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NECESSARY AND SUFFICIENT CONDITIONS FOR THE GEOMETRICAL REGULARIZATION OF SINGULARITIES

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Abstract. We consider singularities of ordinary differential equations similar to the ones which appear in Celestial Mechanics. Using blow-up and time scaling the singular point is replaced by a manifold. In this manifold the flow is smooth, gradient-like and the critical points are (generically) hyperbolic. We look for conditions which assure regularizability of the singularity. Four conditions concerning the invariant manifolds of the critical points on the manifold, the variational equations along these manifolds and the eigenvalues of the linear part of the field at the critical points are necessary. It is proved that they are also sufficient. Several exemples are included.

§1. Introduction. Let $Mq'' = -\nabla V(q)$ be the equation of motion of a system of particles, where M is a positive diagonal matrix (the mass matrix), q the position vector belonging to R^n and V a real analytical function, homogeneous of degree -k, k>0, with an isolated singularity at the origin (V unbounded when q = 0)(see [3]). The symbol ' denotes derivative with respect to time.

We can write down the equations in hamiltonian form, $H(p,q) = \frac{1}{2}p^T M^{-1} p + V(q)$ being the hamiltonian,and then we have

$$q' = M^{-1}p \quad , \quad p' = -\nabla V(q).$$

In order to deal with the singularity, McGehee [7] introduced a blow up and a time scaling through

$$r = (q^T Mq)^{1/2} \quad , \quad s = r^{-1}q \quad , \quad \frac{dt}{d\tau} = r^{k/2+1} \quad .$$

Then s can be seen as a point in a manifold S diffeomorphic to the

S^{2n-1} sphere. The radial velocity $y = s^T p$ and the tangent velocity to S, $x = M^{-1}p - ys$, such that $(s,x) \in TS$, the tangent bundle of S are scaled to

$$u = r^{k/2}x \quad , \quad v = r^{k/2}y \ .$$

Then the corresponding equations are written as

$$\dot{r} = rv \quad , \quad \dot{v} = u^T M u + \frac{k}{2}v^2 + k\,V(s) \quad , \quad \dot{s} = u \quad ,$$
$$\dot{u} = (\frac{k}{2} - 1)vu - (u^T M u)s - k\,V(s)\,s - M^{-1}\nabla V(s) \quad , \qquad (*)$$

where $\dot{} = \frac{d}{d\tau}$.

The energy equation is expressed as $r^k h = \frac{1}{2}(u^T M u + v^2) + V(s)$. The change of variables is meaningful for all $r>0$. In order to have information about the flow near $r=0$ we realize that the flow in the variables (r,v,s,u,τ) extends in a smooth way to the boundary $r=0$. In [3] it is proved that if 0 is a regular value of $V(s)$ then the set $0 = \frac{1}{2}(u^T M u + v^2) + V(s)$ is a smooth manifold heretofore called the total collision manifold, C, or, in general, the singularity manifold. If we can describe the behavior of the flow for $r=0$ we get an approximate description for r small, i.e., for passage near the singularity, due to the smoothness of the vector field. However the physical time t does not change along C and the time needed to reach the singularity is finite despite the new time τ becomes infinite.

We recall some results about the flow on C. We refer to [3] for the proofs.

Let V_S be the potential restricted to the S-sphere. The equations on C are

$$\dot{v} = (1 - \frac{k}{2})u^T M u \quad , \quad \dot{s} = u \quad , \quad \dot{u} = (\frac{k}{2} - 1)vu - (u^T M u)s - \nabla V_S(s) \ .$$

The critical points of the flow on C are given by $u_0 = 0$, $\nabla V_S(s_0) = 0$ (i.e. s_0 is a central configuration of the system of particles, see [15]) $v_0 = \pm\sqrt{-2V_S(s_0)}$. Critical points appear by couples. One of the points, with $v_0 < 0$, corresponds to collision and the other, with $v_0 > 0$, to ejection.

Proposition 1. *If* $k<2$ $(k>2)$ *the flow is gradient-like with respect to the variable* v $(-v)$, *i.e.,* $\dot{v}>0$ $(\dot{v}<0)$ *except at the critical points.*

Let $B = D\nabla V_S(s_0)$ where D means the differential operator and s_0 is a critical point in C, and A the matrix of the linearized equations at the critical point. Let $\mu = (\frac{k}{2} - 1)v_0$. Then the eigenvalues of A are v_0 (related to a change in the r variable), kv_0 (related to a variation of the energy) and $\frac{1}{2}(\mu \pm \sqrt{\mu^2 - 4\lambda})$, where $\lambda \in$ Spec B . Therefore we have the following results.

Proposition 2. *If* $k \neq 2$ *and* V_S *is a Morse function all the critical points in* C *are hyperbolic.*

Proposition 3. *If the critical points in* C *are hyperbolic and they occur in finite number, the set of solutions leading to the singularity or arising from it is the union of a finite number of submanifolds.*

Let us suppose that the energy h is negative. Using scaling thanks to the homogeneity of the potential and the kinetic energy we can suppose $h = -1$. In the space of the variables (r,v,s,u) we have a special class of orbits, the homothetic orbits (see [15]),characterized by the fact that $s = s_0$ = constant. They joint an ejection point to the associated collision point. The equations for a homothetic orbit are

$$s = s_0 \quad , \quad u = 0 \quad , \quad \dot{r} = rv \quad , \quad \dot{v} = -k\,r^k \quad ,$$

i.e. (using $h = -1$)

$$r = \left(\frac{v_0^2}{2}\right)^{1/k}\left(\cosh\left(\frac{kv_0}{2}\tau\right)\right)^{-2/k} \quad , \quad v = -v_0\tanh\left(\frac{kv_0}{2}\tau\right) \quad ,$$

where we take $v_0>0$. These orbits can be seen as heteroclinic connections of critical points. See [1], [11] for more information about these orbits and the dimension of the invariant manifolds at the critical points.

Now we come to the definition of the type of regularization we are looking for. Let z be an initial condition such that the flow ϕ of a vector field through z reaches an isolated singularity p in a finite time t_s . We can suppose that in a neighborhood of $\{\phi_t(z),\ t \in [0,t_s)\}$ there are no more singularities. Let $t_f > t_s$. We consider a sequence of initial conditions (x_n) such that $x_n \to z$ when $n \to \infty$ and we obtain $\phi_{t_f} x_n$. If $\lim_{n\to\infty} \phi_{t_f} x_n$ exists and it is independent of the selected sequence (x_n) we say that the singularity p is regularizable in a geometric sense. Then we can take $\lim_{n\to\infty} \phi_t x_n$ as the natural continuation of $\phi_t z$ for $t > t_s$.

In this way we have continuity with respect to the initial conditions "after" the singularity. The definition of "geometrical" regularization given here is essentially equivalent to the one given by Easton [4].

Our objective is the study of conditions to be impossed on the flow on C in order to get regularization of the singularity.

§2.The Main Theorem. For the sake of defineteness we suppose the flow to be gradient-like with respect to v on C (see proposition 1), i.e., $k<2$. This includes the Newtonian case. If n is the essential number of degrees of freedom of our system (i.e., after all the reductions by symmetry) the dimension of C is $2n-2$.

It is possible that V_S has still some singularities and become unbounded on some submanifolds of C . That is the case if we have some subsingularities inside the main singularity. If these singularities are non-regularizable the same follows for the main singularity : Nearby initial conditions approaching the singularity emerge in a quite different way. Therefore let us suppose that all the subsingularities are regularizable, as it happens, for instance, in problems of triple collision [7,9,10,14]. Then we look at C as a compact manifold, and using again proposition 1 we have that the sets on C such that $V_S(s)$ is unbounded behave like an attractor if $v>0$ or a repellor if $v<0$. These sets need not to be forcely points. For instance, in the triple collision of the planar problem of three bodies, S is diffeomorphic to S^3 and the set of binary collisions is diffeomorphic to three disjoint copies of S^1 . However, after performing a quotient by S^1 (the classical elimination of the node [15]) the set of binary collisions reduces to three disjoint points [9,14] .

Therefore we can suppose that C is compact or can be compactified (see fig.2 of [10] for the isosceles problem and for other problems with $n=2$).

In order to make clear the main difficulties related to the possible regularization, from now on we shall restrict our exposition to the case $n=2$. Higher dimensional cases offer the same essential problems plus some technical details. A full version will appear elsewhere [13].

Let q_i^c, q_i^e, $i \in I$, finite, be the critical points of the flow on C or the repellor/attractor sets just defined. The symbol c (resp.e)stands for collision, $v_0<0$ (resp. ejection, $v_0>0$). Let $W^{u(s)}_{q_i}$ be the unstable (stable) manifolds of this points or sets.

The invariant unstable manifold on C of a collision critical point $W^u_{q_i^c}$ of dimension $d=1$ or 2, together with the stable variable along the r direction at q_i^c originate an invariant strip $\hat{W}^u_{q_i^c}$ of dimension $d+1$. It is not an invariant manifold in the sense of Hartman's theorem.

Let us suppose that an invariant manifold $W^u_{q_i^c}$ coincides with $W^s_{q_j^e}$ for some j . Then the stable directions transversal to $W^u_{q_i^c}$ near q_i^c are carried to the unstable ones near q_j^e . For any δ sufficiently small and any orbit $\gamma \subset W^u_{q_i^c}$ we consider the Poincaré map ψ_γ^δ defined in the following way : Let A be the last point in γ at a distance δ of q_i^c and B be the first point in γ at a distance δ of q_j^e (last and first in the sense in which the orbit is described). At A (resp. B) we take the linear space spanned by the local stable (resp. unstable) directions. Then ψ_γ^δ is the map obtained by flow transportation from the first space to the second one.

Finally, suppose that $W^u_{q_i^c}$ again coincides with $W^s_{q_j^e}$ for some j and both are 1-dimensional. We call a_i^c, b_i^c, c_i^c the eigenvalues at q_i^c, a_i^c being the one related to the r direction. Of course a_i^c, $b_i^c<0$ and $c_i^c>0$. In a similar way we have a_j^e, $b_j^e>0$, $c_j^e<0$.

Now we can state the main theorem.

<u>Theorem</u>. *The following conditions are necessary and sufficient in order to get a geometrical regularization of the singularity :*

1) For all i *there exists* j *such that* $W^u_{q_i^c} \equiv W^s_{q_j^e}$.

2) For such i *and* j $\hat{W}^u_{q_i^c} \equiv \hat{W}^s_{q_j^e}$.

3) *For any* $W^u_{q_i^c}$ *and any* $\gamma \subset W^u_{q_i^c}$ *and for* δ *sufficiently small the Poincaré map* ψ_γ^δ *is independent of* γ *in some sense (see §3).*

4) *For every couple* (i,j) *given by 1) such that* $W^u_{q_i^c}$ *is 1-dimensional, the relation* $b_j^e/a_j^e = b_i^c/a_i^c$ *holds.*

Before going to the proof of the theorem we point out some remarks and exemples.

Remark 1. In order to check 1) one must find saddle connections (mainly numerically). Both branches of the invariant one-dimensional manifold $W^u_{q_i^c}$ must coincide with the branches of $W^s_{q_j^e}$. In general these conditions are not satisfied (except for some degenerate problems).If the initial problem depends on parameters we can find perhaps values of the parameters for which 1) holds (as it happens in [9] for the collinear three body problem). For one parameter families we can find one of the connections but not simultaneously the other (see [10] for the isosceles problem) except if some symmetry appears (see [9] and also [12] for the rectangular quadruple collision in the trapezoidal problem). If the critical points are ordered according to the value of the s coordinate : $q_1 < q_2 < \dots < q_k$ then we must have $W^u_{q_i^c} \equiv W^s_{q_{k+1-i}^e}$ for $i = 1 \div k$ (see [10]). Another interesting case where the condition 1) is not satisfied preventing regularization is the anisotropic Kepler problem [2,5,6].

Remark 2. Condition 2) is always satisfied if 1) is and $h = 0$. Due to the form of the variational equations associated to (*) it is also true in a first order approximation. To check it we need some global computation that can be performed in the following way. The local behavior of $\hat{W}^u_{q_i^c}$ along the r direction at q_i^c is obtained from variational calculations (see [11] for the n-ple collision and variation along the homothetic orbit if $h = -1$). The integration of the local strip just obtained until it reaches $v = 0$ produces a curve. This curve must coincide with the one given by the same process performed backwards from the upper point q_j^e. For the collinear three body problem this computation is carried out in [8] showing that 2) does not hold. Then the assertion in [9] is restricted to $h = 0$ and to isoenergetic variations of the initial conditions. For this case 3) and 4) follow by symmetry.

Remark 3. The condition 3) can be checked using variational equations along γ. It can be expressed as the Cauchy principal value of some integrals along closed curves equal to zero (see [9]). Condition 4) is the easiest one to check because it needs only local computations. Essentially the regularization asks for the r entering direction going to the r leaving direction (and not only lineally) and stable/unstable directions exchanging his role from the lower to the upper connected saddles. Furthermore some quantitative relations are given by 3) & 4).

§3. Proof of the theorem. First we prove the necessity. Let γ_1 an orbit on C such that for $t \to -\infty$ goes to q_i^c (i.e. $\gamma_1 \subset W^u_{q_i^c}$). Under the hypothesis and due to the gradient-like character of the flow, γ_1 reaches some q_j^e (possibly an attractor) for $t \to +\infty$. If another $\gamma_2 \subset W^u_{q_i^c}$ tends to $q_{j'}^e$, for $t \to +\infty$ with $j \neq j'$ then nearby orbits approaching γ_1, γ_2 in the region r>0 leave the neighborhood of the singularity in a quite different way. Therefore $W^u_{q_i^c} \subset W^s_{q_j^e}$. Then the symmetry $(r,v,s,u,\tau) \to (r,-v,s,-u,-\tau)$ implies $W^u_{q_i^c} \supset W^s_{q_j^e}$. If there is some orbit $\gamma_3 \subset W^u_{q_i^c}$ not ending in q_i^e we loss regularization. Therefore condition 1) is necessary.

Now let us suppose that 1) is satisfied. We recall that $\hat{W}^u_{q_i^c}$ and $\hat{W}^s_{q_j^e}$ are tangent along $W^u_{q_i^c} \equiv W^s_{q_j^e}$. If they do not coincide we can choose a sequence (x_n) tending to some point in the homothetic orbit which ends at q_i^c such that :

a) All the points are on the same side with respect to $\hat{W}^u_{q_i^c}$.

b) The points with odd index are on one side with respect to $\hat{W}^s_{q_j^e}$ and the ones with even index are on the other side. The same is true with respect to $W^s_{q_i^c}$.

Then, under the flow, points with index of different parity escape from the neighborhood of q_j^e near different branches of $W^u_{q_j^e}$, preventing regularization.

If 2) is also true we have some kind of splitting of the flow in the flow along C and the one transversal to it.

Let x, y, z be local coordinates near q_i^c along the r direction and along the stable and unstable manifolds of the flow on C, respectively. Consider a sequence of initial conditions of the form $x = \alpha$, $y = \beta + f(n)$, $z = g(n)$, where f, g go to zero when n goes to ∞. We call $\bar{x}$, $\bar{y}$, $\bar{z}$ the coordinates near q_j^e along the r, unstable and stable directions, respectively at the point obtained by flow transportation. The dominant values of $\bar{y}$, $\bar{z}$ when $\bar{x} = \alpha$ can be obtained using linear approximation near q_i^c and q_j^e until z reaches the value δ and the Poincaré map ψ_γ^δ. By condition 2) and the fact that the tangent plane to C at A is carried to the tangent plane to C at B (see the definition of ψ_γ^δ), this map is of the form $(x,y) \to (M_\gamma x, N_\gamma y)$. An easy computation shows

$$\bar{x} = \alpha \ , \quad \bar{y} = N_\gamma(\beta + f(n))\delta^{-\frac{b}{c}+\frac{b'a}{a'c}} g(n)^{\frac{b}{c}-\frac{b'a}{a'c}} M_\gamma^{-\frac{b'}{a'}} \ , \quad \bar{z} = \delta^{1-\frac{ac'}{ca'}} g(n)^{\frac{ac'}{ca'}} M_\gamma^{\frac{c'}{a'}}$$

where we put a,b,c,a',b',c' equal to the absolute value of $a_i^c, b_i^c, c_i^c, a_j^e, b_j^e, c_j^e$, respectively.

In order to have a limit when $n \to \infty$ independent on the way in which f(n), g(n) go to zero we need $\frac{b}{c} \geq \frac{b'a}{a'c}$, i.e., $\frac{b}{a} \geq \frac{b'}{a'}$. This must be true for all i and when i runs over all the collision points, j does over all the ejection points. As the product of all the quotients over i and over j is the same because of the symmetry between each collision point and the ejection point with the same value of s, we obtain condition 4). Then the limit is $\bar{y} = N_\gamma \beta M_\gamma^{-b'/a'}$, $\bar{z} = 0$. Therefore we need $N_\gamma^{a'}/M_\gamma^{b'}$ independent of γ, which is the condition refered to in 3).

Now for the sufficiency we only need to remark that according to proposition 3 we reach the singularity for initial conditions of the form $x = \alpha$, $y = \beta$, $z = 0$ as stated before, whit respect to some of the points q_i^c. Conditions 1) to 4) and the same computation we did assure that we leave with $\bar{x} = \alpha$, $\bar{y} = N\beta M^{-b'/a'}$, $\bar{z} = 0$ as a natural continuation, independent on the way f,g go to zero when $n \to \infty$. We must check that the time is also independent on the initial conditions. But when $f,g \to 0$ the time is the time used in going from $x = \alpha$ to $x = 0$ and from $\bar{x} = 0$ to $\bar{x} = \alpha$ (t does not change on C), i.e., $\Delta t = \alpha^{3/2}(\frac{1}{a'} + \frac{1}{a})\frac{2}{3}$, independent on the initial conditions. Therefore we can choose initial conditions with $x = \alpha$ and take $t_f = t_0 + \Delta t$. This ends the proof of the Theorem.

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