Bender and T. T. Wu, “Anharmonic oscillator,” Phys. Rev. **184**, 1231 (1969).

[3] F.J. Dyson, “Divergence of Perturbation Theory in Quantum Electrodynamics,” Phys. Rev **85**, 631 (1952).

[4] Jean Zinn-Justin, “Quantum Field Theory and Critical Phenomena,” Clarendon Press · Oxford. 2002.

[5] R.P. Feynman, A.R. Hibbs, “Quantum Mechanics and Path Integrals,” McGraw-Hill. New York, 1965.

[6] R. Wong, “Asymptotic Approximation of Integrals,” Society for Industrial and Applied Mathematics. 2001.

[7] Yukio Matsumoto, “An Introduction to Morse Theory,” American Mathematical Society. 2002.

[8] Carl M. Bender and Steven A. Orszag, “Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory,” Springer. New York. 1999.

[9] M. Mariño, “Lectures on non-perturbative effects in large N gauge theories, matrix models and strings,” arXiv:1206.6272 [hep-th].

[10] I. Baldomá, A. Haro, “One dimensional invariant manifolds of Gevrey type in real-analytic maps,” Discrete Contin. Dyn. Syst. Ser. B, **10** (2008), 295-322.

[11] W. Balser, “From divergent power series to analytic functions,” **1582**, Lecture Notes in Mathematics, Springer-Verlag, Berlin, (1994), Theory and application of multisummable power series.

[12] G. N. Watson, “The transformation of an asymptotic series into a convergent series of inverse factorials,” Rendiconti del Circolo Matematico di Palermo **34** (1912), 41-88.

[13] B. Simon and A. Dicke, “Coupling constant analyticity for the anharmonic oscillator,” Annals Phys. **58**, 76 (1970).

[14] Tosio Kato, “Perturbation Theory for Linear Operators,” Springer. 1995.

[15] S. Graffi, V. Grecchi and B. Simon, “Borel Summability: Application To The Anharmonic Oscillator,” Phys. Lett. B **32**, 631 (1970).

[16] C. M. Bender and T. T. Wu, “Anharmonic oscillator. 2: A Study of perturbation theory in large order,” Phys. Rev. D **7**, 1620 (1973).

Appendix A

Mathematica script

Here we present the script for Mathematica that calculates the A_n terms of chapter 3, from equation (3.3). We have included the result for the first 10 coefficients, which are in complete accordance with the ones presented in Bender and Wu's paper [2].

In[1]:=$\Phi(\mathrm{n},\mathrm{x}_{)}:=\sum_{n=0}^{\infty}\exp\left(-\frac{x^2}{4}\right)\lambda^n B(n,x)$

In[2]:=$\Phi\mathrm{n}(\mathrm{n\_},\mathrm{x\_}):=\exp\left(-\frac{x^2}{4}\right)\lambda^n B(n,x)$

In[3]:=$\Phi\mathrm{nm1}(\mathrm{n\_},\mathrm{x\_}):=\exp\left(-\frac{x^2}{4}\right)\lambda^n B(n-1,x)$

In[4]:=$\mathrm{lhs}(\mathrm{n\_}):=\mathrm{Simplify}\left[\frac{\frac{1}{4}x^4\Phi\mathrm{nm1}(n,x)+\frac{1}{4}x^2\Phi\mathrm{n}(n,x)-\frac{\partial}{\partial x}\frac{\partial\Phi\mathrm{n}(n,x)}{\partial x}}{\exp\left(-\frac{x^2}{4}\right)}\right]$

In[5]:=$\mathrm{rhs}(\mathrm{n\_}):=\lambda^n\left(\sum_{p=0}^{n-1}A(n-p)B(p,x)\right)+\frac{1}{2}\lambda^n B(n,x)$

In[6]:=$B(\mathrm{n\_},\mathrm{x\_}):=\sum_{j=1}^{2n}(-1)^n x^{2j}\mathrm{B2i}(n,j)/;n>0$

In[7]:=$B(\mathrm{n\_},\mathrm{x\_}):=1/;-0.5<n<0.5$

In[8]:=$B(\mathrm{n\_},\mathrm{x\_}):=0/;n<0$

In[9]:=$\lambda\mathrm{max}=10;$

In[10]:=$\mathrm{sols}=\{\};\mathrm{solsA}=\{\};$

$\mathrm{Do}\Big[\{\mathrm{temp}=\mathrm{Solve}\big[\mathrm{Coefficient}[\mathrm{lhs}(\mathrm{o}\lambda),x,\mathrm{ox}]=$

$\mathrm{Coefficient}[\mathrm{rhs}(\mathrm{o}\lambda),x,\mathrm{ox}],\mathrm{If}\big[\mathrm{ox}>0,\mathrm{B2i}\left(\mathrm{o}\lambda,\frac{\mathrm{ox}}{2}\right),A(\mathrm{o}\lambda)\big]\big][[1,1]],$

$$\mathrm{If}[\mathrm{ox} > 0, \mathrm{AppendTo}[\mathrm{sols}, \mathrm{temp}/.\,\mathrm{sols}/.\,\mathrm{solsA}],$$

$$\mathrm{AppendTo}[\mathrm{solsA}, \mathrm{temp}/.\,\mathrm{sols}]]\}, \{\mathrm{o}\lambda, 1, \lambda\mathrm{max}\}, \{\mathrm{ox}, 4\mathrm{o}\lambda, 0, -2\}\Big]$$

In[11]:=solsA

Out[11]:= $\Big\{A(1) \to \frac{3}{4}, A(2) \to -\frac{21}{8}, A(3) \to \frac{333}{16}, A(4) \to -\frac{30885}{128},$

$$A(5) \to \frac{916731}{256}, A(6) \to -\frac{65518401}{1024}, A(7) \to \frac{2723294673}{2048},$$

$$A(8) \to -\frac{1030495099053}{32768}, A(9) \to \frac{54626982511455}{65536},$$

$$A(10) \to -\frac{6417007431590595}{262144}\Big\}$$