Sources of Chaos in Celestial Mechanics Problems

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Abstract: The three body restricted problem is introduced. The integrability around certain Libration Points is studied and rejected, providing details and illustrations on the dynamics.

I. INTRODUCTION

The n-body problem in the plane $\mathbb{R}^2$ describes the motion of n points of positive masses $m_1, m_2, \ldots, m_n$ under their mutual gravitational attraction. We will show that, in general, there exist no fixed points of this system, but periodic solutions. Despite the unlikelihood of finding such solutions, many of them have been found by Carles Simó. In this article, we focus on the very simple case of three bodies, and next restricting to a very small third mass. Despite looking like a rather particular case, it already opens a large field of study for two main reasons:

1. It gives rise to the very famous Libration points, already known to Euler and Lagrange.
2. It encloses sources of chaos due to the non-integrable nature of its Hamiltonian.

We will give detail on the existence of such special points $L_1, \ldots, L_5$, and later discuss whether the stable and unstable invariant manifolds of L_3 do splice one with another, thus creating a separatrix around L_4, or else they are not tangent, giving rise to chaotic behaviour [3], destroying the orbits conjectured by Brown [2].

II. SETTING THE THREE BODY PROBLEM

If the coordinates of the i-th body are $\vec{r_i} = (x_i, y_i)$ and also note $\vec{r_{i,j}} = \vec{r_j} - \vec{r_i}$, $r_{i,j} = \|\vec{r_{i,j}}\|$, G the gravitational constant, the equations of motion, according to Newton's law, are

$$\ddot{\vec{r_i}} = \sum_{j=1, j\neq i}^{n} G m_j \vec{r_{i,j}} / r_{i,j}^3 \tag{1}$$

The total energy $E = \sum_{i=1}^{n} m_i (\dot{r}_i)^2/2 - G \sum_{1\leq i<j\leq n}^{n} m_i m_j / r_{i,j}$ is preserved. It is not restrictive to assume $\sum_{j=1}^{n} m_j = 1$ and that the gravitational constant G is equal to 1, using suitable units of time and distance and mass. Indeed, rescaling $t = \alpha T, r_j = \beta R_j, m_j = \gamma M_j$, with

$$\beta^3 = \alpha^2 G \gamma$$

, reminding of the *Kepler's Law*, G cancels in (1), and giving a fixed $\gamma = \sum_{j=1}^{n} m_j$ then $\sum_{j=1}^{n} M_j = \sum_{j=1}^{n} m_j/\gamma = 1$.

Let us show that the system has no fixed points, null velocities and accelerations. There do exist cases of null velocities at a given time, but always accelerating. We also restrict to systems without collisions $\vec{r_i} = \vec{r_j}$, for which the force is undefined. Consider the largest position horizontal coordinate x^*. The resulting forces from each body in $x = x^*$ will attract it leftwards, unless every point is on $x = x^*$, in which case we proceed analogously in the y coordinate.

III. EULER AND LAGRANGE SOLUTIONS

A. The Rotating Frame.

Now we consider the planar three body system in a rotating frame, for instance, with angular velocity equal to 1. That is, we use the synodic coordinates $\vec{s_i} = (u_i, v_i)$ instead of $\vec{r_i} = (x_i, y_i)$, defined by

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

Equivalently $\vec{r}(t) = R(t)\vec{s}(t)$. Motion equations become:

$$\ddot{\vec{r}} = \ddot{R}\vec{s} + 2\dot{R}\dot{\vec{s}} + R\ddot{\vec{s}} = \sum_{j=1, j\neq i}^{n} G m_j R \vec{s_{i,j}} / \|R\vec{s_{i,j}}\|^3$$

, but planar rotations like $R(t) \leftrightarrow e^{it}$, preserve norms, $\|R\vec{s_{i,j}}\| = \|R\|\, s_{i,j} = s_{i,j}$, and its derivative is a multiple of itself $\dot{R}(t) = R(t)R(\pi/2)$, the same rotation composed with a quarter revolution $R(\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Easily, $\ddot{R}(t) = R(t)R(\pi) = -R(t)$, so we can cancel $R(t)$ from $-R\vec{s_i} + 2RR(\pi/2)\dot{\vec{s_i}} + R\ddot{\vec{s_i}} = R \sum_{j=1, j\neq i}^{n} m_j \vec{s_{i,j}} / s_{i,j}^3$

, leading to

$$-\vec{s_i} + 2R(\pi/2)\dot{\vec{s_i}} + \ddot{\vec{s_i}} = \sum_{j=1, j\neq i}^{n} m_j \vec{s_{i,j}} / s_{i,j}^3$$

or in the two components (u_i, v_i),

$$\begin{pmatrix} \ddot{u_i} \\ \ddot{v_i} \end{pmatrix} + 2\begin{pmatrix} -\dot{v_i} \\ \dot{u_i} \end{pmatrix} - \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \sum_{j\neq i} m_j \begin{pmatrix} u_i \\ v_i \end{pmatrix} / s_{i,j}^3 \tag{2}$$

B. Libration Points in the Non Restricted case.

The new equations do have fixed points, called *relative equilibria*. We next show that in the case of 3 bodies and for any choice of the masses, there exist 5 *relative equilibria*, 3 of them collinear ($v_i = 0$) and other 2 triangular, as already known to Euler and Lagrange.

For now, we choose the reference system $\vec{s_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \vec{s_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{s_3} = \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}$ so the mass center is $\vec{s}_{CM} = \vec{s_1}m_1 + \vec{s_2}m_2 + \vec{s_3}m_3 = \begin{pmatrix} m_2 + m_3u_3 \\ m_3v_3 \end{pmatrix}$.

The equations of equilibria for the unknown $\vec{s_3}$ can be obtained by compensating the interaction forces on the third body with the centrifugal force $\alpha^2(\vec{s_3} - \vec{s_{CM}})$, α the angular velocity, rather than neglecting the velocities and accelerations in the equations from (2). The rotation shall be, naturally, around the mass center, explaining the shift by $\vec{s_{CM}}$ in

$$\vec{0} = \frac{\vec{s}_1 - \vec{s}_3}{s_{1,3}^3}m_1 + \frac{\vec{s}_2 - \vec{s}_3}{s_{2,3}^3}m_2 + (\vec{s}_3 - \vec{s}_{CM})\alpha^2$$

Taking again $\alpha = 1$,

$$\vec{0} = -\begin{pmatrix} u_3 \\ v_3 \end{pmatrix}\frac{m_1}{s_{1,3}^3} + \begin{pmatrix} 1-u_3 \\ -v_3 \end{pmatrix}\frac{m_2}{s_{2,3}^3} + \begin{pmatrix} u_3 - m_2 - m_3u_3 \\ v_3 - m_3v_3 \end{pmatrix}$$

For the second component

$$0 = v_3(-\frac{m_1}{s_{1,3}^3} - \frac{m_2}{s_{2,3}^3} + 1 - m_3) = vB$$

1. In the case $v \neq 0$, the non-collinear, we have $B = 0$. But now, for the first component, regrouping terms, $0 = u_3B + \frac{m_2}{s_{2,3}^3} - m_2 \Rightarrow s_{2,3} = 1$
   Also, $0 = B = -\frac{m_1}{s_{1,3}^3} - m_2 + 1 - m_3 \Rightarrow s_{1,3} = 1$.
   The non-collinear solutions are upon the vertex's of equilateral triangles o side 1. $L_{4,5} = (1/2, \pm\sqrt{3}/2)$.
2. The case $v = 0$, collinear, has the quintic equation
   $$0 = -\frac{1}{u_3^2}m_1 + \frac{1}{(1-u_3)^2}m_2 + u_3 - m_2 - m_3u_3$$
   It is straightforward to see by *Bolzano's Theorem* that there exist 3 solutions in the horizontal axis.

C. Restricted Case

Using again the rotating frame, consider the case in which the third mass m_3, is negligible. This is the so-called restricted three-body problem, valid for systems like the Sun-Jupiter-asteroid system, or Sun-Earth-Moon. Usually one puts $m_1 = 1 - \mu$ (the large mass, e.g., the Sun) and $m_2 = \mu$ (the small mass, e.g. Jupiter). Both of them act on the third body, but the effect of this one on the two massive bodies (also known as primaries) can be neglected. Assume that the primaries are in relative equilibrium, therefore obeying the Kepler laws. By writing $u_1 = \mu$, $v_1 = 0$ and $u_2 = \mu - 1, v_2 = 0$ we place the origin at the mass center.

	Sun-Jupiter-Asteroid	Earth-Moon-Spacecraft
μ	$\approx 0.953881180363097 \times 10^{-3}$	≈ 0.0121506683

TABLE I: Important reduced mass parameters.

IV. A HAMILTONIAN FORMULATION

A. Synodic coordinates

Keeping notation with the above, we shall derive a Hamiltonian to better understand and describe the system for the Restricted Three Body Problem. The Lagrangian for the third mass position $\vec{R} = (X, Y)$ in an inertial frame is simply:

$$\mathcal{L}_{inertial}(t, X, Y, \dot{X}, \dot{Y}) = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{1-\mu}{R_1} + \frac{\mu}{R_2}$$

A change to synodic variables $\vec{r} = (x, y)$ by a rotation $R(t)$: $\vec{R} = R(t)\vec{r}$, and $\dot{\vec{R}} = R(t)(\dot{\vec{r}} + R(\pi/2)\vec{r})$ transforms

$$\mathcal{L}_{rotating}(x, y, \dot{x}, \dot{y}) = \frac{(\dot{x} - y)^2 + (\dot{y} + x)^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

The conjugated momenta are
$p_x = \dfrac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} - y$ and $p_y = \dfrac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y} + x$.
The Hamiltonian is thus

$$\begin{aligned}\mathcal{H} &= \dot{x}p_x + \dot{y}p_y - \mathcal{L} \\ &= (p_x + y)p_x + (p_y - x)p_y - \frac{p_x^2 + p_y^2}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \\ &= \frac{p_x^2 + p_y^2}{2} + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}\end{aligned}$$

or also remarkably, if completing squares,

$$\mathcal{H} = \frac{(p_x + y)^2 + (p_y - x)^2}{2} - \frac{x^2 + y^2}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}$$

, which allows to define the effective potential energy $U_{eff} = -\left(\dfrac{x^2 + y^2}{2} + \dfrac{1-\mu}{r_1} + \dfrac{\mu}{r_2}\right)$, plotted in figure IV A. A study of the linear stability is achievable solely from this function, as illustrates the figure, where the Libration Points coincide with local extrema, also confirming its stability nature. We may also observe the sphere of influence of each primary.

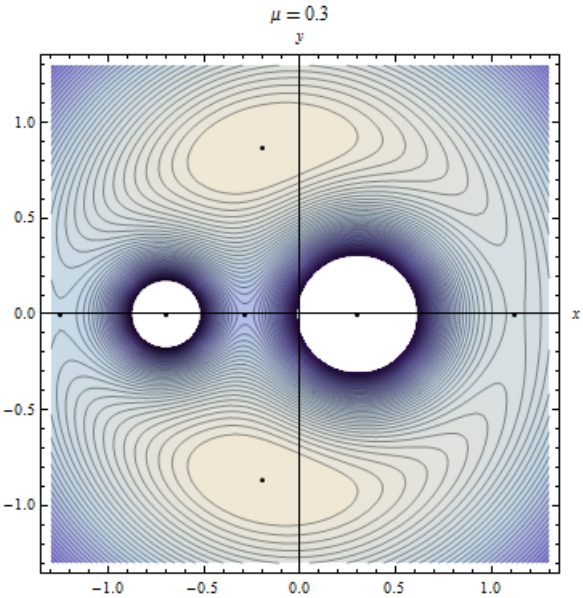

FIG. 1: Libration Points at effective potential local extrema.

Note how the effective potential energy is nothing else than $H - T_{eff}$, $T_{eff} = \dfrac{\dot{x}^2 + \dot{y}^2}{2}$, since Hamiltonian equations dictate

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \tag{3}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \tag{4}$$

$$\dot{p_x} = -\frac{\partial H}{\partial x} = p_y + \frac{(1-\mu)(x-\mu)}{r_1} + \frac{(\mu(x+1-\mu)}{r_2} \tag{5}$$

$$\dot{p_y} = -\frac{\partial H}{\partial y} = p_x + \frac{(1-\mu)y}{r_1} + \frac{y}{r_2} \tag{6}$$

The linear part of the flow is J times the Hamiltonian Hessian, since the field is $\dot{Z} = f(Z) = J\nabla_Z H$

$$J\nabla_{ZZ}H = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ M'-M & M' & 0 & 1 \\ M' & M'-M & -1 & 0 \end{pmatrix}, \quad \text{with}$$

$$\begin{cases} M = \dfrac{1-\mu}{r_1^3} + \dfrac{\mu}{r_2^3} \\ M' = 3\left(\dfrac{(1-\mu)(x-\mu)^2}{r_1^5} + \dfrac{\mu(x+1-\mu)^2}{r_2^5}\right) \end{cases}$$

B. Polar Synodic coordinates

The symmetry of the Hamiltonian suggests a change to polar coordinates $(x, y) = \rho(\cos\delta, \sin\delta)$, but we need to preserve the symplectic structure, that is, introduce a change $C : (\delta, \rho, p_\delta, p_\rho) \mapsto (x, y, p_x, p_y)$ with symplectic differential matrix $D = DC = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$, with $A_{11} = \begin{pmatrix} -\rho\sin\delta & \cos\delta \\ \rho\cos\delta & \sin\delta \end{pmatrix}$, that is, we impose $D^T J D = J$:

$$\begin{pmatrix} A_{11}^T A_{21} - A_{21}^T A_{11} & A_{11}^T A_{22} \\ -A_{22}^T A_{11} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

We are free to take $A_{21} = (0)$ (decorrelating positions and momenta) and we then simply require $A_{22} = A_{11}^{-T} = \begin{pmatrix} -\dfrac{\sin\delta}{\rho} & cos\delta \\ \dfrac{\cos\delta}{\rho} & \sin\delta \end{pmatrix}$. So $C = \begin{pmatrix} \rho\cos\delta \\ \rho\sin\delta \\ A_{11}^{-T}\begin{pmatrix} p_\delta \\ p_\rho \end{pmatrix} \end{pmatrix}$, in other words, we substitute the momenta

$$p_x = -\frac{\sin\delta}{\rho}p_\delta + \cos\delta p_\rho \text{ and } p_y = \frac{\cos\delta}{\rho}p_\delta + \sin\delta p_\rho$$

Then, the parts of the Hamiltonian simplify

$$p_x^2 + p_y^2 = p_\delta^2/\rho + p_\rho^2 \text{ and } yp_x - xp_y = -p_\delta$$

and the the Hamiltonian canonically transforms to

$$\mathcal{H} = \frac{(p_\delta/\rho)^2 + p_\rho^2}{2} - p_\delta - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}. \tag{7}$$

V. ON BROWN'S CONJECTURE

A. Invariant Manifolds

We are interested in the dynamics around the point $L_3 = (x_{L_3}, 0)$, found as the only real solution of the equation compensating gravitational and centrifugal forces

$$\frac{1-\mu}{(x-\mu)^2} + \frac{\mu}{(x+1-\mu)^2} = x \tag{8}$$

From the linear part of the Hamiltonian flow we find that L_3 has eigenvalues $\pm iw \to \pm i$ (center) and $\pm\lambda \approx \pm\sqrt{\dfrac{21\mu}{8}}$ (saddle). The corresponding eigenvectors in the neighbourhood of L_3 in equilibrium as a point in the 4-dimensional phase space form two directions $v_\pm$ perpendicular to oscillations like $e^{\pm iw}$ and two directions $v_{\pm\lambda}$ of ejection/injection at rate $e^{\pm\lambda}$. Continuation of these directions gives the unstable/stable manifolds W^U, W^S, respectively, the points that eventually fall to L_3 backward/forward in time. Integrating back in time an autonomous system $\dfrac{dz}{dt} = f(z)$ is as simple as switching the sign of the field $\tau = -t \Rightarrow \dfrac{dz}{d\tau} = -f(z)$. If one numerically computes W^S, W^U for small μ, as *Brown* conjectured, they seem to coincide: a particle initially in equilibrium in L_3, under a tiny perturbation to the unstable manifold would be ejected from L_3 following an orbit close to $\rho \approx 1$, make one loop around the triangular point L_4 and re-inject into L_3 along the stable manifold. That is, a loop from L_3 to itself preserving energy: a separatrix forming an eight shape with L_3 in the crossing. This is the typical structure of integrable Hamiltonian systems, illustrated in figure V A.

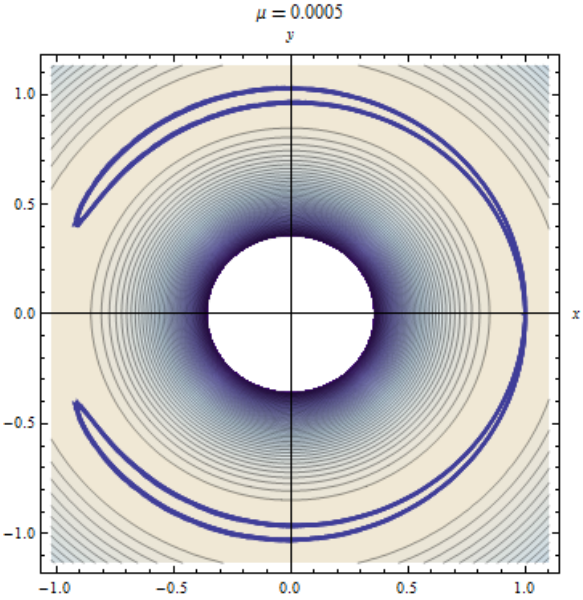

FIG. 2: Apparent separatrix. After many loops, manifolds remain concatenating, behaving like a stable periodical orbit.

However, for larger μ, we observe a lack of coincidence (figure V A) after the first return.

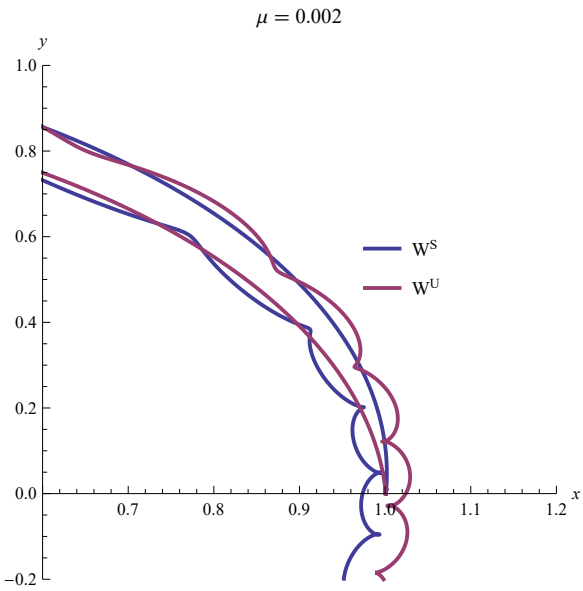

FIG. 3: Lack of coincidence after one return to L_3.

B. Chaos

Observe how in the limit case $\mu \to 0$, the manifolds $W^U, W^S \to \{(\delta, r) \mid \delta \in [0, \delta_{max}], r \approx 1\}$. However, if the perturbation, parametrised by μ, is significant, the return of the unstable manifold passes close to the stable, but already under the influence of the oscillating centre manifold, magnifying the effect of the perturbation. These oscillations will place the returning particle at either side, in (x, y), of W^S. Note first that each eigenvector $v_{\pm\lambda}$ is defined up to sign, and so the crossing of W^U, W^S in L_3 has two pairs of opposed segments exiting L_3. If we denote each semi-manifold $W_\pm = W \cap \{\pm y > 0\}$, then the returning particle, Z_*, passing close to L_3,

- If Z_* lies between W^S_+ and W^U_+, it will be driven again to W^U_+ upwards
- If Z_* lies between W^S_+ and W^U_-, it will now be pushed to W^U_- downwards.

These two radically different behaviours parting from close initial conditions give rise to chaos [3]. The Hamiltonian system is therefore non-integrable. In the next section we will quantify how significant is this lack of integrability.

C. Quantifying the splitting

We take the Poincaré section, a surface transversal to the flow, of $S = \{r(Z)^2 = x^2 + y^2 = 1\}$. We integrate the field $\dot{Z} = f(Z)$ with a Runge-Kutta method forward and backward in time along W^S, W^U and halt when the sign of $\psi(Z) := r(Z)^2 - 1$ changes. From the point of the last two ones closer to S, we solve the equation $\psi(Z(t)) = 0$ by Newton's method: $Z_n - = f(Z_n)\dfrac{\dot{\psi}}{\psi}$, where $\dot{\psi} = 2(x\dot{x} + y\dot{y}) = 2(xp_x + yp_y)$, until convergence. We measure the splitting as the norm in (x, y) plane from $W^U \cap S$ to $W^S \cap S$. We can observe that it decreases with μ in such a way that the $-|\lambda_(\mu)|log(Split)(\mu) \to c$, in other words, that $Split(\mu) \sim exp(-c/\lambda(\mu))$, with $\lambda(\mu) \approx \sqrt{\dfrac{21\mu}{8}}$. This is in perfect accordance with Melnikov theory of splitting in non-integrable problems showing practical stability [1].

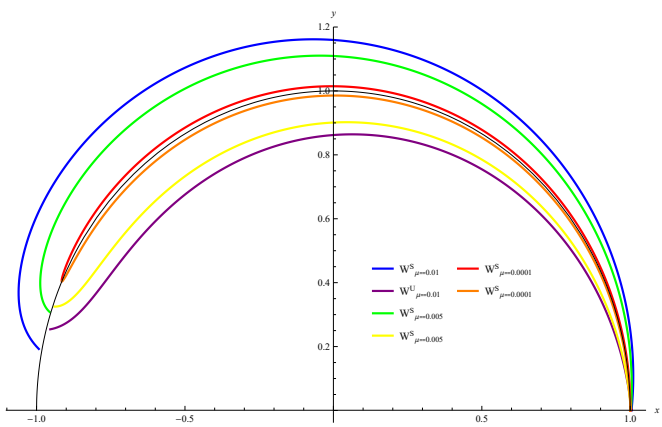

FIG. 4: Illustration of the Splitting.

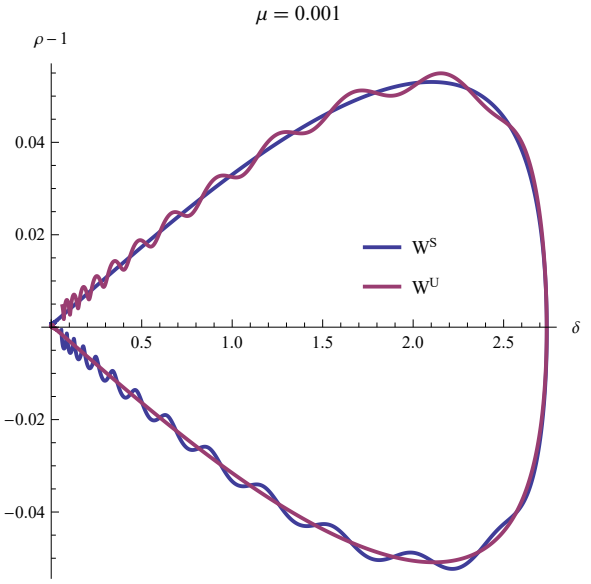

FIG. 5: Separatrix in polar canonical synodic coordinates.

The argument of the point coincidence converges to $2.724\ldots rad$ for $\mu = 10^{-5}$, which coincides reasonably with the expected value $2\arccos\left(\frac{\sqrt{2}-1}{2}\right)$, to be explained. Revisit $H(\delta, \rho, p_\delta, p_\rho)$ from equation (7), and introduce the change ($\delta^* = \delta, w = \rho - 1, p_\delta^* = p_\delta - 1, p_\rho = p_w$) to compensate for the radius $\rho \approx 1$, not giving information.

$H(\delta, w, p_\delta^*, p_w) = -1 + \frac{(p_\delta^*+1)^2}{2(w+1)^2} - p_\delta^* + \frac{p_w^2}{2} + \frac{\mu}{\sqrt{\mu^2 - 2\mu(w+1)\cos(\delta) + (w+1)^2}} - \frac{1+\mu}{\sqrt{\mu^2 - 2\mu(w+1)\cos(\delta) + (w+1)^2}}$

By taking $\epsilon = \sqrt{\mu}$, numerical computations show $w \sim \epsilon, p_\delta^* \sim \epsilon, p_w \sim \epsilon^2$. After using a Taylor expansion in μ of H and selecting first order terms according to the previous, we obtain

$$H = -\frac{3}{2} + \frac{(p_\delta^*)^2}{2} - \mu\left(1 - \cos(\delta) - \frac{1}{\sqrt{2(1+\cos(\delta))}}\right)$$

Adding a suitable constant and switching to $2\cos(\delta/2) = \sqrt{2(1+\cos(\delta))}$, the integrable part of H is

$$H = \frac{p_{\delta^*}^2}{2} - \mu g(\delta) \tag{9}$$

where $g(\delta) = \left(\frac{3}{2} - \cos(\delta) - \frac{1}{2\cos(\delta/2)}\right)$, such that $g(0) = 0$. The two relevant Hamilton equations are

$$\dot{\delta} = \frac{\partial H}{\partial p_{\delta^*}} = p_{\delta^*}, \quad \dot{p_{\delta^*}} = -\frac{\partial H}{\partial \delta} = \mu\left(\sin(\delta) - \frac{\sin(\delta/2)}{(2\cos(\delta/2))^2}\right)$$

Since L_3 ($\delta = 0$) and the Poincaré intersection ($\delta = \delta_{max}$) are extrema of argument δ, then $\dot{\delta} = p_{\delta^*} = 0$. That means the orbit lays on $H = 0$, and thus $g(\delta_{max}) = 0$, which happens for $\delta = 0, \frac{\sqrt{2}-1}{2}$. Observe the similarity of the Hamiltonian equations with those of a classical pendulum $\dot{x} = p_x, \dot{p_x} = -\sin(x)$. Shifting angle π to have the saddle at the origin (not an elliptic point), $\dot{p_x} = sin(x)$. One can transform $H = p_x^2/2 + \mu(\cos x - 1)$ into a classical pendulum $H = p_x^2/2 + \cos x - 1$ just by rescaling time and p_x by $\sqrt{\mu}$. In fact if we plot the orbits in the polar coordinates, figure V C, we observe the classical separatrix of a pendulum plus an oscillating perturbation, with considerable amplitude $\sqrt{\mu}$. This averages well in time and confines the splitting to $e^{-C/\sqrt{\mu}}$. Finally, H has a critical point $H_{p_{\delta^*}} = H_\delta = 0$ at $\delta = \pm 2\pi/3$, (L_4, L_5), around which there exist families of periodical orbits corresponding to its center part. Brown conjectured that such a family of orbits would extend until the separatrix, however, the separatrix doesn't actually exist due to splitting, and so this family of orbits is destroyed at a certain distance from L_3.

VI. CONCLUSIONS

- The conjecture by Brown is rejected numerically, and the results contrasted with theory.
- A quantification of the splitting is provided and modelled. For small perturbation parameters μ, the instability of certain orbits is unnoticeable, a fact that mislead astronomer Brown.

Acknowledgments

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[1] P. Sousa-Silva C. Simó and M. Terra. Practical stability domains near $l_{4,5}$ in the restricted three-body problem: Some preliminary facts. In *Progress and Challenges in Dynamical Systems.*

[2] Jacques Henrard. On brown's conjecture. In *Celestial Mechanics.*

[3] C. Simó. Effective computations in celestial mechanics and astrodynamics. In *Modern Methods of Analytical Mechanics and their Applications.*