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An alternative proof of the characterization of core stability for the assignment game

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Abstract: Solymosi and Raghavan (2001) characterize the stability of the core of the assignment game by means of a property of the valuation matrix. They show that the core of an assignment game is a von Neumann-Morgenstern stable set if and only if its valuation matrix has a dominant diagonal. Their proof makes use of some graph-theoretical tools, while the present proof relies on the notion of buyer-seller exact representative in Núñez and Rafels (2002).

JEL Codes: C71, C78.

Keywords: Assignment game, core, stability, von Neumann-Morgenstern stable set.

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1 Introduction

An assignment game is a model for a two-sided market introduced by Shapley and Shubik (1972). There are buyers, sellers, and a valuation matrix that collects the joint profit of each buyer-seller pair. Once a matching between buyers and sellers that maximizes the total profit in the market is determined, the coalitional game theory studies how to allocate this profit among the agents.

The best known solution concept for coalitional games is the core, (Gillies, 1959), that consists of those allocations that are undominated (according to von Neumann-Morgenstern notion of domination). The first notion of solution that appears in the literature is not the core but the stable sets. Roughly speaking, a stable set is a set of imputations (individually rational allocations) such that an imputation in the set does not dominate another imputation in the set and at the same time every imputation outside the set is dominated by some imputation inside. Lucas (1968) shows that a game may have no stable set. On the other hand, when there exists, the stable set may not be unique. The core of a coalitional game always satisfies the first condition (internal stability) but may fail to satisfy the second one (external stability).

When the core is not externally stable, the argument to dismiss an imputation because it is outside the core is rather weak, since the imputation that dominates it might be also outside the core and hence no better than the first one. This does not happen when the core is externally stable and hence a stable set. Moreover, when the core is a stable set, it is the unique one.

Solymosi and Raghavan (2001) prove that the core of an assignment game with as many buyers as sellers is stable (it is a stable set) if and only if the valuation matrix has a dominant diagonal, that is, the matrix entries associated with an optimal matching are row and column maxima. To prove that, the authors make use of some graph-theoretical arguments.

In some personal notes of Shapley, probably dated around the time of publication of Shapley and Shubik (1972), there appears a conjecture on the existence of a stable set for the assignment game formed by the union of the cores of some specific subgames. The proof in these notes is not complete and the above conjecture was finally proved in Núñez and Rafels (2013).

Now, based on Núñez and Rafels (2002), where a lower bound for the core payoff of a buyer-seller pair is provided, we are able to offer a proof of the characterization of core stability for assignment games that is alternative to the one provided in Solymosi and Raghavan (2001).

2 Preliminaries on the assignment game

Let M and M' be two disjoint finite sets. An *assignment market* (M, M', A) consists of two different sectors, let us say buyers and sellers, denoted by M and M' respectively, and a non-negative matrix $A = (a_{ij})_{\substack{i\in M\\ j\in M'}}$ that represents the potential joint profit obtained by each mixed-pair $(i, j) \in M \times M'$. As in Solymosi and Raghavan (2001) and Núñez and Rafels (2002), we will assume that the assignment market is *square*, that is $|M| = |M'|$.

A *matching* μ between M and M' is a subset of the cartesian product, $M \times M'$, such that each agent belongs to at most one pair. A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for market (M, M', A) if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for all other $\mu' \in \mathcal{M}(M, M')$. We denote by $\mathcal{M}_A(M, M')$ the set of all optimal matchings for market (M, M', A). The corresponding *assignment game*, $(M \cup M', w_A)$, has player set $M \cup M'$ and characteristic function $w_A(S \cup T) = \max_{\mu\in\mathcal{M}(S,T)} \sum_{(i,j)\in\mu} a_{ij}$ for all $S \subseteq M$ and $T \subseteq M'$.

Without loss of generality, assume the main diagonal corresponds to an optimal matching. We use "j" to denote both the j^{th} buyer and the j^{th} seller, since the distinction will be clear from the context.

Given an assignment game $(M \cup M', w_A)$, an allocation is a payoff vector $(u; v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$, where u_l denotes the payoff to buyer $l \in M$ and v_l denotes the payoff to seller $l \in M'$. An *imputation* is a non-negative payoff vector that is efficient, $\sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(M \cup M')$. We denote the set of imputations of an assignment game $(M \cup M', w_A)$ by $I(w_A)$.

Given an optimal matching $\mu \in \mathcal{M}_A(M, M')$, we define the *$\mu$-principal section* of $(M \cup M', w_A)$ as the set of payoff vectors such that $u_i + v_j = a_{ij}$ for all $(i, j) \in \mu$, and the payoff to agents unassigned by μ is zero. We denote it by $B^\mu(w_A)$. In the μ-principal section the only side payments that take place are those among agents matched together by μ. Assume, without loss of generality, that $\mu = \{(i, i) \mid i \in M\}$ is an optimal matching. Notice that the allocation $(a; 0)$, that is $u_i = a_{ii}$ for all $i \in M$ and $v_j = 0$ for all $j \in M'$ always belongs to the μ-principal section. The same happens with the allocation $(0; a)$. We will refer to these two points as the *sector-optimal allocations*.

The *core* of an assignment game, $C(w_A)$, is the set of imputations such that no coalition can improve upon. An imputation, $(u; v) \in I(w_A)$, belongs to the core if for all $(i, j) \in M \times M'$ it holds $u_i + v_j \geq a_{ij}$. Shapley and Shubik (1972) show that an assignment game $(M \cup M', w_A)$ has always a non-empty core.

A binary relation, known as *domination*, is defined on the set of imputations. Given two imputations $(u; v)$ and $(u'; v')$, we say that $(u; v)$ *dominates* $(u'; v')$ if and only if there exists a pair $(i, j) \in M \times M'$ such that $u_i > u'_i$, $v_j > v'_j$ and $u_i + v_j \leq a_{ij}$. We then write $(u; v)\ \text{dom}\,^A_{\{i,j\}}(u'; v')$. We write $(u; v)\ \text{dom}^A(u'; v')$ to denote that $(u; v)$ dominates $(u'; v')$ by means of some pair (i, j).[1]

The core of an assignment game can be defined also by means of this dominance relation. It coincides with the set of undominated imputations. Another solution concept defined by means of domination is the von Neumann-Morgenstern stable set (von Neumann and Morgenstern, 1944).

A subset V of the set of imputations $I(w_A)$ is a von Neumann-Morgenstern stable set if it satisfies *internal stability*, that is, for all $(u; v), (u'; v') \in V$, $(u; v)\ \text{dom}^A(u'; v')$ does not hold; and *external stability*, that is, for all $(u'; v') \in I(w_A) \setminus V$, there exists $(u; v) \in V$ such that $(u; v)\ \text{dom}^A(u'; v')$. Notice that the core always satisfies internal stability whilst external stability may fail.

Solymosi and Raghavan (2001) introduce the *dominant diagonal* property for valua-

[1] For assignment games this dominance relation that only makes use of mixed-pair allocations is equivalent to the usual dominance relation of von Neumann and Morgenstern (1944).

tion matrices. A square valuation matrix A has a dominant diagonal if the profit every agent attains with her optimally matched partner is the most she could achieve with any other partner. That is to say, under the assumption that an optimal matching is on the main diagonal, all diagonal elements are row and column maximum: $a_{ii} \geq \max\{a_{ij}, a_{ji}\}$ for all $(i, j) \in M \times M'$.

It is straightforward to see that a valuation matrix A has a dominant diagonal if and only if the sector optimal allocations $(a; 0)$ and $(0; a)$ belong to the core. Then, it is proved in Solymosi and Raghavan (2001) that the core of a square assignment game $(M \cup M', w_A)$ is a von Neumann-Morgenstern stable set if and only if the valuation matrix A has a dominant diagonal. Their proof is based on some graph-theoretical arguments while we base ours on the properties of the buyer-seller exact representative of an assignment game proposed in Núñez and Rafels (2002). Given any assignment game $(M \cup M', w_A)$, there exists a unique valuation matrix A^r such that $C(w_A) = C(w_{A^r})$ and A^r is the maximum with this property. That is, if any enty in A^r is raised, the resulting market has a different core. The matrix A^r is the buyer-seller exact representative of A, since for all $(i, j) \in M \times M'$ there exists $(u, v) \in C(w_{A^r})$ such that $u_i + v_j = a^r_{ij}$. Notice that for each $(i, j) \in M \times M'$, a^r_{ij} is the lower bound for the joint payoff of agents $i \in M$ and $j \in M'$ in the core.

In Núñez and Rafels (2002), it is provided a formula to obtain the entries a^r_{ij} for all $(i, j) \in M \times M'$ and any given assignment game $(M \cup M', w_A)$. This result will be used in the proof of our Theorem 2.

3 Core stability

In this section we provide an alternative proof of the characterization of core stability for the two-sided assignment game.

To this end, we first adapt to the core a lemma that Shapley provides, without a proof, in his notes for the stable sets of the assignment game. Shapley's lemma states that for any point in a stable set V of an assignment game, there is a monotonic curve passing through this point and connecting the two sector optimal-allocations. Now, under the assumption that the valuation matrix has a dominant diagonal, we prove something similar for the core: for each core allocation there is a monotonic curve through it that is included in the core and connects the two sector-optimal allocations $(a; 0)$ and $(0; a)$. The payoff to any agent in this curve, for a given value of the parameter τ, is computed as the median of three terms.

Lemma 1. *Let $(M \cup M', w_A)$ be a square two-sided assignment game such that its valuation matrix A has a dominant diagonal. Given any $(u; v) \in C(w_A)$, for all $\tau \in \mathbb{R}$ all vectors of the form $(u(\tau); v(\tau))$ where*

$$\begin{aligned} u_i(\tau) &= med\{0, u_i - \tau, a_{ii}\} && \text{for all } i \in M, \\ v_i(\tau) &= med\{0, v_i + \tau, a_{ii}\} && \text{for all } i \in M', \end{aligned} \tag{1}$$

belong to the core $C(w_A)$ of the game.

Proof. Let us assume without loss of generality that $\mu = \{(i, i) \mid i \in M\}$ is an optimal matching. Note first that for $\tau = \max_{i \in M} a_{ii}$, $(u(\tau); v(\tau)) = (0; a)$ and for $\tau = -\max_{i \in M} a_{ii}$,

$(u(\tau); v(\tau)) = (a; 0)$. Now, take $i \in M$ and consider different cases to check that $(u(\tau); v(\tau))$ satisfies the core equality constraints. Notice that since $(u; v) \in C(w_A)$, we have $u_i + v_i = a_{ii}$ for all $i \in M$ and hence

$$v_i + \tau = a_{ii} - u_i + \tau = a_{ii} - (u_i - \tau). \tag{2}$$

We will show that $u_i(\tau) + v_i(\tau) = a_{ii}$ for all $i \in M$.

1. $0 \le u_i - \tau \le a_{ii}$.
   Hence, $0 \le v_i + \tau \le a_{ii}$. Then, $u_i(\tau) + v_i(\tau) = u_i - \tau + v_i + \tau = u_i + v_i = a_{ii}$.

2. $u_i - \tau \le 0$, implies $u_i(\tau) = 0$. By (2) we get $v_i + \tau \ge a_{ii}$. Then, $v_i(\tau) = a_{ii}$. Hence, $u_i(\tau) + v_i(\tau) = a_{ii}$.

3. $u_i - \tau \ge a_{ii}$, implies $u_i(\tau) = a_{ii}$. By (2) we get $v_i + \tau \le 0$. Then, $v_i(\tau) = 0$. Hence, $u_i(\tau) + v_i(\tau) = a_{ii}$.

Take now $i \ne j$ and consider different cases to check that $(u(\tau); v(\tau))$ satisfies the core inequality constraints:

1. $u_i(\tau) = a_{ii}$, implies $u_i(\tau) + v_j(\tau) \ge u_i(\tau) = a_{ii} \ge a_{ij}$ where the last inequality follows from the dominant diagonal assumption. The same follows when $v_j(\tau) = a_{jj}$.

2. $u_i(\tau) = 0$, implies $u_i - \tau \le 0$. Thus, $\tau \ge u_i \ge 0$ and we get $v_j + \tau \ge 0$. Then, $v_j(\tau)$ is either a_{jj} or $v_j + \tau$.

   (a) If $v_j(\tau) = a_{jj}$, then $u_i(\tau) + v_j(\tau) = a_{jj} \ge a_{ij}$ where the last inequality follows from the dominant diagonal assumption.

   (b) If $v_j(\tau) = v_j + \tau$, then $u_i(\tau) + v_j(\tau) = v_j + \tau \ge u_i + v_j \ge a_{ij}$.

3. The proof for the case $v_j(\tau) = 0$ is analogous.

4. $u_i(\tau) = u_i - \tau$ and $v_j(\tau) = v_j + \tau$, implies $u_i(\tau) + v_j(\tau) = u_i - \tau + v_j + \tau = u_i + v_j \ge a_{ij}$.

We have shown that for all cases the core constraints are satisfied, which concludes the proof. □

Next, we show that the core of a two-sided square assignment game is a von Neumann-Morgenstern stable set if and only if its valuation matrix has a dominant diagonal.

Theorem 2. *Let $(M \cup M', w_A)$ be a two-sided square assignment game. Then the following statements are equivalent:*

i A has a dominant diagonal.

ii $C(w_A)$ is a von Neumann-Morgenstern stable set.

Proof. Assume μ is an optimal matching on the main diagonal. Recall that the core of a game is always internally stable. The fact that every allocation outside the principal section is dominated by some core allocation is proved in Shapley's notes and also in Solymosi and Raghavan (2001), but we reproduce the proof for the sake of comprehensiveness. Assume $(x;y) \in I(w_A) \setminus B^{\mu}(w_A)$. Then, $x_i + y_i < a_{ii}$ for some $i \in M$. Thus, $x_i < a_{ii} - y_i$, which implies that there exists $0 \leq x_i < \lambda < a_{ii} - y_i \leq a_{ii}$. By Lemma 1, there exists $(u;v) \in C(w_A)$ with $u_i = \lambda$. Then, $u_i > x_i$ and $u_i < a_{ii} - y_i$ which implies $y_i < a_{ii} - u_i = v_i$. Moreover, $x_i + y_i < a_{ii} = u_i + v_i$. Hence $(u;v)\ \mathrm{dom}^A_{\{i,i\}}(x;y)$.

Now take $(x;y) \in B^{\mu}(w_A) \setminus C(w_A)$. We want to show that it is also dominated by some core allocation. We first need to prove the following claim.

Claim. Given $(x;y) \in B^{\mu}(w_A) \setminus C(w_A)$, there exists a pair $(i,j) \in M \times M'$ and a core allocation $(\bar{u};\bar{v}) \in C(w_A)$ such that $x_i + y_j < a_{ij} = \bar{u}_i + \bar{v}_j$.

Proof. We know that for any two-sided assignment game $(M \cup M', w_A)$ there exists another assignment game $(M \cup M', w_{A^r})$ with the same core (Núñez and Rafels, 2002). Hence, if $(x;y) \notin C(w_A)$ then $(x;y) \notin C(w_{A^r})$. Then, there exists a core allocation $(\bar{u};\bar{v})$ such that

$$x_i + y_j < a^r_{ij} = \bar{u}_i + \bar{v}_j. \tag{3}$$

If $a^r_{ij} = a_{ij}$, the claim is proved. Otherwise, by the definition of A^r, $a^r_{ij} = a_{ik_1} + a_{k_1k_2} + a_{k_2k_3} + ... + a_{k_rj} - a_{k_1k_1} - ... - a_{k_rk_r}$ for some $k_1, ..., k_r \in M \setminus \{i,j\}$ and different.

Since $(\bar{u};\bar{v})$ is a core allocation and the main diagonal is an optimal matching,

$$\begin{aligned} \bar{u}_i + \bar{v}_j &= a_{ik_1} + a_{k_1k_2} + ... + a_{k_rj} - a_{k_1k_1} - ... - a_{k_rk_r} \\ &= a_{ik_1} + a_{k_1k_2} + ... + a_{k_rj} - (\bar{u}_{k_1} + \bar{v}_{k_1}) - ... - (\bar{u}_{k_r} + \bar{v}_{k_r}). \end{aligned} \tag{4}$$

By rearraging (4) we get

$$\bar{u}_i + \bar{v}_{k_1} + \bar{u}_{k_1} + \bar{v}_{k_2} + ... + \bar{u}_{k_r} + \bar{v}_j = a_{ik_1} + ... + a_{k_rj}. \tag{5}$$

Since $(\bar{u};\bar{v}) \in C(w_A)$, any pair $(i,j) \in M \times M'$ satisfies $\bar{u}_i + \bar{v}_j \geq a_{ij}$. Together with (5) we get $\bar{u}_{l_1} + \bar{v}_{l_2} = a_{l_1l_2}$ where $(l_1,l_2) \in \{(i,k_1),(k_1,k_2),...,(k_{r-1},k_r),(k_r,j)\}$.

Since $(x;y) \in B^{\mu}(w_A)$, $a_{l_1l_2} = x_{k_t} + y_{k_t}$ for all $t \in \{1,2,...,r\}$. Now, replacing at (4) and (3) we obtain

$$x_i + y_{k_1} + x_{k_1} + y_{k_2} + ... + x_{k_r} + y_j < a_{ik_1} + a_{k_1k_2} + ... + a_{k_rj}$$

which means that either $x_i + y_{k_1} < a_{ik_1} = \bar{u}_i + \bar{v}_{k_1}$ or $x_{k_l} + y_{k_{l+1}} < a_{k_lk_{l+1}} = \bar{u}_{k_l} + \bar{v}_{k_{l+1}}$ for some $l \in \{1,...,r-1\}$ or $x_{k_r} + y_j < a_{k_rj} = \bar{u}_{k_r} + \bar{v}_j$, which concludes the proof of the claim. □

Continuing with the proof of the theorem, we first consider $(i) \Rightarrow (ii)$. We see, by the claim, that there exists a pair $(i,j) \in M \times M'$ and $(\bar{u};\bar{v}) \in C(w_A)$ such that $x_i + y_j < a_{ij} = \bar{u}_i + \bar{v}_j$. Now, assume without loss of generality $\bar{u}_i > x_i$. If also $\bar{v}_j > y_j$, we get $(\bar{u};\bar{v})\ \mathrm{dom}^A_{\{i,j\}}(x;y)$.

Otherwise, assume $\bar{v}_j \leq y_j$. Since both $(x;y)$ and $(\bar{u};\bar{v})$ belong to $B^{\mu}(w_A)$, $x_j + y_j = \bar{u}_j + \bar{v}_j = a_{jj}$. Then, $\bar{u}_j \geq x_j$. Notice that $\bar{u}_i > x_i + y_j - \bar{v}_j = x_i + (\bar{u}_j + \bar{v}_j - x_j) - \bar{v}_j = x_i + \bar{u}_j - x_j$. Hence, $\bar{u}_j - \bar{u}_i + x_i < x_j$. We want to show that there exists a core allocation that dominates $(x;y)$ via coalition $\{i,j\}$. Hence, we need to lower j^{th} component of $\bar{u}$ in order to increase j^{th} component $\bar{v}$. To this end, consider the monotonic curve defined as in (1) through $(\bar{u};\bar{v})$, and take the point corresponding to $\tau = \bar{u}_i - x_i - \varepsilon$ where $0 < \varepsilon \leq a_{ii} - x_i$. Then, we prove that, for some $\varepsilon > 0$, $(\bar{u}(\tau);\bar{v}(\tau))$ dominates $(x;y)$ via $\{i,j\}$. To this end, we consider some cases:

1. $x_j > 0$. Take $\bar{u}_i(\tau) = \text{med}\{0, x_i + \varepsilon, a_{ii}\}$ and consider two cases:
   (a) $x_i = a_{ii}$. Since, by assumption, $\bar{u}_i > x_i$, we get $a_{ii} = x_i < \bar{u}_i$ which contradicts $(\bar{u}, \bar{v}) \in C(w_A)$.
   (b) $0 < x_i < a_{ii}$. Then, there exists a small enough $\varepsilon_1 > 0$ such that $\bar{u}_i(\tau) = x_i + \varepsilon_1 > x_i$. Take now $\bar{u}_j(\tau) = \text{med}\{0, \bar{u}_j - \bar{u}_i + x_i + \varepsilon, a_{jj}\}$. Note that, since $\bar{u}_j - \bar{u}_i + x_i < x_j \leq a_{jj}$, for $\varepsilon = \varepsilon_2$ small enough we can guarantee that $\bar{u}_j - \bar{u}_i + x_i + \varepsilon_2 < x_j$ and hence $\bar{u}_j(\tau) \neq a_{jj}$. Then, we examine two cases:
      i. $\bar{u}_j(\tau) = \bar{u}_j - \bar{u}_i + x_i + \varepsilon_2$. Then, take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and notice that $\bar{u}_i(\tau) > x_i$, $\bar{u}_j(\tau) < x_j$ or equivalently $\bar{v}_j(\tau) > y_j$ which prove $(\bar{u}(\tau);\bar{v}(\tau)) \text{ dom}^A_{\{i,j\}}(x;y)$.
      ii. $\bar{u}_j(\tau) = 0 < x_j$. Then, $\bar{v}_j(\tau) = a_{jj} > y_j$ and we also get, for $\varepsilon = \varepsilon_1$, that $\bar{u}_i(\tau) > x_i$ and $\bar{v}_j(\tau) > y_j$ and we are done.
2. $x_j = 0$. Since $(x;y) \in B^\mu(w_A)$, $y_j = a_{jj}$. We get from $x_i + y_j < a_{ij}$ that $a_{jj} < a_{ij}$ which contradicts the dominant diagonal assumption on the valuation matrix.

This shows that any $(x;y) \in B^\mu(w_A) \setminus C(w_A)$ is dominated by a core allocation via coalition $\{i,j\}$ which concludes the proof of $(i) \Rightarrow (ii)$.

Next, we prove $(ii) \Rightarrow (i)$. Let us suppose on the contrary, that the core of a two-sided square assignment game $(M \cup M', w_A)$ is a von Neumann-Morgenstern stable set but its corresponding valuation matrix A has not a dominant diagonal. Since A does not have a dominant diagonal, there exists a sector-optimal allocation, let us say $(a;0)$, that does not belong to the core. Since the assumption states that $C(w_A)$ is a von Neumann-Morgenstern stable set, there exists $(u;v) \in C(w_A)$ such that $(u;v) \text{ dom}^A_{\{i,j\}}(a;0)$. Then $u_i > a_{ii}$ which contradicts $(u;v) \in C(w_A)$. □

References

Gillies DB (1959) Solutions to general non-zero-sum games. In: Tucker A, Luce RD (eds) In contributions to the theory of games, IV, in: Annals of Mathematics Studies, vol 40, pp 47–85

Lucas WF (1968) A game with no solution. Bulletin of the American Mathematical Society 74:237–239

Núñez M, Rafels C (2002) Buyer-seller exactness in the assignment game. International Journal of Game Theory 31:423–436

Núñez M, Rafels C (2013) von Neumann-Morgenstern solutions in the assignment game. Journal of Economic Theory 148:1282–1291

Shapley L, Shubik M (1972) The assignment game I: the core. International Journal of Game Theory 1:111–130

Solymosi T, Raghavan TES (2001) Assignment games with stable core. International Journal of Game Theory 30:177–185

von Neumann J, Morgenstern O (1944) The Theory of Games and Economic Behavior. Princeton University Press, Princeton, New Jersey