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Geometric measure theory and Kakeya's problem

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Abstract

The main goal of this work is to define a 2-dimensional Besicovitch set and introduce the Kakeya problem, which asks for the minimal area of a set in which a segment can be rotated continuously in the plane and can lay in all directions, approaching it both from a geometrical point of view and by using techniques of polar reciprocity. We first establish basic results about measures that we use throughout the work, paying special attention to Hausdorff's measure and its properties. We then construct a Besicovitch set and a solution to Kakeya's problem, first geometrically and then by duality and projection arguments before looking into the Hausdorff dimension of such sets.

Introduction

In 1917 Sōichi Kakeya posed the problem of finding the minimal area of a convex set inside which a unit segment can be manouvered to lie in its original position rotated 180° without leaving the set [1]. Matsusaboro Fujiwara and Sōichi Kakeya conjectured that the equilateral triangle of unit height was the smallest possible set [2], conjecture proven by Julius Pál in 1921 [3], and focus changed to solving the same problem dropping the convexity assumption. This question became known as the Kakeya Problem.

In 1919, while working on problems on Riemann integration, Abraham S. Besicovitch constructed a compact set of plane Lebesgue measure zero containing a line segment in every direction, known now as a Besicovitch set. Due to the unstable political situation in Russia his publication did not recieve much attention and it was not until he republished his results in 1928 when it was found that a simple modification to his set yielded a solution of arbitrary small measure to Kakeya's Problem [4].

The greatest subsequent development was made again by Besicovitch in 1964 when he found that, using the techniques of polar reciprocity, the existence of Besicovitch sets follows from a simple dual result to the projection theorems for irregular linearly measurable sets [5]. A further question during more than 50 years was to know what the size of a Besicovitch set was, proven by Roy O. Davies that, in a plane, Besicovitch sets have Hausdorff Dimension 2 [6].

Kakeya's problem admits n -dimensional versions which recently have seen a lot of developement. In 2025 Hong Wang and Joshua Zahl proved that every Besicovitch set in $\mathbb{R}^3$ has Hausdorff dimension 3 [7]. This result was what initially sparked our interest in this problem, even though we focus exclusively on the 2-dimensional version of the problem, which has been extensively studied for decades, basing most of our work on Kenneth Falconer's work [8].

Chapter 1

Concept of measures

The purpose of this chapter is to introduce a series of foundational measures, especially Hausdorff measures, which are the cornerstone of the whole thesis. Due to the scope of our work and the questions we are trying to answer, we will work mostly in Euclidian n -space $\mathbb{R}^n$ even if most results hold in a general metric space setting.

1.1 Hausdorff measures

Definition 1.1. Given a non-empty set $U \subseteq \mathbb{R}^n$ we define the *diameter* of U as

$$\text{diam}(U) = \sup\{|x - y| : x, y \in U\}$$

Definition 1.2. We say that $\{U_i\}$ is a δ -cover of E if $E \subseteq \bigcup_i U_i$ and $0 < |U_i| \leq \delta \quad \forall i$

Definition 1.3. Let $E \subseteq \mathbb{R}^n$ and let $s > 0$. For $\delta > 0$ we define

$$\mathcal{H}_\delta^s(E) = \inf \sum_{i=1}^{\infty} \text{diam}(U_i)^s$$

with the infimum over all countable δ -covers $\{U_i\}$ of E .

Proposition 1.4. $\mathcal{H}_\delta^s$ is a metric outer measure on $\mathbb{R}^n$.

Proof. (i) Let $\{U_i\}$ be a δ -cover of $\emptyset$. Clearly $|U_i| < \varepsilon \quad \forall \varepsilon > 0$ therefore:

$$\mathcal{H}_\delta^s(\emptyset) = \inf \sum_{i=1}^{\infty} \text{diam}(U_i)^s = 0$$

- (ii) Let $A \subseteq B \subseteq \mathbb{R}^n$. Given a fixed $s > 0$ $|x|^s$ increases as $|x|$ increases. Since any δ -cover of B is also a δ -cover of A $\mathcal{H}_\delta^s(A) \leq \mathcal{H}_\delta^s(B)$ follows from the definition.
- (iii) Let $\{A_i\}$ be a collection of subsets of $\mathbb{R}^n$. Let for each i $\{U_{ij}\}$ be a δ -cover of $\{A_i\}$.

$$\sum_{i=1}^{\infty} \mathcal{H}_\delta^s(A_i) = \sum_{i=1}^{\infty} \inf_j \sum_{j=1}^{\infty} \text{diam}(U_{ij})^s = \inf_j \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \text{diam}(U_{ij})^s$$

$\bigcup_{ij} U_{ij}$ is a δ -cover of $\bigcup_i A_i$. Since the infimum is just taken over j we can affirm

$$\sum_{i=1}^{\infty} \mathcal{H}_\delta^s(A_i) \geq \mathcal{H}_\delta^s\left(\bigcup_i A_i\right)$$

- (iv) If δ is the distance between two positively separated sets E, F no set in a δ -cover of $E \cup F$ can intersect both E and F , thus

$$\mathcal{H}_\delta^s(E \cup F) = \mathcal{H}_\delta^s(E) + \mathcal{H}_\delta^s(F)$$

□

Taking $\delta \rightarrow 0$ we obtain the *Hausdorff s -dimensional metrical outer measure* of E . Thus:

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E)$$

Since as δ decreases $\mathcal{H}_\delta^s$ increases the limit exists but may be infinite.

Definition 1.5. The restriction of $\mathcal{H}^s$ to the σ -field of $\mathcal{H}^s$ -measurable sets, which by Theorem A.7 includes the Borel sets, is called the *Hausdorff s -dimensional measure*.

For any E $\mathcal{H}^s(E)$ is non-increasing as s increases from 0 to ∞ . Furthermore, if $s < t$ then $\mathcal{H}_\delta^s(E) \geq \delta^{s-t} \mathcal{H}_\delta^t(E)$ thus if $\mathcal{H}^t(E)$ is positive, then $\mathcal{H}^s(E)$ is infinite. Hence, there is a unique number called the *Hausdorff dimension* of E , written as $\dim E$, such that

$$\begin{aligned} \mathcal{H}^s(E) &= \infty & \text{if } 0 \leq s < \dim E \\ \mathcal{H}^s(E) &= 0 & \text{if } \dim E < s < \infty \end{aligned} \tag{1.1}$$

Definition 1.6. Let ν be an outer measure on X . ν is said to be *regular* if

$$\forall A \subseteq X \quad \exists E \subseteq X$$

with E ν -measurable such that $\nu(A) = \nu(E)$.

If C is a cube of unit side in $\mathbb{R}^n$, by dividing C into k^n subcubes of side $1/k$ in the natural way, we see that if $\delta \geq \sqrt{n}/k$ then $\mathcal{H}_\delta^n(C) \leq k^n \sqrt{n^n}/k^n = \sqrt{n^n} < \infty$. If $s > n$ then $\mathcal{H}^s(C) = 0$ and $\mathcal{H}^s(\mathbb{R}^n) = 0$ since $\mathbb{R}^n$ is expressible as a countable union of such cubes. It follows then:

Proposition 1.7. $0 \leq \dim E \leq n$ for any $E \subseteq \mathbb{R}^n$

Definition 1.8. An $\mathcal{H}^s$ -measurable set $E \subseteq \mathbb{R}^n$ is called an s -set if $0 < \mathcal{H}^s(E) < \infty$

Clearly the Hausdorff dimension of an s -set is s . Let us now see that $\mathcal{H}^s$ is a regular measure as well as how to approximate s -sets.

Definition 1.9. A set is termed a F_σ -set if it is a countable union of closed sets.

Definition 1.10. A set is termed a G_δ -set if it is a countable intersection of open sets.

Note that both F_σ -sets and G_δ are Borel sets.

Theorem 1.11. (i) Let $E \subseteq \mathbb{R}^n$. There is a G_δ -set $G \supseteq E$ with $\mathcal{H}^s(G) = \mathcal{H}^s(E)$. In particular $\mathcal{H}^s$ is an outer measure.

(ii) Let $H \subseteq \mathbb{R}^n$ be $\mathcal{H}^s$ -measurable with $\mathcal{H}^s(H) < \infty$. There is an F_σ -set $F \subseteq H$ such that $\mathcal{H}^s(F) = \mathcal{H}^s(H)$, and so H contains a closed set differing from itself by an arbitrarily small measure.

Proof. (i) If $\mathcal{H}^s(E) = \infty$ then $\mathbb{R}^n$ is an open set with $\mathcal{H}^s(E) = \mathcal{H}^s(\mathbb{R}^n)$ and we are done. Let us therefore assume $\mathcal{H}^s(E) < \infty$. $\forall i \in \mathbb{N}_{>0}$ choose an open $2/i$ -cover of E $\{U_{ij}\}_j$ such that

$$\sum_{j=1}^{\infty} \text{diam}(U_{ij})^s < \mathcal{H}_{1/i}^s(E) + 1/i$$

$G = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} U_{ij}$ is a G_δ -set with $E \subseteq G$. $\{U_{ij}\}_j$ is a $2/i$ -cover of G therefore $\mathcal{H}_{2/i}^s(G) \leq \mathcal{H}_{1/i}^s(E) + 1/i$, taking the limit as $i \rightarrow \infty$ it follows $\mathcal{H}^s(G) = \mathcal{H}^s(E)$. Since G_δ -sets are Borel sets, by Theorem A.7 G_δ -sets are $\mathcal{H}^s$ -measurable and so $\mathcal{H}^s$ is a regular outer measure.

(ii) Using (i) we may find a sequence of open sets $\{O_i\}$ all containing H with $\mathcal{H}^s(\bigcap_{i=1}^{\infty} O_i \setminus H) = \mathcal{H}^s(\bigcap_{i=1}^{\infty} O_i) - \mathcal{H}^s(H) = 0$. Since any open subset of $\mathbb{R}^n$ is a

F_σ -set we may set $O_i = \bigcup_{j=1}^{\infty} F_{ij} \quad \forall i$ where $\{F_{ij}\}_j$ is an increasing sequence of closed sets. Since it is an increasing sequence and measures are continuous from below we have

$$\lim_{j \rightarrow \infty} \mathcal{H}^s(H \cap F_{ij}) = \mathcal{H}^s(H \cap O_i) = \mathcal{H}^s(H)$$

Thus, given $\varepsilon > 0$, we may find a j_i such that

$$\mathcal{H}^s(H \setminus F_{ij_i}) < \varepsilon/2^i \quad \forall i$$

Fixing $F = \bigcap_{i=1}^{\infty} F_{ij_i}$ we have

$$\mathcal{H}^s(F) \geq \mathcal{H}^s(H \cap F) \geq \mathcal{H}^s(H) - \mathcal{H}^s(H \setminus F) \geq \mathcal{H}^s(H) - \sum_{i=1}^{\infty} \mathcal{H}^s(H \setminus F_{ij_i}) > \mathcal{H}^s(H) - \varepsilon$$

Given that $F \subseteq \bigcap_{i=1}^{\infty} O_i$ it follows $\mathcal{H}^s(F \setminus H) \leq \mathcal{H}^s(\bigcap_{i=1}^{\infty} O_i \setminus H) = 0$. Using (i) $F \setminus H$ is contained in a G_δ -set G with $\mathcal{H}^s(F \setminus H) = \mathcal{H}^s(G) = 0$. Since F is closed and G is open $F \setminus G$ is closed and therefore an F_σ -set. $F \setminus G$ is also contained in H with

$$\mathcal{H}^s(F \setminus G) \geq \mathcal{H}^s(F) - \mathcal{H}^s(G) > \mathcal{H}^s(H) - \varepsilon$$

Now taking a countable union of these sets over $\varepsilon = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ we obtain an F_σ -set contained in H and of the same measure as H .

□

The following lemma tells us that when estimating the Hausdorff measure of a set with a cover of sufficiently small sets the result does not differ too much from the actual Hausdorff measure.

Lemma 1.12. *Let E be an $\mathcal{H}^s$ -measurable set with $\mathcal{H}^s(E) < \infty$ and let $\varepsilon > 0$. There exists a $\rho > 0$, dependent only on E and ε , such that for any collection of Borel sets $\{U_i\}_{i=1}^{\infty}$ with $0 < \text{diam}(U_i) \leq \rho$*

$$\mathcal{H}^s(E \cap \bigcup_i U_i) < \sum_i \text{diam}(U_i)^s + \varepsilon$$

holds.

Proof. From the definition of $\mathcal{H}^s$ we may choose ρ such that

$$\mathcal{H}^s(E) < \sum_i \text{diam}(W_i)^s + \frac{1}{2}\varepsilon \quad (1.2)$$

for any ρ -cover $\{W_i\}$ of E . Given Borel sets $\{U_i\}$ with $0 < \text{diam}(U_i) \leq \rho$ we may find a ρ -cover $\{V_i\}$ of $E \setminus \bigcup_i U_i$ such that

$$\mathcal{H}^s(E \setminus \bigcup_i U_i) + \frac{1}{2}\varepsilon > \sum_i \text{diam}(V_i)^s$$

Since $\{U_i\} \cup \{V_i\}$ is a ρ -cover of E we have by (1.2)

$$\mathcal{H}^s < \sum_i \text{diam}(U_i)^s + \sum_i \text{diam}(V_i)^s + \frac{1}{2}\varepsilon$$

Hence

$$\begin{aligned} \mathcal{H}^s(E \cap \bigcup_i U_i) &= \mathcal{H}^s(E) - \mathcal{H}^s(E \setminus \bigcup_i U_i) \\ &< \sum_i \text{diam}(U_i)^s + \sum_i \text{diam}(V_i)^s + \frac{1}{2}\varepsilon - \sum_i \text{diam}(V_i)^s + \frac{1}{2}\varepsilon \\ &= \sum_i \text{diam}(U_i)^s + \varepsilon \end{aligned}$$

□

Finally we introduce the following simple lemma which will prove very useful in the upcoming chapters.

Lemma 1.13. *Let $x, y \in E$. Let $\psi : E \rightarrow F$ be a surjective mapping with*

$$|\psi(x) - \psi(y)| \leq c|x - y|$$

for a constant c . Then $\mathcal{H}^s(F) \leq c^s \mathcal{H}^s(E)$

Proof. Let $\{U_i\}$ be a δ -cover of E . $\forall i$ $\text{diam}(\psi(U_i \cap E)) \leq c \text{diam}(U_i)$ therefore $\{\psi(U_i \cap E)\}$ is a $c\delta$ -cover of F . $\sum_i \text{diam}(\psi(U_i \cap E))^s \leq c^s \sum_i \text{diam}(U_i)^s$ hence $\mathcal{H}_{c\delta}^s(F) \leq c^s \mathcal{H}_\delta^s(E)$. The result follows from taking $\delta \rightarrow 0$. □

1.2 Covering results

The Vitali covering theorem states that given a sufficiently large collection of sets that cover some set E we can select a disjoint subcollection of them that covers almost all of E . It is one of the most useful results in geometric measure theory and one that we will use extensively.

Definition 1.14. let $E \subseteq \mathbb{R}^n$. A collection of sets $\mathcal{V}$ is called a *Vitali class* for E if for each $x \in E$ and for every $\delta > 0$ there exists $U \in \mathcal{V}$ with $x \in U$ and $0 < \text{diam}(U) \leq \delta$

Definition 1.15. A collection of sets is *semidisjoint* if no member of the collection is contained in any other member.

Before proving the Vitali covering theorem we will give the following lemma, a very well known version of the theorem in a simplified and finite setting.

Lemma 1.16. Let $\mathcal{C}$ be a collection of balls contained in a bounded subset of $\mathbb{R}^n$, then we may find a finite or countably infinite disjoint subcollection $\{B_i\}$ such that

$$\bigcup_{B \in \mathcal{C}} B \subseteq \bigcup_i B'_i \quad (1.3)$$

where B'_i is the ball concentric with the ball B_i and of three times the radius. We may also take the collection $\{B'_i\}$ to be semidisjoint.

Proof. We select the $\{B_i\}$ inductively. Let B_1 a ball with maximal diameter in $\mathcal{C}$. Suppose $B_1, B_2, \dots, B_m$ have already been chosen, let $B_{m+1} \in \mathcal{C}$ be a ball with maximal diameter out of every ball not intersecting $\bigcup_{i=1}^m B_i$, if no such ball remains the process terminates. By construction these balls are disjoint, let us see that they also have the desired covering property. If $B \in \mathcal{C}$ then either $B = B_i$ for some i or B intersects some B_i with $\text{diam}(B_i) \geq \text{diam}(B)$. If that were not the case B would have been selected over the first ball B_m for which $\text{diam}(B_m) < \text{diam}(B)$. In either case $B \subseteq B'_i$, yielding (1.3). Removing now B_i from the subcollection if $B'_i \subseteq B'_j$ for any $i \neq j$ we obtain the $\{B'_i\}$ disjoint. Note that B'_i can only be contained in finitely many B'_j since by summing volumes we have $\sum \text{diam}(B_i)^2 < \infty$ and thus $\text{diam}(B_i) \rightarrow 0$ as $i \rightarrow \infty$ if infinitely many balls are selected. $\square$

Finally, the Vitali covering theorem states as follows:

Theorem 1.17. (i) Let $E \subseteq \mathbb{R}^n$ be an $\mathcal{H}^s$ -measurable set and let $\mathcal{V}$ be a Vitali class of closed sets for E . Then we may select a finite or countable disjoint sequence $\{U_i\}$ from $\mathcal{V}$ such that either $\mathcal{H}^s(E \setminus \bigcup_i U_i) = 0$ or $\sum_i \text{diam}(U_i)^s = \infty$

(ii) Given $\varepsilon > 0$, if $\mathcal{H}^s(E) < \infty$ then we may also require

$$\mathcal{H}^s(E) \leq \sum_i \text{diam}(U_i)^s + \varepsilon$$

Proof. Fix $\rho > 0$, we may assume that $\text{diam}(U) \leq \rho$ for all $U \in \mathcal{V}$. We will choose $\{U_i\}$ inductively. Let U_1 be any member of $\mathcal{V}$. Suppose $U_1, \dots, U_m$ have already been chosen, let d_m the supremum of $\text{diam}(U)$ taken over the U in $\mathcal{V}$ which do not intersect any $U_1, \dots, U_m$. If $d_m = 0$ then $E \subseteq \bigcup_1^m U_i$ so (i) follows and the process terminates. Otherwise choose $U_{m+1} \in \mathcal{V}$ such that it is disjoint from $\bigcup_1^m U_i$ and $\text{diam}(U_{m+1}) \geq d_m/2$.

Suppose now that the process continues indefinitely and that

$$\sum_i \text{diam}(U_i)^s < \infty$$

For each i let B_i be a ball with center in U_i with radius $3 \text{diam}(U_i)$. We claim that

$$E \setminus \bigcup_{i=1}^k U_i \subseteq \bigcup_{i=k+1}^{\infty} B_i \quad (1.4)$$

since if $x \in \bigcup_{i=1}^k U_i$ there is a $U \in \mathcal{V}$ not intersecting $U_1, \dots, U_k$ with $x \in U$. Given that $\text{diam}(U_i) \rightarrow 0$, $\text{diam}(U) > 2 \text{diam}(U_m)$ for some m . Given how we have defined the method of selection of $\{U_i\}$, U must intersect U_i for some i with $k < i < m$ for which $\text{diam}(U) \leq 2 \text{diam}(U_i)$, thus $U \in B_i$ and (1.4) follows. Therefore if $\delta > 0$

$$\mathcal{H}_\delta^s(E \setminus \bigcup_{i=1}^{\infty} U_i) \leq \mathcal{H}_\delta^s(E \setminus \bigcup_{i=1}^k U_i) \leq \sum_{i=k+1}^{\infty} \text{diam}(B_i)^s = 6^s \sum_{i=k+1}^{\infty} \text{diam}(U_i)^s$$

provided k is large enough that $\text{diam}(B_i) \leq \delta$ for $i > k$ holds. Hence $\mathcal{H}_\delta^s(E \setminus \bigcup_{i=1}^{\infty} U_i) = 0$ for all $\delta > 0$ so $\mathcal{H}^s(E \setminus \bigcup_{i=1}^{\infty} U_i) = 0$, proving (i).

To obtain (ii) we may suppose that the fixed ρ is the number corresponding to ε and E given by Lemma 1.12. If $\sum_i \text{diam}(U_i)^s = \infty$ then the result is immediate. Otherwise by (i) and Lemma 1.12

$$\begin{aligned} \mathcal{H}^s(E) &= \mathcal{H}^s(E \setminus \bigcup_i U_i) + \mathcal{H}^s(E \cap \bigcup_i U_i) \\ &= 0 + \mathcal{H}^s(E \cap \bigcup_i U_i) < \sum_i \text{diam}(U_i)^s + \varepsilon \end{aligned}$$

□

1.3 Comparable net measures

Net measures are constructed in a similar way to Hausdorff measures but using a restricted class of covering sets $\mathcal{N}$. The class $\mathcal{N}$ is called a *net* of sets and is

chosen such that if $U_1, U_2 \in \mathcal{N}$ then either $U_1 \cap U_2 = \emptyset$, $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. We call this property the *net property*. We also assume that each set in $\mathcal{N}$ is contained in finitely many others. Note that given a collection of sets in $\mathcal{N}$ the net property lets us, by removing those sets completely included in others, find a disjoint sub-collection with the same union.

Let us now construct the s -dimensional net measure $\mathcal{M}^s$ on $\mathbb{R}^n$. Let $\mathcal{N}$ be the collection of all n -dimensional half-open dyadic cubes, i.e.

$$[2^{-k}m_1, 2^{-k}(m_1 + 1)) \times [2^{-k}m_2, 2^{-k}(m_2 + 1)) \times \cdots \times [2^{-k}m_n, 2^{-k}(m_n + 1))$$

where k is a non-negative integer and m_i are integers.

Definition 1.18. Let $E \subseteq \mathbb{R}^n$ and $\delta > 0$. We define

$$\mathcal{M}_\delta^s(E) = \inf \sum_{i=1}^{\infty} \text{diam}(S_i)^s$$

where we take the infimum over all countable δ -covers of E by sets $\{S_i\}$ of $\mathcal{N}$. By the net property it is sufficient to consider covers of E by disjoint collections of sets of $\mathcal{N}$. Letting

$$\mathcal{M}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{M}_\delta^s(E) = \sup_{\delta > 0} \mathcal{M}_\delta^s(E)$$

we obtain a metric outer measure $\mathcal{M}^s$ on $\mathbb{R}^n$. By Theorem A.7 we have that Borel sets are $\mathcal{M}^s$ -measurable and by a similar reasoning as in Theorem 1.11 we may see that $\mathcal{M}^s$ is a regular outer measure.

We now show that $\mathcal{M}^s$ and $\mathcal{H}^s$ are *comparable* measures. This result will prove very useful since it allows us to see that certain results for net measures automatically hold for Hausdorff measures, which normally are harder to work with.

Theorem 1.19. *There exist constants b_n depending only on the dimension n such that $\forall E \subseteq \mathbb{R}^n$, if $0 < \delta < 1$*

$$\mathcal{H}_\delta^s(E) \leq \mathcal{M}_\delta^s(E) \leq b_n \mathcal{H}_\delta^s(E) \quad (1.5)$$

and

$$\mathcal{H}^s(E) \leq \mathcal{M}^s(E) \leq b_n \mathcal{H}^s(E) \quad (1.6)$$

Proof. Since for Hausdorff measures the infimum is taken over a larger class of sets $\mathcal{H}_\delta^s(E) \leq \mathcal{M}_\delta^s(E)$ holds for any E, s and δ . If U is any set with $0 < \text{diam}(U) \leq \delta$ let k be the integer such that

$$2^{-k-1} \leq \text{diam}(U) < 2^{-k} \quad (1.7)$$

and let S be a dyadic cube of side 2^{-k} that intersects U . Then U is contained in the collection of 3^n dyadic cubes of side 2^{-k} and diameter $2^{-k}n^{1/2}$ consisting of S and its immediate neighbours. Now subdividing each of the cubes into 2^{n^2} smaller cubes, by (1.7) U is contained in $b_n = 3^n 2^{n^2}$ dyadic cubes of diameter

$$2^{-k}n^{1/2}2^{-n} \leq 2^{1-n}n^{1/2} \text{diam}(U) \leq \text{diam}(U) \leq \delta$$

Let now $\{U_i\}$ be a δ -cover of E by arbitrary sets. For every i we have $U_i \subseteq \bigcup_{j=1}^{b_n} S_{ij}$

where $\{S_{ij}\}_{j=1}^{b_n}$ is a collection of b_n cubes of diameter at most $\text{diam}(U_i) \leq \delta$. Thus $E \subseteq \bigcup_i \bigcup_j S_{ij}$ and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{b_n} \text{diam}(S_{ij})^s \leq b_n \sum_{i=1}^{\infty} \text{diam}(U_i)^s$$

from which follow the right-side inequality in (1.5) and, by taking $\delta \rightarrow 0$, the inequalities in (1.6). □

If $\{E_j\}$ is any increasing sequence of subsets of $\mathbb{R}^n$ then by the continuity of measures for increasing sequence of sets $\lim_{j \rightarrow \infty} \mathcal{M}^s(E_j) = \mathcal{M}^s(\lim_{j \rightarrow \infty} E_j)$. This conclusion also holds for the outer measures $\mathcal{M}_\delta^s$, result which we will state and use without proving.

Lemma 1.20. *Let $\{E_j\}$ be an increasing sequence of subsets of $\mathbb{R}^n$ and suppose each E_j is a finite union of dyadic cubes. Then*

$$\mathcal{M}_\delta^s(\lim_{j \rightarrow \infty} E_j) = \lim_{j \rightarrow \infty} \mathcal{M}_\delta^s(E_j)$$

Let $E \subseteq \mathbb{R}^2$. We denote by E_x the set of points of E with the first coordinate equal to x . By Fubini's Theorem A.10 if E is $\mathcal{L}^2$ -measurable E_x is $\mathcal{L}^1$ -measurable.

The following lemma sustains the proof of Theorem 1.22, which lets us compare Hausdorff measures of different dimension between sets and subsets and that will become very important when talking about projections.

Lemma 1.21. *Let $A \subseteq \mathbb{R}$, let $\{I_i\}$ be a countable δ -cover of A by dyadic intervals $[2^{-k}m, 2^{-k}(m+1))$ and let $\{a_i\}$ be a sequence of positive numbers. Assume c is a constant such that*

$$\sum_{\{i: x \in I_i\}} a_i > c \quad \forall x \in A \tag{1.8}$$

Then

$$\sum_i a_i \text{diam}(I_i)^s \geq c \mathcal{M}_\delta^s(A) \quad (1.9)$$

Proof. We first assume $\{I_i\}$ and $\{a_i\}$ to be finite. We may reduce slightly each a_i such that we can take each a_i to be rational without losing (1.8). By multiplying by a well chosen factor we may further assume each a_i to be an integer, thus taking a_i copies of the interval I_i for each i lets us assume $a_i = 1$ for all i .

Under these assumptions by (1.8) we have that each x in A lies in at least $\lceil c \rceil$ of the intervals. Since $A \subseteq \bigcup_i I_i$ and $\{I_i\}$ are dyadic intervals with the net property we can choose a non-overlapping set of intervals containing A $\{I_i\}_{i \in \mathcal{I}_1}$. The remaining intervals still cover A at least $\lceil c \rceil - 1$ times so by the same process we can take a non-overlapping subcollection $\{I_i\}_{i \in \mathcal{I}_2}$ that also covers A . Proceeding in this way we are able to obtain $\lceil c \rceil$ collections of non-overlapping intervals $\{I_i\}_{i \in \mathcal{I}_j}$, with $j = 1, 2, \dots, \lceil c \rceil$, each covering A . Since $\text{diam}(I_i) \leq \delta$ for all i

$$\sum_{i \in \mathcal{I}_j} \text{diam}(I_i)^s \geq \mathcal{M}_\delta^s(A)$$

so by summing over all j (1.9) follows in the case of $\{I_i\}$ being a finite collection of intervals.

Finally, if $\{I_i\}_{i=1}^\infty$ is an infinite set of dyadic intervals, for every integer k we set A_k to be

$$A_k = \{x \in \mathbb{R} : \sum_{\substack{i: x \in I_i \\ i \leq k}} a_i > c\}$$

From the finite case we know that

$$\sum_{i=1}^k a_i \text{diam}(I_i)^s \geq c \mathcal{M}_\delta^s(A_k)$$

Each A_k is a finite union of dyadic intervals and the sequence $\{A_k\}$ is increasing with $A \subseteq \bigcup_k A_k = \lim_{k \rightarrow \infty} A_k$. So by Lemma 1.20

$$\sum_{i=1}^\infty a_i \text{diam}(I_i)^s \geq c \lim_{k \rightarrow \infty} \mathcal{M}_\delta^s(A_k) = c \mathcal{M}_\delta^s(\lim_{k \rightarrow \infty} A_k) \geq c \mathcal{M}_\delta^s(A)$$

completing the proof for the infinite case. □

Theorem 1.22. *Let $E \subseteq \mathbb{R}^2$ and let A be a subset of the x -axis. Suppose that if $x \in A$ then for some constant c $\mathcal{H}^t(E_x) > c$ is verified. Then*

$$\mathcal{H}^{s+t}(E) \geq bc\mathcal{H}^s(A)$$

with b depending only on s and t .

Proof. By Theorem 1.19 it is enough to prove the result for $\mathcal{M}$ instead of $\mathcal{H}$. Let $\delta > 0$ and let $\{S_i\}_{i=1}^\infty$ be a collection of dyadic squares forming a $2^{1/2}\delta$ -cover of E .

$$E_x \subseteq \bigcup_{i=1}^\infty (S_i)_x \quad \forall x \in A$$

so

$$\mathcal{M}_\delta^t(E_x) \leq \sum_{i=1}^\infty \text{diam}((S_i)_x)^t$$

If we denote $A_\delta = \{x \in A : \mathcal{M}_\delta^t(E_x) > c\}$ and $\text{proj } S_i$ as the interval of length δ obtained by projecting S_i onto the x -axis then we have for $x \in A_\delta$

$$c < \sum_{i=1}^\infty \text{diam}((S_i)_x)^t = 2^{-\frac{1}{2}t} \sum_{\{i: x \in \text{proj } S_i\}} \text{diam}(S_i)^t$$

hence by taking $I_i = \text{proj } S_i$ and $a_i = \text{diam}(S_i)^t$ and applying Lemma 1.21 we have

$$\sum_{i=1}^\infty \text{diam}(S_i)^{s+t} = 2^{-\frac{1}{2}s} \sum_{i=1}^\infty \text{diam}(S_i)^t \text{diam}(\text{proj } S_i)^s \geq 2^{\frac{1}{2}(s+t)} c \mathcal{M}_\delta^s(A_\delta)$$

This holds for any $2^{1/2}\delta$ -cover of E by dyadic squares $\{S_i\}$ so

$$2^{\frac{1}{2}(s+t)} c \mathcal{M}_\delta^s(A_\delta) \leq \mathcal{M}_{\frac{1}{2}\delta}^{s+t}(E) \leq \mathcal{M}^{s+t}(E)$$

Since as δ decreases to 0 A_δ increases to A , if $\delta \leq \rho$

$$\mathcal{M}_\delta^s(A_\rho) \leq \mathcal{M}_\delta^s(A_\delta) \leq 2^{-\frac{1}{2}(s+t)} c^{-1} \mathcal{M}^{s+t}(E)$$

thus for $\rho > 0$

$$\mathcal{M}^s(A_\rho) \leq 2^{-\frac{1}{2}(s+t)} c^{-1} \mathcal{M}^{s+t}(E)$$

and hence

$$\mathcal{M}^s(A) \leq b^{-1} c^{-1} \mathcal{M}^{s+t}(E)$$

setting $b = 2^{\frac{1}{2}(s+t)}$. □

1.4 Measure potentials

As with last theorem, in this last section of this introductory chapter we give some basic results that are very necessary to sustain the claims we will make in Chapter 3.

Definition 1.23. The *support* of a Borel measure μ , denoted by $\text{supp } \mu$, is the largest closed set S such that $\int f d\mu = 0$ for every continuous function f that vanishes on S .

Clearly, the mapping $\psi(f) = \int_S f d\mu$ is a positive linear functional on $\mathcal{C}(S)$. Riesz representation theorem given below states that the converse is also true.

Theorem 1.24. Let $S \subseteq \mathbb{R}^n$ be a compact set and let ψ be a positive linear functional on the space $\mathcal{C}(S)$ of continuous functions on S . There exists a Borel measure μ supported on S such that $\mu(S) < \infty$ and

$$\psi(f) = \int_S f d\mu \quad \forall f \in \mathcal{F}$$

Proof. Let $f \in \mathcal{C}(D)$. The positivity of ψ tells us that if $|g| \leq 1$ then $\psi(|g|) \leq \psi(1)$. Thus, if we combine this with the linearity of ψ one gets that

$$\begin{aligned} -|f| \leq f \leq |f| &\implies -\psi(|f|) \leq \psi(f) \leq \psi(|f|) \\ &\implies |\psi(f)| \leq \psi(|f|) = \|f\|_\infty \psi\left(\frac{|f|}{\|f\|_\infty}\right) \leq \|f\|_\infty \psi(1) \end{aligned}$$

Making ψ bounded therefore continuous on $\mathcal{C}(S)$. Given that S is compact, we know that the topological dual of $\mathcal{C}(S)$ is precisely the vector space of signed Borel measures supported on S . Thus we can write $\psi(f) = \int f d\mu$ for a positive measure μ defined on a σ -field of subsets of S . $\square$

Definition 1.25. We call a *mass distribution* a Borel measure μ on $\mathbb{R}^n$ of compact support and such that $0 < \mu(\mathbb{R}^n) < \infty$. The *t-energy* of μ is defined as

$$I_t(\mu) = \int \int \frac{d\mu(x) d\mu(y)}{|x - y|^t}$$

Corollary 1.26. Let $E \subseteq \mathbb{R}^n$ be a Souslin set.

- (i) If $I_t(\mu) < \infty$ for some mass distribution μ supported on E then $t \leq \dim E$.
- (ii) If $t < \dim E$ then there exists a mass distribution μ with support on E such that $I_t(\mu) < \infty$.

Proof. This statement is equivalent to saying that if $E \in \mathbb{R}^n$ is a Souslin set then $\dim E = \inf\{t : C_t(E) = 0\} = \sup\{t : C_t(E) > 0\}$, which follows immediately from the definition of *Hausdorff dimension* (1.1). $\square$

Definition 1.27. If f and μ are, respectively, an integrable function and a mass distribution on $\mathbb{R}$, their 1-dimensional Fourier transforms are given by

$$\hat{f}(p) = \frac{1}{\sqrt{2\pi}} \int f(u) e^{iup} du$$

$$\hat{\mu}(p) = \frac{1}{\sqrt{2\pi}} \int e^{iup} d\mu(u)$$

where $p \in \mathbb{R}$. Both $\hat{f}$ and $\hat{\mu}$ are bounded continuous functions.

Lemma 1.28. Let μ be a mass distribution on $\mathbb{R}$ such that

$$0 < \int |\hat{\mu}(p)|^2 dp < \infty$$

then the support of μ has positive Lebesgue measure.

Proof. Plancherel's Theorem, which we will not prove here, states that square-integrable functions Fourier transforms correspond to square-integrable functions. Hence μ is a Borel measure such that, for any continuous function f that vanishes outside some bounded interval

$$\int f(u) d\mu(u) = \int f(u) g(u) du \tag{1.10}$$

where g is a Borel-measurable function with

$$0 < \int |\hat{\mu}(p)|^2 dp = \int |g(u)|^2 du \tag{1.11}$$

If μ were supported by a compact set of Lebesgue measure zero, (1.10) would imply that $g(u) = 0$ for almost every u , which by (1.11) is impossible. $\square$

Chapter 2

Structure of s-sets

This chapter focuses on the local properties of s -sets, especially regarding questions of density. In general we will not consider the case of $s = 0$ since usually 0-sets require an individual treatment, regardless 0-sets are finite sets of points and their properties are quite straightforward. From now on, when using terms as 'measurable' or 'almost all' we will be referring to the measure $\mathcal{H}^s$.

2.1 Notions of density

Let $B_r(x) \subseteq \mathbb{R}^n$ denote the closed ball of center x and radius r .

Definition 2.1. The *upper density* of an s -set E at a point $x \in \mathbb{R}^n$ is defined as

$$\overline{D}^s(E, x) = \overline{\lim}_{r \rightarrow 0} \frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s}$$

The *lower density* of an s -set E at a point $x \in \mathbb{R}^n$ is defined as

$$\underline{D}^s(E, x) = \underline{\lim}_{r \rightarrow 0} \frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s}$$

If $\overline{D}^s(E, x) = \underline{D}^s(E, x)$ we say that the density of E at x exists and we denote it by $D^s(E, x)$.

Definition 2.2. A point $x \in E$ for which $D^s(E, x) = 1$ is called a *regular* point of E . Otherwise x is an *irregular* point of E . An s -set E is termed *regular* if $\mathcal{H}^s$ -almost all of its points are regular and *irregular* if $\mathcal{H}^s$ -almost all of its points are irregular.

Definition 2.3. The *upper convex density* of an s -set E at x is defined as

$$\overline{D}_c^s(E, x) = \overline{\lim}_{r \rightarrow 0} \left\{ \sup \frac{\mathcal{H}^s(E \cap U)}{|U|^s} \right\}$$

with the supremum taken over all *convex* sets U such that $x \in U$ and $0 < |U| < r$.

Since $B_r(x)$ is convex and if $x \in U$ then $U \subseteq B_r(x)$ with $r = \text{diam}(U)$ it follows

$$2^{-s} \overline{D}_c^s(E, x) \leq \overline{D}^s(E, x) \leq \overline{D}_c^s(E, x) \quad (2.1)$$

The following theorem is obvious for compact sets but it is true more generally.

Theorem 2.4. *Let $E \subseteq \mathbb{R}^n$ be an s-set, then $\overline{D}_c^s(E, x) = 0$ for $\mathcal{H}^s$ -almost all $x \notin E$.*

Proof. Fixed $\alpha > 0$ we will show that the measurable set

$$F = \{x \notin E : \overline{D}_c^s(E, x) > \alpha\}$$

has measure zero. By the regularity of $\mathcal{H}^s$, given $\delta > 0$ we are able to find a closed set $E_1 \subseteq E$ such that $\mathcal{H}^s(E \setminus E_1) < \delta$. For a $\rho > 0$ let

$$\begin{aligned} \mathcal{V} = \{U : U \text{ is closed and convex, } 0 < \text{diam}(U) \leq \rho, \\ U \cap E_1 = \emptyset \text{ and } \mathcal{H}^s(E \cap U) > \alpha \text{diam}(U)^s\} \end{aligned} \quad (2.2)$$

By definition 2.3 $\mathcal{V}$ is a Vitali class of closed sets for F and so by the Vitali Theorem 1.17 we may find a disjoint sequence of sets $\{U_i\}$ in $\mathcal{V}$ such that either $\sum \text{diam}(U)^s = \infty$ or $\mathcal{H}^s(F \setminus \bigcup U_i) = 0$ hold. By 2.2

$$\sum \text{diam}(U)^s < \frac{1}{\alpha} \sum \mathcal{H}^s(E \cap U_i) = \frac{1}{\alpha} \mathcal{H}^s(E \cap \bigcup_i U_i) \leq \frac{1}{\alpha} \mathcal{H}^s(E \setminus E_1) < \frac{\delta}{\alpha} < \infty$$

since the $\{U_i\}$ are disjoint and disjoint from E_1 . It follows that $\mathcal{H}^s(F \setminus U_i) = 0$ and thus

$$\mathcal{H}_\rho^s(F) \leq \mathcal{H}_\rho^s(F \setminus \bigcup_i U_i) + \mathcal{H}_\rho^s(F \cap \bigcup_i U_i) \leq \mathcal{H}^s(F \setminus \bigcup_i U_i) + \sum_i \text{diam}(U_i)^s < 0 + \frac{\delta}{\alpha}$$

This holds for any $\delta, \rho > 0$ so $\mathcal{H}^s(F) = 0$. $\square$

Theorem 2.5. *Let $E \subseteq \mathbb{R}^n$ be an s-set, then $\overline{D}_c^s(E, x) = 1$ at $\mathcal{H}^s$ -almost all $x \in E$.*

Proof. (i) We will first see that $\overline{D}_c^s \geq 1$ almost everywhere in E . Fix $\alpha < 1, \rho > 0$ and let

$$\begin{aligned} F = \{x \in E : \mathcal{H}^s(E \cap U) < \alpha \text{diam}(U)^s \text{ for all convex} \\ U \text{ with } x \in U \text{ and } \text{diam}(U) \leq \rho\} \end{aligned} \quad (2.3)$$

$\forall \varepsilon > 0$ we may find a ρ -cover of F by convex sets $\{U_i\}$ such that $\sum \text{diam}(U_i)^s < \mathcal{H}^s(F) + \varepsilon$. Assuming that each U_i contains some point of F and using (2.3) we see

$$\mathcal{H}^s(F) \leq \sum_i \mathcal{H}^s(F \cap U_i) \leq \sum_i \mathcal{H}^s(E \cap U_i) < \alpha \sum_i \text{diam}(U_i)^s < \alpha(\mathcal{H}^s(F) + \varepsilon)$$

Given that we have chosen $\alpha < 1$ and that the last inequality holds for all $\varepsilon > 0$ it follows that $\mathcal{H}^s(F) = 0$. We may define F for any $\rho > 0$ so by definition $\overline{D}_c^s(E, x) \geq \alpha$ for almost all $x \in E$. This is true for all $\alpha < 1$ so it follows $\overline{D}_c^s(E, x) \geq 1$ almost everywhere in E .

(ii) We now show that $\overline{D}_c^s(E, x) \leq 1$ almost everywhere. Let $\alpha > 1$ and let $F = \{x \in E : \overline{D}_c^s(E, x) > \alpha\}$. F is a measurable set of E . Let $F_0 = \{x \in F : \overline{D}_c^s(E \setminus F, x) = 0\}$. By Theorem 2.4 $\mathcal{H}^s(F \setminus F_0) = 0$. By the definition of convex density if $x \in F_0$ $\overline{D}_c^s(F, x) \geq \overline{D}_c^s(E, x) - \overline{D}_c^s(E \setminus F, x) \geq \alpha$ therefore

$$\mathcal{V} = \{U : U \text{ is closed, convex and } \mathcal{H}^s(F \cap U) > \alpha \text{diam}(U)^s\} \quad (2.4)$$

is a Vitali class for F_0 and thus by Vitali's Theorem 1.17, given a $\varepsilon > 0$ we may find a disjoint sequence of sets $\{U_i\}_i \subseteq \mathcal{V}$ with $\mathcal{H}^s(F_0) \leq \sum_i \text{diam}(U_i)^s + \varepsilon$.

Now by (2.4)

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_0) < \frac{1}{\alpha} \sum_i \mathcal{H}^s(F \cap U_i) + \varepsilon \leq \frac{1}{\alpha} \mathcal{H}^s(F) + \varepsilon$$

This holds for any $\varepsilon > 0$ so since $\alpha > 1$ $\mathcal{H}^s(F) = 0$ as we wanted to see. $\square$

Using (2.1) the next two corollary follow immediately.

Corollary 2.6. *Let $E \subseteq \mathbb{R}^n$ be an s -set, then $D^s(E, x) = 0$ at $\mathcal{H}^s$ -almost all $x \notin E$.*

Corollary 2.7. *Let $E \subseteq \mathbb{R}^n$ be an s -set, then $2^{-s} \leq D^s(E, x) \leq 1$ at $\mathcal{H}^s$ -almost all $x \in E$.*

Corollary 2.8. *Let E be an s -set and $F \subseteq E$ a measurable subset. Then $\underline{D}^s(F, x) = \underline{D}^s(E, x)$ and $\overline{D}^s(E, x) = \overline{D}^s(F, x)$ so almost all $x \in F$. Additionally if E is regular (resp. irregular) then any measurable subset of E of positive measure is regular (resp. irregular).*

Proof. Let F be a measurable subset of E . Let $H = E \setminus F$. From Corollary 2.6 we have that for almost all $x \in F$ $D^s(H, x) = 0$. For such x

$$\begin{aligned}\underline{D}^s(E, x) &= \underline{D}^s(F, x) + \underline{D}^s(H, x) = \underline{D}^s(F, x) \\ \overline{D}^s(E, x) &= \overline{D}^s(F, x) + \overline{D}^s(H, x) = \overline{D}^s(F, x)\end{aligned}$$

The result follows immediately from the definition 2.2 of regularity and irregularity. $\square$

2.2 Structure of curves and continua

Regular 1-sets and rectifiable curves are very closely related, in fact, a regular 1-set differs from a countable collection of rectifiable curves by at most a set of measure zero. In the following section we study how these two relate.

Definition 2.9. A *curve* Γ is the image of a continuous injection $\psi : [a, b] \rightarrow \mathbb{R}^n$ with $[a, b]$ a closed interval of $\mathbb{R}$. The *length* of the curve Γ is defined as

$$\mathcal{L}(\Gamma) = \sup \sum_{i=1}^m |\psi(t_i) - \psi(t_{i-1})|$$

with the supremum taken over all dissections of $[a, b]$ $a = t_0 < t_1 < \dots < t_m = b$. If a curve Γ has length $\mathcal{L}(\Gamma) < \infty$ then we say Γ is *rectifiable*.

Lemma 2.10. Let $\psi : [a, b] \rightarrow \mathbb{R}^n$ be a continuous mapping such that $\psi(a) \neq \psi(b)$, then $\psi[a, b]$ contains a curve joining $\psi(a)$ to $\psi(b)$.

Proof. For each point $x \in \mathbb{R}^n$ with two or more preimages by ψ we define I_x as the largest closed interval $[t_1, t_2]$ with $\psi(t_1) = \psi(t_2) = x$. Let $\mathcal{S}$ denote the collection of such intervals contained in no others, then $\mathcal{S}$ consists of countably many disjoint closed intervals and thus we may construct a continuous surjective function $f : [a, b] \rightarrow [0, 1]$ with $f(a) = 0$, $f(b) = 1$ and such that if $t_1 \leq t_2$ then $f(t_1) \leq f(t_2)$ with equality iff t_1 and t_2 lie in a common interval of $\mathcal{S}$. Let us define now $\psi_0 : [0, 1] \rightarrow \mathbb{R}^n$ by $\psi_0(u) = x$ if $f^{-1}(u) = I_x$ for some $I_x \in \mathcal{S}$ and $\psi_0(u) = \psi(f^{-1}(u))$ otherwise. ψ_0 is a continuous injection with $\psi_0(0) = \psi(a)$ and $\psi_0(1) = \psi(b)$. $\square$

Lemma 2.11. Given a curve Γ , $\mathcal{H}^1(\Gamma) = \mathcal{L}(\Gamma)$.

Proof. Let Γ be a curve joining z and w . Let proj denote the orthogonal projection from $\mathbb{R}^n$ onto the straight line through z and w , then if $x, y \in \mathbb{R}^n$ $|\text{proj } x - \text{proj } y| \leq |x - y|$. By Lemma 1.13 and since $[z, w] \subseteq \text{proj } \Gamma$ we have $\mathcal{H}^1(\Gamma) \geq \mathcal{H}^1(\text{proj } \Gamma) \geq \mathcal{H}^1([z, w]) = \mathcal{L}^1([z, w]) = |z - w|$. Suppose now that Γ is defined by $\psi : [a, b] \rightarrow \mathbb{R}^n$. By the observation we have just made we have $\mathcal{H}^1(\psi[t, u]) \geq |\psi(t) - \psi(u)|$ for any t and u so if $a = t_0 < t_1 < \dots < t_n = b$ is a dissection of $[a, b]$

$$\sum_i |\psi(t_i) - \psi(t_{i-1})| \leq \sum_i \mathcal{H}^1(\psi[t_{i-1}, t_i]) = \mathcal{H}^1(\Gamma)$$

since the arcs $\psi[t_{i-1}, t_i]$ are disjoint except for endpoints and thus $\mathcal{L}^1(\Gamma) \leq \mathcal{H}^1(\Gamma)$. Now assume $\mathcal{L}^1(\Gamma) < \infty$ and let ψ parametrize Γ by arc length, being ψ a surjection from $[0, \mathcal{L}^1(\Gamma)]$ to Γ . It is immediate to see that for any x, y $|\psi(x) - \psi(y)| \leq |x - y|$ and thus by Lemma 1.13 we have $\mathcal{H}^1(\Gamma) \leq \mathcal{H}^1([0, \mathcal{L}^1(\Gamma)]) = \mathcal{L}^1(\Gamma)$. $\square$

A consequence of this lemma is that if Γ is a rectifiable curve, then $\mathcal{H}^s = \infty$ if $s < 1$ and $\mathcal{H}^s = 0$ if $s > 1$.

Definition 2.12. A compact and connected set is termed a continuum.

Lemma 2.13. Let E be a continuum and let $x, y \in E$. If $|x - y| = \rho$ then $\mathcal{H}^1(E \cap B_\rho(x)) \geq \rho$ and in particular $\mathcal{H}^1(E) \geq \text{diam}(E)$.

Proof. Let $f : \mathbb{R}^n \rightarrow [0, \infty)$ be defined by $f(z) = |z - x|$. Then f is a continuous mapping such that for any $z, w \in \mathbb{R}^n$

$$|f(z) - f(w)| \leq |z - w| \quad (2.5)$$

The set $f(E \cap B_\rho(x))$ must contain the interval $[0, \rho]$, otherwise, for some r $0 < r < \rho$, $E = (E \cap B_r(x)) \cup (E \setminus B_r(x))$ would decompose E into disjoint closed sets. Now by applying Lemma 1.13 to f we see that

$$\mathcal{H}^1(E \cap B_\rho(x)) \geq \mathcal{H}^1(f(E \cap B_\rho(x))) \geq \mathcal{H}^1([0, \rho]) = \mathcal{L}^1([0, \rho]) = \rho$$

$\square$

We are now in a position to deduce the *basic density property* of rectifiable curves.

Lemma 2.14. A rectifiable curve is a regular 1-set.

Proof. Let Γ be a rectifiable curve. By definition $\mathcal{L}^1(\Gamma) < \infty$ and by Lemma 2.11 $\mathcal{H}^1(\Gamma) < \infty$. Γ contains at least two distinct points so by Lemma 2.13 $\mathcal{H}^1(\Gamma) > 0$ and thus Γ is a 1-set.

Let now x be a point other than an endpoint of Γ and suppose x divides Γ into two rectifiable subcurves Γ_+ and Γ_- . By Lemma 2.13 $\mathcal{H}^1(\Gamma_+ \cap B_\rho(x)) \geq \rho$ and $\mathcal{H}^1(\Gamma_- \cap B_\rho(x)) \geq \rho$ if ρ is sufficiently small, from which we obtain $\mathcal{H}^1(\Gamma \cap B_\rho(x)) \geq 2\rho$. Thus, at all points other than the endpoints $\underline{D}^1(\Gamma, x) \geq 1$. By Corollary 2.7 since Γ is a 1-set $\overline{D}^1(\Gamma, x) \leq 1$ thus $D^1(\Gamma, x)$ exists and is equal to 1 at almost all $x \in \Gamma$. $\square$

A set with the property described in the following lemma is also known as a *Lipschitz set*.

Lemma 2.15. *Let $E \subset \mathbb{R}^n$ be a bounded set such that if $x, y \in E$ the segment $[x, y]$ makes an angle of at most $\phi < \pi/2$ with a fixed line L , then E is a subset of a rectifiable curve.*

Proof. We may assume that E is closed. Let $\Pi(t)$ be the hyperplane perpendicular to L at a distance t from some origin and let a and b be the extreme values of t for which $\Pi(t)$ intersects E . Then $\Pi(t)$ can contain at most one point of E for every t , let $\psi(t)$ denote this point if it exists, if it does not if $a < t < b$ let $\psi(t)$ be the point of $\Pi(t)$ on the line segment joining the points of E closest to $\Pi(t)$ on either side. If $t_1 \neq t_2$ the segment $[\psi(t_1), \psi(t_2)]$ makes an angle of at most ϕ with L , so $|\psi(t_1) - \psi(t_2)| \leq |t_1 - t_2| / \cos \phi$. By Definition 2.9 ψ is a rectifiable curve. $\square$

We now work towards showing that a continuum of finite $\mathcal{H}^1$ -measure is a countable collection of rectifiable curves, within a set of measure zero, and thus is regular. We use pathwise connectivity to achieve this result.

Lemma 2.16. *A continuum E with $\mathcal{H}^1(E) < \infty$ is pathwise connected.*

Proof. Let $z, w \in E$. Since E is connected we may take for each $\varepsilon > 0$ we may find a chain of points $z = x_0, x_1, \dots, x_m = w$ in E with $|x_i - x_{i-1}| \leq \varepsilon$ for $1 \leq i \leq m$. By deleting intermediate points if necessary we may assume that $|x_i - x_j| > \varepsilon$ if $|i - j| \geq 2$ and therefore no point in $\mathbb{R}^n$ lies simultaneously in more than two of the balls $B_{\frac{1}{2}}(x_i)$. Assuming $m \geq 2$ by Lemma 2.13 we have

$$2\mathcal{H}^1(E) \geq \sum_{i=0}^m \mathcal{H}^1(E \cap B_{\frac{1}{2}}(x_i)) \geq \frac{1}{2}m\varepsilon$$

Letting now Γ_ε denote the not self-intersecting polygonal curve obtained by joining the x_i 's we have

$$\mathcal{L}(\Gamma_\varepsilon) = \sum_{i=1}^m |x_i - x_{i-1}| \leq m\varepsilon \leq 4\mathcal{H}^1(E) \quad (2.6)$$

For $0 \leq t \leq 1$ let $\psi_\varepsilon(t)$ be the point on Γ_ε such that the part of Γ_ε lying between $z = \psi_\varepsilon(0)$ and $\psi_\varepsilon(t)$ has length $t\mathcal{L}(\Gamma_\varepsilon)$. Then by (2.6) if $0 \leq t_1 \leq t_2 \leq 1$

$$|\psi_\varepsilon(t_1) - \psi_\varepsilon(t_2)| \leq \mathcal{L}(\psi_\varepsilon[t_1, t_2]) \leq |t_1 - t_2|4\mathcal{H}^1(E) \quad (2.7)$$

The curves $\{\Gamma_\varepsilon\}_{\varepsilon \leq 1}$ all lie in a bounded subset of $\mathbb{R}^n$ so the family of functions $\{\psi_\varepsilon\}_{\varepsilon \leq 1}$ from $[0, 1]$ to $\mathbb{R}^n$ is uniformly bounded and by (2.7) equicontinuous. By the Arzelà-Ascoli Theorem A.11 such a family is compact i.e. there exists a sequence $\varepsilon_j \rightarrow 0$ and a continuous function $\psi : [0, 1] \rightarrow \mathbb{R}^n$ such that ψ_{ε_j} converges uniformly to ψ on $[0, 1]$. Let $\psi[0, 1] = \Gamma$. We have $\psi(0) = z$ and $\psi(1) = w$. If $x \in \Gamma$ then given $\delta > 0$ we may find a $\varepsilon_j < \delta/2$ such that Γ_{ε_j} meets the ball $B_{\frac{1}{2}\delta}(x)$. Since E is closed $x \in E$ and thus $\Gamma \subseteq E$. Finally, if ψ is injective Γ is a curve, otherwise by Lemma 2.10 we may reduce Γ to a curve joining z and w . $\square$

Lemma 2.17. *Let E be a compact pathwise connected set with $\mathcal{H}^1(E) < \infty$ then E consists of a countable union of rectifiable curves together with a set of $\mathcal{H}^1$ -measure zero.*

Proof. Let us define a sequence of curves $\{\Gamma_j\}$ inductively. Let $\Gamma_1 \subseteq E$ be a curve joining two of the most distant points of E . Suppose now that the curves $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ have been defined. Let $x \in E$ be a point at a maximum distance d_k from $\bigcup_{j=1}^k \Gamma_j$, this maximum is attained since every set involved is compact. If $d_k = 0$ the process terminates, the result follows and we are done. If not let $\Gamma_{k+1} \subseteq E$ be a curve joining x to $\bigcup_{j=1}^k \Gamma_j$ with Γ_{k+1} disjoint from $\bigcup_{j=1}^k \Gamma_j$ except for an endpoint. Now by Lemma 2.13 $\mathcal{H}^1(\Gamma_{k+1}) \geq d_k$ thus

$$\sum_{j=1}^{\infty} d_j \leq \sum_{j=1}^{\infty} \mathcal{H}^1(\Gamma_j) \leq \mathcal{H}^1(E) < \infty \quad (2.8)$$

thus $d_j \rightarrow 0$. If $x \in E$ and the distance d from x to $\bigcup_{j=1}^{\infty} \Gamma_j$ is positive then $d_j < d$ for some j and x would have been chosen as the endpoint of Γ_{j+1} so we reach a contradiction. Therefore we conclude that E is the closure of $\bigcup_{j=1}^{\infty} \Gamma_j$. By Lemma 2.11,

$\mathcal{L}(\Gamma_j) = \mathcal{H}^1(\Gamma_j) \leq \mathcal{H}^1(E) < \infty$ thus Γ_j is a rectifiable curve for each j . Finally, for every k $\bigcup_1^k \Gamma_j$ is closed so

$$\mathcal{V} = \{B_r(x) : x \in E \setminus \bigcup_1^k \Gamma_j \text{ and } B_r(x) \cap \bigcup_1^k \Gamma_j = \emptyset\}$$

is a Vitali class of balls for $E \setminus \bigcup_1^k \Gamma_j$. Now let x be the center of some $B \in \mathcal{V}$, since x lies in the closure of $\bigcup_{k+1}^\infty \Gamma_j$ there are points in this union arbitrarily close to x . These points are connected via a sequence of arcs in $\bigcup_{k+1}^\infty \Gamma_j$ to a point of $\bigcup_1^k \Gamma_j$ outside of B , thus by Lemma 2.13

$$\frac{1}{2}|B| \leq \mathcal{H}^1(B \cap \bigcup_{k+1}^\infty \Gamma_j) \quad \forall B \in \mathcal{V}$$

Thus, given $\varepsilon > 0$, by Vitali's Theorem 1.17 we may choose a disjoint sequence of balls $\{B_i\} \subseteq \mathcal{V}$ such that

$$\mathcal{H}^1(E \setminus \bigcup_1^k \Gamma_j) \leq \sum_i |B_i| + \varepsilon \leq 2 \sum_i \mathcal{H}^1(B_i \cap \bigcup_{k+1}^\infty \Gamma_j) + \varepsilon \leq 2 \mathcal{H}^1(\bigcup_{k+1}^\infty \Gamma_j) + \varepsilon$$

thus

$$\mathcal{H}^1(E \setminus \bigcup_1^k \Gamma_j) \leq 2 \mathcal{H}^1(\bigcup_{k+1}^\infty \Gamma_j)$$

By (2.8) the right-hand side of this inequality tends to zero as $k \rightarrow \infty$ so

$$\mathcal{H}^1(E \setminus \bigcup_1^k \Gamma_j) = 0$$

as required. □

Combining these two last lemmas this next Theorem follows immediately.

Theorem 2.18. *Let E be a continuum with $\mathcal{H}^1(E) < \infty$. Then E consists of a countable union of rectifiable curves together with a set of $\mathcal{H}^1$ -measure zero.*

2.3 Structure of 1-sets

Let us now center our attention on 1-sets, or *linearly measurable sets*, on a general note, considering as well non-regular sets.

Definition 2.19. A 1-set contained in a countable union of rectifiable curves is termed a *Y-set*.

Corollary 2.20. A *Y-set* is a regular 1-set.

Proof. Let $E = \bigcup_j E_j$ be a countable disjoint union of s -sets with $\mathcal{H}^s(E) < \infty$. By Corollary 2.8 and Lemma 2.14 a 1-set contained in a rectifiable curve has density 1 almost everywhere. The same is true for a *Y-set* applying Corollary 2.8 to the set E_k . $\square$

Definition 2.21. A 1-set is called a *Z-set* if its intersection with every rectifiable curve is of $\mathcal{H}^1$ -measure zero.

From Theorem 2.18 and the definition of a *Z-set* just given follows the next lemma.

Lemma 2.22. The intersection of a *Z-set* with a continuum of finite $\mathcal{H}^1$ -measure is of measure zero.

By Corollary 2.20 an irregular is a *Z-set*. The rest of the section is dedicated to showing that a *Z-set* has upper density less than 1 almost everywhere and so it is irregular. We first require a result a topological result on removing discs from continua, which refer as the Gruyère cheese lemma.

Lemma 2.23. Let $E \subseteq \mathbb{R}^2$ be a continuum. Let $\{B_i\}_{i=1}^\infty$ be a countable semidisjoint collection of closed discs each contained in E and such that for any $d > 0$ $\text{diam}(B_i) \geq d$ is only true for finitely many i . Then if Γ_i is the perimeter of B_i $F = (E \setminus \bigcup_i B_i) \cup \bigcup_i \Gamma_i$ is a continuum.

Proof. We will first show that F is closed, since $F = (E \setminus \bigcup_i \text{int } B_i) \cup \bigcup_i \Gamma_i$ it is enough to show that $\overline{\bigcup_i \Gamma_i} \subseteq F$. If $x \in \overline{\bigcup_i \Gamma_i} \setminus (E \setminus \bigcup_i \text{int } B_i) \subseteq \bigcup_i \text{int } B_i$ then $x \in \text{int } B_k$ for some k fixed. Let $d > 0$ be the distance from x to the perimeter of B_k . If the perimeter Γ_j intersects $B_{\frac{1}{2}d}(x)$ then B_j meets both $B_{\frac{1}{2}d}(x)$ and $\mathbb{R}^2 \setminus B_k$ since $B_j \not\subseteq B_k$, therefore $\text{diam}(\Gamma_j) \geq d/2$ and thus only finitely many circles Γ_j can

intersect $B_{\frac{1}{2}d}(x)$. Since $x \in \overline{\bigcup_i \Gamma_i}$ there is an n such that $x \in \overline{\bigcup_{i=1}^n \Gamma_i} = \bigcup_{i=1}^n \Gamma_i \subseteq F$ as we wanted to see. To prove now that F is connected suppose that $F = F_1 \cup F_2$ where F_1 and F_2 are disjoint closed sets. Let

$$E_1 = F_1 \cup \bigcup_{\{i: \Gamma_i \subseteq F_1\}} B_i, \quad E_2 = F_2 \cup \bigcup_{\{i: \Gamma_i \subseteq F_2\}} B_i$$

Since every Γ_i is contained in either E_1 or E_2 we have $E_1 \cup E_2 = E$. Since the discs $\{B_i\}$ are semidisjoint and for each i $F \cap B_i \subseteq \bigcup_j \Gamma_j$ we can conclude that $E_1 \cap E_2 = \emptyset$. Let us now see that E_1 and E_2 are closed. If $x \in \partial(\bigcup_{\{i: \Gamma_i \subseteq F_1\}} B_i)$ then

x is either the limit of a sequence of points in $\bigcup_{i=1}^k B_i$ for some k or the limit point of a sequence of discs with boundaries lying in F_1 and with radii tending to zero, in both cases $x \in F_1$ hence E_1 is closed. By the same argument E_2 is also closed, so $E = E_1 \cup E_2$ is a disjoint composition of E into closed sets but since E is a continuum, it is connected thus either E_1 or E_2 is empty therefore either F_1 or F_2 is empty. $\square$

Definition 2.24. Let $x, y \in \mathbb{R}^2$. We denote with $R(x, y) = \text{int}(B_{|x-y|}(x) \cap B_{|x-y|}(y))$ the *common region* of the circle-pair with centers x and y , where a circle-pair is a pair of circles of equal radii where the center of one is on the perimeter of the other and viceversa.

We now use this definition to see that if the common region of a circle-pair with centers in a 1-set E contains a lot of E then it is possible to join up the different components of E by curves which lie in E to a non-negligible extent. The proof of the upper bound for the lower densities of a Z-set relies heavily on this result.

Lemma 2.25. Let $E \subseteq \mathbb{R}^2$ be a 1-set and let $\alpha > 0$. Let E_0 be a compact subset of E with $\mathcal{H}^1(E_0) > 0$ such that $\forall x_1, x_2 \in E_0$ $\mathcal{H}^1(E \cap R(x_1, x_2)) \geq \alpha|x_1 - x_2|$. Then there exists a continuum H such that $0 < \mathcal{H}^1(H \cap E) \leq \mathcal{H}^1(H) < \infty$.

Proof. Using Corollary 2.7 we may find a $\rho_1 > 0$ and a compact set $F \subseteq E_0$ with $\mathcal{H}^1(F) > 0$ such that

$$\mathcal{H}^1(B_r(x) \cap E) \leq 2 \cdot 2r \quad \text{if } x \in F \text{ and } 0 < r \leq \rho_1 \quad (2.9)$$

By corollaries 2.6 and 2.7 there exists a point $y \in F$ and a number ρ with $0 < \rho \leq \rho_1/6$ such that

$$\mathcal{H}^1((E \setminus F) \cap B_r(y)) < 2r \cdot 10^{-3}\alpha \quad (0 < r \leq 3\rho) \quad (2.10)$$

and

$$\mathcal{H}^1(F \cap B_\rho(y)) > \frac{\rho}{2} \quad (2.11)$$

By reducing ρ if necessary we may assume that the perimeter Γ of the disc $B_\rho(y)$ contains some point of F . Now let $\mathcal{C}$ be the family of closed discs given by

$$\mathcal{C} = \{B_r(x) : x \in F \cap B_\rho(y), 0 < r < 2 \text{ and } \mathcal{H}^1((E \setminus F) \cap B_r(x)) \geq \alpha r\} \quad (2.12)$$

Using Lemma 1.16 we may find a null, finite or countable disjoint collection of closed discs $\{B_i\}$ in $\mathcal{C}$ such that $\bigcup_{B \in \mathcal{C}} B \subseteq \bigcup_i B'_i$ where B'_i is the closed disc concentric with B_i and of three times the radius with $\{B'_i\}$ semidisjoint. Let Γ_i be the perimeter of B'_i for each i . Letting

$$G = (F \cap B_\rho(y)) \cup \Gamma \cup \bigcup_i B'_i$$

we define

$$H = (G \setminus \bigcup_i B'_i) \cup \bigcup_i \Gamma_i$$

Our objective now is to show that H is a continuum with the stated properties. We will do so in the following six steps.

(i) **G is closed:** Since both $\Gamma \cap B_\rho(y)$ and Γ are closed it is enough to see that

$\bigcup_i \overline{B'_i} \subseteq G$. If $z \in \bigcup_i \overline{B'_i}$ then either $z \in \bigcup_{i=1}^k B'_i \subseteq G$ for some $k < \infty$ or z is the limit of points from a subsequence of the discs $\{B'_i\}_i$. On summing areas $\sum_i \text{diam}(B_i)^2 < \infty$ so $\text{diam}(B'_i) \rightarrow 0$ as $i \rightarrow \infty$ thus z is the limit point of the centers of these discs which all lie in the closed set $F \cap B_\rho(y)$.

(ii) **G is connected:** Suppose that $G = G_1 \cup G_2$ where G_1 and G_2 are non-empty disjoint closed sets. Any connected subset of G must be in either G_1 or G_2 , assume that $\Gamma \subseteq G_1$. Since each B'_i has center in $B_\rho(y)$ if B'_i contains points on or outside Γ it must intersect Γ and thus be contained in G_1 therefore $G_2 \subseteq \text{int } B_\rho(y)$. Let G'_1 be $G'_1 = G_1 \cup (\mathbb{R}^2 \setminus B_\rho(y))$. G'_1 is closed, connected and disjoint from G_2 . Since both G'_1 and G_2 contain points of F , let $x_1 \in F \cap G'_1$ and $x_2 \in F \cap G_2$ be points that minimize the distance $|x_1 - x_2|$. This infimum is attained and is positive. Since $\Gamma \subseteq G'_1$ contains a point of F we have that $0 < r < 2\rho$. The common region $R(x_1, x_2)$ is disjoint from F , otherwise r would not be the infimum, thus from the hypotheses of the lemma

$$0 < \alpha r \leq \mathcal{H}^1(E \cap R(x_1, x_2)) = \mathcal{H}^1((E \setminus F) \cap R(x_1, x_2)) \leq \mathcal{H}^1((E \setminus F) \cap B_r(x_2))$$

Since $x_2 \in F \cap B_\rho(y)$ and $r < 2\rho$, $B_r(x_2) \in \mathcal{C}$ and so $B_r(x_2) \subseteq \bigcup_i B'_i \subseteq G$, thus $B_r(x_2) = (B_r(x_2) \cap G'_1) \cup (B_r(x_2) \cap G_2)$ is a decomposition of $B_r(x_2)$

into disjoint closed sets with x_1 in the first and x_2 in the second. $B_r(x_2)$ is obviously connected so we have reached an absurd.

(iii) **H is a continuum:** By (i) and (ii) G is a continuum and by lemma 2.23 H is a continuum as well.

(iv) $\sum_i \text{diam}(B'_i) \leq \frac{1}{8}\rho$: By (2.12), the fact that the collection of discs $\{B_i\}$ is disjoint and (2.10) we see that

$$\begin{aligned} \sum_i \text{diam}(B'_i) &= 3 \sum_i \text{diam}(B_i) \leq \frac{6}{\alpha} \sum_i \mathcal{H}^1((E \setminus F) \cap B_i) \\ &\leq \frac{6}{\alpha} \mathcal{H}^1((E \setminus F) \cap B_{3\rho}(y)) \leq \frac{6}{\alpha} 6\rho 10^{-3} \alpha \leq \frac{1}{8}\rho \end{aligned}$$

$\mathcal{H}^1(H) < \infty$: $H \subseteq E \cup \Gamma \cup \bigcup_i \Gamma_i$ using (iv) we have

$$\mathcal{H}^1(H) \leq \mathcal{H}^1(E) + 2\pi\rho + \pi \sum_i \text{diam}(B'_i) < \infty$$

(vi) $\mathcal{H}^1(H \cap E) > 0$: from the definition of H and G we know

$$\begin{aligned} \mathcal{H}^1(H \cap E) &= \mathcal{H}^1((G \setminus \bigcup_i B_i) \cap E) \\ &= \mathcal{H}^1(((F \cap B_\rho(y)) \cup \Gamma) \cap E) - \mathcal{H}^1(\bigcup_i B'_i \cap ((F \cap B_\rho(y)) \cup \Gamma) \cap E) \end{aligned}$$

Since $F \subseteq E_0 \subseteq E$ we have that

$$\mathcal{H}^1(((F \cap B_\rho(y)) \cup \Gamma) \cap E) \geq \mathcal{H}^1(F \cap B_\rho(y))$$

and obviously

$$\mathcal{H}^1(\bigcup_i B'_i \cap ((F \cap B_\rho(y)) \cup \Gamma) \cap E) \leq \mathcal{H}^1(\bigcup_i B'_i \cap E)$$

thus by (2.9), (2.11) and (iv) we have

$$\begin{aligned} \mathcal{H}^1(H \cap E) &\geq \mathcal{H}^1(F \cap B_\rho(y)) - \mathcal{H}^1(E \cap \bigcup_i B'_i) \\ &\geq \mathcal{H}^1(F \cap B_\rho(y)) - \sum_i \mathcal{H}^1(E \cap B'_i) \\ &\geq \mathcal{H}^1(F \cap B_\rho(y)) - 2 \sum_i \text{diam}(B'_i) \\ &\geq \frac{1}{2}\rho - \frac{1}{4}\rho = \frac{1}{4}\rho > 0 \end{aligned}$$

□

The proof of the theorem on the lower densities of Z-sets is now a simple consequence of the geometry of circle-pairs.

Theorem 2.26. *Let $E \subseteq \mathbb{R}^2$ be a Z-set. Then $\underline{D}^1(E, x) \leq \frac{3}{4}$ at almost all $x \in E$.*

Proof. Let us suppose that for some $\alpha > 0$ the set $E_1 = \{x : \underline{D}^1(E, x) > \frac{3}{4} + \alpha\}$ has positive measure. Using Theorem 2.5 we are able to find a $\rho > 0$ and a compact 1-set $E_2 \subseteq E_1$ such that

$$\mathcal{H}^1(E \cap B_r(x)) > \left(\frac{3}{4} + \alpha\right)2r \quad (x \in E_2, 0 < r \leq \rho) \quad (2.13)$$

and

$$\mathcal{H}^1(E \cap U) < (1 + \alpha) \text{diam}(U) \quad (x \in (E_2 \cap U), 0 < \text{diam}(U) \leq 3\rho) \quad (2.14)$$

Let E_0 be a compact subset of E_2 such that $0 < \mathcal{H}^1(E_0) < \infty$ and $\text{diam}(E_0) \leq \rho$ and let $x_1, x_2 \in E_0$. Note that $r = |x_1 - x_2| \leq \rho$ and that any circle intersects E in a set of measure zero since it is a Z-set and thus we have

$$\begin{aligned} \mathcal{H}^1(E \cap R(x_1, x_2)) &= \mathcal{H}^1(E \cap B_r(x_1)) + \mathcal{H}^1(E \cap B_r(x_2)) \\ &\quad - \mathcal{H}^1(E \cap (B_r(x_1) \cup B_r(x_2))) \\ &\geq 2\left(\frac{3}{4} + \alpha\right)2r - (1 + \alpha)3r = \alpha r = \alpha|x_1 - x_2| \end{aligned}$$

where we have used (2.13) and (2.14) with $U = B_r(x_1) \cap B_r(x_2)$. Applying Lemma 2.25 to $E_0 \subseteq E$ we know that there exists a continuum H with $0 < \mathcal{H}^1(H \cap E) \leq \mathcal{H}^1(H) < \infty$, but E is a Z-set and we know this to be impossible by Lemma 2.22 and thus we conclude that if $\alpha > 0$, $\mathcal{H}^1(E_1) = 0$. □

We have shown that regular sets have a lower density of 1 while irregular sets have a lower density of at most $3/4$ everywhere. It is not possible to construct a 1-set with lower density strictly between $3/4$ and 1 on a set of positive measure.

We can now characterize regular and irregular 1-sets.

Theorem 2.27. *An irregular 1-set is a Z-set. A regular 1-set consists of a Y-set together with a set of $\mathcal{H}^1$ -measure zero.*

Proof. Let $E \subseteq \mathbb{R}^2$ be a 1-set. If E is irregular then E intersects every rectifiable curve Γ in a set of measure zero, if this were not the case, $E \cap \Gamma$ would be a Y -set and so by Corollary 2.20 would be regular.

If E is regular then $\underline{D}^1(E, x) = 1$ almost everywhere, so using Theorem 2.26 together with Corollary 2.8 we have that any measurable subset of E of positive measure intersects some rectifiable curve in a set of positive measure. Using this fact we define a sequence of rectifiable curves $\{\Gamma_j\}$ in the following way: Choose Γ_1 so that

$$\mathcal{H}^1(\Gamma_1 \cap E) \geq \frac{1}{2} \sup\{\mathcal{H}^1(\Gamma \cap E) : \Gamma \text{ is rectifiable}\} > 0$$

If we have selected $\Gamma_1, \dots, \Gamma_k$ and $E_k = \bigcup_{j=1}^k \Gamma_j$ has positive measure, let Γ_{k+1} be a rectifiable curve with

$$\mathcal{H}^1(\Gamma_{k+1} \cap E_k) \geq \frac{1}{2} \sup\{\mathcal{H}^1(\Gamma \cap E_k) : \Gamma \text{ is rectifiable}\} > 0$$

Note that the process only terminates if for some k the curves $\{\Gamma_j\}_1^k$ cover almost all of E in which case we are done. Otherwise $\infty > \mathcal{H}^1(E) \geq \sum_k \mathcal{H}^1(\Gamma_{k+1} \cap E_k)$

thus as k increases $\mathcal{H}^1(\Gamma_{k+1} \cap E_k) \rightarrow 0$. If $\mathcal{H}^1(E \setminus \bigcup_{j=1}^{\infty} \Gamma_j) > 0$ then we are able to find a rectifiable curve Γ such that $\mathcal{H}^1(\Gamma \cap (E \setminus \bigcup_{j=1}^{\infty} \Gamma_j)) = d > 0$. But for some k we have that $\mathcal{H}^1(\Gamma_{k+1} \cap E_k) < d/2$ and Γ would have been selected in preference to Γ_{k+1} , hence $\mathcal{H}^1(E \setminus \bigcup_{j=1}^{\infty} \Gamma_j) = 0$ so E consists of the Y -set $\bigcup_{j=1}^{\infty} (E \cap \Gamma_j)$ and a set of measure zero. $\square$

2.4 Angular densities

The densities we have given are isotropic but for some results of the next chapter we will be interested in defining densities that only 'care' about a certain direction, that's why we introduce *angular densities*.

Definition 2.28. Let θ be a unit vector and ϕ an angle. Let then $S(x, \theta, \phi)$ be the closed one-way infinite cone with vertex x and axis in direction θ consisting of the points y for which the segment $[x, y]$ makes at most an angle ϕ with θ . We denote $S_r(x, \theta, \phi) = S(x, \theta, \phi) \cap B_r(x)$. The *upper* and *lower angular densities* of E at x are defined as

$$\overline{D}^s(E, x, \theta, \phi) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^s(E \cap S_r(x, \theta, \phi))}{(2r)^s}$$

$$\underline{D}^s(E, x, \theta, \phi) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^s(E \cap S_r(x, \theta, \phi))}{(2r)^s}$$

Lemma 2.29. *Let $\theta \in \mathbb{R}^2$ be a unit vector perpendicular to a line L . Let P be a parallelogram with sides making angles ϕ to directions $\pm\theta$ and let y and z be opposite vertices of P . Then $|y - z| \leq d / \sin \phi$ where d is the length of the projection of P onto L .*

Proof. Let w be a third vertex of P . We have $|y - z| \leq |y - w| + |w - z|$ and thus $d = (|y - w| + |w - z|) \sin \phi \geq |y - z| \sin \phi$. $\square$

Theorem 2.30. *Let $E \subseteq \mathbb{R}^2$ be an irregular 1-set. Then, given θ and $0 < \phi < \pi/2$, we have*

$$\overline{D}^1(E, x, \theta, \phi) + \overline{D}^1(E, x, -\theta, \phi) \geq \frac{1}{6} \sin \phi \quad (2.15)$$

for almost all $x \in E$.

Proof. Take $\rho, \delta_+, \delta_- > 0$ and let $F_0 = F_0(\rho, \delta_+, \delta_-)$ be the set of $x \in E$ for which both $\mathcal{H}^1(E \cap S_r(x, \theta, \phi)) \leq 2r\delta_+$ and $\mathcal{H}^1(E \cap S_r(x, -\theta, \phi)) \leq 2r\delta_-$ hold for all $r \leq \rho$. We will now show that if F_0 has positive measure then $\delta_+ + \delta_-$ cannot be too small.

If $\mathcal{H}^1(F_0) > 0$ we may find a compact subset $F_1 \subseteq F_0$ with positive measure. Given an $\eta > 0$ we may find, from the definition of $\mathcal{H}^1$, a closed convex set U with $0 < \text{diam}(U) \leq \rho$ and

$$\mathcal{H}^1(F) > (1 - \eta) \text{diam}(U) \quad (2.16)$$

where $F = F_1 \cap U$. From now on we will work inside U . Since F is closed we may choose $y_1, z_1 \in F$ to be among the most distant pair of points in F which have their connecting segment $[y, z]$ at an angle of not more than θ with θ , so that

$$r_1 = |y_1 - z_1| = \sup\{|y - z| : z \in F \cap S(y, \theta, \phi) \text{ and } y \in F \cap S(y, -\theta, \phi)\}$$

where $S(y, \theta, \phi)$ denotes the infinite sector. By the maximality of r_1 we have that

$$F \cap S(y_1, \theta, \phi) = F \cap S_{r_1}(y_1, \theta, \phi)$$

$$F \cap S(z_1, -\theta, \phi) = F \cap S_{r_1}(z_1, -\theta, \phi)$$

Since $r_1 \leq \rho$ and $F \subseteq E$ we conclude from the definition of F_0 that

$$\mathcal{H}^1(F \cap S(y_1, \theta, \phi)) \leq 2r_1\delta_+ \quad (2.17)$$

$$\mathcal{H}^1(F \cap S(z_1, -\theta, \phi)) \leq 2r_1\delta_-$$

Let now P_1 be the closed parallelogram $P_1 = S(y_1, \theta, \phi) \cap S(z_1, -\theta, \phi)$ and let Q_1 be the open region $\text{int}\{S(y_1, \theta, \phi) \cup S(z_1, -\theta, \phi)\}$. From (2.17) we have $\mathcal{H}^1(F \cap Q_1) \leq 2r(\delta_+ + \delta_-)$.

The set $U \setminus Q_1$ is compact and has at most two components, so provided it contains a pair of points of F in the same component with the joining segment at an angle at most ϕ with θ we may, in the same way, obtain a parallelogram P_2 outside Q_1 and an open region Q_2 . Thus for $j \geq 1$ natural we find points $y_j, z_j \in F$ lying in the same component of $U \setminus \bigcup_{i=1}^{j-1} Q_i$, with $r_j = |y_j - z_j|$ as large as possible and with

$$\mathcal{H}^1(F \cap Q_j) \leq 2r_j(\delta_+ + \delta_-) \quad (2.18)$$

where $Q_j = \text{int}\{S(y_j, \theta, \phi) \cup S(z_j, -\theta, \phi)\}$ and $P_j = S(y_j, \theta, \phi) \cap S(z_j, -\theta, \phi)$. This process continues either indefinitely or until no suitable pair of points is left.

Let L be a line perpendicular to θ . If $j > i$ then P_j is disjoint from Q_i so the projections of the parallelograms $\{P_j\}$ onto L are segments with disjoint interiors. Using Lemma 2.29 we see that

$$\sum_j r_j = \sum_j |y_j - z_j| \leq \frac{3 \text{diam}(U)}{\sin \phi} \quad (2.19)$$

Let $y, z \in F \setminus \bigcup_i Q_i$ be distinct points. If y and z lie on opposite sides of some Q_j then $[y, z]$ makes an angle of at least ϕ with θ . If $[y, z]$ were to make an angle of less than ϕ with θ then y and z would have been candidates for y_j and z_j at each stage and so the process of selection could not have terminated and $|y - z| \leq |y_j - z_j|$ for every j . By (2.19), $|y_j - z_j| \rightarrow 0$ so we know this to be impossible. We thus conclude that the angle made with L by the segment joining any pair of points of $F \setminus \bigcup_i Q_i$ is at most $\pi/2 - \phi$. By Lemma 2.15 $F \setminus \bigcup_i Q_i$ is a subset of a rectifiable curve. Thus as $F \setminus \bigcup_i Q_i$ is a subset of E , which by Theorem 2.27 is a Z-set, we know $\mathcal{H}^1(F \setminus \bigcup_i Q_i)$ to be true. Hence using (2.18), (2.19) and (2.16)

$$\begin{aligned} \mathcal{H}^1(F) &\leq \sum_j \mathcal{H}^1(F \cap Q_j) \leq 2(\delta_+ + \delta_-) \sum_j r_j \\ &\leq \frac{6(\delta_+ + \delta_-) \text{diam}(U)}{\sin \phi} \leq \frac{6(\delta_+ + \delta_-)}{(1 - \eta) \sin \phi} \mathcal{H}^1(F) \end{aligned}$$

Given any $\eta > 0$ there is some F with $\mathcal{H}^1(F) > 0$ for which this holds, thus $(\delta_+ + \delta_-) \geq \frac{1}{6} \sin \phi$. If $(\delta_+ + \delta_-) < \frac{1}{6} \sin \phi$ the set $F_0(\delta_+, \delta_-, \rho)$ must have measure zero so the proof is complete. $\square$

Corollary 2.31. *Let $E \subseteq \mathbb{R}^2$ be an irregular 1-set. For almost all $x \in E$*

$$\overline{D}^1(E, x, \theta, \phi) + \overline{D}^1(E, x, -\theta, \phi) \geq \frac{1}{6} \sin \phi > \frac{1}{10} \phi \quad (2.20)$$

for all θ and all $0 < \phi < \pi/2$.

Proof. By Theorem 2.30, inequality (2.15) holds for some countable dense subset of $\{(\theta, \phi)\}$ for almost all x . Since it holds for the rationals and upper angular densities $\overline{D}^1(E, x, \theta, \phi)$ are lower-semicontinuous functions with respect to (θ, ϕ) , we can metrically approximate any pair of (θ, ϕ) by a pair of (θ, ϕ) rationals, in the limit the inequality holds, hence (2.15) holds for all θ and ϕ for almost all x . $\square$

Chapter 3

Construction of Besicovitch and Kakeya sets

3.1 Geometric construction

This section describes the construction of a Besicovitch set that is solution to the Kakeya problem from a geometrical point of view. The basic idea is to form a 'Perron tree', a figure obtained by repeatedly splitting an equilateral triangle of unit height by dividing up the base and sliding them so that the resulting figure reduces its area while still containing unit segments in all directions at angles at least 60° from the base.

Definition 3.1. A set $\mathcal{K} \subseteq \mathbb{R}^n$ of Lebesgue measure zero containing a line segment in every direction is termed a *Besicovitch set*.

Lemma 3.2. Let T_1 and T_2 be adjacent triangles with bases on a line L , with bases of lengths b and of heights h . Take $1/2 < \alpha < 1$ and slide T_2 a distance $2(1 - \alpha)b$ along L so that it overlaps with T_1 . The resulting figure S consists of a triangle T homothetic to triangle $T_1 \cup T_2$, i.e. one can be mapped onto the other through a contraction, with $\mathcal{L}^2(T) = \alpha^2 \mathcal{L}^2(T_1 \cup T_2)$ and two auxiliary triangles (from now on ears). Replacing S with $T_1 \cup T_2$ yields us a reduction in area is $\mathcal{L}^2(T_1 \cup T_2) - \mathcal{L}^2(S) = (1 - \alpha)(3\alpha - 1)\mathcal{L}^2(T_1 \cup T_2)$.

Proof. $T_1 \cup T_2$ has base $2b$ and T has base $2\alpha b$. Since they are similar triangles their areas are related by a α^2 factor. Now let's find the area of the ears. The area of the parallelogram of base $2(1 - \alpha)b$ and height h is $2(1 - \alpha)bh$. Now

$$\mathcal{L}^2(T_1 \cup T_2) - \mathcal{L}^2(T) = (1 - \alpha^2)\mathcal{L}^2(T_1 \cup T_2) = (1 - \alpha^2)bh$$

is the area of the parallelogram without the triangle between the ears so the area of this figure is

$$2(1 - \alpha)b - (1 - \alpha^2)bh = (1 - \alpha)^2bh$$

Since it has a base of $2(1 - \alpha)b$ it has a height of $(1 - \alpha)h$ which is in turn half the base of an ear. Therefore the area of both ears is $2(1 - \alpha)^2hb$. The total area of S is $(3\alpha^2 - 4\alpha + 1)bh$ and thus

$$\mathcal{L}^2(T_1 \cup T_2) - \mathcal{L}^2(S) = (1 - \alpha)(3\alpha - 1)bh = (1 - \alpha)(3\alpha - 1)\mathcal{L}^2(T_1 \cup T_2)$$

□

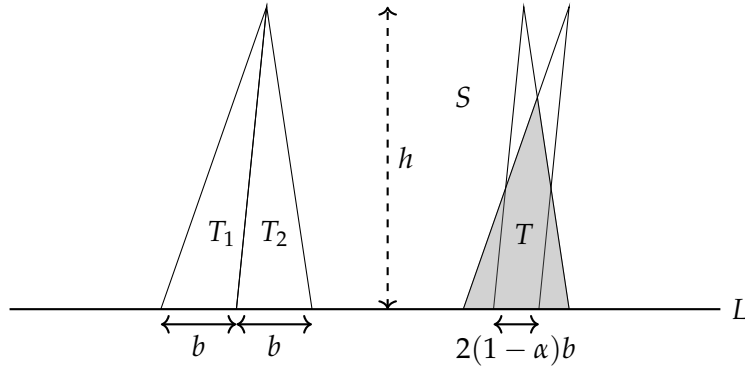


Figure 3.1: Sliding of the triangles in Lemma 3.2.

Theorem 3.3. *Let T be a triangle with base on a line L . Divide the base of T into 2^k equal segments and join each point of division to the opposite vertex to form 2^k elementary triangles $\{T_1, T_2, \dots, T_{2^k}\}$. By choosing k large enough the elementary triangles can be translated along L such that the resulting closed figure S is arbitrarily small. Furthermore, Let $V \supseteq T$ be an open set, this may be achieved such that $S \subseteq V$ is verified.*

Proof. To construct this figure we apply repeatedly Lemma 3.2 for a fixed value of $1/2 < \alpha < 1$ to be determined later. For the first step we operate with consecutive pairs of elementary triangles. $\forall i \ 1 \leq i \leq 2^k$ slide T_{2i} along L relative to T_{2i-1} so that a figure S_i^1 is obtained consisting of a triangle T_i^1 homothetic to $T_{2i} \cup T_{2i-1}$ and two ears. By Lemma 3.2 this can be done such that

$$\mathcal{L}^2(T_i^1) = \alpha^2 \mathcal{L}^2(T_{2i} \cup T_{2i-1})$$

and

$$\mathcal{L}^2(T_{2i} \cup T_{2i-1}) - \mathcal{L}^2(S_i^1) = (1 - \alpha)(3\alpha - 1)\mathcal{L}^2(T_{2i} \cup T_{2i-1})$$

Now for the second step of the construction we work with consecutive S_i^1 . $\forall 1 \leq i \leq 2^{k-2}$ slide S_{2i}^1 relative to S_{2i-1}^1 to get S_i^2 . Since a side of T_{2i-1}^1 is parallel and equal to the opposite side of T_{2i}^1 by Lemma 3.2 we can do this such that S_i^2 includes a triangle T_i^2 verifying

$$\mathcal{L}^2(T_i^2) = \alpha^2[\mathcal{L}^2(T_{2i-1}^1) + \mathcal{L}^2(T_{2i}^1)]$$

The overlap of T_{2i-1}^1 and T_{2i}^1 results in a reduction of area of at least

$$\begin{aligned} & (1 - \alpha)(3\alpha - 1)[\mathcal{L}^2(T_{2i-1}^1) + \mathcal{L}^2(T_{2i}^1)] \\ &= (1 - \alpha)(3\alpha - 1)\alpha^2\mathcal{L}^2(T_{4i-3} \cup T_{4i-2} \cup T_{4i-1} \cup T_{4i}) \end{aligned}$$

Proceeding in this way, at the $(r + 1)$ th step we obtain S_i^{r+1} by sliding along L S_{2i}^r with respect to $S_{2i-1}^r \forall i 1 \leq i \leq 2^{k-r}$ such that the overlap of triangles T_{2i-1}^r and T_{2i}^r results in an area reduction of at least $(1 - \alpha)(3\alpha - 1)\alpha^{2r}$ times the total area of all the elementary triangles moved to form S_{2i-1}^r and S_{2i}^r . At the end of this process we end up with a single figure S_1^k which we term S . Comparing the areas of $\bigcup_i S_i^r$ for successive values of r we obtain

$$\begin{aligned} \mathcal{L}^2(S) &\leq \mathcal{L}^2(T) - (1 - \alpha)(3\alpha - 1)(1 + \alpha^2 + \dots + \alpha^{2(k-1)})\mathcal{L}^2(T) \\ &= (1 - \frac{(3\alpha - 1)(1 - \alpha^{2k})}{(1 + \alpha)})\mathcal{L}^2(T) \end{aligned}$$

By taking α close enough to 1 we keep $(3\alpha - 1)/(1 + \alpha)$ close to 1 and then by taking k large enough we can make $\mathcal{L}^2(S)$ arbitrarily small.

Now let b be the length of the base of the triangle T . Fix T_1 and locate the rest of the elementary triangles in relation to it. Note that at the end of the construction no triangle will have moved a distance greater than b . So by subdividing the original triangle T into triangles of base length ε and applying the process above described on all of these triangles we obtain a figure of arbitrarily small area where none of the elementary triangles has moved more than ε . By choosing ε small enough we can guarantee that this figure will be contained inside the open set V . □

Using Theorem 3.1 we are now able to construct a Besicovitch set.

Theorem 3.4. *There exists a plane set of Lebesgue measure zero which contains a unit segment in every direction.*

Proof. Let F be a set of measure zero containing unit segments in all directions in a 60° sector. Taking the union of F with two congruent copies of itself rotated

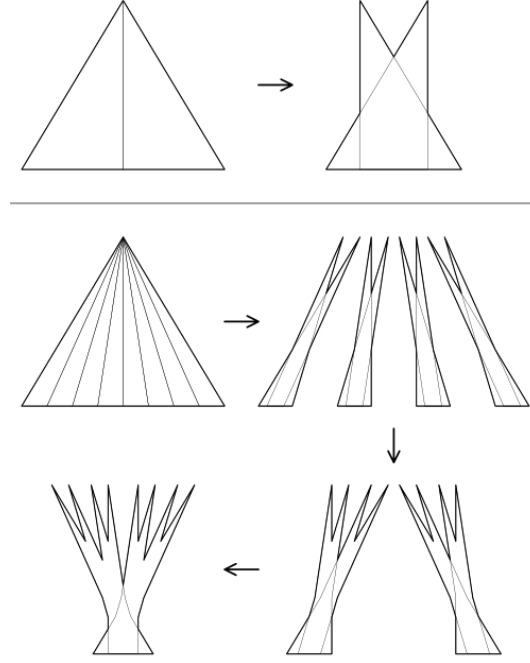


Figure 3.2: Construction of a Perron tree by dividing a triangle into 2^n subtriangles and sliding them. Cases $n = 1$ and $n = 3$ are depicted [9].

60° and 120° respectively we obtain a Besicovitch set. Let us now construct F by applying repeatedly Theorem 3.1.

Let S_1 be an equilateral triangle of unit height with base on a line L . Let V_1 be an open set such that $S_1 \subseteq V_1$ and $\mathcal{L}^2(\bar{V}_1) \leq 2\mathcal{L}^2(S_1)$. By Theorem 3.1 we know we can divide S_1 into a number of elementary triangles of unit height and base on the line L such that by sliding these elementary triangles along L we form a closed set S_2 that verifies both $S_2 \subseteq V_1$ and $\mathcal{L}^2(S_2) \leq 2^{-2}$. Since S_2 is a finite union of triangles there is an open set V_2 such that $S_2 \subseteq V_2 \subseteq V_1$ and $\mathcal{L}^2(\bar{V}_2) \leq 2\mathcal{L}^2(S_2)$. We may now split up S_2 into new elementary triangles which may be slid along L resulting in a new set S_3 contained in V_2 and verifying $\mathcal{L}^2(S_3) \leq 2^{-3}$. We enclose now S_3 inside an open set V_3 with $\mathcal{L}^2(\bar{V}_3) \leq 2\mathcal{L}^2(S_3)$. Continuing to proceed in the same way we obtain a sequence of sets $\{S_i\}$, each being a finite union of elementary triangles with base on L and of unit height, and a sequence of open sets $\{V_i\}$ such that

$$S_i \subseteq V_i \subseteq V_{i-1} \quad \forall i$$

$$\mathcal{L}^2(\bar{V}_i) \leq 2\mathcal{L}^2(\bar{S}_i) \leq 2^{-i+1} \quad \forall i$$

Let F be the closed set $\bigcap_{i=1}^{\infty} \bar{V}_i$. Let us see that F has the desired properties. Obviously $\mathcal{L}^2(F) = 0$. By construction each S_i contains a unit segment in any direction making an angle greater or equal than 60° with L , since $S_i \subseteq V_i$ so do each V_i . Let $\theta \geq 60^\circ$ be some such direction and for every i let M_i be a unit segment in direction θ with $M_i \subseteq \bar{V}_i$. By compactness and taking a subsequence if necessary we can say that $\{M_i\}$ converges to M , a unit segment in direction of θ . Note that $\{\bar{V}_i\}$ is a decreasing sequence with $M_i \subseteq V_j \quad \forall i \geq j$, so given that each $\bar{V}_j$ is closed, $M \subseteq \bar{V}_j \quad \forall j$. Therefore $M \subseteq \bigcap_{i=1}^{\infty} \bar{V}_i = F$ as we wanted to see. $\square$

The following Lemma is required (together with Theorem 3.4) to solve Kakeya's problem.

Lemma 3.5. *Let L_1 and L_2 be parallel lines in the plane. Given $\varepsilon > 0$ there is a set E containing L_1 and L_2 with $\mathcal{L}^2(E) < \varepsilon$ such that a unit segment may be continuously moved from L_1 to L_2 without ever leaving E .*

Proof. Let $x_1 \in L_1$ and $x_2 \in L_2$ be points on the lines. Let M be the segment uniting x_1 and x_2 . Let E be the set consisting of L_1 , L_2 , M and the unit sectors centered at x_i lying between L_i and M . It is immediate to see that by taking x_1 and x_2 as far apart as we need $\mathcal{L}^2(E)$ can be made as small as we desire. Clearly a unit segment might be translated from one line to another via rotating it in the first sector, sliding it along M and rotating it again inside the second sector. $\square$

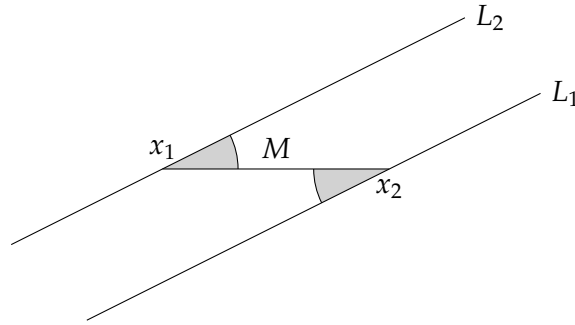


Figure 3.3: Translation of unit segment in a set of arbitrary area.

Finally we solve Kakeya's problem.

Theorem 3.6. *Given $\varepsilon > 0$, there exists a set E with $\mathcal{L}^2(E) < \varepsilon$ inside which a unit segment may be moved continuously to lie in its original position but rotated 180° .*

Proof. We construct a set E_0 with $\mathcal{L}^2(E) < \varepsilon/3$ inside which we can translate a unit segment to a position at an angle 60° from its starting position. Taking three copies of E_0 we can obtain the set E required, applying Lemma 3.5 if translations are needed between the three components.

To construct E_0 , pick an equilateral triangle of unit height T with base on the line L . By Theorem 3.3 we may divide T into $m = 2^k$ elementary triangles and slide them along L to positions $T'_1, \dots, T'_m$ such that $\mathcal{L}^2(\bigcup_1^m T'_i) < \varepsilon/6$. Let $E_1 = \bigcup_1^m T'_i$. For each i one side of T'_i is parallel to other side of T'_{i+1} , thus by Lemma 3.5 we may, for each i , add a set of measure at most $\varepsilon/(6m)$ to E_1 to allow a unit segment to be translated from T'_i to T'_{i+1} . This gives a set E_0 inside which a unit segment may be rotated through 60° of measure, at most,

$$\mathcal{L}^2(E_0) \leq \frac{\varepsilon}{6} + \frac{(m-1)\varepsilon}{6m} < \frac{\varepsilon}{3}$$

as required. $\square$

3.2 The dual approach

In this final section we relate Besicovitch sets to the theory of measurable sets that we have developed throughout the whole thesis. The basic idea is to parametrise lines by points in a way that the projection of a set of points E in some direction is geometrically similar to the intersection of the lines parametrised by E with some fixed line. We may then examine the set formed by the union of the lines by projecting E in various directions.

Definition 3.7. Let $L(x)$ be the polar line to x with respect to the unit circle C , i.e. the line at distance $1/|x|$ from the origin and perpendicular to the radius vector x .

Note that $x \in L(y)$ iff $y \in L(x)$.

Lemma 3.8. Let proj_θ denote the orthogonal projection onto the line L_θ going through the origin in direction θ . $\forall E \subseteq \mathbb{R}^2$ the intersection of the line set $\{L(x) : x \in E\}$ with L_θ is the geometric inverse of the set $\text{proj}_\theta E$ with respect to C .

Proof. Let $a \in E \subseteq \mathbb{R}^2$ and let $\vec{a}$ be the vector starting at the origin with endpoint a . Let φ be the angle between the vector a and the x axis. $|\text{proj}_\theta(\vec{a})| = \frac{1}{|\vec{a}|\cos(\varphi-\theta)}$ with the point laying in L_θ . Now the line set $L(a)$ is a line a distance $\frac{1}{|\vec{a}|}$ from the origin and making a $\pi/2$ angle with $|\vec{a}|$, thus they intersect at distance $\frac{1}{|\vec{a}|\cos(\varphi-\theta)}$ from the origin, with obviously the intersection laying in L_θ . $\square$

Definition 3.9. Let $(a, b) \in \mathbb{R}^2$. We denote $L(a, b)$ as the set of points $\{(x, y) \in \mathbb{R}^2 : y = a + bx\}$

Definition 3.10. Let $E \subseteq \mathbb{R}^2$. We denote by $L(E)$ the line set $\bigcup_{(a,b) \in E} L(a, b)$

Proposition 3.11. Let c be a constant and L_c be the line $x=c$, then

$$L(a, b) \cap L_c = (c, a + bc) = (c, (a, b)(1, c))$$

with " $\cdot$ " the standard scalar product in $\mathbb{R}^2$. Thus if $E \subseteq \mathbb{R}^2$

$$L(E) \cap L_c = \{(c, (a, b) \cdot (1, c)) : (a, b) \in E\}$$

is a set geometrically similar to $\text{proj}_\theta E$ with a ratio of similitude of $\sqrt{1+c^2}$ with $c = \tan(\theta)$.

Proof. Let $(a, b) \in E$. $|\text{proj}_\theta(a, b)| = a \cos \theta + b \sin \theta$ laying in L_θ . Now note that

$$\sqrt{1+c^2} = \sqrt{1+\tan^2 \theta} = \frac{1}{\cos \theta} \sqrt{\cos^2 \theta + \sin^2 \theta} = \frac{1}{\cos \theta}$$

So multiplying $L(a, b) \cap L_c$ by the inverse of the ratio of similitude we have

$$\cos \theta (L(a, b) \cap L_c) = \cos \theta (c, a + bc) = \cos \theta (\tan \theta, a + b \tan \theta) = (\sin \theta, a \cos \theta + b \sin \theta)$$

Since $x = c$ is fixed this intersection lays always on the vertical line L_c . The y-value of $\cos \theta (L(a, b) \cap L_c)$ is the same as the modulus of $\text{proj}_\theta(a, b)$ with respect to the origin. Since both $L(E) \cap L_c$ and $\text{proj}_\theta E$ lay on a straight line we can affirm that these sets are geometrically similar. $\square$

From this particular result it is immediate to see the *duality principle*

$$\mathcal{L}^1(L(E) \cap L_c) = 0 \iff \mathcal{L}^1(\text{proj}_\theta E) = 0 \quad (3.1)$$

Note that if $(a, b) \in \mathbb{R}^2$ then $\text{proj}_{\pi/2}(a, b) = b$ is the gradient of the line $L(a, b)$. Therefore, for any set of points E , if $b \in \text{proj}_{\pi/2}(E)$ then $L(E)$ contains a line of gradient b .

Let us now see the fundamental relationship between linearly measurable sets and line sets.

Lemma 3.12. Let $E \subseteq \mathbb{R}^2$ and Π a k -dimensional subspace of $\mathbb{R}^n$. Then $\mathcal{H}^s(\text{proj}_\Pi E) \leq \mathcal{H}^s(E)$

Proof. The projection mapping does not increase distances, meaning that

$$|\text{proj}_\Pi(x) - \text{proj}_\Pi(y)| \leq |x - y| \quad \forall x, y \in E$$

The results follows immediatly by applying Lemma 1.13. $\square$

Definition 3.13. Let $E \in \mathbb{R}^2$ be a 1-set. Given $x \in E$ and $\rho, \varepsilon, m > 0$ we define the subset $T(x, \rho, \varepsilon, m) \subseteq [0, \pi)$ such that

$$\begin{aligned} \theta \in T(x, \rho, \varepsilon, m) &\iff \exists r \quad 0 < r < \rho \\ &\exists I \quad \theta \in I \subseteq [0, \pi) \quad |I| < \varepsilon \end{aligned}$$

which verify

$$\frac{\mathcal{H}^1(E \cap C_r(x, I))}{r} \geq m|I| \quad (3.2)$$

where $C_r(x, I)$ denotes the double sector consisting of the points of $B_r(x)$ that lie on $L_\theta(x)$, the line through x in the direction θ , for some $\theta \in I$. Note that if E is irregular, $\mathcal{H}^1(E \cap C_r(x, I))$ is continuous in r , from which it is easy to see that $T(x, \rho, \varepsilon, m)$ is a Lebesgue measurable subset of $[0, \pi)$.

Definition 3.14. Let $E \in \mathbb{R}^2$ be a 1-set. A direction θ is a *condensation direction of the first kind* at the point $x \in E$ if $L_\theta(x)$ intersects E infinitely often in every neighbourhood of x .

A direction θ is a *condensation direction of the second kind* at the point $x \in E$ if $\theta \in T$ where

$$T = \bigcap_{\rho > 0} \bigcap_{\varepsilon > 0} \bigcap_{0 < m < \infty} T(x, \rho, \varepsilon, m). \quad (3.3)$$

Definition 3.15. Let $E \in \mathbb{R}^2$ be a 1-set. A point $x \in E$ is called a *point of radiation* of E if almost all θ are first or second kind condensation directions of E at x .

The keypoint of Besicovitch's theorem on the projection of irregular sets is contained in the next lemma, which applies Lebesgue's density theorem in an ingenious manner.

Lemma 3.16. Let $E \in \mathbb{R}^2$ be a closed irregular 1-set in the plane. Then almost all points of E are points of radiation.

Proof. From Corollary 2.31 we know that for $\mathcal{H}^1$ -almost all $x \in E$ and for any interval $I \subseteq [0, \pi)$

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1(E \cap C_r(x, I))}{2r} > \frac{1}{20}|I|$$

Fix x such a point. We will show that x is a point of radiation of E . Let us see first that the set S of condensation directions of the first kind at x is a Lebesgue-measurable subset of $[0, \pi)$. If $S(r)$ is the set of θ for which $L_\theta(x) \cap B_r(x)$ contains points of E other than x then $S(r)$ is

$$S(r) = \bigcup_{\delta > 0} \{\theta : \exists y \in L_\theta \text{ with } \delta \leq |x - y| \leq r\}$$

Taking δ as rational values, $S(r)$ is a countable union of closed sets making it a Borel set. Since

$$S = \bigcap_{j=1}^{\infty} S(1/j)$$

S is also a Borel set and thus Borel measurable.

The objective now is to show that if ρ, ε and m are positive then $\theta \in T(x, \rho, \varepsilon, m)$ for almost all $\theta \notin S$. Let $\theta \in [0, \pi)$ be a direction at which the Lebesgue density of S $\lim_{r \rightarrow 0} (S \cap [\theta - r, \theta + r] / 2r)$ is zero. Taking $\mathcal{L}^1$ to be the Lebesgue measure on $[0, \pi)$ by the Lebesgue density Theorem A.8 this is true for almost all $\theta \notin S$. For all sufficiently small intervals I with $\theta \in I \subseteq [0, \pi)$ we have

$$\mathcal{L}^1(S \cap I) < \frac{|I|}{20m}$$

Let I be such an interval such that $0 < |I| < \varepsilon$. Since $S(r) \searrow S$ as $r \rightarrow 0$ there is a $\rho_1 \leq \rho$ such that if $r < \rho_1$ then

$$\mathcal{L}^1(S(r) \cap I) < \frac{|I|}{20m}$$

By (3.2) we can choose a $r < \rho_1$ such that

$$\mathcal{H}^1(E \cap C_r(x, I)) > \frac{2r|I|}{20} \quad (3.4)$$

If

$$\mathcal{H}^1(E \cap C_r(x, I)) > mr|I| \quad (3.5)$$

is verified, by definition $\theta \in T(x, \rho, \varepsilon, m)$. If not, by the definition of Lebesgue measure and (3.2), there exists a countable collection of disjoint intervals $\{I_i\}$ such that

$$S(r) \cap I \subseteq \bigcup_{i=1} I_i \quad (3.6)$$

and

$$\mathcal{L}^1\left(\bigcup_{i=1} I_i\right) < \frac{|I|}{20m} \quad (3.7)$$

Let now Q be the set of indices i for which

$$\mathcal{H}^1(E \cap C_r(x, I_i)) > mr|I_i| \quad (3.8)$$

By inequality (3.7)

$$\sum_{i \notin Q} \mathcal{H}^1(E \cap C_r(x, I_i)) \leq mr \mathcal{L}^1\left(\bigcup_{i \notin Q} I_i\right) < \frac{r|I|}{20} \quad (3.9)$$

Note that the points of $E \cap C_r(x, I_i)$ all lie in lines $L_\theta(x)$ with $\theta \in S(r) \cap I$ so by (3.6)

$$\bigcup_{i=1}^{\infty} (E \cap C_r(x, I_i)) \supseteq E \cap C_r(x, S(r) \cap I) \supseteq E \cap C_r(x, I)$$

Therefore by (3.9) and (3.4)

$$\begin{aligned} \sum_{i \in Q} \mathcal{H}^1(E \cap C_r(x, I_i)) &= \sum_{i=1}^{\infty} \mathcal{H}^1(E \cap C_r(x, I_i)) - \sum_{i \notin Q} \mathcal{H}^1(E \cap C_r(x, I_i)) \\ &> \mathcal{H}^1(E \cap C_r(x, I)) - \frac{r|I|}{20} > \frac{r|I|}{20} \end{aligned} \quad (3.10)$$

Using (3.8) and by a process of expanding and combining the intervals $\{I_i\}_{i \in Q}$ we can obtain a disjoint collection of open intervals $\{J_j\}_1^\infty$ such that

$$J = \bigcup_{j=1}^{\infty} J_j \supseteq \bigcup_{i \in Q} I_i$$

and

$$\mathcal{H}^1(E \cap C_r(x, J_j)) = mr|J_j| \quad j = 1, 2, \dots \quad (3.11)$$

Since we have previously assumed that (3.5) does not hold, this can be done in such a way that $J \subseteq I$. Now from (3.10) and (3.11) we have

$$\mathcal{L}^1(J) = \sum_{j=1}^{\infty} |J_j| > \frac{|I|}{20m}$$

If $\theta' \in J$ then $\theta' \in J_j$ for some j so by (3.10) $\theta' \in T(x, \rho, \varepsilon, m)$. Hence $J \subseteq T(x, \rho, \varepsilon, m) \cap I$ and therefore

$$\mathcal{L}^1(T(x, \rho, \varepsilon, m) \cap I) > \frac{|I|}{20m}$$

This holds for all sufficiently small intervals I containing θ so we have seen that for almost $\theta \notin S$ either $\theta \in T(x, \rho, \varepsilon, m)$ or the lower Lebesgue density of $T(x, \rho, \varepsilon, m)$ at θ is at least $1/(20m)$. From the Lebesgue density Theorem A.8 we know that

the Lebesgue density of $T(x, \rho, \varepsilon, m)$ at almost all points of its complement is zero so almost all $\theta \notin S$ belong to $T(x, \rho, \varepsilon, m)$. Since this is true for all values of ρ, ε and m and since $T(x, \rho, \varepsilon, m)$ decreases as ρ and ε decrease and as m increases we may take the intersections of (3.3) to be countable and so we have seen that almost all θ not in S are in T . $\square$

Let E be an s -set. Consider the product measure $\mathcal{H}^1 \times \mathcal{L}^1$ on $E \times [0, \pi)$ such that $(\mathcal{H}^1 \times \mathcal{L}^1)(A \times B) = \mathcal{H}^1(A) \times \mathcal{L}^1(B)$ where A and B are Borel sets.

Definition 3.17. Let G_1 and G_2 be the subsets

$$G_i = \{(x, \theta) : \theta \text{ is a condensation direction of the } i\text{-th kind of } E \text{ at } x\} \quad i = 1, 2$$

Lemma 3.18. *If E is a closed irregular 1-set then G_1 and G_2 are Borel subsets of $E \times [0, \pi)$ and therefore $(\mathcal{H}^1 \times \mathcal{L}^1)$ -measurable.*

Proof. (i) Let $0 < r < \rho$. Let us define

$$G_{r,\rho} = \{(x, \theta) : x \in E \text{ and } r \leq |x - y| \leq \rho \text{ for some } y \in E \cap L_\theta(x)\}$$

Since E is closed $G_{r,\rho}$ is closed. Then

$$G_1 = \bigcap_{\rho > 0} \bigcup_{r \leq \rho} G_{r,\rho}$$

is a Borel set since $G_{r,\rho}$ increases as r decreases and decreases as ρ decreases.

(ii) Since E is irregular, $\mathcal{H}^1(E \cap C_r(x, I))$ is continuous in x, r and I . Defining now $T'(x, \rho, \varepsilon, m)$ the same way as $T(x, \rho, \varepsilon, m)$ but taking a strict inequality in (3.2) we see that $\{(x, \theta) : \theta \in T'(x, \rho, \varepsilon, m)\}$ is open in $E \times [0, \pi)$. Now

$$G_2 = \bigcap_{\rho > 0} \bigcap_{\varepsilon > 0} \bigcap_{0 < m < \infty} \{(x, \theta) : \theta \in T'(x, \rho, \varepsilon, m)\}$$

where we are allowed to take T' instead of T since $T(x, \rho, \varepsilon, m)$ is monotonic in m and since we can take these intersections over ρ, ε, m rationals G_2 is a Borel set. $\square$

We can now deduce the projection theorem.

Theorem 3.19. *Let $E \subseteq \mathbb{R}^2$ be an irregular 1-set. Then*

$$\mathcal{L}^1(\text{proj}_\theta E) = 0 \text{ for almost all } \theta \in [0, \pi)$$

Proof. Suppose E is compact. By Lemma 3.18 the set $G = G_1 \cup G_2$ is a $(\mathcal{H}^1 \times \mathcal{L}^1)$ -measurable subset of $E \times [0, \pi)$. By Lemma 3.16 $\mathcal{L}^1\{\theta : (x, \theta) \in G\} = \pi$ for almost all $x \in E$. By Fubini's Theorem A.10 for $\mathcal{L}^1$ -almost all $\theta \in [0, \pi)$ we have $(x, \theta) \in G$ for $\mathcal{H}^1$ -almost all $x \in E$. Let θ be one of these condensation direction for almost all x and let L_ϕ be the line going through the origin at an angle $\phi = \theta + \frac{1}{2}\pi \pmod{\pi}$, i.e. perpendicular to θ . We claim that the projection of E onto L_ϕ has Lebesgue measure zero.

Let $E = E_0 \cup E_1 \cup E_2$ where $\mathcal{H}^1(E_0) = 0$ and E_i are the sets of points of condensation of the i th kind.

(i) By Lemma 3.12 $\mathcal{L}^1(\text{proj}_\phi E_0) = 0$.

(ii) If $x \in E_1$ then $L_\theta(x)$ intersects E infinitely many times so $\mathcal{H}^0(L_\theta(x) \cap E) = \infty$. Setting $\mathbb{R}^2 = L_\theta \times L_\phi$ we may apply Theorem 1.22 with $s = 1$ and $t = 0$ seeing thus

$$\infty > \mathcal{H}^1(E_1) > bc\mathcal{H}^1(\text{proj}_\phi E_1)$$

Since $\mathcal{H}^0(L_\theta(x) \cap E) = \infty$ we can pick c arbitrarily large thus obtaining $\mathcal{H}^1(\text{proj}_\phi E_1) = \mathcal{L}^1(\text{proj}_\phi E_1) = 0$.

(iii) Let m be any positive number and let $\mathcal{V}$ be the class of subintervals of L_ϕ given by

$$\mathcal{V} = \{J : \mathcal{H}^1(x \in E : \text{proj}_\phi x \in J) > m\mathcal{L}^1(J)\}$$

Note that if $\theta \in I$ then $C_r(x, I)$ may be enclosed in a rectangle of width less than $\mathcal{L}^1(I)r$ with one pair of sides in direction θ . Using (3.2) and the definition of E_2 it follows that $\mathcal{V}$ is a Vitali cover for $\text{proj}_\phi E_2$. By Vitali's covering Theorem 1.17 given $\varepsilon > 0$ there exist disjoint intervals $\{J_i\}$ of $\mathcal{V}$ such that

$$\mathcal{L}^1(\text{proj}_\phi E_2) \leq \sum_i \mathcal{L}^1(J_i) + \varepsilon \leq \frac{1}{m} \mathcal{H}^1(E) + \varepsilon$$

Since we may take m arbitrarily large and ε arbitrarily small we conclude $\mathcal{L}^1(\text{proj}_\phi E_2) = 0$.

We have therefore seen $\mathcal{L}^1(\text{proj}_\phi E) = 0$ for almost all ϕ . Now if E is any measurable 1-set, using the regularity of $\mathcal{H}^1$, we may write $E = F \cup G$ where F is compact and $\mathcal{H}^1(G) < \delta$. We have already seen that $\mathcal{L}^1(\text{proj}_\phi F) = 0$ for almost all ϕ thus by Lemma 3.12 $\mathcal{L}^1(\text{proj}_\phi E) = \mathcal{L}^1(\text{proj}_\phi G) \leq \mathcal{H}^1(G) < \delta$. By taking δ arbitrarily small we obtain $\mathcal{H}^1(\text{proj}_\phi E) = 0$ for almost all $\phi \in [0, \pi)$. $\square$

Theorem 3.20. *Let $E \subseteq \mathbb{R}^2$ be a regular 1-set. Then $\mathcal{L}^1(\text{proj}_\theta E) > 0$ for all but at most one value of θ .*

Proof. By Theorem 2.27 a regular 1-set comprises a measurable subset of a countable union of rectifiable curves and a set of measure zero, so it is enough to prove the result if E is a 1-set contained in a rectifiable curve Γ .

Let $x \in E$ be a regular point of both E and Γ . Given $\varepsilon > 0$ we may find r such that both $\mathcal{H}^1(E \cap B_r(x)) > (1 - \varepsilon^2)2r$ and $\mathcal{H}^1(\Gamma \cap B_r(x)) < (1 + \varepsilon)2r$, from which we see that

$$\mathcal{H}^1(E \cap B_r(x)) > (1 - \varepsilon)\mathcal{H}^1(\Gamma \cap B_r(x))$$

and hence

$$\mathcal{H}^1((\Gamma \setminus E) \cap B_r(x)) < \varepsilon \mathcal{H}^1(\Gamma \cap B_r(x))$$

Since Γ is rectifiable and since $\Gamma \cap B_r(x)$ consists of at most a countable number of disjoint arcs, we may choose an arc $\Gamma_0 \subseteq \Gamma \cap B_r(x)$ such that $\mathcal{H}^1(\Gamma_0 \setminus E) < \varepsilon \mathcal{H}^1(\Gamma_0) < 2\varepsilon|y - z|$ where y and z are the endpoints of Γ_0 . Let L_θ be a line making an angle of ϕ with $[y, z]$, where $\cos \phi > 2\varepsilon$. Projecting onto L_θ and using Lemma 3.12 we have

$$\begin{aligned} \mathcal{L}^1(\text{proj}_\theta E) &> |y - z| \cos \phi - \mathcal{L}^1(\text{proj}_\theta(\Gamma_0 \setminus E)) \\ &\geq |y - z| \cos \phi - \mathcal{H}^1(\Gamma_0 \setminus E) > |y - z|(\cos \phi - 2\varepsilon) > 0 \end{aligned}$$

Thus $\mathcal{L}^1(\text{proj}_\theta E) > 0$ except for a set of directions θ contained in an interval of length $2 \arcsin(2\varepsilon)$ for all $\varepsilon > 0$, i.e., except for at most one direction. $\square$

We now prove the most important result on the projection of sets of general dimension, although we will only prove it for two dimensions since we are just working on the plane. In this particular case, 'almost all θ ' refers to the Lebesgue measure.

Theorem 3.21. *Let $E \subseteq \mathbb{R}^2$ be a Souslin set with $\dim E = s$.*

- (i) *If $s \leq 1$ then $\dim(\text{proj}_\theta E) = s$ for almost all $\theta \in [0, \pi)$*
- (ii) *If $s > 1$ then $\mathcal{L}^1(\text{proj}_\theta E) > 0$ for almost all $\theta \in [0, \pi)$*

Proof. (i) By Lemma 3.12, $\dim(\text{proj}_\theta E) \leq s \forall \theta$. If $t < s < 1$, by Corollary 1.26 we may choose a mass distribution μ , with support contained in a compact subset of E , such that $I_t(\mu) < \infty$. The mapping

$$f \mapsto \int f(x \cdot \theta) d\mu(x)$$

is a positive linear functional where θ denotes the unit vector in direction θ , $\cdot$ the usual scalar product and x is identified with $\vec{x}$ the vector from the origin to x . By Theorem 1.24, we may find a mass distribution μ_θ on $\mathbb{R}$ such that

$$\int f(u) d\mu_\theta(u) = \int f(x \cdot \theta) d\mu(x) \quad (3.12)$$

for continuous functions f . Using the monotone convergence theorem for integrals (3.12) holds for any non-negative measurable function.

Now, identifying $u \in \mathbb{R}$ with the point $u\theta \in L_\theta$ we may see μ_θ as a mass distribution on L_θ with support contained in $\text{proj}_\theta E$. Now,

$$I_t(\mu_\theta) = \iint \frac{d\mu_\theta(u) d\mu_\theta(v)}{|u - v|^t} = \iint \frac{d\mu(x) d\mu(y)}{|x \cdot \theta - y \cdot \theta|^t} = \iint \frac{d\mu(x) d\mu(y)}{|(x - y) \cdot \theta|^t}$$

By applying Fubini's Theorem A.10 to $[0, \pi) \times \mathbb{R}^2 \times \mathbb{R}^2$ it is shown that $I_t(\mu_\theta)$ is a measurable function of θ and that

$$\int_0^\pi I_t(\mu_\theta) d\theta = c \iint \frac{d\mu(x) d\mu(y)}{|(x - y) \cdot \theta|^t} = c I_t(\mu) < \infty$$

with

$$c = \int_0^\pi |\theta \cdot \tau|^{-t} d\theta < \infty$$

since the integral is independent of the unit vector τ and $t < 1$. Thus, we have found that $I_t(\mu_\theta) < \infty$ for almost all θ , so from Corollary 1.26 we can conclude that $\dim(\text{proj}_\theta E) \geq t$ holds for all $t < s$, hence $\dim(\text{proj}_\theta E) = s$ for almost all θ .

- (ii) If $s > 1$, by Corollary 1.26 there is a mass distribution μ supported on E with $I_t(\mu) < \infty$. We define again mass distributions μ_θ on L_θ by (3.12), focusing in this case in their Fourier transforms. Using (3.12)

$$\begin{aligned} |\hat{\mu}_\theta(p)|^2 &= \frac{1}{2\pi} \int e^{iup} d\mu_\theta(u) \int e^{-ivp} d\mu_\theta(v) = \frac{1}{2\pi} \iint e^{i(u-v)p} d\mu_\theta(u) d\mu_\theta(v) \\ &= \frac{1}{2\pi} \iint e^{ip(x-y) \cdot \theta} d\mu(x) d\mu(y) \end{aligned}$$

Then

$$|\hat{\mu}_\theta(p)|^2 + |\hat{\mu}_{\theta+\pi}(p)|^2 = \frac{1}{\pi} \iint \cos(p(x-y) \cdot \theta) d\mu(x) d\mu(y)$$

so that

$$\int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} \iint \cos(p|x-y| \cos \theta) d\mu(x) d\mu(y) d\theta$$

Using the definition of the Bessel function

$$J_0(u) = \frac{1}{2\pi} \int_0^{2\pi} \cos(u \cos \theta) d\theta$$

and Fubini's theorem A.10 we obtain

$$\int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta = \iint J_0(p|x-y|) d\mu(x) d\mu(y)$$

Now, if $0 < m < \infty$ Fubini's Theorem gives

$$\begin{aligned} \int_{-m}^m \int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta dp &= \iiint_{-m}^m J_0(p|x-y|) dp d\mu(x) d\mu(y) \\ &= \iiint_{u=-m|x-y|}^{m|x-y|} J_0(u) \frac{1}{|x-y|} du d\mu(x) d\mu(y) \leq b \iint \frac{d\mu(x) d\mu(y)}{|x-y|} = bI_1(\mu) \end{aligned}$$

where b is independent of m . Letting $m \rightarrow \infty$, and again by Fubini's theorem

$$\int_0^{2\pi} \int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp d\theta \leq bI_1(\mu) < \infty$$

so that

$$\int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp < \infty$$

for almost all $\theta \in [0, \pi)$. By lemma 1.28 the support of μ_θ has positive Lebesgue measure for any such θ , since the support of μ_θ is the projection of the support of μ and this is contained in $\text{proj}_\theta E$, $\text{proj}_\theta E$ is measurable and has positive Lebesgue measure for almost all θ . □

The following theorem gives us a fundamental relationship between linearly measurable sets and line sets.

Theorem 3.22. *Let $E \subseteq \mathbb{R}^2$ be a 1-set, then $L(E)$ is an $\mathcal{L}^2$ -measurable subset of $\mathbb{R}^2$. If E is regular then $\mathcal{L}^2(L(E)) > 0$ and if E is irregular then $\mathcal{L}^2(L(E)) = 0$*

Proof. Let us first check the mesurability of $L(E)$. If E is an open set then $L(E)$ is also open and if E is closed $L(E)$ is also closed. Thus if E is a G_δ -set (resp. F_σ -set) $L(E)$ is also a G_δ -set (resp. F_σ -set). Suppose $\mathcal{H}^1(E) = 0$. By Theorem 1.11 $\mathcal{H}^1$ is a regular measure and thus there is a G_δ -set E_0 such that $E \subseteq E_0$ and $\mathcal{H}^1(E) = 0$. By Lemma 3.12 $\mathcal{L}^1(\text{proj}_\theta(E_0)) = 0 \forall \theta$, thus by equation (3.1) $\mathcal{L}^1(L(E_0) \cap L_c) = 0 \forall c$. $L(E_0)$ is a G_δ -set and so is plane measurable thus by Fubini's Theorem $\mathcal{L}^2(L(E_0)) = 0$ and hence $L(E) \subseteq L(E_0)$ is plane measurable with $\mathcal{L}^2(L(E)) = 0$.

Let now $E \subseteq \mathbb{R}^2$ be any 1-set. By regularity $E = E_0 \cup F$ where E_0 is an F_σ -set and $\mathcal{H}^1(F) = 0$. Thus $L(E) = L(E_0) \cup L(F)$ is $\mathcal{L}^2$ -measurable.

If E is an irregular 1-set, by Theorem 3.19 $\mathcal{L}^1(\text{proj}_\theta E) = 0$ for almost all θ . By (3.1) we have $\mathcal{L}^1(L(E) \cap L_c) = 0$ for almost every c . Since $L(E)$ is measurable by Fubini's Theorem we have $\mathcal{L}^2(L(E)) = 0$.

If E is a regular 1-set, by Theorem 3.20 $\mathcal{L}^1(\text{proj}_\theta E) > 0$ for almost all θ thus by the duality principle (3.1) $\mathcal{L}^1(L(E) \cap L_c) > 0$ for almost all c . Since $L(E)$ is measurable by Fubini's Theorem $\mathcal{L}^2(L(E)) > 0$. $\square$

Using duality we can easily construct a set of measure zero containing a whole line, rather than just a segment, in every direction. We will call this set a *line-Besicovitch set*.

Theorem 3.23. *There exists a subset of $\mathbb{R}^2$ of Lebesgue measure zero containing a line in every direction.*

Proof. Let $E \subseteq \mathbb{R}^2$ be an irregular 1-set such that the projection onto the y -axis $\text{proj}_{\pi/2} E$ contains the segment $-1 \leq y \leq 1$. For example, this set might be $E = \{(x, f(x)) : 0 \leq x \leq 1\}$ together with its reflection in the x -axis, where f is the real function on $[0, 1]$ defined by $f(0.x_1x_2\dots) = 0.y_1y_2\dots$ and $0.x_1x_2\dots$ and $0.y_1y_2\dots$ are written in base 4, so that $y_i = 5 - x_i \pmod{4}$. E is a Borel set and, since the projection of E onto the x -axis has unit length we have that $1 \leq \mathcal{H}^1(E)$. E may be covered by 4^k squares of side 4^{-k} so $\mathcal{H}^1(E) \leq \sqrt{2}$ and hence E is a 1-set. Any rectifiable curve intersects E in a set of length zero, thus E is a Z -set and is irregular.

Consider the line set $L(E)$. By Theorem 3.22 $\mathcal{L}^2(L(E)) = 0$. Since $-1 \leq y \leq 1$ is contained in $\text{proj}_{\pi/2} E$, $L(E)$ contains lines of all gradients between -1 and 1 . Taking a copy of $L(E)$, rotating it $\pi/2$ radians and uniting it with the original $L(E)$ we obtain a set with the desired properties. $\square$

We show now that line-Besicovitch sets are forcefully quite large.

Theorem 3.24. *Let $F \subseteq \mathbb{R}^2$ be a set containing a line in every direction. Then F has Hausdorff dimension 2.*

Proof. Since every set is contained in a G_δ -set of the same dimension we may assume F to be a G_δ -set without loss of generality. The set

$$E = \{(a, b) : L(a, b) \subseteq F\} = \bigcap_{r=1}^{\infty} \{(a, b) : L(a, b) \cap B_r(0) \subseteq F \cap B_r(0)\}$$

is also G_δ thus it is Borel measurable. Since F contains a line in every direction $\text{proj}_{\pi/2} E$ is the entire y -axis and thus by Lemma 3.12 $\mathcal{H}^1(E) = \infty$. By Theorem 3.21 $\dim(\text{proj}_\theta E) = 1$ for almost all θ and thus by the duality principle (3.1) $\dim(L(E) \cap L_c) = 1$ for almost all c . By Theorem 1.22 we have $\dim L(E) = 2$. Since $L(E) \subseteq F$ $\dim F = 2$. $\square$

Finally we show that a Besicovitch set is as large as a line-Besicovitch set.

Theorem 3.25. *Let $F \subseteq \mathbb{R}^2$ be a set containing a unit segment in every direction. Then F has Hausdorff dimension 2.*

Proof. Let θ be a unit vector in the direction $\theta \in [0, 2\pi)$. Since F is a Besicovitch set there exists an $x_\theta \in \mathbb{R}^2$ such that the unit segment $I_\theta = \{x_\theta + t\theta \in \mathbb{R}^2 : t \in [0, 1]\}$ is contained in F . Let now

$$H = \bigcup_{c \in \mathbb{Q}^+, v \in \mathbb{Q}^2} (cF + v)$$

be the countable union of all rationally scaled and translated copies of F . By the subadditivity of outer measures (A.16) and the definition of Hausdorff dimension 1.1 we know

$$\dim H = \sup_{c \in \mathbb{Q}^+, v \in \mathbb{Q}^2} \dim(cF + v) \quad (3.13)$$

Note that the Hausdorff dimension is invariant under translations and non-zero scaling thus

$$\dim(cF + v) = \dim F$$

and therefore by (3.13)

$$\dim H = \sup_{c \in \mathbb{Q}^+, v \in \mathbb{Q}^2} \dim(cF + v) = \dim F \quad (3.14)$$

We want to see now that H is a line-Besicovitch set. Let

$$\ell_\theta = \{x_\theta + \lambda\theta \in \mathbb{R}^2 : \lambda \in \mathbb{R}\}$$

Since $c \in \mathbb{Q}^+$ and $v \in \mathbb{Q}^2$ are dense in their respective real spaces, we have

$$\ell_\theta \subseteq \bigcup_{c \in \mathbb{Q}^+, v \in \mathbb{Q}^2} (cI_\theta + v)$$

Since $I_\theta \subseteq H$ it follows that $\ell_\theta \subseteq H$. This holds for every direction θ hence F is a line-Besicovitch set. Finally, by Theorem 3.24 and (3.14)

$$\dim H = \dim F = 2$$

$\square$

Conclusions

In this thesis we have explored the structure of Besicovitch sets arising in the solution of the two-dimensional Kakeya problem. Starting from basic notions in geometric measure theory and further the structure of s -sets, we have developed the necessary tools to construct this solution both via a geometric and a dual point of view, before finally focusing on the Hausdorff dimension of Besicovitch and line-Besicovitch sets.

A central insight of the thesis is the fundamental relationship between linearly measurable sets and line sets which permit us to prove the existence of Besicovitch sets from simple projection theorems for irregular 1-sets.

These ideas culminate in the proof that solutions to the two-dimensional Kakeya problem cannot be too small. In fact, both Besicovitch and line-Besicovitch sets in the plane have necessarily Hausdorff dimension 2.

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Appendix

A.1 Measures and Outer Measures

Definition A.1. A *measure* μ is a function defined on some σ -field $\mathcal{S}$ of subsets of X taking values in $[0, \infty]$ such that

$$\mu(\emptyset) = 0$$

and for every countable sequence of disjoint sets $\{E_j\} \subseteq \mathcal{S}$

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Definition A.2. We say an *outer measure* ν on a set X is a function defined on all subsets of X taking values in $[0, \infty]$ such that

$$\nu(\emptyset) = 0$$

$$A \subseteq B \implies \nu(A) \leq \nu(B) \tag{A.15}$$

$$\nu\left(\bigcup_i A_i\right) \leq \sum_i \nu(A_i) \tag{A.16}$$

for any subsets $\{A_j\}$ of X .

Definition A.3. Let ν be an outer measure on X . Let $E \subseteq X$. We say E is ν -*measurable* or *measurable with respect to the outer measure* ν if

$$\nu(A) = \nu(A \cap E) + \nu(A \setminus E) \quad \forall A \subseteq X$$

We say that E *decomposes* every subset of X additively.

To show that a set E is ν -measurable it is enough to check that

$$\nu(A) \geq \nu(A \cap E) + \nu(A \setminus E) \tag{A.17}$$

since the inequation in the opposite direction is automatically given by (A.16).

Theorem A.4. *Let ν be an outer measure. The collection $\mathcal{M}$ of ν -measurable sets forms a σ -field and the restriction of ν to $\mathcal{M}$ is a measure.*

Proof. We first show that $\mathcal{M}$ is a σ -field. It is not empty since $\emptyset \in \mathcal{M}$. By the symmetry in the definition of ν -measurable A.3 $A \in \mathcal{M}$ iff $X \setminus A \in \mathcal{M}$ thus $\mathcal{M}$ is closed under taking complements. Now let $\{E_i\} \in \mathcal{M}$ and A be any set. Using again definition A.3 we have

$$\begin{aligned} \nu(A) &= \nu(A \cap E_1) + \nu(A \setminus E_1) \\ &= \nu(A \cap E_1) + \nu((A \setminus E_1) \cap E_2) + \nu(A \setminus E_1 \setminus E_2) = \dots \\ &= \sum_{j=1}^k \nu((A \setminus \bigcup_{i=1}^{j-1} E_i) \cap E_j) + \nu(A \setminus \bigcup_{j=1}^k E_j) \end{aligned}$$

Thus for all k we have

$$\nu(A) \geq \sum_{j=1}^k \nu((A \setminus \bigcup_{i=1}^{j-1} E_i) \cap E_j) + \nu(A \setminus \bigcup_{j=1}^k E_j)$$

Therefore

$$\nu(A) \geq \sum_{j=1}^{\infty} \nu((A \setminus \bigcup_{i=1}^{j-1} E_i) \cap E_j) + \nu(A \setminus \bigcup_{j=1}^{\infty} E_j) \quad (\text{A.18})$$

Note that

$$\bigcup_{j=1}^{\infty} (A \setminus \bigcup_{i=1}^{j-1} E_i) \cap E_j = A \cap \bigcup_{j=1}^{\infty} E_j$$

Thus using (A.16) and (A.18) we have

$$\begin{aligned} \nu(A) &\geq \sum_{j=1}^{\infty} \nu((A \setminus \bigcup_{i=1}^{j-1} E_i) \cap E_j) + \nu(A \setminus \bigcup_{j=1}^{\infty} E_j) \\ &\geq \nu(A \cap \bigcup_{j=1}^{\infty} E_j) + \nu(A \setminus \bigcup_{j=1}^{\infty} E_j) \geq \nu(A) \end{aligned}$$

It follows that $\bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$ therefore $\mathcal{M}$ is a σ -field.

Taking now $\{E_j\}$ to be a collection of disjoint sets of $\mathcal{M}$ and setting $A = \bigcup_{j=1}^{\infty} E_j$ in (A.18) we obtain

$$\nu(\bigcup_{j=1}^{\infty} E_j) \geq \sum_{j=1}^{\infty} \nu(E_j)$$

proving that ν is a measure on $\mathcal{M}$. □

Definition A.5. Let (X, d) be a metric space. Let $E, F \subseteq X$ be any two sets positively separated i.e. $d(E, F) > 0$. We say ν is a *metric outer measure* if

$$\nu(E \cup F) = \nu(E) + \nu(F) \quad (\text{A.19})$$

We will now prove Caratheodory's Lemma to later show that the collection of measurable sets of a metric outer measure include the Borel sets.

Lemma A.6. Let ν be a metric outer measure on (X, d) . Let $\{A_j\}$ be an increasing sequence of subsets of X with $A = \lim_{j \rightarrow \infty} A_j$ such that $d(A_j, A \setminus A_{j+1}) > 0 \forall j$. Then $\nu(A) = \lim_{j \rightarrow \infty} \nu(A_j)$

Proof. The inequality " $\geq$ " is automatically given by (A.15). We just need to check

$$\nu(A) \leq \lim_{j \rightarrow \infty} \nu(A_j) \quad (\text{A.20})$$

Note that if $\lim_{j \rightarrow \infty} \nu(A_j) = \infty$ we are done so we may assume going forward that it is not the case. Let $B_1 = A_1$ and $B_j = A_j \setminus A_{j-1}$ for $j \geq 2$. If $j + 2 \leq i$ then $B_j \subseteq A_j$ and $B_i \subseteq A \setminus A_{j+1} \subseteq j + 1$, so B_i and B_j are positively separated. Applying (A.5) (m-1) times we obtain

$$\begin{aligned} \nu\left(\bigcup_{k=1}^m B_{2k-1}\right) &= \sum_{k=1}^m \nu(B_{2k-1}) \\ \nu\left(\bigcup_{k=1}^m B_{2k}\right) &= \sum_{k=1}^m \nu(B_{2k}) \end{aligned}$$

Since both $\bigcup_{k=1}^m B_{2k}$ and $\bigcup_{k=1}^m B_{2k-1}$ are contained in A_{2m} we can assume these two series converge. Thus

$$\begin{aligned} \nu(A) &= \nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \nu\left(A_j \cup \bigcup_{k=j+1}^{\infty} B_k\right) \\ &\leq \nu(A_j) + \sum_{k=j+1}^{\infty} \nu(B_k) \leq \lim_{i \rightarrow \infty} \nu(A_i) + \sum_{k=j+1}^{\infty} \nu(B_k) \end{aligned}$$

where the first inequality is given by (A.16). Since the sum tends to 0 as $j \rightarrow \infty$ (A.20) follows. $\square$

Theorem A.7. If ν is a metric outer measure on (X, d) then all Borel subsets of X are ν -measurable.

Proof. Since the ν -measurable sets form a σ -field and the Borel sets form the smallest σ -field containing the closed subsets it is enough to check that (A.17) holds for any arbitrary set A and a closed E .

Let $A_j = \{x \in A \setminus E : d(x, E) \geq 1/j\}$. Then $d(A \cap E, A_j) \geq 1/j$. Given that ν is a metric outer measure it follows

$$\nu(A \cap E) + \nu(A_j) = \nu((A \cap E) \cup A_j) \leq \nu(A) \quad \forall j \quad (\text{A.21})$$

If $x \in (A \setminus E \setminus A_{j+1}) \exists z \in E$ with $d(x, z) < 1/(j+1)$. If $y \in A_j$ then $d(x, y) \geq d(y, z) - d(x, z) > 1/j - 1/(j+1) > 0$ so $d(A_j, A \setminus E \setminus A_{j+1}) > 0$. The sequence $\{A_j\}$ is increasing and, given that E is closed, $A \setminus E = \bigcup_{j=1}^{\infty} A_j$. Hence applying Lemma A.6 results in $\nu(A \setminus E) \leq \lim_{j \rightarrow \infty} \nu(A_j)$ which using in (A.21) gives us

$$\nu(A) \geq \lim_{j \rightarrow \infty} \nu(A \cap E) + \nu(A_j) \geq \nu(A \cap E) + \nu(A \setminus E)$$

which checks that (A.17) holds. $\square$

The Lebesgue density theorem states as follows:

Theorem A.8. Let E be a $\mathcal{L}^n$ -measurable subset of $\mathbb{R}^n$. Then the Lebesgue density of E at x

$$\lim_{r \rightarrow 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))}$$

exists and equals 1 if $x \in E$ and 0 if $x \notin E$, except for a set of $\mathcal{L}^n$ -measure 0.

Definition A.9. Given a metric space (X, d) we say a set is a *Souslin set* if it is of the form:

$$E = \bigcup_{i_1, i_2, \dots} \bigcap_{k=1}^{\infty} E_i$$

where $E_{i_1, i_2, \dots, i_k}$ is a closed set for each finite sequence $\{i_1, i_2, \dots, i_k\}$ of positive integers.

Fubini's theorem gives us the conditions under which we can compute a double integral as an iterated one, and states as follows:

Theorem A.10. Let X and Y be σ -finite measure spaces and suppose that $X \times Y$ has the product measure. If f is $X \times Y$ integrable then

$$\int_X \left(\int_Y f(x, y) dy \right) dx = \int_Y \left(\int_X f(x, y) dx \right) dy = \int_{X \times Y} f(x, y) d(x, y)$$

The Arzelà-Ascoli theorem states as follows [10]:

Theorem A.11. *If S is compact, then a set in $\mathcal{C}(S)$ is conditionally compact if and only if it is bounded and equicontinuous.*