QUANTUM CORRELATIONS IN ULTRACOLD ROTATING TWO-DIMENSIONAL GASES

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Abstract: We consider an ultracold atomic cloud subjected to an artificial magnetic field. The synthetic field is produced by rotating the atomic gas. We characterize the ground state. In particular we compute the density and the pair-correlation of the atoms. As we increase the rotation the system goes from a condensed many-body state of zero angular momentum to the Laughlin state for bosons for large rotations. The latter is a strongly correlated many-body state with zero interaction energy.

I. INTRODUCTION

In the field of ultracold atomic gases, a particularly interesting setup is that of rapidly rotating Bose-Einstein condensates. The manifestation of quantum vortices in the ultracold clouds as a result of the rotation [1] has been found to have a straight analogy with the fractional quantum Hall effect which considers the emergence of correlated phases between the ground states in two-dimensional charged systems in interaction when undergoing the effects of a perpendicular magnetic field [2].

The equivalence of the two physical descriptions relies in the direct homology between the Coriolis contribution to the force in the co-rotating frame of reference and the force acting on the charged particles, driven by a magnetic field perpendicular to the plane of rotation, which we discuss in Section II. Thus, setting the atomic cloud in rotation can be viewed as generating an artificial perpendicular magnetic field on the atomic gas. This analogy allows us to obtain the Landau quantization of the energy, and specifically, of the eigenstates of the system [1, 3].

In Section III we describe our theoretical framework. First we add the contact interaction and explain how we construct the many-body Fock basis to diagonalize the interacting Hamiltonian. We also introduce the Laughlin wave-functions for bosons, which are known to be exact eigenstates of the Hamiltonian for large rotations.

In Section IV we give our results. Essentially, we study the properties of the ground state of the system as we increase the rotation for a fixed value of the interaction. Finally, in Section V we revise the main conclusions of our work.

II. ARTIFICIAL MAGNETIC FIELD FOR ULTRACOLD ATOMS

In this first section we will introduce the main physics of rotating Bose-Einstein condensates. In Sec. II A we describe the effect of a magnetic field on an electric charge, which we relate after that, in Sec. II B, with the dynamics of a gas in rotation.

A. Charged 2D particle on a magnetic field

Let us take on the following problem: consider a charged particle in the (x, y) plane and a uniform perpendicular magnetic field, $\vec{B} = B\hat{e}_z$. The Hamiltonian can be written as:

$$H_B = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} \,, \tag{1}$$

with m the mass, e the particle charge and c the speed of light. The magnetic field is expressed through the vector potential $\vec{A}$, satisfying $\vec{B} = \vec{\nabla} \times \vec{A}$, allowing us a certain freedom in the choice of $\vec{A}$. For the main purpose of this study, we are interested in maintaining rotational invariance, thus we consider the so-called symmetric gauge:

$$\vec{A} = (A_x, A_y) = \frac{B}{2}(-y, x) \,, \tag{2}$$

which is invariant with respect to rotations around the z axis. With this, the Hamiltonian reads now:

$$H_B = \frac{1}{2m}\left((p_x + \frac{eB}{2c}y)^2 + (p_y - \frac{eB}{2c}x)^2\right) . \tag{3}$$

The two quadratic terms suggest a non-commuting nature, and indeed, defining $\Pi_x \equiv p_x + \frac{eB}{2c}y, \Pi_y \equiv p_y - \frac{eB}{2c}x$, it can be verified that $[\Pi_x, \Pi_y] = i\ \frac{eB\hbar}{c}$ [4]. This looks very similar to the harmonic oscillator, and certainly, if we construct the following creation and annihilation operators:

$$\begin{aligned} a &= \sqrt{\frac{c}{2e\hbar B}}\,(\Pi_x + i\ \Pi_y) \\ a^\dagger &= \sqrt{\frac{c}{2e\hbar B}}\,(\Pi_x - i\ \Pi_y) \ , \end{aligned} \tag{4}$$

we easily obtain:

$$H_B = \hbar\omega\left(a^+ a + \frac{1}{2}\right) . \tag{5}$$

This equation corresponds to the energy levels of a quantum harmonic oscillator. Here we have defined $\omega \equiv \frac{eB}{mc}$, which is no other but the well-known classical cyclotron frequency. This means that the energy spectrum of a charged particle in 2D subjected to a perpendicular magnetic field is that same as the spectrum of a single body in a 1D harmonic oscillator of the cyclotron frequency. The energy of the particle is quantized in units of the cyclotron energy.

B. Rotating trapped condensates

Consider now a Hamiltonian of the form:

$$H_{\rm sp} = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2) - \Omega\ell_z\,. \qquad (6)$$

This can be seen as the single-particle Hamiltonian in the co-rotating frame of reference of an isotropic 2D trap in rotation with frequency Ω around the z axis. This is reflected in the last term, where ℓ_z is the z component of the angular momentum, $\ell_z = (\vec{r}\times\vec{p})_z = xp_y - yp_x$. We can rewrite Eq. (6) in a curious form [1]:

$$H_{\rm sp} = \frac{|\vec{p} - m\vec{\Omega}\times\vec{r}|^2}{2m} + \frac{1}{2}m(x^2+y^2)(\omega^2-\Omega^2)\,. \qquad (7)$$

This last expression looks exactly like the Hamiltonian of a particle subjected to a gauge potential of the form $\vec{A} = \vec{\Omega}\times\vec{r} = \Omega(-y,x,0)$, with $\vec{\Omega} = \Omega\hat{z}$. In this case, the effective magnetic field would be $q\vec{B} = q\vec{\nabla}\times\vec{A} = 2m\vec{\Omega}$. Thus, as seen in Sec. II A, with this Hamiltonian we are mimicking the physics of a charged particle moving in 2D, subjected to a perpendicular magnetic field related to the amount of rotation.

The energy spectrum of the particle under these conditions can be written as [5]:

$$E = \hbar\omega(n_+ + n_- + 1) - (n_+ - n_-)\Omega\hbar\,, \qquad (8)$$

with $n_-, n_+ = 0,1,2,...,\infty$. For a proper treatment of the spectrum let's define:

$$n \equiv n_+ + n_- \quad \text{and} \quad k \equiv n_+ - n_-\,, \qquad (9)$$

which satisfy $k = -n, -n+1, -n+2, ..., n-2, n-1, n$ and $n = 0,1,2,...,\infty$. The energy reads now:

$$E = \hbar\omega + n\hbar\omega - k\Omega\hbar\,. \qquad (10)$$

Here n and k relate to a quantum description of the system, since these are quantum numbers that represent the energy and the momentum levels, respectively, whereas the single term $\hbar\omega$ is the "zero-point energy". The excitation heights are often called Landau Levels and are given by $n-k$ [3], which means a dependence in terms of n_-, and on the other hand it also implies an energy separation gap of $\hbar\omega$ between the different consecutive levels. Within this description, we're interested in studying the lowest eigenstates of the single-particle Hamiltonian, so-called Lowest Landau Levels (LLL), corresponding to $n_- = 0$ for all atoms, and hence $k = n = n_+$. That's accomplishable when $\hbar\omega$ is much larger than the interaction energy between the particles. LLL are given by the Fock-Darwin states, which, if we switch to cylindrical coordinates, $r = \sqrt{x^2+y^2}$ and $\varphi = \tan\frac{y}{x}$, read [1, 6]:

$$\psi_k(r,\varphi) = \frac{1}{\sqrt{\pi k!}}\frac{r^k e^{ik\varphi}}{a_\perp^{k+1}} e^{\frac{-r^2}{2a_\perp^2}}\,, \qquad (11)$$

with $a_\perp = \sqrt{\frac{\hbar}{m\omega}}$, the natural length scale of the harmonic potential, and k the quantum angular momentum:

$$\ell_z\psi_k = -i\hbar\frac{\partial}{\partial\varphi}\psi_k = (-i\hbar)(ik)\,\psi_k = k\hbar\,\psi_k\,. \qquad (12)$$

By rewriting the Hamiltonian in cylindrical coordinates too, $H_{\rm sp} = \frac{-\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\varphi^2}\right] + \frac{1}{2}m\omega^2r^2 + \Omega i\hbar\frac{\partial}{\partial\varphi}\,,$ we recover an already expected result for the energy:

$$H_{\rm sp}\psi_k(r,\varphi) = [\hbar\omega + k\hbar(\omega-\Omega)]\,\psi_k(r,\varphi) \equiv \epsilon_k\psi_k(r,\varphi)\,. \qquad (13)$$

The interesting feature about this single-particle spectrum is the following: for small or moderate Ω, the energy spectrum is clearly reminiscent of the original 2D harmonic oscillator, see Fig. 1. On the other hand, for $\Omega > \omega$, the system becomes unstable and the particle escapes the confinement. That is why, as we can see, if we increase the rotation enough but not too much, such that $\Omega \simeq \omega$, we have a range in which all the single-particle states become nearly-degenerated. In the case of $\omega = \Omega$ each band (in green in the figure) becomes completely degenerated.

III. THEORETICAL FRAMEWORK

In the previous section we worked with only one particle, but now we need to find a general solution for a many-body system that includes interaction. A good way to take in account the interaction which is valid in most ultracold gases is by adding a pair-wise repulsive potential of the form:

$$V_{\rm int} = \frac{g}{2}\sum_{i,j}\delta^{(2)}(z_i - z_j)\,, \qquad (14)$$

with $z = x + iy$ a variable in the complex plane and g is the interaction strength proportional to the s-wave scattering length. Note that, from now on we will write all magnitudes in units of the harmonic oscillator. Energies, distances and frequencies will be given in all cases in units of $\hbar\omega$, $a_\perp$ and ω, respectively. Thus, in Eq. (14), the coordinates are dimensionless and the parameter measuring the interaction g is also dimensionless. Hence, the many-body Hamiltonian becomes:

$$\mathcal{H} = \sum_i H_{\rm sp}(z_i) + \frac{g}{2}\sum_{i,j}\delta^{(2)}(z_i - z_j)\,. \qquad (15)$$

In Sec. III A we translate to Fock's space to treat the problem from the point of view of the second quantization and after that, in Sec. III B, we introduce the Laughlin approach, useful to define the ground state of the system.

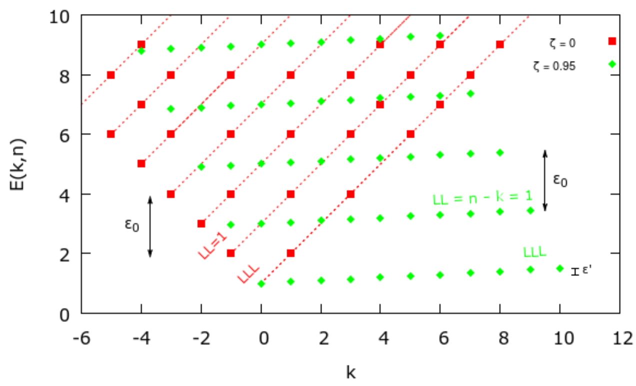

FIG. 1: Energy spectrum in harmonic oscillator units, of a particle given by Eq. (10), for two different values of the rotational frequency, with $\zeta \equiv \frac{\Omega}{\omega}$. Here $\varepsilon_0 = \hbar\omega$ indicates the separation between the excitation energy levels, while $\varepsilon' = \hbar(\omega - \Omega)$ represents the energy gap between two consecutive states, which, for $\Omega \simeq \omega$ results in a compression of the possible up to infinite nearly-degenerated states in the same excitation level. For $\Omega = \omega$, ε' vanishes, leading to an absolute degeneration for each excitation level. In both scenarios there is degeneration of the energy levels, we can see that for each LL, the different energy values match for a given quantum number k. [To avoid confusion due to the superposition of the states, we have omitted some of them for $\zeta = 0$, as we have marked the direction of the LL with the dotted lines.]

A. Second Quantization

Let us now set our problem in the Fock space, which is very useful to characterize many-particle systems. Consider a boson-type system of N components and M single-particle states. The Fock-basis comes defined by:

$$|\alpha\rangle \equiv |n_1, n_2, \dots, n_L\rangle = \frac{\left(a_1^\dagger\right)^{n_1}\left(a_2^\dagger\right)^{n_2}\dots\left(a_L^\dagger\right)^{n_L}|0\rangle}{\sqrt{n_1!n_2!\dots n_L!}}\,, \tag{16}$$

n_i representing the number of particles populating the state ϕ_i. Notice that for axisymmetric Hamiltonians, as in our case, the total angular momentum $L = \sum_i^N \ell_{z,i}$ is a conserved quantity, hence it fixes the possible configurations in the Fock space.

Within this space, the creation and annihilation operators act in the following way:

$$\begin{aligned} a_k^\dagger|n_1,\dots,n_M\rangle &= \sqrt{n_k+1}\,|n_1,\dots,n_k+1,\dots,n_M\rangle \\ a_k|n_1,\dots,n_M\rangle &= \sqrt{n_k}\,|n_1,\dots,n_k-1,\dots,n_M\rangle\,, \end{aligned} \tag{17}$$

satisfying the commutation relation for bosons, $[a_k, a_l^\dagger] = \delta_{k,l}$.

Thus, in second quantization the Hamiltonian can be written as:

$$\mathcal{H} = \sum_{k,l}^{M} H_{\rm sp}^{kl} a_k^\dagger a_l + \frac{g}{2}\sum_{k,l,p,q}^{M} I_{k,l,p,q}\; a_k^\dagger a_l^\dagger a_p a_q\,, \tag{18}$$

where the single-particle part is:

$$H_{\rm sp}^{kl} = \int dz_1 dz_2\; \psi_k^*(z_1)\; H_{\rm sp}\; \psi_l(z_2) = \epsilon_k \delta_{k,l}\,, \tag{19}$$

while the interaction coefficient reads:

$$\begin{aligned} I_{k,l,p,q} &= \int dz_1 dz_2 \psi_k^*(z_1)\psi_l^*(z_2)\delta(z_1 - z_2)\psi_p(z_1)\psi_q(z_2) \\ &= \frac{(k+l)!}{2^{k+l+1}\pi a_\perp^2\sqrt{k!l!p!q!}}\,\delta_{k+l-p-q,0}\,. \end{aligned} \tag{20}$$

B. The Laughlin Wave-Function

The Laughlin state for bosons reads [1, 2, 6, 7]:

$$\Psi_{\mathcal{L}} = \bar{\mathcal{N}}\prod_{i<j}(z_i - z_j)^2 e^{-\sum |z_i|^2/2a_\perp^2}\,, \tag{21}$$

where $\bar{\mathcal{N}}$ is a normalization constant, which is generally difficult to compute. Considering only the expression $\prod_{i<j=1}^N (z_i - z_j)^2$, it can be verified that it presents monomials of power $N(N-1)$, that is, it describes a state for $L = N(N-1)$ [1, 2]. The zeros in this expression indicate that there is a quantum vortex originating in the position z_j leading to a spontaneous breaking of the symmetry, and to a failure of the mean-field theory which considers all the particles occupying the same single-particle state. Another way of viewing the Laughlin state is that it is a state with appropriate quantum correlations such that the atoms completely avoid the contact interaction [2, 6, 8].

IV. RESULTS

In this section, we analyze the behaviour of the system when undergoing rotation with different angular velocity. The procedure is the following: first, we fix the number of particles, N and work in subspaces of well defined total angular momentum L. After that, we construct the Fock-basis and express in it the full Hamiltonian, interactions included. This Hamiltonian is then diagonalized to obtain its eigenenergies and eigenstates. In Section IV A we discuss the properties of the spectrum and in Section IV B we concentrate on the density and pair-correlation of the ground state of the system.

A. Energy spectrum

For given N, L, the many-body Hamiltonian operator obeys $\mathcal{H}|\varphi\rangle = \mathcal{E}_\varphi|\varphi\rangle$, where the eigenvectors are expressed in the Fock-basis, $|\varphi\rangle = \sum_\alpha^D C_{\alpha,\varphi}|\alpha\rangle$, with $D = \frac{L+N+1)!}{(M-1)!N!}$, the dimension of the Fock-space, and $\mathcal{E}_\varphi$, the eigenenergies. One may pay attention to the fact that, due to the fixed total angular momentum, the single-particle contribution has the same value for each state, whereas the interaction term variates from state to state.

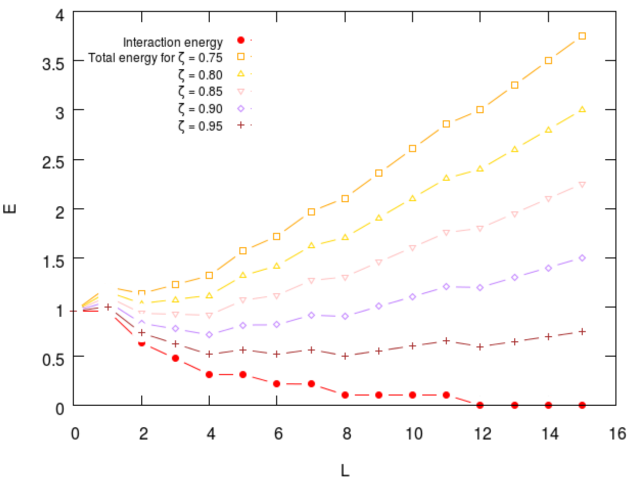

FIG. 2: Lowest energy state for each fixed L sector of a four-particle-system, for different rotational frequencies, with the energy expressed in $\hbar\omega$ units.

We are interested in characterizing the ground state, $|0\rangle = \sum_\alpha^D C_{\alpha,0}|\alpha\rangle$. We fix the interaction strength to $g = 1$ and the number of particles to $N = 4$ for all the subsequent calculations. In Fig. 2 we have represented the evolution of the interaction energy and the total energy of the system as a function of the total angular momentum, for different rotational frequencies. As the figure shows, the ground state of the system is found for different values of L as we increase the rotation frequency. Actually, for values of $\zeta \equiv \frac{\Omega}{\omega} \gtrsim 0.97$ the ground state corresponds to $L = 12$, see Fig. 3, which eventually has zero interaction energy, as seen in Fig. 2. This is the Laughlin state for $N = 4$ discussed above.

Fig. 3 depicts the energy difference between the ground state and the first excited state. For certain values of the rotation, ζ we observe crossings, which correspond to transitions at the ground state level. For instance for $\zeta \simeq 0.84$ the ground state goes from fully condensed with $L = 0$ to a one vortex state $L = 4$. Increasing ζ we find two more transitions at $\zeta \simeq 0.95$ and $\zeta \simeq 0.97$. The latter is the transition to the Laughlin region mentioned above.

B. Density and pair correlation

In second quantization the density could be written as:

$$\rho(r,\varphi) = \sum_{\alpha\beta}^{D}\sum_{kl}^{M} \psi_k^*(r,\varphi)\psi_l^*(r,\varphi)C^*_{\alpha,0}C_{\beta,0}\langle\alpha|a_k^\dagger a_l|\beta\rangle\,, \tag{22}$$

and the pair-correlation is:

$$\eta(r,\varphi,r',\varphi') = \sum_{\alpha\beta}^{D}\sum_{klpq}^{M} \psi_k^*(r,\varphi)\psi_l^*(r',\varphi')\psi_p(r,\varphi) \times \psi_q(r',\varphi')C^*_{\alpha,0}C_{\beta,0}\langle\alpha|a_k^\dagger a_l^\dagger a_p a_q|\beta\rangle\,, \tag{23}$$

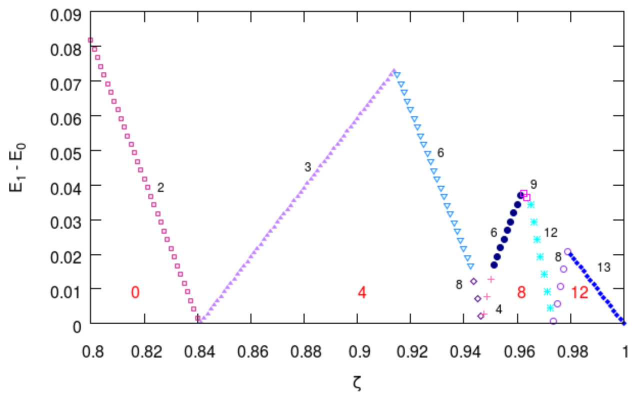

FIG. 3: Evolution of the energy difference between the ground state, with its angular momentum indicated by the red numbers, and the first excited state, with its angular momentum denoted by the smaller black numbers, in terms of the rotational frequency.

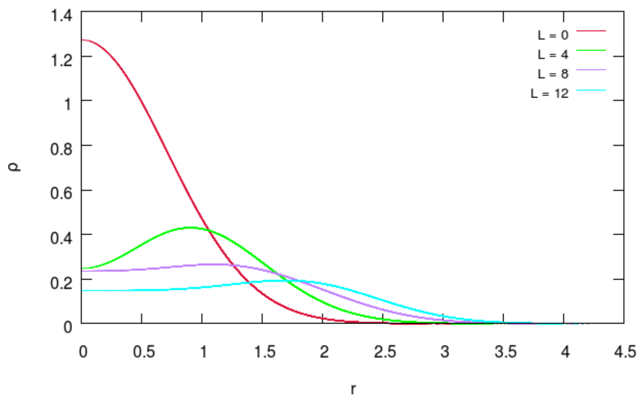

FIG. 4: Density as function of the radius for $N = 4$ and $\zeta = 0.8$ ($L = 0$), $\zeta = 0.9$ ($L = 4$), $\zeta = 0.96$ ($L = 8$) and $\zeta = 0.98$ ($L = 12$). The $L = 12$ corresponds to the Laughlin state of Eq. (21).

both with their proper normalizations, $\int_0^\infty\int_0^{2\pi}\eta(r,\varphi,r',\varphi')rdrr'dr'd\varphi d\varphi' = N(N-1)$ and $\int_0^\infty\int_0^{2\pi}\rho(r,\varphi)rdrd\varphi = N$.

We have presented the density profiles as function of the radius for the ground states in Fig. 4. For $L = 0$ we observe a typical gaussian curve for a bosonic system, nevertheless, for $L = 4, 8$ and 12 the central peak disappears and the function becomes lower and wider. Note that for the Laughlin $L = 12$ the central region is quite flat, which is very characteristic of this state.

Fig. 5 displays the pair-correlation for the angular momentum $L = 0, 6$ and 12. We can see that for $L = 0$ the behaviour is the expected for bosons, as the particles are allowed to occupy the same state, but as the angular momentum increases, the atoms start to build correlations, causing the symmetry to vanish, as can be seen for $L = 6$. Finally, at the Laughlin state, $L = 12$, the probability of finding a second atom at the position of the first one is

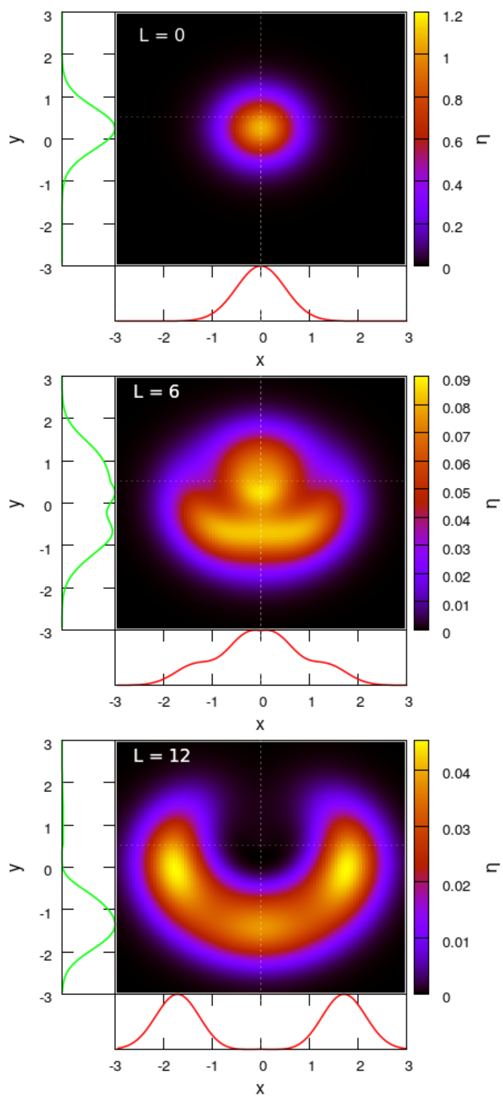

FIG. 5: Colormap of the pair-correlation $\eta(x, y, x', y')$, with one of the particles fixed at $(x', y') = (0, 0.5)$. The upper, middle and lower panels correspond to the lowest energy states for $L = 0, 6$ and 12, respectively. The two side profiles represent the curve at the fixed position $x = 0$ (left) and $y = 0.5$ (down).

zero. This reminds of the Pauli exclusion principle for fermions: for large interaction the bosonic system builds correlations to avoid the contact.

V. SUMMARY AND CONCLUSIONS

We have presented a study of the many-body properties of an ultracold atomic gas under rotation, which can be viewed as a gas of neutral atoms subjected to an artificial magnetic field. Although the computer program was developed for an arbitrary number of particles, we have focused on $N = 4$. The conclusions of this work are, as it follows:

- The ground state of the system has been found to change completely as the angular frequency is increased. In particular, the ground state goes from $L = 0$ to $L = 4$, $L = 8$ and finally $L = 12$ with large rotations. We have seen how the Laughlin state, $L = 12$, has zero interaction energy. In a sense it reminds of the behavior of fermions, which also do not feel the contact interaction.

- An important feature of the Laughlin state is that is is a strongly correlated state. This reflects for instance in clear symmetry breaking patterns in the pair correlations of the state.

- Ultracold quantum gases under rotation are good candidates to look for correlated states such as the ones proposed by Laughlin to understand the fractional quantum Hall effect.

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