Fractional Charge

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Abstract: We study fractional charge in the one-dimensional and the two-dimensional cases. Firstly, we analyse the concept of kink and then we see how fermions in a kink background can carry fractional charge. Secondly, we study the so-called vortices and, as before, we study the coupling with fermions and different models to obtain the fractional charge. At the end of each section we discuss experimental evidence of this phenomenon: polyacetylene chain and graphene.

I. INTRODUCTION

In this work, we study the appearance of fractional charge in low dimensional systems of fermions coupled to non-trivial backgrounds. This phenomenon requires a number of elements that we now list: First, the system must admit degenerate vacua. Then, quite generically, there will be classical solutions interpolating between these vacua. These classical solutions have finite energy and are called solitons. In the background of these solitons, fermions often have zero-energy modes, not present in the ordinary vacua. Finally, when the fermions are charged under a global charge, these novel zero-energy modes carry fractional charge.

We should point out that the mechanism to generate fractional charge is not unique. Another system that displays fractional fermion number is the fractional quantum Hall effect, but the origin of the fractional charge in that system is quite different.

The structure of this work is the following. In section II we present a field theory model in one spatial dimension, and we illustrate it with trans-polyacetylene. In section III we present a field theory model in two spatial dimensions, and we illustrate it with graphene. In section IV we present conclusions and an outlook.

II. ONE-DIMENSIONAL CASE

A. Kinks

In the one-dimensional case, as [1], we have the system with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{4}\lambda\Big(\frac{m^2}{\lambda} - \phi^2\Big)^2 = \frac{1}{2}(\partial_\mu\phi)^2 - U(\phi), \quad (1)$$

where ϕ is a scalar field and $[\lambda] = m^2$. It has $\mathbb{Z}_2$ symmetry: $\phi(x) \to -\phi(x)$. We have total energy from Hamiltonian

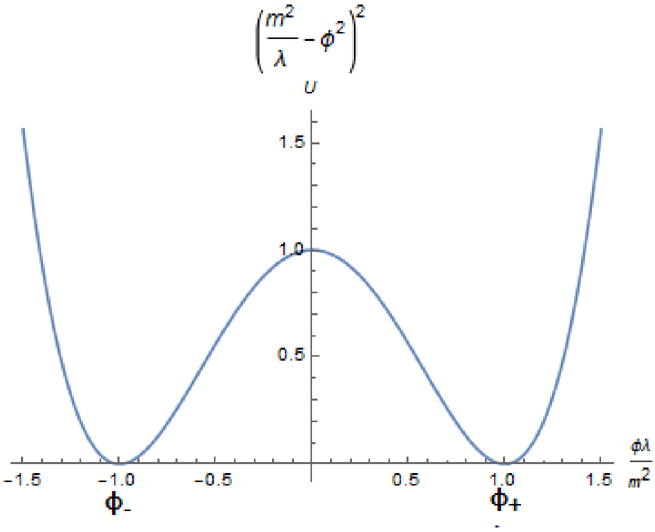

FIG. 1: Potential U(ϕ) with a $\mathbb{Z}_2$ symmetry

$$E[\phi] = \int_{-\infty}^{\infty} \Big(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi'^2 + \frac{1}{4}\lambda\Big(\frac{m^2}{\lambda} - \phi^2\Big)^2\Big)dx. \quad (2)$$

To find the equations of motion we use Euler-Lagrange equations and obtain

$$\partial^\mu\partial_\mu\phi - \lambda\Big(\frac{m^2}{\lambda} - \phi^2\Big)\phi = 0. \quad (3)$$

First, we look for static solutions with the minimum energy. The zero energy solutions are constant field configurations

$$\phi_\pm = \pm\frac{m}{\sqrt{\lambda}}. \quad (4)$$

They break the $\mathbb{Z}_2$ symmetry. Now, let's find other nontrivial static solutions of the equation of motion. These other nontrivial solutions are

$$\phi(x) = \pm\sqrt{\frac{m^2}{\lambda}}\tanh\left(\sqrt{\frac{m^2}{2}}(x - x_0)\right). \quad (5)$$

The positive solution is a soliton and will be called a kink

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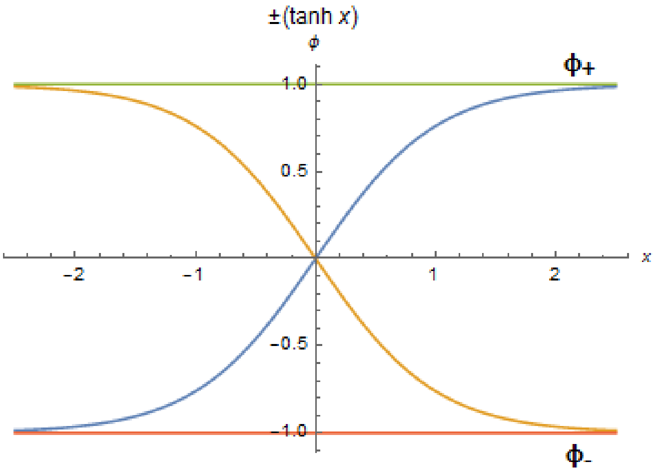

FIG. 2: Kink and antikink solutions that interpolate between the two vacua

and the negative one will be called antikink. We observe that the soliton interpolates between the constant vacua Eq. (4) and it breaks the $\mathbb{Z}_2$ degeneracy. Their total energy is

$$E[\phi] = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{6}$$

We observe that this total energy is higher than the energy of the ground state and is very large at weak coupling. Solitons can be associated with quantic states of extended particles. We have a vacuum sector and a soliton sector. These two are disconnected sectors. The lowest energy state in the kink sector isn't vacua, we call it quantum kink or $|sol\rangle$.

B. Coupling with fermions

Now we add the interaction of one-dimensional fermions with scalar fields, as in [3]. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) + i\bar{\psi}\gamma^\mu\partial_\mu\psi - G\bar{\psi}V(\phi)\psi. \tag{7}$$

In this representation $\gamma^1 = i\sigma^3$, $\gamma^0 = \sigma^1$. Notice that $[G] = L^{-1}$. We choose $V(\phi) = \phi$.

We take the boson mass to be much larger than the fermion mass. Therefore boson is the effective mass of the Dirac equation

$$i\gamma^\mu\partial_\mu\psi - \overbrace{GV(\phi_c)}^{m_{eff}}\psi = 0. \tag{8}$$

If $\psi_r(x) = \begin{pmatrix} u_r(x) \\ v_r(x) \end{pmatrix}$, then

$$\begin{cases} \Big(-\frac{d}{dx} + G\phi_c(x)\Big)v_r(x) = \varepsilon_r u_r(x) \\ \Big(\frac{d}{dx} + G\phi_c(x)\Big)u_r(x) = \varepsilon_r v_r(x). \end{cases} \tag{9}$$

We know that $\{\sigma_3, H\} = 0$, so $|\psi_{-E}\rangle = \sigma_3|\psi_E\rangle$. In other words, for each spinor with energy eigenvalue E there is another with energy eigenvalue $-E$.

Now, we study the zero energy solutions.

$$\begin{aligned} \tfrac{du}{dx} + G\phi_c(x)u &= 0 \\ \tfrac{dv}{dx} - G\phi_c(x)v &= 0. \end{aligned} \tag{10}$$

In general we have

$$\begin{aligned} u(x) &= C_1 e^{-G\int\phi_c(x')dx'} \\ v(x) &= C_2 e^{G\int\phi_c(x')dx'}. \end{aligned} \tag{11}$$

In the case that $\phi_c(x) = \phi_\pm$ we have a zero mode solution, but it isn't normalizable. Therefore, there isn't a fermion zero mode and a gap opens between the minimum energy E and $-E$.

On the other hand, if we have the case that $\phi_c(x) = \pm\sqrt{\frac{m^2}{\lambda}}\tanh\left(\sqrt{\frac{m^2}{2}}(x - x_0)\right)$

we obtain

$$\begin{aligned} u &= C_1\Big(\cosh\Big(\tfrac{m}{\sqrt{2}}(x - x_0)\Big)\Big)^{-G\sqrt{\frac{2}{\lambda}}} \\ v &= C_2\Big(\cosh\Big(\tfrac{m}{\sqrt{2}}(x - x_0)\Big)\Big)^{G\sqrt{\frac{2}{\lambda}}}. \end{aligned} \tag{12}$$

We observe that if one is normalizable, the other one is not. So we can't have the two of them at the same time. Therefore,

$$\psi = \begin{pmatrix} u \\ 0 \end{pmatrix}. \tag{13}$$

This is $|\widetilde{sol}\rangle$. So we have a zero energy solution and a mid-gap state appears. To further explore properties of this state, we promote the fields into operators, as in [1]

$$\Psi(x,t) = b_0 f_0(x) + \sum_{r\geq 1}\Big(b_r e^{-i\varepsilon_r t} f_r^{(+)}(x) + d_r^\dagger e^{i\varepsilon_r t} f_r^{(-)}(x)\Big). \tag{14}$$

We observe that there is only one mode b_0, since f_0 is self-conjugate by $f_0^c = \sigma_3 f_0 = f_0$. We define the associated charge

$$Q = \frac{1}{2}\int(\Psi^\dagger\Psi - \Psi\Psi^\dagger)dx. \tag{15}$$

It is known that

$$\langle f_r|f_s\rangle = \delta_{rs}, \qquad \Big\{b_r, b_{r'}^\dagger\Big\} = \Big\{d_r, d_{r'}^\dagger\Big\} = \delta_{rr'},$$

$$\Big\{b_0, b_0^\dagger\Big\} = 1, \qquad \{b_0, b_r\} = \{b_0, d_r\} = 0$$

and we arrive at the conclusion

$$Q = b_0^\dagger b_0 - \frac{1}{2} + \sum_r b_r^\dagger b_r - \sum_r d_r^\dagger d_r. \tag{16}$$

We apply the charge operator to the solitonic states and we find that they carry fractional charge

$$b_0 \left|sol\right\rangle = b_r \left|sol\right\rangle = d_r \left|sol\right\rangle = 0 \Rightarrow Q \left|sol\right\rangle = -\frac{1}{2} \left|sol\right\rangle \tag{17}$$

and

$$|\widetilde{sol}\rangle = b_0^{\dagger} \left|sol\right\rangle \Rightarrow Q|\widetilde{sol}\rangle = \frac{1}{2}|\widetilde{sol}\rangle. \tag{18}$$

We notice that these fractional charges are eigenvalues and not just expectation values.

C. Trans-polyacetylene

Experimental evidence for fermion number 1/2 to exist was found using a trans-polyacetylene molecule, we follow [4]. Remarkably, this material is a plastic conductor, and not an insulator. This unusual property is caused by solitonic states of the type just described. We have an one-dimensional carbon chain $(CH)_n$ where simple and double bonds alternate. We consider a system with electrons and bosons (phonons) and two minimum energy degenerated states. The state A has the bond configuration simple-double-simple and the state B has double-simple-double.

These states are identified as the vacua $\phi_{\pm}$. The soliton sector may be interpreted as the configuration with a defect. In other words, when there are two zones A and B in the chain, there is a defect in the point where the zone is changed. Where there should be a double bond, there is a simple bond. There are two Dirac points and we

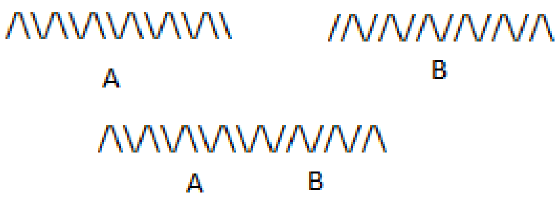

FIG. 3: Polyacetylene chains. There are the two minimum energy states A and B. Below, there is a state with a soliton

linearize the dispersion energy near these points. Now, electrons are described by relativistic effective equations instead of Schrödinger equations. The Dirac equation for polyacetylene, as we see in [5], is

$$\sigma^2 p\Psi_E + \sigma^1 \phi \Psi_E = E\Psi_E, \tag{19}$$

In the absence of defects we find a gap energy between E and $-E$. In the soliton sector we have a zero mode and this state appears at the mid-gap.

There is a complication. Theory is for an infinity chain and we have only finite chains. Then, an extra solution appears localized at the ends of the chain, as we see in [2].

We identify the two ends of the chain to obtain an infinity chain with the two solutions, with two defects. If we add an electron, we obtain state A by finite rearrangement of the electrons. This final configuration has null electric charge, so the two solitons configuration has charge -1. For symmetry, each soliton have charge $-1/2$.

III. TWO-DIMENSIONAL CASE

A. Vortices

A vortex is a time independent stable solution of a set of classical field equations with finite energy in two spatial dimensions; is a two-dimensional soliton.

We start with

$$\mathcal{L} = |\partial_\mu \phi|^2 - V(|\phi|), \tag{20}$$

where ϕ is a complex scalar field. We suppose $V(|\phi|)$ has the form

$$V(|\phi|) = \frac{\lambda}{4}(|\phi|^2 - \frac{1}{2}v^2)^2, \tag{21}$$

where v is real and positive. Here we have a global $U(1)$ symmetry $\phi(x) \to e^{i\alpha}\phi(x)$. The classical ground state of this theory is a constant field configuration and has $|\phi| = v/\sqrt{2}$. There are many vacua, each one labelled by the phase, reflecting $U(1)$ symmetry. The configuration doesn't have finite energy vortex. A way to obtain finite energy soliton is to add an Abelian gauge field. We follow [6]. This is the Abelian Higgs model with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu \phi|^2 - V(|\phi|), \tag{22}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative. There is a local $U(1)$ symmetry: $\phi(x) \to e^{i\alpha(x)}\phi(x)$.

The resulting equations of motion can't be solved analitycally, but the existence of solutions can be rigorously proven. The energy of a static field configuration is the sum of three non-negative terms

$$E = \int d^2r[\frac{1}{2}(E_iE^i + B_iB^i) + D_i\phi D^i\phi^\dagger + V(|\phi|)], \tag{23}$$

each one has to be finite if the total energy is finite. If the third term is finite $V(|\phi|)$ goes to zero at spatial infinity. Therefore

$$\phi(r,\theta) \underset{r\to\infty}{\longrightarrow} (v/\sqrt{2})e^{i\sigma(\theta)}, \tag{24}$$

where $e^{i\sigma(\theta)}$ is an arbitrary phase factor. An application from S^1 to S^1 has a winding number n defined by

$$n = \frac{1}{2\pi}[\sigma(\theta = 2\pi) - \sigma(\theta = 0)]. \tag{25}$$

The winding number is an integer. Since it is an integer, it can't be changed continuously, so it has to be preserved by smooth field deformations that preserve finite energy. We are worried about the second term in Eq. (23) now.

$$\int d^2r \left| \left(\frac{1}{r}\frac{\partial}{\partial\theta} - ieA_\theta \right)\phi \right|^2, \tag{26}$$

it can be finite only if

$$A_\theta \underset{r\to\infty}{\longrightarrow} \frac{1}{er}\frac{d\sigma}{d\theta} + \cdots . \tag{27}$$

The behaviour of A_θ for large values of r allows the field $F_{\mu\nu}$ to fall fast enough at large values of r and the first term of total energy keeps up finite.

B. Coupling with fermions

We will now couple fermions to the previous model, following [7]. The full Lagrangian is

$$\mathcal{L} = \bar\Psi(\gamma^\mu[i\partial_\mu - eA_\mu])\Psi - \frac{1}{2}ig\phi\bar\Psi\Gamma\Psi^c + \frac{1}{2}ig^*\phi^*\bar\Psi^c\Gamma\Psi, \tag{28}$$

where e is the electric charge, g is the scalar coupling constant and the coupled matrix is Γ.

Like before, if there is a spinor with energy E, there is a spinor with energy $-E$ by particle conjugation.

Again, we look for zero-energy solutions of the Dirac equation

$$0 = \boldsymbol{\alpha}\cdot(\boldsymbol{p} - e\boldsymbol{A})\psi - g\phi\sigma^2\psi^*, \tag{29}$$

The solutions in the background of a vortex with n modes are:

$$\psi \underset{n>0}{=} \exp\left\{ \frac{1}{2}\int_0^r d\rho A(\rho) + \frac{1}{2}i(|n|-1) \right\} \times \begin{pmatrix} e^{i\alpha_m}u_m(r)e^{-i[|n|/2-1/2-m]\theta} + e^{-i\alpha_m}v_m(r)e^{i[|n|/2-1/2-m]\theta} \\ 0 \end{pmatrix} \tag{30}$$

and

$$\psi \underset{n<0}{=} \exp\left\{ -\frac{1}{2}\int_0^r d\rho A(\rho) - \frac{1}{2}i(|n|-1) \right\} \times \begin{pmatrix} 0 \\ e^{i\bar\alpha_m}u_m(r)e^{i[|n|/2-1/2-m]\theta} - e^{-i\bar\alpha_m}v_m(r)e^{-i[|n|/2-1/2-m]\theta} \end{pmatrix} \tag{31}$$

Where m is an integer and $|n| - 1 \geq m \geq 0$. The phase α_m doesn't add new degrees of freedom. We conclude that there are $|n|$ linearly independent zero modes. This model doesn't have a well-defined global charge. To display fractional charge we will consider a more sophisticated model.

C. Graphene

We use the model [8] that has the Hamiltonian

$$H = -\sum_{r\in\Lambda_A}\sum_{i=1}^{3}(t + \delta t_{\boldsymbol{r},i})a^\dagger_{\boldsymbol{r}} b_{\boldsymbol{r}+\boldsymbol{s}_i} + H.c., \tag{32}$$

where the spin of the electron is being neglected. The

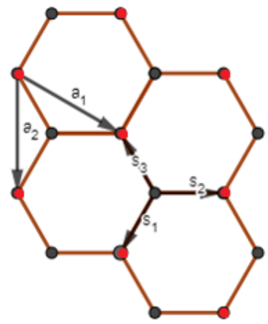

FIG. 4: Graphene lattice. There is a red vertices sublattice Λ_A and black vertices sublattice Λ_B

modulation is

$$\delta t_{\boldsymbol{r},i} = \Delta(\boldsymbol{r})e^{i\boldsymbol{K}_+\cdot\boldsymbol{s}_i}e^{i\boldsymbol{G}\cdot\boldsymbol{r}}/3 + c.c., \tag{33}$$

where $\boldsymbol{G} := \boldsymbol{K}_+ - \boldsymbol{K}_-$ and $K_\pm$ are two Dirac points.

Now we linearize the energy dispersion near Dirac points. To leading order with Eq. (32) and Eq. (33) we obtain

$$\mathcal{H} = \int d^2\boldsymbol{r}\,\Psi^\dagger(\boldsymbol{r})\mathcal{K}_D(\boldsymbol{r})\Psi(\boldsymbol{r}), \tag{34}$$

where $\Psi^\dagger(\boldsymbol{r}) = (u_b^\dagger(\boldsymbol{r}), u_a^\dagger(\boldsymbol{r}), v_a^\dagger(\boldsymbol{r}), v_b^\dagger(\boldsymbol{r}))$ and

$$\mathcal{K}_D = \begin{pmatrix} 0 & -2i\partial_z & \Delta(\boldsymbol{r}) & 0 \\ -2i\partial_{\bar z} & 0 & 0 & \Delta(\boldsymbol{r}) \\ \bar\Delta(\boldsymbol{r}) & 0 & 0 & 2i\partial_z \\ 0 & \bar\Delta(\boldsymbol{r}) & 2i\partial_{\bar z} & 0 \end{pmatrix}, \tag{35}$$

where $z = x + iy$, $\partial_z = (\partial_x - i\partial_y)/2$. We observe that $\mathcal{K}_D$ plays the role of covariant derivative. If $\Delta(\boldsymbol{r}) = \Delta_0$ we conclude that $\varepsilon_\pm(\boldsymbol{p}) = \pm\sqrt{|\boldsymbol{p}|^2 + |\Delta_0|^2}$ and we observe that Δ_0 behaves like a mass. We assume that

$$\Delta(\boldsymbol{r}) = \Delta_0(r)e^{i(\alpha+n\theta)}, \tag{36}$$

where $\Delta_0(r) > 0$ and $n \in \mathbb{Z}$. Again we look for normalizable eigenstates with zero energy. The equations that we obtain are

$$\begin{aligned} (\partial_r - ir^{-1}\partial_\theta)u_a(\boldsymbol{r}) + ie^{i\theta}\Delta(\boldsymbol{r})v_a(\boldsymbol{r}) &= 0 \\ ie^{-i\theta}\bar\Delta(\boldsymbol{r})u_a(\boldsymbol{r}) - (\partial_r + ir^{-1}\partial_\theta)v_a(\boldsymbol{r}) &= 0, \end{aligned} \tag{37}$$

the other two equations can be obtained with the subtitutions $u_a \to u_b$, $v_a \to v_b$, and $\theta \to -\theta$. If $n = -1$ the solution is

$$\begin{aligned} u_a(r,\theta) &= \frac{e^{i(\frac{\alpha}{2}+\frac{\pi}{2})}}{2\sqrt{\pi}} \frac{e^{-\int_0^r dr'\Delta_0(r')}}{\sqrt{\int_0^\infty dr\; re^{-2\int_0^r dr'\Delta_0(r')}}} \\ v_a(r,\theta) &= \bar u_a(r,\theta). \end{aligned} \tag{38}$$

We define the local density of states like $\nu_n(\boldsymbol{r},\varepsilon) := \sum_{\varepsilon'} \psi^\dagger_{\varepsilon'}(\boldsymbol{r})\psi_{\varepsilon'}(\boldsymbol{r})\delta(\varepsilon-\varepsilon')$ and $\delta\nu(\boldsymbol{r},\varepsilon) := \nu_{|n|=1}(\boldsymbol{r},\varepsilon) - \nu_{|n|=0}(\boldsymbol{r},\varepsilon)$. By an unitary transformation, if there is a negative eigenstate, there is a positive energy state. We notice that it is symmetric to zero energy and positive and negative energy eigenstates contribute equally to the local density. With the conservation of total number states we have

$$\int d^2\boldsymbol{r}\Big(2\int_{-\infty}^{0^-}\delta\nu(\boldsymbol{r},\varepsilon)d\varepsilon + |\psi_0(\boldsymbol{r})|^2\Big) = 0. \qquad (39)$$

But the zero mode is normalized to one, so

$$\int d^2\boldsymbol{r}\int_{-\infty}^{0^-}\delta\nu(\boldsymbol{r},\varepsilon)d\varepsilon = -1/2. \qquad (40)$$

The valence band and the conduction band have a deficit of half state. The difference in net charge with and without vortex is $-e/2$. We find vortices in pairs.

In this model there isn't a gauge field and the dynamics of the vortex is unespecified. We can rewrite the Dirac Hamiltonian density following the model [9]

$$\Psi^\dagger\mathcal{K}_D\Psi = \Psi^\dagger(\boldsymbol{\alpha}\cdot\boldsymbol{p} + g\beta[\varphi^r - i\varphi^i\gamma_5])\Psi, \qquad (41)$$

where we have changed Δ by $g\varphi$, g is a coupling strength, $\varphi = \varphi^r + i\varphi^i$ is a complex scalar field and $\beta = \gamma^0$. We note that the interaction part is local chiral gauge invariant, but not the kinetic part. The transformations are $\varphi \to e^{2i\omega}\varphi$ and $\Psi \to e^{i\omega\gamma_5}\Psi$. So we modify it and add a gauge potential

$$\Psi^\dagger\mathcal{K}_D\Psi = \Psi^\dagger(\boldsymbol{\alpha}\cdot[\boldsymbol{p} - \gamma_5\boldsymbol{A}] + g\beta[\varphi^r - i\varphi^i\gamma_5])\Psi, \qquad (42)$$

where $\boldsymbol{A} \to \boldsymbol{A} + \nabla\omega$.

We find φ and $\boldsymbol{A}$ using Nielsen, Olesen/Landau, Ginsburg, Abrikosov equations. We find

$$\varphi_v(\boldsymbol{r}) = \varphi(r)e^{in\theta}$$
$$A^i_v(\boldsymbol{r}) = -n\varepsilon^{ij}\frac{r^j}{r^2}a(r), \qquad (43)$$

where $a(r)$ vanishes at the origin and goes to $1/2$ at large r.

We have now finiteness in vortex energy and we have local chiral $U(1)$ symmetry. The zero modes are not affected by this change.

IV. CONCLUSIONS

We have studied fractional charge in low dimensions. In the one dimensional case we obtain fractional charge theoretically and then we see how it can be realized in condensed matter. In the two dimensional case we construct a two dimensional soliton (vortex) and then we use increasingly sophisticated models in order to obtain a well defined global charge and finiteness in vortex energy.

We can study fractional charge in three and four dimensions where magnetic monopoles and instantons appear, but so far there aren't physical realizations.

We have focused on applications to Condensed Matter, but the phenomena described here has also applications to Mathematics (index theorems [10]), Cosmology (cosmic strings [11]) and string theory.

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