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Abstract: Decentralized rationing problems are those in which the resource is not directly assigned to agents, but first allocated to groups of agents and then divided among their members. Within this framework, we define extensions of the constrained equal awards, the constrained equal losses and the proportional rules. We show that the first two rules do not preserve certain essential properties and prove the conditions under which both rules do preserve those properties. We characterize the extension of the proportional rule as the only solution that satisfies individual equal treatment of equals. We prove that the proportional rule is the only solution that assigns the same allocation regardless of whether the resource is distributed directly to agents or in a decentralized manner (with agents grouped). Finally, we analyse a strategic game based on decentralized rationing problems in which agents can move freely across groups to submit their claims.

JEL Codes: D63.

Keywords: Strategic rationing, decentralized rationing, equal awards rule, equal losses rule, proportional rule, a priori unions.

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1 Introduction

In this chapter we consider how to solve a rationing problem in which the resource cannot be directly assigned to agents. We suppose that agents are grouped and we propose solving this problem by using a two-stage procedure that involves, first, distributing the resource among the groups and, second, dividing it among their members. We call these situations *decentralized rationing problems*, which contrast with *centralized rationing problems*[1] where the resource is directly allocated to agents.

The agents may be grouped for many reasons, including lack of information, geographical proximity, logistics, different typologies of agents, etc. There are many real allocation problems in which the resource to be divided cannot be directly assigned to agents. Consider, for instance, food supplies to refugees and imagine that the refugees (agents) in the region are grouped in several camps (groups). Suppose that the distributor is unable to assign the food supplies (amount of resource) directly to refugees, but that the competent authorities at each camp can collect information about the feeding needs of the refugees resident in that camp. Then, it seems reasonable to assume that the food supplies must first be divided among the refugee camps and, subsequently, the amount received by each camp can be distributed among its resident refugees (members).

The model we analyse corresponds to the *bankruptcy model with a priori unions* (Casas-Méndez et al., 2003). These authors define and characterize a two-stage extension of the adjusted proportional rule. In the same line, Borm et al. (2005) define and characterize a two-stage extension of the constrained equal awards rule[2] within the framework of bankruptcy with a priori unions. Both studies are inspired in the classical paper by Owen (1977), who considers, for cooperative games, the a priori partition of the

[1] We refer to these situations as either standard rationing problems in Chapter 2 or single-issue rationing problems in Chapter 3.

[2] In our model we refer to the same extension of the constrained equal awards rule as the decentralized constrained equal awards rule.

agent set. That is, agents are partitioned in groups according to previously reached agreements.

In spite of proposing a formally equivalent model to that with a priori unions, our model focuses on the need for the central distributor to decentralize the allocation process in several intermediate distribution centers. In Sections 2, 3 and 4, we suppose that agents are already (a priori) attached to centers (groups). However, in Section 5 we propose a strategic game in which agents can move freely across centers. That is, each agent selects the center through which she wants to submit her claim.

A closely related model to the one proposed in this chapter is the *multi-issue allocation (MIA)* model, where each agent makes several claims related to different issues. Two approaches to MIA situations have been studied in the literature. In the first (see, e.g., Calleja et al., 2005; González-Alcón et al., 2007), a rule assigns a single amount to each agent. In the second (e.g., Bergantiños et al., 2010, 2011; Lorenzo-Freire et al., 2010; Moreno-Ternero, 2009), a rule assigns an amount to each agent for each issue and the allocation process is also conducted by means of a two-stage procedure. First, the resource is distributed between issues and, second, it is divided among agents. In fact, each decentralized rationing problem can be easily reinterpreted as a MIA situation (following the second approach), where the groups play the role of issues and each agent (being a member of exactly one group) claims for only one issue.

Within the MIA framework, a two-stage extension of the constrained equal awards rule is axiomatically characterized by Lorenzo-Freire et al. (2010) and Bergantiños et al. (2011). The former also characterize (by using duality relations) a two-stage extension of the constrained equal losses rule. The latter use, in their characterization of the two-stage extension of the constrained equal awards rule, the axioms of *equal treatment within the issues* and *equal treatment between the issues*,[3] among other axioms. The first

[3] Lorenzo-Freire et al. (2010) also use these same axioms in their characterizations, but they call them *equal treatment for the players within an issue* and *equal treatment for the*

of these properties states that if two agents have equal claims for an issue, they should receive equal amounts in this issue. The second states that if the total (aggregate) claim over two issues is the same, the total amount assigned to both issues should coincide. However, if each agent claims for just one issue, a MIA situation can be reinterpreted as a decentralized problem in which issues are identified with groups. In this decentralized context, it is relevant to consider the property of *individual inter-group equal treatment of equals*, which states that if two agents are members of different groups and both have equal claims, they should receive equal amounts. In this chapter, we characterize the decentralized (two-stage extension) proportional rule by using just this last property and *intra-group equal treatment of equals*[4] (see Theorem 1).

The overriding goal in this chapter is to find solutions that assign the same allocation regardless of whether the resource is distributed directly to agents or in a decentralized manner. We name these solutions *decentralized consistent* rules. Regarding this point, in the MIA context, Moreno-Ternero (2009) characterizes the proportional rule as the only anonymous rule that assigns the same allocation directly, or through a two-stage procedure. In Section 4, we prove that decentralized consistency (without anonymity) is enough to characterize the proportional rule (see Theorem 3). In this section we also show that decentralized consistency is equivalent to other concepts studied in the literature, such as strategy-proofness[5] (O'Neill, 1982) and non manipulability[6] (de Frutos, 1999) (see Proposition 5). De Frutos studies *"rules that are immune to strategic manipulations whereby a group of creditors merge (i.e., consolidate their claims) in order to represent a single creditor, or a*

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issues, respectively.

[4]When issues play the role of groups intra-group equal treatment of equals is equivalent to equal treatment within the issues.

[5]Chun (1988) states that "strategy-proofness requires that merging or splitting groups of claimants (an operation which changes the number of claimants) is never globally beneficial to the members of that group."

[6]Non manipulability and strategy-proofness are equivalent.

single creditor splits her claim to represent several creditors." On the other hand, within the MIA framework, Bergantiños et al. (2010) provide a characterization of a two-stage extension of the proportional rule by using the properties of non-advantageous transfer across issues and non-advantageous transfer within issues which adapt the axiom of non-advantageous reallocation (Chun, 1988) to MIA situations.

The remainder of the chapter is organized as follows. In Section 2, we introduce the main notations, describe decentralized rationing problems and define decentralized rationing rules. In Section 3, we analyse (see Propositions 1, 2 and 3) whether the extensions to this model of the constrained equal awards, the constrained equal losses and the proportional rules satisfy some essential properties that characterize their corresponding classical centralized rules (see Moulin, 2000). Moreover, we characterize the extension of the proportional rule on the subdomain of problems with rational claims by using the property of *individual equal treatment of equals*[7] (see Theorem 1). In Section 4, we analyse the conditions under which the constrained equal awards and the constrained equal losses rules satisfy decentralized consistency for some particular partitions (see Theorem 2 and Corollary 1, respectively). Finally, we prove that the only decentralized consistent rule is the proportional rule (see Theorem 3). In Section 5, we propose and analyse a strategic game based on the decentralized model and we check the existence of equilibrium (no agent has an incentive to move from one group to another) when the extension of the proportional distribution is applied as an allocation rule (see Corollary 3). Moreover, we also show the existence of equilibrium under certain conditions when the extensions of the constrained equal awards and of the constrained equal losses rules are applied (see Propositions 6 and 7, respectively). In Section 6, we conclude.

[7] If two agents have equal claims (regardless of their membership), then they should receive equal amounts.

2 Decentralized rationing problems and rules

We first restate the definition of the standard rationing problem or, from now on, the *centralized rationing problem.* As usual, let $\mathbb{N}$ be the set of natural numbers that we identify with the universe of potential agents, and let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. Given $T \in \mathcal{N}$, we denote by t the cardinality of T. On the other hand, given a finite subset of agents $N = \{1, 2, \ldots, n\} \in \mathcal{N}$, a *centralized rationing problem* (r, c) for N aims to distribute $r \geq 0$ among these n agents with claims $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^N_+$. It is assumed that $r \leq \sum_{i \in N} c_i$ since otherwise no rationing problem exists. Let $\mathcal{R}^N$ denote the domain of all centralized rationing problems with agent set N. The family of all centralized rationing problems is $\mathcal{R} = \cup_{N \in \mathcal{N}} \mathcal{R}^N$.

A *centralized rationing rule* F associates to each problem (r, c) a unique allocation $F(r, c) = x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$ such that $0 \leq x_i \leq c_i$, for all $i \in N$, and $\sum_{i \in N} x_i = r$, where x_i represents the payoff to agent $i \in N$.

The three most important rules in centralized rationing problems are the *constrained equal awards* (CEA) rule, which aims to equalize gains; the *constrained equal losses* (CEL) rule, which aims to equalize losses from claims; and, the *proportional* (P) rule, which allocates the amount of resource proportionally to the claims. Formally, for any $(r, c) \in \mathcal{R}^N$, with $N \in \mathcal{N}$, the CEA rule is defined as $CEA_i(r, c) = \min\{c_i, \lambda\}$ for all $i \in N$, the CEL rule is defined as $CEL_i(r, c) = \max\{0, c_i - \lambda\}$ for all $i \in N$ and the P rule is defined as $P_i(r, c) = \lambda \cdot c_i$ for all $i \in N$, where, in each case, $\lambda \in \mathbb{R}_+$ is chosen such that the resultant allocation is efficient.

Now, assume that in a decentralized rationing problem the agent set N has been partitioned in g groups. That is, we consider a partition $\mathcal{P}$ of N formed by a collection of non-empty subsets of N, $\{N_1, N_2, \ldots, N_g\}$, such that $N = \cup_{j \in \{1,2,\ldots,g\}} N_j$ with $N_j \cap N_{j'} = \emptyset$, for all $j \neq j' \in \{1, 2, \ldots, g\}$ and $N_j \neq \emptyset$, for all $j \in \{1, 2, \ldots, g\}$. Let us denote by $G = \{1, 2, \ldots, g\}$ the set of groups. Associated to each group $j \in G$ we have a vector of claims $c^j = (c_i)_{i \in N_j} \in \mathbb{R}^{N_j}_+$, where c_i represents the claim of agent i who belongs to group j.

Definition 1 *Let $N = \{1, 2, \ldots, n\} \in \mathcal{N}$ be a finite set of agents. A* decentralized rationing problem *for N is a triple $(r, c, \mathcal{P})$, where $r \in \mathbb{R}_+$ is the amount of resource, $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}_+^N$ is the claims vector and $\mathcal{P} = \{N_j\}_{j \in G}$ is a partition of the agent set N, such that*

$$r \leq \sum_{j \in G} \sum_{i \in N_j} c_i = \sum_{i \in N} c_i \textit{ (scarcity condition).}$$

Let us denote by $\mathcal{D}^N$ the set of all decentralized rationing problems with agent set N, and by $\mathcal{D} = \cup_{N \in \mathcal{N}} \mathcal{D}^N$ the family of all decentralized rationing problems.

We suppose that the resource cannot be directly assigned to agents. Therefore, we assume that it has to be first allocated to groups and, in a second stage, each group divides the amount received among its members. That is:

1. First, the amount r is allocated to groups according to a centralized rationing rule F^1. This allocation is obtained as the solution to a centralized rationing problem $\big(r, C^{\mathcal{P}} = (C^1, C^2, \ldots, C^g)\big)$, where the groups play the role of agents and the claim related to each group is the total (aggregate) claim of its members, i.e. $C^j = \sum_{i \in N_j} c_i$, for $j = \{1, 2, \ldots, g\}$. Therefore, each group $j \in G$ receives $F_j^1(r, C^{\mathcal{P}})$.

2. Second, the amount that each group $j \in G$ receives, i.e. $F_j^1(r, C^{\mathcal{P}})$, is divided among its members. The allocation is obtained as the solution of a centralized rationing problem $\big(F_j^1(r, C^{\mathcal{P}}), c^j\big)$ according to a centralized rationing rule F^2. Then, each agent $i \in N_j$ receives $F_i^2\big(F_j^1(r, C^{\mathcal{P}}), c^j\big)$.

Next, we formally describe this two-stage procedure.

Definition 2 *Let F^1 and F^2 be two centralized rationing rules. A* decentralized rationing rule *relative to F^1 and F^2 is a function $\mathcal{F}$ that associates to each problem $(r, c, \mathcal{P}) \in \mathcal{D}^N$ a unique allocation $\mathcal{F}(r, c, \mathcal{P}) \in \mathbb{R}_+^N$ as follows: for all $j \in G$ and all $i \in N_j$,*

$$\mathcal{F}_i(r,c,\mathcal{P}) := F_i^2\big(F_j^1(r,C^{\mathcal{P}}),c^j\big).$$

A particular case of this two-stage procedure occurs when both centralized rationing rules (F^1 and F^2) are the same ($F^1 = F^2 = F$). We name these solutions *self-decentralized rationing rules*. Examples of self-decentralized rationing rules are the following.

Definition 3 *The* decentralized constrained equal awards (CEA^d) *rule is the self-decentralized rationing rule that takes* $F^1 = F^2 = CEA$*. The* decentralized constrained equal losses (CEL^d) *rule takes* $F^1 = F^2 = CEL$ *and the* decentralized proportional (P^d) *rule takes* $F^1 = F^2 = P$*.*

Remark 1 *Note that each of these self-decentralized rationing rules generalizes its corresponding centralized rationing rule, in the sense that* $\mathcal{F}(r,c,\mathcal{P})$ *coincides with* $F(r,c)$ *if agents are not divided in groups (there is only one group), i.e.* $\mathcal{P} = \{N\}$ *(see Figure 1 (a)).*

Let us illustrate the application of each of these three decentralized rationing rules with an example.

Example 1 *Consider the five-person and two-group decentralized rationing problem (see Figure 1 (b))*

$$\begin{aligned}(r,c,\mathcal{P}) &= \Big(r, \big(c_1,c_2,c_3,c_4,c_5\big), \big\{N_1,N_2\big\}\Big)\\ &= \Big(150, \big(40,60,10,40,100\big), \big\{\{1,2\},\{3,4,5\}\big\}\Big).\end{aligned}$$

Then, the vector of total claims of the groups is

$$C^{\mathcal{P}} = \big(C^1,C^2\big) = \big(c_1+c_2, c_3+c_4+c_5\big) = (100,150).$$

The allocation assigned by the CEA^d *rule is*

$$CEA^d(r,c,\mathcal{P}) = \big(x_1,x_2,x_3,x_4,x_5\big) = \big(37.5,37.5,10,32.5,32.5\big).$$

Recall that this allocation is the result of a two-stage procedure,[8] *i.e.*

[8]Given a vector $x \in \mathbb{R}^N$ and a subset $T \subseteq N$, we denote by $x_{|T} \in \mathbb{R}^T$ the vector x restricted to the members of T.

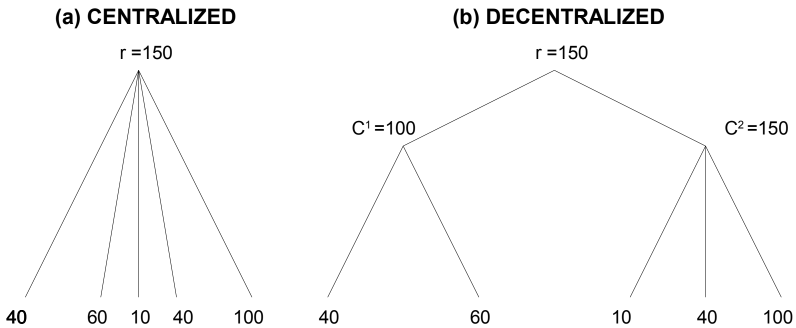

Figure 1: **Centralized and decentralized examples.**

(a) The five-person centralized rationing problem $(r, c) = \big(150, (40, 60, 10, 40, 100)\big)$. **(b)** The five-person and two-group decentralized rationing problem $(r, c, \mathcal{P}) = \big(150, (40, 60, 10, 40, 100), \big\{\{1, 2\}, \{3, 4, 5\}\big\}\big)$.

$$CEA^d(r, c, \mathcal{P})_{|N_j} = CEA\big(CEA_j(r, C^{\mathcal{P}}), c^j\big), \textit{ for all } j \in G.$$

1. *First, the amount of resource is allocated to groups. In this case,*

$$CEA(r, C^{\mathcal{P}}) = CEA\big(r, (C^1, C^2)\big) = CEA\big(150, (100, 150)\big) = (75, 75).$$

2. *Second, the amount that each group receives is divided among its members:*

$$\begin{aligned} \big(x_1, x_2\big) \quad &= CEA\big(CEA_1(r, C^{\mathcal{P}}), c^1\big) = CEA\big(75, (40, 60)\big) = (37.5, 37.5), \\ \big(x_3, x_4, x_5\big) \quad &= CEA\big(CEA_2(r, C^{\mathcal{P}}), c^2\big) = CEA\big(75, (10, 40, 100)\big) \\ &= (10, 32.5, 32.5). \end{aligned}$$

In the same example if we apply the CEL^d and the P^d rules we obtain

$$\begin{aligned} &CEL^d(r, c, \mathcal{P}) = \big(15, 35, 0, 20, 80\big) \textit{ and} \\ &P^d(r, c, \mathcal{P}) = \big(24, 36, 6, 24, 60\big), \textit{ respectively.} \end{aligned}$$

In contrast to decentralized rationing rules, the corresponding centralized rationing rules assign

$$CEA(r,c) = (35, 35, 10, 35, 35),$$
$$CEL(r,c) = (17.5, 37.5, 0, 17.5, 77.5) \text{ and}$$
$$P(r,c) = (24, 36, 6, 24, 60), \text{ respectively.}$$

Notice that $CEA(r,c) \neq CEA^d(r,c,\mathcal{P})$, $CEL(r,c) \neq CEL^d(r,c,\mathcal{P})$, *but* $P(r,c) = P^d(r,c,\mathcal{P})$.

3 Analysis of properties

In the previous section we have studied the decentralization of the CEA, the CEL and the P rules. We selected these rules as they are the most frequently used in the resolution of real rationing situations. Moulin (2000) states[9] the following characterization of these three rules:

> *There are exactly three rules on $\mathcal{R}$ satisfying equal treatment of equals, scale invariance, path-independence, composition and consistency: The constrained equal awards, the constrained equal-losses and the proportional rules.*

In this section, we adapt these five essential properties to the new framework and we show that the CEA^d and the CEL^d rules do not satisfy all these properties. Later, in the next section, we analyse the conditions under which they preserve these properties.

In the centralized framework, *equal treatment of equals* states that if two agents have equal claims, then they should receive equal amounts. Since in the decentralized framework agents are not only characterized by their claims but also by their membership of a group, equal treatment of equals arises from a combination of two subproperties.

[9] Corollary of Theorem 2 on page 662 of Moulin (2000).

First, *intra-group equal treatment of equals*[10] states that if two agents are members of the same group and both have equal claims, then they should receive equal amounts.

Definition 4 *A decentralized rationing rule $\mathcal{F}$ satisfies* intra-group equal treatment of equals *if for all $N \in \mathcal{N}$ and all $(r, c, \mathcal{P}) \in \mathcal{D}^N$, it holds*

$$\text{if } c_i = c_{i'}, \text{ with } i, i' \in N_j \in \mathcal{P}, \text{ then } \mathcal{F}_i(r, c, \mathcal{P}) = \mathcal{F}_{i'}(r, c, \mathcal{P}).$$

It is obvious that a decentralized rationing rule $\mathcal{F}$ relative to F^1 and F^2 satisfies intra-group equal treatment of equals if and only if the centralized rationing rule F^2 satisfies equal treatment of equals.

Second, *individual inter-group equal treatment of equals* states that if two agents are members of different groups and both have equal claims, then they should receive equal amounts.

Definition 5 *A decentralized rationing rule $\mathcal{F}$ satisfies* individual inter-group equal treatment of equals *if for all $N \in \mathcal{N}$ and all $(r, c, \mathcal{P}) \in \mathcal{D}^N$, it holds*

$$\text{if } c_i = c_{i'}, \text{ with } i \in N_j \in \mathcal{P} \text{ and } i' \in N_{j'} \in \mathcal{P},\ j \neq j',$$
$$\text{then } \mathcal{F}_i(r, c, \mathcal{P}) = \mathcal{F}_{i'}(r, c, \mathcal{P}).$$

Two members of different groups might have equal claims, but if the group's characteristics (number of members, members' claims) differ, then these two agents are not identical and, thus, the individual inter-group equal treatment of equals property is usually not satisfied. Indeed, as the reader can verify, in Example 1 in spite of the claims of agents 1 and 4 being equal ($c_1 = c_4 = 40$), the CEA^d rule (the CEL^d rule) suggests different allocations for these agents,

[10] In the context of rationing problems with a priori unions, Casas-Méndez et al. (2003) and Borm et al. (2005) define an equivalent property. They call it *equal treatment within the unions*. On the other hand, in the context of multi-issue allocation situations an equivalent property, when issues play the role of groups, is defined. Lorenzo-Freire et al. (2010) call this *equal treatment within the issues* and Bergantiños et al. (2011) call it *equal treatment for the players within an issue*.

i.e. $CEA_1^d(r,c,\mathcal{P}) = 37.5 \neq 32.5 = CEA_4^d(r,c,\mathcal{P})$ and $CEL_1^d(r,c,\mathcal{P}) = 15 \neq 20 = CEL_4^d(r,c,\mathcal{P})$.

Finally, *individual equal treatment of equals* arises as a result of the combination of the above two properties: if two agents have equal claims (regardless of their membership), then they should receive equal amounts.

Definition 6 *A decentralized rationing rule $\mathcal{F}$ satisfies* individual equal treatment of equals *if it satisfies* intra-group equal treatment of equals *and* individual inter-group equal treatment of equals.

We have checked in Example 1 that the CEA^d and the CEL^d rules do not satisfy individual inter-group equal treatment of equals. Thus, we conclude that these rules do not satisfy individual equal treatment of equals. In fact, we next show (in Theorem 1 below) that, for the case in which all claims are rational numbers, the only self-decentralized rationing rule that satisfies individual equal treatment of equals is the decentralized proportional rule. In order to state the theorem we denote by

$$\mathcal{D}_{\mathbb{Q}}^N := \left\{(r,c,\mathcal{P}) \in \mathcal{D}^N \middle| c_i \in \mathbb{Q}_+, \text{ for all } i \in N\right\}$$

the domain of all decentralized rationing problems with agent set N and all claims being rational numbers. The family of all these problems is denoted by $\mathcal{D}_{\mathbb{Q}} := \cup_{N \in \mathcal{N}} \mathcal{D}_{\mathbb{Q}}^N$.

Theorem 1 *A self-decentralized rationing rule $\mathcal{F}$ on $\mathcal{D}_{\mathbb{Q}}$ satisfies* individual equal treatment of equals *if and only if it is the decentralized proportional rule.*

Proof. The proof of the "if" part is straightforward. Next, we prove the "only if" part. That is, if a self-decentralized rationing rule $\mathcal{F}$ satisfies individual equal treatment of equals, then this rule is the decentralized proportional rule, i.e. $\mathcal{F} = P^d$.

For the sake of simplicity, let us first prove the case where claims are natural numbers, i.e. $c_i \in \mathbb{N}$, for all $i \in N$. We denote by $\mathcal{D}_{\mathbb{N}}^N$ this subset of problems. Let $(r, c, \mathcal{P}) \in \mathcal{D}_{\mathbb{N}}^{N}$ be a decentralized rationing problem and let $\mathcal{F}$ be a self-decentralized rationing rule that satisfies individual equal treatment of equals. We must prove that, for all $i \in N$,

$$\mathcal{F}_i(r, c, \mathcal{P}) = P_i^d(r, c, \mathcal{P}),$$

where $\mathcal{P} = \{N_j\}_{j\in G}$ is an arbitrary partition of N. To this end, consider the associated decentralized rationing problem $\left(r, \widehat{c}, \widehat{\mathcal{P}}\right) \in \mathcal{D}_{\mathbb{N}}^{\widehat{N}}$ where

- $\widehat{N} = \{1, 2, \ldots, \widehat{n}\}$ with $\widehat{n} = \sum_{i\in N} c_i$,
- $\widehat{c}_k = 1$, for all $k \in \widehat{N}$ and
- $\widehat{\mathcal{P}} = \{\widehat{N}_j\}_{j\in G}$ is such that $|\widehat{N}_j| = \sum_{i\in N_j} c_i = C^j$, for all $j \in G$.

Notice that every agent splits her claim into unit claims and, by definition, we have that

$$\widehat{C}^{\widehat{\mathcal{P}}} = (\widehat{C}^1, \widehat{C}^2, \ldots, \widehat{C}^g) = (C^1, C^2, \ldots, C^g) = C^{\mathcal{P}}. \tag{1}$$

Next, we claim that, for all $k \in \widehat{N}$,

$$\mathcal{F}_k\left(r, \widehat{c}, \widehat{\mathcal{P}}\right) = \frac{r}{\widehat{n}} = \frac{r}{\sum_{i\in N} c_i}. \tag{2}$$

To check this, consider two cases:

Case 1: If $\widehat{\mathcal{P}} = \{\widehat{N}\}$. Since all claims are equal ($c_k = 1$, for all $k \in \widehat{N}$) and $\mathcal{F}$ satisfies intra-group equal treatment of equals the proof of this case is done.

Case 2: If $\widehat{\mathcal{P}}$ contains at least two groups. Since all claims are equal ($c_k = 1$, for all $k \in \widehat{N}$) and since $\mathcal{F}$ satisfies individual inter-group equal treatment of equals, the proof of this case is straightforward, and the proof of the claim is done.

At this point, we focus on the resolution of the first stage of the $\mathcal{F}$ rule. In this stage, by (1) and (2), we have that, for all $j \in G$,

$$\begin{aligned} F_j(r, C^{\mathcal{P}}) &= F_j(r, \widehat{C}^{\widehat{\mathcal{P}}}) = \textstyle\sum_{k\in\widehat{N}_j} \mathcal{F}_k(r, \widehat{c}, \widehat{\mathcal{P}}) = |\widehat{N}_j| \cdot \dfrac{r}{\sum_{i\in N} c_i} \\ &= r \cdot \dfrac{C^j}{\sum_{i\in N} c_i} = P_j(r, C^{\mathcal{P}}). \end{aligned} \tag{3}$$

Next, take an arbitrary $j \in G$ and denote $r_j = F_j(r, \widehat{C}^{\widehat{\mathcal{P}}})$. Then, consider the centralized rationing problem $(r_j, \widehat{c}^j) \in \mathcal{R}^{\widehat{N}_j}$, where $\widehat{c}^j = (c_k)_{k \in \widehat{N}_j}$, and associate with it a decentralized rationing problem $\left(r_j, \widetilde{c}, \widetilde{\mathcal{P}}\right) \in \mathcal{D}^{\widetilde{N}}$ where

- $\widetilde{N} = \widehat{N}_j$,
- $\widetilde{c}_k = 1$, for all $k \in \widetilde{N}$ and
- $\widetilde{\mathcal{P}} = \{\widetilde{N}_i\}_{i \in N_j}$ is such that $|\widetilde{N}_i| = c_i$, for all $i \in N_j$.

Notice that,

$$\widetilde{C}^{\widetilde{\mathcal{P}}} = (c_i)_{i \in N_j} = c^j. \tag{4}$$

On the other hand, following the same guidelines used to prove (2), we obtain that, for all $k \in \widetilde{N}$,

$$\mathcal{F}_k(r_j, \widetilde{c}, \widetilde{\mathcal{P}}) = \frac{r_j}{|\widetilde{N}|} = \frac{r_j}{|\widehat{N}_j|} = \frac{r_j}{\sum_{i \in N_j} c_i}. \tag{5}$$

Thus, by (4) and (5), for all $i \in N_j$,

$$\begin{aligned} \mathcal{F}_i(r, c, \mathcal{P}) \; &= F_i(r_j, c^j) = F_i(r_j, \widetilde{C}^{\widetilde{\mathcal{P}}}) = \textstyle\sum_{k \in \widetilde{N}_i} \mathcal{F}_k(r_j, \widetilde{c}, \widetilde{\mathcal{P}}) \\ &= |\widetilde{N}_i| \cdot \frac{r_j}{\sum_{i \in N_j} c_i} = r_j \cdot \frac{c_i}{\sum_{i \in N_j} c_i} = P_i(r_j, c^j). \end{aligned} \tag{6}$$

Since $j \in G$ is an arbitrary group, by (3) and (6), it follows that, for all $j \in G$ and all $i \in N_j$,

$$\mathcal{F}_i(r, c, \mathcal{P}) = F_i(F_j(r, C^{\mathcal{P}}), c^j) = P_i(P_j(r, C^{\mathcal{P}}), c^j) = P_i^d(r, c, \mathcal{P}). \tag{7}$$

We can easily extend this proof to the whole family of decentralized rationing problems with rational claims $(r, c, \mathcal{P}) \in \mathcal{D}^N_{\mathbb{Q}}$ by simply splitting the agents' claims as follows. Notice that for all $i \in N$, $c_i = \frac{a_i}{b_i}$, where $a_i \in \mathbb{N}_+$ and $b_i \in \mathbb{N}_{++}$. Then, if we denote by lm the least common multiple of $(b_i)_{i \in N}$, we have that $c_i = \frac{a_i \cdot lm / b_i}{lm}$, for all $i \in N$, and, thus, $a_i \cdot lm/b_i \in \mathbb{N}_+$, for all $i \in N$. To complete the proof just split the agents' claims into several equal claims of value $\frac{1}{lm}$.

□

The next four properties express the invariance of the solution with respect to certain changes in the parameters of the problem. A rule is *scale invariant* if it is homogeneous of degree one. Thus, changes in the measurement unit will not have any effect on the final allocation.

Definition 7 *A decentralized rationing rule $\mathcal{F}$ satisfies* scale invariance *if for all $N \in \mathcal{N}$, all $(r, c, \mathcal{P}) \in \mathcal{D}^N$ and any positive real number $\theta \in \mathbb{R}_{++}$, it holds that*

$$\mathcal{F}(\theta r, \theta c, \mathcal{P}) = \theta \cdot \mathcal{F}(r, c, \mathcal{P}).$$

The proof of the next proposition can be found in the Appendix.

Proposition 1 *Let $\mathcal{F}$ be a decentralized rationing rule relative to F^1 and F^2 where F^1 and F^2 satisfy scale invariance, then $\mathcal{F}$ also satisfies scale invariance.*

The property of composition was introduced in the centralized rationing model by Young (1988). This property states that the result of directly allocating the amount of resource r is the same as that achieved when first distributing a smaller amount r' and, after that, distributing the remaining quantity $r - r'$ in a new problem in which the claim of each agent $i \in N$ is diminished by the amount initially received, i.e. $c_i - \mathcal{F}_i(r', c, \mathcal{P})$.

Definition 8 *A decentralized rationing rule $\mathcal{F}$ satisfies* composition *if for all $N \in \mathcal{N}$ and all $(r, c, \mathcal{P}) \in \mathcal{D}^N$ it holds*

$$\mathcal{F}(r, c, \mathcal{P}) = \mathcal{F}(r', c, \mathcal{P}) + \mathcal{F}\big(r - r', c - \mathcal{F}(r', c, \mathcal{P}), \mathcal{P}\big),$$

where $0 \leq r' \leq r \leq \sum_{i \in N} c_i$.

The proof of the next proposition is provided in the Appendix.

Proposition 2 *Let $\mathcal{F}$ be a decentralized rationing rule relative to F^1 and F^2 where F^1 and F^2 satisfy composition, then $\mathcal{F}$ also satisfies composition.*

The property of path-independence was introduced in the centralized rationing model by Moulin (1987). This property states that if we apply a decentralized rationing rule $\mathcal{F}$ to a problem $(r', c, \mathcal{P})$ but resource availability diminishes suddenly, that is $r' > r$, the new allocation obtained by applying the same rule again (to the new amount and with the original claims), i.e. $\mathcal{F}(r, c, \mathcal{P})$, is equal to that obtained when using the previous allocation as claims, i.e. $\mathcal{F}\big(r, \mathcal{F}(r', c, \mathcal{P}), \mathcal{P}\big)$.

Definition 9 *A decentralized rationing rule $\mathcal{F}$ satisfies* path-independence *if for all $N \in \mathcal{N}$ and all $(r, c, \mathcal{P}) \in \mathcal{D}^N$ it holds*

$$\mathcal{F}(r, c, \mathcal{P}) = \mathcal{F}\big(r, \mathcal{F}(r', c, \mathcal{P}), \mathcal{P}\big),$$

where $r \leq r' \leq \sum_{i \in N} c_i$.

At this point, let us recall that if a rule satisfies either composition or path-independence, then it is monotonic with respect to r. That is, for all $N \in \mathcal{N}$, all $c \in \mathbb{R}^N_+$ and all r, r':

$$\Big\{r \leq r' \leq \sum_{i \in N} c_i\Big\} \Rightarrow \big\{F(r, c) \leq F(r', c)\big\}. \tag{8}$$

This property is known as *resource monotonicity.*

Proposition 3 *Let $\mathcal{F}$ be a decentralized rationing rule relative to F^1 and F^2 where F^1 and F^2 satisfy path-independence, then $\mathcal{F}$ also satisfies path-independence.*

The proof of this proposition can be found in the Appendix.

Finally, consistency allows us to reduce any problem to any subset of agents. Consistency requires that when we re-evaluate the resource allocation within a subgroup of agents using the same rule, the allocation does not change. Henceforth, we use the following notation: for all $c \in \mathbb{R}^N_+$, all $\mathcal{P} = \{N_j\}_{j \in G}$ of N and all $T \subseteq N$, $c_{|T} = (c_i)_{i \in T} \in \mathbb{R}^T_+$ and $\mathcal{P}_{|T} = \big\{N_j \cap T \big| \; j \in G \text{ and } N_j \cap T \neq \emptyset\big\}$.

Definition 10 *A decentralized rationing rule* $\mathcal{F}$ *is* consistent *if for all* $N \in \mathcal{N}$*, all* $(r, c, \mathcal{P}) \in \mathcal{D}^N$ *and all* $T \subseteq N$ *with* $T \neq \emptyset$ *it holds*

$$\mathcal{F}(r, c, \mathcal{P})_{|T} = \mathcal{F}\Big(r - \sum_{i \in N \setminus T} \mathcal{F}_i(r, c, \mathcal{P}), c_{|T}, \mathcal{P}_{|T}\Big).$$

If T is formed by entire groups, then every decentralized rationing rule ($\mathcal{F}$) relative to consistent centralized rationing rules (F^1 and F^2) satisfies the consistency condition. If T is not formed by entire groups, then there are decentralized rationing rules relative to consistent centralized rationing rules that do not satisfy the consistency condition. For instance, the CEA^d and the CEL^d rules are not consistent. To check this, see Example 1 and imagine that agent 4 leaves the problem with her intended assignment $\big(CEA^d_4(r, c, \mathcal{P}) = x_4 = 32.5$ and $CEL^d_4(r, c, \mathcal{P}) = x_4 = 20\big)$. If we solve the problem

$$\big(r - x_4, c_{|N\setminus\{4\}}, \mathcal{P}_{|N\setminus\{4\}}\big) = \big(r - x_4, (c_1, c_2, c_3, c_5), \big\{\{1, 2\}, \{3, 5\}\big\}\big),$$

we obtain

$$CEA^d\big(r - x_4, c_{|N\setminus\{4\}}, \mathcal{P}_{|N\setminus\{4\}}\big) = \big(x_1, x_2, x_3, x_5\big) = \big(29.375, 29.375, 10, 48.75\big)$$
$$\text{and } CEL^d\big(r - x_4, c_{|N\setminus\{4\}}, \mathcal{P}_{|N\setminus\{4\}}\big) = \big(x_1, x_2, x_3, x_5\big) = \big(20, 40, 0, 70\big).$$

Notice that these payoffs clearly differ from the initial payoffs (see Example 1). Therefore, Moulin's characterization (see page 11) cannot be extended to this framework, since the CEA^d and the CEL^d rules satisfy neither consistency, nor individual equal treatment of equals.

So far we have adapted to the decentralized framework properties that are common to the CEA, the CEL and the P rules in the centralized framework. Now, we extend two properties to the decentralized framework, each of which is considered to be the essence of either the CEA rule or the CEL rule in the centralized framework. Such properties represent dual criteria as to how a solution should treat agents with small enough claims.

The exemption property says that if an agent has a small enough claim (below the average amount of resource), then she should not suffer rationing, and, thus, only agents with larger claims will suffer from rationing. Thus, this property states that the smaller claims are not responsible for scarcity.

Definition 11 *A decentralized rationing rule $\mathcal{F}$ satisfies* exemption *if for all $N \in \mathcal{N}$ and all $(r, c, \mathcal{P}) \in \mathcal{D}^N$, it holds that*

$$\text{if } c_i \leq \frac{r}{n} \text{ then } \mathcal{F}_i(r, c, \mathcal{P}) = c_i.$$

This property is considered the essence of the CEA rule, since it gives priority to those agents with smaller claims. When the CEA rule is extended to the decentralized framework, it does not satisfy this property. Let us illustrate this with an example.

Example 2 *Consider the fourteen-person and three-group decentralized rationing problem $(r, c, \mathcal{P})$ defined as (see Figure 2)*

- $r = 120$,
- $c_i = \begin{cases} 5 & \text{if } i = 1, 2, \ldots, 10, \\ 50 & \text{if } i = 11, 12, \\ 20 & \text{if } i = 13, \\ 80 & \text{if } i = 14 \text{ and} \end{cases}$
- $\mathcal{P} = \Big\{ \{1, 2, \ldots, 10\}, \{11, 12\}, \{13, 14\} \Big\}$.

Then, the vector of total claims of the groups is

$$C^{\mathcal{P}} = \big(C^1, C^2, C^3\big) = \big(c_1 + c_2 + \ldots + c_{10}, c_{11} + c_{12}, c_{13} + c_{14}\big) = (50, 100, 100).$$

The allocation assigned by the CEA^d rule is

$$\begin{aligned} CEA^d(r, c, \mathcal{P}) &= \big(x_1, x_2, \ldots, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\big) \\ &= \big(4, 4, \ldots, 4, 20, 20, 20, 20\big). \end{aligned}$$

Notice that the claim of every member of group 1 is smaller than the average amount of resource, i.e. $c_i = 5 < \frac{r}{n} = \frac{120}{14} = 8.57$, for all $i \in N_1$, but the *payoff assigned by the* CEA^d *rule to each of these agents is lower than her claim, i.e.* $x_i = 4 < c_i = 5$*, for all* $i \in N_1$*. Therefore, we conclude that the* CEA^d *rule does not satisfy the exemption property. However, observe that the claim of agent* 13 *which is well above the average amount of resource, i.e.* $c_{13} = 20 > \frac{r}{n}$*, is entirely satisfied, i.e.* $x_{13} = c_{13}$.

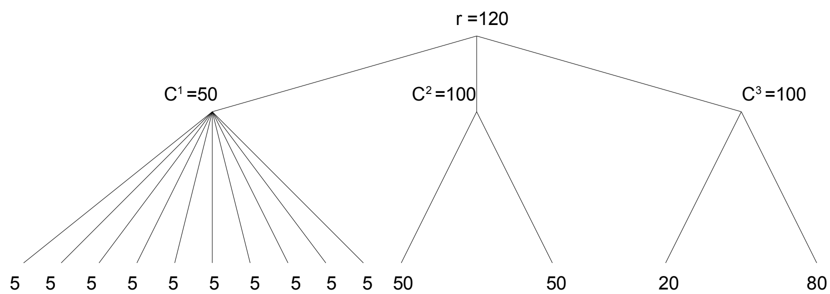

Figure 2: **The fourteen-person and three-group decentralized rationing problem** $(r, c, \mathcal{P})$.

The exclusion property implies the dual principle that the exemption property suggests. That is, an agent with a small enough claim (below that of the average loss) should not receive anything. Thus, the exclusion property states that irrelevant claims should be ignored.[11]

Definition 12 *A decentralized rationing rule* $\mathcal{F}$ *satisfies* exclusion *if for all* $N \in \mathcal{N}$ *and all* $(r, c, \mathcal{P}) \in \mathcal{D}^N$*, it holds that*

$$\text{if } c_i \leq \frac{\sum_{j \in G} \sum_{i \in N_j} c_i - r}{n} = \frac{\sum_{i \in N} c_i - r}{n} \text{ then } \mathcal{F}_i(r, c, \mathcal{P}) = 0.$$

[11] Notice that exemption and exclusion also characterize the CEA and the CEL rules, respectively, for the centralized framework. For instance, the CEA rule is the only rule satisfying exemption, consistency and path-independence and the CEL rule is the only rule satisfying exclusion, consistency and composition (Herrero and Villar, 2001). Let us recall that we use both characterizations to axiomatize the *generalized equal awards* and *generalized equal losses* rules in Chapter 2.

This property is considered the essence of the CEL rule, since it gives priority to those agents with larger claims.[12] When the CEL rule is extended to the decentralized framework, it does not satisfy this property. We can illustrate this by using Example 2. For this fourteen-person and three-group decentralized problem the allocation assigned by the CEL^d rule is

$$\begin{aligned} CEL^d(r,c,\mathcal{P}) &= \big(x_1, x_2, \ldots, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\big) \\ &= \Big(\tfrac{2}{3}, \tfrac{2}{3}, \ldots, \tfrac{2}{3}, \tfrac{85}{3}, \tfrac{85}{3}, 0, \tfrac{170}{3}\Big). \end{aligned}$$

Notice that the claim of every member of group 1 is smaller than the average loss, i.e. $c_i = 5 < \frac{\sum_{j\in G}\sum_{i\in N_j} c_i - r}{n} = \frac{130}{14} = 9.29$, for all $i \in N_1$, but the CEL^d rule assigns to each of these agents a positive payoff, i.e. $x_i = \frac{2}{3} > 0$, for all $i \in N_1$. Therefore, we conclude that the CEL^d does not satisfy the exclusion property. However, observe that agent 13, who has a claim well above the average loss, i.e. $c_{13} = 20 > \frac{\sum_{j\in G}\sum_{i\in N_j} c_i - r}{n}$, does not receive anything, i.e. $x_{13} = 0$.

To conclude this section, we adapt the duality relations between rules to the decentralized framework. Two rules are the dual of each other if one rule distributes the total gain r, in the same way as the other rule distributes the total loss $\sum_{i\in N} c_i - r$.

Definition 13 *Two decentralized rationing rules $\mathcal{F}^*$ and $\mathcal{F}$ are the* dual of each other*, if, for all $N \in \mathcal{N}$ and all $(r,c,\mathcal{P}) \in \mathcal{D}^N$,*

$$\mathcal{F}^*(r,c,\mathcal{P}) = c - \mathcal{F}\Big(\sum_{i\in N} c_i - r, c, \mathcal{P}\Big).$$

The proof of the next proposition is provided in the Appendix.

Proposition 4 *Let $\mathcal{F}$ and $\mathcal{F}^*$ be two self-decentralized*[13] *rationing rules relative to F and F^*, respectively. If F and F^* are the dual of each other, then $\mathcal{F}$ and $\mathcal{F}^*$ are also the dual of each other.*

[12] Note that, since the proportional rule allocates the resource proportionally to the claims, it gives priority neither to smaller nor larger claims.

[13] Notice that this proposition can easily be extended to the more general case of decentralized rationing rules.

Therefore, since the CEA and the CEL rules are the dual of each other and the P rule is self-dual (dual of itself), then the CEA^d and the CEL^d rules are also the dual of each other and the P^d rule is also self-dual. The duality between the CEA^d and the CEL^d rules is crucial for proving Corollary 1.

4 Decentralized consistent rules

In this section we aim to find centralized solutions F that assign the same allocation regardless of whether the resource is distributed directly to agents, i.e. $F(r, c)$, or in a decentralized manner (with agents grouped $(r, c, \mathcal{P})$) taking $F^1 = F^2 = F$. If this occurs, we say that the centralized rule F is a *decentralized consistent* rule. As we have already shown in Example 1, the CEA and the CEL rules do not satisfy decentralized consistency.

Definition 14 *A centralized rationing rule F on $\mathcal{R}$ satisfies* decentralized consistency *if for any arbitrary partition $\mathcal{P} = \{N_j\}_{j \in G}$ of N it holds that*

$$F(r, c) = \mathcal{F}(r, c, \mathcal{P}),$$

where $\mathcal{F}$ is the self-decentralized rationing rule relative to F.

The main implication for a centralized rule of satisfying decentralized consistency is that its corresponding self-decentralized rule preserves all the properties that characterize it in the centralized framework. In Section 3 we have shown that the CEA^d and the CEL^d rules do not preserve some properties that characterize the CEA and the CEL rules in the centralized context. This is because the CEA and the CEL rules are not decentralized consistent rules (see Example 1). Next, we analyse the conditions under which these rules satisfy decentralized consistency for some particular partitions. Previously, we introduce some individual and group features that will be useful in meeting this aim.

Individual features: We say that

- agent i is *marginal* if her claim is smaller than the average amount of resource, i.e. $c_i \leq \frac{r}{n}$.
- agent i is *marginal in losses* if her claim is smaller than the average loss, i.e. $c_i \leq \frac{\sum_{i \in N} c_i - r}{n}$.

Group features: We say that

- two groups $j, j' \in G$ are *homogeneous in the number of agents* if both have the same number of agents, i.e. $|N_j| = |N_{j'}|$.
- group j is *marginal* if its total claim is smaller than the average amount of resource with respect to the g groups, i.e. $C^j \leq \frac{r}{g}$.
- group j is *marginal in losses* if its total claim is smaller than the average loss with respect to the g groups, i.e. $C^j \leq \frac{\sum_{j \in G} C^j - r}{g}$.

Now, imagine a centralized rationing problem without marginal agents. Then, the CEA rule is a *decentralized consistent* rule for a partition $\mathcal{P}$ if and only if all groups in $\mathcal{P}$ are homogeneous in terms of the number of agents.

Theorem 2 *Let $(r, c) \in \mathcal{R}^N$ be a centralized rationing problem, where $c_i > \frac{r}{n}$, for all $i \in N$, and let $(r, c, \mathcal{P}) \in \mathcal{D}^N$ be the decentralized problem associated with a partition $\mathcal{P} = \{N_j\}_{j \in G}$ of N. Then,*

$$CEA(r, c) = CEA^d(r, c, \mathcal{P}) \text{ if and only if } |N_j| = |N_{j'}|, \text{ for all } j, j' \in G.$$

Proof. First, notice that, since $c_i > \frac{r}{n}$, for all $i \in N$,

$$CEA_i(r, c) = \frac{r}{n}, \text{ for all } i \in N. \tag{9}$$

Now, we prove the "if" part. Since $CEA(r, c) = CEA^d(r, c, \mathcal{P})$ and by (9), it follows that, for all $k \in G$,

$$CEA_k(r, C^{\mathcal{P}}) = \sum_{i \in N_k} CEA_i^d(r, c, \mathcal{P}) = |N_k| \cdot \frac{r}{n}. \tag{10}$$

Now, suppose on the contrary that there are two groups $j, j' \in G$ such that $|N_j| > |N_{j'}|$. Then, by (10), we obtain that

$$\begin{aligned} CEA_j(r, C^{\mathcal{P}}) &= \min\{\lambda_1, C^j\} = |N_j| \cdot \frac{r}{n} \\ &> |N_{j'}| \cdot \frac{r}{n} = \min\left\{\lambda_1, C^{j'}\right\} = CEA_{j'}(r, C^{\mathcal{P}}), \end{aligned} \tag{11}$$

where $\lambda_1 \in \mathbb{R}_+$ is such that $\sum_{k\in G} CEA_k(r, C^{\mathcal{P}}) = r$. Hence,

$$\lambda_1 \geq \min\left\{\lambda_1, C^j\right\} > \min\left\{\lambda_1, C^{j'}\right\},$$

which implies that $\lambda_1 > C^{j'}$ and thus, by (11),

$$C^{j'} = |N_{j'}| \cdot \frac{r}{n}. \tag{12}$$

But, since $c_i > \frac{r}{n}$, for all $i \in N$, $C^{j'} = \sum_{i\in N_{j'}} c_i > |N_{j'}| \cdot \frac{r}{n}$, which contradicts (12).

Next, we prove the "only if" part. By (9), we have to prove that

$$CEA_i^d(r, c, \mathcal{P}) = \frac{r}{n}, \text{ for all } i \in N.$$

First, we claim that $CEA_k(r, C^{\mathcal{P}}) = |N_k| \cdot \frac{r}{n}$, for all $k \in G$. Suppose that this is not true; then, by efficiency, there are two groups $j, j' \in G$ such that $CEA_j(r, C^{\mathcal{P}}) > |N_j| \cdot \frac{r}{n}$ and $CEA_{j'}(r, C^{\mathcal{P}}) < |N_{j'}| \cdot \frac{r}{n}$. Therefore, since $CEA_j(r, C^{\mathcal{P}}) = \min\{\lambda_1, C^j\}$, where $\lambda_1 \in \mathbb{R}_+$ is such that $r = \sum_{k\in G} CEA_k(r, C^{\mathcal{P}})$, we have that $\min\{\lambda_1, C^j\} > |N_j| \cdot \frac{r}{n}$. Hence, $\lambda_1 > |N_j| \cdot \frac{r}{n}$. Thus, since $|N_k| = |N_{k'}|$, for all $k, k' \in G$, we obtain that

$$\lambda_1 > |N_k| \cdot \frac{r}{n}, \text{ for all } k \in G. \tag{13}$$

On the other hand, since $c_i > \frac{r}{n}$, for all $i \in N$, it holds that

$$C^k > |N_k| \cdot \frac{r}{n}, \text{ for all } k \in G. \tag{14}$$

Therefore, by efficiency, (13) and (14), we obtain that

$$r = \sum_{k\in G} CEA_k(r, C^{\mathcal{P}}) = \sum_{k\in G} \min\left\{\lambda_1, C^k\right\} > \sum_{k\in G} |N_k| \cdot \frac{r}{n} = r,$$

which is a contradiction.

Finally, take an arbitrary $j \in G$. Then, we have to prove that $CEA_i\big(|N_j| \cdot \frac{r}{n}, c^j\big) = \frac{r}{n}$, for all $i \in N_j$. Now, notice that, since $|N_j| = |N_{j'}|$, for all $j, j' \in G$, $n = \sum_{j \in G} |N_j| = g \cdot |N_j|$. Hence, since $CEA_j(r, C^{\mathcal{P}}) = |N_j| \cdot \frac{r}{n}$ and the allocation $\big(\frac{r}{n}\big)_{i \in N_j} = \big(\frac{r}{g \cdot |N_j|}\big)_{i \in N_j}$ is feasible, the result trivially holds.

□

Notice that, because of the conditions in Theorem 2, the CEA rule allocates the average amount of resource to each agent. Thus, the CEA rule coincides with the *equal awards* (EA) *function*.[14] That is, if $c_i > \frac{r}{n}$, for all $i \in N$, and the partition $\mathcal{P} = \{N_j\}_{j \in G}$ of N is such that $|N_j| = |N_{j'}|$, for all $j, j' \in G$, then $CEA(r, c) = EA(r, c) = CEA^d(r, c, \mathcal{P}) = \big(\frac{r}{n}\big)_{i \in N}$.

Now, we turn to losses and imagine a centralized rationing problem where no agent is marginal in losses. Then, the CEL rule is a *decentralized consistent* rule for a partition $\mathcal{P}$ if and only if all groups in $\mathcal{P}$ are homogeneous in the number of agents.

Corollary 1 *Let* $(r, c) \in \mathcal{R}^N$ *be a centralized rationing problem, where* $c_i > \frac{\sum_{i \in N} c_i - r}{n}$*, for all* $i \in N$*, and let* $(r, c, \mathcal{P}) \in \mathcal{D}^N$ *be the decentralized problem associated with a partition* $\mathcal{P} = \{N_j\}_{j \in G}$ *of* N*. Then,*

$$CEL(r, c) = CEL^d(r, c, \mathcal{P}) \text{ if and only if } |N_j| = |N_{j'}|, \text{ for all } j, j' \in G.$$

This corollary arises from Proposition 4 and Theorem 2.

Analogously to the case of the CEA rule, because of the conditions in Corollary 1, the CEL rule allocates the average loss to each agent. Thus, the CEL rule coincides with the *equal losses* (EL) *function*.[15] That is, if $c_i > \frac{\sum_{i \in N} c_i - r}{n}$, for all $i \in N$, and the partition $\mathcal{P} = \{N_j\}_{j \in G}$ of N is such that

[14] The EA function distributes the available amount of resource equally among agents, i.e. for any $(r, c) \in \mathcal{R}^N$, with $N \in \mathcal{N}$, $EA(r, c) = (\frac{r}{n})_{i \in N}$. Notice that since the EA function is not constrained by claims, it does not necessarily select a feasible allocation.

[15] The EL function assigns the total loss equally to agents, i.e. for any $(r, c) \in \mathcal{R}^N$, with $N \in \mathcal{N}$, $EL(r, c) = (c_i - \frac{\sum_{i \in N} c_i - r}{n})_{i \in N}$. Notice that, the EL function can assign negative payoffs, and thus, it does not necessarily select a feasible allocation.

$|N_j| = |N_{j'}|$, for all $j, j' \in G$, then $CEL(r,c) = EL(r,c) = CEL^d(r,c,\mathcal{P}) = \left(c_i - \frac{\sum_{i\in N} c_i - r}{n}\right)_{i\in N}$.

De Frutos (1999) studies the non manipulability of rules. A rule is non manipulable if no agent has an incentive to split her claim in several claims and no subset of agents has an incentive to merge their claims in a single claim.

Definition 15 *A centralized rationing rule F is a* non manipulable *rule if for all $T, N \in \mathcal{N}$ with $T \subset N$, all $(r,c) \in \mathcal{R}^N$ and all $(r,c') \in \mathcal{R}^T$ such that $c'_i = c_i + \sum_{k\in N\setminus T} c_k$, for some $i \in N$, and $c'_k = c_k$, for all $k \in T \setminus \{i\}$, it holds that*

$$F_i(r,c') = F_i(r,c) + \sum_{k\in N\setminus T} F_k(r,c).$$

The next proposition states that decentralized consistency and non manipulability are equivalent.

Proposition 5 *A centralized rationing rule F is a decentralized consistent rule if and only if it is a non manipulable rule.*

Proof. First, we prove that if F is a decentralized consistent rule, then it is a non manipulable rule. Recall (Definition 14) that we define F as a decentralized consistent rule if for all $N \in \mathcal{N}$, all $(r,c) \in \mathcal{R}^N$ and for any arbitrary partition $\mathcal{P} = \{N_j\}_{j\in G}$ of N it holds that, for all $j \in G$ and all $i \in N_j$,

$$F_i(r,c) = \mathcal{F}_i(r,c,\mathcal{P}) = F_i\big(F_j(r,C^{\mathcal{P}}),c^j\big). \tag{15}$$

Let $T, N \in \mathcal{N}$ be two arbitrary finite sets of agents such that $T \subset N = \{1,2,\ldots,n\}$ with $T \neq \emptyset$. Suppose w.l.o.g. that $T = \{1,2,\ldots,t\}$ and, thus, $N \setminus T = \{t+1, t+2, \ldots, n\}$. Take an arbitrary agent $i \in T$ and consider the partition $\mathcal{P} = \big\{\{1\},\{2\},\ldots,\{i-1\},\{i\} \cup N \setminus T, \{i+1\},\ldots,\{t\}\big\}$ of N; henceforth, we name $N_i = \{i\} \cup N \setminus T$. Notice that the components of the vector $C^{\mathcal{P}}$ are $C^k = c_k$, for all $k \in T \setminus \{i\}$ and $C^i = c_i + \sum_{k \in N \setminus T} c_k$. Hence,

$$
\begin{aligned}
F(r, C^{\mathcal{P}}) &= F\big(r, (C^1, C^2, \ldots, C^{i-1}, C^i, C^{i+1}, \ldots, C^t)\big) \\
&= F\Big(r, \Big(c_1, c_2, \ldots, c_{i-1}, c_i + \sum_{k \in N \setminus T} c_k, c_{i+1}, \ldots, c_t\Big)\Big).
\end{aligned}
$$

On the other hand, since F is a decentralized consistent rule (see (15)) and by efficiency of F, we obtain that

$$
\begin{aligned}
F_i(r, c) + \sum_{k \in N \setminus T} F_k(r, c) &= \mathcal{F}_i(r, c, \mathcal{P}) + \sum_{k \in N \setminus T} \mathcal{F}_k(r, c, \mathcal{P}) \\
&= \sum_{k \in \{i\} \cup N \setminus T} F_k\big(F_i(r, C^{\mathcal{P}}), c^i\big) \\
&= \sum_{k \in \{i\} \cup N \setminus T} F_k\big(F_i(r, C^{\mathcal{P}}), (c_i, c_{t+1}, c_{t+2}, \ldots, c_n)\big) \\
&= F_i(r, C^{\mathcal{P}}).
\end{aligned}
$$

Thus, we conclude that F is non manipulable.

Next, we prove that if F is a non manipulable rule, then it is also a decentralized consistent rule.

First, we claim that if F is non manipulable, then, for any arbitrary partition $\mathcal{P} = \{N_j\}_{j \in G}$ of N, $F_j(r, C^{\mathcal{P}}) = \sum_{i \in N_j} F_i(r, c)$, for all $j \in G$. To check this, take an arbitrary group $j \in G$ and consider the partition $\mathcal{P}_a = \big\{N_j, \{i\}_{i \in N \setminus N_j}\big\}$ of N. Since F is a non manipulable rule, then

$$
\sum_{i \in N_j} F_i(r, c) = F_j(r, C^{\mathcal{P}_a}). \tag{16}
$$

Next, suppose that every agent $i \in N \setminus N_j$ merges in one group. That is, consider the partition $\mathcal{P}_b = \big\{N_j, N_{j'}\big\}$ of N, where $N_{j'} = \cup_{i \in N \setminus N_j} \{i\}$. Then, since F is a non manipulable rule, we have that

$$
\sum_{i \in N_{j'}} F_i(r, C^{\mathcal{P}_a}) = F_{j'}(r, C^{\mathcal{P}_b}), \tag{17}
$$

where $F_{j'}(r, C^{\mathcal{P}_b})$ refers to the payoff assigned to the group j' according to F in the two-person centralized rationing problem $(r, C^{\mathcal{P}_b})$.

Taking into account (16), (17) and by efficiency of F, we obtain

$$\begin{aligned} F_j(r, C^{\mathcal{P}_b}) &= r - F_{j'}(r, C^{\mathcal{P}_b}) = r - \sum_{i \in N_{j'}} F_i(r, C^{\mathcal{P}_a}) = F_j(r, C^{\mathcal{P}_a}) \\ &= \sum_{i \in N_j} F_i(r, c). \end{aligned} \tag{18}$$

At this point, split $N_{j'}$ in $|G| - 1$ groups. That is, consider the partition $\mathcal{P}_c = \{N_k\}_{k \in G} = \mathcal{P}$ of N. Then, since F is a non manipulable rule, we have that

$$F_{j'}(r, C^{\mathcal{P}_b}) = \sum_{k \in G \setminus \{j\}} F_k(r, C^{\mathcal{P}}),$$

and thus, by efficiency,

$$F_j(r, C^{\mathcal{P}_b}) = F_j(r, C^{\mathcal{P}}). \tag{19}$$

Therefore, by (18) and (19), we have that $F_j(r, C^{\mathcal{P}}) = \sum_{i \in N_j} F_i(r, c)$. Since $j \in G$ is an arbitrary group we have

$$F_j(r, C^{\mathcal{P}}) = \sum_{i \in N_j} F_i(r, c), \text{ for all } j \in G, \tag{20}$$

and, thus, the claim is proved.

Next, we have to prove that $F\big(F_k(r, C^{\mathcal{P}}), c^k\big) = F(r, c)_{|N_k}$, for all $k \in G$. Since the only non manipulable rule is the proportional rule (de Frutos, 1999), we have $F = P$ and since the P rule is consistent, it follows that F is also consistent.

Take an arbitrary group $j \in G$. Then

$$\begin{aligned} F(r, c)_{|N_j} &= F\Big(r - \sum_{i \in N \setminus N_j} F_i(r, c), c_{|N_j}\Big) = F\Big(r - \sum_{k \in G \setminus \{j\}} F_k(r, C^{\mathcal{P}}), c^j\Big) \\ &= F\Big(F_j(r, C^{\mathcal{P}}), c^j\Big), \end{aligned}$$

where the first, second and third equalities follow, respectively, from consistency of F, by (20) and by efficiency of F. Therefore, we conclude that F is a decentralized consistent rule.

□

In the context of MIA situations, Moreno-Ternero (2009) proves that the proportional rule is the only anonymous[16] rule that assigns the same allocation directly or via a two-stage procedure. We show that the anonymity requirement can be dropped, in the context of decentralized rationing problems, from his characterization result.

Theorem 3 *A centralized rationing rule F on $\mathcal{R}$ is a decentralized consistent rule if and only if it is the proportional rule.*

Proof. First we prove the "if" part. Let $(r, c) \in \mathcal{R}^N$ be a centralized rationing problem and $\mathcal{P} = \{N_j\}_{j \in G}$ be an arbitrary partition of N. We have to prove that the P rule is a decentralized consistent rule, i.e. $P(r, c) = P^d(r, c, \mathcal{P})$.

By definition of the P^d rule we have that, for all $j \in G$ and all $i \in N_j$,

$$P_i^d(r, c, \mathcal{P}) = P_i\big(P_j(r, C^{\mathcal{P}}), c^j\big). \tag{21}$$

Then, simply applying the P rule, we obtain that, for all $j \in G$ and all $i \in N_j$,

$$\begin{aligned} P_i\big(P_j(r, C^{\mathcal{P}}), c^j\big) &= P_i\left(r \cdot \frac{C^j}{\sum_{j \in G} C^j}, c^j\right) = \left(r \cdot \frac{C^j}{\sum_{j \in G} C^j}\right) \cdot \frac{c_i}{C^j} \\ &= r \cdot \frac{c_i}{\sum_{j \in G} C^j} = r \cdot \frac{c_i}{\sum_{i \in N} c_i} = P_i(r, c). \end{aligned} \tag{22}$$

Therefore, by (21) and (22), we conclude that $P(r, c) = P^d(r, c, \mathcal{P})$, and thus, the "if" part of the proof is done.

Next, we prove the "only if" part. That is, if a centralized rationing rule F is a decentralized consistent rule, then this rule is the proportional one, i.e. $F = P$. By Proposition 5, since F is a decentralized consistent rule, F is a non manipulable rule. Finally, in line with de Frutos (1999) we know that the only non manipulable rule is the P rule, and thus, $F = P$.

[16] Let us denote by Π^N the class of bijections from N into itself. A rule F satisfies anonymity if for all $N \in \mathcal{N}$, all $(r, c) \in \mathcal{R}^N$, all $\pi \in \Pi^N$ and all $i \in N$, it holds that $F_{\pi(i)}\big(r, (c_{\pi(i)})_{i \in N}\big) = F_i(r, c)$.

□

From Theorem 3 and Moulin's characterization we obtain the next corollary.

Corollary 2 *Decentralized consistency implies scale invariance, equal treatment of equals, composition, path-independence and consistency.*

5 Strategic considerations of decentralized rationing rules

In this section we allow agents to move freely from one group to another. Therefore, whether an equilibrium situation exists (no agent has an incentive to move from one group to another) becomes relevant and we need to understand the strategic behaviour of agents.

In the previous section we have seen that the P rule is the only solution that assigns the same allocation, regardless of whether the resource is distributed directly to agents or in a decentralized manner. Thus, all possible situations (regardless of the group to which each agent opts to submit her claim) are of equilibrium when the P^d rule is applied. At this point, we need to determine whether there are other rules than that of the proportional rule that assign at equilibrium the same allocation directly or via a two-stage procedure.

To this end, consider g groups as intermediate distribution centers to which players (agents) can submit their claims. That is, each player selects the distribution center to which she wishes to submit her claim.

A *strategic rationing problem* is a tuple $SR = (N, G, r, c, \mathcal{F})$, where N is the agent set with $|N| = n \geq 2$, G is the set of intermediate distribution centers with $|G| = g \geq 2$, $r \in \mathbb{R}_+$ is the available amount of resource, $c \in \mathbb{R}^N_{++}$ is the claims vector such that $r \leq \sum_{i \in N} c_i$, and $\mathcal{F}$ is a decentralized rationing rule.

Next, we define a non-cooperative game (*decentralized rationing game*) where agents simultaneously choose the distribution center to which they want to submit their claims, and payoffs are determined according to a decentralized rationing rule.

Let $i \in N$ be a player; a strategy s_i of this player consists on selecting one of the g centers available in G. Given a *strategy profile* $s = (s_1, s_2, \ldots, s_n)$, the payoff to a player $i \in N$ is

$$u_i(s) = \mathcal{F}_i\big(r, c, \mathcal{P}(s)\big) = F_i^2\Big(F_{s_i}^1\big(r, C^{\mathcal{P}(s)}\big), c^{s_i}(s)\Big),$$

where $\mathcal{P}(s) = \{N_j(s) | j \in G \text{ and } N_j(s) \neq \emptyset\}$ is the partition of N according to s, with $N_j(s) = \{i \in N | s_i = j\}$, for all $j \in G$.

We denote by S the *set of all strategy profiles*. Let us illustrate decentralized rationing games with an example.

Example 3 *Suppose that a* central distributor *wants to allocate, according to the CEA rule, the amount of resource $r = 30$ among $n = 6$ players with an associated claims vector*

$$c = (c_1, c_2, c_3, c_4, c_5, c_6) = (1, 2, 4, 6, 7, 20).$$

However, suppose that the resource cannot be directly allocated to players. Then, the central distributor decentralizes the allocation process through two intermediate distribution centers, i.e. $g = 2$, where players are not a priori attached to any of these centers, but they can move freely between them. That is, we have the following strategic rationing problem

$$\big(N, G, r, c, \mathcal{F}\big) = \big(\{1, 2, 3, 4, 5, 6\}, \{1, 2\}, 30, (1, 2, 4, 6, 7, 20), CEA^d\big).$$

Consider the following two strategy profiles $s = (1, 1, 1, 1, 1, 2)$ and $s' = (1, 1, 1, 1, 2, 2)$. Then, the corresponding decentralized rationing problems are (see Figure 3):

a. $\big(r, c, \mathcal{P}(s)\big) = \Big(30, (1, 2, 4, 6, 7, 20), \big\{\{1, 2, 3, 4, 5\}, \{6\}\big\}\Big).$

b. $\big(r, c, \mathcal{P}(s')\big) = \Big(30, (1, 2, 4, 6, 7, 20), \big\{\{1, 2, 3, 4\}, \{5, 6\}\big\}\Big).$

Next, we apply the CEA^d rule to determine the payoff vector in each case.

a. $CEA^d\big(r, c, \mathcal{P}(s)\big) = \big(x_1, x_2, x_3, x_4, x_5, x_6\big) = (1, 2, 4, 4, 4, 15).$

b. $CEA^d\big(r, c, \mathcal{P}(s')\big) = \big(x_1, x_2, x_3, x_4, x_5, x_6\big) = (1, 2, 4, 6, 7, 10).$

Notice that s is not an equilibrium of the game since player 5, who claims 7 units and obtains 4 units, has an incentive to move from distribution center 1 to center 2, that is, attaining strategy profile s', where this player obtains all of her claim.

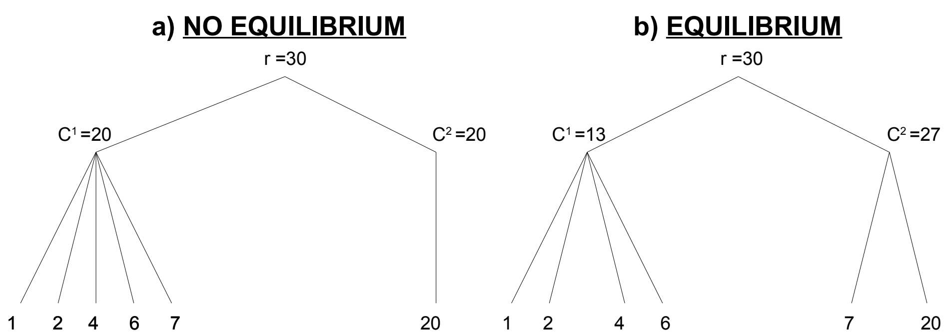

Figure 3: **The six-person and two-group decentralized rationing problems.**

a. $(r, c, \mathcal{P}(s)) = \big(30, (1, 2, 4, 6, 7, 20), \big\{\{1, 2, 3, 4, 5\}, \{6\}\big\}\big)$ and **b.** $(r, c, \mathcal{P}(s')) = \big(30, (1, 2, 4, 6, 7, 20), \big\{\{1, 2, 3, 4\}, \{5, 6\}\big\}\big).$

Let us point out that s' is an equilibrium of the game. Note that, in this equilibrium, players are not uniformly distributed between the centers $\big(|N_1(s')| - |N_2(s')| = 2\big)$. In fact, for this problem we can obtain 22 different equilibria in pure strategies. It is interesting to note that, even when players are uniformly distributed between the centers, this is not necessarily an equilibrium of the game; for instance, consider the strategy $s'' = (1, 1, 2, 2, 2, 1)$.

In this last example, the allocation attached to all these equilibria coincides with the payoff vector proposed by the CEA rule when applied to the centralized problem $(r, c) = \big(30, (1, 2, 4, 6, 7, 20)\big)$, i.e. if s is an equilibrium of the game, then $CEA(r,c) = CEA^d\big(r,c,\mathcal{P}(s)\big)$. However, this does not generally occur, as the following example shows.

Example 4 *In the six-person ($n = 6$) and two-center ($g = 2$) strategic rationing problem $SR = (N, G, r, c, CEA^d)$ where $r = 30$ and*

$$c = (1, 6, 8, 8, 15, 20),$$

there are 8 equilibria, but none of the resulting allocations coincides with the payoff vector proposed by the CEA rule in the corresponding centralized problem.

Next, we analyse the existence of equilibrium for the cases of the P^d, the CEA^d and the CEL^d rules. We know that the proportional rule assigns the same allocation regardless of whether the resource is distributed directly or in a decentralized manner (see Theorem 3). This implies that for any strategic rationing problem where the proportional rule is applied any strategy profile is an equilibrium of the game.

Corollary 3 *For any strategic rationing problem (N, G, r, c, P^d) and its associated decentralized rationing game Γ_{SR}, every strategy profile s is an equilibrium of the game.*

In the case of the CEA^d rule, we prove that for any strategic rationing problem where the number of players is large enough, any strategy profile in which players are distributed among centers as uniformly as possible is an equilibrium of the game.

Proposition 6 *Let (N, G, r, c, CEA^d) be a strategic rationing problem and let Γ_{SR} be the associated decentralized rationing game such that $n > \frac{r}{\min\limits_{i\in N}\{c_i\}} + 2g$. If $s \in S$ is such that $\Big||N_j(s)| - |N_{j'}(s)|\Big| \leq 1$, for all $j, j' \in G$, then*

a. *s is an equilibrium of the game, and*

b. *for all $k \in G$ and all $i \in N_k$,*

$$\left|CEA_i^d\big(r,c,\mathcal{P}(s)\big) - CEA_i(r,c)\right| \leq \max\left\{\frac{r}{g\cdot\left[\frac{n}{g}\right]} - \frac{r}{n}, \frac{r}{n} - \frac{r}{g\cdot\left(\left[\frac{n}{g}\right]+1\right)}\right\},$$

where $\left[\frac{n}{g}\right]$ is the integer part of $\frac{n}{g}$.

The proof of this proposition can be found in the Appendix. The payoff vector at equilibrium proposed by the CEA^d rule does not necessarily coincide with the payoff vector assigned by the CEA rule in the corresponding centralized problem (see Example 4). As a consequence of Proposition 6 *b.*, if the strategy profile s is such that players are distributed among centers as uniformly as possible and the number of players is large enough, then the payoff vectors obtained via centralized and decentralized procedures using the CEA rule approach each other[17] as the number of players rises.

In Proposition 6, we have shown that, for any strategic rationing problem relative to CEA^d, there exists at least one equilibrium under certain conditions.

Conjectures for the CEA^d rule

For the general case we have not reached any definitive conclusions, but we conjecture the following statements for any strategic rationing problem relative to the CEA^d rule:

C1. For any game, there exists at least one equilibrium.

C2. If all players have equal claims ($c_i = \bar{c} \in \mathbb{R}_+$, for all $i \in N$), then every strategy profile s is an equilibrium of the game if and only if the players are distributed among centers as uniformly as possible, i.e. $\big||N_j(s)| - |N_{j'}(s)|\big| \leq 1$, for all $j, j' \in G$.

C3. If no player is marginal ($c_i > \frac{r}{n}$, for all $i \in N$), then the only strategy profiles s that are equilibria are those in which the players are distributed among centers as uniformly as possible.

However, the fact that players are distributed as uniformly as possible does not imply that s is an equilibrium of the game. We can illustrate this point

[17] The difference between these payoffs does not decrease monotonically with respect to n, but with respect to $\left[\frac{n}{g}\right]$.

by analysing the decentralized problem

$$\big(r, (c_1, c_2, c_3), \mathcal{P}(s)\big) = \big(6, (2.1, 2.1, 4), \big\{\{1, 2\}, \{3\}\big\}\big),$$

where players have chosen the strategy profile $s = (s_1, s_2, s_3) = (1, 1, 2)$. Then, $CEA^d\big(r, (c_1, c_2, c_3), \mathcal{P}(s)\big) = (x_1, x_2, x_3) = (1.5, 1.5, 3)$, but, despite $|N_1(s)| - |N_2(s)| = 1$, s is not an equilibrium, since player 2 has an incentive to choose strategy $s'_2 = 2$. This is because

$$\begin{aligned} CEA^d\big(r, c, \mathcal{P}(s')\big) &= CEA^d\big(6, (2.1, 2.1, 4), \big\{\{1\}, \{2, 3\}\big\}\big) \\ &= (x_1, x_2, x_3) = (2.1, 1.95, 1.95). \end{aligned}$$

In the case of the CEL^d rule, we prove that for any strategic rationing problem where no player is marginal in losses, any strategy profile in which all players have chosen the same distribution center is an equilibrium of the game.

Proposition 7 *Let* (N, G, r, c, CEL^d) *be a strategic rationing problem and let* Γ_{SR} *be the associated decentralized rationing game such that* $c_i > \frac{\sum_{i \in N} c_i - r}{n}$, *for all* $i \in N$. *If* $s \in S$ *is such that* $|N_j(s)| = n$, *for some* $j \in G$ *and* $|N_{j'}(s)| = 0$, *for all* $j' \in G \setminus \{j\}$, *then*

a. *s is an equilibrium of the game, and*

b. $CEL^d\big(r, c, \mathcal{P}(s)\big) = CEL(r, c)$.

The proof can be found in the Appendix. Note that statement *b.* in Proposition 7 states that if the strategy profile s is such that all players choose the same distribution center and no player is marginal in losses, then the payoff vectors obtained in a centralized and in a decentralized manner by using the CEL rule coincide.

In Proposition 7, we have shown that, for any strategic rationing problem relative to CEL^d, there exists at least one equilibrium under certain conditions.

Conjectures for the CEL^d rule

For the general case, we have not reached any definitive conclusions, but we conjecture the following statements:

C'1. For any game, there exists at least one equilibrium.

C'2. If all players have equal claims ($c_i = \bar{c} \in \mathbb{R}_+$, for all $i \in N$), then every strategy profile s is an equilibrium of the game if and only if all players have chosen the same distribution center. In this case, we would conclude that if s is an equilibrium of the game, then $CEL(r, c) = CEL^d(r, c, \mathcal{P}(s))$. We also believe that we can extend this characterization to the more general case where no player is marginal in losses.

C'3. In the general case, the strategy profile s such that $|N_j(s)| = n$, for some $j \in G$ and $|N_{j'}(s)| = 0$, for all $j' \in G \setminus \{j\}$, is always an equilibrium, but it is not necessarily the only one.

6 Conclusion

In this chapter we have proved that the only decentralized consistent rule is the P rule (Theorem 3). We have also shown that the CEA and the CEL rules are decentralized consistent rules only under certain restrictive conditions and for some particular partitions. This is because the CEA^d and the CEL^d rules do not satisfy some essential properties, such as individual equal treatment of equals, and they are manipulable rules. In particular, these rules do not satisfy individual inter-group equal treatment of equals, which implies that two agents with equal claims might be discriminated against not only in terms of their claims, but also in terms of their membership. As a consequence, individual interests might enter in conflict with collective (group) interests. If all groups are homogeneous as regards the number of agents and no agent is marginal, then the conflict is avoided (see Theorem 2 and Corollary 1).

Table 1 compares the properties that are satisfied by centralized rationing rules in the centralized framework with those satisfied by their corresponding decentralized rationing rules in the decentralized framework.

	ETE	**SI**	**CMP**	**PI**	**CONS**	**CNT**	**EXE**	**EXC**	**DC**
CEA	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	X	X
CEA^d	X	$\surd$	$\surd$	$\surd$	X	$\surd$	X	X	
CEL	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	X	$\surd$	X
CEL^d	X	$\surd$	$\surd$	$\surd$	X	$\surd$	X	X	
P	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	X	X	$\surd$
P^d	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	X	X	

Table 1: **Properties of centralized and decentralized rationing rules.**

ETE=equal treatment of equals, **SI**=scale invariance, **CMP**=composition, **PI**=path-independence, **CONS**=consistency, **CNT**=continuity, **EXE**=exemption, **EXC**=exclusion and **DC**=decentralized consistency.

As can be seen in Table 1, two of the properties (**ETE** and **CONS**) that characterize the CEA, the CEL and the P rules in the centralized model are not satisfied by the CEA^d and CEL^d rules in the decentralized model. In fact, the only one of these three decentralized rules that still satisfies the properties (adapted to the decentralized framework) required by Moulin's characterization is the P^d rule. Indeed, by Theorem 1, the P^d rule is the only self-decentralized rule (on the subdomain of problems with rational claims) that satisfies individual equal treatment of equals (see Theorem 1).

In future research it might be interesting to decentralize other well-known rationing rules.

Based on the above results, we have also studied a strategic game in which each player selects the distribution center to which she wishes to submit her claim. The P^d rule also arises as an outstanding solution since if the P^d rule is applied, then any strategy profile is an equilibrium (see Corollary 3). In future research it might also be interesting to introduce a cost of access to the intermediate distribution centers. Then, players with greater needs (with larger claims) will be more willing to assume the cost. Thus, the utility can be represented by the difference between the payoff obtained and the cost of access to the intermediate distribution center. Another interesting extension of this strategic model would be to impose the rule by which each intermediate distribution center has maximum capacity to accommodate players or claims.

Appendix

Proof of Proposition 1 Let $(r, c, \mathcal{P}) \in \mathcal{D}^N$ be a decentralized rationing problem and let $\theta \in \mathbb{R}_{++}$ be a positive real number. Then, it follows that, for all $j \in G$,

$$\begin{aligned}\mathcal{F}(\theta r, \theta c, \mathcal{P})_{|N_j} &= F^2\big(F^1_j(\theta r, \theta C^{\mathcal{P}}), \theta c^j\big) = F^2\big(\theta \cdot F^1_j(r, C^{\mathcal{P}}), \theta c^j\big) \\ &= \theta \cdot F^2\big(F^1_j(r, C^{\mathcal{P}}), c^j\big) = \theta \cdot \mathcal{F}(r, c, \mathcal{P})_{|N_j}.\end{aligned}$$

where the second and the third equalities follow from the scale invariance of F^1 and of F^2, respectively. Since this holds for each group $j \in G$, we are done.

□

Proof of Proposition 2 Given a decentralized rationing problem $(r, c, \mathcal{P})$, we must prove that, for all $j \in G$,

$$\mathcal{F}(r, c, \mathcal{P})_{|N_j} = \mathcal{F}(r', c, \mathcal{P})_{|N_j} + \mathcal{F}\big(r - r', c - \mathcal{F}(r', c, \mathcal{P}), \mathcal{P}\big)_{|N_j}, \tag{23}$$

where $0 \leq r' \leq r$. If $r = r'$, the result is straightforward. If $r' < r$ then, for all $j \in G$,

$$\begin{aligned}
\mathcal{F}(r,c,\mathcal{P})_{|N_j} &= F^2(F^1_j(r,C^{\mathcal{P}}),c^j) \\
&= F^2\Big(F^1_j(r',C^{\mathcal{P}}) + F^1_j\big(r-r',C^{\mathcal{P}} - F^1(r',C^{\mathcal{P}})\big),c^j\Big) \\
&= F^2(F^1_j(r',C^{\mathcal{P}}),c^j) \\
&+F^2\big(F^1_j\big(r-r',C^{\mathcal{P}} - F^1(r',C^{\mathcal{P}})\big),c^j - F^2(F^1_j(r',C^{\mathcal{P}}),c^j)\big) \\
&= \mathcal{F}(r',c,\mathcal{P})_{|N_j} + F^2\Big(F^1_j\Big(r-r',C^{\mathcal{P}} \\
&-\Big(\sum_{i\in N_j}\mathcal{F}_i(r',c,\mathcal{P})\Big)_{j\in G}\Big),c^j - \mathcal{F}(r',c,\mathcal{P})_{|N_j}\Big) \\
&= \mathcal{F}(r',c,\mathcal{P})_{|N_j} + \mathcal{F}\big(r-r',c-\mathcal{F}(r',c,\mathcal{P}),\mathcal{P}\big)_{|N_j},
\end{aligned}$$

where the second and third equalities follow, respectively, from the fact that F^1 and F^2 satisfy composition. Therefore, (23) is fulfilled, and we are done. □

Proof of Proposition 3 Given a decentralized rationing problem $(r,c,\mathcal{P})$, we must prove that, for all $j \in G$,

$$\mathcal{F}(r,c,\mathcal{P})_{|N_j} = \mathcal{F}(r,\mathcal{F}(r',c,\mathcal{P}),\mathcal{P})_{|N_j}, \tag{24}$$

where $r \leq r'$. If $r = r'$, the result is straightforward. If $r < r'$ then, for all $j \in G$,

$$\begin{aligned}
\mathcal{F}(r,c,\mathcal{P})_{|N_j} &= F^2(F^1_j(r,C^{\mathcal{P}}),c^j) = F^2\Big(F^1_j\big(r,F^1(r',C^{\mathcal{P}})\big),c^j\Big) \\
&= F^2\Big(F^1_j\big(r,F^1(r',C^{\mathcal{P}})\big),F^2\Big(F^1_j\big(r',F^1(r',C^{\mathcal{P}})\big),c^j\Big)\Big) \\
&= F^2\Big(F^1_j\big(r,F^1(r',C^{\mathcal{P}})\big),F^2\big(F^1_j(r',C^{\mathcal{P}}),c^j\big)\Big) \\
&= F^2\Big(F^1_j\Big(r,\big(\textstyle\sum_{i\in N_j}\mathcal{F}_i(r',c,\mathcal{P})\big)_{j\in G}\Big),\mathcal{F}(r',c,\mathcal{P})_{|N_j}\Big) \\
&= \mathcal{F}(r,\mathcal{F}(r',c,\mathcal{P}),\mathcal{P})_{|N_j},
\end{aligned}$$

where the second and fourth equalities follow from the path-independence of F^1 and the third follows from the path-independence of F^2 and the *resource monotonicity* of F^1 (see (8)) since $r < r'$ and thus $F^1_j\big(r',F^1(r',C^{\mathcal{P}})\big) \geq F^1_j\big(r,F^1(r',C^{\mathcal{P}})\big)$, for all $j \in G$. Therefore, (24) is fulfilled, and we are done.

□

Proof of Proposition 4 First of all, since F^* is the dual rule of F, it holds that $F_i^*(r,c) = c_i - F_i(\sum_{i\in N} c_i - r, c)$, for all $i \in N$.

Let $(r,c,\mathcal{P}) \in \mathcal{D}^N$ be a decentralized rationing problem. Then, since F^* is the dual rule of F, it follows that, for all $j \in G$,

$$F_j^*(r, C^{\mathcal{P}}) = C^j - F_j\Big(\sum_{i\in N} c_i - r, C^{\mathcal{P}}\Big). \tag{25}$$

Next, take an arbitrary $j \in G$. Then, since F^* is the dual rule of F, we obtain that, for all $i \in N_j$,

$$\begin{aligned}\mathcal{F}_i^*(r,c,\mathcal{P}) \quad &= F_i^*(F_j^*(r,C^{\mathcal{P}}), c^j) = c_i - F_i\big(C^j - F_j^*(r,C^{\mathcal{P}}), c^j\big)\\ &= c_i - F_i\Big(F_j\Big(\textstyle\sum_{i\in N} c_i - r, C^{\mathcal{P}}\Big), c^j\Big)\\ &= c_i - \mathcal{F}_i(\textstyle\sum_{i\in N} c_i - r, c, \mathcal{P}),\end{aligned}$$

where the third equality follows from (25). Thus, $\mathcal{F}$ and $\mathcal{F}^*$ are the dual of each other and we are done.

□

Proof of Proposition 6 Let us first prove part a. We denote $\widehat{c} = \min\limits_{i\in N}\{c_i\}$. Notice that, by reordering the initial assumption $\big(n > \frac{r}{\widehat{c}} + 2g\big)$, we have that

$$\widehat{c} \cdot \Big(\frac{n}{g} - 2\Big) > \frac{r}{g}. \tag{26}$$

Then, since $c_i \geq \widehat{c} > 0$, for all $i \in N$, we get that, for all $i \in N$,

$$c_i \geq \widehat{c} > \frac{r}{g \cdot \big(\frac{n}{g} - 2\big)} = \frac{r}{n - 2\cdot g} > \frac{r}{n}, \tag{27}$$

and, thus, we conclude that no player is marginal.

Now, consider a strategy profile $s \in S$ such that $\big||N_j(s)| - |N_{j'}(s)|\big| \leq 1$, for all $j, j' \in G$ and take an arbitrary player $i^* \in N$ such that $s_{i^*} = j$. Next, consider a strategy profile $s' \in S$ such that $s'_{i^*} = j'$ and $s'_i = s_i$, for all $i \in N \setminus \{i^*\}$. Then, we consider two cases:

Case 1: $|N_j(s)| = |N_{j'}(s)| = \widetilde{N}$, for all $j, j' \in G$. First, by (26), for all $k \in G$,

$$C^k(s) = \sum_{i \in N_k(s)} c_i \geq \widehat{c} \cdot |N_k(s)| = \widehat{c} \cdot \frac{n}{g} > \widehat{c} \cdot \left(\frac{n}{g} - 2\right) > \frac{r}{g}. \tag{28}$$

Thus, there are no marginal groups. Hence, for all $k \in G$,

$$CEA_k\big(r, C^{\mathcal{P}(s)}\big) = \frac{r}{g}. \tag{29}$$

Furthermore, by the hypothesis of this case and since $\mathcal{P}(s)$ is a partition of N, it holds that $n = \sum_{k \in G} |N_k(s)| = g \cdot \widetilde{N}$. Hence, by (27), we obtain that, for all $i \in N$,

$$c_i \geq \widehat{c} > \frac{r}{n} = \frac{r}{g \cdot \widetilde{N}}. \tag{30}$$

Therefore, by (29) and (30), we have that, for all $k \in G$ and all $i \in N_k$,

$$\begin{aligned} CEA_i^d\big(r, c, \mathcal{P}(s)\big) &= CEA_i\big(CEA_k\big(r, C^{\mathcal{P}(s)}\big), c^k(s)\big) \\ &= CEA_i\left(\frac{r}{g}, c^k(s)\right) = \frac{r}{g \cdot |N_k(s)|} = \frac{r}{g \cdot \widetilde{N}} = \frac{r}{n}. \end{aligned} \tag{31}$$

On the other hand, notice that

$$|N_{j'}(s')| = |N_{j'}(s)| + 1 = |N_j(s)| + 1. \tag{32}$$

Hence, by (26), we have that

$$C^j(s') = \sum_{i \in N_j(s')} c_i \geq \widehat{c} \cdot |N_j(s')| = \widehat{c} \cdot \left(\frac{n}{g} - 1\right) > \widehat{c} \cdot \left(\frac{n}{g} - 2\right) > \frac{r}{g}. \tag{33}$$

Notice that this expression implies that $|N_j(s')| \geq 1$. Then, by (28), (32), (33) and since $s'_i = s_i$, for all $i \in N \backslash \{i^*\}$, it holds that $C^k(s') = \sum_{i \in N_k(s')} c_i > \frac{r}{g}$, for all $k \in G$ (there are no marginal groups). Thus, for all $k \in G$,

$$CEA_k\big(r, C^{\mathcal{P}(s')}\big) = \frac{r}{g}. \tag{34}$$

At this point, by (30) and (32), we have that, for all $i \in N_{j'}(s')$,

$$c_i \geq \widehat{c} > \frac{r}{n} = \frac{r}{g \cdot \widetilde{N}} = \frac{r}{g \cdot |N_{j'}(s)|} > \frac{r}{g \cdot |N_{j'}(s')|}.$$

Hence, from the last expression, by (31), (32) and (34), we conclude that

$$\begin{aligned} CEA^d_{i^*}\big(r,c,\mathcal{P}(s')\big) &= CEA_{i^*}\big(CEA_{j'}\big(r,C^{\mathcal{P}(s')}\big),c^{j'}(s')\big) \\ &= CEA_{i^*}\Big(\frac{r}{g},c^{j'}(s')\Big) = \frac{r}{g\cdot|N_{j'}(s')|} \\ &< \frac{r}{g\cdot|N_j(s)|} = \frac{r}{n} = CEA^d_{i^*}\big(r,c,\mathcal{P}(s)\big), \end{aligned}$$

Thus, player i^* has no incentive to move from distribution center j.

Case 2: There exist two distribution centers $j, j' \in G$ such that $|N_j(s)| - |N_{j'}(s)| = 1$. Notice that, $|N_j(s)| \geq |N_k(s)|$, for all $k \in G$, and $|N_{j'}(s)| \leq |N_k(s)|$, for all $k \in G$. First, by (26), we have that, for all $k \in G$,

$$C^k(s) = \sum_{i\in N_k(s)} c_i \geq \widehat{c}\cdot|N_k(s)| \geq \widehat{c}\cdot\Big(\frac{n}{g}-1\Big) > \widehat{c}\cdot\Big(\frac{n}{g}-2\Big) > \frac{r}{g}. \qquad (35)$$

Thus, there are no marginal groups. Hence, for all $k \in G$,

$$CEA_k\big(r,C^{\mathcal{P}(s)}\big) = \frac{r}{g}. \qquad (36)$$

Then, we consider two subcases:

Subcase 2a: $s_{i^*} = j$ and the strategy profile s' is such that $s'_{i^*} = k'$, where $k' \neq j \in G$, and $s'_i = s_i$, for all $i \in N \setminus \{i^*\}$.

First, notice that, since $|N_j(s)| \geq |N_k(s)|$, for all $k \in G$, and $n = \sum_{k\in G}|N_k(s)|$, it holds that $g\cdot|N_j(s)| > n$, and, thus, by (27),

$$c_i \geq \widehat{c} > \frac{r}{n} > \frac{r}{g\cdot|N_j(s)|}, \text{ for all } i \in N_j(s). \qquad (37)$$

Hence, by (36) and (37), we obtain that

$$\begin{aligned} CEA^d_{i^*}\big(r,c,\mathcal{P}(s)\big) &= CEA_{i^*}\big(CEA_j\big(r,C^{\mathcal{P}(s)}\big),c^j(s)\big) = CEA_{i^*}\Big(\frac{r}{g},c^j(s)\Big) \\ &= \frac{r}{g\cdot|N_j(s)|}. \end{aligned} \qquad (38)$$

On the other hand, notice that

$$|N_{k'}(s')| \geq |N_j(s)|. \qquad (39)$$

Moreover, by (35) and since $s'_i = s_i$, for all $i \in N \setminus \{i^*\}$, we have that $C^k(s') = \sum_{i \in N_k(s')} c_i > \frac{r}{g}$, for all $k \in G$. Thus,

$$CEA_k\big(r, C^{\mathcal{P}(s')}\big) = \frac{r}{g}, \text{ for all } k \in G. \tag{40}$$

At this point, since $|N_{k'}(s')| \geq |N_k(s')|$, for all $k \in G$, and $n = \sum_{k \in G} |N_k(s')|$, it holds that $g \cdot |N_{k'}(s')| > n$. Hence, by (27), we obtain that, for all $i \in N_{k'}(s')$,

$$c_i \geq \widehat{c} > \frac{r}{n} > \frac{r}{g \cdot |N_{k'}(s')|}.$$

Hence, by (38), (39) and (40), we have that

$$\begin{aligned} CEA^d_{i^*}\big(r, c, \mathcal{P}(s')\big) &= CEA_{i^*}\big(CEA_{k'}\big(r, C^{\mathcal{P}(s')}\big), c^{k'}(s')\big) \\ &= CEA_{i^*}\Big(\frac{r}{g}, c^{k'}(s')\Big) = \frac{r}{g \cdot |N_{k'}(s')|} \\ &\leq \frac{r}{g \cdot |N_j(s)|} = CEA^d_{i^*}\big(r, c, \mathcal{P}(s)\big). \end{aligned}$$

Therefore, we conclude that player i^* has no incentive to move from distribution center j.

Subcase 2b: $s_{i^*} = j'$ and the strategy profile s' is such that $s'_{i^*} = k'$, where $k' \neq j \in G$, and $s'_i = s_i$, for all $i \in N \setminus \{i^*\}$.

First, notice that if $CEA^d_{i^*}(r, c, \mathcal{P}(s)) = c_{i^*}$, then i^* has no incentive to move from j'. Otherwise,

$$CEA^d_{i^*}\big(r, c, \mathcal{P}(s)\big) \geq \frac{r}{g \cdot |N_{j'}(s)|}. \tag{41}$$

Second, by definition of s and s',

$$|N_{k'}(s')| > |N_{j'}(s)|. \tag{42}$$

Hence, by (26), we obtain that

$$C^{j'}(s') = \sum_{i \in N_{j'}(s')} c_i \geq \widehat{c} \cdot \Big(|N_{j'}(s)| - 1\Big) > \widehat{c} \cdot \Big(\Big(\frac{n}{g} - 1\Big) - 1\Big) > \frac{r}{g}.$$

From this last expression and since $|N_{j'}(s')| < |N_k(s')|$, for all $k \in G$, it holds that $C^k(s') > \frac{r}{g}$, for all $k \in G$. Thus,

$$CEA_k\big(r, C^{\mathcal{P}(s')}\big) = \frac{r}{g}, \text{ for all } k \in G. \tag{43}$$

On the other hand, since $|N_{k'}(s')| \geq |N_k(s')|$, for all $k \in G$, and since $n = \sum_{k \in G} |N_k(s')|$, it holds that $g \cdot |N_{k'}(s')| > n$. Hence, by (27), we obtain that, for all $i \in N_{k'}(s')$,

$$c_i \geq \widehat{c} > \frac{r}{n} > \frac{r}{g \cdot |N_{k'}(s')|}. \tag{44}$$

Finally, by (41), (42), (43) and (44), we have that

$$\begin{aligned} CEA^d_{i^*}\big(r, c, \mathcal{P}(s')\big) &= CEA_{i^*}\big(CEA_{k'}\big(r, C^{\mathcal{P}(s')}\big), c^{k'}(s')\big) \\ &= CEA_{i^*}\Big(\frac{r}{g}, c^{k'}(s')\Big) = \frac{r}{g \cdot |N_{k'}(s')|} \\ &< \frac{r}{g \cdot |N_{j'}(s)|} \leq CEA^d_{i^*}\big(r, c, \mathcal{P}(s)\big). \end{aligned}$$

Therefore, we conclude that player i^* has no incentive to move from the distribution center and, thus, the proof of part $a.$ is done.

Next, we prove part $b.$ First, notice that, since no agent is marginal (see (27)),

$$CEA_i(r, c) = \frac{r}{n}, \text{ for all } i \in N. \tag{45}$$

Moreover, observe that

$$\max\left\{\frac{r}{g \cdot \left[\frac{n}{g}\right]} - \frac{r}{n}, \frac{r}{n} - \frac{r}{g \cdot \left(\left[\frac{n}{g}\right] + 1\right)}\right\} \geq 0.$$

If $|N_j(s)| = |N_{j'}(s)|$, for all $j, j' \in G$, then, since no player is marginal (see (27)), we can apply Theorem 2, obtaining that $CEA^d_i(r, c, \mathcal{P}(s)) - CEA_i(r, c) = 0$, for all $i \in N$. Otherwise, there exist two distribution centers $j, j' \in G$ such that $|N_j(s)| - |N_{j'}(s)| = 1$, and, thus, $|N_j(s)| = \left[\frac{n}{g}\right] + 1$ and $|N_{j'}(s)| = \left[\frac{n}{g}\right]$. Then, by (26), we have that, for all $k \in G$,

$$C^k(s) = \sum_{i \in N_k(s)} c_i \geq \widehat{c} \cdot |N_k(s)| > \widehat{c} \cdot \Big(\frac{n}{g} - 2\Big) > \frac{r}{g},$$

and thus, for all $k \in G$,

$$CEA_k\big(r, C^{\mathcal{P}(s)}\big) = \frac{r}{g}. \tag{46}$$

By using (27), we obtain that, for all $i \in N$,

$$c_i \geq \widehat{c} > \frac{r}{g \cdot \left(\frac{n}{g} - 2\right)} > \frac{r}{g \cdot \left[\frac{n}{g}\right]} > \frac{r}{g \cdot \left(\left[\frac{n}{g}\right] + 1\right)}. \tag{47}$$

Hence, by (46) and (47), we conclude that, for all $k \in G$ and all $i \in N_k$,

$$\begin{aligned} CEA_i^d\big(r, c, \mathcal{P}(s)\big) &= CEA_i\Big(CEA_k\big(r, C^{\mathcal{P}(s)}\big), c^k(s)\Big) \\ &= CEA_i\Big(\frac{r}{g}, c^k(s)\Big) = \frac{r}{g \cdot |N_k(s)|}. \end{aligned}$$

Therefore, by (45), if $i \in N_k(s)$ is such that $|N_k(s)| = \left[\frac{n}{g}\right]$, then

$$CEA_i^d(r, c, \mathcal{P}(s)) - CEA_i(r, c) = \frac{r}{g \cdot \left[\frac{n}{g}\right]} - \frac{r}{n} > 0.$$

Otherwise, if $i \in N_k(s)$ is such that $|N_k(s)| = \left[\frac{n}{g}\right] + 1$, then

$$CEA_i(r, c) - CEA_i^d(r, c, \mathcal{P}(s)) = \frac{r}{n} - \frac{r}{g \cdot \left(\left[\frac{n}{g}\right] + 1\right)} > 0.$$

This ends the proof of part b.

□

Proof of Proposition 7 Let us first prove part a. We denote $\widehat{c} = \min\limits_{i \in N}\{c_i\}$. Now, notice that, since $n \geq 2$ and no player is marginal in losses, i.e.

$$c_i \geq \widehat{c} > \frac{\sum_{i \in N} c_i - r}{n}, \tag{48}$$

we have that $\sum_{i \in N} c_i - r < n \cdot \widehat{c} \leq \big(n + (n-2)\big) \cdot \widehat{c} = 2(n-1) \cdot \widehat{c}$. Hence, by reordering the last expression, we obtain that

$$\widehat{c} \cdot (n-1) > \frac{\sum_{i \in N} c_i - r}{2}. \tag{49}$$

Now, consider a strategy profile s such that $|N_j(s)| = n$, for some $j \in G$ and $|N_{j'}(s)| = 0$, for all $j' \in G \setminus \{j\}$. Next, take an arbitrary player $i^* \in N$

and consider a strategy profile $s' \in S$ such that $s'_{i^*} = j'$ and $s'_i = s_i = j$, for all $i \in N \setminus \{i^*\}$.

At this point, notice that, since $|N_j(s)| = n$ and by efficiency of the CEL rule, we have that $CEL_j\big(r, C^{\mathcal{P}(s)}\big) = r$. Thus, by (48) and since $c^j(s) = (c_i)_{i\in N_j(s)} = (c_i)_{i\in N} = c$, we obtain that, for all $i \in N_j(s) = N$,

$$\begin{aligned} CEL^d_i\big(r, c, \mathcal{P}(s)\big) &= CEL_i\Big(CEL_j\big(r, C^{\mathcal{P}(s)}\big), c^j(s)\Big) = CEL_i(r, c) \\ &= c_i - \frac{\sum_{i\in N} c_i - r}{n}. \end{aligned} \tag{50}$$

Observe that s' is such that $N_{j'}(s') = \{i^*\}$ and $N_j(s') = N \setminus \{i^*\}$. Then, we consider two cases:

Case 1: $C^{j'}(s') = c_{i^*} \leq \frac{\sum_{i\in N} c_i - r}{2}$. Hence, by the exclusion[18] property of the CEL rule, it holds that $CEL_{j'}\big(r, C^{\mathcal{P}(s)}\big) = 0$. Thus, since $N_{j'}(s') = \{i^*\}$, by (48) and (50), we have that $CEL^d_{i^*}\big(r, c, \mathcal{P}(s')\big) = CEL_{j'}\big(r, C^{\mathcal{P}(s)}\big) = 0 < c_{i^*} - \dfrac{\sum_{i\in N} c_i - r}{n} = CEL^d_{i^*}\big(r, c, \mathcal{P}(s)\big)$. Therefore, player i^* has no incentive to move from distribution center j.

Case 2: $C^{j'}(s') = c_{i^*} > \frac{\sum_{i\in N} c_i - r}{2}$. Notice that, since $N_j(s') = N \setminus \{i^*\}$ and by (49), we have that

$$C^j(s') = \sum_{i\in N_j(s')} c_i = \sum_{i\in N} c_i - c_{i^*} \geq \widehat{c} \cdot (n-1) > \frac{\sum_{i\in N} c_i - r}{2}.$$

Hence, $CEL_j\big(r, C^{\mathcal{P}(s')}\big) = C^j(s') - \frac{\sum_{i\in N} c_i - r}{2}$ and $CEL_{j'}\big(r, C^{\mathcal{P}(s')}\big) = C^{j'}(s') - \frac{\sum_{i\in N} c_i - r}{2}$. Then, since $N_{j'}(s') = \{i^*\}$, $n \geq 2$ and by (50), we have that

$$\begin{aligned} CEL^d_{i^*}\big(r, c, \mathcal{P}(s')\big) &= C^{j'}(s') - \frac{\sum_{i\in N} c_i - r}{2} = c_{i^*} - \frac{\sum_{i\in N} c_i - r}{2} \\ &\leq c_{i^*} - \frac{\sum_{i\in N} c_i - r}{n} = CEL^d_{i^*}\big(r, c, \mathcal{P}(s)\big). \end{aligned}$$

Therefore, player i^* has no incentive to move from the distribution center j and, thus, the proof of part $a.$ is done.

[18]A centralized rationing rule F satisfies exclusion if for any problem (r, c), if $c_i \leq \frac{\sum_{i\in N} c_i - r}{n}$, then $F_i(r, c) = 0$.

Finally, notice that the proof of part $b.$ follows directly from the definition of the CEL^d rule.

□

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