Discrete Schur-constant models

A. Castañer[a], M.M. Claramunt[a], C. Lefèvre[b], S. Loisel[c,*]

[a] *Universitat de Barcelona, Departament de Matemàtica Econòmica, Financera i Actuarial, 690 Avinguda Diagonal, E-08034 Barcelona, Spain*

[b] *Université Libre de Bruxelles, Département de Mathématique, Campus de la Plaine C.P. 210, B-1050 Bruxelles, Belgium*

[c] *Université de Lyon, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France*

Abstract

This paper introduces a class of Schur-constant survival models, of dimension n, for arithmetic non-negative random variables. Such a model is defined through a univariate survival function that is shown to be n-monotone. Two general representations are obtained, by conditioning on the sum of the n variables or through a doubly mixed multinomial distribution. Several other properties including correlation measures are derived. Three processes in insurance theory are discussed for which the claim interarrival periods form a Schur-constant model.

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1. Introduction

Schur-constant models play a special role in the analysis of lifetime data. Their properties have been studied by several authors including Barlow and Mendel (1993), Caramellino and Spizzichino (1994), Nelsen (2005), Chi et al. (2009) and Nair and Sankaran (2014). Traditionally, the lifetimes considered are absolutely continuous random variables valued in $\mathbb{R}_+$. The present work aims to discuss Schur-constant models for discrete survival data valued in $\mathbb{N}_0 = \{0, 1, \ldots\}$.

Let $(X_1, \ldots, X_n)$ be a vector of n (≥ 2) arithmetic non-negative random variables, called lifetimes. It is said to have a Schur-constant joint survival function if for all $(x_1, \ldots, x_n) \in \mathbb{N}_0^n$,

$$P(X_1 \geq x_1, \ldots, X_n \geq x_n) = S(x_1 + \ldots + x_n), \tag{1.1}$$

where S is an admissible function from $\mathbb{N}_0$ to $[0, 1]$. Clearly, such a survival function S is both Schur-convex and Schur-concave (see Marshall et al. (2011)), hence the appellation of Schur-constant.

By (1.1), the n variables X_i of this vector are exchangeable. Moreover, any subvector is also Schur-constant. As in the continuous case, a Schur-constant model translates a no-aging property, i.e. the residual lifetimes of any two components, $X_i - x_i$ and $X_j - x_j$ say, have the same conditional distributions, even if they have different ages x_i and x_j:

$$\begin{aligned} P(X_i - x_i \geq t | X_1 \geq x_1, \ldots, X_n \geq x_n) &= S(x_1 + \ldots + x_n + t)/S(x_1 + \ldots + x_n) \\ &= P(X_j - x_j \geq t | X_1 \geq x_1, \ldots, X_n \geq x_n). \end{aligned}$$

*Corresponding author

Email addresses: acastaner@ub.edu (A. Castañer), mmclaramunt@ub.edu (M.M. Claramunt), clefevre@ulb.ac.be (C. Lefèvre), stephane.loisel@univ-lyon1.fr (S. Loisel)

Concerning the function S, putting $x_2 = \ldots = x_n = 0$ in (1.1) gives $P(X_1 \geq x_1) = S(x_1)$, so that S is at least a univariate survival function. In fact, S is a multivariate survival function, which means that $S(0) = 1$, $S(\infty) = 0$ and the probability mass associated by S to any rectangle in $\mathbb{N}_0^n$ is nonnegative.

As a first result, we will show that this admissibility condition is equivalent to the property of n-monotonicity of S on $\mathbb{N}_0$. A function $f(x)$: $\mathbb{N}_0 \to \mathbb{R}$ is said to be n-monotone if it satisfies

$$(-1)^j \, \Delta^j f(x) \geq 0, \quad j = 0, \ldots, n, \tag{1.2}$$

where Δ is the forward difference operator (i.e. $\Delta f(x) = f(x+1) - f(x)$) and Δ^j is its j-th iterated. Multiple monotone functions on $\mathbb{N}_0$ have received little attention so far in the literature. Recently, Lefèvre and Loisel (2013) have studied the property of monotonicity for probability distributions, in the continuous and discrete cases.

It is worth indicating that the multiple monotonicity on $\mathbb{R}_+$ is a much more standard concept. Williamson (1956) has investigated in detail the properties of such functions when $n \geq 1$ is an integer (as here) or even any real; see also Lévy (1962) and Gneiting (1999). In probability, n-monotonicity of continuous distributions corresponds to the so-called beta$(1, n)$-unimodality, defined for n real ≥ 0 (Bertin et al. (1997), page 72). In statistics, the estimation problem of n-monotone densities when n is an integer ≥ 0 has been studied by Balabdaoui and Wellner (2007), for instance. As shown by McNeil and Nešlehová (2009), an Archimedean generator yields a n-dimensional copula if and only if this generator is n-monotone on $\mathbb{R}_+$; see also e.g. Genest and Rivest (1993), Albrecher et al. (2011) and Constantinescu et al. (2011). In Lefèvre and Utev (2013), it is proved that symmetric n-monotone densities are preserved by convolution provided $n \in [0, 1]$.

The paper is organized as follows. In Section 2, we show that a Schur-constant model requires the n-monotonicity of S, and we derive different joint life time distributions. In Section 3, we provide two representations of a Schur-constant model, by conditioning on the sum $X_1 + \ldots + X_n$ or through a doubly mixed multinomial distribution. In Section 4, we prove that an infinite sequence is Schur-constant when the joint distributions are of mixed geometric form. In Section 5, we present some parametric functions S that are monotone with various degrees. In Section 6, we obtain simple expressions for the usual correlation coefficients. In Section 7, we discuss three processes in insurance for which the claim interarrival periods form a Schur-constant model. The paper ends with a short Appendix.

2. Joint lifetime distributions

We start by deriving a necessary and sufficient condition for the function S in (1.1) to be a multivariate survival function. Firstly, the lemma below characterizes the survival function $\bar{F}(x_1, \ldots, x_n)$ of an arbitrary $\mathbb{N}_0^n$-valued random vector $(X_1, \ldots, X_n)$. The result is well-known, but a short proof is given for reasons of completeness. Let $\bar{F}(x_{i_1}, \ldots, x_{i_j})$ be the survival function of any subvector $(X_{i_1}, \ldots, X_{i_j})$, $1 \leq j \leq n$. Forward difference operators are defined as follows: g being a real function on $\mathbb{N}_0$, then for any integers $h_i \geq 1$, $1 \leq i \leq n$,

$$\Delta_{i,h_i} g(x_1, \ldots, x_n) = g(x_1, \ldots, x_i + h_i, \ldots, x_n) - g(x_1, \ldots, x_i, \ldots, x_n);$$

by convenience, we write $\Delta_{i,1} g = \Delta_i g$.

Lemma 2.1. *A function $\bar{F}(x_1, \ldots, x_n)$: $\mathbb{N}_0^n \to [0, 1]$ is the survival function of a $\mathbb{N}_0^n$-valued random vector $(X_1, \ldots, X_n)$ if and only if $\bar{F}(0, \ldots, 0) = 1$, $\bar{F}(x_1, \ldots, x_n) = 0$ if $x_i = \infty$ for at least one i, and*

$$(-1)^j \, \Delta_{i_1} \ldots \Delta_{i_j} \bar{F}(x_{i_1}, \ldots, x_{i_j}) \geq 0, \quad j = 1, \ldots, n. \tag{2.1}$$

Proof. Applying Δ_{i,h_i}, with $h_i \geq 1$, to $\bar{F}(x_i)$ gives

$$-\Delta_{i,h_i}\bar{F}(x_i) = P(X_i \geq x_i) - P(X_i \geq x_i + h_i) = P(x_i \leq X_i < x_i + h_i).$$

More generally, for $1 \leq j \leq n$ and $h_{i_1}, \ldots, h_{i_j} \geq 1$,

$$(-1)^j\,\Delta_{i_1,h_{i_1}}\ldots\Delta_{i_j,h_{i_j}}\bar{F}(x_{i_1}, \ldots, x_{i_j}) = P(x_{i_k} \leq X_{i_k} < x_{i_k} + h_{i_k},\, k = 1, \ldots, j), \tag{2.2}$$

which is obviously nonnegative, hence (2.1) by taking $h_{i_1} = \ldots = h_{i_j} = 1$. Conversely, it is immediate that a function $\bar{F}$ well normalized and fulfilling the condition (2.1) may be considered as the survival function of a random vector $(X_1, \ldots, X_n)$. $\diamond$

Now, let us go back to the Schur-constant model for which $\bar{F}(x_1, \ldots, x_n) = S(x_1 + \ldots + x_n)$. As for (1.2), put $\Delta S(x) = S(x+1) - S(x)$ with Δ^j its j-th iterated. Evidently,

$$\Delta_i\bar{F}(x_{i_1}, \ldots, x_{i_j}) = \Delta S(x_{i_1} + \ldots + x_{i_j}),$$

so that the condition (2.1) becomes

$$(-1)^j\,\Delta^j S(x) \geq 0, \quad j = 1, \ldots, n. \tag{2.3}$$

This yields the following characterization result.

Proposition 2.2. *A function $S(x)$: $\mathbb{N}_0 \to [0,1]$ is the Schur-constant survival function of a $\mathbb{N}_0^n$-valued random vector $(X_1, \ldots, X_n)$ if and only if $S(0) = 1$, $S(\infty) = 0$ and S is n-monotone on $\mathbb{N}_0$.*

In other words, S is simply a univariate survival function that is n-monotone on $\mathbb{N}_0$.

Let $\{p(x),\, x \in \mathbb{N}_0\}$ denote the probability mass function (p.m.f.) associated to S. Since $\Delta^j S(x) = -\Delta^{j-1}p(x)$, $j \geq 1$, the condition (2.3) is equivalent to

$$(-1)^j\,\Delta^j p(x) \geq 0, \quad j = 0, \ldots, n-1.$$

Proposition 2.3. *A function $p(x)$: $\mathbb{N}_0 \to [0,1]$ is the Schur-constant p.m.f. of a $\mathbb{N}_0^n$-valued random vector $(X_1, \ldots, X_n)$ if and only if the $p(x)$'s are of sum 1 and p is $(n-1)$-monotone on $\mathbb{N}_0$.*

From (1.1) and (2.2), we directly obtain simple formulas for various probablities on subvectors $(X_1, \ldots, X_j)$, $1 \leq j \leq n$. Some cases of interest are listed below.

Proposition 2.4. *For $1 \leq j \leq n$ and $(x_1, \ldots, x_j) \in \mathbb{N}_0^j$,*

$$P(x_1 \leq X_1 < x_1 + h_1, \ldots, x_j \leq X_j < x_j + h_j) = (-1)^j\,\Delta_{1,h_1}\ldots\Delta_{j,h_j}S(x_1 + \ldots + x_j), \tag{2.4}$$

$$P(X_1 = x_1, \ldots, X_j = x_j) = (-1)^j\,\Delta^j S(x_1 + \ldots + x_j), \tag{2.5}$$

$$P(X_1 = x_1, \ldots, X_{j-1} = x_{j-1}, X_j \geq x_j) = (-1)^{j-1}\,\Delta^{j-1}S(x_1 + \ldots + x_j). \tag{2.6}$$

For the sequel, it is useful to consider the associated partial sums $T_j = X_1 + \ldots + X_j$, $1 \leq j \leq n$.

Proposition 2.5. *For $1 \leq k \leq j \leq n$ and $0 \leq t_{j-k+1} \leq \ldots \leq t_j$,*

$$P(T_{j-k+1} = t_{j-k+1}, \ldots, T_j = t_j) = (-1)^j \, \Delta^j S(t_j) \binom{t_{j-k+1} + j - k}{j-k}. \tag{2.7}$$

In particular,

$$P(T_j = t_j) = (-1)^j \, \Delta^j S(t_j) \binom{t_j + j - 1}{j-1}, \tag{2.8}$$

$$P(T_1 = t_1, \ldots, T_j = t_j) = (-1)^j \, \Delta^j S(t_j), \tag{2.9}$$

which also yields

$$P(T_1 = t_1, \ldots, T_{j-1} = t_{j-1} | T_j = t_j) = 1 / \binom{t_j + j - 1}{j-1}. \tag{2.10}$$

Proof. In terms of $(T_1, \ldots, T_j)$, we have

$$P(T_{j-k+1} = t_{j-k+1}, \ldots, T_j = t_j) = \sum_{t_1 \leq \ldots \leq t_{j-k} : \, t_{j-k} \leq t_{j-k+1}} P(T_1 = t_1, \ldots, T_{j-k} = t_{j-k}, \ldots, T_j = t_j).$$

The sum in the r.h.s. can then be expressed in terms of $(X_1, \ldots, X_j)$ as

$$\sum_{x_1, \ldots, x_{j-k} : \, x_1 + \ldots + x_{j-k+1} = t_{j-k+1}} P(X_1 = x_1, \ldots, X_{j-k} = x_{j-k}, \ldots, X_j = t_j - t_{j-1}).$$

By (2.5), the probabilities $P(X_1 = x_1, \ldots, X_j = t_j - t_{j-1})$ are all equal to $(-1)^j \Delta^j S(t_j)$. Remember that the number of ways to put b indistinguishable balls in n urns is equal to $\binom{b+n-1}{n-1}$. Thus, the number of terms in the sum above is obtained by taking $b = t_{j-k+1}$ and $n = j-k+1$, which gives $\binom{t_{j-k+1}+j-k}{j-k}$. Formula (2.7) now follows. It gives (2.8) for $k = 1$ and (2.9) for $k = j$, with (2.10) as a consequence. $\diamond$

Formula (2.10) means that given $T_j = t_j$, the $j-1$ previous arrival times are obtained by throwing $j-1$ balls in $t_j + 1$ urns (which correspond to the instants $0, \ldots, t_j$). In the continuous case, $[T_1, \ldots, T_{j-1} | T_j = t_j]$ is distributed as the order statistics of a sample of $j-1$ independent $(0, t_j)$-uniform random variables (e.g. Theorem 2.1 of Chi et al. (2009)).

3. Representations of Schur-constancy

Our purpose in this Section is to provide general representations that are valid for any discrete Schur-constant model. Put $\binom{a}{b} = 0$ when $a < b$.

Proposition 3.1. *For $1 \leq j \leq n$ and $x_1, \ldots, x_j, z \geq 0$ with $x_1 + \ldots + x_j \leq z$,*

$$P(X_1 \geq x_1, \ldots, X_j \geq x_j | T_n = z) = \binom{z - (x_1 + \ldots + x_j) + n - 1}{n-1} / \binom{z + n - 1}{n-1}, \tag{3.1}$$

so that for $1 \leq j \leq n-1$,

$$P(X_1 = x_1, \ldots, X_j = x_j | T_n = z) = \binom{z - (x_1 + \ldots + x_j) + n - j - 1}{n-j-1} / \binom{z + n - 1}{n-1}. \tag{3.2}$$

Thus, the function S can be represented as

$$S(x_1 + \ldots + x_n) = E\left[\binom{Z - (x_1 + \ldots + x_n) + n - 1}{n-1} / \binom{Z + n - 1}{n-1} \right], \tag{3.3}$$

where the variable Z is distributed as T_n, i.e. with a p.m.f. given by (2.8) where $j = n$.

Proof. By definition and from (2.8),

$$P(X_1 \geq x_1, \ldots, X_j \geq x_j | T_n = z) = \frac{P(X_1 \geq x_1, \ldots, X_j \geq x_j, T_n = z)}{(-1)^n \, \Delta^n S(z) \binom{z+n-1}{n-1}}.$$

The numerator can be expressed as

$$\sum_{y_1, \ldots, y_n : \, y_1 \geq x_1, \ldots, y_j \geq x_j \text{ and } y_1 + \ldots + y_n = z} P(X_1 = y_1, \ldots, X_j = y_j, \ldots, X_n = y_n),$$

in which, by (2.5), the probabilities are all equal to $(-1)^n \, \Delta^n S(x)$. One easily sees that the number of ways to put b indistinguishable balls in n urns with at least x_1 balls in urn 1, …, x_j balls in urn j, is equal to $\binom{b-(x_1+\ldots+x_j)+n-1}{n-1}$. This leads to formula (3.1).

To get (3.2), it suffices to apply (2.5) and the fact that

$$\Delta \binom{z-x+n-1}{n-1} = -\binom{z-x+n-2}{n-2},$$

where Δ operates on x. Finally, (3.1) where $j = n$ gives (3.3). $\diamond$

This result is the discrete analogue of a representation obtained for the continuous model (see Proposition 2.3 in Caramellino and Spizzichino (1994) and Theorem 2.1 in Chi et al. (2009)).

A different but equivalent characterization of Schur-constancy is derived below.

Proposition 3.2. *The model $(X_1, \ldots, X_n)$ is Schur-constant if its joint distribution is of doubly mixed multinomial ($\mathcal{M}M$) form, namely*

$$(X_1, \ldots, X_n) =_d \mathcal{M}M(Z; U_1, \ldots, U_n). \tag{3.4}$$

Here, Z represents the random number of experiments and is distributed as T_n, while $(U_1, \ldots, U_n)$ represents the vector of randomized cell probabilities, independent of T_n and with a joint survival function that is (continuous) Schur-constant and defined by

$$P(U_1 \geq u_1, \ldots, U_n \geq u_n) = [1 - (u_1 + \ldots + u_n)]_+^{n-1}, \quad u_1, \ldots, u_n \in (0, 1). \tag{3.5}$$

Proof. We are going to show that under (3.4) and (3.5), the p.m.f. of $(X_1, \ldots, X_j)$ conditionally on Z is given by (3.2), for $1 \leq j \leq n-1$. Indeed, we have, for all $x_1, \ldots, x_j \geq 0$ with $x_1 + \ldots + x_j \leq z$,

$$P(X_1 = x_1, \ldots, X_j = x_j | Z = z) = \frac{z!}{x_1! \ldots x_j! (z - x_1 \ldots - x_j)!} (n-1) \ldots (n-j)$$
$$\int_{u_1, \ldots, u_j \geq 0 \text{ and } u_1 + \ldots + u_j \leq 1} u_1^{x_1} \ldots u_j^{x_j} (1 - u_1 - \ldots - u_j)^{z - x_1 - \ldots - x_j + n - j - 1} \, du_1 \ldots du_j. \tag{3.6}$$

Now, consider an integral of the form

$$\int_{0 \leq u_1, \ldots, u_{k-1} \leq 1 \text{ and } u_k = 1 - u_1 - \ldots - u_{k-1}} u_1^{\alpha_1 - 1} \ldots u_k^{\alpha_k - 1} \, du_1 \ldots du_{k-1},$$

for reals $\alpha_1, \ldots, \alpha_k > 0$. That integral is the multinomial Beta function and is equal to

$$\Gamma(\alpha_1) \ldots \Gamma(\alpha_k) / \Gamma(\alpha_1 + \ldots + \alpha_k).$$

This is a known identity, which is also easily proved by induction. Going back to the integral in (3.6), we see that it corresponds to the particular case where $k = j+1$, $\alpha_1 = x_1+1, \ldots, \alpha_j = x_j+1$ and $\alpha_{j+1} = z - x_1 - \ldots - x_j + n - j$. Thus, we can write that

$$\int_{u_1,\ldots,u_j \geq 0 \text{ and } u_1+\ldots+u_j \leq 1} u_1^{x_1} \ldots u_j^{x_j} (1 - u_1 - \ldots - u_j)^{z-x_1-\ldots-x_j+n-j-1} \, du_1 \ldots du_j$$
$$= \frac{x_1! \ldots x_j! (z - x_1 - \ldots - x_j + n - j - 1)!}{(z + n - 1)!}. \tag{3.7}$$

Substituting (3.7) in (3.6) then yields

$$P(X_1 = x_1, \ldots, X_j = x_j | Z = z) = \frac{z! \, (n-1) \ldots (n-j) \, (z - x_1 - \ldots - x_j + n - j - 1)!}{(z - x_1 - \ldots - x_j)! \, (z + n - 1)!},$$

and after multiplication by $(n-j-1)!/(n-j-1)!$ $(= 1)$,

$$P(X_1 = x_1, \ldots, X_j = x_j | Z = z) = \binom{z - (x_1 + \ldots + x_j) + n - j - 1}{n - j - 1} \Big/ \binom{z + n - 1}{n - 1},$$

i.e. the desired formula (3.2). $\diamond$

In the continuous case, the vector $(X_1/Z, \ldots, X_n/Z)$ is independent of Z and is Schur-constant with survival function $(1-x)_+^{n-1}$. Note that in the discrete case, $(X_1/Z, \ldots, X_n/Z)$ is not independent of Z since the X_i's are valued in $\{0, \ldots, Z\}$.

4. The geometric special model

Firstly, we show below that the no-aging property of Schur-constant models is a generalization of the lack of memory property for geometric random variables.

Proposition 4.1. *Let $(X_1, \ldots, X_n)$ be a Schur-constant random vector. Then, the components X_i, $1 \leq i \leq n$, are independent if and only if they are geometrically distributed.*

Proof. If the X_i's are independent, (1.1) implies that

$$S(x_1 + \ldots + x_n) = S(x_1) \ldots S(x_n),$$

for all $(x_1, \ldots, x_n) \in \mathbb{N}_0^n$. Since S is non-increasing with $S(0) = 1$, we then obtain by induction that $S(x) = q^x$, $x \in \mathbb{N}_0$, for some $0 < q \leq 1$. The converse is obvious. $\diamond$

Now, let us consider an infinite discrete Schur-constant model, i.e. (1.1) holds for all $n \geq 2$. Since the sequence $\{X_i,\, i \geq 1\}$ is exchangeable, de Finetti theorem asserts that the X_i's are conditionally i.i.d. given the σ-algebra $\mathcal{G}$ of permutable events (e.g. Chow and Teicher (1988), section 7.3). The Schur-constant property allows us to make quite explicit the mixture structure involved.

Proposition 4.2. *An infinite sequence of random variables $\{X_i,\, i \geq 1\}$ with finite mean is Schur-constant if and only if for all j, $(X_1, \ldots, X_j)$ has a mixed geometric distribution, namely*

$$P(X_1 \geq x_1, \ldots, X_j \geq x_j) = E\left[\left(\frac{\Theta}{\Theta + 1}\right)^{x_1+\ldots+x_j}\right], \quad j \geq 1, \tag{4.1}$$

where

$$\Theta = \lim_{n\to\infty} T_n/n \;\; a.s. \tag{4.2}$$

Proof. The sufficiency is immediate and omitted. Now, from (3.1), we can write that

$$
\begin{aligned}
P(X_1 \geq x_1, \ldots, X_j \geq x_j) &= E\left[\binom{T_n - (x_1 + \ldots + x_j) + n - 1}{n-1} \Big/ \binom{T_n + n - 1}{n-1}\right] \\
&= E\left[\frac{(T_n - (x_1 + \ldots + x_j) + n - 1) \ldots (T_n - (x_1 + \ldots + x_j) + 1)}{(T_n + n - 1) \ldots (T_n + 1)} \mathbf{1}(T_n \geq x_1 + \ldots + x_j)\right] \\
&= E\left[\prod_{k=1}^{n-1}\left(1 - \frac{x_1 + \ldots + x_j}{T_n + k}\right) \mathbf{1}(T_n \geq x_1 + \ldots + x_j)\right], \quad 1 \leq j \leq n, \qquad (4.3)
\end{aligned}
$$

where $\mathbf{1}(.)$ denotes the indicator function.

The strong law of large numbers for an exchangeable sequence $\{X_i,\, i \geq 1\}$ with finite mean asserts that the random variable $\Theta_n \equiv T_n/n$ converges a.s. to a variable Θ which is distributed as $E(X_1|\mathcal{G})$ (Chow and Teicher (1988), section 9.2). This provides us with the assertion (4.2). Moreover, by the dominated convergence theorem, (4.3) yields

$$
P(X_1 \geq x_1, \ldots, X_j \geq x_j) = E\left[\lim_{n\to\infty} \prod_{k=1}^{n-1}\left(1 - \frac{x_1 + \ldots + x_j}{n\Theta_n + k}\right) \mathbf{1}(n\Theta_n \geq x_1 + \ldots + x_j)\right],
$$

so that the assertion (4.1) will follow if the limit in $[\ldots]$ is equal to $[\Theta/(\Theta+1)]^{x_1+\ldots+x_j}$. To prove this, we rewrite the limit as

$$
\begin{aligned}
&\lim_{n\to\infty} \exp\left[\sum_{k=1}^{n-1} \ln\left(1 - \frac{x_1 + \ldots + x_j}{n\Theta_n + k}\right)\right] \mathbf{1}(n\Theta_n \geq x_1 + \ldots + x_j) \\
&\quad = \lim_{n\to\infty} \exp\left[-\sum_{k=1}^{n-1} \frac{x_1 + \ldots + x_j}{n\Theta_n + k} + o(1)\right] \mathbf{1}(n\Theta_n \geq x_1 + \ldots + x_j) \\
&\quad = \lim_{n\to\infty} \exp\left[-(x_1 + \ldots + x_j) \ln\left(\frac{n\Theta_n + n - 1}{n\Theta_n}\right) + o(1)\right] \mathbf{1}(n\Theta_n \geq x_1 + \ldots + x_j),
\end{aligned}
$$

using to the approximation

$$
\sum_{l=a}^{b} \frac{1}{l} = \ln\left(\frac{b}{a}\right) + o(1) \text{ as } a, b \to \infty.
$$

From (4.2), we then get the announced variable as the above limit. ⋄

We notice that for the continuous case, results similar to Propositions 4.1 and 4.2 hold with respect to the exponential distribution (see Theorem 1 in Nelsen (2005) and Corollary 2.3 in Chi et al. (2009)).

5. Monotone survival functions

In this Section, we present some parametric survival functions S that are monotone of various degrees. Before this, we come back shortly on two general characterizations for such functions.

5.1. Representations

By Proposition 2.1, the function S in a Schur-constant model is a n-monotone survival function. Recently, Lefèvre and Loisel (2013) proved that such a function admits a general representation (see Proposition 2.5 with $t = n - 1$ and formula (2.10) for $i = 0$ in that paper). Specifically, there exists a random variable Z valued in $\mathbb{N}_0$ for which S can be expressed as

$$S(x) = E\left\{\binom{Z-x+n-1}{n-1}\Big/\binom{Z+n-1}{n-1}\right\}, \quad x \in \mathbb{N}_0, \tag{5.1}$$

and the p.m.f. of Z is univoquely determined from S by

$$P(Z = z) = (-1)^n \binom{z+n-1}{n-1} \Delta^n S(z). \tag{5.2}$$

This result provides us with another method to derive the representation (3.3) for a Schur-constant model. Indeed, comparing (2.8) and (5.2), we see that Z has the same distribution as T_n. Furthermore, inserting (5.1) in (1.1) then yields the formula (3.3).

A different representation for such a function S is also given by Lefèvre and Loisel (2013) (see their formulas (2.13) and (2.16)). More precisely, S corresponds to the survival function of a random variable X whose distribution is of doubly mixed binomial ($\mathcal{M}B$) form, namely

$$X =_d \mathcal{M}B(Z, 1 - U^{1/(n-1)}), \tag{5.3}$$

where Z is the random number of experiments and $1 - U^{1/(n-1)}$ is the random parameter, U being a $(0, 1)$-uniform random variable independent of Z.

We note that, as expected, the formula (5.3) is in fact a consequence of the representation (3.4), (3.5) for Schur-constant models.

5.2. Bernoulli model

Let X be a Bernoulli random variable with parameter p. Its survival function is

$$S(0) = 1, \quad S(1) = p \quad \text{and} \quad S(x) = 0, \quad x \geq 2.$$

Proposition 5.1.

$S(x)$ is n-monotone iff $p \leq 1/n$.

Proof. It suffices to observe that for all $j \geq 0$,

$$\begin{aligned} \Delta^j S(0) &= (-1)^j(-jp + 1), \\ \Delta^j S(1) &= (-1)^j p, \end{aligned}$$

with $\Delta^j S(x) = 0$ for $x \geq 2$, hence the assertion for n-monotonicity. $\diamond$

Note that if $1/(n + 1) < p \leq 1/n$, S is n-monotone but not $(n + 1)$-monotone. From (5.2), we see that the corresponding variable Z has a Bernoulli distribution with parameter np.

For illustration, consider n successive time intervals of unitary length. Denote by X_i the indicator of the claim occurence in interval i, $1 \leq i \leq n$. The model (1.1) with X binomial describes a situation where the n claim indicators are exchangeable and of probability p, and at most one claim can arise during the whole period $(0, n)$. This could arise, for example, in reliability with one-shot device testing and in life insurance with monthly death risk estimation on the basis of yearly reports.

5.3. Stop-loss model

Let X be a random variable with a survival function of stop-loss type defined by

$$S(x) = (k-x)_+^t/k^t, \quad x \in \mathbb{N}_0, \tag{5.4}$$

where k and t are positive integers.

To begin with, we point out that the function S can be expanded as a mean of combinatorial terms. The proof is given in the Appendix.

Lemma 5.2.

$$\frac{(k-x)_+^t}{t!} = \sum_{i=0}^{t-1} \alpha_i(t) \binom{k-x+i}{t}, \tag{5.5}$$

where $\{\alpha_i(t),\, 0 \le i \le t-1\}$ *is a symmetric p.m.f. which is computed recursively by*

$$\alpha_i(t) = \alpha_{i-1}(t-1)\frac{t-i}{t} + \alpha_i(t-1)\frac{i+1}{t}, \quad t = 2, 3, \ldots, \tag{5.6}$$

with $\alpha_0(1) = 1$ *and* $\alpha_{-1}(t-1) = 0$.

We are now ready to establish the monotonicity property satisfied by S.

Proposition 5.3.

$$S(x) \text{ is } (t+1)\text{-monotone},$$

and the p.m.f. of the corresponding variable Z *is*

$$P(Z = z) = \alpha_{z+t-k}(t)\binom{z+t}{t}\frac{t!}{k^t}, \quad \max(0, k-t) \le z \le k-1. \tag{5.7}$$

Proof. Note that $\Delta\binom{a-x}{t} = -\binom{a-x-1}{t-1}$ for $t \ge 1$, while $\Delta\binom{a-x}{0} = -\mathbf{1}(x = a)$ where $\mathbf{1}(.)$ is the indicator function. From (5.5), we thus get

$$\Delta^j (k-x)_+^t = t!\,(-1)^j \sum_{i=0}^{t-1} \alpha_i(t) \binom{k-x+i-j}{t-j}, \quad 0 \le j \le t,$$

and for $j = t+1$,

$$\begin{aligned} \Delta^{t+1}(k-x)_+^t &= t!\,(-1)^{t+1} \sum_{i=0}^{t-1} \alpha_i(t)\, \mathbf{1}(x = k+i-t) \\ &= t!\,(-1)^{t+1}\, \alpha_{x+t-k}(t)\, \mathbf{1}[\max(0, k-t) \le x \le k-1]. \end{aligned} \tag{5.8}$$

From (5.4), we then deduce that $(-1)^j \Delta^j S(x) \ge 0$ for $0 \le j \le t+1$, i.e. $S(x)$ is (t+1)-monotone. Now, from (5.2) with $n = t+1$, we have

$$P(Z = z) = (-1)^{t+1}\,[\Delta^{t+1}(k-z)_+^t]\,\frac{1}{k^t}\binom{z+t}{t}.$$

Using the formula (5.8) we then deduce the announced result (5.7). ⋄

Note that the function S is not $(t+2)$-monotone since by (5.8),

$$(-1)^{t+2}\Delta^{t+2} S(x) = -t! \sum_{i=0}^{t-1} \alpha_i(t)\, \Delta\delta_{x,k+i-t},$$

which is not always nonnegative. For instance, it reduces to $-t!\alpha_0(t) < 0$ when $x = k - 1 - t$.

Tables 1 and 2 below give the p.m.f. of Z for the first values of t when $k = 3$ or 10.

Table 1: P.m.f. $\{P(Z = z)\}$ when $t = 1, \ldots, 7$ and $k = 3$.

$t \quad \backslash \quad z$	0	1	2
1			1
2		$1/3$	$2/3$
3	$1/3^3$	$16/3^3$	$10/3^3$
4	$11/3^4$	$55/3^4$	$15/3^4$
5	$66/3^5$	$156/3^5$	$21/3^5$
6	$302/3^6$	$399/3^6$	$28/3^6$
7	$1191/3^7$	$960/3^7$	$36/3^7$

Table 2: P.m.f. $\{P(Z = z)\}$ when $t = 1, \ldots, 7$ and $k = 10$.

$t \quad \backslash \quad z$	3	4	5	6	7	8	9
1							1
2						0.45	0.55
3					0.12	0.66	0.22
4				0.021	0.363	0.5445	0.0715
5			0.00252	0.12012	0.52272	0.33462	0.02002
6		0.00021	0.026334	0.279048	0.518232	0.171171	0.005005
7	0.000012	0.00396	0.0943272	0.4145856	0.4087512	0.07722	0.001144

5.4. Simple models

Many parametric models are possible for a discrete survival function. In general, however, it is not easy to check the degree of monotonicity verified by S. Some examples are briefly reported below.

Power-type model. Let X be a random variable with survival function

$$S(x) = [1 - (x/k)^t]_+, \quad x \in \mathbb{N}_0,$$

where k is a positive integer and t a positive real.

Proposition 5.4.

$S(x)$ is 2-monotone iff $t \leq 1$.

Proof. We see that

$$\Delta^2 S(x) = [-(x+2)^t + 2(x+1)^t - x^t]/k^t, \quad 0 \leq x \leq k-2,$$
$$\Delta^2 S(k-1) = 1 - (k-1)^t/k^t,$$

with $\Delta^2 S(x) = 0$ for $x \geq k$. So, when $x \geq k - 1$, $\Delta^2 S(x) \geq 0$ for all t. When $0 \leq x \leq k - 2$, this condition means $(x + 1)^t \geq [(x + 2)^t + x^t]/2$, which is true iff $t \leq 1$. The result follows. ⋄

In that case, S is not 3-monotone since if $t = 1$ for instance, $\Delta^3 S(k - 2) = 1/k > 0$.

Gompertz model. Let X be a random variable with survival function

$$S(x) = \exp[\theta(1 - e^x)], \quad x \in \mathbb{N}_0,$$

where θ is a positive real.

First, we define a sequence of reals $\{\theta_j, \, j = 2, 3, \dots\}$ by

$$\theta_j = \max\{\theta > 0 : f_j(\theta) \equiv \sum_{k=0}^{j} \binom{j}{k} (-1)^k \exp(-\theta e^k) = 0\}. \tag{5.9}$$

Using Mathematica 8.0 for instance, it can be seen that $f_j(\theta) > 0$ when $\theta > \theta_j$, and $\theta_{j+1} > \theta_j$ for all $j = 2, 3, \dots$ Thus, $f_1(\theta), \dots, f_n(\theta) > 0$ iff $\theta > \theta_n$. More details are given in the Appendix.

Proposition 5.5.

$S(x)$ is n-monotone iff $\theta \geq \theta_n$.

Proof. We have, for $j \geq 0$,

$$(-1)^j \Delta^j S(x) = \sum_{k=0}^{j} \binom{j}{k} (-1)^k S(x + k) = \sum_{k=0}^{j} \binom{j}{k} (-1)^k \exp[\theta(1 - e^{x+k})].$$

For $j = 1$, this is positive. Thus, the n-monotonicity condition requires that

$$f_{j,x}(\theta) \equiv \sum_{k=0}^{j} \binom{j}{k} (-1)^k \exp(-\theta e^{x+h}) \geq 0, \quad 2 \leq j \leq n. \tag{5.10}$$

When $x = 0$, $f_{j,0}(\theta) = f_j(\theta)$ defined in (5.9), so that (5.10) is fulfilled iff $\theta \geq \theta_n$. For $x > 0$, when $\theta > \theta_n$, then $\theta e^x > \theta_n$ and thus (5.10) is again satisfied. $\diamond$

Note that $S(x)$ is not (n+1)-monotone when $\theta_n \leq \theta < \theta_{n+1}$.

Other cases. We present below the p.m.f. of a few parametric models for which the function S is at least 2-monotone. In fact, it seems that S is ∞-monotone but we have not been able to prove it so far.

Logarithmic model (of parameter $\theta \in (0, 1)$):

$$p(x) = -c\theta^{x+1}/(x + 1) \text{ where } c = 1/ln(1 - \theta), \quad x \in \mathbb{N}_0.$$

Bendford model (of parameter b integer ≥ 3):

$$p(x) = c \ln[(x + 2)/(x + 1)] \text{ where } c = 1/\ln(b), \; 0 \leq x \leq b - 2, \text{ and } p(x) = 0, \; x > b - 2.$$

Pareto model (of parameter $\rho > 0$):

$$p(x) = c/(1 + x)^{1+\rho} \text{ where } c = 1/\sum_{k=1}^{\infty}(1/k)^{1+\rho}, \quad x \in \mathbb{N}_0.$$

6. Correlation measures

Various dependence properties and association measures, as well as their links with aging properties, are widely discussed for the continuous Schur-constant model, especially in the bivariate case. The reader is referred e.g. to Nair and Sankaran (2014) and the references therein. Much of these studies can be adapted to the present discrete model. For brevity reasons, here we focus mainly on the study of the Pearson linear correlation coefficient.

By exchangeability, all the X_i's have the same mean and variance, μ and σ^2 say (assumed to exist), and the same Pearson correlation coefficient ρ. As Schur-constancy is expressed in terms of the survival function S of X_1, one expects that ρ is related to some parameters of X_1 alone. We will see that ρ is indeed a function of μ and σ^2. Let us begin by showing how to calculate these two parameters. We can use either the p.m.f. of X_1 (i.e. (2.5) with $j = 1$), or the characterization (3.4), (3.5) where the mean and variance of Z are denoted by μ_Z and σ_Z^2.

Proposition 6.1. *In terms of S,*

$$\mu = \sum_{x=0}^{\infty} S(x+1), \tag{6.1}$$

$$\sigma^2 = 2\sum_{x=0}^{\infty} xS(x+1) - \mu^2 + \mu, \tag{6.2}$$

and in terms of μ_Z and σ_Z^2,

$$\mu = \mu_Z/n, \tag{6.3}$$

$$\sigma^2 = 2\sigma_Z^2/n(n+1) + \mu_Z^2(n-1)/n^2(n+1) + \mu_Z(n-1)/n(n+1). \tag{6.4}$$

Proof. The k-th descending factorial moment of X_1, $k \geq 1$, is given by

$$\begin{aligned} E(X_{1,[k]}) &= \sum_{x=0}^{\infty} x_{[k]}[S(x) - S(x+1)] \\ &= \sum_{x=0}^{\infty} (x+1)_{[k]} S(x+1) - \sum_{x=0}^{\infty} x_{[k]} S(x+1) \\ &= k\sum_{x=0}^{\infty} x_{[k-1]} S(x+1). \end{aligned} \tag{6.5}$$

Taking $k = 1$ and 2 in (6.5) then yields the first two formulas (6.1) and (6.2).

Let us derive the following two formulas. By (3.4), (3.5) (or (5.3)), we know that $X_1 =_d \mathcal{M}B(Z, U_1)$ where U_1 is independent of Z and $P(U_1 \geq u_1) = (1-u_1)_+^{n-1}$. Since $E(U_1) = 1/n$,

$$\mu = E(Z)E(U_1) = \mu_Z/n,$$

as stated in (6.3). Now, applying a standard conditional argument, we get (in obvious notation)

$$\begin{aligned} \sigma^2 &= var\{E[B(Z,U_1)|Z,U_1]\} + E\{var[B(Z,U_1)|Z,U_1]\} \\ &= var(ZU_1) + E[ZU_1(1-U_1)] \\ &= var[E(ZU_1|Z)] + E[var(ZU_1|Z)] + E[ZU_1(1-U_1)] \\ &= var[ZE(U_1)] + E[Z^2 var(U_1)] + E[ZU_1(1-U_1)] \\ &= [E(U_1)]^2\sigma_Z^2 + var(U_1)E(Z^2) + E(Z)E[U_1(1-U_1)]. \end{aligned} \tag{6.6}$$

It is directly checked that $E(U_1^2) = 2/n(n+1)$, so that

$$var(U_1) = (n-1)/n^2(n+1),$$
$$E[U_1(1-U_1)] = (n-1)/n(n+1).$$

Substituting this in (6.6) leads to

$$\sigma^2 = \sigma_Z^2/n^2 + E(Z^2)(n-1)/n^2(n+1) + E(Z)(n-1)/n(n+1).$$

Finally, writing $1/n^2 = 2/n(n+1) - (n-1)/n^2(n+1)$, we obtain the formula (6.4). $\diamond$

We are in a position to provide the expression of ρ.

Proposition 6.2. *In terms of μ and σ^2,*

$$\rho = (\sigma^2 - \mu^2 - \mu)/2\sigma^2, \tag{6.7}$$

and in terms of μ_Z and σ_Z^2,

$$\rho = \frac{n\sigma_Z^2 - \mu_Z^2 - n\mu_Z}{2n\sigma_Z^2 + (n-1)\mu_Z^2 + n(n-1)\mu_Z}. \tag{6.8}$$

Proof. By (2.5), we have

$$E(X_1X_2) = \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty} x_1x_2P(X_1 = x_1, X_2 = x_2) = \sum_{x_1=0}^{\infty} x_1\,[\sum_{x_2=0}^{\infty} x_2\Delta^2 S(x_1+x_2)].$$

The sum $[\ldots]$ above is easily checked to reduce to $S(x_1+1)$. Therefore,

$$E(X_1X_2) = \sum_{x_1=0}^{\infty} x_1 S(x_1+1) = E(X_{1,[2]})/2,$$

by virtue of (6.5). We then deduce that ρ is given by formula (6.7). Let us now establish (6.8). Of course, we could evaluate $cov(X_1, X_2)$ by arguing as above for σ^2. A simpler method, however, consists in using (6.7) where (6.3) is substituted for μ and (6.4) for σ^2. After an elementary calculation, we then obtain the desired formula. $\diamond$

From (6.7) or (6.8), we see that ρ can be positive or not. This is not surprising in view of the representation (6.3): the common factor of Z tends to generate positive correlation while the negative dependence between U_1 and U_2 tends to generate negative correlation. In fact, $\rho > 0$ if $\sigma^2 > \mu^2 + \mu$ or $n\sigma_Z^2 > \mu_Z^2 + n\mu_Z$; roughly, when σ^2 (σ_Z^2) is large enough with respect to μ (μ_Z). We also notice that, as expected, ρ is an increasing function of σ_Z^2 (and σ^2) when n and μ_Z are kept fixed. Moreover, it is clear that $-1 \leq \rho < 1/2$.

Let us recall that in the continuous case, ρ can be expressed by the following two formulas:

$$\rho = (\kappa^2 - 1)/2\kappa^2,$$

where $\kappa = \sigma/\mu$ is the variation coefficient of X (Nelsen (2005)) and, when $n = 2$,

$$\rho = (2\kappa_Z^2 - 1)/(4\kappa_Z^2 + 1),$$

where $\kappa_Z = \sigma_Z/\mu_Z$ is the variation coefficient of Z (Chi et al. (2009)). Here too, $-1 \leq \rho < 1/2$. We observe, however, that in the discrete case, ρ is a function of the mean and variance of X or Z, and not only of their variation coefficient.

An alternative measure of association between two random variables is provided by the Kendall τ coefficient. The variant named τ_b is an adjustment of τ to deal with discrete random variables (e.g. Agresti (2013); see also Nešlehová (2007)). Its population version is defined as follows: let (X_1, X_2) and (Y_1, Y_2) be two i.i.d. random vectors with the same marginals, then

$$\tau_b = \frac{P[(X_1 - Y_1)(X_2 - Y_2) > 0] - P[(X_1 - Y_1)(X_2 - Y_2) < 0]}{\sqrt{P(X_1 \neq Y_1)P(X_2 \neq Y_2)}}. \tag{6.9}$$

For a Schur-constant model, τ_b can be expressed by the formula (6.10) below.

Proposition 6.3. *In terms of S,*

$$\tau_b = \frac{4\sum_{k=0}^{\infty}(k+1)S(k+2)\Delta^2 S(k) + 2\sum_{k=0}^{\infty}[\Delta S(k)]^2 - \sum_{k=0}^{\infty}(k+1)[\Delta^2 S(k)]^2 - 1}{1 - \sum_{k=0}^{\infty}[\Delta S(k)]^2}. \tag{6.10}$$

Proof. First, we note that

$$P[(X_1 - Y_1)(X_2 - Y_2) < 0] = 1 - P[(X_1 - Y_1)(X_2 - Y_2) > 0] - P(X_1 = Y_1 \text{ or } X_2 = Y_2),$$

so that (6.9) can be rewritten as

$$\tau_b = \frac{2P[(X_1 - Y_1)(X_2 - Y_2) > 0] + P(X_1 = Y_1 \text{ or } X_2 = Y_2) - 1}{\sqrt{P(X_1 \neq Y_1)P(X_2 \neq Y_2)}}. \tag{6.11}$$

As (Y_1, Y_2) is an independent copy of (X_1, X_2),

$$P[(X_1 - Y_1)(X_2 - Y_2) > 0] = 2\sum_{y_1=0}^{\infty}\sum_{y_2=0}^{\infty} P(X_1 \geq y_1 + 1, X_2 \geq y_2 + 1)P(Y_1 = y_1, Y_2 = y_2).$$

For a Schur-constant model, we then get from (1.1) and (2.5)

$$\begin{aligned} P[(X_1 - Y_1)(X_2 - Y_2) > 0] &= 2\sum_{y_1=0}^{\infty}\sum_{y_2=0}^{\infty} S(y_1 + y_2 + 2)\Delta^2 S(y_1 + y_2) \\ &= 2\sum_{k=0}^{\infty}(k+1)S(k+2)\Delta^2 S(k), \end{aligned} \tag{6.12}$$

after putting $k = x_1 + x_2$. In a similar way, we obtain

$$P(X_1 \neq Y_1) = 1 - P(X_1 = X_2) = 1 - \sum_{k=0}^{\infty}[\Delta S(k)]^2, \tag{6.13}$$

and

$$\begin{aligned} P(X_1 = Y_1 \text{ or } X_2 = Y_2) &= 2P(X_1 = Y_1) - P(X_1 = Y_1, X_2 = Y_2) \\ &= 2\sum_{k=0}^{\infty}[\Delta S(k)]^2 - \sum_{y_1=0}^{\infty}\sum_{y_2=0}^{\infty}[\Delta^2 S(y_1 + y_2)]^2 \\ &= 2\sum_{k=0}^{\infty}[\Delta S(k)]^2 - \sum_{k=0}^{\infty}(k+1)[\Delta^2 S(k)]^2. \end{aligned} \tag{6.14}$$

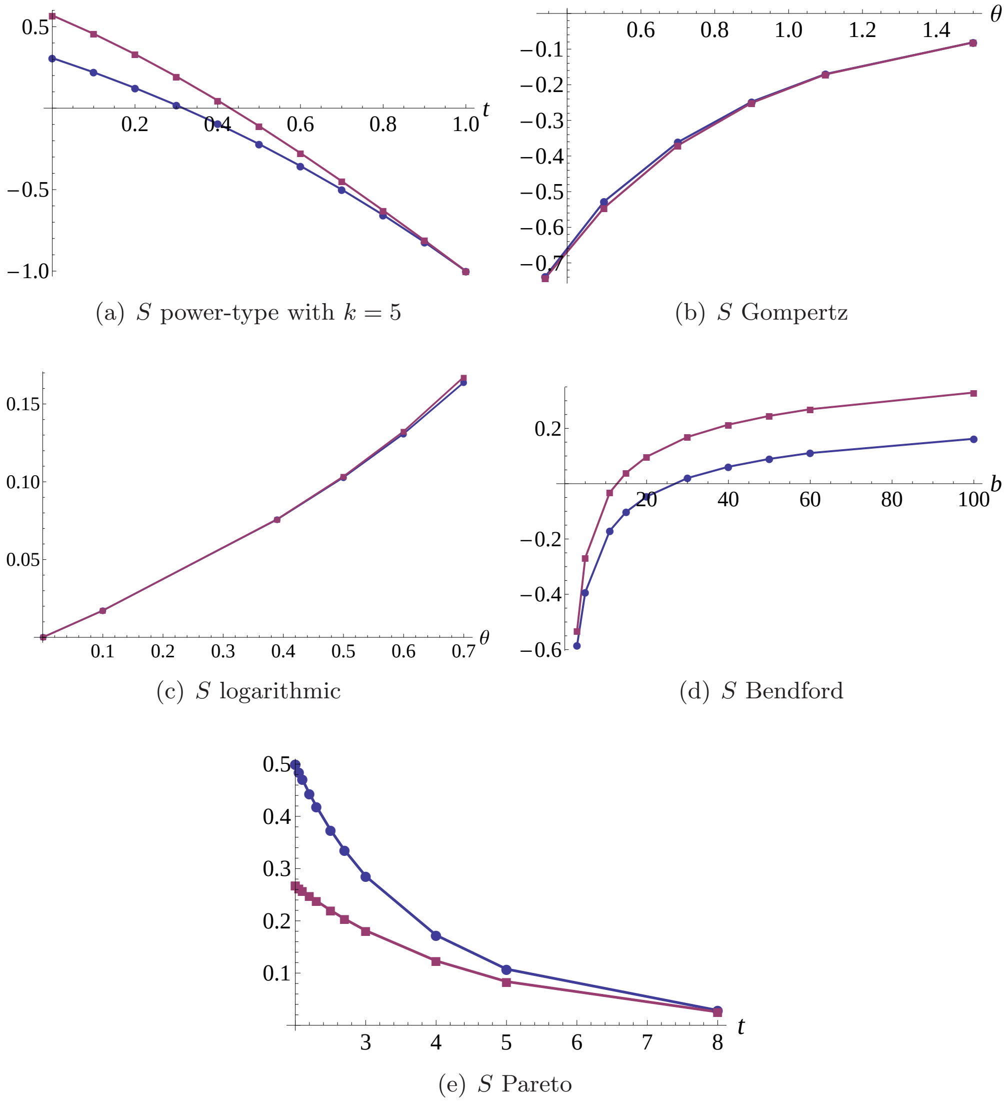
Inserting (6.12), (6.13), (6.14) in (6.11) then yields (6.10). $\diamond$

Let us examine the Schur-constant models, of dimension n, generated by the functions S of Section 5. First, for the Bernoulli case, (6.7) yields $\rho = -p/(1-p)$, regardless of n. From (6.9) and using (2.5), we also see that $\tau_b = \rho$. Now, for the stop-loss case, we have computed ρ from (6.7) and τ_b from (6.10) for several values of t and k. By Propositin 5.3, the Schur-constant model is here of dimension $n = t + 1$. Table 3 shows that the values of the two parameters are negative and increase with t (or n). We note that when $t = 1$, S reduces to the survival function of a uniform on $(0,1)$; it is then easily checked that $X_2 =_d k - X_1$, which explains the value -1 obtained for both coefficients.

Table 3: Coefficients ρ and τ_b when S is of stop-loss form.

t \ k		2	3	4	5	6	7
1	ρ	-1	-1	-1	-1	-1	-1
	τ_b	-1	-1	-1	-1	-1	-1
2	ρ	-0.333333	-0.421053	-0.454545	-0.470588	-0.479452	-0.484848
	τ_b	-0.333333	-0.391304	-0.395349	-0.391304	-0.386139	-0.381295
3	ρ	-0.142857	-0.250000	-0.286713	-0.303571	-0.312693	-0.318182
	τ_b	-0.142857	-0.245283	-0.253886	-0.250889	-0.246213	-0.241742
4	ρ	-0.066667	-0.166400	-0.202267	-0.219214	-0.228526	-0.234180
	τ_b	-0.066667	-0.167773	-0.185030	-0.185698	-0.182908	-0.179570
5	ρ	-0.032258	-0.114078	-0.149812	-0.167343	-0.177121	-0.183102
	τ_b	-0.032258	-0.115436	-0.141662	-0.146555	-0.145914	-0.143810
6	ρ	-0.015873	-0.078354	-0.113867	-0.131973	-0.142235	-0.148566
	τ_b	-0.015873	-0.079042	-0.110539	-0.119374	-0.120903	-0.120059
7	ρ	-0.007874	-0.053562	-0.087780	-0.106196	-0.116865	-0.123524
	τ_b	-0.007874	-0.053850	-0.086681	-0.098756	-0.102342	-0.102746

We have also considered the other functions S for generating bivariate Schur-contant models (i.e. with $n = 2$). As seen before, S is 2-monotone in the power-type case when $t \leq 1$, in the Gompertz case when $\theta \geq \theta_2 = 0.340983$ and in the logarithmic, Bendford and Pareto cases for any parameter value. Figure 1 gives ρ and τ_b in these different situations. We observe that the dependence can be positive or negative, and that the two parameters are often very close.



(a) S power-type with $k = 5$

(b) S Gompertz

(c) S logarithmic

(d) S Bendford

(e) S Pareto

Figure 1: Coefficients ρ (circles, blue line) and τ_b (squares, red line) for different S when $n = 2$.

7. Schur-constant interarrival models

In this Section, we are going to discuss three processes in insurance theory for which the claim interarrival periods form a Schur-constant model: a claim counting process, a random payment process and an insurance risk process, respectively.

7.1. Claim counting process

Let us introduce an associated counting process defined by

$$N(t) = \sum_{i=1}^{n} I(T_i \leq t), \quad t \in \mathbb{N}_0,$$

where, as before, $T_i = X_1 + \ldots + X_i$ and $\{X_1, \ldots, X_n\}$ is a Schur-constant model.

In an insurance context, suppose that a maximum number of n claims can arise in a portfolio. Let T_i denote the claim arrival time of the i-th claim. Then, $N(t)$ represents the total number of claims that occur until time t.

Proposition 7.1. *For $t \geq 0$,*

$$P[N(t) = k] = (-1)^k \, \Delta^k S(t+1) \binom{t+k}{k}, \quad 0 \leq k \leq n-1, \tag{7.1}$$

and $P[N(t) = n] = P(T_n \leq t)$ is obtained from (2.8). For $0 \leq t_1 \leq \ldots \leq t_k \leq t$,

$$P[T_1 = t_1, \ldots, T_k = t_k | N(t) = k] = 1/\binom{t+k}{k}, \quad 1 \leq k \leq n-1. \tag{7.2}$$

Proof. Clearly, $P[N(t) = 0] = P(X_1 > t) = S(t+1)$. For $1 \leq k \leq n-1$,

$$\begin{aligned} P[N(t) = k | T_1 = t_1, \ldots, T_k = t_k] &= P(T_{k+1} \geq t+1 | T_1 = t_1, \ldots, T_k = t_k) \\ &= P(X_{k+1} \geq t+1-t_k | X_1 = t_1, \ldots, X_k = t_k - t_{k-1}) \\ &= \Delta^k S(t+1)/\Delta^k S(t_k), \end{aligned} \tag{7.3}$$

by virtue of (2.5) and (2.6). Using (2.9), we then get

$$\begin{aligned} P[N(t) = k] &= \sum_{t_1 \leq \ldots \leq t_k \leq t} P[N(t) = k | T_1 = t_1, \ldots, T_k = t_k] \, P(T_1 = t_1, \ldots, T_k = t_k) \\ &= \sum_{t_1 \leq \ldots \leq t_k \leq t} [\Delta^k S(t+1)/\Delta^k S(t_k)] \, (-1)^k \, \Delta^k S(t_k) \\ &= (-1)^k \, \Delta^k S(t+1) \, A_k, \end{aligned}$$

where A_k counts the cases satisfying $t_1 \leq \ldots \leq t_k \leq t$. Since A_k is equal to $\binom{t+k}{k}$, (7.1) follows.

Applying Bayes' rule yields

$$\begin{aligned} P[T_1 = t_1, \ldots, T_k = t_k | N(t) = k] &= \frac{P[N(t) = k | T_1 = t_1, \ldots, T_k = t_k] \, P(T_1 = t_1, \ldots, T_k = t_k)}{P[N(t) = k]} \\ &= \frac{[\Delta^k S(t+1)/\Delta^k S(t_k)] \, (-1)^k \, \Delta^k S(t_k)}{(-1)^k \, \Delta^k S(t+1) \binom{t+k}{k}}, \end{aligned}$$

thanks to (2.9), (7.1) and (7.3), so that formula (7.2) follows. ⋄

Formula (7.2) means that given $N(t) = k$ with $k \, (\leq n-1)$, the arrival times of these k events are obtained by throwing k indistinguishable balls in $t+1$ urns (the instants $0, \ldots, t$). Note that when $k = n$, the probability in (7.3) is equal to 1 by definition; this case differs from the others, of course.

For the continuous model, formulas of this type are derived by Chi et al. (2009) in Lemma 7.1 and Theorem 2.4. In particular, $[T_1, \ldots, T_k | N(t) = k]$, $1 \leq k \leq n-1$, is then distributed as the order statistics of a sample of k independent $(0, t)$-uniform random variables.

Proposition 7.2. *In an infinite discrete Schur-constant model, $N(t)$ has a mixed negative binomial ($\mathcal{M}NB$) distribution, namely*

$$N(t) =_d \mathcal{M}NB[t+1, 1/(\Theta+1)], \tag{7.4}$$

where Θ is defined in (4.2).

Proof. By (4.1), $S(x) = E[(\Theta/(\Theta+1))^x]$ for an infinite Schur-constant model. Substituting this in (7.1) and since $\Delta q^x = -(1-q)q^x$, we then get

$$P[N(t)=k] = \binom{t+k}{k} E\left[\left(\frac{1}{\Theta+1}\right)^k \left(\frac{\Theta}{\Theta+1}\right)^{t+1}\right], \quad k \geq 0.$$

In other words, $N(t)$ has the mixed distribution stated in (7.4). $\diamond$

7.2. Random payment process

Much research is devoted to the evaluation of the present value of random payments at random times (e.g. Léveillé and Garrido (2001), Chi et al. (2009), Garrido et al. (2010), Woo and Cheung (2013)).

Here we consider a compound Schur-constant sum of discounted claims expressed as

$$R(t) = \sum_{i=1}^{N(t)} C_i \prod_{j=1}^{T_i} v_j = \sum_{i=1}^{n} I(T_i \leq t)\, C_i \prod_{j=1}^{T_i} v_j, \quad t \in \mathbb{N}_0,$$

where T_i represents the i-th payment time, C_i is the claim amount at that time and v_j ($\in (0,1]$) is a deterministic discount factor for the period $(j-1, j)$; of course, $\prod_{j=1}^{0} \equiv 1$. Here too, $T_i = X_1 + \ldots + X_i$ where $\{X_1, \ldots, X_n\}$ is a discrete Schur-constant model. The C_i's are assumed to be i.i.d. positive random variables, independent of the T_j's.

Our purpose is to determine the Laplace transform of $R(t)$, i.e. $L_{R(t)}(\lambda) = E\{\exp[-\lambda R(t)]\}$ with $\lambda > 0$. Let $L_C(\lambda)$ be the Laplace transform of C_i.

Proposition 7.3.

$$\begin{aligned} L_{R(t)}(\lambda) &= S(t+1) + \sum_{k=1}^{n-1} (-1)^k \Delta^k S(t+1) \sum_{0\leq t_1 \leq \ldots \leq t_k \leq t} \prod_{i=1}^{k} L_c(\lambda \prod_{j=1}^{t_i} v_j) + (-1)^n \Delta^n S(0)\, [L_C(\lambda)]^n \\ &+ \sum_{t_n=1}^{t} (-1)^n \Delta^n S(t_n)\, L_c(\lambda \prod_{j=1}^{t_n} v_j) \sum_{0\leq t_1 \leq \ldots \leq t_{n-1} \leq t_n} \prod_{i=1}^{n-1} L_c(\lambda \prod_{j=1}^{t_i} v_j). \end{aligned} \tag{7.5}$$

Proof. Evidently,

$$\begin{aligned} L_{R(t)}(\lambda) &= \sum_{k=0}^{n} E[e^{-\lambda R(t)} I(N(t)=k)] = P[N(t)=0] \\ &+ \sum_{k=1}^{n-1} P[N(t)=k]\, E[e^{-\lambda R(t)} | N(t)=k] + E[e^{-\lambda R(t)} I(N(t)=n)]. \end{aligned} \tag{7.6}$$

For the terms with $1 \leq k \leq n-1$ in the second sum of (7.6), we obtain, using (7.2),

$$\begin{aligned} E[e^{-\lambda R(t)} | N(t)=k] &= E[e^{-\lambda \sum_{i=1}^{k} C_i \prod_{j=1}^{T_i} v_j} | N(t)=k] \\ &= \frac{1}{\binom{t+k}{k}} \sum_{0\leq t_1 \leq \ldots \leq t_k \leq t} E(e^{-\lambda \sum_{i=1}^{k} C_i \prod_{j=1}^{t_i} v_j}) \\ &= \frac{1}{\binom{t+k}{k}} \sum_{0\leq t_1 \leq \ldots \leq t_k \leq t} \prod_{i=1}^{k} L_c(\lambda \prod_{j=1}^{t_i} v_j), \end{aligned} \tag{7.7}$$

since the C_i's are i.i.d. random variables.

For the last term in (7.6) where $k = n$, we have, since $[N(t) = n]$ means $(T_n \le t)$,

$$\begin{aligned} E[e^{-\lambda R(t)} I(N(t) = n)] &= P(T_n = 0)\, E(e^{-\lambda \sum_{i=1}^{n} C_i}) \\ &\quad + \sum_{t_n=1}^{t} P(T_n = t_n)\, E(e^{-\lambda \sum_{i=1}^{n} C_i \prod_{j=1}^{T_i} v_j} | T_n = t_n). \end{aligned} \tag{7.8}$$

Using (2.10) with $j = n$, we express the conditional expectation $E(\ldots)$ in (7.8) as

$$\begin{aligned} E(\ldots) &= E(e^{-\lambda C_n \prod_{j=1}^{t_n} v_j})\, E(e^{-\lambda \sum_{i=1}^{n-1} C_i \prod_{j=1}^{T_i} v_j} | T_n = t_n) \\ &= E(e^{-\lambda C_n \prod_{j=1}^{t_n} v_j})\, \frac{1}{\binom{t_n+n-1}{n-1}} \sum_{0 \le t_1 \le \ldots \le t_{n-1} \le t_n} E(e^{-\lambda \sum_{i=1}^{n-1} C_i \prod_{j=1}^{t_i} v_j}) \\ &= \frac{1}{\binom{t_n+n-1}{n-1}} L_c(\lambda \prod_{j=1}^{t_n} v_j) \sum_{0 \le t_1 \le \ldots \le t_{n-1} \le t_n} \prod_{i=1}^{n-1} L_c(\lambda \prod_{j=1}^{t_i} v_j). \end{aligned} \tag{7.9}$$

It remains to insert (7.7), (7.8) and (7.9) in (7.6) and then to use (7.1) for the p.m.f. of $N(t)$ and (2.8) with $j = n$ for the p.m.f. of T_n. $\diamond$

Example. Suppose that the claim amounts C_i are exponentially distributed with parameter 1. Since $L_C(\lambda) = 1/(1+\lambda)$, formula (7.5) gives

$$\begin{aligned} L_{R(t)}(\lambda) = S(t+1) &+ \sum_{k=1}^{n-1} (-1)^k\, \Delta^k S(t+1)\, V(k,t) + (-1)^n\, \Delta^n S(0) \left(\frac{1}{1+\lambda}\right)^n \\ &+ \sum_{t_n=1}^{t} (-1)^n\, \Delta^n S(t_n)\, V(t_n)\, V(n-1, t_n), \end{aligned}$$

where

$$V(t_i) = 1/(1 + \lambda \prod_{j=1}^{t_i} v_j), \quad 1 \le i \le n,$$

$$V(k,\tau) = \sum_{0 \le t_1 \le \ldots \le t_k \le \tau} \prod_{i=1}^{k} V(t_i), \quad 1 \le k \le n-1, \tau \in \mathbb{N}_0.$$

The claim interarrival periods $(X_1, \ldots, X_n)$ form a Schur-constant model. For illustration, we first consider the Bernoulli case of Section 5. Then, $t_i = 0$ or 1 for all i, which greatly simplifies the calculations. So, we easily obtain the following formula: for $t = 0$,

$$L_{R(0)}(\lambda) = p + \frac{p}{\lambda}\left(1 - \frac{1}{(1+\lambda)^{n-1}}\right) + \frac{1-np}{(1+\lambda)^n},$$

and for $t = 1$ (or $t \ge 1$),

$$L_{R(t)}(\lambda) = \frac{1-np}{(1+\lambda)^n} + \frac{p}{(1+\lambda)^{n-1}\lambda(1-v_1)}\left((\frac{1+\lambda}{1+\lambda v_1})^n - 1\right).$$

Table 4 gives $P[R(t) = 0]$ and several quantiles $R_\alpha(t)$ for different values of n when $p = 0.08$ and $v_1 = 0.95$. Note that, as expected, the quantiles increase with n and t.

Table 4: $P[R(t)=0]$ and $R_\alpha(t)$ with S of Bernoulli form when $p = 0.08$, $v_1 = 0.95$.

	n	2	3	4	5	6	7
	$P[R(0)=0]$	0.08	0.08	0.08	0.08	0.08	0.08
	$R_{0.50}(0)$	1.45497	2.24537	2.96722	3.60988	4.15995	4.60201
	$R_{0.95}(0)$	4.55266	5.96314	7.24594	8.43023	9.53011	10.5449
	$R_{0.99}(0)$	6.45200	8.08879	9.56817	10.9353	12.2084	13.3923
$t \geq 1$	$P[R(t)=0]$	0	0	0	0	0	0
	$R_{0.50}(t)$	1.66802	2.65219	3.63464	4.61386	5.58950	6.56139
	$R_{0.95}(t)$	4.71638	6.24742	7.67994	9.04840	10.3722	11.6609
	$R_{0.99}(t)$	6.60129	8.34365	9.95497	11.4758	12.9384	14.3540

Next, we consider a bivariate Schur-constant model ($n = 2$) generated by a stop-loss function S where $k = 4$ and $t = 1$ or 2. Figure 2 shows the distribution function of $R(t)$ when $v_j = 0.95$ for all j.

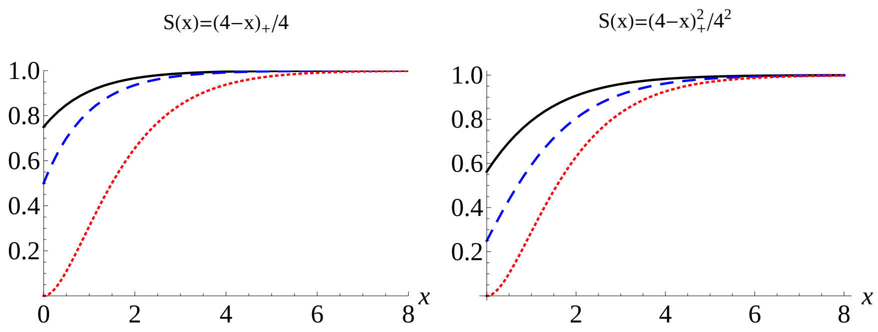


Figure 2: Distribution function of $R(t)$ with S of stop-loss form when $n = 2$ and all $v_j = 0.95$, for $t = 0$ (thick black line), $t = 1$ (dashed blue line), $t \geq 3$ (dotted red line).

7.3. Insurance risk process

A large number of works are devoted to the evaluation of the ruin probability for an insurance over a finite or infinite horizon (see e.g. the books by Seal (1978), Dickson (2005), Asmussen and Albrecher (2010)). Let us consider a discrete-time risk model in which claims occur according to a Schur-constant counting process $N(t)$. The successive claim amounts, C_i say, are independent of the claim arrival process (but may be interdependent); their partial sums are denoted by $A_i = C_1 + \ldots + C_i$, $i \geq 1$. The premium flow is deterministic (but may be nonstationary); the cumulated premiums until time t are given by the nondecreasing function $h(t)$ ($h(0) \geq 0$ being the initial reserves). Thus, the reserves process is written as

$$U(t) = h(t) - A_{N(t)}, \quad \text{where} \quad A_{N(t)} = \sum_{i=1}^{N(t)} C_i, \quad t \in \mathbb{N}_0.$$

Ruin occurs when the reserves $U(t)$ become negative, i.e. as soon as $A_{N(t)} > h(t)$. Let $\phi(t)$ be the probability of non-ruin until time t. We derive below a formula for computing $\phi(t)$.

Proposition 7.4.

$$\phi(t) = S(t+1) + \sum_{k=1}^{n-1} (-1)^k \, \Delta^k S(t+1) \sum_{0 \le t_1 \le \ldots \le t_k \le t} P[A_1 \le h(t_1), \ldots, A_k \le h(t_k)]$$
$$+ \sum_{t_n=0}^{t} (-1)^n \, \Delta^n S(t_n) \sum_{0 \le t_1 \le \ldots \le t_{n-1} \le t_n} P[A_1 \le h(t_1), \ldots, A_n \le h(t_n)]. \tag{7.10}$$

Proof. By definition, $\phi(t)$ can be expressed as

$$\begin{aligned} \phi(t) &= P[N(t)=0] + \sum_{k=1}^{n} P[\text{non-ruin until time } t, N(t)=k] \\ &= S(t+1) + \sum_{k=1}^{n-1} P[N(t)=k] \, P[A_1 \le h(T_1), \ldots, A_k \le h(T_k) | N(t)=k] \\ &\quad + P[A_1 \le h(T_1), \ldots, A_n \le h(T_n), N(t)=n]. \end{aligned} \tag{7.11}$$

For $1 \le k \le n-1$, we get from (5.2) that

$$P[A_1 \le h(T_1), \ldots, A_k \le h(T_k) | N(t)=k] = \frac{1}{\binom{t+k}{k}} \sum_{0 \le t_1 \le \ldots \le t_k \le t} P[A_1 \le h(t_1), \ldots, A_k \le h(t_k)]. \tag{7.12}$$

For $k=n$, we write

$$\begin{aligned} &P[A_1 \le h(T_1), \ldots, A_n \le h(T_n), N(t)=n] \\ &\quad = \sum_{t_n=0}^{t} P(T_n = t_n) \, P[A_1 \le h(T_1), \ldots, A_n \le h(T_n) | T_n = t_n], \end{aligned} \tag{7.13}$$

and by virtue of (2.10),

$$\begin{aligned} &P[A_1 \le h(T_1), \ldots, A_n \le h(T_n) | T_n = t_n] \\ &\quad = \frac{1}{\binom{t_n+n-1}{n-1}} \sum_{0 \le t_1 \le \ldots \le t_{n-1} \le t_n} P[A_1 \le h(t_1), \ldots, A_n \le h(t_n)]. \end{aligned} \tag{7.14}$$

Combining (7.11), (7.12), (7.13), (7.14) and using (5.1), (2.8), we then deduce formula (7.10). ◇

To apply (7.10), it remains to evaluate probabilities of the form $P[A_1 \le h(t_1), \ldots, A_k \le h(t_k)]$. Clearly, this can be cumbersome in practice, as for the traditional models.

Example. Suppose that the claim amounts C_i are exponentially distributed with parameter 1. Then, each A_k has an Erlang$(k,1)$ distribution, i.e.

$$P(A_k \le x) = 1 - \sum_{j=0}^{k-1} \frac{1}{j!} e^{-x} x^j, \quad x > 0. \tag{7.15}$$

For the claim interarrival periods, consider again a Schur-constant model $(X_1, \ldots, X_n)$ with S of Bernoulli type. From (7.10), we then obtain the following formula: for $t=0$,

$$\phi(0) = p + p \sum_{k=1}^{n-1} P[A_k \le h(0)] + (1-np) P[A_n \le h(0)],$$

and for $t = 1$ (or $t \geq 1$),

$$\phi(t) = (1 - np)P[A_n \leq h(0)] + pP[A_n \leq h(1)] + p\sum_{k=1}^{n-1} P[A_{n-k} \leq h(0), A_n \leq h(1)],$$

in which we get, after some calculations and using (7.15),

$$P[A_{n-k} \leq h(0), A_n \leq h(1)] = P[A_{n-k} \leq h(0)] - \frac{e^{-h(1)}}{(n-k-1)!}\sum_{j=0}^{k-1}\sum_{i=0}^{j} \frac{(-1)^{j-i}h(1)^i h(0)^{n-k+j-i}}{(n-k+j-i)!i!(j-i)!}.$$

This result is illustrated in Table 5 for different values of n when $p = 0.08$, $h(0) = 4$ and $h(1) = 8$.

Table 5: Probability $\phi(t)$ with S of Bernoulli form when $p = 0.08$, $h(0) = 4$, $h(1) = 8$.

n	2	3	4	5	6	7
$t = 0$	0.921609	0.810250	0.677401	0.560180	0.478908	0.433062
$t \geq 1$	0.921260	0.808184	0.669609	0.538648	0.431727	0.346796

We also reconsider a bivariate model with S of stop-loss type where $k = 4$ and $t = 1$ or 2. Figure 3 shows the probability $\phi(t)$ in function of t when $h(0) = 1$, $h(1) = 2$, $h(2) = 3$ and $h(3) = 4$.

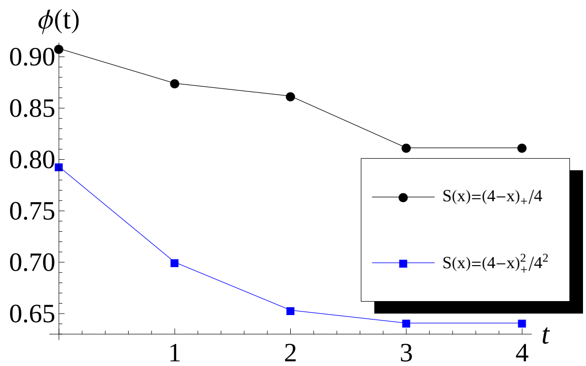


Figure 3: Probability $\phi(t)$ with S of stop-loss form when $n = 2$ and $h(0) = 1$, $h(1) = 2$, $h(2) = 3$, $h(3) = 4$.

8. Appendix

The coefficients $\alpha_i(t)$ (in Section 5.3)

We first derive the expansion stated in Lemma 5.3. The argument is inspired from the proof of Lemma 3.3 in Denuit et al. (2002).

Proof of Lemma 5.3. Observe that (5.5) is true for $t = 1$ with $\alpha_0(1) = 1$. Proceeding by induction, let us consider the case $t+1$, $t \geq 1$. Clearly,

$$\begin{aligned} \frac{(k-x)_+^{t+1}}{(t+1)!} &= \frac{(k-x)_+}{t+1}\frac{(k-x)_+^t}{t!} \\ &= \sum_{i=0}^{t-1} \alpha_i(t)\,\frac{k-x}{t+1}\binom{k-x+i}{t}, \end{aligned} \tag{8.1}$$

by assumption and since $k-x$ may be substituted to $(k-x)_+$ in the first equality. Now, we notice that $k-x$ can be rewritten as

$$k-x=\frac{t-i}{t+1}(k-x+i+1)+\frac{i+1}{t+1}(k-x+i-t),$$

so that

$$\frac{k-x}{t+1}\binom{k-x+i}{t}=\frac{t-i}{t+1}\binom{k-x+i+1}{t+1}+\frac{i+1}{t+1}\binom{k-x+i}{t+1}. \tag{8.2}$$

Inserting (8.2) in (8.1) (and changing the index i to $i+1$ in the first sum) yields

$$\begin{aligned}\frac{(k-x)_+^{t+1}}{(t+1)!} &= \sum_{i=1}^{t}\alpha_{i-1}(t)\frac{t+1-i}{t+1}\binom{k-x+i}{t+1}+\sum_{i=0}^{t-1}\alpha_i(t)\frac{i+1}{t+1}\binom{k-x+i}{t+1}\\ &= \sum_{i=0}^{t}\left[\alpha_{i-1}(t)\frac{t+1-i}{t+1}+\alpha_i(t)\frac{i+1}{t+1}\right]\binom{k-x+i}{t+1}, \end{aligned} \tag{8.3}$$

after putting $\alpha_{-1}(t)=0=\alpha_t(t)$. By (8.3), we thus see that the expansion (5.5) holds too for $t+1$ where the $\alpha_i(t+1)$'s correspond to the terms $[\ldots]$ above. In other words, the coefficients $\alpha_i(t)$ satisfy the recurrence (5.6). Again by induction, we get that the $\alpha_i(t)$'s are positive, of sum 1 and symmetric (i.e. $\alpha_i(t)=\alpha_{t-1-i}(t)$). $\diamond$

Table 6 gives the coefficients $\{\alpha_i(t),\, 0\le i\le t-1\}$ in (5.5) for the first values of t. Observe that, as indicated before, they form a symmetric p.m.f.

Table 6: Coefficients $\{\alpha_i(t)\}$ when $t=1,\ldots,7$.

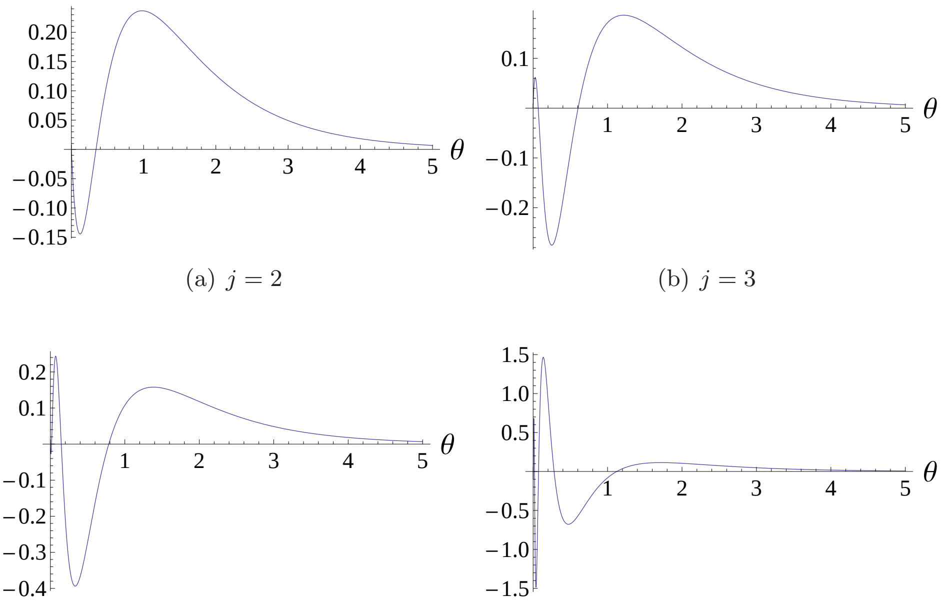
$t \quad \backslash \quad i$	0	1	2	3	4	5	6
1	1						
2	1/2	1/2					
3	1/3!	4/3!	1/3!				
4	1/4!	11/4!	11/4!	1/4!			
5	1/5!	26/5!	66/5!	26/5!	1/5!		
6	1/6!	57/6!	302/6!	302/6!	57/6!	1/6!	
7	1/7!	120/7!	1191/7!	2416/7!	1191/7!	120/7!	1/7!

It can be shown that $\alpha_i(t)$ is provided by the following explicit formula:

$$\begin{aligned}\alpha_i(t) \quad = \quad & \frac{1}{t!}\sum_{s_i=0}^{t-i-1}(i+1)^{s_i}\,(t-i-s_i)\sum_{s_{i-1}=0}^{t-i-1-s_i} i^{s_{i-1}}\,(t-i-s_i-s_{i-1})\\ & \sum_{s_{i-2}=0}^{t-i-1-s_i-s_{i-1}}(i-1)^{s_{i-2}}\,(t-i-s_i-s_{i-1}-s_{i-2})\ldots\\ & \sum_{s_1=0}^{t-i-1-s_i-s_{i-1}-\ldots-s_2} 2^{s_1}\,(t-i-s_i-s_{i-1}-\ldots-s_1)\,,\quad 1\le i\le t-1.\end{aligned}$$

The roots θ_j (in Section 5.4)

The functions $f_j(\theta)$, $j \geq 2$, introduced in (5.9) can be analyzed using Mathematica 8.0. In Figure 4 below, they are plotted for different values of j.



(a) $j = 2$ (b) $j = 3$

(c) $j = 4$ (d) $j = 7$

Figure 4: Functions $f_j(\theta)$ when $j = 2, 3, 4, 7$.

Observe that $f_j(\theta)$ has j real roots; they are given in Table 7. The largest root corresponds to θ_j defined in (5.9).

Table 7: Roots of $f_j(\theta)$ when $j = 2, 3, 4, 7$.

j							
2	0	$\theta_2 = 0.340983$					
3	0	0.068210	$\theta_3 = 0.603576$				
4	0	0.015599	0.146406	$\theta_4 = 0.783918$			
7	0	0.0002416	0.003448	0.017788	0.072867	0.281368	$\theta_7 = 1.1232$

We also notice that $f_j(\theta) > 0$ for $\theta > \theta_j$ and $\theta_{j+1} > \theta_j$. In fact, these properties are found to be true for all $j \geq 2$.

Future extension of the model

The Schur-constant property implies the exchangeability of the X_i's, and in particular the identity between the marginal distributions. This assumption may be restrictive or unrealistic in certain fields of applications. This is the case, for instance, in survival analysis for the study of risks in competition. In a forthcoming paper, we will develop a Schur-constant model that is rescaled to take into account the heterogeneity between the different risks.

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