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Constrained multi-issue rationing problems

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Abstract: We study a variant of the multi-issue rationing model, where agents claim for several issues. In this variant, the available amount of resource intended for each issue is constrained to an amount fixed a priori according to exogenous criteria. The aim is to distribute the amount corresponding to each issue taking into account the allocation for the rest of issues (issue-allocation interdependence). We name these problems constrained multi-issue allocation situations (CMIA). In order to tackle the solution to these problems, we first reinterpret some single-issue egalitarian rationing rules as a minimization program based on the idea of finding the feasible allocation as close as possible to a specific reference point. We extend this family of egalitarian rules to the CMIA framework. In particular, we extend the constrained equal awards rule, the constrained equal losses rule and the reverse Talmud rule to the multi-issue rationing setting, which turn out to be particular cases of a family of rules, namely the extended α-egalitarian family. This family is analysed and characterized by using consistency principles (over agents and over issues) and a property based on the Lorenz dominance criterion.

JEL Codes: D63.

Keywords: rationing, multi-issue, reverse Talmud rule, equal losses rule, egalitarian family.

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1 Introduction

A standard rationing situation[1] (O'Neill, 1982; Aumann and Maschler, 1985) is an allocation problem in which an amount of a (perfectly divisible) resource (e.g., money) must be distributed among several agents, each of whom has a claim on the resource. The problem arises when the amount to be divided is not enough to satisfy all claims.[2]

Calleja et al. (2005) extend this model to incorporate situations in which the available amount of resource must be allocated among several agents, each of whom has several claims related to different issues. Quoting Calleja et al. (2005, page 731): *"An issue constitutes a reason on the basis of which the estate is to be divided"*. In this case, the problem also arises when the amount of resource is not enough to satisfy all claims. These authors called these problems *multi-issue allocation situations* (MIA).

In the literature, there are two basic approaches to MIA situations:

1. The first approach (e.g., Calleja et al., 2005; González-Alcón et al., 2007; Hinojosa and Mármol, 2014; Hinojosa et al., 2012, 2014; Ju et al., 2007; Lorenzo-Freire et al., 2007) provides rules that assign a single payoff to each agent, but do not specify which proportion of this payoff corresponds to each issue, which is not relevant in this approach.

2. The second approach (e.g., Bergantiños et al., 2010a, 2010b, 2011; Borm et al., 2005; Lorenzo-Freire et al. 2010; Moreno-Ternero, 2009) provides two-stage rules. In the first stage, the total amount of resource is allocated to issues. That is, it is specified which proportion of the total amount of resource will be assigned to satisfy the claims for each specific issue. This is done by solving a single-issue rationing problem, where the issues play the role of agents and the claim related to each issue is the sum of agents' claims for the corresponding issue. In a second stage, the amount assigned to each issue (in the first stage) is

[1] In this chapter, we call these situations *single-issue rationing problems*.

[2] For further reading see the surveys undertaken by Thomson (2003, 2015).

distributed among agents. The allocation within an issue is done separately from the allocation within another issue by solving a single-issue rationing problem where only the claims of the agents for the specific issue are taken into account.

Formally, a MIA situation is described by means of a set of agents $N = \{1, 2, \ldots, n\}$, a set of issues $M = \{1, 2, \ldots, m\}$, a list of claims $\mathbf{c} = (c_i^j)_{\substack{i \in N \\ j \in M}}$, being c_i^j the claim of agent i for the issue j, and an available amount of resource r such that $r \leq \sum_{j \in M} \sum_{i \in N} c_i^j$. In the aforementioned first approach, a single-point solution to a problem is a payoff vector $(x_1, x_2, \ldots, x_n)$ where x_i is the (total) payoff to agent $i \in N$, without specifying which proportion of this payoff corresponds to each issue. In the second approach, the solution is an allocation vector $\mathbf{x} = (x_i^j)_{\substack{i \in N \\ j \in M}} \in \mathbb{R}_+^{N \times M}$, being x_i^j the payoff to agent i for the issue j with no other restriction than efficiency, i.e. $\sum_{j \in M} \sum_{i \in N} x_i^j = r$ and claims boundedness.

In this chapter, we face multi-issue rationing problems (where agents claim for several issues) from a different point of view. There are many interesting economic situations in which each agent also has several claims related to different issues, but the proportion of the available amount of resource intended for each issue has already been fixed *a priori* according to exogenous criteria. The aim of our approach is to focus on the (total) payoff to agents (as in the first approach), but also specifying its distribution across issues in a fair way, with the constraint that the proportion of the total resource intended for each issue is fixed previously to the rationing process.

For instance, imagine that a Central Government has already decided how to spent its total budget (amount of resource) into different budget items (issues) such as public education, health care, grants to industrial sectors, organization of events, security, etc. Then, suppose that the Regional Institutions (agents) have several demands (claims) related to these items. At this point, let us remark that the *a priori* allocation of the total budget to each item (issue) might be related to the demand of each Regional Institution for each item, but it might response to other criteria such as assigning a large proportion of the total budget to basic or essential services, or to strategic sectors, etc. In any case, we assume that this distribution is already given. That is, the budget intended for each item is *constrained* to an amount and constitutes an exogenous parameter of the model. The solution is then an allocation vector that specifies the payoff to each agent for each issue, but respecting the *a priori* allocation of the total resource among the different issues. We call these kind of problems *constrained multi-issue allocation situations (CMIA)*. Formally, in a $CMIA$ problem an arbitrator must allocate different amounts $r^1 \geq 0$, $r^2 \geq 0$, …, $r^m \geq 0$ (as many as issues) of (an homogeneous) resource among several agents $N = \{1, 2, \ldots, n\}$ with claims $(c_i^j)_{\substack{i \in N \\ j \in M}}$ over these issues $M = \{1, 2, \ldots, m\}$. A solution assigns to each problem a payoff for each agent within each issue $\mathbf{x} = (x_i^j)_{\substack{i \in N \\ j \in M}}$ such that, for each issue $j = 1, 2, \ldots, m$, the amount of resource is entirely allocated to agents $\sum_{i=1}^{n} x_i^j = r^j$. Finally, the total payoff assigned to each agent $i \in N$ is the sum of the partial payoffs allocated to the agent relative to the different issues, i.e. $X_i = \sum_{j \in M} x_i^j$, for all $i \in N$.

As mentioned above, in the second approach to MIA problems (Bergantiños et al., 2010a; Moreno-Ternero, 2009) the allocation within an issue does not depend on the allocation for other issues. As a consequence of this, the fairness considerations that are often associated to classical single-issue rationing rules can not be applied to compare total payoffs to agents. In what follows, we will try to define and analyse solutions, for our model, that cannot be only considered fair within each issue, but also fair from the point of view of the total payoff to agents by solving simultaneously several rationing problems. To illustrate this point, let us consider the following example.

Example 1 *Imagine a funds allocation problem to two research groups (group* 1 *and group* 2*) who claim for financial support for the next three academic years. The amount of money intended for each period is* $r^1 = r^2 = r^3 = 150$. *The claims of the groups for years 1, 2 and 3 are respectively* $(c_1^1, c_2^1) =$

$(140, 20)$, $(c_1^2, c_2^2) = (140, 20)$ and $(c_1^3, c_2^3) = (20, 260)$. This situation can be interpreted as a $CMIA$ problem where there are two agents (the research groups), three issues (the three periods) and the budget corresponding to each period has been already fixed by the institution supporting the grants. The outcome (or solution) to this problem is a finance plan for the three-year period.

In order to solve this inter-temporal allocation problem, imagine we are trying to equalize gains as much as possible. We can tackle this problem (among other options) either by considering issues one-by-one, or by aggregating claims and resources and solving it as a single-issue rationing problem. Let's analyse these two cases.

1. *Considering issues one-by-one. In order to equalize gains, we compute the constrained equal awards solution*[3] *in each of these single-issue rationing problems separately. That is,*

   $x^1 = CEA(r^1, (c_1^1, c_2^1)) = (130, 20)$, $x^2 = CEA(r^2, (c_1^2, c_2^2)) = (130, 20)$ *and* $x^3 = CEA(r^3, (c_1^3, c_2^3)) = (20, 130)$,

   corresponding to years 1, 2 *and* 3, *respectively.*

   Although the aggregate three-period claim (the total claim) is the same for both agents, i.e. $C_1 = c_1^1 + c_1^2 + c_1^3 = 300 = c_2^1 + c_2^2 + c_2^3 = C_2$, *the aggregate three-period payoff (the total payoff) to agent* 2 *is smaller than the aggregate three-period payoff to agent* 1, *i.e.* $X_2 = x_2^1 + x_2^2 + x_2^3 = 170 < 280 = x_1^1 + x_1^2 + x_1^3 = X_1$. *This example suggests that solving independently (one-by-one) the allocation problem within each issue might lead to "*unfair*" global allocations. It seems reasonable that the distribution for an issue should be influenced by the amount received for the rest of issues.*

2. *Aggregating claims and resources. The aggregate (total) claims of agents are respectively* $C_1 = c_1^1 + c_1^2 + c_1^3 = 300$ *and* $C_2 = c_2^1 + c_2^2 + c_2^3 = 300$.

[3]For a formal definition, see page 9.

If we aggregate the available resources for each issue we obtain $r = r^1 + r^2 + r^3 = 450$. *Then, if we apply the CEA rule to the problem* $(r, (C_1, C_2))$ *we obtain* $(X_1, X_2) = CEA(450, (300, 300)) = (225, 225)$.

It remains to assign this total payoff of agents within issues. However, you can check that there is no feasible allocation $\big(x_1^j + x_2^j = r^j$, *for all* $j \in \{1, 2, 3\}$, *and* $0 \leq x_i^j \leq c_i^j$, *for all* $i \in \{1, 2\}$ *and all* $j \in \{1, 2, 3\}\big)$ $\mathbf{x} = (x_i^j)_{\substack{i\in\{1,2\}\\ j\in\{1,2,3\}}}$ *such that* $X_1 = x_1^1 + x_1^2 + x_1^3 = 225$ *and* $X_2 = x_2^1 + x_2^2 + x_2^3 = 225$.

This example not only suggests that it does not seem appropriate to distribute amounts of resource separately within each issue (as in case **1***), but we cannot also dismiss issue constraints (as in case* **2***). The allocation within an issue must be dependent on the allocation within other issues. To this end and continuing with the same example, the reader can check that the allocation given by* $(x_1^1, x_2^1) = (130, 20)$, $(x_1^2, x_2^2) = (130, 20)$ *and* $(x_1^3, x_2^3) = (0, 150)$ *is more egalitarian in the aggregate (total) payoffs* ($X_1 = 260$ *and* $X_2 = 190$) *than any other feasible allocation. This allocation coincides with the extension, defined later, of the constrained equal awards rule to the multi-issue framework.*

The extension of some classical rules that we propose in this chapter are based on finding the solution of an optimization program. To this aim, in Section 2 we reinterpret some of the main rules for single-issue rationing situations as a minimization program and we introduce a family of single-issue egalitarian rules, each of which picks out the feasible allocation that minimizes the distance to a reference point. In the same section, we also introduce a subfamily of these rules, namely the β-egalitarian family of rules, which coincides with the $RTAL$ *family* of rules (Thomson, 2008 and van den Brink et al., 2013). In Section 3, we focus on $CMIA$ situations and extend the aforementioned families of rules to the multi-issue context. Finally, we characterize axiomatically a subfamily of the extended egalitarian rules by means of consistency principles (over issues and over agents) and a property based on the Lorenz-domination criterion. In Section 4, we conclude.

2 Single-issue rationing problems

In this section we focus on single-issue (standard) rationing problems. First, we reinterpret some single-issue rationing solutions as a minimization program. Second, we introduce a family of egalitarian rules. Lastly, we study whether the analysed single-issue rationing rules follows some egalitarian principles.

2.1 Single-issue rationing rules as a minimization program

In order to afford the multi-issue rationing problems, we first reinterpret some of the main rules for single-issue rationing problems (where agents claim for only one issue) as an optimization program based on the idea of finding a feasible allocation as close as possible to a reference point.

We denote by $\mathbb{N}$ the set of natural numbers that we identify with the universe of potential agents, and by $\mathcal{N}$ the family of all finite subsets of $\mathbb{N}$. Given $S \in \mathcal{N}$, we denote by s the cardinality of S.

A single-issue rationing problem is a pair (r, c) where $r \geq 0$ is the available amount of resource and $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}_+^N$ is the vector of claims being c_i the claim of agent $i \in N$. The scarcity condition requires that the total claim is at least as large as the available amount of resource, i.e. $r \leq \sum_{i \in N} c_i = C$. Let $\mathcal{R}^N$ denotes the domain of all single-issue rationing problems with agent set N. The family of all single-issue rationing problems is $\mathcal{R} = \cup_{N \in \mathcal{N}} \mathcal{R}^N$.

For any problem $(r, c) \in \mathcal{R}^N$ with $N \in \mathcal{N}$, the feasible set $\mathcal{D}(r, c)$ is defined as

$$\mathcal{D}(r,c) := \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} \sum_{i \in N} x_i = r \text{ and} \\ 0 \leq x_i \leq c_i \text{ for all } i \in N \end{array} \right\}.$$

It includes those efficient allocations such that no agent gets more than her claim. A single-issue rationing rule selects a unique payoff vector $F(r,c) = x$ within the feasible set $\mathcal{D}(r,c)$.

Next, we recall two classical solutions that will be used later in this section. For any single-issue rationing problem $(r,c) \in \mathcal{R}^N$, with $N \in \mathcal{N}$, the *constrained equal awards* (CEA) rule is defined as $CEA_i(r,c) = \min\{c_i, \lambda\}$ for all $i \in N$, and the *constrained equal losses* (CEL) rule is defined as $CEL_i(r,c) = \max\{0, c_i - \lambda\}$ for all $i \in N$, where, in both cases, $\lambda \in \mathbb{R}_+$ is chosen such that the resultant allocation is efficient.

In the more general context of bargaining problems,[4] Pfingsten and Wagener (2004) point out that solutions can be viewed as *social compromises* in the sense that they select those elements among the feasible outcomes that come as close as possible to some ideal but not achievable outcome. Closeness is defined by suitably defined metrics or quasi metrics on the outcome space. In this chapter we will also follow this idea of minimizing the distance to a point; however, this point must not coincide necessarily with the ideal outcome (claims vector). As we next see, several rationing rules can be reinterpreted as solutions which select the feasible allocation in $\mathcal{D}(r,c)$ that minimizes the euclidean distance to a given reference point.

For instance, the CEA rule can be viewed as the solution to the following minimization program:

$$\min_{x \in \mathcal{D}(r,c)} \sum_{i \in N} x_i^2.$$

That is, the rule picks out the feasible allocation in $\mathcal{D}(r,c)$ that is the closest to the origin (zero vector). Looking at Figure 1 (a) it is straightforward to

[4] Chun and Thomson (1992) study a bargaining model with claims. They consider the claims vector as the utopia point and also consider disagreement points other than the origin.

realize that the same allocation is selected when we minimize the distance from any vector with equal components (not necessarily the origin). The proof of this result can be found in Proposition 6 in Appendix A.[5]

On the other hand, the CEL rule can be viewed as the solution of the minimization program $\min\limits_{x\in\mathcal{D}(r,c)} \sum_{i\in N}(x_i - c_i)^2$. In this case, losses are split equally from the claims vector (see Figure 1 (b) and Proposition 7 in Appendix A).

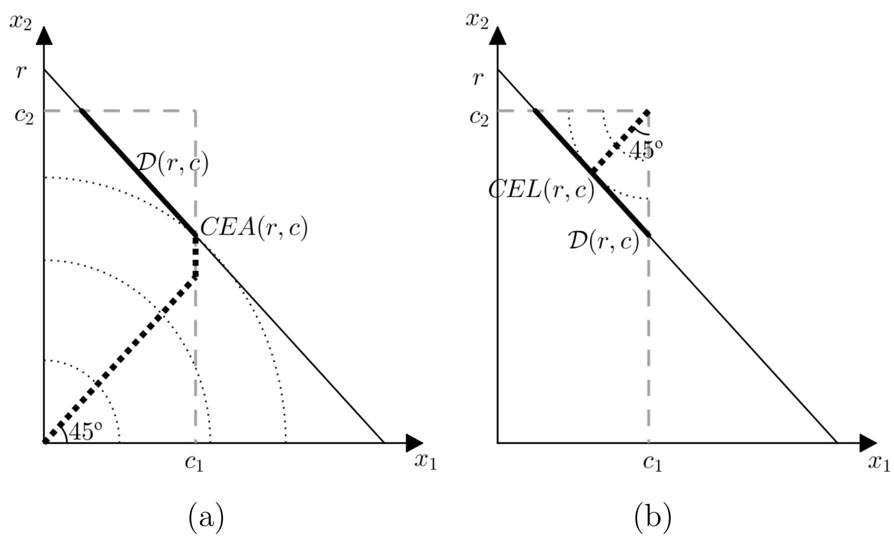

Figure 1: **The CEA and the CEL rules as minimization programs.**

(a) The CEA rule selects the feasible allocation that minimizes the euclidean distance to the origin. (b) The CEL rule selects the feasible allocation that is the closest to the claims vector.

We aim to generalize this idea by selecting the feasible allocation that is the closest to a reference point.

Definition 1 *Given $N \in \mathcal{N}$, a* reference function ${}_N\alpha : \mathbb{R}^N_+ \to \mathbb{R}^N$ *associates*

[5]In the same line, Schummer and Thomson (1997) characterize the CEA rule as the only solution minimizing the variance with respect to the equal split solution.

to any vector of claims $c \in \mathbb{R}^N$ *a* reference point

$$_N\alpha(c) = \big({_N\alpha_1}(c), {_N\alpha_2}(c), \ldots, {_N\alpha_n}(c)\big) = (a_1, a_2, \ldots, a_n) = a.$$

Moreover, we denote by $\alpha = ({_N\alpha})_{N \in \mathcal{N}}$ *the collection of reference functions relative to each* $N \in \mathcal{N}$*, and we name it* reference system α*.*

Let us illustrate this idea with some examples of reference systems.

Example 2 *For all* $N \in \mathcal{N}$ *and all* $c \in \mathbb{R}^N_+$*,*

1. $_N\alpha(c) = c$*. In this case, the reference point coincides with the claims vector.*

2. $_N\alpha(c) = (0, 0, \ldots, 0)$*. In this case, the reference point coincides with the zero vector (origin).*

3. $_N\alpha(c) = \begin{cases} (0,0,\ldots,0) & \text{if } \sum_{i \in N} c_i \leq 100 \\ c & \text{if } \sum_{i \in N} c_i > 100 \end{cases}$*. In this case, the reference point coincides with the zero vector (origin), if the sum of claims is bellow a threshold equal to* 100*, and with the claims vector, otherwise.*

4. $_N\alpha_i(c) = p_i \in \mathbb{R}$*, for all* $i \in N$*. In this case, the reference point is constant and independent of the claims.*

Given a reference system α and an agent set $N \in \mathcal{N}$, the α-egalitarian rule selects the feasible allocation that minimizes the euclidean distance to the corresponding reference point. Notice that other distances could be used.

Definition 2 *Given a reference system* α *and* $N \in \mathcal{N}$*, the* α*-egalitarian rule* E^α *assigns to any problem* $(r, c) \in \mathcal{R}^N$*, the unique feasible payoff vector* $E^\alpha(r, c) \in \mathbb{R}^N_+$ *defined as follows:*

$$\{E^\alpha(r,c)\} := \operatorname*{arg\,min}_{x \in \mathcal{D}(r,c)} \sum_{i \in N} (x_i - a_i)^2,$$

where $a_i = {_N\alpha_i}(c)$*, for all* $i \in N$*.*

Note that, each α-egalitarian rule is well-defined since $\sum_{i \in N}(x_i - a_i)^2$ is a continuous and strictly convex function, and $\mathcal{D}(r,c)$ is a compact and convex set and, consequently, the minimization program has a unique solution. Each rule contained in this family follows an egalitarian principle since it distributes as equal as possible the excess or the loss from the corresponding reference point $a = {}_N\alpha(c)$.

Every α-egalitarian rule satisfies some fundamental properties as:

- *resource monotonicity:* if the amount of resource increases, then no agent gets less payoff.
- *path-independence:* if applying the rule we obtain an initial allocation but resource availability suddenly diminishes, the allocation that comes from applying the rule to the new problem with the smaller level of resource and with the original claims, is equal to the allocation that comes from applying the rule taking into account the initial allocation as claims.

Formal definitions of each of these properties and the proofs that show that every α-egalitarian rule satisfies these properties are given in Definitions 21 and 25 and Propositions 8 and 11 in Appendix A.

Other properties hold under some additional requirements:

- *claims monotonicity:* if the claim of an agent increases, then her payoff does not decrease.

  E^{α} satisfies claims monotonicity if the reference system α is constant and independent of the claims (see Case **4.** of Example 2).
- *consistency:* if we apply a rule to a subgroup of agents, then the rule recommends the same initial assignment.

E^α satisfies consistency if the reference system α is consistent (for a formal definition of a consistent reference system see Definition 24 in Appendix A).

The definitions of these two properties and the proofs that show that α-egalitarian rules satisfy these properties under the corresponding conditions can be found in Definitions 22 and 23 and Propositions 9 and 10 in Appendix A.

Finally, let us remark that there are α-egalitarian rules that do not satisfy the classical property of *equal treatment of equals* (if two agents have equal claims, then they should receive equal amounts). This is because the allocation suggested by an α-egalitarian rule does not depend only of the claims, but also on the reference system α. In the next subsection, we introduce a subfamily of the α-egalitarian family of rules that satisfies the classical property of equal treatment of equals.

2.2 The β-egalitarian family of rules

An interesting subfamily of the α-egalitarian family of rules is the one that proposes to take as reference point a proportion $\beta \in [0,1]$ of the claims, i.e. for all $N \in \mathcal{N}$, the reference system assigns ${}_N\alpha_i(c) = \beta \cdot c_i$, for all $i \in N$. We denote each rule contained in this subfamily as $\mathcal{E}^\beta$ and we name this subfamily β-egalitarian.

Definition 3 *Given $\beta \in [0,1]$ and $N \in \mathcal{N}$, the β-egalitarian rule $\mathcal{E}^\beta$ assigns to any problem $(r,c) \in \mathcal{R}^N$, the unique feasible payoff vector $\mathcal{E}^\beta(r,c) \in \mathbb{R}^N_+$ defined as follows:*

$$\{\mathcal{E}^\beta(r,c)\} := \underset{x \in \mathcal{D}(r,c)}{\arg\min} \sum_{i \in N} (x_i - \beta \cdot c_i)^2.$$

The β-egalitarian family of rules allows to interpret the allocation problem from different points of view. As Young (1994) points out on page 73 of his book "*Equity in Theory and Practice*":

"*There are two ways of looking any solution to a claims problem. One is to focus on the amount each agents gets, that is, the gain. The other is to look at how much each claimant fails to get, that is, on the loss*".

The β-egalitarian family embodies the two ways in which agents perceive an allocation. If the amount to divide r is larger than the total claim $C = \sum_{i \in N} c_i$ weighted by β, i.e. $r > \beta \cdot C$, agents perceive that they gain something from the reference point $\beta \cdot c$; if it is smaller, i.e. $r < \beta \cdot C$, agents perceive that they lose something from the reference point $\beta \cdot c$. Otherwise, if $r = \beta \cdot C$, then the reference point is feasible, i.e. $\beta \cdot c \in \mathcal{D}(r, c)$, and thus, since the closest point of the feasible set to $\beta \cdot c$ is itself, it follows that $\mathcal{E}^\beta(r, c) = \beta \cdot c$ and so the rule proposes the proportional allocation with respect to claims, i.e. $\mathcal{E}^\beta(r, c) = \left(\frac{r \cdot c_i}{C}\right)_{i \in N}$.

The allocation selected by a β-egalitarian rule obviously depends on how we select $\beta \in [0, 1]$. We suggest the following two interpretations of the parameter β:

1. The first interpretation states that β reflects the shared expectation of the agents about what proportion of their claims will be satisfied. In this case, the value of β depends on exogenous factors (e.g., estimated scarcity of the resource, current situation of the local economy, market situation) that we suppose are common knowledge for all agents. Consider, for instance, a firm that goes bankrupt, and imagine each creditor expects to recoup forty percent of her claim, i.e. $\beta = 0.4$. Therefore, if the total assets are not enough to cover the expectations of the creditors, i.e. $r < 0.4 \cdot C$, then each creditor perceives what is left to satisfy her expectation as a loss. On the other hand, if the total assets exceed the expectations of the creditors, i.e. $r > 0.4 \cdot C$, then each creditor perceives what exceeds her expectation as a gain.

2. The second interpretation states that β reflects a minimum threshold, common for all agents. That is, if an agent is assigned below this threshold, then the situation of this agent worsens. Consider, for instance, a food supplies to refugees and suppose that the distributor estimates the ideal amount of food that each refugee should ingest (claim) taking into account several factors (e.g., age, sex, state of health) which determine her physical condition. Then, $\beta \cdot c_i$ reflects the nutrients that the refugee i requires to ensure that her health will not get worse (e.g., subsistence level). Therefore, if the amount of food to be distributed in the refugee camp is insufficient to satisfy the basic needs of the refugees, i.e. $r < \beta \cdot C$, then each refugee perceives what remains to cover her basic needs as a loss. On the other hand, if the total amount of food is enough to cover the basic needs of the refugees, i.e. $r > \beta \cdot C$, then each refugee perceives what exceeds to her basic needs as an extra gain.[6]

The β-egalitarian family offers a compromise between the CEA and the CEL rules. That is, both rules can be recovered at the extreme values of β: if $\beta = 0$, any assignment is viewed as a gain and the rule coincides with the CEA rule, i.e. if $\beta = 0$, then $\mathcal{E}^\beta = CEA$ (see Figure 1 (a) on page 17). If $\beta = 1$, any assignment smaller than the claim is viewed as a loss and the rule coincides with the CEL rule, i.e. if $\beta = 1$, then $\mathcal{E}^\beta = CEL$ (see Figure 1 (b) on page 17).

Another example of β-egalitarian rule is the *reverse Talmud* (RT) rule (Chun et al., 2001) which is an hybrid of the CEA and the CEL rules. It is defined as

$$\begin{aligned} RT_i(r,c) := {} & CEL_i\left(\min\left\{r, \frac{\sum_{i\in N} c_i}{2}\right\}, \frac{c}{2}\right) \\ & + CEA_i\left(\max\left\{0, r - \frac{\sum_{i\in N} c_i}{2}\right\}, \frac{c}{2}\right), \text{ for all } i \in N. \end{aligned}$$

[6] We can consider many other cases as, for instance, a distribution of irrigation water during a drought among a group of farmers (agents). Suppose that the farmers' needs (claims) are estimated by using factors as the crop extension, kinds and number of plants, etc. In this case, β reflects the smaller fraction over the needs of water to avoid that the harvest decreases.

This solution can be interpreted as a β-egalitarian rule when the parameter is $\beta = \frac{1}{2}$, and thus, the reference point is half of the claims.[7] That is, $\mathcal{E}^{\frac{1}{2}} = RT$ (see Figure 2 (a)). Another generalization of the RT rule is a family of rules called $RTAL \equiv \left\{RT^{\beta}\right\}_{\beta\in[0,1]}$ (van den Brink et al., 2013). These rules[8] are defined as follows: given $\beta \in [0,1]$, we have

$$RT^{\beta}(r,c) := CEL\left(\min\left\{r, \beta C\right\}, \beta c\right) + CEA\left(\max\left\{0, r - \beta C\right\}, \beta c\right).$$

In fact, it is straightforward to check that the family of β-egalitarian rules coincides with the $RTAL$ family. This coincidence allows to reinterpret the $RTAL$ family as an optimization program from the perspective of the distance minimization (see Figure 2 (b)).[9]

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[7] The *reverse Talmud* rule is the "reverse" of a classical solution to these problems, the *Talmud* rule (Aumann and Maschler, 1985) which is defined as $T_i(r,c) := CEA_i\left(\min\left\{r, \frac{\sum_{i\in N} c_i}{2}\right\}, \frac{c}{2}\right) + CEL_i\left(\max\left\{0, r - \frac{\sum_{i\in N} c_i}{2}\right\}, \frac{c}{2}\right)$, for all $i \in N$. The difference between both rules is that the CEA and the CEL rules switch roles in their corresponding definitions. Aumann and Maschler (1985) give an interpretation of how the *Talmud* rule works. Quoting these authors: "*the half-way point is a psychological watershed. If you get more than half your claim, your mind focuses on the full debt, and your concern is with the size of your loss. If you get less than half, your mind writes off the debt entirely, and is "happy" with whatever it can get; your concern is with your award*".

We can apply a similar argument to justify the *reverse Talmud* rule:"*If you get more than half your claim, your mind considers that you have already received quite a lot, and any additional amount above half of the claim is considered as an extra gain. If you get less than half, your mind considers this fact disappointing, and focuses on the debt with respect to half of this claim*".

[8]The $RTAL$ rules are also a subfamily of the class of CIC rules developed by Thomson, 2008.

[9]Other rules like the *Talmud* (T), the *Pineles* (PI) and the *dual Pineles* (PI^*) can be explained from the perspective of the distance minimization by a combination of two β-egalitarian rules, since these rules are hybrids or "double" applications of the CEA and the CEL rules. Specifically,

$$\begin{aligned}
T(r,c) &:= \mathcal{E}^0\left(\min\left\{r, \tfrac{C}{2}\right\}, \tfrac{c}{2}\right) + \mathcal{E}^1\left(\max\left\{0, r - \tfrac{C}{2}\right\}, \tfrac{c}{2}\right), \\
PI(r,c) &:= \mathcal{E}^0\left(\min\left\{r, \tfrac{C}{2}\right\}, \tfrac{c}{2}\right) + \mathcal{E}^0\left(\max\left\{0, r - \tfrac{C}{2}\right\}, \tfrac{c}{2}\right) \text{ and} \\
PI^*(r,c) &:= \mathcal{E}^1\left(\min\left\{r, \tfrac{C}{2}\right\}, \tfrac{c}{2}\right) + \mathcal{E}^1\left(\max\left\{0, r - \tfrac{C}{2}\right\}, \tfrac{c}{2}\right).
\end{aligned}$$

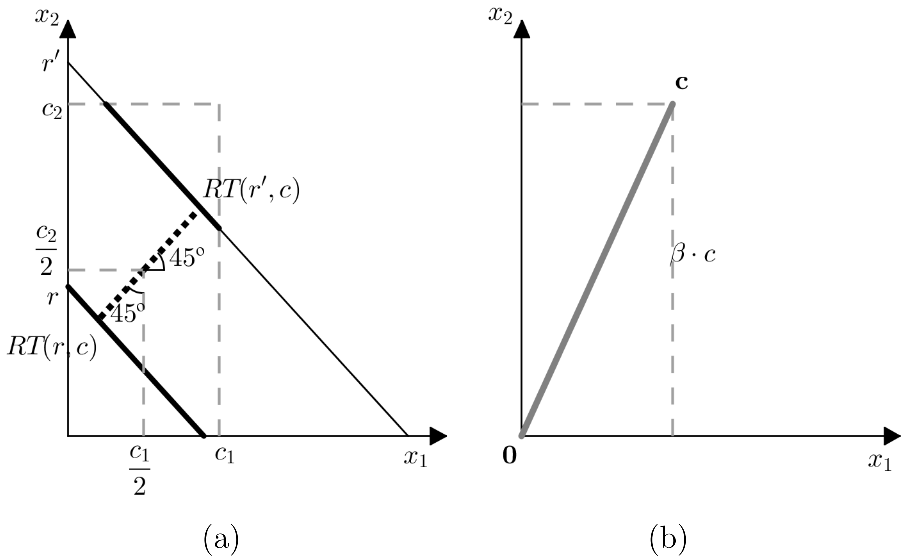

Figure 2: **The RT rule as a minimization program and all possible $\beta \cdot c$.** (a) The RT rule takes as reference point half of claims $\frac{c}{2}$. For the level of resource r (that does not cover half of claims) the selected feasible allocation is viewed as a loss from $\frac{c}{2}$. For the level of resource r' (that exceeds half of claims) the selected feasible allocation is viewed as a gain from $\frac{c}{2}$. (b) All possible locations of the reference points $\beta \cdot c$ (on the straight line joining the origin and the claims vector c).

A feature of the β-egalitarian rules is that there is a duality relationship between pairs of rules within this family. The duality approach relates two rules as follows; two rules are the dual of each other if one divides what is available in the same way as the other divides what is missing: formally, F^* and F are dual if $F^*(r,c) = c - F(\ell, c)$, for all $(r,c) \in \mathcal{R}^N$, where ℓ is the total loss, i.e. $\ell = C - r$. For instance, the CEA and the CEL rules are the dual of each other.

Each β-egalitarian rule has its corresponding β^*-egalitarian dual rule, i.e.

$$\mathcal{E}^{\beta}(r,c) = c - \mathcal{E}^{\beta^*}(\ell, c).$$

This statement was already proved by Thomson (2008) for the more general class of CIC rules. Next, we prove this duality approach from the perspective of the distance minimization.

Proposition 1 *$\mathcal{E}^{\beta}$ and $\mathcal{E}^{\beta^*}$ are dual rules if and only if $\beta^* = 1 - \beta$.*

Proof. Let $\mathcal{E}^{\beta}(r,c) = x^*$ and $\mathcal{E}^{\beta^*}(\ell,c) = l^*$. First of all, notice that

$$\begin{aligned} \min_{x\in\mathcal{D}(r,c)} \sum_{i\in N} (x_i - \beta\cdot c_i)^2 &= \min_{x\in\mathcal{D}(r,c)} \sum_{i\in N} (\beta\cdot c_i - x_i)^2 \\ = \min_{x\in\mathcal{D}(r,c)} \sum_{i\in N} (c_i - x_i - (1-\beta)\cdot c_i)^2 &= \min_{l\in\mathcal{D}(\ell,c)} \sum_{i\in N} (l_i - (1-\beta)\cdot c_i)^2, \end{aligned} \tag{1}$$

where $\mathcal{D}(\ell,c) = \left\{ l \in \mathbb{R}^N_+ \,|\, \exists x \in \mathcal{D}(r,c) \text{ such that } l = c - x \right\}$.

Let us prove first the "only if" part. Suppose that $\beta^* = 1-\beta$. Then, by (1), we obtain that

$$\min_{x\in\mathcal{D}(r,c)} \sum_{i\in N} (x_i - \beta\cdot c_i)^2 = \min_{l\in\mathcal{D}(\ell,c)} \sum_{i\in N} (l_i - \beta^*\cdot c_i)^2.$$

We claim that $l^* = c - x^*$. To check it, let us suppose on the contrary that $l^* = c - z$, where $z \in \mathcal{D}(r,c)$ and $z \neq x^*$. In this case, we would obtain

$$\begin{aligned} \textstyle\sum_{i\in N} (z_i - \beta\cdot c_i)^2 &= \textstyle\sum_{i\in N} (c_i - z_i - (1-\beta)\cdot c_i)^2 = \sum_{i\in N} ({l^*}_i - \beta^*\cdot c_i)^2 \\ &< \textstyle\sum_{i\in N} (c_i - x^*_i - \beta^*\cdot c_i)^2 = \sum_{i\in N} (x^*_i - \beta\cdot c_i)^2, \end{aligned}$$

where the strict inequality follows from the uniqueness of the solution and the last equality follows from (1). Thus, this contradicts the fact that $x^* = \mathcal{E}^{\beta}(r,c)$. Therefore, we conclude that $l^* = c - x^*$ and $\mathcal{E}^{\beta^*}(\ell,c) = c - \mathcal{E}^{\beta}(r,c)$.

Next, we prove the "if" part. Let $\mathcal{E}^{\beta}$ and $\mathcal{E}^{\beta^*}$ rules be the dual of each other, i.e. $l^* = c - x^*$. Since $\mathcal{E}^{\beta}(r,c) = x^*$, we have that

$$\begin{aligned} \min_{x\in\mathcal{D}(r,c)} \sum_{i\in N} (x_i - \beta\cdot c_i)^2 &= \sum_{i\in N} (x^*_i - \beta\cdot c_i)^2 = \sum_{i\in N} (l^*_i - (1-\beta)\cdot c_i)^2 \\ &\geq \min_{l\in\mathcal{D}(\ell,c)} \sum_{i\in N} (l_i - (1-\beta)\cdot c_i)^2 = \min_{x\in\mathcal{D}(r,c)} \sum_{i\in N} (x_i - \beta\cdot c_i)^2, \end{aligned}$$

where the second and the last equalities follows from (1). Therefore,

$$\sum_{i\in N}(l_i^* - (1-\beta)\cdot c_i)^2 = \min_{l\in\mathcal{D}(\ell,c)} \sum_{i\in N}(l_i - (1-\beta)\cdot c_i)^2.$$

Thus, we conclude that $\mathcal{E}^{\beta}(r,c) = x^* = c - l^* = c - \mathcal{E}^{(1-\beta)}(\ell,c)$.

□

As it was shown by Thomson (2008) for the class of CIC rules, Proposition 1 implies that the β-egalitarian family is closed under duality which means that if whenever it contains a rule, it also contains its dual. Indeed, since $\mathcal{E}^{\beta} \neq \mathcal{E}^{\beta^*}$ with $\beta^* = 1-\beta$ for all $\beta \neq \frac{1}{2}$, the unique self-dual (dual of itself) rule within this family is the $RT = \mathcal{E}^{\frac{1}{2}}$ rule.

2.3 Egalitarian criteria

Parametric rules were first analysed by Young (1987). A rule is parametric if the payoff received by an agent depends on a function of his claim and of a parameter related to r. Young characterizes the family of parametric rules as the only that satisfies *consistency*, *equal treatment of equals* and *continuity* (small changes in the parameter should not lead to large changes in the payoff vector). All rules within the β-egalitarian family satisfy these three properties,[10] but as we have noted above, there are α-egalitarian rules that do not satisfy equal treatment of equals and consistency, and thus, they are not parametric rules.[11]

Each α-egalitarian rule follows an egalitarian principle from its corresponding reference system α, although some of these rules do not meet the equal treatment of equals property.

To analyse the equity of an allocation, let us introduce one of the most relevant egalitarian criterion used to evaluate income distributions: the clas-

[10]The proofs that show that each β-egalitarian rule satisfies consistency and equal treatment of equals can be found in Propositions 10 and 12, respectively, in Appendix A. The reader may check easily that each β-egalitarian rule is continuous.

[11]There are also parametric rules that are not contained in the α-egalitarian family; for instance, consider the *proportional* or the *Talmud* rules.

sical Lorenz criterion. In the literature of rationing problems, this criterion is commonly applied to compare awards vectors generated by different rules.[12] The classical Lorenz ordering is based on successive sums of ordered awards as follows: given two vectors $x, y \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, we say that x *Lorenz-dominates* y $\left(x \succ_{\mathcal{L}} y\right)$ if, when the coordinates of these vectors are rearranged in a decreasing order (from large to small), and denoting the vectors obtained by $\tilde{x}$ and $\tilde{y}$, we have $\sum_{i=1}^{k} \tilde{x}_i \leq \sum_{i=1}^{k} \tilde{y}_i$, for all $k = 1, 2, \ldots, n-1$, with at least one strict inequality. Usually, if we consider two awards vectors $x, y \in \mathbb{R}^N$, we say that x is more egalitarian *in awards* than y, when x Lorenz-dominates y. That is, the largest payoff in x is smaller than the largest payoff in y, i.e. $\tilde{x}_1 \leq \tilde{y}_1$, the sum of the two largest payoffs in x is smaller than the sum of the two largest payoffs in y, i.e. $\tilde{x}_1 + \tilde{x}_2 \leq \tilde{y}_1 + \tilde{y}_2$, and so on and so forth.

We use the Lorenz criterion to compare not pairs of vectors, but the corresponding difference vectors with respect to a reference point.

Given a reference point $a \in \mathbb{R}^N$, we associate to any vector $x \in \mathbb{R}^N$ its corresponding *difference vector*, i.e. $d^a(x) = a - x = (a_1 - x_1, a_2 - x_2, \ldots, a_n - x_n) = \left(d^a_1(x), d^a_2(x), \ldots, d^a_n(x)\right) \in \mathbb{R}^N$. We use this to compare *à la* Lorenz two difference vectors.

Definition 4 *Given a reference point $a \in \mathbb{R}^N$ and two vectors $x, y \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, we say that x* Lorenz-dominates y from a $\left(x \succ^a_{\mathcal{L}} y\right)$ *if*

$$d^a(x) \succ_{\mathcal{L}} d^a(y).$$

Notice that, if $a = (0, 0, \ldots, 0)$, then we recover the definition of the classical Lorenz criterion. In this case, x Lorenz-dominates y $\left(x \succ^0_{\mathcal{L}} y\right)$ means that x is more egalitarian than y from the origin.

[12]There are however examples in the rationing literature where Lorenz criterion is used to measure equality from points of view other than awards. For instance, Arín and Benito (2012) compare, by means of the Lorenz criterion, sets of weighted vectors of awards and losses and Kasajima and Velez (2011) compare, also by means of the Lorenz criterion, claims vectors which, in their view, should be reflected in awards and losses.

Given a reference point $a \in \mathbb{R}^N$ and taking two feasible allocations $x, y \in \mathbb{R}^N_+$, we have that x Lorenz-dominates y *from* a, when, being $\widetilde{d^a(x)}$ and $\widetilde{d^a(y)}$ the corresponding difference vectors with their coordinates rearranged in a decreasing order, the largest difference in x is smaller than the largest difference in y, i.e. $\widetilde{d^a_1(x)} \leq \widetilde{d^a_1(y)}$, the sum of the two largest differences in x is smaller than the sum of the two largest differences in y, i.e. $\widetilde{d^a_1(x)} + \widetilde{d^a_2(x)} \leq \widetilde{d^a_1(y)} + \widetilde{d^a_2(y)}$, and so on and so forth. When this happens, we say that x *is more egalitarian than* y *from* a.

A particular case occurs when $a = c$. In this case, the vector of differences is $d^c(x) = c - x$ and it evaluates the loss of each agent from her claim. If $x \succ^c_{\mathcal{L}} y$, then we can say that x *Lorenz-dominates* y *in losses* or from the claims vector.

Next, we can extend the idea of Lorenz-domination *from a reference point* to the domain of rules. We say that a rule F Lorenz-dominates a rule F' *from a reference system* if for all problem, the allocation proposed by F Lorenz-dominates *from* the corresponding *reference point* to the allocation proposed by F'. Formally,

Definition 5 *Given a reference system α and two rules F and F', we say that a rule F* Lorenz-dominates *a rule F'* from the reference system α $\left(F \succ^\alpha_{\mathcal{L}} F'\right)$ *if, for each $N \in \mathcal{N}$ and any $(r, c) \in \mathcal{R}^N$,*

$$F(r, c) \succ^a_{\mathcal{L}} F'(r, c),$$

where $a = {}_N\alpha(c)$.

Chun et al. (2001) state that the CEA rule Lorenz-dominates any other rule from the origin, i.e. $CEA \succ^0_{\mathcal{L}} F$, for any arbitrary rule F, or, what is the same, the CEA rule is more egalitarian in awards (from the origin) than any other rule. However, as we have discussed previously in Subsections 2.1 and 2.2, there are different ways in which a rule might follow an egalitarian principle. Recall, for instance, that the CEL rule follows an egalitarian principle in losses from the claims vector. Thus, it is natural to expect that the CEL rule be more egalitarian in losses than any other rule, i.e. $CEL \succ_{\mathcal{L}}^{c} F$, for any arbitrary rule F. Also, given a reference system α, the corresponding E^{α} rule Lorenz-dominates any other rule from the reference system α, i.e. $E^{\alpha} \succ_{\mathcal{L}}^{\alpha} F$, for any arbitrary rule F. Thus, we say that the E^{α} rule is more egalitarian *from* α than any other rule.

The Lorenz-domination from a reference point will be crucial to characterize egalitarian solutions in the multi-issue context.

3 $CMIA$ problems

Now, we turn to multi-issue rationing problems, where agents claim for several issues and the amount of resource intended for each issue is constrained to an amount fixed *a priori* according to exogenous criteria. In the first part of this section we deal with the formal analysis of *constrained multi-issue allocation problem (CMIA)*. In the second part we propose egalitarian solutions for this model. In the last part we carry out an axiomatic analysis of these solutions.

3.1 Notation and definitions

Recall that the set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ denotes the universe of potential agents (a set with an infinite number of elements) and let $\mathcal{N}$ be the set of all non-empty finite subsets of $\mathbb{N}$. A set $N = \{1, 2, \ldots, n\} \in \mathcal{N}$ describes a finite set of agents. The set $\mathbb{M} = \{1, 2, \ldots\}$ denotes the set of potential issues and $\mathcal{M}$ refers to the set of all non-empty finite subsets of $\mathbb{M}$. The set $M = \{1, 2, \ldots, m\} \in \mathcal{M}$ describes a finite set of issues.

Definition 6 *Let $N \in \mathcal{N}$ be a set of agents and $M \in \mathcal{M}$ be a set of issues. A* constrained multi-issue allocation problem ($CMIA$) *is a pair* $(\mathbf{r}, \mathbf{c})$*, where $\mathbf{r} = (r^j)_{j \in M} \in \mathbb{R}_+^M$ is the vector of resources (one per issue) and*

$\mathbf{c} = (c_i^j)_{\substack{i\in N\\ j\in M}} \in \mathbb{R}_+^{N\times M}$ *is the vector of claims such that, for all issue* $j \in M$,

$$r^j \leq \sum_{i\in N} c_i^j \text{ (scarcity conditions (one per issue)).}$$

The quantity r^j represents the amount of resource intended for the issue j and c_i^j represents the amount that the agent $i \in N$ claims according to the issue $j \in M$. The class of $CMIA$ problems with set of agents N and set of issues M is denoted by $\mathcal{MR}^{N\times M}$ and we write the family of all these problems as $\mathcal{MR} = \cup_{\substack{N\in\mathcal{N}\\ M\in\mathcal{M}}} \mathcal{MR}^{N\times M}$. Notice that a single-issue rationing problem is a $CMIA$ problem with $|M| = 1$.

The amount of resource intended for each issue is fixed *a priori* (previously to the rationing process) which makes the difference with respect to the classical multi-issue allocation (MIA) model (e.g., Calleja et al., 2005; Moreno-Ternero, 2009). Thus, in a $CMIA$ problem, an allocation $\boldsymbol{v} = (v_i^j)_{\substack{i\in N\\ j\in M}} \in \mathbb{R}^{N\times M}$ is efficient if the amount of resource intended for each issue is entirely assigned to agents, i.e. $\sum_{i\in N} v_i^j = r^j$, for all $j \in M$. We denote by $v^j = (v_1^j, v_2^j, \ldots, v_n^j)$ the vector relative to issue $j \in M$ and by $\boldsymbol{v}_{|T} = (v_i^j)_{\substack{i\in T\\ j\in M}} \in \mathbb{R}^{T\times M}$ the vector $\boldsymbol{v}$ restricted to the members of T.

Definition 7 *A rule $\mathcal{F}$ on $\mathcal{MR}$ is a function which associates to each $CMIA$ problem $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N\times M}$, with $N \in \mathcal{N}$ and $M \in \mathcal{M}$, a unique payoff vector $\mathbf{x} = \mathcal{F}(\mathbf{r}, \mathbf{c}) \in \mathbb{R}_+^{N\times M}$ within the feasible set $\mathcal{D}(\mathbf{r}, \mathbf{c})$ which includes those efficient allocations such that no agent gets more than her claim. That is,*

$$\mathcal{D}(\mathbf{r}, \mathbf{c}) := \left\{ \mathbf{x} = \left(x_i^j\right)_{\substack{i\in N\\ j\in M}} \in \mathbb{R}_+^{N\times M} \;\middle|\; \begin{array}{l} \sum_{i\in N} x_i^j = r^j, \text{ for all } j \in M \text{ and} \\ x_i^j \leq c_i^j, \text{ for all } i \in N \text{ and all } j \in M \end{array} \right\}$$

We denote by $\mathcal{F}_i^j(\mathbf{r}, \mathbf{c}) \in \mathbb{R}_+$ the payoff to agent $i \in N$ according to the issue $j \in M$.

3.2 Egalitarian multi-issue allocation rules

Next we extend the α-egalitarian family of rules to the multi-issue framework. Similarly to the case of single-issue rationing problems we define a reference point, for the multi-issue framework, as follows.

Definition 8 *Given $N \in \mathcal{N}$ and $M \in \mathcal{M}$, a reference function ${}_N^M\boldsymbol{\alpha} : \mathbb{R}^{N\times M} \to \mathbb{R}^{N\times M}$ associates to any vector of claims $\mathbf{c} \in \mathbb{R}_+^{N\times M}$ a reference point ${}_N^M\boldsymbol{\alpha}(\mathbf{c}) = \left({}_N^M\alpha_i^j(\mathbf{c})\right)_{\substack{i\in N\\ j\in M}} = \left(a_i^j\right)_{\substack{i\in N\\ j\in M}} = \boldsymbol{a} \in \mathbb{R}^{N\times M}$ being a_i^j the reference of agent i according to issue j.*

Moreover, we denote by $\boldsymbol{\alpha} = ({}_N^M\boldsymbol{\alpha})_{\substack{N\in\mathcal{N}\\ M\in\mathcal{M}}}$ the collection of reference functions relative to each $N \in \mathcal{N}$ and each $M \in \mathcal{M}$, and we call it multi-issue reference system $\boldsymbol{\alpha}$*.*

Payoffs to agents within each issue will be evaluated (as gains or losses) with respect to reference points. If no confusion arises, we write $\boldsymbol{\alpha} = 0$ ($\boldsymbol{\alpha} = 1$) to mean the multi-issue reference system $\boldsymbol{\alpha}$ such that ${}_N^M\boldsymbol{\alpha}(\mathbf{c}) = \mathbf{0} = (0, 0, \ldots, 0)$ $\left({}_N^M\boldsymbol{\alpha}(\mathbf{c}) = \mathbf{1} = (1, 1, \ldots, 1)\right)$, for all $N \in \mathcal{N}$ and all $M \in \mathcal{M}$.

In Example 1 we have discussed that there is a trade off between solving separately the allocation within each issue and solving the problem when we focus on the total payoff to agents. The approach we adopt in this chapter is to prioritize the total payoff to agents and, after that, focusing on the allocation within each issue. This assumption implies that the extensions of the α-egalitarian rules are defined by means of two stages:

1. In the first stage, the extension of the α-egalitarian rule selects the *feasible vector of total payoffs* $\left(\sum_{j\in M} x_i^j\right)_{i\in N}$ which minimizes the euclidean distance to the aggregate reference point $\left(\sum_{j\in M} a_i^j\right)_{i\in N}$. This can be viewed as the solution of the following minimization program:

$$\arg\min_{\mathbf{x}\in\mathcal{D}(\mathbf{r},\mathbf{c})} \sum_{i\in N} \Big(\sum_{j\in M} x_i^j - \sum_{j\in M} a_i^j \Big)^2.$$

   We denote the set of solutions of this program by $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})$. Notice that this set is non-empty, compact and convex, since we are minimizing a convex function over a compact and convex domain. The set $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$ contains, in general, more than one allocation vector, but all of them have the same *vector of total payoffs*. That is,

$$\text{for all } \mathbf{x},\mathbf{y} \in D^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c}), \text{ it holds } \sum_{j\in M} x_i^j = \sum_{j\in M} y_i^j, \text{ for all } i \in N. \quad (2)$$

Notice that, if we denote the vector of total payoffs to agents as $X = (X_i)_{i\in N} = (\sum_{j\in M} x_i^j)_{i\in N}$, the function to be minimized in the first stage can be rewritten as a function of only n variables

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i\in N}\left(\sum_{j\in M}\left(x_i^j - a_i^j\right)\right)^2 = \sum_{i\in N}\left(X_i - \sum_{j\in M} a_i^j\right)^2 \\ &= F(X_1, X_2, \ldots, X_n). \end{aligned}$$

The function F is strictly convex and the optimization program

$$\begin{aligned} &\mathit{min}\ F(X) \\ &\text{s.t. } X \in \left\{Y \in \mathbb{R}^N_+ \middle| \exists \mathbf{x} \in \mathcal{D}(\mathbf{r},\mathbf{c}) :\ Y_i = \sum_{j\in M} x_i^j \text{ for all } i \in N\right\} \end{aligned}$$

has a unique solution $X^* = (X_i^*)_{i\in N}$. Since $f(\mathbf{x}) = F(X)$, for all $\mathbf{x} \in \mathcal{D}(\mathbf{r},\mathbf{c})$, and X^* is unique, it follows that $\sum_{j\in M} x_i^j = X_i^*$, for all $i \in N$ and all $\mathbf{x} \in \mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$.

2. In the second stage the rule selects, among the set of allocations obtained in the first one, a payoff vector that aims to distribute payoffs across issues in an egalitarian way from $\boldsymbol{a} = {}^M_N\boldsymbol{\alpha}(\mathbf{c})$. This is carried out by solving the following optimization program

$$\operatorname*{arg\,min}_{\mathbf{x}\in\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})} \sum_{i\in N}\sum_{j\in M}\left(x_i^j - a_i^j\right)^2,$$

where the differences between payoffs and references (for all agents according to each issue) are overall minimized. The solution is unique since the function $\sum_{i\in N}\sum_{j\in M}\left(x_i^j - a_i^j\right)^2$ is strictly convex and the domain $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$ is also compact and convex.

Definition 9 *Given a multi-issue reference system* $\boldsymbol{\alpha}$*,* $N \in \mathcal{N}$ *and* $M \in \mathcal{M}$*, the* extended $\boldsymbol{\alpha}$-egalitarian *rule* $\widehat{E}^{\boldsymbol{\alpha}}$ *assigns to any problem* $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N \times M}$*, the unique feasible payoff vector* $\widehat{E}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c}) \in \mathbb{R}_+^{N \times M}$ *defined as follows:*

$$\{\widehat{E}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})\} := \underset{\mathbf{x} \in \mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})}{\arg\min} \sum_{i \in N} \sum_{j \in M} \left(x_i^j - a_i^j\right)^2,$$

where $a_i^j = {}_N^M\alpha_i^j(\mathbf{c})$*, for all* $i \in N$ *and all* $j \in M$*, and*

$$\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c}) := \underset{\mathbf{x} \in \mathcal{D}(\mathbf{r}, \mathbf{c})}{\arg\min} \sum_{i \in N} \Big(\sum_{j \in M} x_i^j - \sum_{j \in M} a_i^j \Big)^2.$$

Let us illustrate the aplication of an extended $\boldsymbol{\alpha}$-egalitarian rule with an example.

Example 3 *Consider the inter-temporal allocation problem described in Example 1 where years are interpreted as issues. Suppose that the reference for an agent relative to each year (issue) is given by the three-year average claim of the agent (truncated by her own claim). That is, for all* $i = \{1, 2\}$ *and all* $j = \{1, 2, 3\}$*,*

$$a_i^j = {}_N^M\alpha_i^j(\mathbf{c}) = \min\left\{\frac{c_i^1 + c_i^2 + c_i^3}{3}, c_i^j\right\}.$$

Therefore, the reference point corresponding to each year is $(a_1^1, a_2^1) = (100, 20)$*,* $(a_1^2, a_2^2) = (100, 20)$ *and* $(a_1^3, a_2^3) = (20, 100)$*. The extended* $\boldsymbol{\alpha}$*-egalitarian solution is computed as follows:*

(i) *In a* **first stage***, we minimize* $\left[x_1^1 + x_1^2 + x_1^3 - (100 + 100 + 20)\right]^2 + \left[x_2^1 + x_2^2 + x_2^3 - (20 + 20 + 100)\right]^2$ *subject to* $\mathbf{x} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$*. The solution to this problem is any payoff vector within the set*

$$\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c}) = \left\{ \mathbf{x} \in \mathcal{D}(\mathbf{r}, \mathbf{c}) \;\middle|\; \begin{array}{l} x_1^1 + x_1^2 + x_1^3 = 265 \\ x_2^1 + x_2^2 + x_2^3 = 185 \end{array} \right\}.$$

Notice that the set $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})$ *contains more than one vector, for instance,*

$$(x_1^1, x_2^1; x_1^2, x_2^2; x_1^3, x_2^3) = (132.5, 17.5; 132.5, 17.5; 0, 150), \text{ or}$$
$$(x_1^1, x_2^1; x_1^2, x_2^2; x_1^3, x_2^3) = (131, 19; 131, 19; 3, 147).$$

(ii) *In a* **second stage***, we minimize* $(x_1^1 - 100)^2 + (x_1^2 - 100)^2 + (x_1^3 - 20)^2 + (x_2^1 - 20)^2 + (x_2^2 - 20)^2 + (x_2^3 - 100)^2$ *subject to* $\mathbf{x} \in \mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})$. *Solving this program we obtain the next unique allocation*

$$\widehat{E}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c}) = (x_1^1, x_2^1; x_1^2, x_2^2; x_1^3, x_2^3) = (130, 20; 130, 20; 5, 145).$$

As in the single-issue case, an interesting subfamily of the extended $\boldsymbol{\alpha}$-egalitarian family is the one which proposes to take as reference point proportions of the claims.

Definition 10 *Given* $\boldsymbol{\beta} = (\beta^j)_{j \in M} \in [0,1]^M$, $N \in \mathcal{N}$ *and* $M \in \mathcal{M}$, *the extended* $\boldsymbol{\beta}$*-egalitarian rule* $\widehat{\mathcal{E}}^{\boldsymbol{\beta}}$ *assigns to any problem* $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N \times M}$, *the unique feasible payoff vector* $\widehat{\mathcal{E}}^{\boldsymbol{\beta}}(\mathbf{r}, \mathbf{c})$ *defined as follows:*

$$\widehat{\mathcal{E}}^{\boldsymbol{\beta}}(\mathbf{r}, \mathbf{c}) := \widehat{E}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c}),$$

where ${}_N^M\alpha_i^j(\mathbf{c}) = \beta^j \cdot c_i^j$, *for all* $i \in N$ *and all* $j \in M$.

Notice that, $\beta^j \in [0,1]$ represents the proportion of the claims according to the issue $j \in M$.

Some interesting cases of extended $\boldsymbol{\beta}$-egalitarian rules are those that arise when $\boldsymbol{\beta} = \mathbf{0} = (0, 0, ..., 0)$, $\boldsymbol{\beta} = \mathbf{1} = (1, 1, ...1)$, or $\boldsymbol{\beta} = \frac{\mathbf{1}}{\mathbf{2}} = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. In case $\boldsymbol{\beta} = \mathbf{0}$, the corresponding reference point is $\boldsymbol{a} = \mathbf{0} = (0, 0, \dots, 0)$ and the payoff assigned to any agent for any issue is considered entirely as a gain. In case $\boldsymbol{\beta} = \mathbf{1}$, the corresponding reference point is $\boldsymbol{a} = \mathbf{c}$ and what is left from achieving full claims are perceived as a loss. In case $\boldsymbol{\beta} = \frac{\mathbf{1}}{\mathbf{2}}$, half of the claims becomes the reference point. Then, in parallel to the single-issue case, some examples of extended $\boldsymbol{\beta}$-egalitarian rules are the following.

Definition 11 *The* extended constrained equal awards $(\widehat{CEA})$ *rule is the extended* $\boldsymbol{\beta}$*-egalitarian rule that takes* $\beta^j = 0$, *for all* $j \in M$, *i.e.* $\widehat{CEA} := \widehat{\mathcal{E}}^{\mathbf{0}}$. *The* extended constrained equal losses $(\widehat{CEL})$ *rule takes* $\beta^j = 1$, *for all* $j \in M$, *i.e.* $\widehat{CEL} := \widehat{\mathcal{E}}^{\mathbf{1}}$, *and the* extended reverse Talmud $(\widehat{RT})$ *rule takes* $\beta^j = \frac{1}{2}$, *for all* $j \in M$, *i.e.* $\widehat{RT} := \widehat{\mathcal{E}}^{\frac{\mathbf{1}}{\mathbf{2}}}$.

Notice that, if there is only one issue, i.e. $|M| = 1$, each of these rules corresponds to the definition of its corresponding single-issue solution. Let us illustrate the application of these three extended $\boldsymbol{\beta}$-egalitarian rules with an example.

Example 4 *Consider the two-person and three-issue CMIA problem*

$$\big((r^1, r^2, r^3), (c_1^1, c_2^1; c_1^2, c_2^2; c_1^3, c_2^3)\big) = \big((150, 150, 150), (130, 50; 90, 100; 80, 150)\big).$$

As the reader may check, the allocations assigned by the $\widehat{CEA}$*, the* $\widehat{CEL}$ *and the* $\widehat{RT}$ *rules are*

$$\widehat{CEA}(\mathbf{r}, \mathbf{c}) = (x_1^1, x_2^1; x_1^2, x_2^2; x_1^3, x_2^3) = (100, 50; 62.5, 87.5; 62.5, 87.5),$$
$$\widehat{CEL}(\mathbf{r}, \mathbf{c}) = (x_1^1, x_2^1; x_1^2, x_2^2; x_1^3, x_2^3) = (115, 35; 70, 80; 40, 110) \text{ and}$$
$$\widehat{RT}(\mathbf{r}, \mathbf{c}) = (x_1^1, x_2^1; x_1^2, x_2^2; x_1^3, x_2^3) = (100, 50; 70, 80; 55, 95).$$

Notice that the allocation suggested by the $\widehat{CEL}$ *rule is* the most egalitarian possible distribution in losses*, since the total loss assigned to both agents is the same, i.e.* $c_1^1+c_1^2+c_1^3-(x_1^1+x_1^2+x_1^3) = 75 = c_1^2+c_2^2+c_2^3-(x_2^1+x_2^2+x_2^3)$*, and the losses assigned to both agents within each issue coincide, i.e.* $c_1^1 - x_1^1 = 15 = c_2^1 - x_2^1$*,* $c_1^2 - x_1^2 = 20 = c_2^2 - x_2^2$ *and* $c_1^3 - x_1^3 = 40 = c_2^3 - x_2^3$*. In fact, these allocations coincide with the distribution that results of applying separately the* CEL *rule within each issue; but this is just a coincidence and, in general, it does not hold.*

3.3 Axiomatic analysis

In the previous subsection, we have introduced extended $\boldsymbol{\alpha}$-egalitarian rules as natural extensions of the corresponding single-issue rationing rules viewed as a minimization program. Next, we analyse some properties that are satisfied by these rules and we provide an axiomatic characterization for a subfamily of the extended $\boldsymbol{\alpha}$-egalitarian family of rules.

The first property that we study is *consistency*. In a broad sense, consistency requires that the solution to a full problem does not change if we apply the same solution to a properly defined subproblem. Consistency has been extensively analysed within the area of the design of allocation rules, playing a central role in the corresponding literature. It was introduced in the context of single-issue rationing problems by Aumann and Maschler (1985). Quoting Thomson (2012), page 392,

> "*Consistency is, first of all, the desire for something like regularity, coherence and predictability. (...) Moreover, properties of this type are often thought of as fundamental for a fair society.*"

In the multi-issue context the concept of subproblem can be understood either as a reduction in the number of agents, or as a reduction in the number of issues. We first focus on the reduction of the number of agents (consistency over agents). The consistency principle has been described by some authors not just only as a robustness principle, but also as a fairness principle (see the survey undertaken by Thomson, 2011). It is a powerful property that expresses the invariance of a solution with respect to any change in population, linking the solution for a given society N and its subsocieties, all $S \subseteq N$. In the context of multi-issue rationing its formal definition is as follows.

Definition 12 *A rule $\mathcal{F}$ on $\mathcal{MR}$ is* consistent over agents *if for all $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N \times M}$ and all $T \subseteq N$, $T \neq \varnothing$, it holds*

$$\mathcal{F}(\mathbf{r}, \mathbf{c})_{|T} = \mathcal{F}\Big(\big(r^j - \sum_{i \in N \setminus T} x_i^j\big)_{j \in M}, \mathbf{c}_{|T}\Big),$$

where $x = \mathcal{F}(\mathbf{r}, \mathbf{c})$.

Notice that a rule is consistent over agents if the payoff assigned to each agent in a subset $T \subseteq N$ remains unaltered if we re-evaluate this payoff according to the same rule when the agents in $N \setminus T$ leave with their intended allocations.

To ensure that each extended $\boldsymbol{\alpha}$-egalitarian rule satisfies consistency over agents we need that its multi-issue reference system $\boldsymbol{\alpha}$ is also consistent. The definition follows.

Definition 13 *A multi-issue reference system* $\boldsymbol{\alpha}$ *is* consistent *if, for all* $T, N \in \mathcal{N}$ *with* $T \subseteq N$ *and all* $M \in \mathcal{M}$ *we have* ${}^{M}_{N}\alpha_i^j(\mathbf{z})_{|T} = {}^{M}_{T}\alpha_i^j(\mathbf{z}_{|T})$, *for all* $\mathbf{z} \in \mathbb{R}^{N\times M}$, *all* $j \in M$ *and all* $i \in T$.

Proposition 2 *If the multi-issue reference system* $\boldsymbol{\alpha}$ *is consistent, then the corresponding* $\widehat{E}^{\boldsymbol{\alpha}}$ *rule is consistent over agents.*

Proof. Let $(\mathbf{r}, \mathbf{c}) \in \mathcal{R}^{N\times M}$ be an arbitrary $CMIA$ problem and let us write $\mathbf{x} = \widehat{E}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})$, where $\boldsymbol{\alpha}$ is an arbitrary consistent multi-issue reference system such that ${}^{M}_{N}\boldsymbol{\alpha}(\mathbf{c}) = (a_i^j)_{\substack{i\in N\\ j\in M}}$. By definition of the rule,

$$\sum_{i\in N}\sum_{j\in M}(x_i^j - a_i^j)^2 < \sum_{i\in N}\sum_{j\in M}(z_i^j - a_i^j)^2, \tag{3}$$

for all $\mathbf{z} \in \mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})$ such that $\mathbf{z} \neq \mathbf{x}$. Let $T \subsetneq N$ be an arbitrary subset of agents and $\mathbf{y} = \widehat{E}^{\boldsymbol{\alpha}}\Big(\big(r^j - \sum_{i\in N\setminus T} x_i^j\big)_{j\in M}, \mathbf{c}_{|T}\Big)$. Suppose on the contrary that

$$\mathbf{x}_{|T} = \widehat{E}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})_{|T} \neq \mathbf{y}.$$

To reach a contradiction, let us first prove

$$\mathbf{x}_{|T} \in \mathcal{D}^{\boldsymbol{\alpha}}\Big(\big(r^j - \sum_{i\in N\setminus T} x_i^j\big)_{j\in M}, \mathbf{c}_{|T}\Big). \tag{4}$$

To check this, suppose that $\mathbf{x}_{|T} \notin \mathcal{D}^{\boldsymbol{\alpha}}\Big(\big(r^j - \sum_{i\in N\setminus T} x_i^j\big)_{j\in M}, \mathbf{c}_{|T}\Big)$. Then, since $\mathbf{y} \in \mathcal{D}^{\boldsymbol{\alpha}}\Big(\big(r^j - \sum_{i\in N\setminus T} x_i^j\big)_{j\in M}, \mathbf{c}_{|T}\Big)$ and $\boldsymbol{\alpha}$ is consistent $\Big({}^{M}_{N}\boldsymbol{\alpha}(\mathbf{c})_{|T} = {}^{M}_{T}\boldsymbol{\alpha}(\mathbf{c}_{|T}) = (a_i^j)_{\substack{i\in T\\ j\in M}}\Big)$, we would obtain

$$\sum_{i\in T}\Big(\sum_{j\in M}\big(y_i^j - a_i^j\big)\Big)^2 < \sum_{i\in T}\Big(\sum_{j\in M}\big(x_i^j - a_i^j\big)\Big)^2 \text{ and thus,}$$

$$\sum_{i\in T}\Big(\sum_{j\in M}\big(y_i^j-a_i^j\big)\Big)^2+\sum_{i\in N\setminus T}\Big(\sum_{j\in M}\big(x_i^j-a_i^j\big)\Big)^2<\sum_{i\in N}\Big(\sum_{j\in M}\big(x_i^j-a_i^j\big)\Big)^2. \quad (5)$$

However, since $\big(\mathbf{y},\mathbf{x}_{|N\setminus T}\big)\in\mathcal{D}(\mathbf{r},\mathbf{c})$, the inequality (5) contradicts that $\mathbf{x}\in\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$, and (4) holds.

Now, since $\boldsymbol{\alpha}$ is consistent, $\widehat{E}^{\boldsymbol{\alpha}}$ assigns a unique solution and taking (4) into account, we have

$$\sum_{i\in T}\sum_{j\in M}(y_i^j-a_i^j)^2<\sum_{i\in T}\sum_{j\in M}(x_i^j-a_i^j)^2. \quad (6)$$

On the other hand, by (4), we know that

$$\mathbf{x}_{|T}\in\mathcal{D}^{\boldsymbol{\alpha}}\Big(\big(r^j-\sum_{i\in N\setminus T}x_i^j\big)_{j\in M},\mathbf{c}_{|T}\Big),$$

which implies that

$$\sum_{i\in T}\Big(\sum_{j\in M}\big(y_i^j-a_i^j\big)\Big)^2=\sum_{i\in T}\Big(\sum_{j\in M}\big(x_i^j-a_i^j\big)\Big)^2 \text{ and thus,}$$

$$\sum_{i\in T}\Big(\sum_{j\in M}\big(y_i^j-a_i^j\big)\Big)^2+\sum_{i\in N\setminus T}\Big(\sum_{j\in M}\big(x_i^j-a_i^j\big)\Big)^2=\sum_{i\in N}\Big(\sum_{j\in M}\big(x_i^j-a_i^j\big)\Big)^2. \quad (7)$$

Finally, from $\big(\mathbf{y},\mathbf{x}_{|N\setminus T}\big)\in\mathcal{D}(\mathbf{r},\mathbf{c})$ and (7), we have that $(\mathbf{y};\mathbf{x}_{|N\setminus T})\in\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$ and

$$\sum_{i\in T}\sum_{j\in M}(y_i^j-a_i^j)^2+\sum_{i\in N\setminus T}\sum_{j\in M}(x_i^j-a_i^j)^2<\sum_{i\in N}\sum_{j\in M}(x_i^j-a_i^j)^2,$$

where the inequality follows from (6), contradicting (3). This conclude the proof.

□

Note that, since the multi-issue reference system corresponding to any extended $\boldsymbol{\beta}$-egalitarian rule is consistent and this family is contained in the extended $\boldsymbol{\alpha}$-egalitarian class, then every $\widehat{\mathcal{E}}^{\boldsymbol{\beta}}$ rule is consistent over agents. However, they are not the only extended $\boldsymbol{\alpha}$-egalitarian consistent (over agents) rules. For instance, observe that ${}_N^M\alpha_i^j(\mathbf{c}) = \beta^j \cdot c_i^j + k$, for all $i \in N$ and all $j \in M$, where $k \in \mathbb{R}$, is also a consistent multi-issue reference system.

In order to connect multi-issue rationing rules with single-issue rationing rules it is important to check whether the allocation proposed by a multi-issue rule coincides for any issue with the output of a single-issue rationing rule applied to a reduced problem to the corresponding issue. Specifically, we study the issue-consistency of rules with respect to single-issue α-egalitarian rules.

Definition 14 *Let $\mathcal{F}$ be a rule on $\mathcal{MR}$ and $\boldsymbol{\alpha}$ be a multi-issue reference system. We say $\mathcal{F}$ is* one-issue $\boldsymbol{\alpha}$-consistent *if for all $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N\times M}$ with $N \in \mathcal{N}$ and $M \in \mathcal{M}$, we have that, for each $j \in M$,*

$$x^j = E^\theta(r^j, c^j),$$

where $(x^j)_{j\in M} = \mathbf{x} = \mathcal{F}(\mathbf{r}, \mathbf{c})$ and θ is a (single-issue) reference system satisfying

$$_N\theta(c^j) = \left(a_i^j - \sum_{k\in M\setminus\{j\}} \left(x_i^k - a_i^k \right) \right)_{i\in N},$$

where $a_i^k = {}_N^M\alpha_i^k(\mathbf{c})$, for all $i \in N$ and all $k \in M$.

A rule $\mathcal{F}$ is one-issue $\boldsymbol{\alpha}$-consistent if it predicts an egalitarian allocation within each issue relative to a new reference system θ, that depends on $\boldsymbol{\alpha}$. The new reference point associated within each issue $_N\theta(c^j)$ is the result of subtracting from the initial reference point a^j the net amounts (w.r.t. the references) received by agents for other issues $\sum_{k\in M\setminus\{j\}} \left(x^k - a^k\right)$. This adjustment means that agents' perception about the allocation for a certain issue j depends on what they have received for the rest of issues $M \setminus \{j\}$. Let us illustrate this point with an example.

Example 5 *Consider the 2-person and 2-issue CMIA problem, where*

$$(r^1, r^2) = (3, 3) \text{ and } c_i^j = 2, \text{ for all } i \in N = \{1, 2\} \text{ and all } j \in M = \{1, 2\}.$$

Let us suppose that a rule $\mathcal{F}$ assigns to this problem the allocation

$$\mathcal{F}(\mathbf{r},\mathbf{c}) = (x_1^1, x_2^1; x_1^2, x_2^2) = (2,1;1,2). \tag{8}$$

Let us check that this allocation satisfies one-issue $\mathbf{1}$-consistency,[13] *i.e. taking as initial reference point $\boldsymbol{a} = \mathbf{1} = (a_1^1, a_2^1; a_1^2, a_2^2) = (1,1;1,1)$.*

If we focus on issue $j = 1$, the corresponding initial reference point is $a^1 = (a_1^1, a_2^1) = (1,1)$. Suppose that agents have already received a net payoff for issue $j = 2$ equal to $(x_1^2 - a_1^2, x_2^2 - a_2^2) = (0,1)$. Then, the initial reference point a^1 is modified as follows:

$${}_N\theta(c^1) = \big(a_1^1 - (x_1^2 - a_1^2), a_2^1 - (x_2^2 - a_2^2)\big) = (1-0, 1-1) = (1,0).$$

Therefore, as the reader can check,

$$\{(x_1^1, x_2^1)\} \;\; = \{\widehat{E}^{{}_N\theta(c^1)}(r^1, c^1)\} = \operatorname*{arg\,min}_{y \in \mathcal{D}(3,(2,2))} (y_1^1 - 1)^2 + (y_2^1 - 0)^2 = \{(2,1)\},$$

and thus, the allocation for issue $j = 1$ does not change $\big($see (8)$\big)$. A similar argument can be used to show that the allocation for issue $j = 2$ does not change. Thus, one-issue $\mathbf{1}$-consistency holds. A graphical representation of this fact can be found in Figure 3.

The next proposition characterizes the set of allocations that are the outcome of a rule satisfying one-issue $\boldsymbol{\alpha}$-consistency.

Proposition 3 *Let $\mathcal{F}$ be a multi-issue rule and $\boldsymbol{\alpha}$ a multi-issue reference system. Then the following two statements are equivalent:*

1. *$\mathcal{F}$ is* one-issue $\boldsymbol{\alpha}$-consistent.

2. *For each $N \in \mathcal{N}$, each $M \in \mathcal{M}$ and any $(\mathbf{r},\mathbf{c}) \in \mathcal{MR}^{N\times M}$,*

$$\mathcal{F}(\mathbf{r},\mathbf{c}) \in \mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c}) = \operatorname*{arg\,min}_{\mathbf{x}\in\mathcal{D}(\mathbf{r},\mathbf{c})} \sum_{i\in N} \Big(\sum_{j\in M} \big(x_i^j - a_i^j\big)\Big)^2,$$

where $a_i^j = {}_N^M\alpha_i^j(\mathbf{c})$, for all $i \in N$ and all $j \in M$.

[13] See page 24 for the definition of one-issue $\mathbf{1}$-consistency.

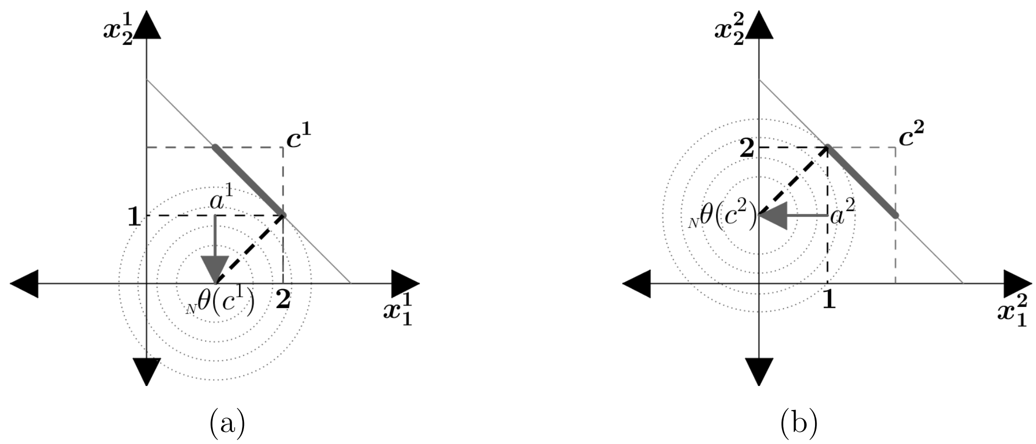

Figure 3: **One-issue 1-consistency.**

(a) For issue $j = 1$ the initial reference point moves downwards from a^1 to $_N\theta(c^1)$ subtracting the net payoffs received for issue $j = 2$, i.e. $_N\theta(c^1) = \left(a_1^1 - (x_1^2 - a_1^2), a_2^1 - (x_2^2 - a_2^2)\right) = (1, 0)$. (b) For issue $j = 2$ the initial reference point moves to the left from a^2 to $_N\theta(c^2)$ subtracting the net payoffs received for issue $j = 1$, i.e. $_N\theta(c^2) = \left(a_1^2 - (x_1^1 - a_1^1), a_2^2 - (x_2^1 - a_2^1)\right) = (1 - 1, 1 - 0) = (0, 1)$.

Proof. (2.$\Rightarrow$ 1.) Let $\mathbf{x}^* = \mathcal{F}(\mathbf{r}, \mathbf{c}) \in \underset{\mathbf{x}\in\mathcal{D}(\mathbf{r},\mathbf{c})}{\arg\min} \sum_{i\in N} \left(\sum_{k\in M} \left(x_i^k - a_i^k\right)\right)^2$, where $a_i^k = {}_N^M\alpha_i^k(\mathbf{c})$, for all $i \in N$ and all $k \in M$, and take an arbitrary issue $j \in M$. Then, for all $x^j \in \mathcal{D}(r^j, c^j)$,

$$\sum_{i\in N}\left(\sum_{k\in M}\left({x^*}_i^k - a_i^k\right)\right)^2 \leq \min_{x\in\mathcal{D}(r^j,c^j)} \sum_{i\in N}\left(x_i^j + \sum_{k\in M\setminus\{j\}} {x^*}_i^k - \sum_{k\in M} a_i^k\right)^2.$$

Therefore, ${x^*}^j = \underset{x\in\mathcal{D}(r^j,c^j)}{\arg\min} \sum_{i\in N}\left(x_i^j - a_i^j + \sum_{k\in M\setminus\{j\}}\left({x^*}_i^k - a_i^k\right)\right)^2$ for all $j \in M$. Taking a reference system θ such that

$$_N\theta(c^j) = \left(a_i^j - \sum_{k\in M\setminus\{j\}}\left(x_i^k - a_i^k\right)\right)_{i\in N}$$

we conclude that ${x^*}^j = E^\theta(r^j, c^j)$.

(1.$\Rightarrow$ 2.) Let $\mathbf{x}^* = \mathcal{F}(\mathbf{r}, \mathbf{c})$. For any arbitrary $j \in M$ and by one-issue $\boldsymbol{\alpha}$-consistency, we have that

$$x^{*j} = E^{\theta}(r^j, c^j) \text{ with } {}_N\theta(c^j) = \Big(a_i^j - \sum_{k \in M\setminus\{j\}} \big(x_i^k - a_i^k\big)\Big)_{i \in N}.$$

Hence, $x^{*j} = \underset{x \in \mathcal{D}(r^j, c^j)}{\arg\min} \sum_{i \in N} \Big(x_i^j - a_i^j + \sum_{k \in M\setminus\{j\}} \big(x^{*k}_i - a_i^k\big)\Big)^2$.

This minimization problem is a convex non-linear program. Then x^{*j} satisfies the Karush-Kuhn-Tucker necessary conditions.[14] That is, there exist $\lambda^{*j} \in \mathbb{R}^N$, $\mu^{*j} \in \mathbb{R}^N$ and $\delta^{*j} \in \mathbb{R}$ such that

- $$\begin{aligned}\nabla_{x^j}\mathscr{L}(x^{*j}, \lambda^{*j}, \mu^{*j}, \delta^{*j}) = \Big(\; & 2\big(x_i^j - a_i^j + \textstyle\sum_{k \in M\setminus\{j\}}(x^{*k}_i - a_i^k)\big) \\ & -\lambda^{*j} + \mu^{*j} + \delta^{*j}\Big)_{i \in N} = \vec{0},\end{aligned}$$

  $$\begin{aligned}\text{where } \mathscr{L}(x^j, \lambda^j, \mu^j, \delta^j) \; &= \textstyle\sum_{i \in N}\Big(\big(x_i^j - a_i^j + \sum_{k \in M\setminus\{j\}}(x^{*k}_i - a_i^k)\big)^2 \\ & -\big(\lambda_i^j(-x_i^j)\big) + \big(\mu_i^j(x_i^j - c_i^j)\big)\Big) \\ & +\delta^j\big(\textstyle\sum_{i \in N} x_i^j - r^j\big).\end{aligned}$$

- $x^{*j} \in \mathcal{D}(r^j, c^j)$.

- $\lambda^{*j}_i \geq 0$ for all $i \in N$ and $\mu^{*j}_i \geq 0$ for all $i \in N$.

- $\lambda^{*j}_i\big(-x^{*j}_i\big) = 0$ for all $i \in N$, $\mu^{*j}_i\big(x_i^j - c_i^j\big) = 0$ for all $i \in N$ and $\delta^{*j}\big(\sum_{i \in N} x_i^j - r^j\big) = 0$.

On the other hand, since the minimization program

$$\underset{\mathbf{x} \in \mathcal{D}(\mathbf{r}, \mathbf{c})}{\arg\min} \sum_{i \in N}\left(\sum_{j \in M}\big(x_i^j - a_i^j\big)\right)^2$$

is also a convex non-linear program, we have that there exist $\lambda = \big(\lambda_i^k\big)_{\substack{i \in N \\ k \in M}} \in \mathbb{R}^{N \times M}$, $\mu = \big(\mu_i^k\big)_{\substack{i \in N \\ k \in M}} \in \mathbb{R}^{N \times M}$ and $\delta = (\delta^k)_{k \in M} \in \mathbb{R}^M$ such that

[14] As the reader may verify, the regular conditions are satisfied since all the constrains are linear (Karlin conditions - see Borrell (1989) page 204).

- $\nabla_x \mathscr{L}(x,\lambda,\mu,\delta) = \Big(2\Big(\sum_{k\in M}\big(x_i^k - a_i^k\big)\Big) - \lambda + \mu + \delta\Big)_{i\in N} = \vec{0}$,

  where $\mathscr{L}(x,\lambda,\mu,\delta) = \sum_{i\in N}\Big(\big(\sum_{k\in M}(x_i^k - a_i^k)\big)^2 - \sum_{k\in M}\big(\lambda_i^k(-x_i^k)\big)$ $+ \sum_{k\in M}\big(\mu_i^k(x_i^k - c_i^k)\big)\Big) + \sum_{k\in M}\delta^k\big(\sum_{i\in N} x_i^k - r^k\big)$.

- $\mathbf{x} \in \mathcal{D}(\mathbf{r},\mathbf{c})$.

- $\lambda_i^k \geq 0$ for all $i \in N$ and all $k \in M$ and $\mu_i^k \geq 0$ for all $i \in N$ and all $k \in M$.

- $\lambda_i^k(-x_i^k) = 0$ for all $i \in N$ and all $k \in M$, $\mu_i^k(x_i^k - c_i^k) = 0$ for all $i \in N$ and all $k \in M$ and $\delta^k(\sum_{i\in N} x_i^k - r^k) = 0$ for all $k \in M$.

However, when

$$\lambda = (\lambda^{*1}, \lambda^{*2}, \ldots, \lambda^{*m}),\ \mu = (\mu^{*1}, \ldots, \mu^{*m}) \text{ and } \delta = (\delta^{*1}, \delta^{*2}, \ldots, \delta^{*m}),$$

we observe that $\mathbf{x}^* = (x_i^{*j})_{\substack{i\in N\\ j\in M}}$ is a critical point of the minimization program. Since this is a convex program, the Karush-Kuhn-Tucker necessary conditions are also sufficient. Therefore, we conclude that $\mathbf{x}^*$ minimizes this program, and thus,

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathcal{D}(\mathbf{r},\mathbf{c})} \sum_{i\in N}\Big(\sum_{j\in M}\big(x_i^j - a_i^j\big)\Big)^2.$$

□

Proposition 3 states that any one-issue $\boldsymbol{\alpha}$-consistent solution must be taken out from the set $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$. Since this set is obtained by minimizing the euclidean distance from the *aggregate reference point* $\big(\sum_{j\in M} a_i^j\big)_{i\in N}$ to the *vector of total payoffs* $\big(\sum_{j\in M} x_i^j\big)_{i\in N}$, we can conclude that one-issue $\boldsymbol{\alpha}$-consistent solutions are egalitarian in aggregate from $\big(\sum_{j\in M} a_i^j\big)_{i\in N}$.

On the other hand, we know by (2) that all allocations within the set $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$ have the same vector of total payoffs, that is, for all $\mathbf{x},\mathbf{y} \in \mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})$, $\sum_{j\in M} x_i^j = \sum_{j\in M} y_i^j$, for all $i \in N$. In order to discriminate between allocations that have the same vector of total payoffs, let us introduce the concept of *vector of payoff differences*.

Definition 15 *Let* $\mathbf{x} = (x_i^j)_{\substack{i\in N\\ j\in M}} \in \mathbb{R}^{N\times M}$ *be a multi-issue payoff vector. The vector of payoff differences relative to* $\mathbf{x}$, $\Delta_{\mathbf{x}} \in \mathbb{R}_+^M$, *is defined as follows:*

$$\Delta_{\mathbf{x}} := \left(\Delta_{\mathbf{x}}^j\right)_{j\in M} = \left(\max_{i\in N} x_i^j - \min_{i\in N} x_i^j\right)_{j\in M} \in \mathbb{R}_+^M.$$

Moreover, we denote by $\widetilde{\Delta_{\mathbf{x}}}$ the vector obtained from $\Delta_{\mathbf{x}}$ by rearranging its coordinates in a decreasing order (from large to small).[15]

The payoff difference $\Delta_{\mathbf{x}}^j$ expresses how different are the payoffs to agents according to $\mathbf{x}$ within the issue j. The smaller payoff difference, the more egalitarian allocation. This idea was already suggested by Schummer and Thomson (1997) for single-issue rationing problems [Proposition 3, page 336]:

> "*For any bankruptcy problem, the difference between the largest amount received by any agent and the smallest such amount is strictly smaller at the CEA allocation than at any other feasible allocation at which no agent receives more than his claim.*"

The vector $\widetilde{\Delta_{\mathbf{x}}}$ puts the stress on larger differences assigning them to the first components of the vector.

As we have pointed out before, the Lorenz criterion is widely used to compare payoff vectors with the same efficiency level. In the multi-issue context, we will apply this criterion to discriminate between pairs of payoff vectors $\mathbf{x}$ and $\mathbf{z}$ that assign the same vector of total payoffs, i.e. $\sum_{j\in M} x_i^j = \sum_{j\in M} z_i^j$, for all $i \in N$.

Definition 16 *Given a problem* $(\mathbf{r},\mathbf{c}) \in \mathcal{MR}^{N\times M}$ *and two payoff vectors* $\mathbf{x},\mathbf{z} \in \mathcal{D}(\mathbf{r},\mathbf{c})$ *such that*[16] $\sum_{j\in M} z_i^j = \sum_{j\in M} x_i^j$, *for all* $i \in N$, *we say that* $\mathbf{x}$ multi-issue Lorenz-dominates $\mathbf{z}$ in awards $\left(\mathbf{x} \succ_{\mathbf{m}}^{\mathbf{0}} \mathbf{z}\right)$ *if*

$$\sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{x}}}^j \leq \sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{z}}}^j, \ \textit{for all } k \in M,$$

[15]This means that $\widetilde{\Delta_{\mathbf{x}}}^1 = \Delta_{\mathbf{x}}^{j_1} \geq \widetilde{\Delta_{\mathbf{x}}}^2 = \Delta_{\mathbf{x}}^{j_2} \geq \ldots \geq \widetilde{\Delta_{\mathbf{x}}}^m = \Delta_{\mathbf{x}}^{j_m}$, where $\vartheta = (j_1, j_2, \ldots, j_m)$ is a permutation of elements of $M = \{1, 2, \ldots, m\}$.

[16]Notice that, by efficiency, $\sum_{i\in N} z_i^j = r^j = \sum_{i\in N} x_i^j$, for all $j \in M$.

with at least one strict inequality.[17]

This property basically states that $\mathbf{x}$ multi-issue Lorenz-dominates $\mathbf{z}$ in awards, when the largest payoff difference in $\mathbf{x}$ is smaller than the largest payoff difference in $\mathbf{z}$, i.e. $\widetilde{\Delta_{\mathbf{x}}}^1 \leq \widetilde{\Delta_{\mathbf{y}}}^1$, the sum of the two largest payoff differences in $\mathbf{x}$ is smaller than the two largest payoff differences in $\mathbf{z}$, i.e. $\widetilde{\Delta_{\mathbf{x}}}^1 + \widetilde{\Delta_{\mathbf{x}}}^2 \leq \widetilde{\Delta_{\mathbf{y}}}^1 + \widetilde{\Delta_{\mathbf{y}}}^2$, and so on and so forth. When this happens, we can say that $\mathbf{x}$ is more egalitarian *in awards* across issues than $\mathbf{z}$. Notice that multi-issue Lorenz-domination only discriminates when agents claim for at least two issues, $|M| \geq 2$.

Let us illustrate the multi-issue Lorenz criterion with an example.

Example 6 *Consider the* 3*-person and* 3*-issue* $CMIA$ *problem defined by* $r^1 = 40$*,* $r^2 = 60$ *and* $r^3 = 40$ *with claims* $c^1 = (40, 40, 40)$*,* $c^2 = (40, 20, 40)$ *and* $c^3 = (20, 50, 20)$*. Table 1 describes the allocation for each agent and each issue according to payoff vectors* $\mathbf{x}$ *and* $\mathbf{z}$*:*

$$\mathbf{x} = (x_1^1, x_2^1, x_3^1; x_1^2, x_2^2, x_3^2; x_1^3, x_2^3, x_3^3) = (10, 10, 20; 10, 30, 20; 10, 20, 10) \textit{ and}$$
$$\mathbf{z} = (z_1^1, z_2^1, z_3^1; z_1^2, z_2^2, z_3^2; z_1^3, z_2^3, z_3^3) = (0, 20, 20; 15, 30, 15; 15, 10, 15).$$

Notice that, the total payoff to agent 1 is the same in both payoff vectors, and equal to **30***; the same remark applies to agents 2 and 3, being the total payoffs* **60** *and* **50***, respectively. The* $\boxed{\text{red box}}$ *indicates the maximum payoff within the corresponding issue (column). The* $\boxed{\text{blue box}}$ *indicates the minimum payoff within the corresponding issue (column). The vectors of payoff differences* $\Delta_{\mathbf{x}}$ *and* $\Delta_{\mathbf{y}}$ *are indicated at the bottom of Table 1. Rearranging in a decreasing order the components of these vectors we get*

$$\widetilde{\Delta_{\mathbf{x}}} = (20, 10, 10) \textit{ and } \widetilde{\Delta_{\mathbf{y}}} = (20, 15, 5).$$

Since $\widetilde{\Delta_{\mathbf{x}}}^1 = 20 = \widetilde{\Delta_{\mathbf{z}}}^1$, $\widetilde{\Delta_{\mathbf{x}}}^1 + \widetilde{\Delta_{\mathbf{x}}}^2 = 30 < \widetilde{\Delta_{\mathbf{z}}}^1 + \widetilde{\Delta_{\mathbf{x}}}^2 = 35$ *and* $\widetilde{\Delta_{\mathbf{x}}}^1 + \widetilde{\Delta_{\mathbf{x}}}^2 + \widetilde{\Delta_{\mathbf{x}}}^3 = \widetilde{\Delta_{\mathbf{z}}}^1 + \widetilde{\Delta_{\mathbf{x}}}^2 + \widetilde{\Delta_{\mathbf{z}}}^3 = 40$*, and by Definition 16, we conclude that* $\mathbf{x} \succ_{\mathbf{m}}^{\mathbf{0}} \mathbf{z}$.

[17] Notice that, in general, $\sum_{j \in M} \Delta_{\mathbf{x}}{}^j$ does not coincide with $\sum_{j \in M} \Delta_{\mathbf{z}}{}^j$.

x:

AGENT		ISSUE 1	ISSUE 2	ISSUE 3	
	1	$\boxed{10}$	$\boxed{10}$	$\boxed{10}$	**30**
	2	$\boxed{10}$	$\boxed{30}$	$\boxed{20}$	**60**
	3	$\boxed{20}$	20	$\boxed{10}$	**50**
		40	60	40	
	$\Delta_{\mathbf{x}} =$	(10,	20,	10)	

z:

AGENT		ISSUE 1	ISSUE 2	ISSUE 3	
	1	$\boxed{0}$	$\boxed{15}$	$\boxed{15}$	**30**
	2	20	$\boxed{30}$	$\boxed{10}$	**60**
	3	$\boxed{20}$	$\boxed{15}$	$\boxed{15}$	**50**
		40	60	40	
	$\Delta_{\mathbf{z}} =$	(20,	15,	5)	

Table 1: **A comparison between two vectors of payoff differences.**

Next, we extend the idea of multi-issue Lorenz-domination in awards to the domain of rules. We say that a rule $\mathcal{F}$ multi-issue Lorenz-dominates a rule $\widehat{\mathcal{F}}$ in awards if, for any problem, the allocation proposed by $\mathcal{F}$ multi-issue Lorenz-dominates in awards the allocation proposed by $\widehat{\mathcal{F}}$.

Definition 17 *Given two rules $\mathcal{F}$ and $\widehat{\mathcal{F}}$, we say that* a rule $\mathcal{F}$ multi-issue Lorenz-dominates a rule $\widehat{\mathcal{F}}$ in awards $\left(\mathcal{F}\succ^{\mathbf{0}}_{\mathbf{m}}\widehat{\mathcal{F}}\right)$ *if, for all $N \in \mathcal{N}$, all $M \in \mathcal{M}$ and any $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N\times M}$,*

$$\mathcal{F}(\mathbf{r}, \mathbf{c})\succ^{\mathbf{0}}_{\mathbf{m}}\widehat{\mathcal{F}}(\mathbf{r}, \mathbf{c}).$$

Let us remark that the definition of multi-issue Lorenz-domination in awards implies that two payoff vectors can only be compared if they assign the same vector of total payoffs. This means that a rule $\mathcal{F}$ might multi-issue Lorenz-dominate another rule $\widehat{\mathcal{F}}$ in awards if, for any problem $(\mathbf{r}, \mathbf{c})$, $\sum_{j\in M}\mathcal{F}_i^j(\mathbf{r}, \mathbf{c}) = \sum_{j\in M}\widehat{\mathcal{F}}_i^j(\mathbf{r}, \mathbf{c})$, for all $i \in N$.

Next, we focus on multi-issue Lorenz-domination in awards between payoff vectors for the two-person case ($|N| = 2$). The following proposition states that, for two-person problems, the allocation suggested by the $\widehat{CEA}$ rule multi-issue Lorenz-dominates in awards any other payoff vector with the same vector of total payoffs.

Proposition 4 *Let it be* $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N\times M}$ *with* $|N| = 2$ *and let* $\mathbf{x} = \widehat{CEA}(\mathbf{r}, \mathbf{c})$*. Then, it holds*

$$\mathbf{x} \succ^{\mathbf{0}}_{\mathbf{m}} \mathbf{z}, \text{ for all } \mathbf{z} \in \mathcal{D}(\mathbf{r}, \mathbf{c}) \text{ with } \sum_{j\in M} z_i^j = \sum_{j\in M} x_i^j, \text{ for all } i \in N.$$

Proof. Assume $N = \{1, 2\}$. We must prove that, for $k = 1, 2, \ldots, m$,

$$\sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{x}}}^j \leq \sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{z}}}^j, \tag{9}$$

for all $\mathbf{z} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$ with $\sum_{j\in M} z_i^j = \sum_{j\in M} x_i^j$, for all $i \in N$, with at least one strict inequality.

First of all, we claim that the sum of all payoff differences relative to $\mathbf{x}$ is smaller than the sum of all payoff differences relative to $\mathbf{z}$.

Claim 1 $\sum_{j\in M} \Delta_{\mathbf{x}}^j \leq \sum_{j\in M} \Delta_{\mathbf{z}}^j$*, for all* $\mathbf{z} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$ *with* $\sum_{j\in M} z_i^j = \sum_{j\in M} x_i^j$*, for all* $i \in N$*.*

The proof of this claim is consigned into Appendix B.

To check (9) for any $k \in \{1, 2, \ldots, m-1\}$, let us suppose, w.l.o.g., that the vector of differences $\Delta_{\mathbf{x}}$ is already ordered in a non-increasing way, $\widetilde{\Delta_{\mathbf{x}}} = \Delta_{\mathbf{x}}$. As there might exist ties between some differences, we suppose additionally that

$$\begin{aligned} &\text{if} \quad \Delta_{\mathbf{x}}^j = \Delta_{\mathbf{x}}^{j'} \text{ for some } j, j' \in M \text{ with } j < j', \\ &\text{then,} \quad \text{either } \left[\Delta_{\mathbf{x}}^j \leq \Delta_{\mathbf{z}}^j\right] \text{ or } \left[\Delta_{\mathbf{x}}^j > \Delta_{\mathbf{z}}^j \text{ and } \Delta_{\mathbf{x}}^{j'} > \Delta_{\mathbf{z}}^{j'}\right]. \end{aligned} \tag{10}$$

Next, we prove by induction on $k \in \{1, 2, \ldots, m-2\}$ that

$$\sum_{j=1}^{k} \Delta_{\mathbf{x}}^j \leq \sum_{j=1}^{k} \Delta_{\mathbf{z}}^j, \text{ for all } k \in \{1, 2, \ldots, m-1\}.$$

(a) First of all, we prove the induction hypothesis for $k = 1$. That is, $\Delta_{\mathbf{x}}^1 \leq \Delta_{\mathbf{z}}^1$. Suppose, on the contrary, that

$$\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{z}}^1 \geq 0. \tag{11}$$

From here, we get $\Delta_{\mathbf{x}}^1 > 0$ which means that $x_1^1 \neq x_2^1$. Suppose, w.l.o.g., that

$$x_1^1 > x_2^1. \tag{12}$$

Since $\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{z}}^1$ and by efficiency, we have that

$$x_1^1 > z_1^1 \text{ and } x_2^1 < z_2^1. \tag{13}$$

Since $\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{z}}^1$ and by Claim 1, there exists $j' \in M \setminus \{1\}$ such that $\Delta_{\mathbf{x}}^{j'} < \Delta_{\mathbf{z}}^{j'}$. Moreover, since $\Delta_{\mathbf{x}}$ is ordered in a non-increasing way, we have that $\Delta_{\mathbf{x}}^1 \geq \Delta_{\mathbf{x}}^{j'}$. Now, taking into account (10), since $\Delta_{\mathbf{x}}^{j'} < \Delta_{\mathbf{z}}^{j'}$ and $\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{z}}^1$, we can deduce that this inequality is strict, i.e. $\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{x}}^{j'}$.

On the other hand, since $\Delta_{\mathbf{x}}^{j'} < \Delta_{\mathbf{z}}^{j'}$, we have that $x^{j'} \neq z^{j'}$. At this point we consider two cases:

Case 1: $x_1^{j'} < z_1^{j'}$ and thus, by efficiency, $x_2^{j'} > z_2^{j'}$. In this case, define $\widehat{\mathbf{x}} \in \mathbb{R}_+^{N\times M}$ as follows:

$$\widehat{x}_1^1 = x_1^1 - \epsilon,\ \widehat{x}_2^1 = x_2^1 + \epsilon,\ \widehat{x}_1^{j'} = x_1^{j'} + \epsilon,\ \widehat{x}_2^{j'} = x_2^{j'} - \epsilon \text{ and } \widehat{x}_i^j = x_i^j, \text{ else,} \tag{14}$$

where $0 < \epsilon < \min\left\{x_1^1 - z_1^1, z_2^1 - x_2^1, z_1^{j'} - x_1^{j'}, x_2^{j'} - z_2^{j'}, \frac{x_1^1 + x_2^{j'} - \left(x_1^{j'} + x_2^1\right)}{2}\right\}$.

In order to check that ϵ exists notice that: in case $x_1^{j'} \leq x_2^{j'}$, since $x_1^1 > x_2^1$ (see (12)), we have that $x_1^1 + x_2^{j'} > x_1^{j'} + x_2^1$; in case $x_1^{j'} > x_2^{j'}$, since $x_1^1 > x_2^1$ (see (12)) and $\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{x}}^{j'}$, we also obtain that $x_1^1 + x_2^{j'} > x_1^{j'} + x_2^1$. Let us remark that, by definition of ϵ, $\widehat{\mathbf{x}} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$ and $\sum_{j\in M} \widehat{x}_i^j = \sum_{j\in M} x_i^j$, for all $i \in N$. Now, from the definition of $\widehat{\mathbf{x}}$ it holds that

$$\begin{aligned}\sum_{i\in N}\sum_{j\in M}(\widehat{x}_i^j)^2 &= \sum_{i\in N}\sum_{j\in M}(x_i^j)^2 + 2\cdot\epsilon\left(2\cdot\epsilon - \left(x_1^1 + x_2^{j'} - \left(x_1^{j'} + x_2^1\right)\right)\right)\\ &< \sum_{i\in N}\sum_{j\in M}(x_i^j)^2,\end{aligned}$$

where the inequality follows from $\epsilon < \frac{x_1^1 + x_2^{j'} - \left(x_1^{j'} + x_2^1\right)}{2}$. Therefore, we reach a contradiction with the fact that $\mathbf{x} = \widehat{CEA}(\mathbf{r}, \mathbf{c})$ and thus we conclude that $\Delta_{\mathbf{x}}^1 \leq \Delta_{\mathbf{z}}^1$.

Case 2: $x_1^{j'} > z_1^{j'}$ and thus, by efficiency, $x_2^{j'} < z_2^{j'}$. Since $x_1^{j'} > z_1^{j'}$, $x_1^1 > z_1^1$ (see (13)) and $\sum_{j \in M} x_1^j = \sum_{j \in M} z_1^j$, there exists $j'' \in M \setminus \{1, j'\}$ such that $x_1^{j''} < z_1^{j''}$. At this point we consider two subcases:

2.1 In case $x_1^{j''} \leq x_2^{j''}$ and since $x_1^1 > x_2^1$ (see (12)), we have that $x_1^1 + x_2^{j''} > x_1^{j''} + x_2^1$. Thus, we can define the payoff vector $\widehat{\mathbf{x}}$ as in (14), but replacing j' by j'', and we reach the same contradiction.

2.2 In case $x_1^{j''} > x_2^{j''}$, since $x_1^{j''} < z_1^{j''}$ and by efficiency, we obtain that $x_2^{j''} > z_2^{j''}$. Hence, we have that $z_1^{j''} > x_1^{j''} > x_2^{j''} > z_2^{j''}$ and thus, $\Delta_{\mathbf{x}}^{j''} < \Delta_{\mathbf{z}}^{j''}$. Moreover, since we are supposing that $\Delta_{\mathbf{x}}$ is ordered in a non-increasing way, we have that $\Delta_{\mathbf{x}}^1 \geq \Delta_{\mathbf{x}}^{j''}$. In fact, taking into account (10), since $\Delta_{\mathbf{x}}^{j''} < \Delta_{\mathbf{z}}^{j''}$ and $\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{z}}^1$ (see (11)), we can deduce that this inequality is strict, i.e. $\Delta_{\mathbf{x}}^1 > \Delta_{\mathbf{x}}^{j''}$. Hence, since we are supposing that $x_1^{j''} > x_2^{j''}$ and $x_1^1 > x_2^1$ (see (12)), we obtain that $x_1^1 + x_2^{j''} > x_1^{j''} + x_2^1$. Thus, we can again define the payoff vector $\widehat{\mathbf{x}}$ as in (14), but replacing j' by j'', and we reach the same contradiction.

Therefore, we conclude that $\Delta_{\mathbf{x}}^1 \leq \Delta_{\mathbf{z}}^1$.

(b) Assume the induction hypothesis holds up to $k-1$, for some $k \in \{2, 3, \ldots, m-1\}$ and suppose that

$$\sum_{j=1}^{k-1} \Delta_{\mathbf{x}}^j \leq \sum_{j=1}^{k-1} \Delta_{\mathbf{z}}^j. \tag{15}$$

Next, we prove that $\sum_{j=1}^{k} \Delta_{\mathbf{x}}^j \leq \sum_{j=1}^{k} \Delta_{\mathbf{z}}^j$. Suppose, on the contrary, that

$$\sum_{j=1}^{k} \Delta_{\mathbf{x}}^j > \sum_{j=1}^{k} \Delta_{\mathbf{z}}^j. \tag{16}$$

Hence, by (15), it holds

$$\Delta_{\mathbf{x}}^k > \Delta_{\mathbf{z}}^k \geq 0, \tag{17}$$

which implies that $\Delta_{\mathbf{x}}^k > 0$ and thus, $x_1^k \neq x_2^k$. Suppose, w.l.o.g., that

$$x_1^k > x_2^k. \tag{18}$$

Then, since we are supposing that $\Delta_{\mathbf{x}}^k > \Delta_{\mathbf{z}}^k$ (see (17)) and by efficiency, we have that

$$x_1^k > z_1^k \text{ and } z_2^k > x_2^k. \tag{19}$$

Next, we divide the subset $M \setminus \{k+1, k+2, \ldots, m\}$ in four subsets as follows:

$$\widehat{M}_1 = \left\{ j \in M \setminus \{k+1, k+2, \ldots, m\} | \; x_1^j \geq z_1^j \text{ and } x_1^j > x_2^j \right\},$$
$$\widehat{M}_2 = \left\{ j \in M \setminus \{k+1, k+2, \ldots, m\} | \; x_1^j \geq z_1^j \text{ and } x_1^j < x_2^j \right\},$$
$$\widehat{M}_3 = \left\{ j \in M \setminus \{k+1, k+2, \ldots, m\} | \; x_1^j < z_1^j \right\} \text{ and}$$
$$\widehat{M}_4 = \left\{ j \in M \setminus \{k+1, k+2, \ldots, m\} | \; x_1^j = x_2^j \right\},$$

Notice that $\widehat{M}_1 \cup \widehat{M}_2 \cup \widehat{M}_3 \cup \widehat{M}_4 = M \setminus \{k+1, k+2, \ldots, m\}$ and $\widehat{M}_k \cap \widehat{M}_{k'} = \emptyset$, for all $k \neq k' \in \{1, 2, 3, 4\}$, since the following claim holds.

Claim 2 *If $x_1^j < z_1^j$ for some $j \in \{1, 2, \ldots, k-1\}$, then $x_1^j > x_2^j$.*

The proof of this claim is consigned into Appendix B.

We next show that $\sum_{j=1}^k \Delta_{\mathbf{x}}^j < \sum_{j=1}^k \Delta_{\mathbf{z}}^j$. To this aim let us denote[18] $d^j = |x_1^j - z_1^j|$, for all $j \in M$. Then,

[18]Notice that, by efficiency, $|x_1^j - z_1^j| = |x_2^j - z_2^j|$, for all $j \in M$.

$$\begin{aligned}
\sum_{j=1}^{k} \Delta_{\mathbf{x}}^{j} &= \sum_{j\in\widehat{M}_1} (x_1^j - x_2^j) + \sum_{j\in\widehat{M}_2} (x_2^j - x_1^j) + \sum_{j\in\widehat{M}_3} (x_1^j - x_2^j) \\
&= \sum_{j\in\widehat{M}_1} \big(x_1^j - d^j - (x_2^j + d^j) + 2\cdot d^j\big) \\
&+ \sum_{j\in\widehat{M}_2} \big(x_2^j + d^j - (x_1^j - d^j) - 2\cdot d^j\big) \\
&+ \sum_{j\in\widehat{M}_3} \big(x_1^j + d^j - (x_2^j - d^j) - 2\cdot d^j\big) = \sum_{j\in\widehat{M}_1} (z_1^j - z_2^j) + \sum_{j\in\widehat{M}_2} (z_2^j - z_1^j) \\
&+ \sum_{j\in\widehat{M}_3} (z_1^j - z_2^j) + 2\Big(\sum_{j\in\widehat{M}_1} d^j - \sum_{j\in\widehat{M_2}\cup\widehat{M}_3} d^j\Big) \\
&< \sum_{j\in\widehat{M}_1\cup\widehat{M}_3} (z_1^j - z_2^j) + \sum_{j\in\widehat{M}_2} (z_2^j - z_1^j) \le \sum_{j\in\widehat{M}_1\cup\widehat{M}_2\cup\widehat{M}_3} \big|z_1^j - z_2^j\big| \le \sum_{j=1}^{k} \Delta_{\mathbf{z}}^{j},
\end{aligned}$$

where the strict inequality follows from the next claim.

Claim 3 $\sum_{j\in\widehat{M}_1} d^j < \sum_{j\in\widehat{M_2}\cup\widehat{M}_3} d^j$.

The proof of this claim is also consigned into Appendix B.

We have reached to a contradiction with the assumption $\sum_{j=1}^{k} \Delta_{\mathbf{x}}^{j} > \sum_{j=1}^{k} \Delta_{\mathbf{z}}^{j}$ (see (16)). Then, we obtain that $\sum_{j=1}^{k} \Delta_{\mathbf{x}}^{j} \le \sum_{j=1}^{k} \Delta_{\mathbf{z}}^{j}$. Taking this fact into account and by Claim 1, we conclude that

$$\sum_{j=1}^{k} \Delta_{\mathbf{x}}^{j} \le \sum_{j=1}^{k} \Delta_{\mathbf{z}}^{j}, \text{ for all } k \in M. \tag{20}$$

Next, we prove that at least one of these inequalities is strict. Let us suppose, on the contrary, that all of them are equalities, i.e. $\sum_{j=1}^{k} \Delta_{\mathbf{x}}^{k} = \sum_{j=1}^{k} \Delta_{\mathbf{z}}^{k}$, for all $k \in M$, and thus,

$$\Delta_{\mathbf{x}}^{j} = \Delta_{\mathbf{z}}^{j}, \text{ for all } j \in M. \tag{21}$$

Since $\mathbf{x} \neq \mathbf{z}$, there exists $j^* \in M$ such that $x^{j^*} \neq z^{j^*}$, w.l.o.g.

$$x_1^{j^*} > z_1^{j^*} \text{ and, by efficiency, } x_2^{j^*} < z_2^{j^*}. \tag{22}$$

Hence, since $\Delta_{\mathbf{x}}^{j^*} = \Delta_{\mathbf{z}}^{j^*}$ and by efficiency of both allocations, we have that $x_1^{j^*} \neq x_2^{j^*}$. Then, we obtain that $x_1^{j^*} > x_2^{j^*}$, otherwise, i.e. $x_1^{j^*} < x_2^{j^*}$, and, by (22), we have that $z_2^{j^*} > x_2^{j^*} > x_1^{j^*} > z_1^{j^*}$, which means that $\Delta_{\mathbf{x}}^{j^*} < \Delta_{\mathbf{z}}^{j^*}$, reaching a contradiction with (21).

On the other hand, since $x_1^{j^*} > z_1^{j^*}$ and $\sum_{j \in M} x_1^j = \sum_{j \in M} z_1^j$, there exists $j' \in M$ such that

$$x_1^{j'} < z_1^{j'} \text{ and, by efficiency, } x_2^{j'} > z_2^{j'}. \tag{23}$$

Hence, since $\Delta_{\mathbf{x}}^{j'} = \Delta_{\mathbf{z}}^{j'}$ and by efficiency of both allocations, $x_1^{j'} \neq x_2^{j'}$. Then, we claim that $x_1^{j'} < x_2^{j'}$; otherwise, $x_1^{j'} > x_2^{j'}$ and, by (23), we would have that $z_1^{j'} > x_1^{j'} > x_2^{j'} > z_2^{j'}$, which would imply that $\Delta_{\mathbf{x}}^{j^*} < \Delta_{\mathbf{z}}^{j^*}$, reaching a contradiction with (21).

At this point we can define a vector $\widehat{\mathbf{x}}$ as in (14), but replacing 1 by j^*, and we reach the same contradiction. Then, there exists at least one strict inequality in (20).

Finally, since $\widetilde{\Delta_{\mathbf{x}}}^j = \Delta_{\mathbf{x}}^j$, for all $j \in M$, we have that, for all $k \in M$,

$$\sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{x}}}^j = \sum_{j=1}^{k} \Delta_{\mathbf{x}}^j \leq \sum_{j=1}^{k} \Delta_{\mathbf{z}}^j \leq \sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{z}}}^j,$$

where the first inequality follows from (20). Then, we conclude that

$$\sum_{j=1}^{k} \widetilde{\Delta_{x}}^j \leq \sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{z}}}^j, \text{ for all } k \in M,$$

and, since at least one of these inequalities is strict, $x\mathcal{L}^{\mathbf{0}}z$, and the proof is done.

□

From this proposition we can state two corollaries.

Corollary 1 *For any two-person CMIA problem* $(\mathbf{r},\mathbf{c}) \in \mathcal{MR}^{N\times M}$ *with* $M \in \mathcal{M}$*,* $N \in \mathcal{N}$ *and* $|N| = 2$*, the* $\widehat{CEA}$ *rule* multi-issue Lorenz-dominates in awards *any other rule* $\mathcal{F}$ $\left(\widehat{CEA}\succ^{\mathbf{0}}_{\mathbf{m}}\mathcal{F}\right)$ *satisfying, for all* $i \in N$*,*

$$\sum_{j\in M} \mathcal{F}_i^j(\mathbf{r},\mathbf{c}) = \sum_{j\in M} \widehat{CEA}_i^j(\mathbf{r},\mathbf{c}).$$

Corollary 2 *The* $\widehat{CEA}$ *rule is* multi-issue Lorenz undominated in awards*.*

By using the consistency properties and the multi-issue Lorenz-domination, we can characterize the $\widehat{CEA}$ rule.

Theorem 1 *A rule* $\mathcal{F}$ *on* $\mathcal{MR}$ *is one-issue* 0*-consistent*[19]*, consistent over agents and multi-issue Lorenz undominated in awards for any two-person problem if and only if* $\mathcal{F}$ *is the* $\widehat{CEA}$ *rule.*

Proof. Since the $\widehat{CEA}$ rule is contained in the extended $\boldsymbol{\beta}$-egalitarian family of rules, the $\widehat{CEA}$ rule is consistent over agents. Moreover, by Proposition 3 it is also one-issue 0-consistent, and by Corollary 2 it is Lorenz undominated in awards for any two-person problem. Next, let us check the uniqueness of the rule. Let $\mathcal{F}$ be a rule satisfying these properties, but suppose on the contrary that $\mathcal{F} \neq \widehat{CEA}$. Then, there exists a $CMIA$ problem $(\mathbf{r},\mathbf{c}) \in \mathcal{MR}^{N\times M}$ such that $\mathbf{z} = \mathcal{F}(\mathbf{r},\mathbf{c}) \neq \widehat{CEA}(\mathbf{r},\mathbf{c}) = \mathbf{x}$. Since both rules are consistent over agents, there exist two agents $i', i'' \in N$ such that

$$\begin{aligned} \mathbf{z}_{|\{i',i''\}} &= \mathcal{F}\Big(\big(r^j - \textstyle\sum_{i\in N\setminus\{i',i''\}} z_i^j\big)_{j\in M}, \mathbf{c}_{|\{i',i''\}}\Big) \\ &\neq \widehat{CEA}\Big(\big(r^j - \textstyle\sum_{i\in N\setminus\{i',i''\}} x_i^j\big)_{j\in M}, \mathbf{c}_{|\{i',i''\}}\Big) = \mathbf{x}_{|\{i',i''\}}. \end{aligned} \tag{24}$$

Moreover, since both rules are one-issue 0-consistent, using Proposition 3 and by the implication given by (2), we obtain that

$$\sum_{j\in M} x_{i'}^j = \sum_{j\in M} z_{i'}^j \text{ and } \sum_{j\in M} x_{i''}^j = \sum_{j\in M} z_{i''}^j. \tag{25}$$

[19]See page 24 for the definition of one-issue 0-consistency.

Finally, by (25) and Corollary 1, we know that $\mathbf{x}_{|\{i',i''\}} \succ^{\mathbf{0}}_{\mathbf{m}} \mathbf{z}_{|\{i',i''\}}$, but this contradicts that $\mathcal{F}$ is multi-issue Lorenz undominated in awards for any two-person problem.

$\square$

The properties used in Theorem 1 are logically independent, as we see next.

- The rule $\mathcal{F}'$ that assigns within each issue $j \in M$ the vector

  $$\left(\mathcal{F}'^j_1(\mathbf{r},\mathbf{c}), \mathcal{F}'^j_2(\mathbf{r},\mathbf{c}), \ldots, \mathcal{F}'^j_n(\mathbf{r},\mathbf{c})\right) = CEA(r^j, c^j),$$

  is consistent over agents and multi-issue Lorenz undominated for any two-person problem, but it is not one-issue $\mathsf{0}$-consistent.

- The rule $\mathcal{F}''$ defined as

  $$\mathcal{F}''(\mathbf{r},\mathbf{c}) = \begin{cases} \underset{\mathbf{x}\in\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})}{\arg\min} \sum_{i\in N}\sum_{j\in M} \left(x_i^j\right)^2 & \text{if } |N| = 2 \\ \underset{\mathbf{x}\in\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})}{\arg\min} \sum_{i\in N}\sum_{j\in M} \left(x_i^j - c_i^j\right)^2 & \text{if } |N| \neq 2 \end{cases},$$

  where $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c}) = \underset{\mathbf{x}\in\mathcal{D}(\mathbf{r},\mathbf{c})}{\arg\min} \sum_{i\in N}(\sum_{j\in M} x_i^j)^2$, is multi-issue Lorenz undominated for any two-person problem and one-issue $\mathsf{0}$-consistent, but it does not satisfy consistency over agents.

- The rule $\mathcal{F}'''$ defined as

  $$\mathcal{F}'''(\mathbf{r},\mathbf{c}) = \underset{\mathbf{x}\in\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c})}{\arg\min} \sum_{i\in N}\sum_{j\in M} \left(x_i^j - c_i^j\right)^2,$$

  where $\mathcal{D}^{\boldsymbol{\alpha}}(\mathbf{r},\mathbf{c}) = \underset{\mathbf{x}\in\mathcal{D}(\mathbf{r},\mathbf{c})}{\arg\min} \sum_{i\in N}(\sum_{j\in M} x_i^j)^2$, is one-issue $\mathsf{0}$-consistent and consistent over agents, but it is multi-issue Lorenz-dominated by the $\widehat{CEA}$ rule for any two-person problem.

Analogously to the single-issue case, we can define multi-issue Lorenz-domination from a reference point $\boldsymbol{a}$. To this aim we define the vector of net payoff differences from a reference point $\boldsymbol{a}$.

Definition 18 *Let* $\mathbf{x} \in \mathbb{R}^{N\times M}$ *be a multi-issue payoff vector and* $\boldsymbol{a} \in \mathbb{R}^{N\times M}$ *be a reference point. The vector of* net payoff differences of $\mathbf{x}$ from $\boldsymbol{a}$, $\Delta_{\mathbf{x}}^{\boldsymbol{a}} \in \mathbb{R}_+^M$, *is defined as follows:*

$$\Delta_{\mathbf{x}}^{\boldsymbol{a}} = \left(\max_{i\in N}\{x_i^j - a_i^j\} - \min_{i\in N}\{x_i^j - a_i^j\}\right)_{j\in M} \in \mathbb{R}_+^M.$$

Moreover, we denote by $\widetilde{\Delta_{\mathbf{x}}^{\boldsymbol{a}}}$ *the vector obtained from* $\Delta_{\mathbf{x}}^{\boldsymbol{a}}$ *by rearranging its coordinates in a decreasing order (from large to small).*

Following the same line as before, we use the vector of net payoff differences to evaluate when a payoff vector is more egalitarian across issues from a reference point $\boldsymbol{a}$ than any other payoff vector with the same vector of total payoffs.

Definition 19 *Given a problem* $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N\times M}$, *a reference point* $\boldsymbol{a} \in \mathbb{R}^{N\times M}$ *and two payoff vectors* $\mathbf{x}, \mathbf{z} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$ *such that* $\sum_{j\in M} z_i^j = \sum_{j\in M} x_i^j$, *for all* $i \in N$, *we say that* $\mathbf{x}$ multi-issue Lorenz-dominates $\mathbf{z}$ from $\boldsymbol{a}$ $\left(\mathbf{x} \succ_{\mathbf{m}}^{\boldsymbol{a}} \mathbf{z}\right)$ *if*

$$\sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{x}}^{\boldsymbol{a}}}^j \leq \sum_{j=1}^{k} \widetilde{\Delta_{\mathbf{z}}^{\boldsymbol{a}}}^j, \text{ for all } k \in M,$$

with at least one strict inequality.[20]

We also apply the concept of multi-issue Lorenz-domination to rules.

Definition 20 *Given a multi-issue reference system* $\boldsymbol{\alpha}$ *and two rules* $\mathcal{F}$ *and* $\widehat{\mathcal{F}}$, *we say that a rule* $\mathcal{F}$ multi-issue Lorenz-dominates *a rule* $\widehat{\mathcal{F}}$ from *the multi-issue reference system* $\boldsymbol{\alpha}$ $\left(\mathcal{F} \succ_{\mathbf{m}}^{\boldsymbol{\alpha}} \widehat{\mathcal{F}}\right)$, *if, for all* $N \in \mathcal{N}$, *all* $M \in \mathcal{M}$ *and any* $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N\times M}$,

$$\mathcal{F}(\mathbf{r}, \mathbf{c}) \succ_{\mathbf{m}}^{\boldsymbol{a}} \widehat{\mathcal{F}}(\mathbf{r}, \mathbf{c}),$$

where $\boldsymbol{a} = {}_N^M\boldsymbol{\alpha}(\mathbf{c})$.

[20] Notice that, if $\boldsymbol{a} = (0, 0, \ldots, 0)$, then $\widehat{E}^{\boldsymbol{a}} = \widehat{CEA}$ and Definition 16 arises from this definition.

The next proposition and theorem mimic the results obtained for the $\widehat{CEA}$ rule to other multi-issue reference systems other than $\boldsymbol{\alpha} = \mathsf{0}$.

Proposition 5 *Let* $(\mathbf{r}, \mathbf{c}) \in \mathcal{MR}^{N \times M}$ *with* $|N| = 2$ *and* $\mathbf{x} = \widehat{E}^{\boldsymbol{\alpha}}(\mathbf{r}, \mathbf{c})$*, then*

$$\mathbf{x} \succ_{\mathbf{m}}^{\boldsymbol{a}} \mathbf{z}, \text{ for all } \mathbf{z} \in \mathcal{D}(\mathbf{r}, \mathbf{c}) \text{ with } \sum_{j \in M} z_i^j = \sum_{j \in M} x_i^j, \text{ for all } i \in N,$$

where $\boldsymbol{a} = {}_{N}^{M}\boldsymbol{\alpha}(\mathbf{c})$.

The last result of this section characterizes the whole family of the extended $\boldsymbol{\alpha}$-egalitarian rules with a consistent multi-issue reference system.

Theorem 2 *For an arbitrary consistent multi-issue reference system* $\boldsymbol{\alpha}$*, a rule* $\mathcal{F}$ *on* $\mathcal{MR}$ *is consistent over agents, one-issue* $\boldsymbol{\alpha}$*-consistent and multi-issue Lorenz undominated from* $\boldsymbol{\alpha}$ *for any two-person problem if and only if* $\mathcal{F}$ *is the* $\widehat{E}^{\boldsymbol{\alpha}}$ *rule.*

The proofs of Proposition 5 and Theorem 2 can be obtained following the same guidelines of the proof of Proposition 4 and Theorem 1, respectively.

4 Conclusions

Section 2 introduces the idea of reinterpreting a single-issue rationing rule as a distance minimization program from a reference point. In the same section, we analyse a family of rules which have as reference point a proportion of the claims. This is the family of β-egalitarian rules which is a generalization of the RT rule and coincides with the $RTAL$ family (van den Brink et al., 2013; Thomson, 2008). As it is known, the RT rule is the "reverse" of the T rule (see footnote 7 on page 16). Similarly, the "reverse" of the $RTAL$ family is the $TAL \equiv \left\{T^{\theta}\right\}_{\theta \in [0,1]}$ family of rules (Moreno-Ternero and Villar, 2006; Thomson, 2008) which is a generalization of the T rule. Given $\theta \in [0, 1]$, the corresponding T^{θ} rule can be explained from the perspective of the distance minimization by a combination of two β-egalitarian rules as follows:

$$T^{\theta}(r, c) = \mathcal{E}^{0}\big(\min\left\{r, \theta \cdot C\right\}, \theta \cdot c\big) + \mathcal{E}^{1}\big(\max\left\{0, r - \theta \cdot C\right\}, (1 - \theta) \cdot c\big).$$

For further research, we aim to extend to the $CMIA$ framework the TAL family of rules.

The E^{α} rule, when the reference system α is constant and independent of the claims[21] is directly related to the *generalized equal awards* (GEA) rule (see Chapter 2). Recall that the GEA rule is a generalization of the CEA rule for *rationing problems with ex-ante conditions* $\big((r, c, \delta) \in \mathcal{RC}^N\big)$ in which agents, in contrast to the single-issue rationing problems, are not only identified by their respective claims, but also by some exogenous ex-ante conditions (initial stock of resource or net worth of agents), other than claims. The GEA rule follows an egalitarian principle from minus the vector of ex-ante conditions ($-\delta \in \mathbb{R}^N$). Thus, if we take minus the vector of ex-ante conditions as reference point, i.e. ${}_N\alpha(c) = -\delta$, then the α-egalitarian rule E^{α} is equivalent to the GEA rule (see Proposition 13 in Appendix A).

On the other hand, this chapter can also shed some light in the search of a dynamic intertemporal rationing problem (when periods are represented by issues). Indeed, in the conclusions of Chapter 2 we state that "*our model can be applied to allocate resources in other contexts, for instance, those in which the same group of agents faces a sequence of rationing problems at different periods of time. The distribution in the current period is influenced by the amount received in previous periods, which can be considered as an ex-ante condition for the current rationing problem.*" In this way, the reference point in each period (issue) would not be constant and might be determined by the previous allocations (see the interpretation of one-issue $\boldsymbol{\alpha}$-consistent rules on page 32).

In future research, it might be possible to characterize the extended $\boldsymbol{\beta}$-egalitarian family of rules, by using adapted properties of the ones that characterize the $RTAL$ family of rules (Arín et al., 2016; van den Brink et al., 2013).

[21]Let us recall that a reference system α is constant and independent of the claims if ${}_N\alpha_i(c) = p_i \in \mathbb{R}$, for all $i \in N$ and all $N \in \mathcal{N}$.

5 Appendix A

Proposition 6 *For all $N \in \mathcal{N}$ and all $(r,c) \in \mathcal{R}^N$ it holds that*

$$\{CEA(r,c)\} = \arg\min_{x \in \mathcal{D}(r,c)} \sum_{i \in N} {x_i}^2.$$

Proof. First of all, notice that $x = CEA(r,c)$ if and only if, for all $i \neq j \in N$,

$$\text{if } x_i < x_j, \text{ then } x_i = c_i. \tag{26}$$

Let x^* be the unique solution of the minimization program $\min\limits_{x \in \mathcal{D}(r,c)} \sum_{i \in N} {x_i}^2$ and suppose on the contrary that $x^* \neq CEA(r,c)$. By (26) there exist at least two agents $i,j \in N$, such that $x_i^* < x_j^*$, but $x_i^* < c_i$. Next, define $x' \in \mathbb{R}_+^N$ as follows: $x_i' = x_i^* + \epsilon$, $x_j' = x_j^* - \epsilon$ and $x_k' = x_k^*$ else, where $0 < \epsilon < \min\{c_i - x_i^*, x_j^* - x_i^*\}$. By definition of ϵ, it follows $x' \in \mathcal{D}(r,c)$. Moreover, notice that

$$\begin{aligned} \textstyle\sum_{k \in N} {x_k'}^2 &= \textstyle\sum_{k \in N \setminus \{i,j\}} {x_k^*}^2 + (x_i^* + \epsilon)^2 + (x_j^* - \epsilon)^2 \\ &= \textstyle\sum_{k \in N} {x_k^*}^2 + 2\epsilon\big(\epsilon - (x_j^* - x_i^*)\big) < \sum_{k \in N} {x_k^*}^2, \end{aligned}$$

where the inequality follows from $\epsilon < x_j^* - x_i^*$. Therefore, we reach a contradiction with the fact that $\{x^*\} = \arg\min \sum_{i \in N} x_i^2$, and thus, we conclude that $x^* = CEA(r,c)$.

□

Proposition 7 *For all $N \in \mathcal{N}$ and all $(r,c) \in \mathcal{R}^N$ it holds that*

$$\{CEL(r,c)\} = \arg\min_{x \in \mathcal{D}(r,c)} \sum_{i \in N} (x_i - c_i)^2.$$

Proof. First of all, notice that $x = CEL(r,c)$ if and only if, for all $i \neq j \in N$,

$$\text{if } x_i > 0 \text{ and } x_j > 0, \text{ then } c_i - x_i = c_j - x_j. \tag{27}$$

Let x^* be the unique solution of the minimization program $\min\limits_{x \in \mathcal{D}(r,c)} \sum_{i \in N}(x_i - c_i)^2$. Suppose, on the contrary, that $x^* \neq CEL(r,c)$. By (27) there exist at least two agents $i, j \in N$, such that $x_i^* > 0$ and $x_j^* > 0$, but (w.l.o.g.) $c_i - x_i^* > c_j - x_j^*$. Next, define $x' \in \mathbb{R}_+^N$ as follows: $x_i' = x_i^* + \epsilon$, $x_j' = x_j^* - \epsilon$ and $x_k' = x_k^*$ else, where $0 < \epsilon < \min\{c_i - x_i^*, x_j^*, c_i - x_i^* - (c_j - x_j^*)\}$. By definition of ϵ, we have $x' \in \mathcal{D}(r, c)$. The remaining of the proof follows the same guidelines of the proof of Proposition 6.

□

Definition 21 *A rationing rule F satisfies* resource monotonicity *if for all $N \in \mathcal{N}$ and all $(r, c) \in \mathcal{R}^N$ with $\sum_{i \in N} c_i \geq r' \geq r$ it holds*

$$F(r', c) \geq F(r, c).$$

Proposition 8 *For an arbitrary reference system α, the corresponding E^α rule satisfies resource monotonicity.*

Proof. Let α be a reference system, $(r, c) \in \mathcal{R}^N$ be a single-issue rationing problem and $x = E^\alpha(r, c)$. Then, take r' such that $\sum_{i \in N} c_i \geq r' \geq r$ and let $x' = E^\alpha(r', c)$. If $r' = r$, the result is straightforward. If $r' > r$, suppose on the contrary that there exists $i \in N$ such that $x_i' < x_i$. Then, by efficiency, i.e. $\sum_{k \in N} x_k' = r' > r = \sum_{k \in N} x_k$, there exists $j \in N \setminus \{i\}$ such that $x_j' > x_j$. Next, we claim that

$$x_j - a_j \geq x_i - a_i. \tag{28}$$

where $a = {}_N\alpha(c)$. Suppose on the contrary that $x_j - a_j < x_i - a_i$ and define $\check{x} \in \mathbb{R}^N$ as follows: $\check{x}_i = x_i - \epsilon_1$, $\check{x}_j = x_j + \epsilon_1$, and $\check{x}_k = x_k$ else, where $0 < \epsilon_1 < \min\{x_i, c_j - x_j, x_i - a_i - (x_j - a_j)\}$. Let us remark that, by definition of ϵ_1, $\check{x} \in \mathcal{D}(r, c)$. Notice that

$$\begin{aligned}
\sum_{k \in N} (\check{x}_k - a_k)^2 &= \sum_{k \in N \setminus \{i,j\}} (x_k - a_k)^2 + (x_i - \epsilon_1 - a_i)^2 + (x_j + \epsilon_1 - a_j)^2 \\
&= \sum_{k \in N} (x_k - a_k)^2 + 2\epsilon_1 \Big(\epsilon_1 - \big(x_i - a_i - (x_j - a_j)\big)\Big) \\
&< \sum_{k \in N} (x_k - a_k)^2,
\end{aligned}$$

where the inequality follows from $\epsilon_1 < x_i - a_i - (x_j - a_j)$. Therefore, we reach a contradiction with $x = E^{\alpha}(r, c)$, and thus, (28) holds.

Taking into account (28), we next prove that

$$x'_j - a_j > x'_i - a_i \tag{29}$$

Indeed, since $x'_j > x_j$, $x_i > x'_i$ and (28), we have that

$$x'_j - a_j \quad > x_j - a_j \geq x_i - a_i > x'_i - a_i,$$

and thus, (29) holds. Finally, let us define $\widehat{x} \in \mathbb{R}^N$ as follows: $\widehat{x}_i = x'_i + \epsilon_2$, $\widehat{x}_j = x'_j - \epsilon_2$ and $\widehat{x}_k = x'_k$ else, where $0 < \epsilon_2 < \min\{x'_j - x_j, c_i - x'_i, x'_j - a_j - (x'_i - a_i)\}$. Notice that, by (29), ϵ_2 is well-defined. Let us remark that, by definition of ϵ_2, $\widehat{x} \in \mathcal{D}(r', c)$. However, notice that

$$\begin{aligned}
\sum_{k \in N} (\widehat{x}_k - a_k)^2 \quad &= \sum_{k \in N \setminus \{i,j\}} (x'_k - a_k)^2 + (x'_i + \epsilon_2 - a_i)^2 + (x'_j - \epsilon_2 - a_j)^2 \\
&= \sum_{k \in N} (x'_k - a_k)^2 + 2\epsilon_2 \Big(\epsilon_2 - \big(x'_j - a_j - (x'_i - a_i) \big) \Big) \\
&< \sum_{k \in N} (x'_k - a_k)^2,
\end{aligned}$$

where the inequality follows from $\epsilon_2 < x'_j - a_j - (x'_i - a_i)$. Therefore, we reach a contradiction with the fact that $x' = E^{\alpha}(r', c)$. We conclude that $x'_i \geq x_i$, for all $i \in N$ and $E^{\alpha}(r', c) \geq E^{\alpha}(r, c)$.

$\square$

Definition 22 *A rationing rule F satisfies* claims monotonicity *if for all $N \in \mathcal{N}$, all $(r, c) \in \mathcal{R}^N$ and all $(r, c') \in \mathcal{R}^N$ such that $c \neq c'$ with $c_i > c'_i$ and $c_k = c'_k$, for all $k \in N \setminus \{i\}$, it holds*

$$F_i(r, c') \geq F_i(r, c).$$

Proposition 9 *For any arbitrary* constant and independent of the claims *reference system α, the corresponding E^{α} rule satisfies claims monotonicity.*

Proof. Let α be a constant and independent of the claims reference system. Then, take two claims vectors c, $c' \in \mathbb{R}^N_+$ such that $c'_i > c_i$, for some $i \in N$, $c'_k = c_k$, for all $k \in N \setminus \{i\}$ and take $r \leq \sum_{k \in N} c_k \leq \sum_{k \in N} c'_k$. Then, denote $x = E^\alpha(r, c)$ and $x' = E^\alpha(r, c')$ and suppose on the contrary that $x'_i < x_i$. Since $x' \in \mathcal{D}(r, c)$ and $x = E^\alpha(r, c)$, we have that $\sum_{k \in N}(x_k - p_k)^2 < \sum_{k \in N}(x'_k - p_k)^2$, reaching a contradiction with $x' = E^\alpha(r, c')$, since $x \in \mathcal{D}(r, c')$. Therefore, we conclude that $x'_i \geq x_i$.

□

Definition 23 *A rationing rule F is* consistent *if for all $(r, c) \in \mathcal{R}^N$ and all $T \subseteq N$, $T \neq \emptyset$, it holds*

$$F(r, c)_{|T} = F\Big(r - \sum_{i \in N \setminus T} F_i(r, c), c_{|T}\Big).$$

Definition 24 *A reference system α is consistent if, for all $T, N \in \mathcal{N}$ with $T \subseteq N$ we have ${}_N\alpha_i(z)_{|T} = {}_T\alpha_i(z_{|T})$, for all $z \in \mathbb{R}^N$ and all $i \in T$.*

Proposition 10 *For an arbitrary* consistent *reference system α, the corresponding E^α rule is also consistent.*

Proof. Let $(r, c) \in \mathcal{R}^N$ be a rationing problem and denote $x^* = E^\alpha(r, c)$, where α is an arbitrary consistent reference system with ${}_N\alpha(c) = a \in \mathbb{R}^N$. Then, by definition, we have that

$$\sum_{i \in N}(x^*_i - a_i)^2 < \sum_{i \in N}(x_i - a_i)^2, \tag{30}$$

for all $x \in \mathcal{D}(r, c)$ such that $x \neq x^*$.

Let $T \subseteq N$ be an arbitrary sub-coalition and let us suppose on the contrary that

$$x^*_{|T} \neq E^\alpha\Big(r - \sum_{i \in N \setminus T} E^\alpha_i(r, c), c_{|T}\Big) = y.$$

Now, since α is consistent, taking (30) into account and since E^α assigns a unique solution, we have that $\sum_{i \in T}(y_i - a_i)^2 < \sum_{i \in T}(x^*_i - a_i)^2$. However,

$(y; x^*_{|N\setminus T}) \in \mathcal{D}(r, c)$ and thus

$$\sum_{i\in T}(y_i - a_i)^2 + \sum_{i\in N\setminus T}(x^*_i - a_i)^2 < \sum_{i\in N}(x^*_i - a_i)^2,$$

which contradicts $x^* = E^\alpha(r, c)$. Therefore, we conclude that

$$x^*_{|T} = E^\alpha\Big(r - \sum_{i\in N\setminus T} x^*_i, c_{|T}\Big).$$

□

Definition 25 *A rationing rule F satisfies* path-independence *if for all $N \in \mathcal{N}$, all $(r, c) \in \mathcal{R}^N$ and all $r' > 0$ such that $\sum_{i\in N} c_i \geq r' \geq r$ it holds*

$$F(r, c) = F(r, F(r', c)).$$

Proposition 11 *For an arbitrary reference system α, the corresponding E^α rule satisfies path-independence.*

Proof. Let α be a reference system, $(r, c) \in \mathcal{R}^N$ be a single-issue rationing problem and $x = E^\alpha(r, c)$. Moreover, take r' such that $\sum_{i\in N} c_i \geq r' \geq r$ and write $x' = E^\alpha(r', c)$ and $\widehat{x} = E^\alpha(r, x')$. If $r' = r$, the result is straightforward. If $r' > r$, since E^α rule satisfies resource monotonicity (Proposition 8), we have that $x_i \leq x'_i \leq c_i$ and $\widehat{x}_i \leq x'_i \leq c_i$, for all $i \in N$. Hence, $\widehat{x} \in \mathcal{D}(r, c)$ which implies that, since $x = E^\alpha(r, c)$,

$$\sum_{k\in N}(x_k - a_k)^2 < \sum_{k\in N}(\widehat{x}_k - a_k)^2, \tag{31}$$

where $a = {}_N\alpha(c)$. However, we have that $x \in \mathcal{D}(r, x')$ which implies that $\sum_{k\in N}(\widehat{x}_k - a_k)^2 < \sum_{k\in N}(x_k - a_k)^2$ since $\widehat{x} = E^\alpha(r, x')$, reaching a contradiction with (31). Therefore, we conclude that E^α satisfies path-independence.

□

Definition 26 *A rationing rule F satisfies* equal treatment of equals *if, for all $N \in \mathcal{N}$ and all $(r, c) \in \mathcal{R}^N$, it holds*

$$\text{if } c_i = c_j, \text{ then } F_i(r, c) = F_j(r, c).$$

Proposition 12 *For an arbitrary $\beta \in [0,1]$, the corresponding $\mathcal{E}^\beta$ rule satisfies equal treatment of equals.*

Proof. Let $(r,c) \in \mathcal{R}^N$ be a rationing problem and denote $x = \mathcal{E}^\beta(r,c)$. Suppose on the contrary that there exist $i,j \in N$, $i \neq j$, such that $c_i = c_j$, but (w.l.o.g.) $x_i > x_j$. Then, define $x' \in \mathbb{R}^N_+$ as follows: $x'_i = x_i - \epsilon$, $x'_j = x_j + \epsilon$ and $x'_k = x_k$ else, where $0 < \epsilon < \min\{x_i - x_j, c_j - x_j\}$. By definition of ϵ, we have $x' \in \mathcal{D}(r,c)$. However, notice that

$$\begin{aligned}&\sum_{k\in N}(x'_k - \beta\cdot c_k)^2 = \sum_{k\in N\setminus\{i,j\}}(x_k-\beta\cdot c_k)^2 + (x_i - \epsilon - \beta\cdot c_i)^2\\ &+ (x_j + \epsilon - \beta\cdot c_j)^2 = \sum_{k\in N}(x_k - \beta\cdot c_k)^2 + 2\epsilon\big(\epsilon - (x_i - x_j)\big)\\ &< \sum_{k\in N}(x_k - \beta\cdot c_k)^2,\end{aligned}$$

where the inequality follows from $\epsilon < x_i - x_j$. Therefore, we reach a contradiction with the fact that $x = \mathcal{E}^\beta(r,c)$ and we conclude $x_i = x_j$.

□

Proposition 13 *Let α be a reference system and $(r,c,\delta) \in \mathcal{RC}^N$ be a* rationing problem with ex-ante conditions*. Then,*

$$\text{if } {}_N\alpha(c) = -\delta, \text{ then } GEA(r,c,\delta) = E^\alpha(r,c).$$

Proof. Let $(r,c,\delta) \in \mathcal{RC}^N$ with $N \in \mathcal{N}$. First of all, recall[22] that $z^* = GEA(r,c,\delta)$ if and only if

$$\text{for all } i,j \in N \text{ with } i \neq j, \text{ if } z^*_i + \delta_i < z^*_j + \delta_j, \text{ then either } z^*_j = 0, \text{ or } z^*_i = c_i. \tag{32}$$

Let x^* be the solution of the minimization program $\min\limits_{x\in\mathcal{D}(r,c)} \sum_{i\in N}(x_i+\delta_i)^2$ and suppose on the contrary that $x^* \neq z^* = GEA(r,c,\delta)$. By (32) there exist at least two agents $i,j \in N$, such that $x^*_i + \delta_i < x^*_j + \delta_j$, but $x^*_i < c_i$ and

[22]See the first proposition of Chapter 2.

$x^*_j > 0$. Then, define $x' \in \mathbb{R}^N_+$ as follows: $x'_i = x^*_i + \epsilon$, $x'_j = x^*_j - \epsilon$ and $x'_k = x^*_k$ else, where $0 < \epsilon < \min\{c_i - x^*_i, x^*_j, x^*_j + \delta_j - (x^*_i + \delta_i)\}$. Then, notice that

$$\begin{aligned}\sum_{k\in N}(x'_k + \delta_k)^2 &= \sum_{k\in N\setminus\{i,j\}}(x^*_k + \delta_k)^2 + (x^*_i + \epsilon + \delta_i)^2 + (x^*_j - \epsilon + \delta_j)^2 \\ &= \sum_{k\in N}(x^*_k + \delta_k)^2 + 2\epsilon\big(\epsilon - (x^*_j + \delta_j - (x^*_i + \delta_i))\big) \\ &< \sum_{k\in N}(x^*_k + \delta_k)^2,\end{aligned}$$

where the inequality follows from $\epsilon < x^*_j + \delta_j - (x^*_i + \delta_i)$. Therefore, we reach a contradiction with the fact that $\{x^*\} = \underset{x\in\mathcal{D}(r,c)}{\arg\min} \sum_{i\in N}(x_i + \delta_i)^2$, and thus, we conclude that $E^\alpha(r,c) = x^* = z^* = GEA(r,c,\delta)$.

□

6 Appendix B

Proof of Claim 1 Let us recall that we are supposing that $N = \{1,2\}$ and $\mathbf{x} = \widehat{CEA}(\mathbf{r},\mathbf{c})$. First, given two feasible payoffs $\mathbf{y}, \mathbf{y}' \in \mathcal{D}(\mathbf{r},\mathbf{c})$, let us divide the set of issues M in three subsets

$$\begin{aligned}&M_1(\mathbf{y},\mathbf{y}') = \{j \in M | y^j_1 > {y'}^j_1\},\\ &M_2(\mathbf{y},\mathbf{y}') = \{j \in M | y^j_2 > {y'}^j_2\} \text{ and}\\ &M_3(\mathbf{y},\mathbf{y}') = M \setminus M_1(\mathbf{y},\mathbf{y}') \cup M_2(\mathbf{y},\mathbf{y}').\end{aligned}$$

Let $\mathbf{z} \in \mathcal{D}(\mathbf{r},\mathbf{c})$ be a payoff vector such that $\sum_{j\in M}\Delta^j_{\mathbf{z}} \le \sum_{j\in M}\Delta^j_{\mathbf{y}}$, for all $\mathbf{y} \in \mathcal{D}(\mathbf{r},\mathbf{c})$ with $\sum_{j\in M} y^j_i = \sum_{j\in M} z^j_i = \sum_{j\in M} x^j_i$, for all $i \in N$, but suppose on the contrary that $\sum_{j\in M}\Delta^j_{\mathbf{z}} < \sum_{j\in M}\Delta^j_{\mathbf{x}}$, which implies that $\mathbf{x} \neq \mathbf{z}$. Since $\mathbf{x} \neq \mathbf{z}$ and by efficiency ($x^j_1 + x^j_2 = r^j = z^j_1 + z^j_2$, for all $j \in M$), we have that

$$M_1(\mathbf{x},\mathbf{z}) \neq \emptyset \text{ and } M_2(\mathbf{x},\mathbf{z}) \neq \emptyset. \tag{33}$$

Then, we consider two cases:

Case 1: There exist $j' \in M_1(\mathbf{x},\mathbf{z})$ and $j'' \in M_2(\mathbf{x},\mathbf{z})$ such that $x^{j'}_1 > x^{j'}_2$ and $x^{j''}_1 < x^{j''}_2$. Then, define $\widehat{\mathbf{x}} \in \mathbb{R}^{N\times M}_+$ as follows: $\widehat{x}^{j'}_1 = x^{j'}_1 - \epsilon$, $\widehat{x}^{j'}_2 = x^{j'}_2 + \epsilon$,

$\widehat{x}_1^{j''} = x_1^{j''} + \epsilon$, $\widehat{x}_2^{j''} = x_2^{j''} - \epsilon$ and $\widehat{x}_i^j = x_i^j$ else, where

$$0 < \epsilon < \min\left\{ x_1^{j'} - z_1^{j'}, z_2^{j'} - x_2^{j'}, z_1^{j''} - x_1^{j''}, x_2^{j''} - z_2^{j''}, \frac{x_1^{j'} + x_2^{j''} - \left(x_1^{j''} + x_2^{j'}\right)}{2} \right\}$$

Let us remark that ϵ is well-defined: first, since $j' \in M_1(\mathbf{x}, \mathbf{z})$, we have that $x_1^{j'} > z_1^{j'}$ and thus, by efficiency, $x_2^{j'} < z_2^{j'}$; second, since $j'' \in M_2(\mathbf{x}, \mathbf{z})$, we have that $x_2^{j''} > z_2^{j''}$ and thus, by efficiency $x_1^{j''} < z_1^{j''}$; finally, since $x_1^{j'} > x_2^{j'}$ and $x_2^{j''} > x_1^{j''}$, we have that $x_1^{j'} + x_2^{j''} > x_1^{j''} + x_2^{j'}$. Notice that, by definition of ϵ, $\widehat{\mathbf{x}} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$ with $\sum_{j \in M} \widehat{x}_i^j = \sum_{j \in M} x_i^j$, for all $i \in N$. Now, observe that

$$\begin{aligned} \sum_{i \in N} \sum_{j \in M} (\widehat{x}_i^j)^2 &= \sum_{i \in N} \sum_{j \in M} (x_i^j)^2 + 2 \cdot \epsilon \left(2 \cdot \epsilon - \left(x_1^{j'} + x_2^{j''} - \left(x_1^{j''} + x_2^{j'}\right)\right)\right) \\ &< \sum_{i \in N} \sum_{j \in M} (x_i^j)^2, \end{aligned}$$

where the inequality follows from $\epsilon < \frac{x_1^{j'} + x_2^{j''} - \left(x_1^{j''} + x_2^{j'}\right)}{2}$. However, this contradicts the fact that $\mathbf{x} = \widehat{CEA}(\mathbf{r}, \mathbf{c})$.

Case 2: For all $j' \in M_1(\mathbf{x}, \mathbf{z})$ and all $j'' \in M_2(\mathbf{x}, \mathbf{z})$, either $x_1^{j'} \leq x_2^{j'}$, or $x_1^{j''} \geq x_2^{j''}$. Then, we define recursively a sequence of feasible payoff vectors

$${}^0\mathbf{x}, {}^1\mathbf{x}, {}^2\mathbf{x}, \ldots, {}^k\mathbf{x}, \ldots, {}^{\mathcal{K}}\mathbf{x},$$

following the next procedure.

For $k = 0$, set ${}^0\mathbf{x} = \mathbf{x}$.

For $k = 1, 2, \ldots, \mathcal{K}$.

If ${}^{k-1}\mathbf{x} = \mathbf{z}$. Stop.

If ${}^{k-1}\mathbf{x} \neq \mathbf{z}$, take $j' \in M_1({}^{k-1}\mathbf{x}, \mathbf{z})$ such that $|{}^{k-1}x_1^{j'} - z_1^{j'}| \leq |{}^{k-1}x_1^j - z_1^j|$, for all $j \in M_1({}^{k-1}\mathbf{x}, \mathbf{z})$ and $j'' \in M_2({}^{k-1}\mathbf{x}, \mathbf{z})$ such that[23] $|{}^{k-1}x_1^{j''} - z_1^{j''}| \leq |{}^{k-1}x_1^j - z_1^j|$, for all $j \in M_2({}^{k-1}\mathbf{x}, \mathbf{z})$. Notice that,

[23] Notice that, by efficiency, $|{}^{k-1}x_1^j - z_1^j| = |{}^{k-1}x_2^j - z_2^j|$, for all $j \in M$.

$M_1({}^{k-1}\mathbf{x}, \mathbf{z}) \neq \emptyset$ and $M_2({}^{k-1}\mathbf{x}, \mathbf{z}) \neq \emptyset$, since ${}^{k-1}\mathbf{x} \neq \mathbf{z}$ and ${}^{k-1}\mathbf{x} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$, and thus, ${}^{k-1}\mathbf{x}$ is efficient (${}^{k-1}x_1^j + {}^{k-1}x_2^j = r^j = z_1^j + z_2^j$, for all $j \in M$). Then,

- in case $|{}^{k-1}x_1^{j'} - z_1^{j'}| \leq |{}^{k-1}x_1^{j''} - z_1^{j''}|$, define ${}^k\mathbf{x} \in \mathbb{R}^{N\times M}$ as follows:[23]

$$\begin{aligned}
&{}^k x_1^{j'} = {}^{k-1}x_1^{j'} - |{}^{k-1}x_1^{j'} - z_1^{j'}| = z_1^{j'}, \\
&{}^k x_2^{j'} = {}^{k-1}x_2^{j'} + |{}^{k-1}x_1^{j'} - z_1^{j'}| = z_2^{j'}, \\
&{}^k x_1^{j''} = {}^{k-1}x_1^{j''} + |{}^{k-1}x_1^{j'} - z_1^{j'}|, \\
&{}^k x_2^{j''} = {}^{k-1}x_2^{j''} - |{}^{k-1}x_1^{j'} - z_1^{j'}| \text{ and} \\
&{}^k x_i^j = {}^{k-1}x_i^j, \text{ else;}
\end{aligned} \tag{34}$$

- in case $|{}^{k-1}x_1^{j'} - z_1^{j'}| > |{}^{k-1}x_1^{j''} - z_1^{j''}|$, define ${}^k\mathbf{x} \in \mathbb{R}^{N\times M}$ as follows:

$$\begin{aligned}
&{}^k x_1^{j'} = {}^{k-1}x_1^{j'} - |{}^{k-1}x_1^{j''} - z_1^{j''}|, \\
&{}^k x_2^{j'} = {}^{k-1}x_2^{j'} + |{}^{k-1}x_1^{j''} - z_1^{j''}|, \\
&{}^k x_1^{j''} = {}^{k-1}x_1^{j''} + |{}^{k-1}x_1^{j''} - z_1^{j''}| = z_1^{j''}, \\
&{}^k x_2^{j''} = {}^{k-1}x_2^{j''} - |{}^{k-1}x_1^{j''} - z_1^{j''}| = z_2^{j''} \\
&\text{and } {}^k x_i^j = {}^{k-1}x_i^j, \text{ else.}
\end{aligned} \tag{35}$$

Notice that, ${}^k\mathbf{x} \in \mathcal{D}(\mathbf{r}, \mathbf{c})$ and $\sum_{j\in M} {}^k x_i^j = \sum_{j \in M} x_i^j$, for all $i \in N$, and go to the next step.

Observe that, the procedure is well-defined, since, either, $M_1({}^{k-1}\mathbf{x}, \mathbf{z}) \supsetneq M_1({}^k\mathbf{x}, \mathbf{z})$ if case (34) holds, or $M_2({}^{k-1}\mathbf{x}, \mathbf{z}) \supsetneq M_2({}^k\mathbf{x}, \mathbf{z})$ if case (35) holds. Note that, the procedure stops at stage $\mathcal{K} \leq m-1$ when ${}^{\mathcal{K}}\mathbf{x} = \mathbf{z}$, and thus, $M_1({}^{\mathcal{K}}\mathbf{x}, \mathbf{z}) = \emptyset$ and $M_2({}^{\mathcal{K}}\mathbf{x}, \mathbf{z}) = \emptyset$.

Now, it can be proved that

Subclaim 3.1.1 $\sum_{j\in M} |{}^{k-1}x_1^j - {}^{k-1}x_2^j| \leq \sum_{j \in M} |{}^k x_1^j - {}^k x_2^j|$ *for all* $k \in \{1, \ldots, \mathcal{K}\}$.

The proof of this result can be found just below. Finally, since ${}^{\mathcal{K}}\mathbf{x} = \mathbf{z}$ and ${}^{0}\mathbf{x} = \mathbf{x}$, and by Subclaim 3.1.1, we conclude that $\sum_{j\in M} \Delta^j_{\mathbf{x}} \leq \sum_{j\in M} \Delta^j_{\mathbf{z}}$ reaching a contradiction with the initial hypothesis, and we are done.

$\square$

Proof of Subclaim 3.1.1 First of all, let us recall that we are under hypothesis of Case 2 of Claim 1. That is,

$$\begin{gathered}\text{for all } j' \in M_1(\mathbf{x},\mathbf{z}) \text{ and all } j'' \in M_2(\mathbf{x},\mathbf{z}),\\ \text{either } x_1^{j'} \leq x_2^{j'}, \text{ or } x_1^{j''} \geq x_2^{j''}.\end{gathered} \tag{36}$$

Let $k \in \{1, 2, \ldots, \mathcal{K}\}$, and let $j' \in M_1({}^{k-1}\mathbf{x}, \mathbf{z})$ and $j'' \in M_2({}^{k-1}\mathbf{x}, \mathbf{z})$ be the agents selected in the step k of the recursive definition of ${}^{k}\mathbf{x}$. Then, we consider two cases:

Case a: $|{}^{k-1}x_1^{j'} - z_1^{j'}| \leq |{}^{k-1}x_1^{j''} - z_1^{j''}|$ and thus ${}^{k}\mathbf{x}$ is defined as in (34). First, denote $d = |{}^{k-1}x_1^{j'} - z_1^{j'}|$ and consider three subcases:

Subcase a.1: ${}^{k-1}x_1^{j'} \leq {}^{k-1}x_2^{j'}$ and ${}^{k-1}x_1^{j''} < {}^{k-1}x_2^{j''}$.

Then, by (34), we have that

$$\begin{aligned}&\sum_{j\in M} |{}^{k-1}x_1^{j} - {}^{k-1}x_2^{j}| = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^{j} - {}^{k}x_2^{j}| + ({}^{k-1}x_2^{j'} - {}^{k-1}x_1^{j'})\\ &+ ({}^{k-1}x_2^{j''} - {}^{k-1}x_1^{j''}) = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^{j} - {}^{k}x_2^{j}| + \Big({}^{k-1}x_2^{j'} + d - ({}^{k-1}x_1^{j'} - d)\Big)\\ &+ \Big({}^{k-1}x_2^{j''} - d - ({}^{k-1}x_1^{j''} + d)\Big) = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^{j} - {}^{k}x_2^{j}| + ({}^{k}x_2^{j'} - {}^{k}x_1^{j'})\\ &+ ({}^{k}x_2^{j''} - {}^{k}x_1^{j''}) \leq \sum_{j\in M} |{}^{k}x_1^{j} - {}^{k}x_2^{j}|\end{aligned}$$

and we are done.

Subcase a.2: ${}^{k-1}x_1^{j'} \leq {}^{k-1}x_2^{j'}$ and ${}^{k-1}x_1^{j''} \geq {}^{k-1}x_2^{j''}$.

Then, by (34) and since $d > 0$, we obtain that

$$
\begin{aligned}
&\sum_{j\in M} |{}^{k-1}x_1^j - {}^{k-1}x_2^j| = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^j - {}^{k}x_2^j| + ({}^{k-1}x_2^{j'} - {}^{k-1}x_1^{j'}) \\
&+ ({}^{k-1}x_1^{j''} - {}^{k-1}x_2^{j''}) < \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^j - {}^{k}x_2^j| + \Big({}^{k-1}x_2^{j'} + d - ({}^{k-1}x_1^{j'} - d)\Big) \\
&+ \Big({}^{k-1}x_1^{j''} + d - ({}^{k-1}x_2^{j''} - d)\Big) = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^j - {}^{k}x_2^j| + ({}^{k}x_2^{j'} - {}^{k}x_1^{j'}) \\
&+ ({}^{k}x_1^{j''} - {}^{k}x_2^{j''}) \leq \sum_{j\in M} |{}^{k}x_1^j - {}^{k}x_2^j|
\end{aligned}
$$

and we are done.

Subcase a.3: ${}^{k-1}x_1^{j'} > {}^{k-1}x_2^{j'}$ and ${}^{k-1}x_1^{j''} \geq {}^{k-1}x_2^{j''}$. Then, by (34), it holds that

$$
\begin{aligned}
&\sum_{j\in M} |{}^{k-1}x_1^j - {}^{k-1}x_2^j| = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^j - {}^{k}x_2^j| + ({}^{k-1}x_1^{j'} - {}^{k-1}x_2^{j'}) \\
&+ ({}^{k-1}x_1^{j''} - {}^{k-1}x_2^{j''}) = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^j - {}^{k}x_2^j| + \Big({}^{k-1}x_1^{j'} - d - ({}^{k-1}x_2^{j'} + d)\Big) \\
&+ \Big({}^{k-1}x_1^{j''} + d - ({}^{k-1}x_2^{j''} - d)\Big) = \sum_{j\in M\setminus\{j',j''\}} |{}^{k}x_1^j - {}^{k}x_2^j| + ({}^{k}x_1^{j'} - {}^{k}x_2^{j'}) \\
&+ ({}^{k}x_1^{j''} - {}^{k}x_2^{j''}) \leq \sum_{j\in M} |{}^{k}x_1^j - {}^{k}x_2^j|.
\end{aligned}
$$

and we are done.

Notice that, the remaining combination,

$$
{}^{k-1}x_1^{j'} > {}^{k-1}x_2^{j'} \text{ and } {}^{k-1}x_1^{j''} < {}^{k-1}x_2^{j''}, \tag{37}
$$

is not possible. To check it, let us remark that if $k = 1$ then ${}^{k}\mathbf{x} = {}^{0}\mathbf{x} = \mathbf{x}$ which contradicts (36). On the other hand, if $k \geq 2$, then, by (34), we obtain that

$$
\begin{aligned}
z_1^{j'} = {}^{k}x_1^{j'} < {}^{k-1}x_1^{j'} \leq {}^{k-2}x_1^{j'} \leq \ldots \leq {}^{0}x_1^{j'} = x_1^{j'} \text{ and} \\
z_2^{j'} = {}^{k}x_2^{j'} > {}^{k-1}x_2^{j'} \geq {}^{k-2}x_2^{j'} \geq \ldots \geq {}^{0}x_2^{j'} = x_2^{j'},
\end{aligned} \tag{38}
$$

and

$$\begin{aligned} z_1^{j''} \geq {}^k x_1^{j''} > {}^{k-1}x_1^{j''} \geq {}^{k-2}x_1^{j''} \geq \ldots \geq {}^0 x_1^{j''} = x_1^{j''} \text{ and} \\ z_2^{j''} \leq {}^k x_2^{j''} < {}^{k-1}x_2^{j''} \leq {}^{k-2}x_2^{j''} \leq \ldots \leq {}^0 x_2^{j''} = x_2^{j''}. \end{aligned} \tag{39}$$

Therefore, by (37) and (38) we obtain that $x_1^{j'} > x_2^{j'}$ and, by (37) and (39) we have that $x_1^{j''} < x_2^{j''}$. Since $j' \in M_1(\mathbf{x}, \mathbf{z})$ and $j'' \in M_2(\mathbf{x}, \mathbf{z})$ we reach a contradiction with (36). We conclude that (37) is not possible.

Case b: $|{}^{k-1}x_1^{j'} - z_1^{j'}| > |{}^{k-1}x_1^{j''} - z_1^{j''}|$ and thus ${}^k\mathbf{x}$ is defined as in (35). Analogously, this case is solved in the same way than Case a, but taking $d = |{}^{k-1}x_1^{j''} - z_1^{j''}|$. And we are done.

$\square$

Proof of Claim 2 Suppose on the contrary that there exists $j' \in \{1, 2, \ldots, k-1\}$ such that $x_1^{j'} < z_1^{j'}$, but $x_1^{j'} \leq x_2^{j'}$. Hence, by (18), it holds that $x_1^k + x_2^{j'} > x_1^{j'} + x_2^k$ and, by (19), we can define the payoff vector $\widehat{\mathbf{x}}$ as in (14), but replacing 1 by k, and we reach the same contradiction. Therefore, we conclude that $x_1^{j'} > x_2^{j'}$ and we are done.

$\square$

Proof of Claim 3 We start the proof by analysing two subclaims:

Subclaim 3.3.1 $x_1^j \geq z_1^j$, for all $j \in \{k, k+1, \ldots, m\}$.

Proof. Suppose on the contrary that there exists $j' \in \{k, k+1, \ldots, m\}$ such that $x_1^{j'} < z_1^{j'}$. In case $x_1^{j'} \leq x_2^{j'}$, since $x_1^k > x_2^k$ (see (18)), it holds that $x_1^k + x_2^{j'} > x_1^{j'} + x_2^k$ and, by (19), we can define the payoff vector $\widehat{\mathbf{x}}$ as in (14), but replacing 1 by k, and we reach the same contradiction. In the remaining case $x_1^{j'} > x_2^{j'}$, since $x_1^{j'} < z_1^{j'}$ and by efficiency, we obtain that $x_2^{j'} > z_2^{j'}$. Hence, we have that $z_1^{j'} > x_1^{j'} > x_2^{j'} > z_2^{j'}$ and thus, $\Delta_{\mathbf{x}}^{j'} < \Delta_{\mathbf{z}}^{j'}$. Moreover, since $j' > k$ and since we are supposing that $\Delta_{\mathbf{x}}$ is ordered in a non-increasing way, we have that $\Delta_{\mathbf{x}}^k \geq \Delta_{\mathbf{x}}^{j'}$. Furthermore, taking into account (10), since $\Delta_{\mathbf{x}}^{j'} < \Delta_{\mathbf{z}}^{j'}$ and $\Delta_{\mathbf{x}}^k > \Delta_{\mathbf{z}}^k$ (see (17)), we can deduce that this inequality is strict, i.e. $\Delta_{\mathbf{x}}^k > \Delta_{\mathbf{x}}^{j'}$. Finally, since we are supposing that $x_1^{j'} > x_2^{j'}$ and $x_1^k > x_2^k$ (see (18)), we obtain that $x_1^k + x_2^{j'} > x_1^{j'} + x_2^k$ and we can define the payoff vector $\widehat{\mathbf{x}}$ as in (14), but replacing 1 by k, and we reach the same contradiction. Thus, the proof of the subclaim is done.

$\Diamond$

Subclaim 3.3.2 There exists $j' \in \{k+1, k+2, \ldots, m\}$ such that $d^{j'} > 0$.

Proof. Since we are supposing that $\sum_{j=1}^k \Delta_{\mathbf{x}}^j > \sum_{j=1}^k \Delta_{\mathbf{z}}^j$ (see (16)) and by Claim 1, there exists $j' \in \{k+1, k+2, \ldots, m\}$ such that $\Delta_{\mathbf{x}}^{j'} < \Delta_{\mathbf{z}}^{j'}$. This implies that $x^{j'} \neq z^{j'}$ which means that $d^{j'} > 0$.

$\Diamond$

Next, taking into account that $\mathbf{x} \neq \mathbf{z}$, let us define

$$S_1 = \{j \in M | x_1^j < z_1^j\} \text{ and}$$
$$S_2 = \{j \in M | x_1^j \geq z_1^j\},$$

as a partition of M.

Since $\sum_{j \in M} x_1^j = \sum_{j \in M} z_1^j$, it follows that

$$\sum_{j \in S_1} d^j = \sum_{j \in S_2} d^j. \tag{40}$$

Next, we show that $S_2 = M \setminus \widehat{M}_3$ and thus, $S_1 = \widehat{M}_3$. Claim 2 implies that $x_1^j \geq z_1^j$, for all $j \in \widehat{M}_4$. Hence, by Subclaim 3.3.1, and by the definitions of $\widehat{M_1}$ and $\widehat{M_2}$, it holds that $x_1^j \geq z_1^j$, for all $j \in M \setminus \widehat{M_3}$ which means that $S_2 = M \setminus \widehat{M_3}$. Hence, by (40),

$$\sum_{j \in \widehat{M_3}} d^j = \sum_{j \in M \setminus \widehat{M_3}} d^j.$$

Finally, we conclude that

$$\begin{aligned}\sum_{j \in \widehat{M_2} \cup \widehat{M_3}} d^j &\geq \sum_{j \in \widehat{M_3}} d^j = \sum_{j \in \widehat{M_1} \cup \widehat{M_2} \cup \widehat{M_4}} d^j + \sum_{j \in M \setminus \{1,2,\ldots,k\}} d^j \\ &> \sum_{j \in \widehat{M_1} \cup \widehat{M_2} \cup \widehat{M_4}} d^j \geq \sum_{j \in \widehat{M_1}} d^j,\end{aligned}$$

where the strict inequality follows from Subclaim 3.3.2 and thus, the proof of the claim is done.

$\square$

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